Navier-Stokes

12 de fevereiro de 2012

$$
\begin{aligned}
\mathrm{Re}\, u_k \partial_k u_c - \partial_k \left(\partial_k u_c + \partial_c u_k \right) + \partial_c p = f_c \qquad & \boldsymbol{x} \in \Omega \\
\partial_k u_k = 0 \qquad & \boldsymbol{x} \in \Omega \\
\left[-p\delta_{kc} + \left(\partial_k u_c + \partial_c u_k \right) \right] n_k = t_c \qquad & \boldsymbol{x} \in \Gamma_1 \\
u_c = v_c \qquad & \boldsymbol{x} \in \Gamma_2
\end{aligned}
$$

FV:

$$
\begin{aligned}
\int_\Omega \mathrm{Re} \left(u_k \partial_k u_c \right) \mathcal{N}^i + \int \partial_k \mathcal{N}^i \left(\partial_k u_c + \partial_c u_k \right) - \int p \partial_c \mathcal{N}^i = \\
= \int f_c \mathcal{N}^i + \int_{\Gamma_1} t_c \mathcal{N}^i
\end{aligned}
$$

$$
\int \partial_k u_k \mathcal{Q}_i = 0
$$

Newton $\dfrac{\partial \mathcal{F}(\boldsymbol{q})}{\partial \boldsymbol{q}} \Delta \boldsymbol{q} = -\mathcal{F}(\boldsymbol{q})$:

$$
\begin{aligned}
A_{ij}^{(c,d)} = \mathrm{Re} \int_\Omega \left(\mathcal{N}^j \partial_d u_c + \delta_{cd} \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^j \right) \mathcal{N}^i + \int_\Omega \left(\delta_{cd} \boldsymbol{\nabla} \mathcal{N}^i \cdot \boldsymbol{\nabla} \mathcal{N}^j + \partial_d \mathcal{N}^i \partial_c \mathcal{N}^j \right) \\
G_{ij}^{(c)} = -\int_\Omega \mathcal{Q}^j \partial_c \mathcal{N}^i \\
C_{ij}^{(d)} = -\int_\Omega \mathcal{Q}^i \partial_d \mathcal{N}^j \\
R_i = \int f_c \mathcal{N}^i + \oint t_c \mathcal{N}^i
\end{aligned}
$$

onde $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{p})$, e denotando

$$
\frac{\partial \mathcal{F}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{bmatrix} \dfrac{\partial \mathcal{F}_u}{\partial \boldsymbol{u}} & \dfrac{\partial \mathcal{F}_u}{\partial \boldsymbol{p}} \\[2ex] \dfrac{\partial \mathcal{F}_p}{\partial \boldsymbol{u}} & \dfrac{\partial \mathcal{F}_p}{\partial \boldsymbol{p}} \end{bmatrix}
$$

onde

$$\begin{aligned}\frac{\partial \mathcal{F}_u}{\partial \boldsymbol{u}} &= \left[A_{ij}^{(c,d)}\right]\\ \frac{\partial \mathcal{F}_u}{\partial \boldsymbol{p}} &= \left[G_{ij}^{(c)}\right]\\ \frac{\partial \mathcal{F}_p}{\partial \boldsymbol{u}} &= \left[C_{ij}^{(d)}\right]\\ \frac{\partial \mathcal{F}_p}{\partial \boldsymbol{p}} &= [0]\end{aligned}$$

\partial_{u_{n}}\left(A{}_{nn}u_{n}\right)

1 Estabilizações

1.1 Condensação das bolhas

resíduo:

$$\begin{aligned}\mathcal{F}_n = A_{nn}u_n + A_{nb}u_b + G_{np}p - f_n &= 0\\ \mathcal{F}_b = A_{bn}u_n + A_{bb}u_b + G_{bp}p - f_b &= 0\\ \mathcal{F}_p = D_{pn}u_n + D_{pb}u_b &= 0\end{aligned}$$

onde os subscritos n e b correspondem aos coeficientes dos nós e das bolhas respectivamente.

$$\begin{aligned}A|_{ij} &= \mathrm{Re}\int_\Omega \delta_{cd}\widetilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathcal{N}^j\mathcal{N}^i + \int_\Omega\left(\delta_{cd}\boldsymbol{\nabla}\mathcal{N}^i\cdot\boldsymbol{\nabla}\mathcal{N}^j + \partial_d\mathcal{N}^i\partial_c\mathcal{N}^j\right), \qquad c,d = 1,..,d\\ G|_{ij} &= -\int_\Omega \mathcal{N}^j\partial_c\mathcal{N}^i\\ D|_{ij} &= -\int_\Omega \mathcal{N}^i\partial_c\mathcal{N}^j\end{aligned}$$

onde $\widetilde{\boldsymbol{u}}$ é a velocidade com os coeficientes das bolhas anuladas.

Newton $\dfrac{\partial\mathcal{F}(q)}{\partial q}\Delta q = -\mathcal{F}(q)$:

$$\begin{aligned}\begin{bmatrix}\partial_{u_n}\left(A_{nn}u_n\right) + \partial_{u_n}\left(A_{nb}u_b\right) & A_{nb} & G_{np}\\ \partial_{u_n}\left(A_{bn}u_n\right) + \partial_{u_n}\left(A_{bb}u_b\right) & A_{bb} & G_{bp}\\ D_{pn} & D_{pb} & 0\end{bmatrix}\begin{bmatrix}\Delta u_n\\ u_b\\ \Delta p\end{bmatrix}^{k+1} &= -\begin{bmatrix}A_{nn} & 0 & G_{np}\\ A_{bn} & 0 & G_{bp}\\ D_{pn} & 0 & 0\end{bmatrix}\begin{bmatrix}u_n\\ u_b\\ p\end{bmatrix}^k + \begin{bmatrix}f_n\\ f_b\\ 0\end{bmatrix}\\ &= -\begin{bmatrix}\widetilde{\mathcal{F}}_n\\ \widetilde{\mathcal{F}}_b\\ \widetilde{\mathcal{F}}_p\end{bmatrix}\end{aligned}$$

Lembrar que

$$
\begin{aligned}
\partial_{u_n}\left(A_{nn}u_n\right)|_{ij} &= \mathrm{Re}\int_\Omega \left(\mathcal{N}^j\partial_d\tilde{u}_c + \delta_{cd}\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathcal{N}^j\right)\mathcal{N}^i + \text{rigidez}\\
\partial_{u_n}\left(A_{nb}u_b\right)|_{ij} &= \mathrm{Re}\int_\Omega \mathcal{N}^j\partial_d u_c^{bub}\mathcal{N}^i + \text{rigidez}\\
\partial_{u_n}\left(A_{bn}u_n\right)|_{ij} &= \mathrm{Re}\int_\Omega \left(\mathcal{N}^j\partial_d\tilde{u}_c + \delta_{cd}\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathcal{N}^j\right)\mathcal{B} + \text{rigidez}\\
\partial_{u_n}\left(A_{bb}u_b\right)|_{ij} &= \mathrm{Re}\int_\Omega \mathcal{N}^j\partial_d u_c^{bub}\mathcal{B} + \text{rigidez}
\end{aligned}
$$

Denotando

$$
\begin{aligned}
S_{nn} :&= \partial_{u_n}\left(A_{nn}u_n\right) + \partial_{u_n}\left(A_{nb}u_b\right)\\
S_{bn} :&= \partial_{u_n}\left(A_{bn}u_n\right) + \partial_{u_n}\left(A_{bb}u_b\right)
\end{aligned}
$$

então, como $u_b^{(k+1)} = A_{bb}^{-1}\left(-S_{bn}\Delta u_n - G_{bp}\Delta p - \widetilde{\mathcal{F}}_b\right)$, vem que

$$
\left[\begin{array}{cc} S_{nn} - A_{nb}A_{bb}^{-1}S_{bn} & G_{np} - A_{nb}A_{bb}^{-1}G_{bp} \\ D_{pn} - D_{pb}A_{bb}^{-1}S_{bn} & -D_{pb}A_{bb}^{-1}G_{bp}\end{array}\right]\left[\begin{array}{c}\Delta u_n\\ \Delta p\end{array}\right]^{k+1} = -\left[\begin{array}{c}\widetilde{\mathcal{F}}_n - A_{nb}A_{bb}^{-1}\widetilde{\mathcal{F}}_b\\ \widetilde{\mathcal{F}}_p - D_{pb}A_{bb}^{-1}\widetilde{\mathcal{F}}_b\end{array}\right]
$$

Os coeficientes $u_b^{(k)}$ são calculador no elemento como:

$$
u_b^{(k)} = -A_{bb}^{-1}\widetilde{\mathcal{F}}_b
$$

No caso discretização temporal com o método θ, é preciso fazer

$$
u_b^{(k),n+\theta} = \theta u_b^{(k)} + (1-\theta)u_b^{old}
$$

Pode-se adotar $u_b^{old} = 0$.

1.2 GSL

Formulação GSL para elementos P_1-P_1:

$$
\begin{aligned}
\mathrm{FV}_{\text{momento}} + \sum_K \tau_K\left(\mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{w},\ \mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}p - \boldsymbol{f}\right) &= 0, \quad \forall\boldsymbol{w}\in W\\
\mathrm{FV}_{\text{incomp}} - \sum_K \tau_K\left(\boldsymbol{\nabla}q,\ \mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}p - \boldsymbol{f}\right) &= 0, \quad \forall q\in Q
\end{aligned}
$$

Newton:

$$
\left[\begin{array}{cc} A+B & G+C\\ D+H & E\end{array}\right]\left[\begin{array}{c}\Delta u\\ \Delta p\end{array}\right]^{k+1} = -\left[\begin{array}{c}\mathcal{F}_u + \sum_K \tau_K\left(\mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}\mathcal{N}^i,\ \mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}u_c + \partial_c p - f_c\right)\\ \mathcal{F}_p - \sum_K \tau_K\left(\boldsymbol{\nabla}\mathcal{N}^i,\ \mathrm{Re}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}p - \boldsymbol{f}\right)\end{array}\right]
$$

onde A, G, e D vêm da formulação de Galerkin e

$$\begin{aligned}
B|_{ij} &= \tau_K \mathrm{Re}^2 \int_{\Omega_K} \left(\mathcal{N}^j \partial_d \mathcal{N}^i \boldsymbol{u} \cdot \boldsymbol{\nabla} u_c + \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^i \mathcal{N}^j \partial_d u_c + \delta_{cd} \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^i \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^j \right) \\
&\quad + \tau_K \mathrm{Re} \int_{\Omega_K} \mathcal{N}^j \partial_d \mathcal{N}^i \left(\partial_c p - f_c \right) \\
C|_{ij} &= \tau_K \mathrm{Re} \int_{\Omega_K} \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^i \partial_c \mathcal{N}^j \\
H|_{ij} &= -\tau_K \mathrm{Re} \int_{\Omega_K} \left(\mathcal{N}^j \partial_c \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^i + \boldsymbol{u} \cdot \boldsymbol{\nabla} \mathcal{N}^j \partial_c \mathcal{N}^i \right) \\
E|_{ij} &= -\tau_K \int_{\Omega_K} \boldsymbol{\nabla} \mathcal{N}^i \cdot \boldsymbol{\nabla} \mathcal{N}^j
\end{aligned}$$

$$\begin{aligned}
B_{ij} \ : \ \frac{\partial}{\partial \mathbf{u}^j} \left(\rho \mathbf{c}^{n+\theta} \cdot \nabla N^i \right) \left(\rho \delta_t \mathbf{u} + \rho \nabla \mathbf{u}^{n+\theta} \cdot \mathbf{c}^{n+\theta} + \nabla p^{n+1} - \rho \boldsymbol{g}^{n+\theta} \right) &= \\
\theta \rho N^j \mathbf{R} \otimes \nabla N^i + \rho \mathbf{c}^{n+\theta} \cdot \nabla N^i \frac{\partial \mathbf{R}}{\partial \mathbf{u}}
\end{aligned}$$

onde

$$\begin{aligned}
\mathbf{R} &= \rho \delta_t \mathbf{u} + \rho \nabla \mathbf{u}^{n+\theta} \cdot \mathbf{c}^{n+\theta} + \nabla p^{n+1} - \rho \boldsymbol{g}^{n+\theta}, \\
\frac{\partial \mathbf{R}}{\partial \mathbf{u}} &= \mathbb{I} \frac{\rho}{\Delta t} N^j + \rho \theta \left(N^j \nabla \mathbf{u}^{n+\theta} + \mathbb{I} \mathbf{c}^{n+\theta} \cdot \nabla N^j \right).
\end{aligned}$$

também,

$$\begin{aligned}
H_{ij} \ : \ -\frac{\partial}{\partial \mathbf{u}^j} \nabla M^i \cdot \left(\rho \delta_t \mathbf{u} + \rho \nabla \mathbf{u}^{n+\theta} \cdot \mathbf{c}^{n+\theta} + \nabla p^{n+1} - \rho \boldsymbol{g}^{n+\theta} \right) &= \\
-\nabla M^i \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{u}}
\end{aligned}$$

também

$$\begin{aligned}
C_{ij} \ : \ \frac{\partial}{\partial p^j} \left(\rho \mathbf{c}^{n+\theta} \cdot \nabla N^i \right) \left(\rho \delta_t \mathbf{u} + \rho \nabla \mathbf{u}^{n+\theta} \cdot \mathbf{c}^{n+\theta} + \nabla p^{n+1} - \rho \boldsymbol{g}^{n+\theta} \right) &= \\
\rho \mathbf{c}^{n+\theta} \cdot \nabla N^i \nabla M^j
\end{aligned}$$

2 ALE para gota oscilante

$$\int \left(\partial_t \boldsymbol{u} + \left(\boldsymbol{v} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u} \right) \cdot \boldsymbol{w} + \int \nu \boldsymbol{\nabla} \boldsymbol{w} : \left(\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla}^T \boldsymbol{u} \right) - \int \bar{p} \boldsymbol{\nabla} \cdot \boldsymbol{w} = \int \boldsymbol{f} + \int_\Gamma \bar{\gamma} \boldsymbol{P} : \boldsymbol{\nabla} \boldsymbol{w}$$

$$\int q \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

onde $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_m$, $\bar{p} = p/\rho$ e $\boldsymbol{P} = I - \boldsymbol{n}\boldsymbol{n}$. Atenção: $\bar{\gamma} = \gamma/\rho$.

em componentes, discretizado no tempo e estabilizando:

$$
\begin{aligned}
FVG_U + GLS_U =& 0 \\
FVG_P + GLS_P =& 0
\end{aligned}
$$

onde

$$
\begin{aligned}
FVG_U &= \int \left(\delta_t^n u_c + (\boldsymbol{v}^n \cdot \boldsymbol{\nabla})\, u_c^{n+\theta}\right) N^i + \int \nu \boldsymbol{\nabla} N^i \cdot \left(\boldsymbol{\nabla} u_c^{n+\theta} + \partial_c \boldsymbol{u}^{n+\theta}\right) - \int \overline{p}^{n+1} \partial_c N^i - \\
&- \int f_c^{n+1} - \int_\Gamma \bar{\gamma} \left(\boldsymbol{P}^n \cdot \boldsymbol{\nabla} N^i\right)|_c - \Delta t \int_\Gamma \bar{\gamma} \left(\boldsymbol{\nabla} \boldsymbol{u}^{n+1} \cdot \boldsymbol{P} \cdot \boldsymbol{\nabla} N^i\right)|_c \\
GLS_U &= \sum_K \tau_K \left(\boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^i, \delta_t^n u_c + \boldsymbol{v}^n \cdot \boldsymbol{\nabla} u_c^{n+\theta} + \partial_c \overline{p}^{n+1} - f_c^{n+1}\right) \\
FVG_P &= \int N^i \boldsymbol{\nabla} \cdot \boldsymbol{u}^{n+1} \\
GLS_P &= \sum_K \tau_K \left(\boldsymbol{\nabla} N^i, \delta_t^n \boldsymbol{u} + (\boldsymbol{v}^n \cdot \boldsymbol{\nabla})\, \boldsymbol{u}^{n+\theta} + \boldsymbol{\nabla} \overline{p}^{n+1} - \boldsymbol{f}^{n+1}\right)
\end{aligned}
$$

Newton

$$
\left[\begin{array}{cc} A+B & G+C \\ D+H & E \end{array}\right] \left[\begin{array}{c} \Delta u^{n+1} \\ \Delta p^{n+1} \end{array}\right] \;=\; - \left[\begin{array}{c} \mathcal{F}_u \\ \mathcal{F}_p \end{array}\right]
$$

onde

$$
\begin{aligned}
A|_{ij} &= \int_\Omega \delta_{cd} \left(\frac{N^j}{\Delta t} + \theta \boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^j\right) N^i + \int_\Omega \theta \nu \left(\delta_{cd} \boldsymbol{\nabla} N^i \cdot \boldsymbol{\nabla} N^j + \partial_d N^i \partial_c N^j\right) - \\
&\quad - \Delta t \int_\Gamma \delta_{cd} \bar{\gamma} \boldsymbol{\nabla} N^j \cdot \boldsymbol{P} \cdot \boldsymbol{\nabla} N^i \\
G|_{ij} &= - \int_\Omega N^j \partial_c N^i \\
D|_{ij} &= \int_\Omega N^i \partial_d N^j
\end{aligned}
$$

e as estabilizações

$$
\begin{aligned}
B|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \delta_{cd} \boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^i \left(\frac{N^j}{\Delta t} + \theta \boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^j\right) \\
C|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^i \partial_c N^j \\
H|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \partial_d N^i \left(\frac{N^j}{\Delta t} + \theta \boldsymbol{v}^n \cdot \boldsymbol{\nabla} N^j\right) \\
E|_{ij} &= \sum_K \tau_K \int_{\Omega_K} \boldsymbol{\nabla} N^i \cdot \boldsymbol{\nabla} N^j
\end{aligned}
$$

3 Ale explícito + Crouzeix-Raviart

Depois da eliminação de bolhas:

$$\begin{bmatrix} A_{nn} - A_{nb}A_{bb}^{-1}A_{bn} & G_{np} - A_{nb}A_{bb}^{-1}G_{bp} \\ D_{pn} - D_{pb}A_{bb}^{-1}A_{bn} & -D_{pb}A_{bb}^{-1}G_{bp} \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p \end{bmatrix}^{k+1} = - \begin{bmatrix} F_n - A_{nb}A_{bb}^{-1}F_b \\ F_p - D_{pb}A_{bb}^{-1}F_b \end{bmatrix}$$

Separando o gradiente da pressão de seu valor no centro:

$$\begin{aligned} G_{np} &= \begin{bmatrix} G_{nc} & G_{nx} \end{bmatrix} \\ G_{bp} &= \begin{bmatrix} G_{bc} & G_{bx} \end{bmatrix} = \begin{bmatrix} 0 & G_{bx} \end{bmatrix} \end{aligned}$$

e

$$\begin{aligned} D_{pn} &= \begin{bmatrix} D_{cn} & D_{xn} \end{bmatrix}^T \\ D_{pb} &= \begin{bmatrix} D_{cb} & D_{xb} \end{bmatrix}^T = \begin{bmatrix} 0 & D_{xb} \end{bmatrix}^T \end{aligned}$$

tem-se que

$$D_{pb}A_{bb}^{-1}G_{bp} = \begin{bmatrix} 0 & 0 \\ 0 & D_{xb}A_{bb}^{-1}G_{bx} \end{bmatrix} =< \text{no elemento} >= \begin{bmatrix} 0 & 0 \\ 0 & \alpha^2 A_{bb}^{-1} \end{bmatrix}$$

onde foi usado que, no elemento:

$$D_{xb}|_{i,j} = -\int x_i \partial_j B = \int B \partial_j x_i = \int B \delta_{ij} \equiv \alpha \delta_{ij}.$$

O sistema fica:

$$\begin{bmatrix} A_{nn} - A_{nb}A_{bb}^{-1}A_{bn} & G_{nc} & G_{nx} - A_{nb}A_{bb}^{-1}G_{bx} \\ D_{cn} & 0 & 0 \\ D_{xn} - D_{xb}A_{bb}^{-1}A_{bn} & 0 & -D_{xb}A_{bb}^{-1}G_{bx} \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p_c \\ \Delta p_x \end{bmatrix}^{k+1} = - \begin{bmatrix} F_n - A_{nb}A_{bb}^{-1}F_b \\ F_c \\ F_x - D_{xb}A_{bb}^{-1}F_b \end{bmatrix}$$

então

$$\begin{aligned} \Delta p_x &= G_{bx}^{-1}A_{bb}D_{xb}^{-1}\left[\left(D_{xn} - D_{xb}A_{bb}^{-1}A_{bn}\right)\Delta u_n + F_x - D_{xb}A_{bb}^{-1}F_b\right] \\ &= G_{bx}^{-1}\left(A_{bb}D_{xb}^{-1}D_{xn} - A_{bn}\right)\Delta u_n + G_{bx}^{-1}A_{bb}D_{xb}^{-1}F_x - G_{bx}^{-1}F_b \end{aligned}$$

substituindo:

$$\begin{bmatrix} A_{nn} + \left(G_{nx}G_{bx}^{-1}A_{bb} - A_{nb}\right)D_{xb}^{-1}D_{xn} - G_{nx}G_{bx}^{-1}A_{bn} & G_{nc} \\ D_{cn} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_n \\ \Delta p_c \end{bmatrix}^{k+1} = - \begin{bmatrix} R \\ F_c \end{bmatrix}$$

onde

$$R = F_n + (G_{nx}G_{bx}^{-1}A_{bb} - A_{nb})D_{xb}^{-1}F_x - G_{nx}G_{bx}^{-1}F_b$$

que no elemento pode ser construído como

$$\left[\begin{array}{cc} A_{nn}+\frac{1}{\alpha}\left(\frac{1}{\alpha}G_{nx}A_{bb}D_{xn}-A_{nb}D_{xn}-G_{nx}A_{bn}\right) & G_{nc} \\ D_{cn} & 0 \end{array}\right]\left[\begin{array}{c} \Delta u_n \\ \Delta p_c \end{array}\right]^{k+1}=-\left[\begin{array}{c} R \\ F_c \end{array}\right]$$

e

$$R=F_n+\frac{1}{\alpha}\left(\frac{1}{\alpha}G_{nx}A_{bb}F_x-A_{nb}F_x-G_{nx}F_b\right)$$

$$\begin{aligned} \mathcal{F}_n=A_{nn}u_n+A_{nb}u_b+G_{nc}p_c+G_{nx}p_x-f_n &= 0 \\ \mathcal{F}_b=A_{bn}u_n+A_{bb}u_b+G_{bc}p_c+G_{bx}p_x-f_b &= 0 \\ \mathcal{F}_c=D_{cn}u_n+D_{cb}u_b &= 0 \\ \mathcal{F}_x=D_{xn}u_n+D_{xb}u_b &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{F}_n=\left(A_{nn}+\frac{1}{\alpha^2}G_{nx}A_{bb}D_{xn}-\frac{1}{\alpha}A_{nb}D_{xn}-\frac{1}{\alpha}G_{nx}A_{bn}\right)u_n+G_{nc}p_c+G_{nx}\frac{1}{\alpha}f_b-f_n &= 0 \\ \mathcal{F}_c=D_{cn}u_n &= 0 \end{aligned}$$

forma compacta:

$$\begin{aligned} \mathcal{F}_n=F_n+\frac{1}{\alpha}\left(\frac{1}{\alpha}G_{nx}A_{bb}F_x-A_{nb}F_x-G_{nx}F_b\right) &= 0 \\ \mathcal{F}_c=D_{cn}u_n &= 0 \end{aligned}$$

onde

$$F_x=D_{xn}u_n;\qquad F_n=A_{nn}u_n+G_{nc}p_c-f_n;\qquad ;F_b=A_{bn}u_n-f_b$$

lembrando

$$\begin{aligned} \partial_{u_n}\left(A_{nn}u_n\right)|_{ij} &= \mathrm{Re}\int_\Omega\left(\mathcal{N}^j\partial_d\tilde{u}_c+\delta_{cd}\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathcal{N}^j\right)\mathcal{N}^i+\text{rigidez} \\ \partial_{u_n}\left(A_{nb}u_b\right)|_{ij} &= \mathrm{Re}\int_\Omega\mathcal{N}^j\partial_d u_c^{bub}\mathcal{N}^i+\text{rigidez} \\ \partial_{u_n}\left(A_{bn}u_n\right)|_{ij} &= \mathrm{Re}\int_\Omega\left(\mathcal{N}^j\partial_d\tilde{u}_c+\delta_{cd}\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathcal{N}^j\right)\mathcal{B}+\text{rigidez} \\ \partial_{u_n}\left(A_{bb}u_b\right)|_{ij} &= \mathrm{Re}\int_\Omega\mathcal{N}^j\partial_d u_c^{bub}\mathcal{B}+\text{rigidez} \end{aligned}$$

então

$$\frac{\partial}{\partial u_n}\mathcal{F}_n=\partial_u F_n+\frac{1}{\alpha}\left(\frac{1}{\alpha}G_{nx}A_{bb}D_{xn}-A_{nb}D_{xn}-G_{nx}\partial_u F_b\right)-\frac{1}{\alpha}G_{nx}\partial_u(A_{bb}u_b)+\partial_u(A_{nb}u_b)$$

e então:

$$\frac{\partial}{\partial u_n}\mathcal{F}_n=S_{nn}+\frac{1}{\alpha}\left(\frac{1}{\alpha}G_{nx}A_{bb}D_{xn}-A_{nb}D_{xn}-G_{nx}S_{bn}\right)$$