MSO 203B - PDE Lecture 5 : (Fourier Series II)

f is piecewise continuous on $[-L,L]$. The Fourier series corresponding to f is given by

Please prove it.

$$f(x) \approx a_0 + \sum_{1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \longrightarrow (*)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

Remark:- (i) If f is an even fxn on $[-L,L]$ s.t $f(x+2L) = f(x)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx \;;\; a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \;;\; b_n = 0. \quad \forall n\in\mathbb{N}.$$

$$f(x) \approx a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$ (Fourier Cosine Series).

$\|^{ly}$ for f odd.

$a_n = 0 \quad \forall n \in \mathbb{N}\cup\{0\}$ and $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

$$f(x) \approx \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$ (Fourier Sine series).

Ex#1 :- $f(x) = |\sin x|$. in $[-\pi, \pi]$.

period $= \pi$. $L = \pi$.

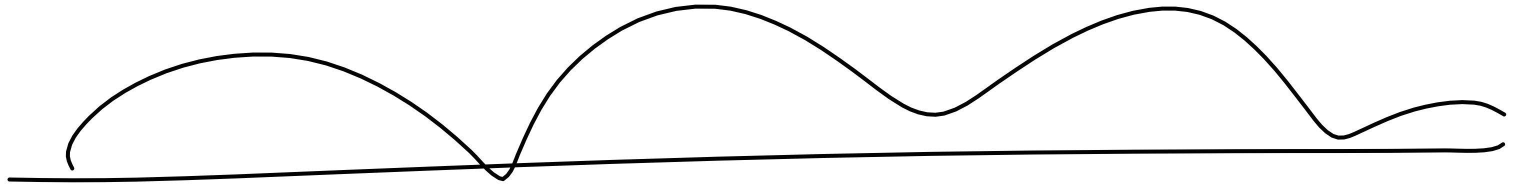
$$a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos(nx)\, dx \qquad ; n \geq 2 \text{ and } n = 0$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx.$$

$$= \frac{2\left[1 + (-1)^n\right]}{\pi(1 - n^2)}$$

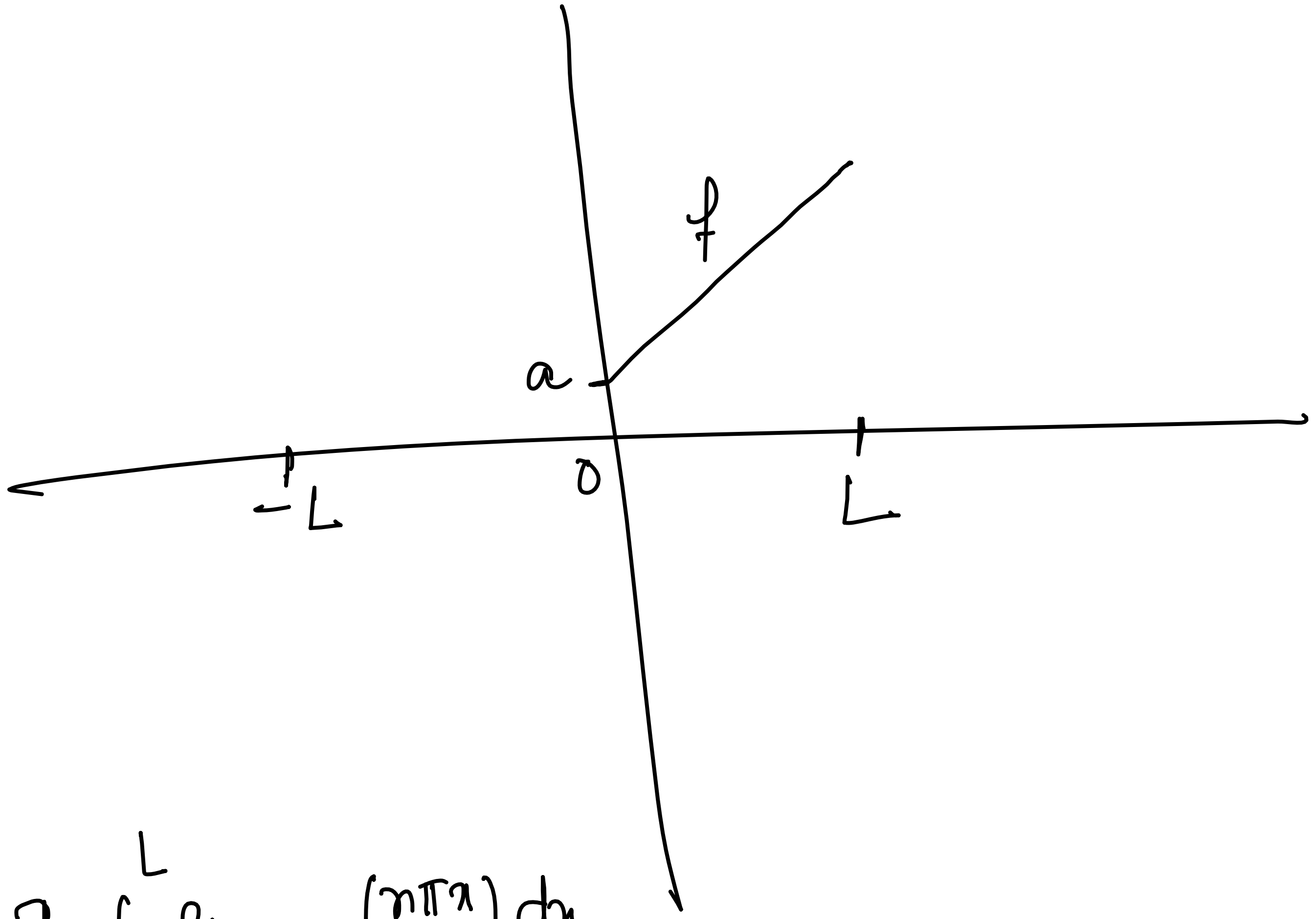
$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = 0$$

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{1 - 4k^2}.$$



Half range expansion

Let f is defined on $[0,L]$ s.t f is piecewise continuous.



Even

Extend f s.t $\tilde{f}$ is an even fxn on $[-L,L]$

$$\tilde{f}(x) = \begin{cases} f(x) \;; & x \in [0,L] \\ f(-x) \;; & x \in [-L,0) \end{cases}$$

So, $\tilde{f}$ is an p.c fxn on $[-L,L]$ which is even

$$\tilde{f}(x) = a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

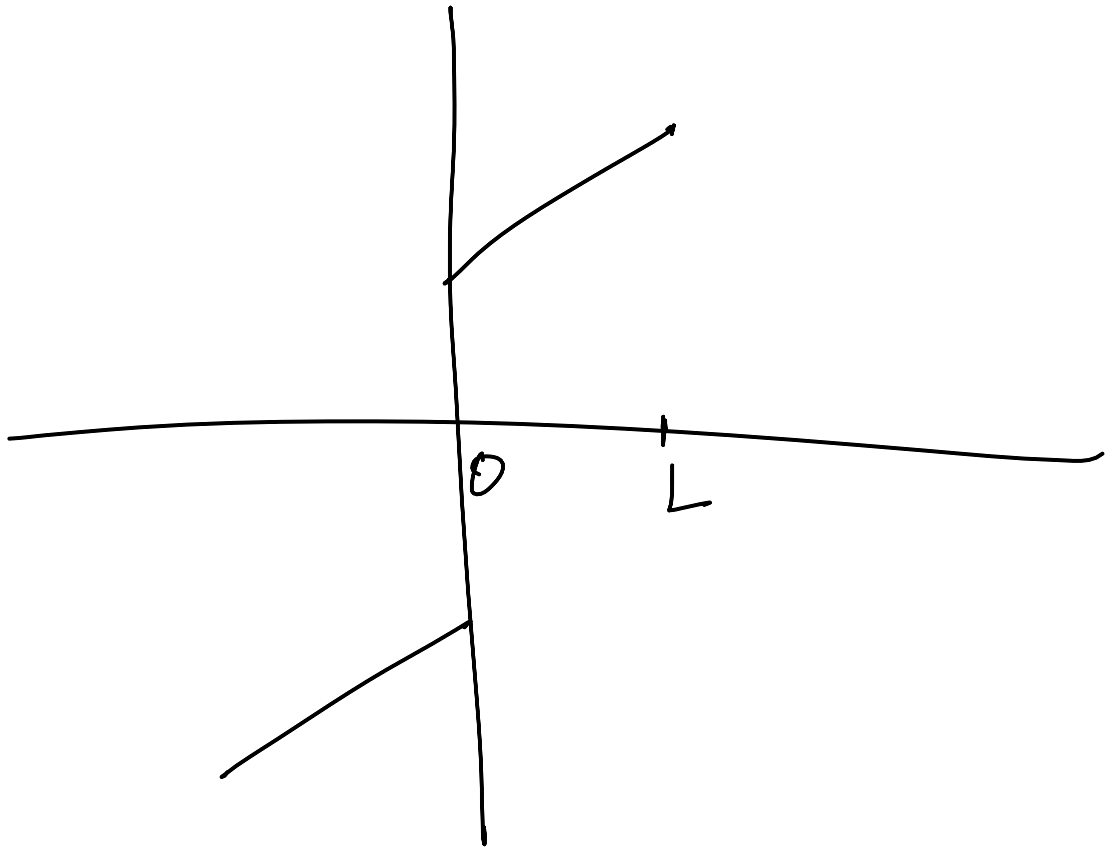
where $a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \tilde{f}(x)\,dx = \frac{1}{L}\int_0^L \tilde{f}(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx.$$

Odd Extension

f is p.c on $[0,L]$.

$$\tilde{f}(x) := \begin{cases} f(x) & ; x \in [0,L] \\ -f(-x) & ; x \in [-L,0) \end{cases}$$



So, $\tilde{f}$ is an odd fxn defined on $[-L,L]$ and p.c.

$$\tilde{f}(x) = \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

s.t $b_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$

Q:- Given $f : [a,b] \xrightarrow{p.c} \mathbb{R}$, can we talk about the Fourier series of 'f'

Define $g(x) = f(x+a)$.

So, $g : [0, b-a] \longrightarrow \mathbb{R}$ is p.c. $\blacksquare$
(DIY)

Parseval Identity :-

f is piecewise continuous on $[-\pi, \pi]$ and also f is square integrable on $[-\pi, \pi]$.

If $f(x) = a_0 + \sum_{1}^{\infty} [a_n \cos nx + b_n \sin nx] dx$

$$f(x) \cdot f(x) = a_0^2 + 2a_0 \sum_{1}^{\infty} [a_n \cos nx + b_n \sin nx] + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big[a_n a_m \underset{\shortparallel}{\cos(nx)} \cos(mx)$$

$$+ b_n b_m \sin(nx) \sin(mx)$$

$$+ a_n b_m \underset{\substack{\shortparallel \\ 0}}{\cos(nx)} \sin(mx) + a_m b_n \overset{\substack{0 \\ \shortparallel}}{\cos(mx)} \sin(nx) \Big]$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 \int_{-\pi}^{\pi} 1 \cdot dx + 0 + \sum \Big[a_n^2 \int_{-\pi}^{\pi} \cos^2 nx \, dx + b_n^2 \int_{-\pi}^{\pi} \sin^2 nx \, dx \Big]$$

$$= 2\pi a_0^2 + \pi \sum a_n^2 + \pi \sum b_n^2$$

$$\boxed{\Rightarrow 2a_0^2 + \sum_{1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx}$$ — Parseval Identity

Defn :- f is said to be a square integrable fxn if

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$$

Ex :- Any continuous fxn on $[-\pi, \pi]$ is square integrable

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq M^2 \int_{-\pi}^{\pi} dx$$

$$= 2\pi M^2$$

Q :- Is all p.c fxn on $[-\pi, \pi]$ square integrable