MSO 203B - PDE    Lecture 9 : Method of Characteristics 3

$$y u_x + x u_y = u \implies (u_x, u_y, -1)\cdot(y, x, u) = 0$$

$$u(0,y) = y^3.$$

Projected Data curve $\Gamma(r) = (0, r) \leftarrow$ in $\mathbb{R}^2$

Data curve :- $(\Gamma, \phi) = (0, r, r^3)$. : $\phi(\omega) = \omega^3$.

$(\Gamma, \phi) \subset S$ = Graph of $u$.

Let $C(s)$ be the char. curve

$$x'(r,s) = y \qquad x(r,0) = 0 \quad \text{—(I)}$$

$$y'(r,s) = x \qquad y(r,0) = r \quad \text{—(II)}$$

$$z'(r,s) = z \qquad z(r,0) = r^3.$$

$$z(r,s) = \psi(r)e^s.$$

$$r^3 = z(r,0) = \psi(r)e^0.$$

$$\implies \psi(r) = r^3$$

$$z(r,s) = r^3 e^s.$$

$$\frac{dy}{dx} = \frac{x}{y} \qquad \left(\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx}\right).$$

$$\implies y^2 - x^2 = \pi(r)$$

$$r^2 - 0 = \pi(r)$$

$$\boxed{\begin{aligned} y^2 - x^2 &= r^2 \\ z &= r^3 e^s \end{aligned}}$$

given $r \in \mathbb{R}$ Alternative

$$x_r'(s) = y \qquad x_r(0) = 0.$$

$$y_r'(s) = x \qquad y_r(0) = r.$$

$$\Downarrow$$

$$\text{\_\_\_\_\_\_\_\_\_\_} \;\ldots$$

$$z_r'(s) = z \qquad z_r(0) = r^3.$$

$$\Downarrow$$

$$z(s) = Ae^s.$$

$$z_r(s) = A(r)e^s.$$

$$z_r(s) = r^3 e^s.$$

$$u_t + a u_x = 0$$

$$y^2 - x^2 = r^2$$

$$z = r^3 e^s.$$

$$\tilde{r}\cosh^2 s - \tilde{r}\sinh^2 s = \tilde{r}$$

$$\{\underbrace{r\sinh s}_{x},\ \underbrace{r\cosh s}_{y}\} \leftarrow \text{Projected Char.}$$

$$r = \sqrt{y^2 - x^2}.$$

$$s = \tanh^{-1}\left(\frac{x}{y}\right).$$

$$u(x,y) := z(r,s) = (y^2 - x^2)^{3/2} \tanh^{-1}\left(\frac{x}{y}\right).$$

Question: (2) Is the soln unique?

(1) Does there exist a soln?

$$\# \quad a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

$$u\big|_\Gamma = \phi$$

$$\Gamma(r) = (\gamma_1(r), \gamma_2(r)).$$

(*)

Defn:- $\Gamma$ is called the non-characteristics. if $\Gamma$ is nowhere tangent to the projected char curves. ie,

$\Big(a(r_1(r), r_2(r), u(r_1(r), r_2(r)), b(r_1(r), r_2(r), u(r_1(r), r_2(r))\Big) \cdot \big(-r_2'(r), r_1'(r)\big) \neq 0.$

$\Rightarrow \Big(a(r_1(r), r_2(r), \phi(r)), b(r_1(r), r_2(r), \phi(r))\Big) \cdot \big(-r_2'(r), r_1'(r)\big) \neq 0.$

Th :- (Existence & Uniqueness).

If $\Gamma$ is a non-char to the eqn (*) then $\exists$ a unique soln $u \in C^1$ near the data curve $\Gamma$.

$y' = f(x,y)$

$y(x_0) = y_0$

$\left|\frac{\partial f}{\partial y}\right| < M.$

Transport Eqn :-

$u_t + \tilde{a} u_x = 0$

$u(x,0) = \phi(x).$

$\Rightarrow u(x,t) = \phi(x - \tilde{a} t)$

$\Gamma(r) = \{(r,0) : r \in \mathbb{R}\}$

Non-Char $\Rightarrow (\tilde{a}, 1) \cdot (0,1) = 1 \neq 0$

$a(r_1(r), r_2(r)) = \tilde{a}$

$b(r_1(r), r_2(r)) = 1.$

## Burger's Eqn :-

$u_t + u u_x = 0$

$u(x,0) = \phi(x)$.

$a(r_1(r), r_2(r), u(r_1(r), r_2(r))) = a(r, 0, \phi(r)) =$

$F(r) = (r, 0) = (r_1(r), r_2(r))$

Non. Ch $:\Rightarrow \left(\phi(r), 1\right) \cdot (0, 1) = 1 \neq 0$

Char Eq

$x'(r,s) = z$

$x(r,0) = r$

$\Downarrow$

$x'(r,s) = \phi(r)$

$\Rightarrow x(r,s) = \phi(r)s + A(r)$.

$r = x(r,0) = \phi(r)\cdot 0 + A(r)$

$\therefore x(r,s) = \phi(r)s + r$.

$\Rightarrow x = zs + r$

$\Rightarrow x = zt + r \quad$ —— (1)

$t'(r,s) = 1$

$t(r,0) = 0$

$\Rightarrow t(r,s) = s + B(r)$

$\Rightarrow t(r,s) = s$

$z'(r,s) = 0$

$z(r,0) = \phi(r)$.

$\Downarrow$

$z(r,s) = \phi(r)$

Soln $\quad u(x,y) := z(r,s) = \phi(r)$

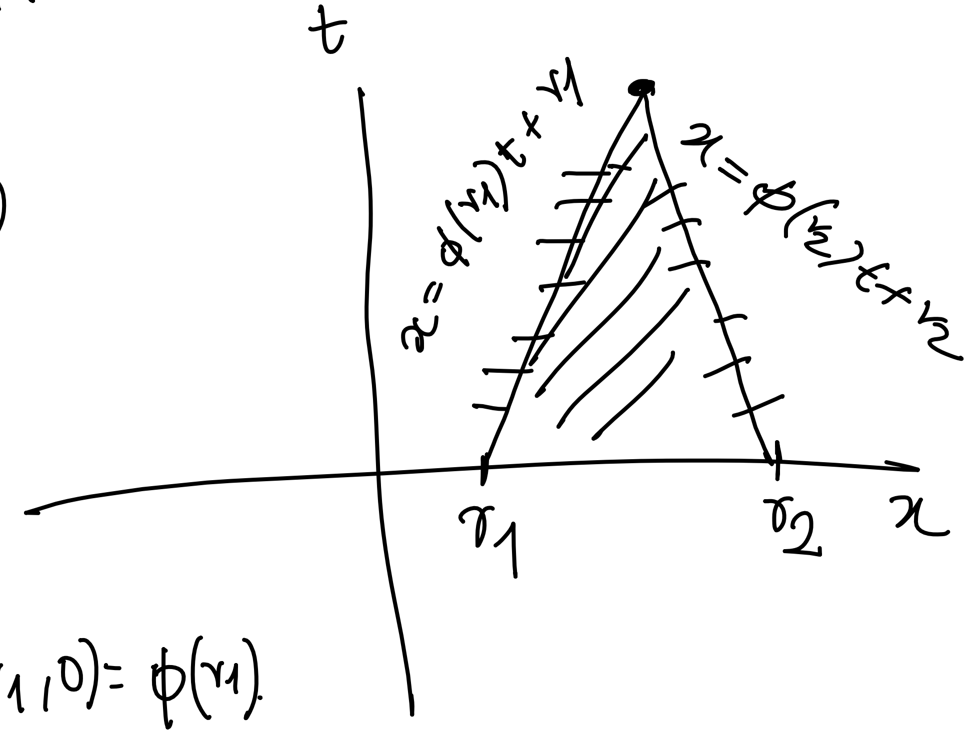
$= \phi(x - ut)$

Projected Ch :-

$x = \phi(r)t + r$

✓ $r_1 > r_2$

$\phi(r_1) < \phi(r_2)$




$u(r_1, 0) = \phi(r_1)$

$u(r_2, 0) = \phi(r_2)$