

# Navier-Stokes

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$$\begin{aligned} \operatorname{Re} u_k \partial_k u_c - \partial_k (\partial_k u_c + \partial_c u_k) + \partial_c p &= f_c & \mathbf{x} \in \Omega \\ \partial_k u_k &= 0 & \mathbf{x} \in \Omega \\ [-p \delta_{kc} + (\partial_k u_c + \partial_c u_k)] n_k &= t_c & \mathbf{x} \in \Gamma_1 \\ u_c &= v_c & \mathbf{x} \in \Gamma_2 \end{aligned}$$

FV:

$$\begin{aligned} \int_{\Omega} \operatorname{Re} (u_k \partial_k u_c) \mathcal{N}^i + \int \partial_k \mathcal{N}^i (\partial_k u_c + \partial_c u_k) - \int p \partial_c \mathcal{N}^i &= \\ = \int f_c \mathcal{N}^i + \int_{\Gamma_1} t_c \mathcal{N}^i \end{aligned}$$

$$\int \partial_k u_k \mathcal{Q}_i = 0$$

Newton  $\frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} \Delta \mathbf{q} = -\mathcal{F}(\mathbf{q})$ :

$$A_{ij}^{(c,d)} = \operatorname{Re} \int_{\Omega} (\mathcal{N}^j \partial_d u_c + \delta_{cd} \mathbf{u} \cdot \nabla \mathcal{N}^j) \mathcal{N}^i + \int_{\Omega} (\delta_{cd} \nabla \mathcal{N}^i \cdot \nabla \mathcal{N}^j + \partial_d \mathcal{N}^i \partial_c \mathcal{N}^j)$$

$$G_{ij}^{(c)} = - \int_{\Omega} \mathcal{Q}^j \partial_c \mathcal{N}^i$$

$$C_{ij}^{(d)} = - \int_{\Omega} \mathcal{Q}^i \partial_d \mathcal{N}^j$$

$$R_i = \int f_c \mathcal{N}^i + \oint t_c \mathcal{N}^i$$

onde  $\mathbf{q} = (\mathbf{u}, \mathbf{p})$ , e denotando

$$\frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \mathcal{F}_u}{\partial \mathbf{u}} & \frac{\partial \mathcal{F}_u}{\partial \mathbf{p}} \\ \frac{\partial \mathcal{F}_p}{\partial \mathbf{u}} & \frac{\partial \mathcal{F}_p}{\partial \mathbf{p}} \end{bmatrix}$$

onde

$$\begin{aligned}\frac{\partial \mathcal{F}_u}{\partial \mathbf{u}} &= [A_{ij}^{(c,d)}] \\ \frac{\partial \mathcal{F}_u}{\partial \mathbf{p}} &= [G_{ij}^{(c)}] \\ \frac{\partial \mathcal{F}_p}{\partial \mathbf{u}} &= [C_{ij}^{(d)}] \\ \frac{\partial \mathcal{F}_p}{\partial \mathbf{p}} &= [0]\end{aligned}$$

$$\partial_{\mathbf{u}} \left( A_{nn} u_n + A_{nb} u_b + G_{np} p - f_n \right)$$

## 1 Estabilizações

### 1.1 Condensação das bolhas

resíduo:

$$\begin{aligned}\mathcal{F}_n &= A_{nn}u_n + A_{nb}u_b + G_{np}p - f_n = 0 \\ \mathcal{F}_b &= A_{bn}u_n + A_{bb}u_b + G_{bp}p - f_b = 0 \\ \mathcal{F}_p &= D_{pn}u_n + D_{pb}u_b = 0\end{aligned}$$

onde os subscritos  $n$  e  $b$  correspondem aos coeficientes dos nós e das bolhas respectivamente.

$$\begin{aligned}A|_{ij} &= \text{Re} \int_{\Omega} \delta_{cd} \tilde{\mathbf{u}} \cdot \nabla \mathcal{N}^j \mathcal{N}^i + \int_{\Omega} (\delta_{cd} \nabla \mathcal{N}^i \cdot \nabla \mathcal{N}^j + \partial_d \mathcal{N}^i \partial_c \mathcal{N}^j), \quad c, d = 1, \dots, d \\ G|_{ij} &= - \int_{\Omega} \mathcal{N}^j \partial_c \mathcal{N}^i \\ D|_{ij} &= - \int_{\Omega} \mathcal{N}^i \partial_c \mathcal{N}^j\end{aligned}$$

onde  $\tilde{\mathbf{u}}$  é a velocidade com os coeficientes das bolhas anuladas.

$$\text{Newton} \frac{\partial \mathcal{F}(q)}{\partial q} \Delta q = -\mathcal{F}(q):$$

$$\begin{aligned}\begin{bmatrix} \partial_{u_n} (A_{nn}u_n) + \partial_{u_n} (A_{nb}u_b) & A_{nb} & G_{np} \\ \partial_{u_n} (A_{bn}u_n) + \partial_{u_n} (A_{bb}u_b) & A_{bb} & G_{bp} \\ D_{pn} & D_{pb} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_n \\ u_b \\ \Delta p \end{bmatrix}^{k+1} &= - \begin{bmatrix} A_{nn} & 0 & G_{np} \\ A_{bn} & 0 & G_{bp} \\ D_{pn} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_b \\ p \end{bmatrix}^k + \begin{bmatrix} f_n \\ f_b \\ 0 \end{bmatrix} \\ &= - \begin{bmatrix} \tilde{\mathcal{F}}_n \\ \tilde{\mathcal{F}}_b \\ \tilde{\mathcal{F}}_p \end{bmatrix}\end{aligned}$$

Lembrar que

$$\begin{aligned}
\partial_{u_n} (A_{nn} u_n) |_{ij} &= \operatorname{Re} \int_{\Omega} (\mathcal{N}^j \partial_d \tilde{u}_c + \delta_{cd} \tilde{\mathbf{u}} \cdot \nabla \mathcal{N}^j) \mathcal{N}^i + \text{rigidez} \\
\partial_{u_n} (A_{nb} u_b) |_{ij} &= \operatorname{Re} \int_{\Omega} \mathcal{N}^j \partial_d u_c^{bub} \mathcal{N}^i + \text{rigidez} \\
\partial_{u_n} (A_{bn} u_n) |_{ij} &= \operatorname{Re} \int_{\Omega} (\mathcal{N}^j \partial_d \tilde{u}_c + \delta_{cd} \tilde{\mathbf{u}} \cdot \nabla \mathcal{N}^j) \mathcal{B} + \text{rigidez} \\
\partial_{u_n} (A_{bb} u_b) |_{ij} &= \operatorname{Re} \int_{\Omega} \mathcal{N}^j \partial_d u_c^{bub} \mathcal{B} + \text{rigidez}
\end{aligned}$$

Denotando

$$\begin{aligned}
S_{nn} : &= \partial_{u_n} (A_{nn} u_n) + \partial_{u_n} (A_{nb} u_b) \\
S_{bn} : &= \partial_{u_n} (A_{bn} u_n) + \partial_{u_n} (A_{bb} u_b)
\end{aligned}$$

então, como  $u_b^{(k+1)} = A_{bb}^{-1} \left( -S_{bn} \Delta u_n - G_{bp} \Delta p - \tilde{\mathcal{F}}_b \right)$ , vem que

$$\begin{bmatrix} S_{nn} - A_{nb} A_{bb}^{-1} S_{bn} & G_{np} - A_{nb} A_{bb}^{-1} G_{bp} \\ D_{pn} - D_{pb} A_{bb}^{-1} S_{bn} & -D_{pb} A_{bb}^{-1} G_{bp} \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p \end{bmatrix}^{k+1} = - \begin{bmatrix} \tilde{\mathcal{F}}_n - A_{nb} A_{bb}^{-1} \tilde{\mathcal{F}}_b \\ \tilde{\mathcal{F}}_p - D_{pb} A_{bb}^{-1} \tilde{\mathcal{F}}_b \end{bmatrix}$$

Os coeficientes  $u_b^{(k)}$  são calculados no elemento como:

$$u_b^{(k)} = -A_{bb}^{-1} \tilde{\mathcal{F}}_b$$

## 1.2 GSL

Formulação GSL para elementos P<sub>1</sub>-P<sub>1</sub>:

$$\operatorname{FV}_{\text{momento}} + \sum_K \tau_K (\operatorname{Re} \mathbf{u} \cdot \nabla \mathbf{w}, \operatorname{Re} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}) = 0, \quad \forall \mathbf{w} \in W$$

$$\operatorname{FV}_{\text{incomp}} - \sum_K \tau_K (\nabla q, \operatorname{Re} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}) = 0, \quad \forall q \in Q$$

Newton:

$$\begin{bmatrix} A + B & G + C \\ D + H & E \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta p \end{bmatrix}^{k+1} = - \begin{bmatrix} \mathcal{F}_u + \sum_K \tau_K (\operatorname{Re} \mathbf{u} \cdot \nabla \mathcal{N}^i, \operatorname{Re} \mathbf{u} \cdot \nabla u_c + \partial_c p - f_c) \\ \mathcal{F}_p - \sum_K \tau_K (\nabla \mathcal{N}^i, \operatorname{Re} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}) \end{bmatrix}$$

onde  $A$ ,  $G$ , e  $D$  vêm da formulação de Galerkin e

$$\begin{aligned}
B|_{ij} &= \tau_K \text{Re}^2 \int_{\Omega_K} (\mathcal{N}^j \partial_d \mathcal{N}^i \mathbf{u} \cdot \nabla u_c + \mathbf{u} \cdot \nabla \mathcal{N}^i \mathcal{N}^j \partial_d u_c + \delta_{cd} \mathbf{u} \cdot \nabla \mathcal{N}^i \mathbf{u} \cdot \nabla \mathcal{N}^j) \\
&\quad + \tau_K \text{Re} \int_{\Omega_K} \mathcal{N}^j \partial_d \mathcal{N}^i (\partial_c p - f_c) \\
C|_{ij} &= \tau_K \text{Re} \int_{\Omega_K} \mathbf{u} \cdot \nabla \mathcal{N}^i \partial_c \mathcal{N}^j \\
H|_{ij} &= -\tau_K \text{Re} \int_{\Omega_K} (\mathcal{N}^j \partial_c \mathbf{u} \cdot \nabla \mathcal{N}^i + \mathbf{u} \cdot \nabla \mathcal{N}^j \partial_c \mathcal{N}^i) \\
E|_{ij} &= -\tau_K \int_{\Omega_K} \nabla \mathcal{N}^i \cdot \nabla \mathcal{N}^j
\end{aligned}$$

## 2 ALE para gota oscilante

$$\begin{aligned}
\int (\partial_t \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u}) \cdot \mathbf{w} + \int \nu \nabla \mathbf{w} : (\nabla \mathbf{u} + \nabla^T \mathbf{u}) - \int \bar{p} \nabla \cdot \mathbf{w} &= \int \mathbf{f} + \int_{\Gamma} \bar{\gamma} \mathbf{P} : \nabla \mathbf{w} \\
&\quad \int q \nabla \cdot \mathbf{u} = 0
\end{aligned}$$

onde  $\mathbf{v} = \mathbf{u} - \mathbf{u}_m$ ,  $\bar{p} = p/\rho$  e  $\mathbf{P} = I - \mathbf{n}\mathbf{n}$ . Atenção:  $\bar{\gamma} = \gamma/\rho$ .  
em componentes, discretizado no tempo e estabilizando:

$$\begin{aligned}
FVG_U + GLS_U &= 0 \\
FVG_P + GLS_P &= 0
\end{aligned}$$

onde

$$\begin{aligned}
FVG_U &= \int (\delta_t^n u_c + (\mathbf{v}^n \cdot \nabla) u_c^{n+\theta}) N^i + \int \nu \nabla N^i \cdot (\nabla u_c^{n+\theta} + \partial_c \mathbf{u}^{n+\theta}) - \int \bar{p}^{n+1} \partial_c N^i - \\
&\quad - \int f_c^{n+1} - \int_{\Gamma} \bar{\gamma} (\mathbf{P}^n \cdot \nabla N^i)|_c - \Delta t \int_{\Gamma} \bar{\gamma} (\nabla \mathbf{u}^{n+1} \cdot \mathbf{P} \cdot \nabla N^i)|_c \\
GLS_U &= \sum_K \tau_K (\mathbf{v}^n \cdot \nabla N^i, \delta_t^n u_c + \mathbf{v}^n \cdot \nabla u_c^{n+\theta} + \partial_c \bar{p}^{n+1} - f_c^{n+1}) \\
FVG_P &= \int N^i \nabla \cdot \mathbf{u}^{n+1} \\
GLS_P &= \sum_K \tau_K (\nabla N^i, \delta_t^n \mathbf{u} + (\mathbf{v}^n \cdot \nabla) \mathbf{u}^{n+\theta} + \nabla \bar{p}^{n+1} - \mathbf{f}^{n+1})
\end{aligned}$$

Newton

$$\begin{bmatrix} A+B & G+C \\ D+H & E \end{bmatrix} \begin{bmatrix} \Delta u^{n+1} \\ \Delta p^{n+1} \end{bmatrix} = - \begin{bmatrix} \mathcal{F}_u \\ \mathcal{F}_p \end{bmatrix}$$

onde

$$\begin{aligned}
A|_{ij} &= \int_{\Omega} \delta_{cd} \left( \frac{N^j}{\Delta t} + \theta \mathbf{v}^n \cdot \nabla N^j \right) N^i + \int_{\Omega} \theta \nu \left( \delta_{cd} \nabla N^i \cdot \nabla N^j + \partial_d N^i \partial_c N^j \right) - \\
&\quad - \Delta t \int_{\Gamma} \delta_{cd} \bar{\gamma} \nabla N^j \cdot \mathbf{P} \cdot \nabla N^i \\
G|_{ij} &= - \int_{\Omega} N^j \partial_c N^i \\
D|_{ij} &= \int_{\Omega} N^i \partial_d N^j
\end{aligned}$$

e as estabilizações

$$\begin{aligned}
B|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \delta_{cd} \mathbf{v}^n \cdot \nabla N^i \left( \frac{N^j}{\Delta t} + \theta \mathbf{v}^n \cdot \nabla N^j \right) \\
C|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \mathbf{v}^n \cdot \nabla N^i \partial_c N^j \\
H|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \partial_d N^i \left( \frac{N^j}{\Delta t} + \theta \mathbf{v}^n \cdot \nabla N^j \right) \\
E|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \nabla N^i \cdot \nabla N^j
\end{aligned}$$

### 3 Ale explícito + Crouzeix-Raviart

Depois da eliminação de bolhas:

$$\begin{bmatrix} A_{nn} - A_{nb} A_{bb}^{-1} A_{bn} & G_{np} - A_{nb} A_{bb}^{-1} G_{bp} \\ D_{pn} - D_{pb} A_{bb}^{-1} A_{bn} & -D_{pb} A_{bb}^{-1} G_{bp} \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p \end{bmatrix}^{k+1} = - \begin{bmatrix} F_n - A_{nb} A_{bb}^{-1} F_b \\ F_p - D_{pb} A_{bb}^{-1} F_b \end{bmatrix}$$

Separando o gradiente da pressão de seu valor no centro:

$$\begin{aligned}
G_{np} &= \begin{bmatrix} G_{nc} & G_{nx} \end{bmatrix} \\
G_{bp} &= \begin{bmatrix} G_{bc} & G_{bx} \end{bmatrix} = \begin{bmatrix} 0 & G_{bx} \end{bmatrix}
\end{aligned}$$

e

$$\begin{aligned}
D_{pn} &= \begin{bmatrix} D_{cn} & D_{xn} \end{bmatrix}^T \\
D_{pb} &= \begin{bmatrix} D_{cb} & D_{xb} \end{bmatrix}^T = \begin{bmatrix} 0 & D_{xb} \end{bmatrix}^T
\end{aligned}$$

tem-se que

$$D_{pb} A_{bb}^{-1} G_{bp} = \begin{bmatrix} 0 & 0 \\ 0 & D_{xb} A_{bb}^{-1} G_{bx} \end{bmatrix} = \text{< no elemento >} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha^2 A_{bb}^{-1} \end{bmatrix}$$

onde foi usado que, no elemento:

$$D_{xb}|_{i,j} = - \int x_i \partial_j B = \int B \partial_j x_i = \int B \delta_{ij} \equiv \alpha \delta_{ij}.$$

O sistema fica:

$$\begin{bmatrix} \tilde{A}_{nn} & G_{nc} & G_{nx} - A_{nb}A_{bb}^{-1}G_{bx} \\ D_{cn} & 0 & 0 \\ D_{xn} - D_{xb}A_{bb}^{-1}A_{bn} & 0 & -D_{xb}A_{bb}^{-1}G_{bx} \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p_c \\ \Delta p_x \end{bmatrix}^{k+1} = - \begin{bmatrix} F_n - A_{nb}A_{bb}^{-1}F_b \\ F_c \\ F_x - D_{xb}A_{bb}^{-1}F_b \end{bmatrix}$$

então

$$\begin{aligned} \Delta p_x &= G_{bx}^{-1}A_{bb}D_{xb}^{-1}[(D_{xn} - D_{xb}A_{bb}^{-1}A_{bn})\Delta u_n + F_x - D_{xb}A_{bb}^{-1}F_b] \\ &= G_{bx}^{-1}(A_{bb}D_{xb}^{-1}D_{xn} - A_{bn})\Delta u_n + G_{bx}^{-1}A_{bb}D_{xb}^{-1}F_x - G_{bx}^{-1}F_b \end{aligned}$$

substituindo:

$$\begin{bmatrix} A_{nn} + (G_{nx}G_{bx}^{-1}A_{bb} - A_{nb})D_{xb}^{-1}D_{xn} - G_{nx}G_{bx}^{-1}A_{bn} & G_{nc} \\ D_{cn} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p_c \end{bmatrix}^{k+1} = - \begin{bmatrix} R \\ F_c \end{bmatrix}$$

onde

$$R = F_n + (G_{nx}G_{bx}^{-1}A_{bb} - A_{nb})D_{xb}^{-1}F_x - G_{nx}G_{bx}^{-1}F_b$$

que no elemento pode ser construído como

$$\begin{bmatrix} A_{nn} + \frac{1}{\alpha} \left( \frac{1}{\alpha} G_{nx}A_{bb}D_{xn} - A_{nb}D_{xn} - G_{nx}A_{bn} \right) & G_{nc} \\ D_{cn} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p_c \end{bmatrix}^{k+1} = - \begin{bmatrix} R \\ F_c \end{bmatrix}$$

e

$$R = F_n + \frac{1}{\alpha} \left( \frac{1}{\alpha} G_{nx}A_{bb}F_x - A_{nb}F_x - G_{nx}F_b \right)$$