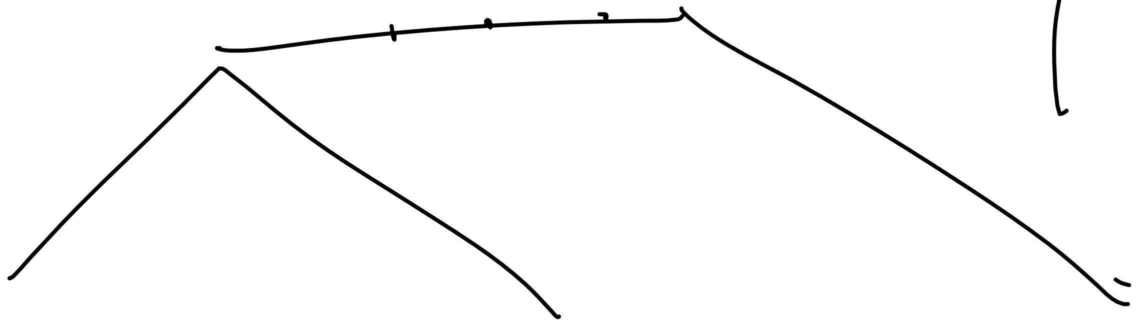
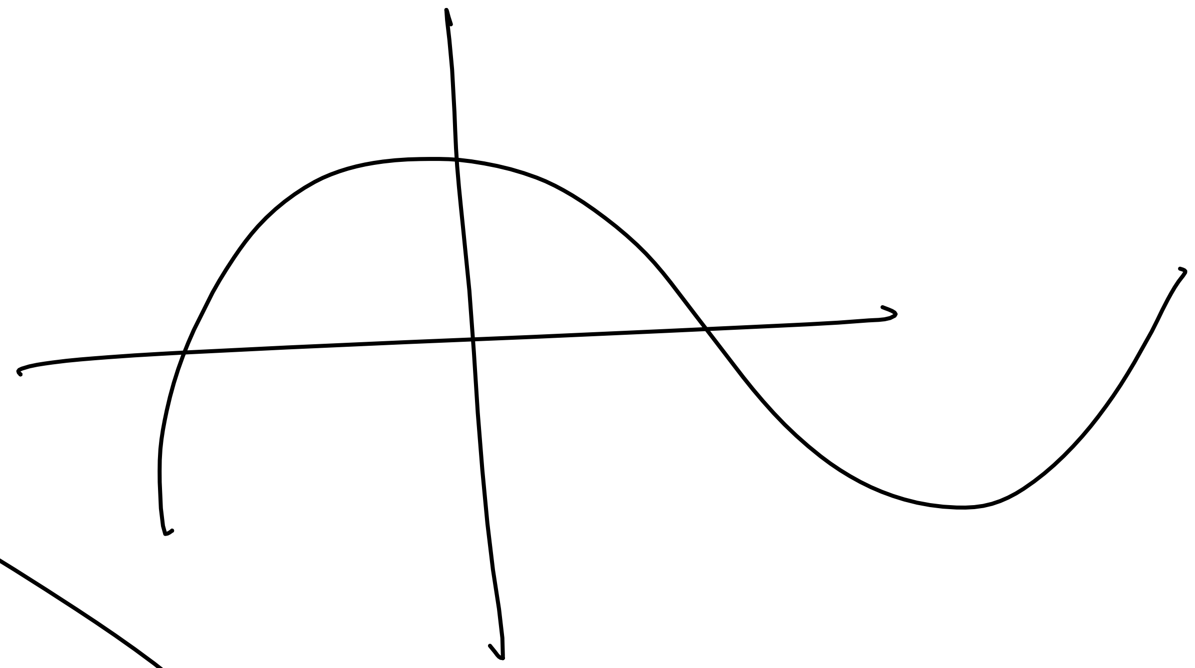
MSO 203B (PDE) $\boxed{\text{Lecture-8 : Method of Characteristics-II}}$

$a(x,y,u)\,u_x + b(x,y,u)\,u_y + c(x,y,u) = 0$ in $\mathbb{R}^2$ ($u:\mathbb{R}^2 \to \mathbb{R}$ is the unknown).

$u|_\Gamma = \phi$





Transport Equation

$u_t + a u_x = 0$ —— (1)

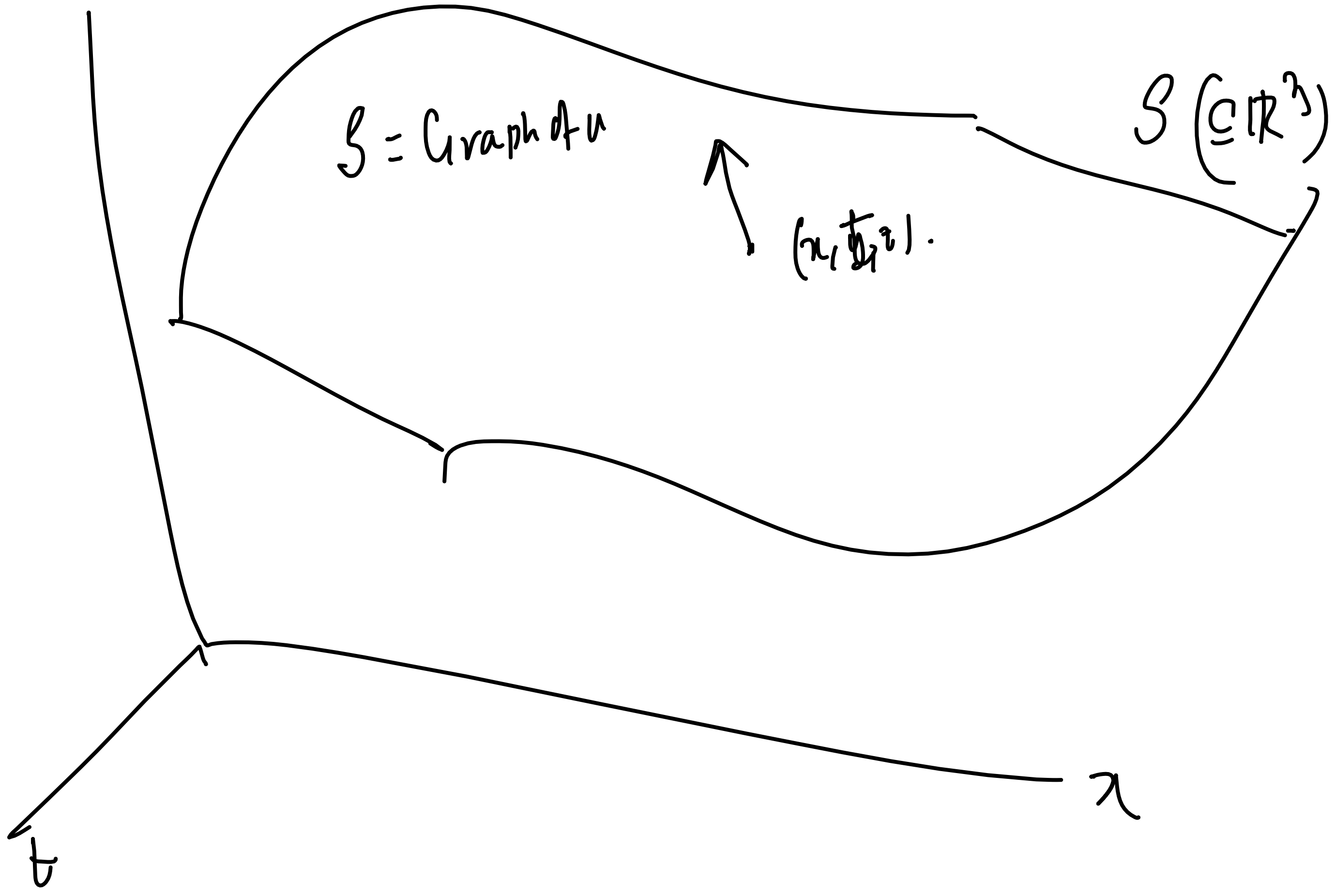
$\boxed{u|_\Gamma = \phi \times}$

Assume $u \in C^1$ solving (1)

S = graph of u.

$\hat{n}(x,y,z)$ = ~~unit~~ outward normal

$= (u_x, u_t, -1)$



From (1),

$$(u_x, u_t, -1)\cdot(a, 1, 0) = 0$$

Find a curve C s.t the tangent vector to C at (x,y,z) is $(a,1,0)$.

$C(s) = (x(s), t(s), z(s)) \subset S$

$\dot{x}(s) = a \quad ; \quad \dot{t}(s) = 1 \quad ; \quad \dot{z}(s) = 0$ = Characteristic eqn.

$x(s) = as + c_1$ — (I)

$t(s) = s + c_2$ — (II)

$z(s) = c_3$ — (III)

$x(s) = at(s) + c_4$

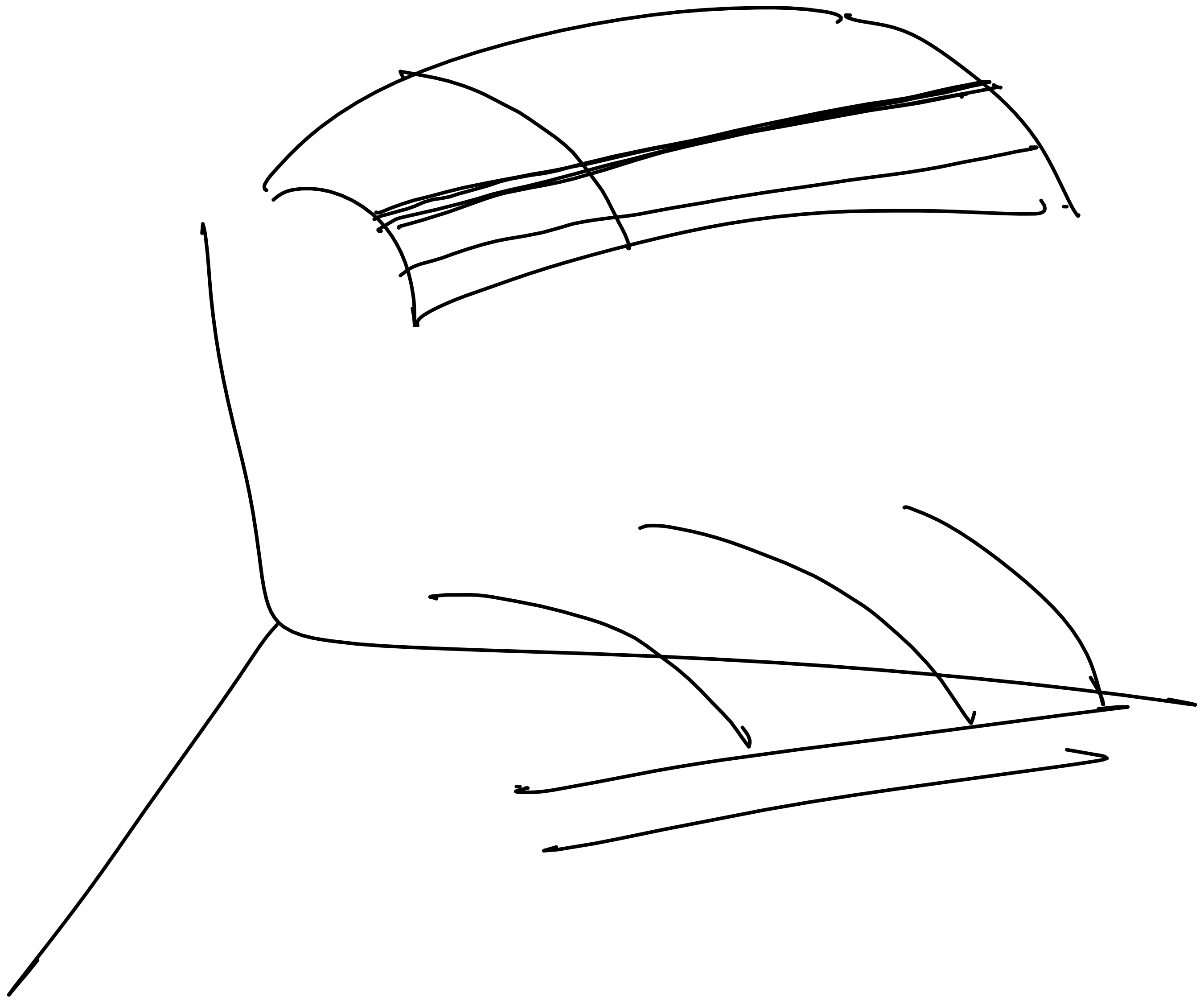
$\Rightarrow x - at = c_4$

& $z = c_3$

Define, $u(x,t) = f(x - at)$ for any $f \in C^1$.

$u : \mathbb{R}^2 \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$



$u_t + a u_x = 0$

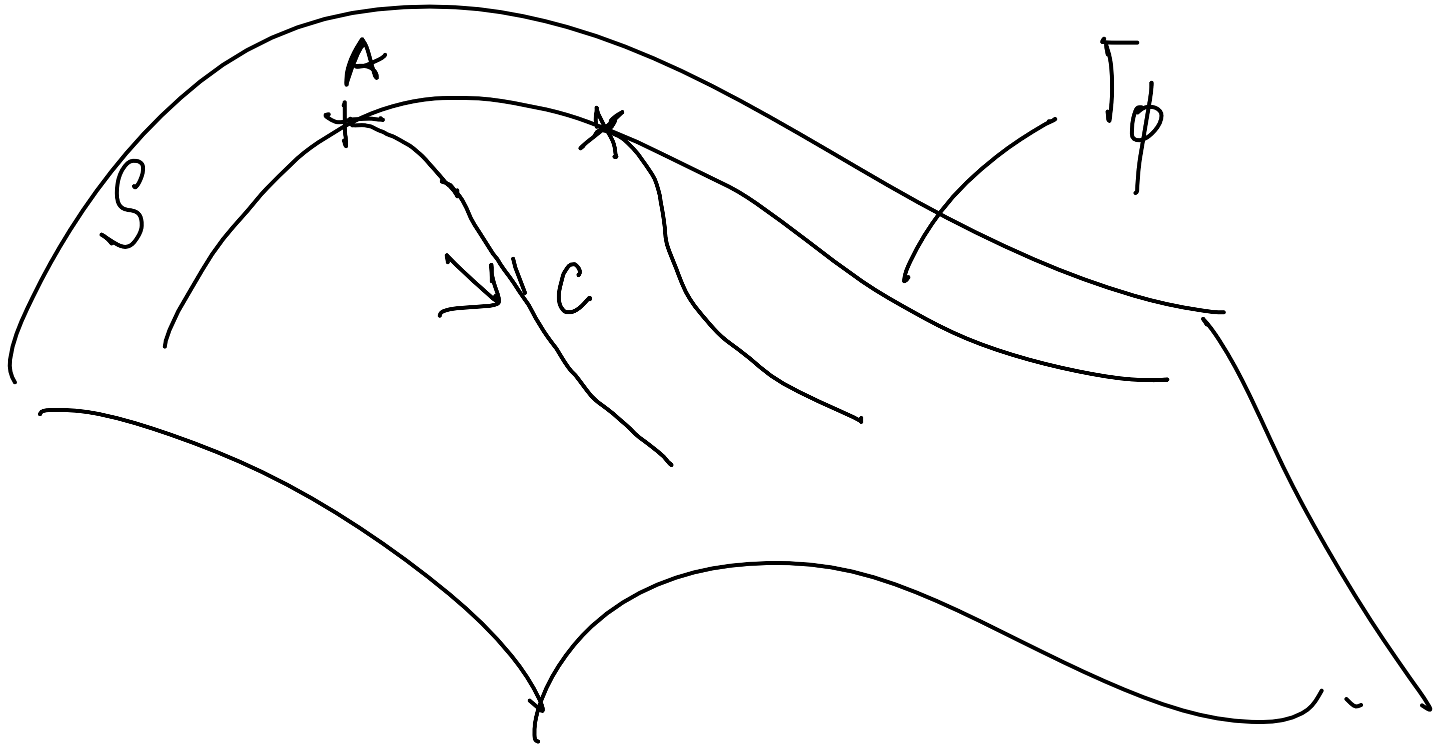
$u(x,0) = \phi(x)$. data

If one parametrizes the data curve $\Gamma(r) = (\gamma_1(r), \gamma_2(r)) = (r, 0)$

$A\ (r, 0, \phi(r))$.

$\Gamma_\phi = (\Gamma(r), \phi(r)) = (r, 0, \phi(r))$

We are looking for a surface S s.t. $(r, 0, \phi(r))$ must lie on S.



Char Curve $C(r,s) = (x(r,s), y(r,s), z(r,s))$

Char Eqn :- For a fixed $r \in \mathbb{R}$.

$x'(r,s) = a.$

$x(r,0) = r$

$t'(r,s) = 1$

$t(r,0) = 0$

$z'(r,s) = 0$.

$z(r,0) = \phi(r)$

$x(r,s) = as + g(r)$

$r = x(r,0) = a0 + g(r)$

$\Rightarrow g(r) = r$.

$\Rightarrow x(r,s) = as + r$ — (I)

$t(r,s) = s + h(r)$.

$0 = t(r,0) = 0 + h(r)$

$\Downarrow$

$t(r,s) = s$ — (II)

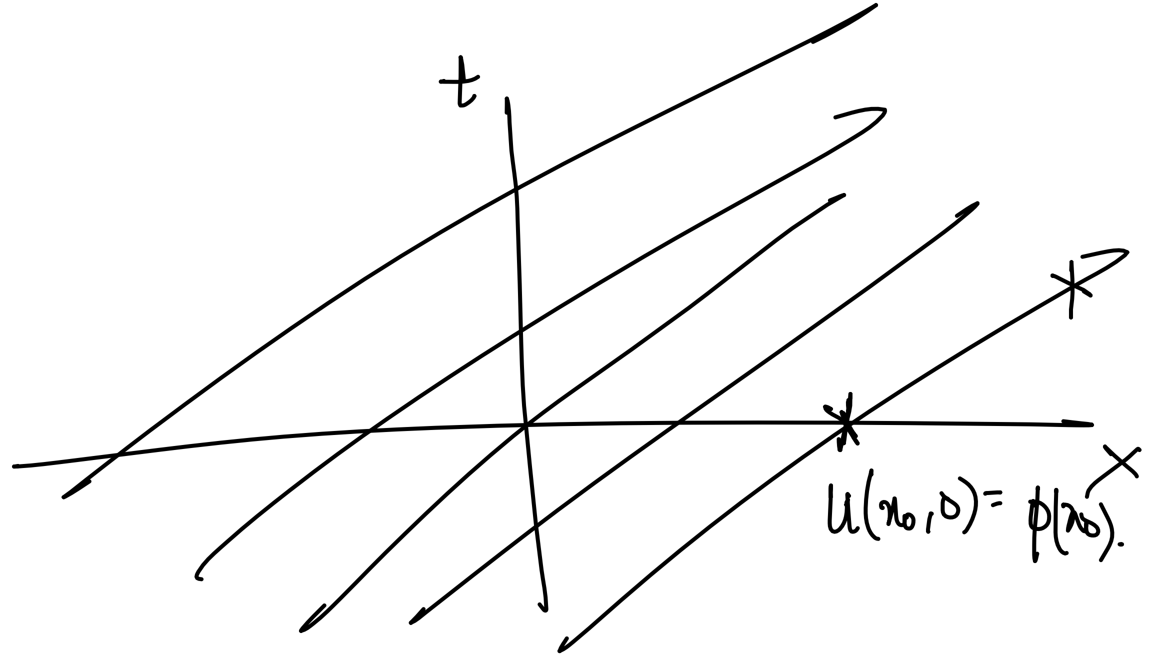
$z(r,s) = \phi(r)$

From (I) & (II) $\Rightarrow$ $x = at + r$ $\Rightarrow$ $r = x - at$

Define,

$u(x,t) := z(r,s) = \phi(x-at)$ ✓

$$\boxed{\begin{aligned} u_x &= \phi' \\ u_t &= -a\phi' \\ u_t + a u_x &= 0 \end{aligned}}$$



Q:- Given an linear PDE (1st Order) with data curve, can we always find a soln.

$u(x,t) = \phi(x-at)$ solves, $\quad u_t + a u_x = 0$

$u(x,0) = \phi(x)$.

Remark:- (I) When projected on $\{z=0\}$ the char curves are called the projected characteristic curve

(II) The solution is constant along the characteristics.

Ex:- $u_t + a u_x = 0 \longrightarrow u(x,t) = f(x-at)$

$u|_\Gamma = \phi$ where $\Gamma = \{x - at = 5\}$