The Monotone Convergence Theorem and a First Look at Infinite Series

Note 1

7f744b7eecb54041a6e188d2283abcff

A sequence (a_n) is {{c2::increasing}} if {{c1::$a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.
}}

Note 2

cb73357863a14f808fcb79e9f2888e9d

A sequence (a_n) is {{c2::decreasing}} if {{c1::$a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.
}}

Note 3

428c29af1f87467cba4605f856da5dc0

A sequence (a_n) is {{c2::monotone}} if {{c1::it is either increasing or decreasing.}}

Note 4

f0effd26705b4fe2850675b4a8b69fa7

If a sequence is {{c3::monotone}} and {{c2::bounded,}} then {{c1::it converges.
}}

Note 5

f04966660a1d453499de164d33c3efd9

If a sequence is monotone and bounded, then it converges.

«{{c1::Monotone Convergence Theorem}}»

Note 6

fe52926982cd479399d0e77cf6fbb8ae

What is the key idea in the proof of the Monotone Convergence Theorem?

> The limit equals to $\sup \{a_n \mid n \in \mathbb{N}\}$

Note 7

b7b0d33916a74554bee0bb1e829b7a20

Let {{c3::(a_n) be a sequence.}} {{c2::An infinite series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} = b_1 + b_2 + b_3 + \cdots .$$

}}

Note 8

024782c9319a441f91dfd2c8e8aac542

Let $\sum_{n=1}^{\infty} a_n$ be a series. We define the corresponding {{c2::sequence of partial sums}} by {{c1::

$$m \mapsto a_1 + a_2 + \cdots + a_m.$$

}}

Note 9

56563c7563df42c0a111a49ad4ae24ae

Let $\sum_{n=1}^{\infty} a_n$ be a series. {{c2::The sequence of partial sums}} is usually denoted {{c1::(s_m).}}

Note 10

dc59f9b31fff4dcb9113d42da885c946

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A}} if {{c1::the sequence of partial sums converges to A.}}

Note 11

356961ddcb85482c8155d43bd6d8061c

Let $\sum_{n=1}^{\infty} a_n$ be a series. If {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A,}} we write

{{c1::

$$\sum_{n=1}^{\infty} a_n = A.$$

}}

Note 12

4819e0996d5d4eeb8ab8df01f58c8efe

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

> Yes.

Note 13

64c293a1a2f74541ba8e3ffa23fb54b2

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. What is the key idea in the proof?

> $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$.

Note 14

cd5ca73daf014641b49c5445adcd69b5

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

▌ No.

Note 15

184fe5e5e62b4c3f8a49c4ea6d26c240

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What is the key idea in the proof?

▌ Find a lower bound using powers of two.

Note 16

4608dd8499934012aadc1209fb34ec1e

{{c2::$\sum_{n=1}^{\infty} \frac{1}{n}$}} is called {{c1::the harmonic series.}}

Note 17

ccea4c33507e4d5f9387c996a8bb13ad

Let (a_n) be {{c5::a decreasing sequence}} and {{c4::$a_n \leq 0$.}} Then

$$\text{\{\{c2::} \sum_{n=1}^{\infty} a_n \text{ converges\}\}\{\{c3::} \iff \text{\}\}\{\{c1::} \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.\}\}}$$

«{{c6::Cauchy Condensation Test}}»

Note 18

88287ba71bd545459ba16b4e2ca5cb69

Let (a_n) be a decreasing sequence and $a_n \leq 0$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

What is the key idea in the proof?

▌ Group the element of a partial sum in chunks of size 2^m.

Note 19

7dfc9afff8a045caa6549458d3264c8d

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ {{c2::converges}} {{c3::if and only if}} {{c1::$p > 1$.}}

Note 20

666661971092437289591809 63a362d4

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. What is the key idea in the proof?

> The Cauchy Condensation Test and the convergence of geometric series.

Properties of Infinite Series

Note 1

51836e3c068e46888891ad60f449bd6

Let $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$. Under which condition does

$$\sum_{k=1}^{\infty} ca_k$$

converge?

> Always.

Note 2

548101004aba462b8e81b2c4f7cbd1b9

If $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$, then $\sum_{k=1}^{\infty} ca_k =$ {{c1::cA}}.

Note 3

30607fca749d4ea9814ec7460a102865

Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Under which condition does

$$\sum_{k=1}^{\infty} a_k + b_k$$

converge?

> Always.

Note 4

4f1064d2b18d4e889fa4e80010f532b1

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\sum_{k=1}^{\infty} a_k + b_k = \text{\{\{c1::}A + B.\text{\}\}}$$

Note 5

6795efea2a204bfb90bf19f3ac01f60a

The series $\sum_{k=1}^{\infty} a_k$ {{c5::converges}} {{c4::if and only if,}} given {{c3:: $\epsilon > 0$,}} there exists {{c2::an $N \in \mathbf{N}$}} such that whenever {{c2::$n > m \geq N$ }} it follows that {{c1::

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

}}

Note 6

f83e35fa266b4b71ae674a5ae53196aa

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

«{{c1::Cauchy Criterion}}»

Note 7

255fd1a8d1ca40ddbe4706f396dcaad5

What is the key idea in the proof of the Cauchy Criterion for Series?

> Cauchy Criterion for the sequence of partial sums.

Note 8

2ccccd666d0d4025a48baaa6ac297e88

If the series $\sum_{k=1}^{\infty} a_k$ {{c2::converges,}} then {{c1::$(a_k) \to 0$.}}

Note 9

e553a27c1b0240b4a08a2d2e1291a1c5

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. What is the key idea in the proof?

> Apply the Cauchy Criterion with $n = m + 1$.

Note 10

0314d6d2761e4bd1b24b1b858e9c5086

Assume (a_k) and (b_k) are sequences satisfying {{c3::$0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.}} If $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::converges,}} then $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::converges.}}

Note 11

03fddbcdb39340e0a421d24fe7298f2e

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::diverges,}} then $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::diverges.}}

Note 12

d6553b70220c4348a7c7692a58f91271

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«{{c1::Comparison Test}}»

Note 13

7f40a1b03ff44e75af1465ca5e329e3e

What is the key idea in the proof of the Comparison Test for Series?

> Use the Cauchy Criterion explicitly.

Note 14

1f9364c8930f4fedbfb3501d9a92ee2e

Statements about {{c2::convergence}} of sequences and series are immune to {{c1::changes in some finite number of initial terms.}}

Note 15

89c3e03f687b4c4aa41185f6c668d327

A series is called {{c2::geometric}} if it is of the form {{c1::

$$\sum_{k=0}^{\infty} ar^k.$$

}}

Note 16

4d18a586f7754236bac47a23a54ede43

The series $\sum_{k=0}^{\infty} ar^k$ {{c2::converges}} {{c3::if and only if}} {{c1::$|r| < 1$.}}

Note 17

f7ab1e58f37b4580a558de06c51dc6f7

Given $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \text{\{\{c1::}\frac{a}{1-r}.\text{\}\}}$$

Note 18

c409ec230f6741b796ea4ef3e8813d9c

Given $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. What is the key idea in the proof?

> Rewrite partial sums.

Note 19

28dc84fd3d384adea7a15102e07c644a

If {{c2::the series $\sum_{k=1}^{\infty} |a_k|$ converges,}} then {{c1::$\sum_{k=1}^{\infty} a_k$ converges.
}}

«{{c3::Absolute Convergence Test}}»

Note 20

fb10bc5e919347ffa66da221bf832aa3

What is the key idea in the proof of the Absolute Convergence Test?

> The Cauchy Criterion and the Triangle Inequality.

Note 21

998d23f7cbbb49ed885b7ef2f62bb629

Let (a_k) be {{c4::a decreasing sequence}} and {{c3::$(a_k) \to 0$.}} Then {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

}} {{c1::converges.}}

Note 22

df767d19abbf4031899b4a87577b2625

Let (a_k) be a decreasing sequence and $(a_k) \to 0$. Then

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges.

«{{c1::Alternating Series Test}}»

Note 23

5023b0a2f0ca4300bfa09b61e0ec0a9c

{{c1::An alternating series}} is a series of the form {{c2::

$$\sum_{k=0}^{\infty}(-1)^k a_k,$$

}} where {{c3::all $a_k > 0$.}}

Note 24

cb8249219a644a12b50a90701e47e548

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges absolutely,}} if {{c1:: $\sum_{k=1}^{\infty} |a_k|$ converges.
}}

Note 25

c07bf73c30a04766803b1c0fae6b38d9

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges conditionally,}} if {{c1::it converges and does not converge absolutely.}}

Note 26

f54a6f91b89f42c7b548ace2e106608d

A series $\sum_{k=1}^{\infty} a_k$ is said to be {{c2::positive}} if {{c1::$a_k \geq 0$ for all $k \in \mathbf{N}$.}}

Note 27

c5acade4dde342f8b7ac4acec2278ac6

Any {{c2::positive}} convergent series must {{c1::converge absolutely.
}}

Note 28

e85b9eb09cfa4056b868f983703a571c

May a positive series diverge?

> Only to $+\infty$.

Note 29

b65eba46e51c438e933833ad313a4cf8

A {{c2::positive}} series converges {{c3::if and only if}} {{c1::the sequence of partial sums (s_n) is bounded.}}

Note 30

4ef68f3ca3544ea98fd3c54340c65ce5

Let $\sum_{k=1}^{\infty} a_k$ be a series and {{c3::$f : \mathbf{N} \to \mathbf{N}$ be 1–1 and onto.}} {{c2:: The series $\sum_{k=1}^{\infty} a_{f(k)}$}} is called {{c1::a rearrangement of $\sum_{k=1}^{\infty} a_k$. }}

Note 31

4071d910f5e6410cb2b01dfc73ae48da

If a series {{c2::converges absolutely,}} then {{c3::any rearrangement of this series}} {{c1::converges to the same limit.}}

Note 32

057430cb21934da7ac9bc037ba169eb5

If a series converges absolutely, then any rearrangement of this series converges to the same limit. What is the key idea in the proof?

> Substitute the original series' initial terms for the rearrangement's partial sum.

Note 33

d572332d7e36407ab1531e824f794b4b

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the original series' initial terms are substituted from the rearrangement's partial sum?

> So as to use the definition of convergence and the Cauchy Criterion for absolute convergence.

Note 34

574ee484bcf94971932baee731b90c95

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the rearrangement's terms are taken for the partial sum?

> So as to contain the initial terms of the original sequence.

Note 35

c50d4f3043cb4ca38411c1b1dc20ae26

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::s_n}} to be {{c1::the original series' partial sum.}}

Note 36

2f9195ab94ee4143800fc5300d10d80f

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::t_n}} to be {{c1::the rearrangement' partial sum.}}

Note 37

1bacf92272b04fc98d69ac25f5fcdfe2

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, what do we show about $t_m - s_N$?

> $|t_m - s_N| < \varepsilon$

Note 38

6e8705bf5bd84118a85ac3eb8a1d5e28

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, why is it that $|t_m - s_N| < \varepsilon$?

> Due to the Cauchy Criterion.

Note 39

8ffac6aca55141b29861f55f5d1dd8fb

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how do you show $|t_m - A| < \varepsilon$?

> $|t_m - s_N + s_N - A|$ and the triangle inequality.