A Theory of Demand for Products Distinguished by Place of Production

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I. Introduction and Summary

INTERNATIONAL TRADE flows are commonly identified and classified on the basis of three characteristics: the kind of merchandise involved, the country (or region) of the seller, and the country (or region) of the buyer. In theories of demand for tradable goods, it is frequently assumed that merchandise of a given kind supplied by sellers in one country is a perfect substitute for merchandise of the same kind supplied by any other country. This assumption implies—leaving aside any factors that lead buyers to spend more for a given item than necessary—that elasticities of substitution between these supplies are infinite and that the corresponding price ratios are constants. While the importance of lags in buyers' responses, and other such "imperfections" in buyers' behavior, need not be overlooked, an appeal to them as the sole basis for changes in relative prices of directly competing merchandise would appear to be neither realistic nor attractive theoretically. A preferable approach would be to recognize explicitly that any world model of feasible dimensions would identify few, if any, kinds of merchandise for which the perfect-substitutability assumption is tenable.

Accordingly, this paper presents a general theory of demand for products that are distinguished not only by their kind—e.g., machinery, chemicals—but also by their place of production. Thus French machinery, Japanese machinery, French chemicals, and Japanese chemicals might be 4 different products distinguished in the model. Such products are distinguished from one another in the sense that they are assumed to be imperfect substitutes in demand. Not only is each good, such as chemicals, different from any other good but also each good is assumed to be differentiated (from the buyers' viewpoint) according to the suppliers' area of residence. If the model distinguished 10 goods and

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20 supplying areas, the number of products distinguished in the model would be 200.[1]

The geographic areas that serve as a basis for distinguishing products by origin are also used as a basis for identifying different sources of demand. For example, if France is identified as 1 of the 20 supplying areas, the model would contain a function expressing French demand for each of the 200 products. There would be 10 French demands for domestic products and 190 French import demands. Conversely, for each French product, there would be 1 domestic demand and 19 export demands.

The problem confronted in this paper is that of systematically simplifying the product demand functions to the point where they are relevant to the practical purposes of estimation and forecasting.[2] Starting with the general Hicksian model, the exposition runs through a sequence of progressively more restrictive assumptions leading to a specification of the product demand functions which, though highly simplified, preserves the relationships between demand, income, and prices that are apt to be quantitatively significant.

The fundamental modification of the basic Hicksian model is the assumption of independence—an assumption whose implications have already been explored in other branches of demand theory and in capital theory.[3] In its present application, the assumption of independence states, roughly, that buyers' preferences for different products of any given kind (e.g., French chemicals, Japanese chemicals) are independent of their purchases of products of any other kind.[4] By this assumption, for example, an increase in purchases of French machinery does not change the buyers' relative evaluation of French chemicals and Japanese chemicals. Given the assumption of independence, the quantity of each good demanded by each country (e.g., French demand

[1] In the second paragraph of Section I and throughout the rest of the paper, a distinction is made between "goods" and "products." "Goods" are distinguished only by kind (that is, by the kinds of wants or needs they serve), whereas "products" are distinguished both by kind and by place of production. The geographic and commodity dimensions of the model are spelled out more formally in Section II.

[2] See Section VI, pages 170–71.

[3] Seminal contributions were made by Robert M. Solow, "The Production Function and the Theory of Capital," *The Review of Economic Studies*, Vol. XXIII (1955–56), pp. 101–108, and by R. H. Strotz, "The Empirical Implications of a Utility Tree," *Econometrica*, Vol. 25 (April 1957). A fuller development is given by I. F. Pearce, Chapters 4 and 5, in his *A Contribution to Demand Analysis* (Oxford University Press, 1964), pp. 133–230. H.A.J. Green provides a good review of the relevant literature in his book, *Aggregation in Economic Analysis* (Princeton University Press, 1964).

[4] The assumption of independence is stated in more precise terms in Section III, page 164.

for chemicals-in-general) can in principle be measured unambiguously.[5] In other words, there exist demands for *groups* of competing products. Following conventional terminology, each such demand can be called a market. There would be, for example, the French market for chemicals; and chemicals supplied by different countries or areas (including France, of course) could be said to compete in that market. Moreover, demand for any particular product (e.g., French demand for Japanese chemicals) can be rigorously expressed as a function of the size of the corresponding market (e.g., French demand for chemicals-in-general) and of relative prices of the competing products.

It is next assumed that each country's market share is unaffected by changes in the size of the market as long as relative prices in that market remain unchanged. On this additional assumption, the size of the market is a function of money income and of the prices of the various goods (e.g., the price of chemicals-in-general, the price of machinery-in-general).[6] Combining this function with the product demand function described above, the demand for any product becomes a function of money income, the price of each good, and the price of that product relative to prices of other products in the same market. (Prices of products competing in other markets are influential only insofar as they determine the prices of goods.)

If there are a large number of products competing in the market (in other words, if the number of supplying areas identified in the model is large), then further ways of simplifying the product demand functions are needed if they are to be of much relevance to practical research. The approach suggested in this study (Section IV) is to assume that (a) elasticities of substitution between products competing in any market are constant—that is, they do not depend on market shares, and (b) the elasticity of substitution between any two products competing in a market is the same as that between any other pair of products competing in the same market. These assumptions yield a specific form for the relation between demand for a product, the size of the corresponding market, and relative prices; and the only price parameter in this function is the (single) elasticity of substitution in that market.

Differentiation of the demand functions yields an analysis of changes in demand for any given product (Section V). The percentage change in demand for any product depends additively on the growth of the market in which it competes and on the percentage change in the product's share in that market. The change in the product's market

[5] See page 164.
[6] See pages 165–66.

share will depend in a specific way on the change in the product's price relative to the average change in prices of products in the market. The growth of the market will depend mainly on the change in income and on the income elasticity of demand for the respective good (i.e., the class of products of which the given product is a member). In order that market growth depend exclusively on this income effect (and not on the prices of goods), certain additional assumptions are needed (see p. 170). The study concludes with a brief discussion of the relevance of demand theory to some of the current research in the area of trade analysis and forecasting.

II. Geographic and Commodity Dimensions of the Model

Any large model of the world economy would make use of some vector of countries or other geographic areas, $C = (C_1, C_2, \ldots, C_m)$, as well as some vector of goods, $X = (X_1, X_2, \ldots, X_n)$. The present demand model stipulates, in addition, that each good is differentiated in use (e.g., in demand) according to where it is produced. Any "good," X_i, refers to a group of "products," each supplied by a different country or area; that is, $X_i = (X_{i1}, X_{i2}, \ldots, X_{im})$, where X_{ij} is assumed to be an imperfect substitute for $X_{ik} (j \neq k)$ from the viewpoint of buyers in any country or area, C_i. For later reference, it will be useful to set out these specifications in the following format:

$$\begin{aligned} X = {} & (X_{11}, X_{12}, \ldots, X_{1m}, X_{21}, X_{22}, \\ & \ldots, X_{2m}, \ldots, X_{n1}, X_{n2}, \ldots, X_{nm}) \\ \equiv {} & (X_1, X_2, \ldots, X_n), \text{ where} \\ X_i \equiv {} & (X_{i1}, X_{i2}, \ldots, X_{im}), \text{ for } i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

The top line of this formulation, which may be called the product vector, can also be presented as a product matrix:

$$\begin{array}{c|cccc} \text{goods} \downarrow \diagdown & \multicolumn{4}{l}{\text{supplying countries} \longrightarrow} \\ & 1 & 2 & \ldots\ldots\ldots & m \\ \hline 1 & X_{11} & X_{12} & \ldots\ldots\ldots & X_{1m} \\ 2 & X_{21} & X_{22} & \ldots\ldots\ldots & X_{2m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ n & X_{n1} & X_{n2} & \ldots\ldots\ldots & X_{nm} \,. \end{array}$$

The country vector $C=(C_1,C_2, \ldots ,C_m)$ also lists the various sources of demand. The demand of buyers in any country, C_i, for any product, X_{ij} is here called a product demand, and the demand side of the model is described by all such functions. Since there are m demands for each product, and since there are mn products, the demand side of the model is comprised of m^2n product demands, of which mn are domestic demands and $mn(m-1)$ are export (or import) demands. (Import demands are not residual demands, depending on domestic supply functions, as is the case in models which presume that goods are homogeneous over different sources of supply. In the present model, the analysis of *ex ante* demand—domestic, import, and export—requires no particular assumptions about supply functions.)

III. Market Demands and Product Demands: An Application of the Assumption of Independence

Ex ante demand functions state relationships that must exist among certain variables if buyers are to be satisfied. Buyers' satisfaction entails getting the most for their money, given the available selection of products and their prices. Demand functions may thus be viewed as statements of conditions under which an index of buyers' satisfaction is as high as limited incomes and given prices permit. Given such an index, U, these conditions or demand functions can be derived by maximizing U subject to a budget constraint.[7]

The general approach to the derivation of the product demand functions identified in the previous section is to express U as a function of all mn products; that is, using (1), $U=U(X)$.[8] Then, given a corresponding price vector,

$$P=P_{11},P_{12}, \ldots ,P_{1m},P_{21},P_{22}, \ldots ,P_{2m}, \ldots ,P_{n1},P_{n2}, \ldots ,P_{nm}, \quad (2)$$

and national money expenditure, D, $U(X)$ is maximized subject to the budget constraint $D=PX'$. Once U is specified, the first-order condi-

[7] Each of the four simplifying assumptions introduced in Section I can be interpreted as a certain restriction on, or specification of, the index U. Thus, by using the maximization procedure to derive the product demand functions, a formal link is established between general demand functions and the highly simplified relationships mentioned in Section I.

[8] The rest of this paper deals with the demand of any single country and does not consider the aggregation of demands across markets. Hence, no notation is attached to U, or to other variables mentioned later, to identify the country whose demand is referred to.

tions, together with the budget constraint, imply this country's *mn* demand functions, each one having the general form

$$X_{ij}=X_{ij}(D,P_{11},P_{12}, \ldots ,P_{1m},P_{21},P_{22}, \ldots ,$$
$$P_{2m}, \ldots ,P_{n1},P_{n2}, \ldots ,P_{nm}), \tag{3}$$

for all i and all j.

Of course, the close association between products of the same kind is not reflected at all in the general form of (3). The problem at hand is to specify U in such a way that the information implicit in the product classification scheme is fully utilized, to the end that the product demand functions may be appropriately simplified. The first and most fundamental step is to specify U in such a way that the demand for any good, X_i, can be measured unambiguously.

Professor Solow asked the question (albeit in a rather different context),[9] Under what condition can U be "collapsed" in the following way?

$$\begin{aligned} U= U(X_{11},X_{12}, \ldots ,X_{1m},X_{21},X_{22}, \\ \ldots ,X_{2m}, \ldots ,X_{n1},X_{n2}, \ldots ,X_{nm}) \\ \equiv U'(X_1,X_2, \ldots ,X_n), \text{ where} \\ X_i \equiv \phi_i(X_{i1},X_{i2}, \ldots ,X_{im}), \text{ for } i=1,2, \ldots ,n.^{10} \end{aligned} \tag{4}$$

If U can be so collapsed, all combinations of $X_{i1},X_{i2}, \ldots ,X_{im}$ which yield any given value of X_i are equally good, and that given value is a specific quantity of X_i-in-general. In other words, if (4) is true, demands for goods—here called market demands—can be measured unambiguously. The necessary and sufficient condition for collapsing U is that marginal rates of substitution between any two products of the *same* kind must be independent of the quantities of the products of all *other* kinds.[11] In other words, buyers' relative evaluation (at the margin) of different products competing in a given market must not be affected by their purchases in other markets. This is the assumption of independence.[12]

[9] See Solow, *op. cit.*

[10] Compare with equation (1).

[11] The proof is in Solow, *op. cit.*, and in an early work by Leontief referred to in the Solow article.

[12] In theory, the assumption of independence might be viewed as tautological; for independence could well be taken as a *defining* characteristic of products distinguished by their kind (that is, by the kind of want or need they serve). Following this approach, an alternative, more basic assumption would be necessary, namely, that products can be rigorously classified by kind in the first place. In practice, however, goods must be identified within the framework of some available classification scheme (such as the Standard International Trade Classification). Given this constraint, independence is not necessarily tautological. How far the

Given the assumption of independence, which leads to (4), demand for any particular product X_{ij} can be written as a function of X_i and of relative product prices in the i^{th} market.[13] If the demand for X_i can in turn be related to income and appropriate price variables, then a manageable specification of (3) is within reach.

The price variables on which X_i will depend are, naturally, the prices of goods, or P_i $(i=1,2, \ldots, n)$, and P_i is a function of prices of products in the i^{th} market, just as X_i is a function of quantities of these products. But P_i cannot be just any function of product prices; the prices of goods must be such that the demand for the i^{th} good, which they explain, is consistent with the optimum selection of products in the i^{th} market. More exactly, the demand for X_i as determined by income and prices of goods must be the same as the value of ϕ_i implied by all demands for products in the i^{th} market as determined by direct reference to income and product prices.

This condition is fulfilled if

$$P_i = P_{i1} \div \frac{\partial \phi_i}{\partial X_{i1}} = P_{i2} \div \frac{\partial \phi_i}{\partial X_{i2}} = \ldots = P_{im} \div \frac{\partial \phi_i}{\partial X_{im}}, \quad \text{for } i=1,2, \ldots, n.^{14} \tag{5}$$

Note that (5) implies that

$$\frac{P_{i1}}{P_{i2}} = \frac{\dfrac{\partial \phi_i}{\partial X_{i1}}}{\dfrac{\partial \phi_i}{\partial X_{i2}}}, \text{ and } \frac{P_{i2}}{P_{i3}} = \frac{\dfrac{\partial \phi_i}{\partial X_{i2}}}{\dfrac{\partial \phi_i}{\partial X_{i3}}}, \text{ etc.,}$$

which are the first-order conditions for optimum mix of products in the i^{th} market.

It is not clear from inspection of (5) that P_i depends only on product prices. In fact, to ensure that P_i is independent of X_i, it must be assumed that ϕ_i is linear and homogeneous. Then the partial derivatives in (5) depend only on ratios of quantities of products demanded in the i^{th}

assumption restricts the realism of the model depends on the extent to which the different items in the goods vector correspond to more or less discrete wants or needs. Within the limitation imposed by the available classification scheme, the analyst may attempt to select a vector of goods that renders the independence assumption as realistic as possible. It should be noted, however, that supply factors may suggest to him a rather different choice of goods. The identification of goods, for purposes of a complete model, cannot be guided in practice by any single criterion.

[13] The function may be derived from the conditions for minimizing the money cost of purchasing any given volume of X_i, given some specification of ϕ_i. The derivation is analogous to that of the demand for a factor of production as a function of output and relative factor prices.

[14] See Solow, *op. cit.*

market, and these ratios in turn depend only on ratios of the product prices; hence, P_i is only a function of $P_{i1}, P_{i2}, \ldots, P_{im}$. The assumption that the quantity index functions are linear and homogeneous is the second restriction (the first being the assumption of independence) that has been placed on U. This second restriction means that market shares must depend only on relative prices of the products in the market; shares must not depend on the size of the market itself.

An important property of P_i as given in (5) is that $P_i X_i = \sum_{k=1}^{m} P_{ik} X_{ik} =$ money expenditure in the i^{th} market.[15] This fact leads directly to the budget constraint,

$$D \equiv \sum_{i=1}^{n} \sum_{k=1}^{m} P_{ik} X_{ik} = \sum_{i=1}^{n} P_i X_i,$$

which, along with U' in (4), determines the demand for X_i. To summarize, the demand for any good, X_i, can be obtained by maximizing $U'(X_1, X_2, \ldots, X_n)$ subject to the constraint $D = \sum_{i=1}^{n} P_i X_i$. Then, the demand for any product, X_{ij}, can be obtained by minimizing the cost of purchasing the volume of X_i just determined; that is, the expression $\sum_{k=1}^{m} P_{ik} X_{ik}$ is minimized subject to the constraint $X_i = \phi_i(X_{i1}, X_{i2}, \ldots, X_{im})$. The resulting demand functions are

$$X_i = X_i(D, P_1, P_2, \ldots, P_n), \text{ where } X_i \text{ is any good, and} \tag{6}$$

$$X_{ij} = X_{ij}\left(X_i, \frac{P_{ij}}{P_{i1}}, \frac{P_{ij}}{P_{i2}}, \ldots, \frac{P_{ij}}{P_{im}}\right), \text{ where } X_{ij} \text{ is any product.} \tag{7}$$

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[15] Equation (5) implies that

$$P_{i1} X_{i1} = P_i \frac{\partial \phi_i}{\partial X_{i1}} X_{i1};$$

$$P_{i2} X_{i2} = P_i \frac{\partial \phi_i}{\partial X_{i2}} X_{i2};$$

$$\vdots \qquad \vdots$$

$$P_{im} X_{im} = P_i \frac{\partial \phi_i}{\partial X_{im}} X_{im}.$$

Therefore,

$$\sum_{k=1}^{m} P_{ik} X_{ik} = P_i \sum_{k=1}^{m} \frac{\partial \phi_i}{\partial X_{ik}} X_{ik} = P_i X_i,$$

using the assumption that ϕ_i is linear and homogeneous, together with Euler's theorem.

By substituting (6) into (7), a particular reformulation of (3) is obtained. In this reformulation the role of prices is handled by only $m+n$ variables, compared with mn variables in (3): product prices outside the i^{th} market affect X_{ij} only insofar as they determine price levels in other markets (that is, $P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$). This, specifically, is the sense in which the two restrictions on U serve to utilize the information implicit in (1). And these restrictions—both very mild—open the door to more powerful simplifying assumptions, such as those discussed below.

IV. Product Demand Functions Assuming a Single, Constant Elasticity of Substitution in Each Market

If many countries or areas were identified in the model, equations (7), in the above form, would probably be too complicated to be of practical use. A way to simplify them is to introduce the assumptions that (a) elasticities of substitution in each market are constant and (b) the elasticity of substitution between any two products competing in a market is the same as that between any other pair of products competing in the same market. In terms of the index U (equation 4), these assumptions are equivalent to the specification that the ϕ_i's are constant-elasticity-of-substitution (CES) functions, having the general form

$$X_i \equiv \phi_i(X_{i1}, X_{i2}, \ldots, X_{im}) = \left[b_{i1}X_{i1}^{-\rho_i} + b_{i2}X_{i2}^{-\rho_i} + \ldots + b_{im}X_{im}^{-\rho_i}\right]^{-\frac{1}{\rho_i}}.\text{[16]} \quad (8)$$

Given (8), it can be shown that equations (7) have the form

$$X_{ij} = b_{ij}^{\sigma_i} X_i \left(\frac{P_{ij}}{P_i}\right)^{-\sigma_i}, \quad (9)$$

where σ_i is the elasticity of substitution in the i^{th} market and b_{ij} is a constant.[17]

In equation (9) and in all those that follow, X_{ij} can be interpreted either as the demand for the i^{th} good supplied by country j or as the demand for the i^{th} good supplied by the j^{th} *group* of countries. For example, the j^{th} group could be all foreign countries; in this instance, (9) would express the demand for total imports of the i^{th} good (X_{ij}) as a function of demand for the i^{th} good wherever produced (X_i) and of the ratio of the average import price (P_{ij}) to the average price level in the market (P_i). This flexible interpretation of the variables in (9) is the

[16] This function is linear and homogeneous, as required.
[17] The derivation of (9) from (8) is in Part I of the Appendix.

consequence of two properties of ϕ_i as specified in (8): (a) the marginal rate of substitution between any pair of products competing in the i^{th} market is independent of demand for any other product(s) in that market, so that ϕ_i in (8) can be "collapsed" in the manner of equation (4); (b) ϕ_i in (8) is linear and homogeneous, so that the index functions appearing in any "collapsed" form of (8) must be linear and homogeneous also. Hence there exists an unambiguous demand for any subset of products in the i^{th} market, and this demand can be related to the over-all market demand just as X_i, under strictly analogous conditions, can be related to total income.

Equation (9) can be written in a variety of useful ways; for example,

$$P_{ij}X_{ij}=b_{ij}^{\sigma_i}(P_iX_i)\left(\frac{P_{ij}}{P_i}\right)^{1-\sigma_i}, \tag{10}$$

which relates money demand for X_{ij} to the size of the corresponding market measured in value terms, and

$$\frac{X_{ij}}{X_i}=b_{ij}^{\sigma_i}\left(\frac{P_{ij}}{P_i}\right)^{-\sigma_i}, \text{ or} \tag{11}$$

$$\frac{P_{ij}X_{ij}}{P_iX_i}=b_{ij}^{\sigma_i}\left(\frac{P_{ij}}{P_i}\right)^{1-\sigma_i}, \tag{12}$$

which expresses the market share as the dependent variable. As is clear from (12), value shares are constant if $\sigma_i=1$. (In this special case, ϕ_i is a Cobb-Douglas function with parameters b_{ik}.) If $\sigma_i>1$, a relative fall (or increase) in P_{ij} yields an increase (or decrease) in the market share of X_{ij}. The reverse is true if $\sigma_i<1$. Ordinarily, it would be expected that σ_i exceeds unity: an "improvement in competitiveness" should yield an increased share, and vice versa. But the logic of the model (as distinct from some particular interpretation of the variables) places no such restriction on this elasticity.

V. Analysis of Changes [18]

Total differentiation of the market demand function (6) and the product demand function (9) yields the following relation between the percentage change in demand for X_{ij}, in value terms, and percentage changes in income and price variables:

[18] The argument of this section is more fully developed in Part II of the Appendix.

$$\frac{d(P_{ij}X_{ij})}{P_{ij}X_{ij}} = \epsilon_i \frac{dD}{D} - (\eta_i - 1)\frac{dP_i}{P_i} + \sum_{k \neq i} \eta_{i/k} \frac{dP_k}{P_k} - (\sigma_i - 1)\left(\frac{dP_{ij}}{P_{ij}} - \frac{dP_i}{P_i}\right), \quad (13)$$

[19]

where ϵ_i is the income elasticity of demand for X_i, η_i is the direct price elasticity of demand for X_i, and $\eta_{i/k}$ is the cross elasticity of demand for X_i with respect to P_k $(k=1,2, \ldots, i-1, i+1, \ldots, n)$. The first three terms together measure the growth of the market (in value) for X_{ij}, while the fourth term measures the percentage change in X_{ij}'s share of the market.

The change in demand for X_{ij} can also be analyzed in more traditional terms by replacing the price level in the i^{th} market with product prices and market shares. As is demonstrated in the Appendix,[20] one of the properties of P_i is that the percentage change in this variable is equal to an average of changes in component product prices, weighted by market shares; specifically,

$$\frac{dP_i}{P_i} = \sum_{k=1}^{m} S_{ik} \frac{dP_{ik}}{P_{ik}}, \text{ where } S_{ik} = \frac{P_{ik}X_{ik}}{P_iX_i}.$$

Substituting this summation for $\frac{dP_i}{P_i}$ in the second and fourth terms of (13), one obtains (after a little shuffling of terms) the following result:

$$\frac{d(P_{ij}X_{ij})}{P_{ij}X_{ij}} = \epsilon_i \frac{dD}{D} - \Big[(1 - S_{ij})(\sigma_i - 1) + S_{ij}(\eta_i - 1)\Big]\frac{dP_{ij}}{P_{ij}} + \sum_{k \neq j}\Big[S_{ik}(\sigma_i - 1) - S_{ik}(\eta_i - 1)\Big]\frac{dP_{ik}}{P_{ik}} + \sum_{k \neq i} \eta_{i/k} \frac{dP_k}{P_k}. \quad (14)$$

Here, the growth of demand for X_{ij} is divided into the following components: an income effect (first term), an "own price" effect (second term), the effect of prices of closely related products (third term), and the effect of all other prices (fourth term). The bracketed coefficient of $\frac{dP_{ij}}{P_{ij}}$ is the direct price elasticity of demand for X_{ij}, in value terms, while the bracketed coefficient of $\frac{dP_{ik}}{P_{ik}}$ represents the cross elasticity of demand for X_{ij} with respect to the price of any other product competing in the

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[19] $\sum_{k \neq i}$ indicates summation over all k, for $k = 1,2, \ldots, i-1, i+1, \ldots, n$.

[20] See page 174, footnote 29.

same market. Conversely, of course, the cross elasticity of demand for X_{ik} with respect to P_{ij} would be given by $[S_{ij}(\sigma_i - 1) - S_{ij}(\eta_i - 1)]$.

The analysis of changes into market-expansion factors and share-adjustment factors (equation 13) is of greater relevance to current research than the more conventional breakdown shown in (14)—see Section VI. On the other hand, (14) is useful because it focuses attention on how changes in individual product prices affect trade, and in particular on the role played by market shares.[21]

In the event that equations (13) or (14) are yet too complicated to suit practical purposes, the feasibility of introducing two further simplifying assumptions may be considered. The first of these is that the elasticity of demand for X_i (η_i) equals unity, and the second is that the third term of (13), or the fourth term of (14), is small enough to be ignored. The first of these assumptions, which implies that expenditure on the i^{th} class of products is independent of price changes in the i^{th} market, focuses attention on the effects of changes in *relative* prices in the market and abstracts from any (presumably small) effects of changes in the general *level* of prices in the market. The second assumption would not be unreasonable if changes in price levels in other markets are very small, or if such changes are apt to have offsetting effects on demand for X_i.[22] On these additional assumptions, (13) and (14) reduce to the relatively simple formula

$$\frac{d(P_{ij}X_{ij})}{P_{ij}X_{ij}} = \epsilon_i \frac{dD}{D} - (\sigma_i - 1)\left(\frac{dP_{ij}}{P_{ij}} - \frac{dP_i}{P_i}\right). \tag{15}$$

VI. Conclusion

In much of the current research in the area of trade analysis and forecasting, the change in any particular trade flow is viewed as the sum

[21] See pages 174–75 of the Appendix. This role of market shares is examined in detail by the author in "The Geographic Pattern of Trade and the Effects of Price Changes," which will be published in a subsequent issue of *Staff Papers*. The partial elasticities in (14) formally resemble the formulas derived by Hicks and Allen. (See J. R. Hicks and R. G. D. Allen, "A Reconsideration of the Theory of Value: Part II.—A Mathematical Theory of Individual Demand Functions," *Economica*, New Series, Vol. I (1934), pp. 201–202 and 208–11.) Similar formulas have been derived and applied in the context of international trade by Professor P. J. Verdoorn. For his early work in this area, see Annex A of his contribution to the annual papers of the Dutch Economic Society, 1952, and also Appendix A of "The Intra-bloc Trade of Benelux," in *Economic Consequences of the Size of Nations*, ed. by E. A. G. Robinson (London, 1960), pp. 319–21.

[22] One might be tempted to assume that the cross elasticities, $\eta_{i/k}$, are zero. However, if $\eta_{i/k} = 0$ and $\eta_i = 1$, then ϵ_i must equal unity, since, by definition, $\eta_i = \sum_{k \neq i} \eta_{i/k} + \epsilon_i$. See Paul Anthony Samuelson, *Foundations of Economic Analysis* (Harvard University Press, 1961), p. 105. Ordinarily, it would be intolerably restrictive to assume that the income elasticity is perforce equal to 1.

of two components: the change that would occur if the given seller country were to maintain its share in the market (that is, its share of total sales to the given buyer country in the given commodity class), and the deviation of actual sales from constant-shares sales. Within this framework, forecasting of trade is essentially a two-step process in which (a) forecasts of growth in the various markets, together with a base-period matrix, yield a constant-shares matrix for the projection period, and (b) this constant-shares matrix is modified to take account of factors expected to yield gains or losses in shares. In retrospective analysis, the role of these factors is evaluated by comparing actual sales in particular markets—or, more frequently, groups of markets—with constant-shares sales. This general approach to trade analysis and forecasting might conveniently be called the modified-shares approach.[23]

Is the breakdown of changes in trade flows into the two components merely a matter of accounting that seems useful for certain purposes but which has no causal significance, no roots in buyers' behavior? Or can the traditional theory of buyers' behavior provide a satisfactory rationalization of the modified-shares approach? Can a few assumptions tie theory and practice together?

This paper shows that the modified-shares approach does not require any radical departures from the traditional theory of buyers' behavior. Starting with the fundamental assumption that products of different countries competing in the same market are imperfect substitutes, the study shows how a powerful and reasonably realistic specialization of the function describing buyers' behavior—the specialization indicated by equations (4) and (8)[24]—leads to quite simple demand relationships embodying the constant-share and share-adjustment components. The modified-shares approach may find foundations on the demand side that are different from those proposed in this paper, including the market imperfections referred to at the outset. But the assumption that products are distinguished by place of production is a very convenient point of departure toward a rigorous theory of market growth and share adjustment.

[23] For an introduction to the research in this area, together with an appraisal of alternative frameworks for the analysis of international trade, see Grant B. Taplin, "Models of World Trade," *Staff Papers,* Vol. XIV (1967), pp. 433–55. Forecasting methods involving the modified-shares approach are currently being developed at the International Monetary Fund and elsewhere. Regarding retrospective analysis, studies of export performance measured by changes in average export shares appear from time to time in recent Fund Annual Reports.

[24] See pages 164 and 167.

MATHEMATICAL APPENDIX

I. Derivation of Product Demands Assuming a Single, Constant Elasticity of Substitution in Each Market

Section IV introduces the simplifying assumption that any given quantity-index function, ϕ_i, has the generalized CES form;[25] that is (to repeat),

$$X_i = \phi_i(X_{i1}, X_{i2}, \ldots, X_{im}) = \left[b_{i1} X_{i1}^{-\rho_i} + b_{i2} X_{i2}^{-\rho_i} + \ldots + b_{im} X_{im}^{-\rho_i} \right]^{-\frac{1}{\rho_i}}, \quad (8)$$

where $\sum_{k=1}^{m} b_{ik} = 1$, and where ρ_i is a constant greater than -1. The demand for any product competing in the i^{th} market, X_{ij}, can then be expressed as a specific function of X_i and of relative prices (see equation 9). The derivation of this function is given below.

If any given quantity of the i^{th} good is to be obtained at least money cost, the following conditions must hold:

$$\frac{\dfrac{\partial \phi_i}{\partial X_{ij}}}{\dfrac{\partial \phi_i}{\partial X_{ik}}} = \frac{b_{ij}}{b_{ik}} \left(\frac{X_{ik}}{X_{ij}} \right)^{1+\rho_i} = \frac{P_{ij}}{P_{ik}}, \qquad k = 1,2, \ldots, m.^{26} \quad (16)$$

That is, marginal rates of substitution between competing products must equal the corresponding ratios of their prices.

Solving (16) for X_{ik},

$$X_{ik} = X_{ij} \left(\frac{b_{ik} P_{ij}}{b_{ij} P_{ik}} \right)^{\frac{1}{1+\rho_i}}, \qquad k = 1,2, \ldots, m. \quad (17)$$

Using this equation, (8) can be expressed as a relation between X_i, X_{ij}, and the prices. Rearrangement of this relation, to show X_{ij} as the dependent variable, yields the desired product demand function. First, however, (17) might conveniently be rewritten in terms of the elasticity of substitution. By rearranging (17),

$$\frac{X_{ij}}{X_{ik}} = \left(\frac{b_{ik}}{b_{ij}} \right)^{-\frac{1}{1+\rho_i}} \left(\frac{P_{ij}}{P_{ik}} \right)^{-\frac{1}{1+\rho_i}}, \qquad k = 1,2, \ldots, m,$$

from which it follows that the elasticity of substitution between X_{ij} and any other product competing in the market is equal to the constant $\dfrac{1}{1+\rho_i}$.[27] To simplify

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[25] The CES form has been used elsewhere in the literature in a somewhat similar framework; see, for example, Harry G. Johnson, "The Costs of Protection and Self-Sufficiency," *The Quarterly Journal of Economics,* Vol. LXXIX (1965), pp. 358–62.

[26] There are, of course, only $m - 1$ nontrivial first-order conditions. To except the trivial case where $j = k$ would unduly complicate the notation.

[27] The elasticity of substitution between X_{ij} and X_{ik} is, by definition,

$$-\frac{\partial \left(\dfrac{X_{ij}}{X_{ik}} \right) \left(\dfrac{P_{ij}}{P_{ik}} \right)}{\partial \left(\dfrac{P_{ij}}{P_{ik}} \right) \left(\dfrac{X_{ij}}{X_{ik}} \right)}.$$

notation, let $\frac{1}{1+\rho_i} \equiv \sigma_i$, the elasticity of substitution in the i^{th} market. Equation (17) then becomes

$$X_{ik} = X_{ij}\left(\frac{b_{ik}P_{ij}}{b_{ij}P_{ik}}\right)^{\sigma_i}, \qquad k = 1,2, \ldots, m, \tag{18}$$

where $0 < \sigma_i < \infty$. (The limiting cases have the following interpretations: if $\sigma_i = 0$, the products are perfect complements; if $\sigma_i = \infty$, the products are perfect substitutes.)

Now, substituting (18) into (8), and writing ρ_i in terms of σ_i

$$X_i = \left\{\sum_{k=1}^{m} b_{ik}\left[X_{ij}\left(\frac{b_{ik}P_{ij}}{b_{ij}P_{ik}}\right)^{\sigma_i}\right]^{\frac{\sigma_i - 1}{\sigma_i}}\right\}^{\frac{\sigma_i}{\sigma_i - 1}} \tag{19}$$

$$= b_{ij}^{-\sigma_i}X_{ij}\left[\sum_{k=1}^{m} b_{ik}^{\sigma_i}\left(\frac{P_{ij}}{P_{ik}}\right)^{\sigma_i - 1}\right]^{\frac{\sigma_i}{\sigma_i - 1}}.$$

Then, solving (19) for X_{ij}

$$X_{ij} = b_{ij}^{\sigma_i}X_i\left[\sum_{k=1}^{m} b_{ik}^{\sigma_i}\left(\frac{P_{ij}}{P_{ik}}\right)^{\sigma_i - 1}\right]^{\frac{\sigma_i}{1-\sigma_i}}. \tag{20}$$

Equation (20) may be viewed as a particular specification of (7).

Simplification of (20) can be effected by relating the complex, relative-price term to the price level in the market, P_i. P_i has a particular form, corresponding to the specification of ϕ_i. Using (5) and (8),

$$P_i = P_{ij} \div \frac{\partial\phi_i}{\partial X_{ij}} = P_{ij}b_{ij}^{-1}X_{ij}^{\frac{1}{\sigma_i}}X_i^{-\frac{1}{\sigma_i}}.$$

Then, substituting from (19)

$$P_i = P_{ij}b_{ij}^{-1}X_{ij}^{\frac{1}{\sigma_i}}b_{ij}X_{ij}^{-\frac{1}{\sigma_i}}\left[\sum_{k=1}^{m} b_{ik}^{\sigma_i}\left(\frac{P_{ij}}{P_{ik}}\right)^{\sigma_i - 1}\right]^{\frac{1}{1-\sigma_i}} \tag{21}$$

$$= P_{ij}\left[\sum_{k=1}^{m} b_{ik}^{\sigma_i}\left(\frac{P_{ij}}{P_{ik}}\right)^{\sigma_i - 1}\right]^{\frac{1}{1-\sigma_i}}.$$

Therefore,

$$\left(\frac{P_{ij}}{P_i}\right)^{-\sigma_i} = \left[\sum_{k=1}^{m} b_{ik}^{\sigma_i}\left(\frac{P_{ij}}{P_{ik}}\right)^{\sigma_i - 1}\right]^{\frac{\sigma_i}{1-\sigma_i}}. \tag{22}$$

Substituting (22) into (20), equation (9) is obtained. To repeat,

$$X_{ij} = b_{ij}^{\sigma_i}X_i\left(\frac{P_{ij}}{P_i}\right)^{-\sigma_i}, \tag{9}$$

where X_{ij} is any country's demand for any product, expressed in volume.

II. Analysis of Changes

Total differentiation of (9) and (6) yields the relationship between changes in X_{ij} and changes in the explanatory variables. Starting with (9),

$$dX_{ij} = \frac{\partial X_{ij}}{\partial X_i} dX_i + \frac{\partial X_{ij}}{\partial P_{ij}} dP_{ij} + \frac{\partial X_{ij}}{\partial P_i} dP_i$$

$$= \frac{\partial X_{ij}}{\partial X_i} dX_i - \sigma_i X_{ij} P_{ij}^{-1} dP_{ij} + \sigma_i X_{ij} P_i^{-1} dP_i.$$

Dividing through by X_{ij},

$$\frac{dX_{ij}}{X_{ij}} = \frac{\partial X_{ij} X_i}{\partial X_i X_{ij}} \frac{dX_i}{X_i} - \sigma_i \frac{dP_{ij}}{P_{ij}} + \sigma_i \frac{dP_i}{P_i} \tag{23}$$

$$= \frac{dX_i}{X_i} - \sigma_i \left(\frac{dP_{ij}}{P_{ij}} - \frac{dP_i}{P_i} \right),$$

noting that the partial elasticity of X_{ij} with respect to X_i equals unity. The first term represents the growth of the market for X_{ij}; the second term represents the percentage change in X_{ij}'s share of the market. The growth of the market can, of course, be analyzed by differentiating (6).

$$\frac{dX_i}{X_i} = \epsilon_i \frac{dD}{D} - \eta_i \frac{dP_i}{P_i} + \sum_{k \neq i} \eta_{i/k} \frac{dP_k}{P_k},^{28} \tag{24}$$

where ϵ_i is the income elasticity of demand for X_i, η_i is the direct price elasticity of demand for X_i, and $\eta_{i/k}$ is the cross elasticity of demand for X_i with respect to P_k ($k = 1,2, \ldots, i-1, i+1, \ldots, n$). And substituting (24) into (23),

$$\frac{dX_{ij}}{X_{ij}} = \epsilon_i \frac{dD}{D} - \eta_i \frac{dP_i}{P_i} + \sum_{k \neq i} \eta_{i/k} \frac{dP_k}{P_k} - \sigma_i \left(\frac{dP_{ij}}{P_{ij}} - \frac{dP_i}{P_i} \right). \tag{25}$$

It can be shown that $\frac{dP_i}{P_i} = \sum_{k=1}^{m} S_{ik} \frac{dP_{ik}}{P_{ik}}$, where $S_{ik} = \frac{P_{ik} X_{ik}}{P_i X_i}$ = the market share of X_{ik} in value terms.[29] Thus the effects on X_{ij} of changes in prices of products

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[28] $\sum_{k \neq i}$ indicates summation over all k, for $k = 1,2, \ldots, i-1, i+1, \ldots, n$.

[29] Since ϕ_i in (8) is linear and homogeneous,

$$X_i = \sum_{k=1}^{m} \frac{\partial \phi_i}{\partial X_{ik}} X_{ik} = \sum_{k=1}^{m} \frac{P_{ik}}{P_i} X_{ik}, \text{ using (5).}$$

Therefore,

$$P_i = \sum_{k=1}^{m} \frac{X_{ik}}{X_i} P_{ik}.$$

Then

$$\frac{dP_i}{P_i} = \sum_{k=1}^{m} \frac{P_{ik} X_{ik}}{P_i X_i} \frac{dP_{ik}}{P_{ik}} + \sum_{k=1}^{m} \frac{P_{ik} X_{ik}}{P_i X_i} \frac{d\left(\frac{X_{ik}}{X_i}\right)}{\frac{X_{ik}}{X_i}}, \text{ and}$$

the second term, a weighted average of percentage changes in market shares, is zero.

competing in the i^{th} market depend not only on σ_i and η_i but also on market shares. This role of market shares can be seen more clearly if the second and fourth terms of (25) are expanded in the following way:

$$-\sigma_i\left(\frac{dP_{ij}}{P_{ij}}-\frac{dP_i}{P_i}\right)-\eta_i\frac{dP_i}{P_i}=-\sigma_i\left(\frac{dP_{ij}}{P_{ij}}-\sum_{k=1}^{m}S_{ik}\frac{dP_{ik}}{P_{ik}}\right)-\eta_i\sum_{k=1}^{m}S_{ik}\frac{dP_{ik}}{P_{ik}}$$

$$=-\sigma_i\left(\frac{dP_{ij}}{P_{ij}}-S_{ij}\frac{dP_{ij}}{P_{ij}}-\sum_{k\neq j}S_{ik}\frac{dP_{ik}}{P_{ik}}\right)-\eta_i\left(S_{ij}\frac{dP_{ij}}{P_{ij}}+\sum_{k\neq j}S_{ik}\frac{dP_{ik}}{P_{ik}}\right)^{80}$$

$$=-(1-S_{ij})\sigma_i\frac{dP_{ij}}{P_{ij}}+\sum_{k\neq j}S_{ik}\sigma_i\frac{dP_{ik}}{P_{ik}}-S_{ij}\eta_i\frac{dP_{ij}}{P_{ij}}-\sum_{k\neq j}S_{ik}\eta_i\frac{dP_{ik}}{P_{ik}}$$

$$=-\left[(1-S_{ij})\sigma_i+S_{ij}\eta_i\right]\frac{dP_{ij}}{P_{ij}}+\sum_{k\neq j}\left[S_{ik}\sigma_i-S_{ik}\eta_i\right]\frac{dP_{ik}}{P_{ik}}.$$

Substituting into (25),

$$\frac{dX_{ij}}{X_{ij}}=\epsilon_i\frac{dD}{D}-\left[(1-S_{ij})\sigma_i+S_{ij}\eta_i\right]\frac{dP_{ij}}{P_{ij}}+\sum_{k\neq j}\left[S_{ik}\sigma_i-S_{ik}\eta_i\right]\frac{dP_{ik}}{P_{ik}}$$
$$+\sum_{k\neq i}\eta_{i/k}\frac{dP_k}{P_k}. \qquad (26)$$

The bracketed coefficient of $\frac{dP_{ij}}{P_{ij}}$ in (26) is the direct, partial elasticity of demand for X_{ij}, and the bracketed coefficient of $\frac{dP_{ik}}{P_{ik}}$ is the cross elasticity of demand for X_{ij} with respect to the price of any other product in the i^{th} class. The direct elasticity of demand for X_{ij} is inversely related to its market share (assuming that $\sigma_i > \eta_i$). Intuitively, the more important is X_{ij} in the market, the smaller will be the percentage gain or loss from the substitution associated with a given change in its price—and the larger the percentage change in demand for all other products in the market. By the same token, the cross elasticity of demand for X_{ij}, with respect to the price of any other product in the same market, is directly related to the market share of that product (again, if $\sigma_i > \eta_i$).

The two bracketed coefficients in (26) each have two terms, reflecting the fact that a change in the price of a product affects demand for that product, or any other product in the same market, in two different ways. First, the price change alters *relative* prices of products in that market, bringing about a substitution effect measured by the first term. Second, the price change alters the price *level* in the market, bringing about a change in the size of the market itself, and this market-expansion effect is measured by the second term. In (25), market-expansion factors and share-adjustment factors are clearly separated, while in (26) they are scrambled in such a way as to focus attention on the effects of particular price changes.

The percentage change in demand for X_{ij} in value terms may be obtained by adding $\frac{dP_{ij}}{P_{ij}}$ to both sides of (25) or (26). After considerable manipulation it is found that this change, $\frac{d(P_{ij}X_{ij})}{P_{ij}X_{ij}}$, is the same as shown in (25) or (26) except that σ_i and η_i are replaced by $(\sigma_i - 1)$ and $(\eta_i - 1)$, respectively.

[80] $\sum_{k\neq j}$ indicates summation over all k, for $k = 1,2, \ldots, j-1, j+1, \ldots, m$.

In some practical applications, it may be feasible to introduce the further simplifying assumptions that $\eta_i = 1$ and that the fourth term of (26), or the third term of (25), is quantitatively insignificant. In this event, the percentage change in demand for X_{ij} in value terms reduces to a relatively simple formula. Starting from (26)

$$\frac{d(P_{ij}X_{ij})}{P_{ij}X_{ij}} = \epsilon_i \frac{dD}{D} - (1 - S_{ij})(\sigma_i - 1)\frac{dP_{ij}}{P_{ij}} + \sum_{k \neq j} S_{ik}(\sigma_i - 1)\frac{dP_{ik}}{P_{ik}}$$

$$= \epsilon_i \frac{dD}{D} - (\sigma_i - 1)\left(\frac{dP_{ij}}{P_{ij}} - \sum_{k=1}^{m} S_{ik}\frac{dP_{ik}}{P_{ik}}\right) \tag{27}$$

$$= \epsilon_i \frac{dD}{D} - (\sigma_i - 1)\left(\frac{dP_{ij}}{P_{ij}} - \frac{dP_i}{P_i}\right).$$

The first term represents the growth of the market for X_{ij}, in value terms, and the second term represents the percentage change in the product's value share of the market. Of course, this result can also be derived directly from (25).

Une théorie de la demande de produits différenciés d'après leur origine

Résumé

Cette étude offre un appui théorique à certaines pratiques de recherche selon lesquelles la variation d'un flux commercial particulier entre pays est considérée comme la résultante de deux éléments : la modification qui se produirait si le pays fournisseur donné devait conserver sa part du marché, et l'écart entre les ventes effectives et celles qui s'effectueraient si sa part du marché demeurait constante. Ces pratiques comprennent la méthode de prévision des échanges dans laquelle : 1) les prévisions de l'expansion des divers marchés, combinées avec une matrice relative à la période courante, fournissent une matrice pour la période future établie sur la base de parts du marché constantes; 2) cette matrice établie sur la base de parts du marché constantes est modifiée pour tenir compte de facteurs censés engendrer des gains ou des pertes de parts. Dans la présente étude, on fait valoir que l'analyse des modifications des flux commerciaux en deux éléments représentant, l'un des parts du marché constantes et l'autre des parts du marché modifiées, n'intéresse pas seulement la comptabilité, mais qu'elle peut certainement être rattachée purement et simplement à la théorie traditionnelle du comportement des consommateurs. On part de l'hypothèse que les produits sont différenciés non seulement d'après leur espèce mais également d'après leur origine. Autrement dit, on suppose que des produits originaires de différents pays et offerts concurrement sur le même marché ne sont pas susceptibles de se remplacer parfaitement. Il est démontré ensuite qu'une spécialisation poussée et suffisamment réaliste de la fonction de bien-être de Hicks permet d'obtenir des équations assez simples de la demande englobant les deux éléments mentionnés plus haut. Cette spécialisation se fonde sur l'hypothèse "d' indépendance" (telle qu'elle a été formulée par R. M. Solow, R. H. Strotz et d'autres) ainsi que sur celle selon laquelle les élasticités du remplacement entre produits offerts concurremment sur un marché quelconque donné sont constantes et égales.

Una teoría de la demanda de productos distinguiéndolos según el lugar de producción

Resumen

Este estudio ofrece un apoyo teórico a ciertas prácticas de investigación según las cuales a la variación en una corriente determinada de intercambio entre países se la considera como la suma de dos componentes: la variación que ocurriría si un país vendedor dado mantuviera su participación en el mercado, y la desviación de las ventas efectivas con respecto a las ventas que tendrían lugar de permanecer constantes las participaciones. Dichas prácticas incluyen la previsión de los intercambios, en la cual 1) los pronósticos del crecimiento en diversos mercados, junto con la matriz de un período de base, resultan en una matriz de participaciones constantes para el período futuro, y 2) se modifica esta matriz de participaciones constantes para tener en cuenta a los factores que se espere que produzcan pérdidas o ganancias en las participaciones. En este trabajo se mantiene que el análisis de las variaciones en las corrientes de intercambio comercial, separando el componente de participaciones constantes y el de ajuste de las participaciones, es más que una simple cuestión de contabilidad, y en realidad se le puede ligar sencilla y rigurosamente a la teoría tradicional del comportamiento del consumidor. El punto de partida es el supuesto de que se establecen distinciones no solamente según la clase de los productos sino también según el lugar de producción de los mismos. Es decir, se supone que los productos de distintos países que compiten en el mismo mercado son sustitutos imperfectos. Se demuestra luego que una especialización eficaz y bastante realista de la función hicksiana del bienestar lleva a relaciones de demanda muy sencillas en las que se incluyen los componentes de la participación constante y del ajuste de las participaciones. Esta especialización exige el supuesto de "independencia" (que formularan R. M. Solow, R. H. Strotz, *et al.*) y el supuesto de que las elasticidades de sustitución entre los productos que compiten en un mercado determinado son constantes e iguales.

In statistical matter (except in the *résumés* and *resúmenes*) throughout this issue,

Dots (. . .) indicate that data are not available;

A dash (—) indicates that the figure is zero or less than half the final digit shown, or that the item does not exist;

A single dot (.) indicates decimals;

A comma (,) separates thousands and millions;

"Billion" means a thousand million;

A short dash (–) is used between years or months (e.g., 1955–58 or January–October) to indicate a total of the years or months inclusive of the beginning and ending years or months;

A stroke (/) is used between years (e.g., 1962/63) to indicate a fiscal year or a crop year;

Components of tables may not add to totals shown because of rounding.