MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-X

A. Illustrative Discussion Problems

1. Let random vector (X, Y) have the joint p.m.f.

$$f_{X,Y}(x,y) = P(X = x, Y = y) = \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta^{x+y}(1-2\theta)^k, \; x, y = 0, 1, 2, \ldots,$$

where $k \geq 1$ is an integer and $0 < \theta < \frac{1}{2}$ (we say that (X, Y) has bivariate negative binomial distribution). Find the marginal p.m.f. of X, the conditional distribution of Y given $X = x$ ($x \in \{0, 1, \ldots\}$) and the p.m.f. of $Z = X + Y$.

2. Let the bivariate r.v. (X, Y) have joint p.d.f.

$$f_{X,Y}(x,y) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}x^{\theta_1-1}y^{\theta_2-1}(1-x-y)^{\theta_3-1}, \; x > 0, \; y > 0, \; x+y < 1,$$

where $\theta_i > 0, \; i = 1, 2, 3$ (we say that (X, Y) has bivariate beta distribution). Find the marginal p.d.f. of X, the conditional p.d.f. of X given $Y = y$ ($y \in (0, 1)$) and the p.d.f. of $Z = X + Y$.

3. Let $X_1, \ldots, X_n$ be i.i.d. A.C. r.v.s with common p.d.f. $f(\cdot)$ and common d.f. $F(\cdot)$. For $r \in \{1, \ldots, n\}$, let $X_{r:n}$ = r-th smallest of $X_1, \ldots, X_n$ (so that $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are order statistics of random sample $X_1, \ldots, X_n$).
(a) Find the joint d.f. of r.v. $(X_{1:n}, \ldots, X_{n:n})$ and hence show that it is A.C. with a joint p.d.f.

$$g(y_1, \ldots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f(y_i), & \text{if } y_1 < y_2 < \cdots < y_n \\ 0, & \text{otherwise} \end{cases}.$$

(b) (**Relationship between beta and binomial distribution**) Let $X \sim \text{Bin}(n, p)$, where $p \in (0, 1)$ and $n \in \mathbb{N}$. Using integration by parts, show that

$$P(\{X \geq r\}) = \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \frac{1}{\beta(r, n-r+1)} \int_0^p t^{r-1}(1-t)^{n-r} dt, \; r \in \mathbb{N}, \; r \leq n.$$

Hence show that if $X \sim \text{Bin}(n, p)$, then $P(X \geq r) \leq \binom{n}{r} p^r$.
(c) Using the fact that, for any real number x, the event $\{X_{r:n} \leq x\}$ occurs iff at least r among $X_1, \ldots, X_n$ are $\leq x$, find the d.f. of $X_{r:n}, r = 1, \ldots, n$. Using the

relation derived in (b) show that $X_{r:n}$ is A.C. with p.d.f.

$$g_r(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad -\infty < x < \infty.$$

4. (a) (**Poisson approximation to binomial distribution**) Show that $\lim_{n\to\infty} \binom{n}{r} p_n^r (1-p_n)^{n-r} = e^{-\lambda}\lambda^r/r!$, $r = 0, 1, 2, \ldots$, provided $\lim_{n\to\infty} np_n = \lambda$ (i.e., for large n and small p, so that $np = \lambda > 0$, Bin(n,p) probabilities can be approximated by Poisson(λ) probabilities);
(b) The probability of hitting a target is 0.001 for each shot. Find the approximate probability of hitting a target at least twice in 5000 shots.

5. Let X be a non-negative and integer valued r.v., satisfying $P(X=0) = p \in (0,1)$ and having lack of memory property $P(X \geq n+m | X \geq n) = P(X \geq m)$, $n, m \in \{1, 2, \ldots, \}$. Show that $X \sim$ Geometric(p).

6. (a) (**Binomial approximation to hypergeometric distribution**) Show that a hypergeometric distributed can be approximated by a Bin(n,p) distribution provided N is large $(N \to \infty)$, a is large $(a \to \infty)$ and $a/N \to p$, as $N \to \infty$;
(b) Out of 120 applicants for a job, only 80 are qualified for the job. Out of 120 applicants, five are selected at random for the interview. Find the probability that at least two selected applicants will be qualified for the job. Using Problem 6 (a) find an approximate value of this probability.

7. (a) If the m.g.f. of a r.v. X is $(1/3 + 2e^t/3)^5$, $-\infty < t < \infty$, find $P(X \in \{2,3\})$;
(b) If the m.g.f. of a r.v. X is $e^{4(e^t-1)}$, $-\infty < t < \infty$, find $P(E(X) - 2\sqrt{\text{Var}(X)} < X < E(X) + 2\sqrt{\text{Var}(X)})$.

8. (**Relationship between gamma and Poisson distribution**) Let $X \sim$ Poisson(λ) and let k be a positive integer. Show that $P(X \leq k) = (k!)^{-1}\int_\lambda^\infty e^{-t}t^k dt$.

9. Each child in a family is equally likely to be a boy or a girl. Find the minimum number of children the family should have so that the probability of it having at least a boy and at least a girl is at least 0.95.

10. **(Relation between binomial and negative binomial probabilities)** Let n be a positive integer, $r \in \{1, 2, \ldots, n\}$ and let $p \in (0,1)$. Using probabilistic arguments and also otherwise show that

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = p^r \sum_{k=0}^{n-r} . \binom{r+k-1}{k} (1-p)^k,$$

i.e., for $r \in \{1, 2, \ldots, n\}$, $P(\text{Bin}(n,p) \geq r) = P(\text{NB}(r,p) \leq n-r)$.

11. A mathematician carries at all times two match boxes, one in his left pocket and one in his right pocket. To begin with each match box contains n matches. Each time the mathematician needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician for the first time discovers that one of the match boxes is empty. Find the probability that at that moment the other box contains exactly k matches, where $k \in \{0, 1, \ldots, n\}$.

12. **(Poisson approximation to negative binomial distribution)** Show that

$$\lim_{r\to\infty} \binom{r+k-1}{k} p_r^r (1-p_r)^k = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, \ldots,$$

provided $\lim_{r\to\infty} p_r = 1$ and $\lim_{r\to\infty}(r(1-p_r)) = \lambda > 0$.

13. (a) Let $F(\cdot)$ be the d.f. of the r.v. X, where $P(X = 1) = p = 1 - P(X = 0)$. Find the distribution of $Y = F(X)$. Is $Y \sim \text{U}(0,1)$?
(b) Let the r.v. X have the p.d.f. $f(x) = 6x(1-x)$, if $0 \leq x \leq 1$, $= 0$, otherwise. Show that $Y = X^2(3-2X) \sim \text{U}(0,1)$.

14. (a) If $X \sim \text{U}(0,1)$, show that of $Y = -\lambda \ln X \sim \text{Exp}(\lambda)$, where $\lambda > 0$;
(b) Let X_1 and X_2 be independent $\text{N}(0,1)$ r.v.s and let $Y = \frac{X_1}{X_2}$. Show that Y has the Cauchy distribution. What is the distribution of $Y = X^{-1}$?
(c) Let X and Y be i.i.d. r.v.s. with common p.d.f. $f(x) = \frac{c}{1+x^4}$, $-\infty < x < \infty$, where c is the normalizing constant. Find the p.d.f. of $\frac{X}{Y}$.

15. (a) If $X \sim \text{N}(12, 16)$, find $P(X \geq 20)$ (use $\Phi(2) = 0.9772$);
(b) If $X \sim \text{N}(\mu, \sigma^2)$, $P(9.6 \leq X \leq 13.8) = 0.7008$ and $P(X \geq 9.6) = 0.8159$, find μ, σ^2 and $P(X \geq 13.8 | X \geq 9.6)$ (use $\Phi(0.9) = 0.8159$ and $\Phi(1.2) = 0.8849$).

16. For $x > 0$, show that $(x^{-1} - x^{-3})\phi(x) < 1 - \Phi(x) < x^{-1}\phi(x)$ (Hint: Use integration by parts in $(2\pi)^{\frac{1}{2}}(1 - \Phi(x)) = \int_x^\infty t^{-1}(te^{-\frac{t^2}{2}})dt$).

17. Let X be a r.v. with $E(X^m) = m!$, $m = 1, 2, \ldots$, and suppose that the m.g.f. of X exists in a neighborhood of zero. Find the p.d.f./p.m.f. of X.

18. (a) Let $X \sim \text{Gamma}(\alpha_1, \theta)$ and $Y \sim \text{Gamma}(\alpha_2, \theta)$ be independent. Show that $U = X+Y$ and $V = \frac{X}{X+Y}$ are independently distributed; also find their distributions.
(b) Let $W \sim \text{Beta}(\alpha_1, \alpha_2)$ and $T \sim \text{Gamma}(\alpha_1 + \alpha_2, \theta)$ be independent. Using (a), show that $WT \sim \text{Gamma}(\alpha_1, \theta)$.

19. Let $X \sim \text{N}(0,1)$ and $Y \sim \text{N}(0,1)$ be independent. Find p.d.f.s of $\frac{X}{|Y|}$ and $\frac{|X|}{|Y|}$.

20. Let $X_1, \ldots X_n$ denote a random sample from the Exp(1) distribution. Find the marginal distributions of $Y_1, \ldots, Y_n$, where

$$Y_i = \frac{\sum_{j=1}^{i} X_j}{\sum_{j=1}^{i+1} X_j}, \; i = 1, \ldots, n-1, \quad Y_n = X_1 + \ldots + X_n.$$

Are $Y_1, \ldots, Y_n$ independent?

21. (a) Let $X_1, \ldots, X_n$ $(n \geq 3)$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$ distribution. Find $E\left(\frac{\overline{X}}{S}\right)$.
(b) Let X_1 and X_2 be i.i.d. $\mathrm{N}(0, 1)$ r.v.s. Find the distribution of $Z = \frac{X_1+X_2}{|X_1-X_2|}$.

22. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the set of order statistics associated with a random sample of size n (≥ 2) from the Exp(1) distribution.
(a) Let $Z_1 = nX_{(1)}$, $Z_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, $i = 2, \ldots, n$. Show that $Z_1, \ldots, Z_n$ are i.i.d. Exp(1) r.v.s.
(b) Using (a) or otherwise, find $E(X_{(r)})$, $\mathrm{Var}(X_{(r)})$ and $\mathrm{Cov}(X_{(r)}, X_{(s)})$, $r < s$.
(c) Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for any $s > r$;
(d) Find the p.d.f. of $X_{(r+1)} - X_{(r)}$;

23. Let $X_1, \ldots, X_n$ be i.i.d. nonnegative A.C. r.v.s. with common d.f. $F(\cdot)$. If $E(|X_1|) < \infty$ and $M_n = \max(X_1, \ldots, X_n)$, show that

$$E(M_n) = E(M_{n-1}) + \int_0^\infty (F(x))^{n-1} (1 - F(x))\, dx.$$

24. Let X and Y be independent and identically distributed $\mathrm{N}(0, \sigma^2)$ r.v.s. Let $U = aX + bY$ and $V = bX - aY$ $(a \neq 0, b \neq 0)$.
(a) Show that U and V are normally distributed independent random variables;
(b) Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are independent $\mathrm{N}(0, \sigma^2)$ random variables.

25. Let X be a random variable and let α and β be real numbers such that $-\infty < \alpha < \beta < \infty$. Show that $X \sim \mathrm{U}(\alpha, \beta)$ if, and only if, $P(\{\alpha < X \leq \beta\}) = 1$ and $P(\{X \in I\}) = P(\{X \in J\})$, for any pair of subintervals I, J of (α, β) having the same length.

B. Practice Problems

1. Let $\underline{X} = (X_1, X_2)$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{36}, & \text{if } x_1 = 1, 2, 3, x_2 = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = X_1 + \frac{X_2}{2}$. Find the p.m.f. of Y.

2. (a) Let $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$ be independent r.v.s. For $t \in \{0, 1, \ldots, \min(n_1, n_2)\}$, find the conditional distribution and conditional mean of X given $X + Y = t$.

(b) Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent r.v.s. For $t \in \{0, 1, \ldots, \}$, find the conditional distribution and conditional mean of X given $X + Y = t$.

3. Suppose that the number, X, of eggs laid by a bird has the Poisson(λ) distribution, and the probability that an egg would finally develop is $p \in (0, 1)$; here $\lambda > 0$. Under the assumption of independence of development of eggs, show that the number, Y, of eggs surviving has the Poisson(λp) distribution. Find the conditional distribution of X given $Y = y$.

4. (a) If $X \sim \text{Poisson}(\lambda)$, find $E((2 + X)^{-1})$;

(b) If $X \sim \text{Geometric}(p)$ and r is a positive integer, find $E(\min(X, r))$.

5. Let $X \sim \text{U}(0, \theta)$, where θ is a positive integer. Let $Y = X - [X]$, where $[x]$ is the largest integer $\leq x$. Show that $Y \sim \text{U}(0, 1)$.

6. If $X \sim \text{U}(0, \theta)$, where $\theta > 0$, find the distribution of $Y = \min(X, \theta/2)$. Calculate $P(\theta/4 < Y < \theta/2)$.

7. Let $X \sim \text{N}(0, 1)$ and let $Y = X$, if $|X| \leq 1$, $= -X$, if $|X| > 1$. Find the distribution of Y.

8. Let X and Y be i.i.d. $\text{U}(-1, 1)$ r.v.s. Find the probability that $x^2 + Xx + Y = 0 > 0$, for every real number x.

9. Let X and Y be respective arrival times of two friends A and B who agree to meet at a spot and wait for the other only for t minutes. Suppose that X and Y are i.i.d. Exp(λ). Show that probability of A and B meeting each other is $1 - e^{-\lambda t}$.

10. Let X and Y be independent r.v.s. with respective p.d.f.s. $f_X(x) = \frac{1}{3}$, if $1 \leq x \leq 4$, $= 0$, otherwise, and $f_Y(y) = e^{-(y-2)}$, if $y \geq 2$ and $= 0$, otherwise. Find the d.f. of $T = \frac{X}{Y}$ and hence find the p.d.f. of T.

11. Let X and Y be i.i.d. $\mathrm{U}(0,1)$ r.v.s. Find the marginal p.d.f.s of
(a) $X+Y,\ X-Y,\ \frac{X+Y}{X-Y},\ |X-Y|$;
(b) $\min(X,Y),\ \max(X,Y),\ \frac{\min(X,Y)}{\max(X,Y)}$;
(c) X^2+Y^2.

12. Let X and Y be i.i.d. $\mathrm{N}(0,1)$ random variables. Define the random variables R and Θ by $X = R\cos\Theta$, $Y = R\sin\Theta$.
(a) Show that R and Θ are independent with $\frac{R^2}{2} \sim \mathrm{Exp}(1)$ and $\Theta \sim \mathrm{U}(0,2\pi)$.
(b) Show that X^2+Y^2 and $\frac{X}{Y}$ are independently distributed.
(c) Show that $\sin\Theta$ and $\sin 2\Theta$ are identically distributed and hence find the pdf of $T = \frac{XY}{\sqrt{X^2+Y^2}}$.
(d) Find the distribution of $U = \frac{3X^2Y-Y^3}{X^2+Y^2}$.
(e) Let U_1 and U_2 be i.i.d. $\mathrm{U}(0,1)$ r.v.s. Then show that $X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ are i.i.d. $\mathrm{N}(0,1)$ r.v.s. (This is known as the Box-Muller transformation).

13. Let X_1, X_2, X_3 be i.i.d. $\mathrm{Gamma}(m,1)$ random variables. Let $Z_1 = X_1+X_2+X_3$, $Z_2 = \frac{X_2}{X_1+X_2+X_3}$ and $Z_3 = \frac{X_3}{X_1+X_2+X_3}$.
(a) Show that Z_1 and (Z_2, Z_3) are independent and find marginal p.d.f.s of Z_1, Z_2 and Z_3.
(b) Find $E(Z_1^2 Z_2 Z_3)$.

14. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics associated with a random sample of size n (≥ 2) from the $\mathrm{U}(0,1)$ distribution. Let $Y_i = \frac{X_{(i)}}{X_{(i+1)}}$, $i = 1,\ldots,n-1$, and $Y_n = X_{(n)}$. Show that $Y_1,\ldots,Y_n$ are independent and find the p.d.f. of Y_i, $i = 1,\ldots,n$.

MSO 201a : Probability and Statistics
2016-2017-II Semester
Assignment-X
Solutions

$\boxed{\text{Problem No. 1}}$ We know that, for a positive integer m and $x \in (-1,1)$

$$(1-x)^{-m} = 1 + \binom{m}{1}x + \binom{m+1}{2}x^2 + \binom{m+2}{3}x^3 + \cdots = \sum_{j=0}^{\infty}\binom{m+j-1}{j}x^j.$$

Clearly the support of X is $S_X = \{0,1,2,\ldots\}$ and for $x \in S_X$

$$f_X(x) = P(X=x) = \sum_{y=0}^{\infty} P(X=x, Y=y) = \frac{\lfloor x+k-1}{\lfloor x \, \lfloor k-1}\theta^x (1-2\theta)^k \sum_{y=0}^{\infty}\binom{x+y+k-1}{y}\theta^y$$

$$= \binom{x+k-1}{k-1}\theta^x(1-2\theta)^k(1-\theta)^{-(x+k)}$$

Thus

$$f_X(x) = \begin{cases} \binom{x+k-1}{k-1}\left(\frac{\theta}{1-\theta}\right)^x\left(1-\frac{\theta}{1-\theta}\right)^k, & \text{if } x = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\longrightarrow NB\left(k, \frac{1-2\theta}{1-\theta}\right)$$

For $x \in \{0,1,2,\ldots\}$,

$$f_{Y|X}(y|x) = \frac{P(X=x, Y=y)}{P(X=x)} = \begin{cases} \binom{x+y+k-1}{y}(1-\theta)^{x+k}\theta^y, & \text{if } y = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\longrightarrow NB(x+k, 1-\theta).$$

Let $Z = X+Y$. Then $S_Z = \{0,1,2,\ldots\}$ and for $z \in S_Z$,

$$f_Z(z) = P(X+Y=z) = \sum_{\substack{x=0 \\ x+y=z}}^{\infty}\sum_{y=0}^{\infty}\frac{\lfloor x+y+k-1}{\lfloor x \, \lfloor y \, \lfloor k-1}\theta^{x+y}(1-2\theta)^k$$

$$= \frac{\lfloor z+k-1 \; \theta^z(1-2\theta)^k}{\lfloor k-1 \, \lfloor z}\sum_{x=0}^{z}\binom{z}{x} = \binom{z+k-1}{z}\theta^z(1-2\theta)^k 2^z$$

Thus

$$f_Z(z) = \begin{cases} \binom{z+k-1}{z}(2\theta)^z(1-2\theta)^k, & \text{if } z = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases} \longrightarrow NB(k, 1-2\theta).$$

$\boxed{\text{Problem No. 2}}$ Let $U = h_1(X,Y) = X$ and $Z = h_2(X,Y) = X+Y$. Then $S_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x>0, y>0, x+y<1\}$, the transformation $\underline{h} = (h_1, h_2): \mathbb{R}^2 \to \mathbb{R}^2$ is 1-1 with inverse transformation $(h_1^{-1}(u,z), h_2^{-1}(u,z))$ where $h_1^{-1}(u,z) = u$, $h_2^{-1}(u,z) = z-u$, the Jacobian

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Also $(u,z) \in \underline{h}(S_{X,Y}) \iff u>0, z-u>0, u+z-u<1$

$$\Rightarrow 0<u<z<1. \text{ Thus}$$

$$\underline{h}(S_{X,Y}) = \{(u,z) \in \mathbb{R}^2 : 0<u<z<1\}.$$

and the joint p.d.f. of (U, Z) is

$$f_{U,Z}(u,z) = f_{X,Y}(u, z-u)\,|J|\, I_{h(S_{X,Y})}(u,z) = \begin{cases} \dfrac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3} u^{\theta_1-1}(z-u)^{\theta_2-1}(1-z)^{\theta_3-1}, & \text{if } 0<u<z<1 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, for $u \notin (0,1)$, $f_X(u) = f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\,dz = 0$.

For $u \in (0,1)$

$$f_X(u) = f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\,dz = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3} u^{\theta_1-1}\int_u^1 (z-u)^{\theta_2-1}(1-z)^{\theta_3-1}\,dz$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3} u^{\theta_1-1}\int_0^1 ((1-u)(1-z))^{\theta_2-1}((1-u)z)^{\theta_3-1}(1-u)\,dz$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3} u^{\theta_1-1}(1-u)^{\theta_2+\theta_3-1}\int_0^1 z^{\theta_3-1}(1-z)^{\theta_2-1}\,dz = \frac{1}{B(\theta_1, \theta_2+\theta_3)} u^{\theta_1-1}(1-u)^{\theta_2+\theta_3-1}.$$

Thus $X \sim \text{Beta}(\theta_1, \theta_2+\theta_3)$.

By symmetry $Y \sim \text{Beta}(\theta_2, \theta_1+\theta_3)$. Thus, for $y \in (0,1)$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{B(\theta_1,\theta_3)}\dfrac{1}{1-y}\left(\dfrac{x}{1-y}\right)^{\theta_1-1}\left(1-\dfrac{x}{1-y}\right)^{\theta_3-1}, & \text{if } 0<x<1-y \\ 0, & \text{otherwise} \end{cases}$$

Also

$f_Z(z) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\,du$. Clearly, for $z \notin (0,1)$, $f_Z(z) = 0$.

For $z \in (0,1)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\,du = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3}(1-z)^{\theta_3-1}\int_0^z u^{\theta_1-1}(z-u)^{\theta_2-1}\,du$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma\theta_1\,\Gamma\theta_2\,\Gamma\theta_3}(1-z)^{\theta_3-1}\int_0^1 (zu)^{\theta_1-1}(z-zu)^{\theta_2-1} z\,du = c\, z^{\theta_1+\theta_2-1}(1-z)^{\theta_3-1}$$

$\Rightarrow Z \sim \text{Beta}(\theta_1+\theta_2, \theta_3)$

Problem No. 3 (a) Let S_n denote the set of all permutations $\underline{p} = (p_1, \ldots, p_n)$ of $(1, \ldots, n)$. Since $\underline{X}$ is A.C.

$$\sum_{\underline{p}\in S_n} P(X_{p_1} < X_{p_2} < \cdots < X_{p_n}) = 1 \quad \text{---}(A_1)$$

Also $X_1, \ldots, X_n$ are i.i.d $\Rightarrow X_1, \ldots, X_n$ are exchangeable

$\boxed{2/17}$ $\Rightarrow (X_1, \ldots, X_n) \stackrel{d}{=} (X_{p_1}, \ldots, X_{p_n})\ \forall\, \underline{p} \in S_n$

$\longrightarrow (A_2)$

The joint d.f. of $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ is

$$G(y_1, \ldots, y_n) = P(X_{1:n} \le y_1, X_{2:n} \le y_2, \ldots, X_{n:n} \le y_n)$$

$$= \sum_{\beta \in S_n} P(X_{1:n} \le y_1, X_{2:n} \le y_2, \ldots, X_{n:n} \le y_n, X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n}) \quad \text{(using } A_1\text{)}$$

$$= \sum_{\beta \in S_n} P(X_{\beta_1} \le y_1, X_{\beta_2} \le y_2, \ldots, X_{\beta_n} \le y_n, X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n})$$

$$= \sum_{\beta \in S_n} P(X_1 \le y_1, X_2 \le y_2, \ldots, X_n \le y_n, X_1 < X_2 < \cdots < X_n) \quad \text{(using } A_2\text{)}$$

$$= \underline{\lfloor n} \, P(X_1 \le y_1, X_2 \le y_2, \ldots, X_n \le y_n, X_1 < X_2 < \cdots < X_n)$$

$$= \underline{\lfloor n} \underset{x_1 < x_2 < \cdots < x_n}{\int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n}} f_{\underline{X}}(x_1, \ldots, x_n) \, dx_n \cdots dx_1$$

$$= \underline{\lfloor n} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} \left(\prod_{i=1}^{n} f(x_i) \right) I_A(\underline{x}) \, dx_n \cdots dx_1 \quad \ldots (A_3)$$

where $A = \{ \underline{x} \in \mathbb{R}^n : -\infty < x_1 < x_2 < \cdots < x_n < \infty \}$.

From A_3 it follows that $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ is A.C. with joint p.d.f.

$$g(y_1, \ldots, y_n) = \left(\prod_{i=1}^{n} f(y_i) \right) I_A(\underline{y}) = \begin{cases} \underline{\lfloor n} \prod_{i=1}^{n} f(y_i), & \text{if } -\infty < y_1 < \cdots < y_n < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(b) For $p \in (0,1)$, $n \in \mathbb{N}$, $r \in \mathbb{N}$ and $r \le n$, let

$$I_{r,n}(p) = \frac{1}{\beta(r, n-r+1)} \int_0^p t^{r-1} (1-t)^{n-r} \, dt.$$

Integration by parts gives

$$I_{r,n}(p) = \binom{n}{r} p^r (1-p)^{n-r} + \frac{\underline{\lfloor n}}{\underline{\lfloor r} \, \underline{\lfloor n-r-1}} \int_0^p t^r (1-t)^{n-r-1} \, dt$$

$$= \binom{n}{r} p^r (1-p)^{n-r} + I_{r+1,n}(p)$$

$$= \binom{n}{r} p^r (1-p)^{n-r} + \binom{n}{r+1} p^{r+1} (1-p)^{n-r-1} + I_{r+2,n}(p)$$

$$= \binom{n}{r} p^r (1-p)^{n-r} + \binom{n}{r+1} p^{r+1} (1-p)^{n-r-1} + \cdots + \binom{n}{n-1} p^{n-1} (1-p) + I_{n,n}(p)$$

$$= \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

(c) For $x \in \mathbb{R}$ and $r = 1, 2, \ldots, n$

$$F_{X_{r:n}}(x) = P(X_{r:n} \le x) = P(\text{at least } r \text{ of } X_1, \ldots, X_n \le x)$$
$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, \ldots, X_n \le x)$$

Fix $x \in \mathbb{R}$. Consider a sequence of n trials where i-th trial is designated as success if $\{X_i \le x\}$ and failure otherwise.

$X_1, \ldots, X_n$ are independent $\Rightarrow$ n trials are independent

$X_1 \ldots X_n$ are identically distributed $\Rightarrow$ probability of success in each trials is the same $= P(X_i \le x) = F(x)$

Thus

$$F_{X_{r:n}}(x) = \sum_{i=r}^{n} P(i \text{ successes in } n \text{ trials})$$
$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i}$$
$$= \frac{1}{\beta(r, n-r+1)} \int_0^{F(x)} t^{r-1} (1-t)^{n-r}\, dt$$
$$= \frac{1}{\beta(r, n-r+1)} \int_{-\infty}^{x} [F(y)]^{r-1} (1-F(y))^{n-r} f(y)\, dy$$

[on making the transformation $t = F(y) \Rightarrow dt = f(y)\,dy$]

$\Rightarrow$ $X_{r:n}$ is A.C. with p.d.f.

$$f_{X_{r:n}}(x) = \frac{1}{\beta(r, n-r+1)} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x), \quad -\infty < x < \infty$$
$$= \frac{\lfloor n}{\lfloor r-1 \; \lfloor n-r} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x), \quad -\infty < x < \infty.$$

Problem No. 4 (a) For $r \in \{0, 1, \ldots, n\}$ let

$$A_n = \binom{n}{r} p_n^r (1-p_n)^{n-r} = \frac{1}{\lfloor r} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)(np_n)^r (1-p_n)^n (1-p_n)^{-r}, \quad n \in \mathbb{N}$$

Let $t_n = n \ln(1-p_n) = n\left[-p_n - \frac{p_n^2}{2} \cdot \frac{1}{1-\xi_n}\right]$, where $0 < \xi_n < p_n$, $n = 1, 2, \ldots$

$np_n \to \lambda \Rightarrow p_n \to 0$ and $\xi_n \to 0 \Rightarrow t_n \to -\lambda \Rightarrow e^{t_n} = (1-p_n)^n \to e^{-\lambda}$

Therefore $A_n \xrightarrow{n\to\infty} \frac{e^{-\lambda}\lambda^r}{\lfloor r}$, $r = 0, 1, 2, \ldots$

(b) Define, X = # of times the target is hit in 5000 shots

Then $X \sim Bin(5000, .001)$. $n = 5000$ is large and $p = 0.001$ is small. Using (a) (with $\lambda = np = 5$)

Required probability $= P(X\geq 2) = 1-P(X=0) - P(X=1)$

$$= 1-e^{-5} - e^{-5}\times 5 = 1-6e^{-5}.$$

$\boxed{\text{Problem No. 5}}$ We have

$$P(X\geq n+m) = P(X\geq n)\,P(X\geq m), \quad \forall\, m, n \in \{1, 2, \ldots\}$$

$$P(X\geq n+1) = P(X\geq n)\,P(X\geq 1) = (1-p)\,P(X\geq n), \quad n=1, 2, \ldots$$

$P(X\geq 2) = (1-p)\,P(X\geq 1) = (1-p)^2$. By induction $P(X\geq n) = (1-p)^n, \forall n\in\mathbb{N}$

$\Rightarrow P(X=n) = P(X\geq n) - P(X\geq n+1) = p(1-p)^n, \quad n=1, 2, \ldots$

$$\Rightarrow P(X=n) = \begin{cases} p(1-p)^n, & \text{if } n=0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \Rightarrow X\sim \text{Geometric}(p).$$

$\boxed{\text{Problem No. 6}}$ (a) Let $X\sim \text{Hyp}(a, n, N)$. The support of X is $S_X = \{m\in\mathbb{N} : \max(0, n-N+a) \leq m \leq \min(n, a)\}$. Note that

$n-N+a = N\left(\frac{n}{N} - 1 + \frac{a}{N}\right) \to -\infty$ and $a = N\frac{a}{N} \to \infty$ as $N\to\infty$

Thus, as $N\to\infty$, $\max(0, n-N+a) \to 0$ and $\min(n, a)\to n$.

Also, for $k\in S_X$

$$f_X(k) = \frac{\binom{a}{k}\binom{N-a}{n-k}}{\binom{N}{n}} = \binom{n}{k}\left\{\prod_{j=0}^{k-1}\frac{a-j}{N-j}\right\}\left\{\prod_{j=0}^{n-k-1}\frac{N-a-j}{N-k-j}\right\}$$

$$\xrightarrow{N\to\infty} \binom{n}{k}\left\{\prod_{j=0}^{k-1} p\right\}\left\{\prod_{j=0}^{n-k-1}(1-p)\right\} = \binom{n}{k}p^k(1-p)^{n-k}.$$

Thus

$$\lim_{N\to\infty} f_X(k) = \begin{cases}\binom{n}{k}p^k(1-p)^{n-k}, & \text{if } k=0, 1, 2, \ldots, n \\ 0, & \text{otherwise}\end{cases}.$$

(b) Let $X=$ # of applicants in the selected sample that are qualified for the job.

$X\sim \text{Hyp}(80, 5, 120)$

Required probability $= P(X\geq 2) = 1-P(X=0) - P(X=1)$

$$= 1 - \frac{\binom{80}{0}\binom{40}{5}}{\binom{120}{5}} - \frac{\binom{80}{1}\binom{40}{4}}{\binom{120}{5}}$$

$N=120$ is large, $a=80$ is large, $p=\frac{a}{N} = \frac{2}{3}$. By (a) $X \overset{\text{approx}}{\sim} \text{Bin}(5, \frac{2}{3})$. Approximate Required probability

$$= 1-P(X=0) - P(X=1)$$

$$= 1-\binom{5}{0}\left(\tfrac{2}{3}\right)^0\left(\tfrac{1}{3}\right)^5 - \binom{5}{1}\left(\tfrac{2}{3}\right)^1\left(\tfrac{1}{3}\right)^4.$$

$\boxed{\text{Problem No. 7}}$ (a) Clearly $X \sim Bin(5, \frac{2}{3})$

$$P(X \in \{2, 3\}) = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 + \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{40}{81}.$$

(b) Clearly $X \sim Poisson(4)$. Thus $E(X) = Var(X) = 4$.

$$P\left(E(X) - 2\sqrt{Var(X)} < X < E(X) + 2\sqrt{Var(X)}\right) = P(0 < X < 8)$$
$$= \sum_{j=1}^{7} \frac{e^{-4}4^j}{\lfloor j}$$

$\boxed{\text{Problem No. 8}}$ For $k = 1, 2, \dots$, let

$$I_k(\lambda) = \frac{1}{\lfloor k} \int_\lambda^\infty e^{-t} t^k \, dt.$$

Integrating by parts we get

$$I_k(\lambda) = \frac{e^{-\lambda}\lambda^k}{\lfloor k} + I_{k-1}(\lambda) = \frac{e^{-\lambda}\lambda^k}{\lfloor k} + \frac{e^{-\lambda}\lambda^{k-1}}{\lfloor k-1} + I_{k-2}(\lambda)$$
$$= \sum_{j=1}^{k} \frac{e^{-\lambda}\lambda^j}{\lfloor j} + I_0(\lambda) = \sum_{j=0}^{k} \frac{e^{-\lambda}\lambda^j}{\lfloor j} = P(X \le k).$$

$\boxed{\text{Problem No. 9}}$ Suppose that the family has n children and X = # of boys in the family. Then

$$X \sim Bin(n, \tfrac{1}{2}).$$

$P(\text{the family has atleast a boy and a girl}) = P(X \ge 1, n - X \ge 1)$

$$= P(1 \le X \le n-1) = 1 - P(X=0) - P(X=n) = 1 - \left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{2}\right)^n = 1 - \frac{1}{2^{n-1}}$$

We require $1 - \frac{1}{2^{n-1}} \geq 0.95 \Leftrightarrow 2^n \ge 40 \Leftrightarrow n \ge 6$

$\boxed{\text{Required Answer} = 6}$

$\boxed{\text{Problem No. 10}}$ Consider a sequence of independent Bernoulli trials with probability of success in each trial being p. Then

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = P(\text{at least } r \text{ successes in } n \text{ Bernoulli trials})$$
$$= P\left(\bigcup_{k=0}^{n-r} \{r\text{th success in } (r+k)\text{th trial}\}\right)$$
$$= \sum_{k=0}^{n-r} P(r\text{th success in } (r+k)\text{th trial})$$
$$= \sum_{k=0}^{n-r} \underbrace{\binom{r+k-1}{r-1} p^{r-1} (1-p)^k}_{(r-1) \text{ successes in first } (r+k-1) \text{ trials}} \cdot \underbrace{p}_{\text{success in } (r+k)\text{th trial}}$$
$$= RHS$$

Direct Method: Let $q = 1-p$. We need to show that

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = p^r \sum_{k=0}^{n-r} \binom{r+k-1}{k} (1-p)^k$$

$$\Leftrightarrow \quad \sum_{k=r}^{n} \binom{n}{k} (1-q)^{k-r} q^{n-k} = \sum_{k=0}^{n-r} \binom{r+k-1}{k} q^k .$$

LHS = polynomial of degree $n-r = \sum_{k=0}^{n-r} c_k q^k$

It suffices to show that $c_k = \binom{r+k-1}{k}$, $k = 0, 1, 2, \ldots, n-r$

For $k \in \{0, 1, 2, \ldots, n-r\}$

$$c_k = \text{Coefficient of } q^k \text{ in } \sum_{j=r}^{n} \binom{n}{j} (1-q)^{j-r} q^{n-j} = \sum_{j=0} \underbrace{\binom{n}{n-j}}_{=\binom{n}{j}} (1-q)^{n-j-r} q^j$$

$$= \text{Coefficient of } q^k \text{ in } \sum_{j=0}^{k} \binom{n}{j} (-1)^{k-j} \binom{n-j-r}{k-j}$$

$$= \sum_{j=0}^{k} \binom{n}{k-j} (-1)^j \binom{n-k-r+j}{j}$$

But

$$(-1)^j \binom{n-k-r+j}{j} = (-1)^j \frac{(n-k-r+j)(n-k-r+j-1)\cdots(n-k-r+1)}{\lfloor j}$$

$$= \frac{(-n+k+r-j)(-n+k+r-j+1)\cdots(-n+k+r-1)}{\lfloor j}$$

$$= \binom{-n+k+r-1}{j}$$

$$\Rightarrow \quad c_k = \sum_{j=0}^{k} \binom{n}{k-j} \binom{-n+k+r-1}{j} = \binom{k+r-1}{k}.$$

$\boxed{\text{Problem No. 11}}$ Let us call the event of choosing box 1 as success and that of choosing box 2 as failure. Then we have a sequence of Bernoulli trials with probability of success in each trial being $p = \frac{1}{2}$.

Required probability = P(when box 1 is found empty, box 2 has k matches)

+ P(when box 2 is found empty box 1 has k matches)

= P($n-k$ failures precede the $(n+1)$th success)

+ P($n-k$ successes precede the $(n+1)$th failure)

$$= \binom{2n-k}{n} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^{n-k} \times \tfrac{1}{2} + \binom{2n-k}{n} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^{n-k} \times \tfrac{1}{2} = \binom{2n-k}{n} \left(\tfrac{1}{2}\right)^{2n-k}$$

Problem No. 12 We will use the Stirling approximation

$$\lim_{n\to\infty} \frac{\lfloor n}{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}} = 1, \quad \text{i.e. } \lfloor n \approx \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}} \text{ for large } n$$

Thus
$$\text{LHS} = \frac{\lfloor r+k-1}{\lfloor k \, \lfloor r-1} p_r^r (1-p_r)^k$$

$$\simeq \frac{\sqrt{2\pi}\, e^{-(r+k-1)} (r+k-1)^{r+k-\frac{1}{2}}}{\lfloor k \, \sqrt{2\pi}\, e^{-(r-1)} (r-1)^{r-\frac{1}{2}}} \left(1-\frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^k, \text{ for large } r$$

$$= \frac{e^{-k}\lambda^k}{\lfloor k} \left(1-\frac{\lambda}{r}\right)^r \frac{\left(1+\frac{k-1}{r}\right)^{r+k-\frac{1}{2}}}{\left(1-\frac{1}{r}\right)^{r-\frac{1}{2}}}$$

$$\simeq \frac{e^{-k}\lambda^k}{\lfloor k}\, e^{-\lambda}\, \frac{e^{k-1}}{e^{-1}} \quad \text{(for large } r\text{)}$$

$$= \frac{e^{-\lambda}\lambda^k}{\lfloor k}$$

Problem No. 13 (a)
$$F(x) = \begin{cases} 0, & \text{if } x<0 \\ 1-p, & \text{if } 0\le x<1 \\ 1, & \text{if } x\ge 1 \end{cases} \Rightarrow Y = F(X) = \begin{cases} 1-p & \text{if } X=0 \\ 1, & \text{if } X=1 \end{cases}$$

Clearly Y is discrete with p.m.f.
$$f_Y(y) = P(Y=y) = \begin{cases} 1-p, & \text{if } y=1-p \\ p, & \text{if } y=1 \\ 0, & \text{otherwise} \end{cases}. \quad Y \not\sim U(0,1).$$

(b) The d.f. of X is
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x<0 \\ x^2(3-2x), & \text{if } 0\le x<1 \\ 1, & \text{if } x\ge 1 \end{cases} \Rightarrow Y = F(X), \text{ with probability one}$$

$\Rightarrow Y \sim U(0,1)$ (since X is continuous)

Problem No. 14 $S_X = [0,1]$
$$f_X(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0, & \text{o.w.} \end{cases}$$

$Y = h(X) = -\lambda \ln X$, where $h(x) = -\lambda \ln x$, $x>0$.

$h: (0,\infty) \to (0,\infty)$ is 1-1 with inverse transformation $h^{-1}(y) = e^{-\frac{y}{\lambda}}$.

$h((0,1)) = (0,\infty)$. Thus the p.d.f. of Y is

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{(0,\infty)}(y)$$

$$= f_X(e^{-\frac{y}{\lambda}}) \left| -\frac{1}{\lambda} e^{-\frac{y}{\lambda}} \right| I_{(0,\infty)}(y) = \begin{cases} \frac{1}{\lambda} e^{-\frac{y}{\lambda}}, & \text{if } y>0 \\ 0, & \text{o.w.} \end{cases}$$

(b) The joint p.d.f. of (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} e^{-\frac{(x_1^2+x_2^2)}{2}}, \quad (x_1,x_2) \in S_{X_1X_2} = \mathbb{R}^2.$$

Consider the 1-1 transformation $(X_1,X_2) \to (Y,Z)$ defined by

$$Y = \frac{X_1}{X_2}, \quad Z = X_2$$

i.e. $X_1 = YZ, \quad X_2 = Z$

$$S_{YZ} = \{(y,z): -\infty < yz < \infty, -\infty < z < \infty\} = \mathbb{R}^2.$$

$$J = \begin{vmatrix} z & y \\ 0 & 1 \end{vmatrix} = z$$

Therefore joint p.d.f. of (Y,Z) is

$$f_{Y,Z}(y,z) = f_{X_1,X_2}(yz, z)\,|z|\, I_{\mathbb{R}^2}(y,z)$$

$$= \frac{|z|}{2\pi} e^{-\frac{z^2}{2}(1+y^2)}, \quad -\infty < y, z < \infty.$$

The p.d.f. of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} |z|\, e^{-\frac{z^2}{2}(1+y^2)}\,dz = \frac{1}{\pi}\cdot\frac{1}{1+y^2}, \quad -\infty<y<\infty$$

$Y \sim$ Cauchy distribution or Student t distribution with 1 d.f.

(c)
$$f_{X,Y}(x,y) = \frac{c^2}{(1+x^4)(1+y^4)}, \quad (x,y) \in S_{XY} = \mathbb{R}^2$$

Consider the 1-1 transformation $(X,Y) \to (U_1,U_2)$ defined by $U_1 = \frac{X}{Y}$, $U_2 = Y$, i.e., $X = U_1U_2$, $Y = U_2$.

$$S_{U_1U_2} = \{(u_1,u_2): -\infty < u_1u_2 < \infty, -\infty < u_2 < \infty\} = \mathbb{R}^2$$

$$J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

Therefore

$$f_{U_1U_2}(u_1,u_2) = \frac{c^2|u_2|}{(1+u_1^4u_2^4)(1+u_2^4)}, \quad -\infty < u_1, u_2 < \infty.$$

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} f_{U_1U_2}(u_1,u_2)\,du_2 = 2c^2 \int_0^{\infty} \frac{u_2}{(1+u_1^4u_2^4)(1+u_2^4)}\,du_2$$

$$= 2c^2 \int_0^{\infty} \frac{u_2}{1-u_1^4}\left(\frac{1}{1+u_2^4} - \frac{u_1^4}{1+u_1^4u_2^4}\right)du_2$$

$$= \frac{2c^2}{1-u_1^4}\int_0^{\infty}\frac{u_2}{1+u_2^4}\,du_2 - \frac{2c^2u_1^4}{1-u_1^4}\int_0^{\infty}\frac{u_2}{1+u_1^4u_2^4}\,du_2$$

$$= \frac{c^2}{1-u_1^4}\int_0^{\infty}\frac{dz}{1+z^2}\,dz - \frac{c^2u_1^4}{1-u_1^4}\times\frac{1}{u_1^2}\int_0^{\infty}\frac{dz}{1+z^2} = \frac{c^2\pi\,(1-u_1^2)}{2(1-u_1^4)}$$

$$= \frac{c\pi}{2}\cdot\frac{1}{1+u_1^2} = \frac{1}{\pi}\cdot\frac{1}{1+u_1^2}, \quad \left(c = \frac{2}{\pi^2}, \text{ since } \int_{-\infty}^{\infty} f_{U_1}(u_1)\,du_1 = 1\right)$$

$-\infty < u_1 < \infty.$

Problem No. 15 (a) $P(X \geq 20) = 1 - P(X \leq 20) = 1 - \Phi\left(\frac{20-12}{4}\right) = 1 - \Phi(2)$

$= 1 - 0.9772 = 0.228$

(b) $P(9.6 \leq X \leq 13.8) = 0.7008 \Rightarrow P(X \leq 13.8) - (1 - P(X \geq 9.6)) = 0.7008$

$\Rightarrow P(X \leq 13.8) = 0.7008 + .1841 = .8849$

$\Rightarrow \Phi\left(\frac{13.8-\mu}{\sigma}\right) = 0.8849 \Rightarrow \frac{13.8-\mu}{\sigma} = 1.2 \Rightarrow \mu + 1.2\sigma = 13.8$ ....(A)

$P(X \geq 9.6) = .8159 \Rightarrow 1 - \Phi\left(\frac{9.6-\mu}{\sigma}\right) = .8159 \Rightarrow \Phi\left(-\frac{9.6-\mu}{\sigma}\right) = .8159$

$\Rightarrow -\frac{9.6-\mu}{\sigma} = 0.9 \Rightarrow \mu - 0.9\sigma = 9.6$ .... (B)

(A) + (B) $\Rightarrow \mu = 9.6, \ \sigma = 3.5$

$$P(X \geq 13.8 \mid X \geq 9.6) = 1 - P(X < 13.8 \mid X \geq 9.6) = 1 - \frac{P(9.6 \leq X < 13.8)}{P(X \geq 9.6)}$$

$$= 1 - \frac{.7008}{.8159} = .1411.$$

Problem No. 16 $\sqrt{2\pi}\,(1-\Phi(x)) = \int_x^{\infty} e^{-\frac{t^2}{2}}\,dt = \int_x^{\infty} \underbrace{\frac{1}{t}}_{DF} \underbrace{(te^{-\frac{t^2}{2}})}_{IF}\,dt$

Using $\frac{d}{dt}\left(-e^{-\frac{t^2}{2}}\right) = te^{-t^2/2}$ we get

$$\sqrt{2\pi}\,(1-\Phi(x)) = \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^{\infty}\frac{1}{t^2}e^{-t^2/2}\,dt \leq \frac{e^{-x^2/2}}{x}$$

Also

$$\sqrt{2\pi}\,(1-\Phi(x)) = \frac{e^{-x^2/2}}{x} - \int_x^{\infty}\frac{1}{t^3}\left(te^{-t^2/2}\,dt\right) = \frac{e^{-\frac{x^2}{2}}}{x} - \frac{e^{-\frac{x^2}{2}}}{x^3} + \int_x^{\infty}\frac{3}{t^4}e^{-\frac{t^2}{2}}\,dt$$

$$\geq \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3}.$$

Problem No. 17 $M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{\lfloor j} E(X^j) = \sum_{j=0}^{\infty} t^j = (1-t)^{-1}, \quad |t|<h, \; h>0$

$\Rightarrow X \sim Exp(1) \Rightarrow X$ is A.C. with p.d.f. $f_X(x) = \begin{cases} e^{-x}, & \text{if } x>0 \\ 0, & \text{o.w.} \end{cases}$

Problem No. 18 $f_{X,Y}(x,y) = \frac{1}{\Gamma\alpha_1 \Gamma\alpha_2} \frac{x^{\alpha_1-1} y^{\alpha_2-1}}{\theta^{\alpha_1+\alpha_2}} e^{-\frac{x+y}{\theta}}, \quad (x,y) \in S_{X,Y} = [0,\infty)^2$

(a) Consider the 1-1 transformation $(X,Y) \to (U,V)$ defined by

$U = X+Y, \quad V = \frac{X}{X+Y} \Rightarrow X = UV, \quad Y = U(1-V).$

$S_{U,V} = \{(u,v): uv \geq 0, \; u(1-v) \geq 0\} = \{(u,v): u \geq 0, \; 0 \leq v \leq 1\}$
$= [0,\infty) \times [0,1].$

$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$

Therefore

$f_{U,V}(u,v) = \frac{1}{\Gamma\alpha_1 \Gamma\alpha_2 \, \theta^{\alpha_1+\alpha_2}} (uv)^{\alpha_1-1} (u(1-v))^{\alpha_2-1} e^{-\frac{u}{\theta}} |-u| \, I_{S_{U,V}}(u,v)$

$= \begin{cases} \left(\frac{1}{\Gamma\alpha_1+\alpha_2} u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\theta}}\right)\left(\frac{1}{\beta(\alpha_1,\alpha_2)} v^{\alpha_1-1} (1-v)^{\alpha_2-1}\right), & u>0, \; 0<v<1 \\ 0, & \text{o.w.} \end{cases}$

$\Rightarrow U \sim Gamma(\alpha_1+\alpha_2, \theta)$ and $V \sim Beta(\alpha_1, \alpha_2)$ are independent.

(b) Using (a), we can write $W = \frac{X}{X+Y}$ and $T = X+Y$, where $X \sim Gamma(\alpha_1, \theta)$ and $Y \sim Gamma(\alpha_2, \theta)$ are independent. Then $WT = X \sim Gamma(\alpha_1, \theta)$.

Problem 19 $X \sim N(0,1)$ and $Y^2 \sim \chi^2_1$ are independent

$\Rightarrow \frac{X}{\sqrt{\frac{Y^2}{1}}} \sim t_1$ (Student t distribution with 1 d.f)

$\Rightarrow Z = \frac{X}{|Y|} \sim t_1$ (or Cauchy distribution)

The p.d.f. of Z is $f_Z(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2}, \quad -\infty<z<\infty$

Thus $U = \frac{|X|}{|Y|} = |Z|$ has p.d.f. $f_U(u) = \begin{cases} \frac{2}{\pi} \cdot \frac{1}{1+u^2}, & \text{if } u>0 \\ 0, & \text{o.w} \end{cases}$

(use Jacobian method)

$\boxed{\text{Problem No. 20}}$ $f_{\underline{X}}(x_1,\dots,x_n) = \prod_{i=1}^{n} f(x_i) = e^{-\sum_{i=1}^{n} x_i}, \quad \underline{x} \in S_{\underline{X}} = [0, \infty)^n$

Consider the 1-1 transformation $(x_1,\dots,x_n) \to (Y_1,\dots,Y_n)$ defined by

$$Y_1 = \frac{X_1}{X_1+X_2}, \quad Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}, \quad Y_3 = \frac{X_1+X_2+X_3}{X_1+X_2+X_3+X_4}, \dots, \quad Y_{n-1} = \frac{\sum_{i=1}^{n-1} X_i}{\sum_{i=1}^{n} X_i},$$

$$Y_n = \sum_{i=1}^{n} X_i, \quad \text{i.e.}$$

$$X_1 = \prod_{i=1}^{n} Y_i, \quad X_2 = \left(\prod_{i=2}^{n} Y_i\right)(1-Y_1), \quad X_3 = \left(\prod_{i=3}^{n} Y_i\right)(1-Y_2), \dots, \quad X_{n-1} = Y_{n-1}Y_n(1-Y_{n-2}),$$

$$X_n = Y_n(1-Y_{n-1}).$$

$$S_{\underline{Y}} = \left\{\underline{y} : \prod_{i=1}^{n} y_i \geq 0, \left(\prod_{i=2}^{n} y_i\right)(1-y_1) \geq 0, \left(\prod_{i=3}^{n} y_i\right)(1-y_2) \geq 0, \dots, y_{n-1}y_n(1-y_{n-2}) \geq 0, \; y_n(1-y_{n-1}) \geq 0\right\}$$

$$= \{\underline{y} : 0 \leq y_i \leq 1, \; i=1,\dots,n-1, \; y_n \geq 0\} = [0,1]^{n-1} \times [0,\infty)$$

$$J = \begin{vmatrix} y_2\cdots y_n & y_1y_3\cdots y_n & y_1y_2y_4\cdots y_n & \cdots & y_1\cdots y_{n-3}y_{n-1}y_n & y_1\cdots y_{n-2}y_n & y_1\cdots y_{n-1} \\ -y_2\cdots y_n & y_3\cdots y_n(1-y_1) & y_2y_4\cdots y_n(1-y_1) & \cdots & y_2\cdots y_{n-3}y_{n-1}y_n(1-y_1) & y_2\cdots y_{n-2}y_n(1-y_1) & y_2\cdots y_{n-1}(1-y_1) \\ 0 & -y_3\cdots y_n & y_4\cdots y_n(1-y_2) & \cdots & y_3\cdots y_{n-3}y_{n-1}y_n(1-y_2) & y_3\cdots y_{n-2}y_n(1-y_2) & y_3\cdots y_{n-1}(1-y_2) \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & y_{n-1}y_n(1-y_{n-3}) & y_{n-2}y_n(1-y_{n-3}) & \cdots \\ 0 & 0 & 0 & \cdots & -y_{n-1}y_n & y_n(1-y_{n-2}) & y_{n-1}(1-y_{n-2}) \\ 0 & 0 & 0 & \cdots & 0 & -y_n & 1-y_{n-1} \end{vmatrix}$$

$$= (-1)^{n+1}(-y_2y_3\cdots y_n)(-y_3\cdots y_n)\cdots(-y_{n-1}y_n)(-y_n)$$

(on performing $R_1 \to R_1 + \cdots + R_n$ and expanding)

$$= y_2 y_3^2 \cdots y_{n-1}^{n-2} y_n^{n-1}$$

Thus

$$f_{\underline{Y}}(\underline{y}) = y_2 y_3^2 \cdots y_{n-1}^{n-2} y_n^{n-1} e^{-y_n} I_{S_{\underline{Y}}}(\underline{y}) = \prod_{i=1}^{n} g_i(y_i) I_{S_i}(y_i)$$

where

$$g_i(y) = \begin{cases} i y^{i-1}, & \text{if } 0<y<1 \\ 0, & \text{o.w.} \end{cases}, \quad i=1,\dots,n-1, \qquad g_n(y) = \begin{cases} \dfrac{e^{-y} y^{n-1}}{\lfloor n-1}, & \text{if } y>0 \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow Y_1, \ldots Y_n$ are independent with $f_{Y_i}(y) = \begin{cases} i\, y^{i-1}, & \text{for } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$,

$i=1,\ldots,n-1$ and $f_{Y_n}(y) = \begin{cases} \frac{1}{\Gamma n} e^{-y} y^{n-1}, & \text{if } y>0 \\ 0, & \text{o.w.} \end{cases}$

$\boxed{\text{Problem No. 21}}$ (a) We know that $\bar{X}$ and S^2 are independent, $\bar{X} \sim N(\mu, \sigma^2)$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Therefore

$$E\left(\frac{\bar{X}}{S}\right) = E\left(\bar{X}\cdot\frac{1}{S}\right) = E(\bar{X})\, E\left(\frac{1}{S}\right) = \mu\, E\left(\frac{1}{S}\right)$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow E\left(\frac{\sigma}{\sqrt{n-1}\, S}\right) = E\left(\frac{1}{\sqrt{\chi^2_{n-1}}}\right) = \frac{1}{2^{\frac{n-1}{2}}\Gamma\frac{n-1}{2}} \int_0^\infty \frac{1}{\sqrt{x}} e^{-\frac{x}{2}} x^{\frac{n-1}{2}-1}\, dx$$

$$= \frac{2^{\frac{n-2}{2}}\Gamma\frac{n-2}{2}}{2^{\frac{n-1}{2}}\Gamma\frac{n-1}{2}} \Rightarrow E\left(\frac{1}{S}\right) = \frac{1}{\sigma}\sqrt{\frac{n-1}{2}}\ \frac{\Gamma\frac{n-2}{2}}{\Gamma\frac{n-1}{2}}$$

$$\Rightarrow E\left(\frac{\bar{X}}{S}\right) = \sqrt{\frac{n-1}{2}}\ \frac{\Gamma\frac{n-2}{2}}{\Gamma\frac{n-1}{2}}\ \frac{\mu}{\sigma}$$

(b) Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Then

$$M_{Y_1,Y_2}(t_1,t_2) = E\left(e^{t_1Y_1+t_2Y_2}\right) = E\left(e^{(t_1+t_2)X_1} e^{(t_1-t_2)X_2}\right)$$
$$= E\left(e^{(t_1+t_2)X_1}\right) E\left(e^{(t_1-t_2)X_2}\right) \quad (X_1 \text{ and } X_2 \text{ independent})$$
$$= M_{X_1}(t_1+t_2)\, M_{X_2}(t_1-t_2)$$

$$\Rightarrow M_{Y_1,Y_2}(t_1,t_2) = e^{\frac{(t_1+t_2)^2}{2}} e^{\frac{(t_1-t_2)^2}{2}} = e^{\frac{2t_1^2}{2}} e^{\frac{2t_2^2}{2}}, \quad (t_1,t_2)\in\mathbb{R}^2$$

$\Rightarrow Y_1$ and Y_2 are independent, $Y_1 \sim N(0,2)$, $Y_2 \sim N(0,2)$

$\Rightarrow \frac{Y_1}{\sqrt{2}} \sim N(0,1)$, $\frac{Y_2}{\sqrt{2}} \sim N(0,1)$ are independent

$\Rightarrow \frac{Y_1}{\sqrt{2}} \sim N(0,1)$ and $\frac{Y_2^2}{2} \sim \chi^2_1$ are independent

$$\Rightarrow Z = \frac{Y_1}{|Y_2|} = \frac{\frac{Y_1}{\sqrt{2}}}{\sqrt{\frac{Y_2^2}{2}}} = \frac{N(0,1)}{\sqrt{\chi^2_1/1}} \ \Big\} \text{ independent}$$

$\Rightarrow Z \sim t_1$ (t distribution with 1 d.f. or Cauchy dist)

$$\Rightarrow f_Z(z) = \frac{1}{\pi}\cdot\frac{1}{1+z^2}, \quad -\infty < z < \infty$$

Problem No. 22 $f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} f(x_i) = e^{-\sum_{i=1}^{n} x_i}$, $\underline{x} \in S_{\underline{X}} = (0,\infty)^n$.

By Problem 3, joint p.d.f. of $(X_{(1)}, \ldots, X_{(n)})$ is

$$g(y_1, \ldots, y_n) = \begin{cases} \lfloor n \, e^{-\sum_{i=1}^{n} y_i}, & \text{if } 0 < y_1 < \ldots < y_n < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Consider the 1-1 transformation $(X_{(1)}, \ldots, X_{(n)}) \to (Z_1, \ldots, Z_n)$ defined by $Z_1 = n X_{(1)}$, $Z_2 = (n-1)(X_{(2)} - X_{(1)})$, $Z_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, $i = 3, \ldots, n$, i.e.,

$$X_{(1)} = \frac{Z_1}{n}, \quad X_{(2)} = \frac{Z_1}{n} + \frac{Z_2}{n-1}, \quad X_{(3)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \frac{Z_3}{n-2}, \ldots,$$

$$X_{(n)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + \frac{Z_{n-1}}{2} + Z_n$$

$$S_{\underline{Z}} = \left\{ \underline{z} : 0 \le \frac{z_1}{n} < \frac{z_1}{n} + \frac{z_2}{n-1} \le \frac{z_1}{n} + \frac{z_2}{n-1} + \frac{z_3}{n-2} \le \cdots \le \frac{z_1}{n} + \frac{z_2}{n-1} + \cdots + \frac{z_{n-1}}{2} + z_n \right\}$$

$$= \{ \underline{z} : z_1 \ge 0, \; z_2 \ge 0, \; \ldots, \; z_n \ge 0 \}$$

$$J = \begin{vmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & \frac{1}{3} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{\lfloor n}$$

Thus

$$f_{\underline{Z}}(\underline{z}) = \frac{1}{\lfloor n} \lfloor n \, e^{-\sum_{i=1}^{n} z_i} \, I_{S_{\underline{Z}}}(\underline{z})$$

$$= \begin{cases} \prod_{i=1}^{n} e^{-z_i}, & \text{if } z_i > 0, \; i = 1, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow$ $Z_1, Z_2, \ldots, Z_n$ are i.i.d. $\text{Exp}(1)$

(b) $$X_{(r)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + \frac{Z_r}{n-r+1}$$

$$X_{(s)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + \frac{Z_r}{n-r+1} + \frac{Z_{r+1}}{n-r} + \cdots + \frac{Z_s}{n-s+1} \quad (r < s)$$

$\Rightarrow E(X_{(r)}) = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}$

$Var(X_{(r)}) = \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-r+1)^2}$

$Cov(X_{(r)}, X_{(s)}) = Cov\left(\frac{Z_1}{n} + \cdots + \frac{Z_r}{n-r+1}, \frac{Z_1}{n} + \cdots + \frac{Z_s}{n-s+1}\right)$

$= Var(X_{(r)}) = \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-r+1)^2}$

(c) $X_{(r)} = \sum_{i=1}^{r} \frac{Z_i}{n-i+1}, \quad X_{(s)} - X_{(r)} = \sum_{i=r+1}^{s} \frac{Z_i}{n-i+1}, \quad 1 \le r < s \le n$

From (a) it follows that $(Z_1, \ldots, Z_r)$ and $(Z_{r+1}, \ldots, Z_s)$ are independent $\Rightarrow$ $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent

(d) $Z_{r+1} = (n-r)(X_{(r+1)} - X_{(r)}) \sim Exp(1)$ (by (a))

$$\Rightarrow f_{Z_{r+1}}(z) = \begin{cases} e^{-z} & \text{if } z > 0 \\ 0, & \text{o.w.} \end{cases}$$

Therefore the p.d.f. of $S = X_{(r+1)} - X_{(r)}$ is

$$f_S(s) = \begin{cases} (n-r)\, e^{-(n-r)s}, & \text{if } s > 0 \\ 0, & \text{o.w.} \end{cases}$$

$\boxed{\text{Problem No. 23}}$ By Problem 3 M_n is of A.C. with p.d.f.

$$f_{M_n}(x) = n F^{n-1}(x) f(x), \quad -\infty < x < \infty,$$

where f is the p.d.f. of X_1.

Clearly the d.f. of M_n is

$$F_{M_n}(x) = P(M_n \le x) = P(X_i \le x,\ i = 1, \ldots, n)$$
$$= \prod_{i=1}^{n} P(X_i \le x) = [F(x)]^n, \quad -\infty < x < \infty.$$

Also $P(M_n \ge 0) = 1$. Therefore

$$E(M_n) = \int_0^\infty [1 - F_{M_n}(x)]\,dx = \int_0^\infty [1 - (F(x))^n]\,dx$$
$$= \int_0^\infty [1 - (F(x))^{n-1}]\,dx + \int_0^\infty (F(x))^{n-1}(1 - F(x))\,dx$$
$$= E(M_{n-1}) + \int_0^\infty (F(x))^{n-1}(1 - F(x))\,dx.$$

Problem No. 24 (a)

$M_{U,V}(t_1, t_2) = E(e^{t_1 U + t_2 V}) = E(e^{(t_1 a + t_2 b)X} e^{(t_1 b - t_2 a)Y})$

$= E(e^{(t_1 a + t_2 b)X}) E(e^{(t_1 b - t_2 a)Y})$ (X and Y are indept)

$= M_X(t_1 a + t_2 b) \; M_Y(t_1 b - t_2 a)$

$= e^{\frac{(t_1 a + t_2 b)^2 \sigma^2}{2}} \; e^{\frac{(t_1 b - t_2 a)^2 \sigma^2}{2}}$

$= e^{\frac{(a^2+b^2)\sigma^2 t_1^2}{2}} \; e^{\frac{(a^2+b^2)\sigma^2 t_2^2}{2}}$

$= M_{U,V}(t_1, 0) \; M_{U,V}(0, t_2), \quad (t_1, t_2) \in \mathbb{R}^2$

$\Rightarrow U \sim N(0, (a^2+b^2)\sigma^2)$ and $V \sim N(0, (a^2+b^2)\sigma^2)$ are i.i.d.

(b) Follows from (a) by taking $a = b = \frac{1}{\sqrt{2}}$.

Problem No. 25 First suppose that $X \sim U(\alpha, \beta)$. Then

$$f_X(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{o.w.} \end{cases}$$

Clearly $P(\alpha < X \leq \beta) = \int_\alpha^\beta f_X(x)\,dx = 1$

Also if $I = (c, d)$ (or $(c, d]$, $[c, d]$, $[c, d)$) $\subseteq (\alpha, \beta)$ then

$P(X \in I) = \int_c^d f_X(x)\,dx = \frac{d-c}{\beta - \alpha} \rightarrow$ depends only on length of I

Conversely suppose that $P(\alpha < X \leq \beta) = 1$ and $P(X \in I) = P(X \in J)$ for any pair of subintervals $I, J \subseteq (\alpha, \beta)$ having the same length. Then clearly

$F_X(\alpha) = P(X \leq \alpha) = 1 - P(X > \alpha) \leq 1 - P(\alpha < X \leq \beta) = 0$

$\Rightarrow F_X(\alpha) = 0$. Also $F_X(\beta) = P(X \leq \beta) \geq P(\alpha < X \leq \beta) = 1$

$\Rightarrow F_X(\beta) = 1$.

Thus $F_X(x) = 0 \;\; \forall\, x \leq \alpha$ and $F_X(x) = 1, \;\; \forall\, x \geq \beta$.

For $\alpha < x < \beta$, $F_X(x) = P(X \leq x) = P(\alpha < X \leq \alpha + (x - \alpha)) = h(x - \alpha)$,

where $h(\lambda) = P(\alpha < X \leq \alpha + \lambda)$, $0 < \lambda \leq \beta - \alpha$.

$h(\lambda_1+\lambda_2) = P(\alpha < X \le \alpha+\lambda_1+\lambda_2) = P(\alpha < X \le \alpha+\lambda_1) + P(\alpha+\lambda_1 < X \le \alpha+\lambda_1+\lambda_2)$

$= h(\lambda_1) + P(\alpha < X \le \alpha+\lambda_2)$ (since $P(X \in I)$ depends only on length of I)

$= h(\lambda_1)+h(\lambda_2)$, $\lambda_i > 0$, $i=1,2$, $\lambda_1+\lambda_2 \le \beta-\alpha$

Define $G(x) = h((\beta-\alpha)x)$, $0 < x \le 1$.

Then $G(x_1+x_2) = G(x_1)+G(x_2)$, $x_i > 0$, $i=1,2$, $x_1+x_2 \le 1$

By induction

$G(x_1+x_2+\cdots+x_n) = G(x_1)+G(x_2)+\cdots+G(x_n)$, $x_i > 0$, $i=1(1)n$

$x_1+\cdots+x_n \le 1$, $n \ge 2$..

$\Rightarrow G(x) = n\,G(\frac{x}{n})$, $0 < x \le 1$ (by taking $x_i = \frac{x}{n}$, $i=1,\ldots$) ... (A)

$G(my) = m\,G(y)$, $0 < y \le \frac{1}{m}$ (by taking $m=n$, and $x_i = y$, $i=1(1)m$) .... (B)

Now let $m,n \in \{1,2,\ldots\}$ and $\frac{m}{n} \in (0,1)$ (i.e. $m < n$)

$G(\frac{m}{n}) = m\,G(\frac{1}{n})$ (from (B), since $\frac{1}{n} < \frac{1}{m}$)

$= \frac{m}{n}\,G(1)$ (from (A))

$\Rightarrow G(r) = r\,G(1)$, $\forall$ rational $r \in (0,1)$

Fix $x \in (0,1]$. Choose sequences $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$ of rationals in $(0,1)$ such that $u_n \uparrow x$ and $v_n \downarrow x$.

Note that $G\uparrow$ and therefore

$G(u_n) \le G(x) \le G(v_n)$, $\forall n \ge 1$

$\Rightarrow u_n G(1) \le G(x) \le v_n G(1)$ $\forall n \ge 1$

$\Rightarrow x\,G(1) \le G(x) \le x\,G(1)$ (on taking limits)

$\Rightarrow G(x) = x\,G(1)$, $\forall x \in (0,1)$

$\Rightarrow h((\beta-\alpha)x) = x\,h(\beta-\alpha) = x$, $\forall x \in (0,1]$

Since $h(\beta-\alpha) = P(\alpha < X \le \beta) = 1$

$\Rightarrow h(\lambda) = \frac{\lambda}{\beta-\alpha}$, $\forall\ 0 < \lambda \le \beta-\alpha$

$\Rightarrow F_X(\alpha+\lambda) = F_X(\alpha+\lambda) - F_X(\alpha) = P(\alpha < X \le \alpha+\lambda) = h(\lambda) = \frac{\lambda}{\beta-\alpha}$, $\forall\ 0 < \lambda \le \beta-\alpha$

$\Rightarrow F_X(x) = \frac{x-\alpha}{\beta-\alpha}$, $\forall\ \alpha < x \le \beta$

$$\Rightarrow F_X(x) = \begin{cases} 0, & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \le x < \beta \\ 0, & \text{o.w.} \end{cases} \Rightarrow X \sim U(\alpha,\beta)$$

$\boxed{17/17}$