Thresholds, 2AFC and Not Signal Detection Theory: A Tutorial Introduction

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1 Introduction

This chapter is about how to find out how good an observer (or a neuron) is at detecting a stimulus, from a purely statistical perspective. This, it turns out, is also how to find out how good an observer is at detecting the *difference*, or discriminating, between two stimuli. These stimuli could be sounds, colours, velocities, or rabbits; but we will use the example of brightness here. The reason that detection and discrimination can be examined in the same framework is this. An observer who can detect a very dim light can (almost by definition) also discriminate between that dim light and no light.

The plan is to estimate how good an observer is at detecting a fixed brightness difference, as in Figure 1. As we increase the brightness difference, we naturally expect performance to increase, as in Figure 2b. Note that we might expect performance to increase in a step-like manner, as in Figure 2a. The reason performance does not increase like this is (as we shall see) because of a single ubiquitous factor in all sensory systems: *noise.*

The plan is *not* to step through all the details of signal detection theory (SDT) in order to estimate observer sensitivity in terms of a quantity known as d' (d-prime). Instead we will simply show why using the most common psychophysical method (2AFC) yields an estimate of d' (see below for an account of d' 2AFC).

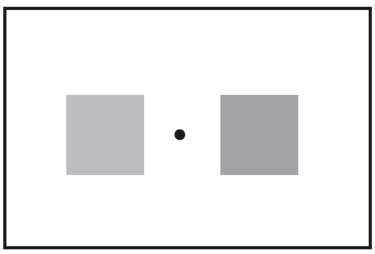


Figure 1: **Stimulus pair**. The observer's task is to choose with stimulus is brighter, the one on the left, or the one on the right?

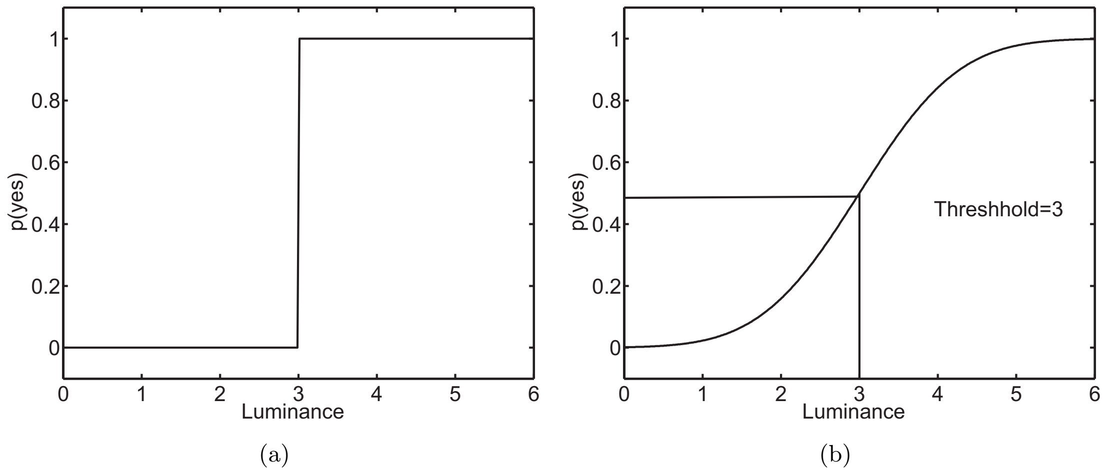


Figure 2: **Transition from not seeing to seeing**.
a) In an ideal world, the probability of seeing a brief flash of light would jump from zero to one as the light increased in luminance.
b) In practice, the probability of a yes response increases gradually, and defines a smooth S-shaped or sigmoidal psychometric function. The threshold is taken to be the mid-point of this curve: the luminance at which the probability of seeing the light is 0.5. Most quantities on the abscissa (horizontal axis) in plots of this kind are given in log units, so that each increment (e.g., from 3 to 4) along this axis implies a multiplicative change in magnitude. For example, if the log scale being used was to the base 10 then a single step on the abscissa would mean the luminance increased by a factor of ten.

2 Measuring performance using 2AFC

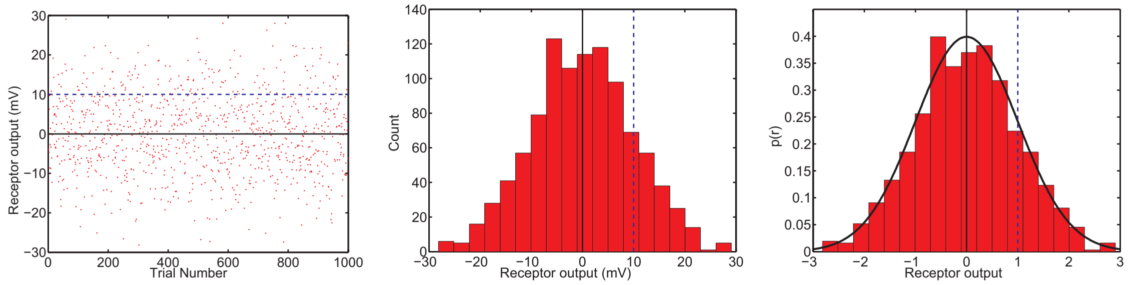
So, there you are sitting in a psychology laboratory, staring at spot a computer screen. Suddenly, two squares appear, one on each side of a *fixation point*, the spot in Figure 1. Your mission is to decide which square is brighter. Of course, as this is a psychology laboratory, sometimes the squares are equally bright; but you still have to choose the one that you think *appears* to be brighter. This is called a *two-alternative forced choice* procedure, or 2AFC, for short.

Given that you have to choose one of the stimuli, what is the probability of choosing the brighter stimulus? If we use the variable s to denote brightness then we know that either $s = s_1$ (dark stimulus) which we will call class C_1, or $s = s_2$ (bright stimulus) which we will call class C_2.

Now, let us suppose that your response depends on the output of a single receptor or neuron which is sensitive to brightness, but which has a noisy output r with a Gaussian distribution. Suppose you look at the darker of the two stimuli, which happens to be black, so that the neuron's output is actually its baseline 'idling' output. For example, if the same black stimulus is presented 1000 times then the responses of this neuron could look like Figure 3a. A histogram of these responses is given in Figure 3b, which is a good approximation to a Gaussian curve, as shown in Figure 3c. Specifically, if $s = s_1$ then the distribution of r values is defined by the conditional *probability density function* (pdf)

$$p(r|s_1) = k_1 \, e^{-(r-\mu_1)^2/(2\sigma_1^2)}, \tag{1}$$

where μ_1 is the distribution's mean, and σ_1 is the standard deviation of the distribution of r



(a) (b) (c)

Figure 3: **Noisy neurons**.
a) If we measured the output r of a single photoreceptor over 1000 trials then we would obtain values like those plotted here, because r varies randomly from trial to trial. Note that this receptor is assumed to be in total darkness here. The probability that r is greater than some value μ (set to 10mV here) is given by the proportion of dots above the blue dashed line, and is written as $p(r > \mu)$.
b) Histogram of r values measured over 1000 trials shows that the mean receptor output is $\mu = 10$ mV and that the variation around this mean value has a standard deviation of 10mV. The probability $P(r > \mu)$ is given by the proportion of histogram area to the right of μ, indicated here by the blue dashed vertical line.
c) The histogram in **b** is a good approximation to a Gaussian or normal distribution of r values, as indicated by the solid (black) curve. Notice that this distribution has been normalized to unit area. This means that the values plotted on the ordinate (vertical axis) have been adjusted to that the area under the curves adds up to unity. The resulting distribution is called a *probability density function* or pdf.

values, and $k_1 = (2\pi\sigma_1^2)^{-1/2}$ is a constant.

When the eyes are looking at the darker stimulus, the resultant distribution of output values is known as the *noise distribution*. If, as in this example, the eyes are looking at a stimulus with *no* light then it is not hard to see why this is so.

Similarly, when the eyes are looking at the brighter stimulus, the resultant distribution of output values is known as the *signal distribution*. In this case, $s = s_2$, and the distribution of r values is defined by the pdf

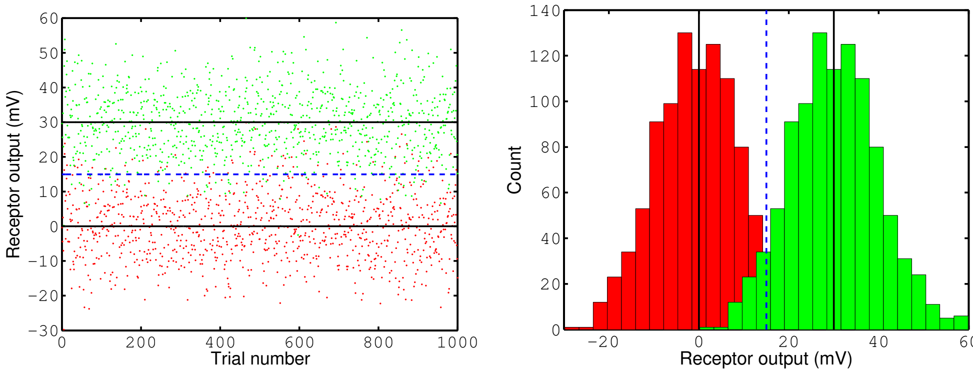
$$p(r|s_2) = k_2 \, e^{-(r-\mu_2)^2/(2\sigma_2^2)}, \tag{2}$$

with mean μ_2, and with σ_2. The responses of the neuron if $s = s_1$ are also shown in Figure 4a, alongside responses if $s = s_2$, and a histogram of each set of responses are overlaid in Figure 4b.

For simplicity we assume that $\sigma_2 = \sigma_1$, so that $k_1 = k_2$, and we will write these as σ and k from now on.

2.1 The ideal observer and d'

On each trial, the observer is presented with two stimuli, and therefore has two values of r: r_1 which is generated by s_1, and r_2 generated by s_2. But note that the observer does not know which response was generated by which stimulus. However, if the observer makes the simple assumption that the larger of the two responses is r_2, and therefore that this response was generated by s_2, then the observer's choice is *correct* (ie $s = s_2$) only if $r_2 > r_1$. Indeed, this turns out to be the optimal rule for minimising the number of incorrect choices:



(a) (b)

Figure 4: **Signal and noise distributions**.
a) Measured values of receptor response r to the dark stimulus s_1 (lower red dots) and to the bright stimulus (upper green dots), with the mean of each set of r values given by a horizontal black line.
b) Histograms of the two sets of dots shown in **a**, with means now given by vertical black lines. The dashed line denotes the point μ mid-way between the means of the two distributions.

choose the stimulus associated with the largest value of r.

An observer making use of this rule is known as an *ideal observer*.

Clearly, for a fixed brightness difference, the larger the distance between μ_1 and μ_2, the easier it is to tell which response is from which distribution. This distance is actually measured in units of standard deviation, and is called d-prime:

$$d' = \frac{\mu_2 - \mu_1}{\sigma}. \quad (3)$$

Indeed, d' is an excellent measure of the sensitivity of an observer, because it indicates how well the observer can discriminate between receptor/neuronal outputs caused by the environment and outputs caused by noisy receptors.

Given that the observer makes the correct choice s_2 only when $r_2 > r_1$, the proportion $P(r_2 > r_1)$ of trials on which the observer is correct is the proportion of trials on which $r_2 > r_1$. Therefore, $P(r_2 > r_1)$ is the probability of a correct response using the rule: *choose the stimulus with the largest response*. In essence, this means comparing the two *unlabelled* responses (because the observer does not know which is which), and choosing that stimulus (ie either s_1 or s_2) associated with the largest response.

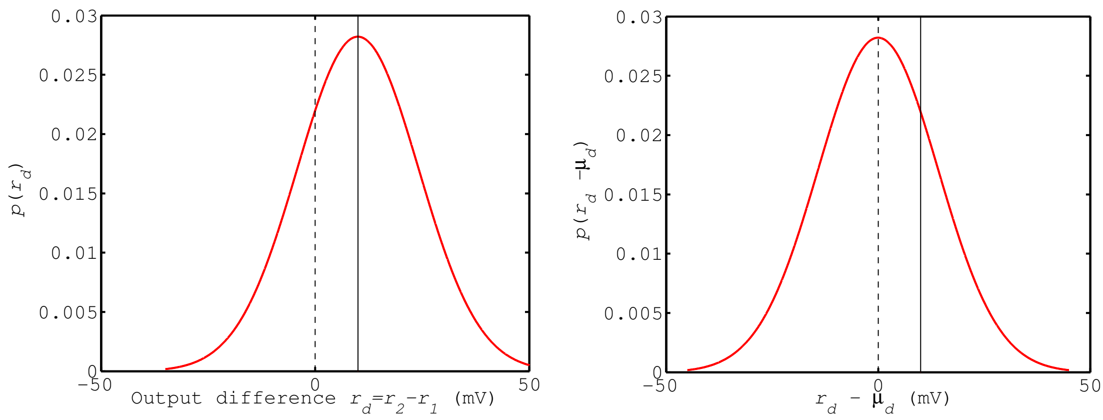
2.2 Difference Distribution

The probability $P(r_2 > r_1)$ of choosing s_2 is derived as follows. We begin by defining a new variable, the difference

$$r_d = r_2 - r_1. \quad (4)$$

If $r_2 > r_1$ then $r_d > 0$, and therefore the pdf

$$p(r_2 > r_1) = p(r_d > 0). \quad (5)$$



(a) (b)

Figure 5: **Distribution of differences**.
a) The distribution of differences $r_d = r_2 - r_1$, for two Gaussian distributions such as those in Figure (4). This distribution has a mean of $\mu_d = \mu_2 - \mu_1$ (solid black line), and a standard deviation $\sigma_d = \sqrt{(\sigma_1^2 + \sigma_2^2)}$. The probability $P(r_d > 0)$ that $r_d > 0$ is given by the area to the right of the dashed black line.
b) If the curve in a) is shifted to the left by an amount μ_d, then the shifted curve has a mean of zero. Now $P(r_d > 0)$ is given by the area to the left of the solid black line located at μ_d.

If r_1 and r_2 are Gaussians with means μ_1 and μ_2, and a common standard deviation $\sigma = \sigma_1 = \sigma_2$, then the pdf $p(r_d)$ shown in Figure (5) has a mean

$$\mu_d = (\mu_2 - \mu_1), \tag{6}$$

and standard deviation[1]

$$\sigma_d = \sigma\sqrt{2}, \tag{7}$$

so that the pdf of the difference is

$$p(r_d) = k_d\, e^{-(r_d-\mu_d)^2/(2\sigma_d^2)}, \tag{8}$$

where $k_d = (2\pi\sigma_d^2)^{-1/2}$.

Now, if the rule is to choose s_2 whenever $r_d > 0$ then the probability that s_2 is the correct choice is

$$P(r_d > 0) = k_d \int_{y=0}^{\infty} e^{-(y-\mu_d)^2/(2\sigma_d^2)}\, dy, \tag{9}$$

as shown in Figure 5a. Notice that the area to the right of zero for the pdf with mean μ_d (Figure 5a) is the same as the area to the left of μ_d for the same shifted pdf with mean zero (Figure 5b), so we can re-write Equation (9) as

$$P(r_d > 0) \quad = \quad k_d \int_{y=-\infty}^{\mu_d} e^{-y^2/(2\sigma_d^2)}\, dy. \tag{10}$$

All we have done here is to translate the pdf along the abscissa by an amount $-\mu_d$, so that it

[1]See Appendix to why this is so.

now has zero mean. Because

$$\mu_d/\sigma = \mu_d/(\sigma_d\sqrt{2}) \quad (11)$$
$$= d', \quad (12)$$

it follows that

$$\sigma_d = \sigma_1\sqrt{2}, \quad (13)$$

so we can re-write Equation (10) as

$$P(r_d > 0) = k_d \int_{-\infty}^{\mu_d} e^{-y^2/(2\times 2\sigma_1^2)}\, dy. \quad (14)$$

Now we can 'trade' units of the the upper limit for units of standard deviation. By definition, if the distribution has zero mean and a standard deviation of $\sigma_d = \sigma_1\sqrt{2}$ then μ_d is $\mu_d/(\sigma_1\sqrt{2})$ standard deviations below the mean of zero. If we re-express this upper limit in units of standard deviation then we can set the standard deviation of the integrand to unity (because the new upper limit will still be $\mu_d/(\sigma_1\sqrt{2})$ standard deviations above the mean of zero)

$$P(r_d > 0) = k_d \int_{-\infty}^{\mu_d/(\sigma_1\sqrt{2})} e^{-y^2/2}\, dy. \quad (15)$$

If we now set $\sigma_1 = 1$ (which requires us to change k_d to k) then we have

$$P(r_d > 0) = k \int_{-\infty}^{d'/\sqrt{2}} e^{-y^2/2}\, dy \quad (16)$$
$$= \Phi(d'/\sqrt{2}), \quad (17)$$

where Φ is the standard symbol for the cumulative distribution function (cdf) of a Gaussian pdf. For convenience, if define the measured proportion correct as

$$P_c = P(r_d > 0) \quad (18)$$

then we can find d' from

$$d' = \Phi^{-1}(P_c)\sqrt{2}, \quad (19)$$

where Φ^{-1} is the inverse of the Gaussian cdf.

This is an important result because it allows us to obtain a measure d' of an observer's sensitivity from proportion of correct choices in a 2AFC procedure. For example, if the difference between two squares like those in Figure 1 is adjusted until the proportion of correct observer responses $P_c = 0.76$ then (using Equation (19)) this implies $d' = 1$.

3 Absolute Threshold: Same result, different derivation

We can obtain the same result by considering the probability of *detecting* a dim flash of light. As this amounts to detecting the *difference* between a dim light and no light, it is equivalent to the discrimination task described above, and we should not be surprised to find it yields the same result.

In this case, the observer is presented with a light or no light on each trial. The observer's task is to decide if the neuronal response r belongs to the pdf associated with no light, or to the pdf associated with a dim light.

For consistency, we retain the same notation as before. Thus, the no-light trials have a Gaussian pdf $p(r|\mu_1)$ of r values with mean μ_1, and the light trials have a pdf $p(r|\mu_2)$ with a mean of μ_2, and both share a common standard deviation σ.

We assume that the observer chooses to classify r as s_1 only if $r < r_{crit}$, where r_{crit} is some critical value of r, as shown in Figure 6. There exists a value r_{crit} which minimises the number of incorrect choices. If s_1 and s_2 are equally likely then the way to minimise the number of incorrect choices is to set r_{crit} to be mid-way between μ_1 and μ_2, a quantity we define as

$$\mu = (\mu_2 + \mu_1)/2. \tag{20}$$

So, we set $r_{crit} = \mu$.

If $s = s_1$ and if $r < \mu$ then the observer classifies r as belonging to class C_1. The probability that C_1 is the correct class is the probability that $r < \mu$ given that $s = s_1$

$$P(r < \mu|s_1) = k_1 \int_{-\infty}^{\mu} e^{-(r-\mu_1)^2/(2\sigma^2)}\, dr. \tag{21}$$

Similarly, the probability that C_2 is the correct class is the probability that $r > \mu$ given that $s = s_2$

$$P(r > \mu|s_2) = k_2 \int_{\mu}^{\infty} e^{-(r-\mu_2)^2/(2\sigma^2)}\, dr. \tag{22}$$

Notice that (assuming a common σ value)

$$P(r < \mu|s_1) \quad = \quad P(r > \mu|s_2). \tag{23}$$

So, the overall probability of a correct response (ie for any value of r) is

$$\begin{aligned} P_c &= [P(r < \mu|s_1) + P(r > \mu|s_2)]/2 && (24)\\ &= P(r < \mu|s_1) && (25)\\ &= P(r > \mu|s_2). && (26) \end{aligned}$$

Given that

$$d' = (\mu_2 - \mu_1)/\sigma, \tag{27}$$

the distance between the two pdf means is

$$\mu_2 - \mu_1 = d'\sigma. \tag{28}$$

Now, μ is half way between μ_1 and μ_2, so that

$$\begin{aligned} \mu &= \mu_1 + (\mu_2 - \mu_1)/2 && (29)\\ &= d'\mu_1 + \sigma/2. && (30) \end{aligned}$$

Substituting Equation (30) in the upper limit of Equation (21), the probability of a correct response is

$$P_c = k \int_{-\infty}^{\mu_1 + d'\sigma/2} e^{-(r-\mu_1)^2/(2\sigma^2)}\, dr. \tag{31}$$

If we set $\mu_1 = 0$ then the probability of a correct response is

$$P_c \quad = \quad k \int_{-\infty}^{d'\sigma/\sqrt{2}} e^{-r^2/(2\sigma^2)}\, dr, \tag{32}$$

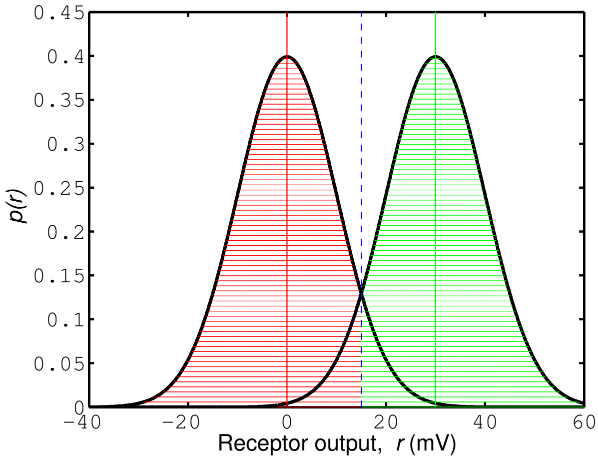


Figure 6: **Setting the optimal criterion**.
When the light is off the neuronal response r belongs to the left hand (noise) distribution with mean 0mV, and when the light is on, r belongs to the right hand (signal) distribution with mean 30mV. An observer can minimise the number of incorrect responses by stating that a light was seen only if the neuronal response r is greater than a critical value. If the two distributions have identical standard deviations then this critical value μ is mid-way between the means of the two distributions, indicated by the vertical dashed line. In this case, the probability of correctly stating the light is on is given by green shaded the area to the right of μ, and the probability of correctly stating the light is off is given by red shaded the area to the left of μ.

and if we set $\sigma = 1$ then we have

$$P_c = k \int_{-\infty}^{d'/\sqrt{2}} e^{-r^2/2}\, dr \quad (33)$$

$$= \Phi(d'/\sqrt{2}). \quad (34)$$

Why did we just do all that? Because we now have the means to relate the observed probability P_c of a correct response to d', the ability to separate the signal and noise distributions. More importantly, we can also deduce the signal to noise ratio as $SNR = (d')^2$.

4 Appendix

4.1 The mean of the distribution of differences

Given two distributions with means

$$\mu_1 = \mathrm{E}[r_1] \quad (35)$$

$$\mu_2 = \mathrm{E}[r_2], \quad (36)$$

the mean of the distribution of differences is

$$\mathrm{E}[r_2 - r_1] = \mathrm{E}[r_2] - \mathrm{E}[r_1] \quad (37)$$

$$= \mu_2 - \mu_2. \quad (38)$$

If we define $r_d = r_2 - r_1$ and $\mu_d = \mu_2 - \mu_2$ then

$$\mathrm{E}[r_d] = \mu_d. \tag{39}$$