MSO203B (Lecture 3) $\Rightarrow$ Sturm Liouville Theory 2

$$Ly = (py')' + qy \quad ; x \in I \qquad L : C^2(I) \xrightarrow[\text{linear operator}]{} C(I)$$

Case 1

weight

$$Ly + \lambda r y = 0 \quad \text{--- (I)}$$

$$\left.\begin{array}{l} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \ ; \ \alpha_1^2 + \alpha_2^2 \neq 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \ ; \ \beta_1^2 + \beta_2^2 \neq 0 \end{array}\right\} \text{(II)}$$

Here y is unknown, $\lambda \in \mathbb{C}$

p, q, r are smooth fns on I.

(I) + (II) gives us Regular SL BVP provided p and r are positive in I.

Case 2:-

$$Ly + \lambda r y = 0 \quad \text{--- (III)}$$

$$\left.\begin{array}{l} y(a) = y(b) \\ y'(a) = y'(b) \end{array}\right\} \text{--- (IV)}$$

(III) + (IV) is called a periodic S-L provided provided $p(a) = p(b)$ with $p, r > 0$ in I

transform a General eqn to self-adjoint form

$$\tilde{L}y = a(x)y'' + b(x)y' + c(x)y.$$

$$F(x)a(x)y'' + F(x)b(x)y' + c(x)F(x)y = \boxed{F(x)\tilde{L}(y) := \tilde{L}_0(y)}$$

I want $\tilde{L}_0$ to look like L.

$$\Rightarrow \quad F(x)\,b(x) = [F(x)\,a(x)]'$$

$$\Rightarrow \quad Fb = F'a + Fa' \Rightarrow F'a = F(b - a').$$

Self-adjoint form

$$(py')' + qy = Ly$$

$$py'' + p'y' + qy = Ly$$

$F'a = F(b - a')$.

$F(x) = \exp\left(\int_x \frac{b-a'}{a}\,dx\right)$.

$\boxed{y, w \in C^2 \sim C^\infty}$

Lagrange Identity :-

$L[y] = (py')' + qy$. $\longrightarrow \times w$

$L[w] = (pw')' + qw$. $\longrightarrow \times y$.

$$wL[y] - yL[w] = \underline{w(py')' + q\cancel{yw} - (pw')'y} - q\cancel{wy}$$

$$= [w(py') - (pw')y]'$$

$$\int_a^b [wL[y] - yL[w]]\,dx = \int_a^b [p(wy' - yw')]'\,dx$$

$$= p(wy' - yw')\Big|_a^b.$$

$[w(py') - (pw')y]'$

$w'\cancel{py'} + \underline{w(py')' - (pw')'y} - \cancel{pw'y'}$

Property:- (i) The eigenvalues of a RSLBVP are always real. **Remarks:-** Same holds for PSLBVP

(λ, u) – Eigen pair s.t $\lambda \in \mathbb{C}$.

$$\left.\begin{aligned} & Lu + \lambda w u = 0 \\ & \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ & \beta_1 u(b) + \beta_2 u'(b) = 0. \end{aligned}\right\} \text{— (I)}$$

Taking complex conjugate

$$\left.\begin{aligned} & L\bar{u} + \bar{\lambda} w \bar{u} = 0 \\ & \alpha_1 \bar{u}(a) + \alpha_2 \bar{u}'(a) = 0. \\ & \beta_1 \bar{u}(b) + \beta_2 \bar{u}'(b) = 0 \end{aligned}\right\} \text{— (II)}.$$

$$\int_a^b \left[\bar{u}\, L(u) - u\, L(\bar{u})\right] dx = p(\bar{u}u' - u\bar{u}')\Big|_a^b$$

$Lu = -\lambda w u.$

$p(b)\bar{u}(b)u'(b) - p(b)u(b)\bar{u}'(b)$

$\boxed{- p(a)\bar{u}(a)u'(a) + p(a)u(a)\bar{u}'(a)}$

$\alpha_1 u(a) + \alpha_2 u'(a) = 0$

$\alpha_1 \bar{u}(a) + \alpha_2 \bar{u}'(a) = 0$

$$\begin{vmatrix} u(a) & u'(a) \\ \bar{u}(a) & \bar{u}'(a) \end{vmatrix} = 0$$

$\Rightarrow \boxed{u(a)\bar{u}'(a) - u'(a)\bar{u}(a) = 0}$

$$\int_a^b (-\lambda w u\bar{u} + \bar{\lambda} w u \bar{u})\, dx = 0 \Rightarrow (\bar{\lambda} - \lambda)\int_a^b w u \bar{u}\, dx = 0$$

$$\Rightarrow (\bar{\lambda} - \lambda)\int_a^b w|u|^2\, dx = 0 \Rightarrow \underline{\lambda = \bar{\lambda}}$$

Property 2 :- Assume λ and $\bar{\lambda}$ are two distinct eigenvalue of a RSLBVP and let u and $\bar{u}$ are the corresponding eigenfunctions. $\Rightarrow$ u and $\bar{u}$ are mutually orthogonal to each other w.r.t 'w'.

($\bar{u}$ or $\bar{\lambda}$ are not complex conjugate).

$(\lambda, u) \leftarrow$ Eigenpair

$(\bar{\lambda}, \bar{u}) \leftarrow$ Eigenpair

$$Lu + \lambda r u = 0 \quad \text{--- (I)} \times \bar{u}$$

$$L\bar{u} + \bar{\lambda} r \bar{u} = 0 \quad \text{--- (II)} \cdot \times u$$

$$\int_a^b [\bar{u}Lu - uL\bar{u}]\,dx = p(\bar{u}u' - u\bar{u}')\Big|_a^b$$

$$\Rightarrow \int_a^b (-\lambda r u\bar{u} + \bar{\lambda} r u\bar{u})\,dx = 0$$

$$\Rightarrow (\bar{\lambda} - \lambda)\int_a^b r u\bar{u}\,dx = 0$$

$$\Rightarrow \int_a^b r \underset{''}{u}\,\underset{''}{\bar{u}}\,dx = 0.$$

Defn :- We say u and v are mutually orthogonal to each other w.r.t the weight fcn w if

$$\int_a^b uvw\,dx = 0$$

In $\mathbb{R}^2$; $a \cdot b = 0$.

In X; $\int_a^b uv\cancel{w}\,dx = 0$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx =$$

Property 3 :- Eigenvalues of a RSLBVP are simple.

Remark :- This property does not hold Periodic SLBVP

$$\lambda \begin{cases} u \\ \bar{u} \end{cases}$$

$$Lu + \lambda r u = 0 \quad \text{— (I)}$$

$$L\bar{u} + \lambda r \bar{u} = 0 \quad \text{— (II)}$$

$$\bar{u} L u = -\lambda r u \bar{u}$$

$$u L \bar{u} = -\lambda r \bar{u} u$$

Lagrange Identity

$$(\bar{u} L u - u L \bar{u}) = [p(\bar{u} u' - u \bar{u}')]'$$

$$\Rightarrow [p(\bar{u} u' - u \bar{u}')]' = 0$$

$$\Rightarrow p(\bar{u} u' - u \bar{u}') = c = 0 \quad \text{(From the Bdry condition)}$$

$$\Rightarrow p\, W(u, \bar{u}) = 0 \Rightarrow W(u, \bar{u}) = 0 \Rightarrow u = c\bar{u}$$

$$p(\bar{u} u' - u \bar{u}')(a) = 0$$

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Existence of Eigenvalues :-

$\exists$ an infinite sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ ~~of~~ of a RSLBVP / PSLBVP such that.

$$-\infty < \lambda_1 < \lambda_2 < \cdots\cdots < \lambda_n < \cdots\cdots$$

For RSLBVP :- $\lim_{n\to\infty} \lambda_n = \infty$