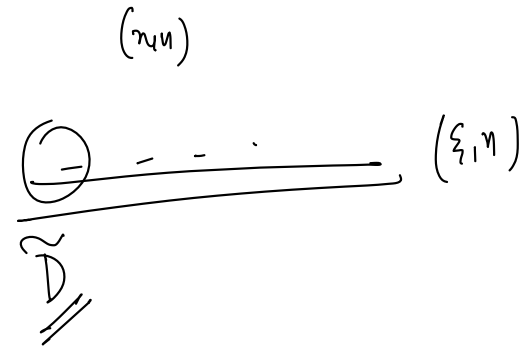
Lecture 10 : MSO 203B (Canonical Form)

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{--- (1)}$$

Here $A, B, \ldots$ are smooth fns in (x,y)



Classification

$\Delta := B^2 - 4AC$

If $\Delta > 0$ - (1) is Hyperbolic

If $\Delta = 0$ - (1) is Parabolic

If $\Delta < 0$ - (1) is Elliptic

Ex 1:- $u_{tt} - u_{xx} = 0$ (Wave Eqn).

$A = -1,\ B = 0,\ C = 1,\ D = E = F = G = 0$.

$\Delta := B^2 - 4AC = -4 \cdot (-1) \cdot 1 = 4 > 0$

Hyperbolic

Ex 2:- $u_{xx} + u_{yy} = 0$ (Laplace Eqn).

$A = 1,\ B = 0,\ C = 1, \ldots$

$\Delta := B^2 - 4AC = -4 < 0$

- Elliptic Eqn

Ex 3:- $u_t - u_{xx} = 0$ (Heat Eqn)

$A = -1,\ B = 0,\ C = 0,\ D = 0,\ E = 1$.

$B^2 - 4AC = 0$ - Parabolic Eqn

Change of variable :-

$(x,y) \xmapsto{F} (\xi(x,y), \eta(x,y))$ is called a C.O.V if F is smooth and $JF(x,y) \neq 0$.

$$JF(x,y) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 .$$

Theorem. The nature of the eqn under a C.O.V remains invariant.

$w: \mathbb{R}^2 \to \mathbb{R}$

Define $w(\xi,\eta) := u(x,y)$.

$\nabla w = (w_\xi, w_\eta) \cdot (\xi_x, \eta_x)$

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$u_y = w_\xi \xi_y + w_\eta \eta_y$.

(A) $u_{xx} = [w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x]\xi_x + w_\xi \xi_{xx} + [w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x]\eta_x + w_\eta \eta_{xx}$

(C) $u_{yy} = [w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y]\xi_y + w_\xi \xi_{yy} + [w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y]\eta_y + w_\eta \eta_{yy}$.

(B) $u_{xy} = [w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y]\xi_x + w_\xi \xi_{xy} + w_\eta \eta_{xy} + [w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y]\eta_x$

Our transformed eqn is

$$\tilde{A} w_{\xi\xi} + \tilde{B} w_{\xi\eta} + \tilde{C} w_{\eta\eta} + \tilde{D} w_{\xi} + \tilde{E} w_{\eta} + \tilde{F} w = \tilde{G}. \quad \text{———} \quad (11)$$

$$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2.$$

$$\tilde{B} = 2A\xi_x\eta_x + (\xi_x\eta_y + \eta_x\xi_y)B + 2C\xi_y\eta_y$$

$$\tilde{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y.$$

$$\tilde{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y.$$

$$\tilde{F} = F \;/\; \hat{G} = G.$$

$$\tilde{\Delta} := \tilde{B}^2 - 4\tilde{A}\tilde{C}.$$

$$= J^2(B^2 - 4AC) \quad (\text{Check}).$$

$$= J^2\Delta.$$

$$\boxed{J^2 = \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}^2}$$

Case 1:- (Hyperbolic Eqn).

$\tilde{A} = \tilde{C} = 0$

$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ — (3)

$A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ — (4)

(3) + (4) $\Rightarrow$ ξ & η are roots of the eqn $A\theta_x^2 + B\theta_x\theta_y + C\theta_y^2 = 0$.

$\Rightarrow A\dfrac{\theta_x^2}{\theta_y^2} + B\dfrac{\theta_x}{\theta_y} + C = 0 \quad (\theta_y \neq 0)$.

$\Rightarrow \dfrac{\theta_x}{\theta_y} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ — $:= p$

$\theta_x = p\theta_y \Rightarrow \theta_x - p\theta_y = 0$.

$x'(s) = 1 \mid y'(s) = -p \quad (z'(s) = 0$

$\dfrac{dy}{dx} = -p \Rightarrow \dfrac{dy}{dx} = \dfrac{B \mp \sqrt{B^2 - 4AC}}{2A}$ ← Char Curves

This implies θ is constant along the Char. Curves.

$\therefore$ Char Curves are given as $\phi_1(x,y)=0$ and $\phi_2(x,y)=0$.

Choose $\xi(x,y)=\phi_1(x,y)$

$\eta(x,y)=\phi_2(x,y)$.