Uniform Convergence of a Sequence of Functions

Note 1

1bf1a79b9eba47cf852e1a9c7468c5f7

Let (f_n) be {{c1::a sequence of function on a set A.}} We say {{c3::(f_n) converges pointwise on A to a function f}} if {{c2::for all $x \in A$

$$\Big(f_n(x)\Big) \underset{n\to\infty}{\longrightarrow} f(x).$$

}}

Note 2

f11dc20a5619424cafc97ab1b4d64b5f

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges pointwise on A to f,}} we write

$$\text{\{\{c1::}f_n \to f\text{\}\}} \quad \text{or} \quad \text{\{\{c1::}\lim_{n\to\infty} f_n = f.\text{\}\}}$$

Note 3

6f3f051b9e0741dcbd85037d47c4fd19

Let $f_n(x) = \frac{x^2+nx}{n}$.

$$\lim_{n\to\infty} f_n(x) = \text{\{\{c1::}x.\text{\}\}}$$

Note 4

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Let $f_n(x) = x^n,\ f_n : [0,1] \to \mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = \text{\{\{c1::} \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x = 1. \end{cases} \text{\}\}}$$

Note 5

7218c9c8b0f04d4887dc2345da75c6c6

Let (f_n) be a sequence of function on a set A. We say {{c2::(f_n) converges uniformly on A to a function f}} if {{c1::

$$\forall \epsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall n \ge N$$
$$|f_n - f| < \epsilon.$$

}}

Note 6

c80f9e6c9feb486fb69c66c740b4fa7b

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges uniformly on A to f,}} we write {{c1::

$$f_n \rightrightarrows f.$$

}}

Note 7

77ef924775b2453cb303b726f3081917

What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

> The dependence of N on x.

Note 8

42d2e1017eac4382878c195aa5a4c54d

What is the visual behind the uniform convergence of a sequence of functions?

> Eventually every f_n is completely contained in the ϵ-strip.

Note 9

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Which is stronger, uniform or pointwise convergence?

> Uniform convergence is stronger.

Note 10

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Uniform convergence implies {{c1::pointwise convergence.}}

Note 11

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Let (f_n) be a sequence of function on a set A.

{{c2::$f_n \rightrightarrows f$}} {{c3:: $\iff$ }} {{c1:: $\sup |f_n - f| \xrightarrow[n\to\infty]{} 0.$}}

(in terms of sup)

Note 12

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Let (f_n) be a sequence of function on a set A. {{c3::Then (f_n) converges uniformly on A}} {{c4::if and only if}}

$$\text{\{\{c1::}\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N\text{\}\}}$$
$$\text{\{\{c2::}|f_n - f_m| < \varepsilon.\text{\}\}}$$

Note 13

2b9e4671775a43e9aa4a6b4d581b1658

Let (f_n) be a sequence of function on a set A. Then $f_n \rightrightarrows f$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N$$
$$|f_n - f_m| < \varepsilon.$$

«{{c1::Cauchy Criterion}}»

Note 14

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What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

> Follows immediately from the definition.

Note 15

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What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

> Define a candidate for the limit and prove by definition.

Note 16

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

> Use the pointwise limit.

Note 17

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

> Due to the Cauchy Criterion for sequences.

Note 18

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have $f_n \to f$. How do you show that $f_n \rightrightarrows f$?

> Take the limit of the inequality from the Cauchy Criterion.

Note 19

baab958475694fc08316e2031a57fa58

Let $f_n \to f$ on a set A and $c \in A$. If {{c3::the convergence is uniform }} and {{c2::all f_n are continuous at c,}} then {{c1::f is continuous at c.}}

Note 20

a026cf3ddb2f4d5b9a94b36b2bc20ef9

Let $f_n \to f$ on a set A and $c \in A$. If the convergence is uniform and all f_n are continuous at c, then f is continuous at c.

«{{c1::Continuous Limit Theorem}}»

Note 21

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What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

> Triple triangle inequality after adding and subtracting f_N.

Note 22

06425162bee447479d3a4f5c71c9cf2a

Let $f_n \to f$ on a set A and $c \in A$. If {{c1::the convergence is uniform and all f_n are continuous at c,}} then

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \text{\{\{c2::}} \lim_{n\to\infty}\lim_{x\to c} f_n(x).\text{\}\}}$$

Note 23

05371b5a401f4756bc04fc154476e2c4

Let $f_n \to f$ on a set A. If each f_n is continuous, but f is discontinuous, then {{c1::the convergence is not uniform.}}

Note 24

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Give an example of a sequence of functions $f_n \to f$ such that

- each f_n is continuous almost everywhere; and
- f is nowhere continuous.

> Step-by-step construction of the Dirichlet's function.

Note 25

81c5e1a2081241d1973bb2cacde92627

Assume $f_n \to f$ on a set A and each f_n is uniformly continuous. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is uniformly continuous.}}

Note 26

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Assume $f_n \to f$ on a set A and each f_n is bounded. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is bounded.}}

Note 27

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Assume $f_n \to f$ on a set A and each f_n has a finite number of discontinuities. If $f_n \rightrightarrows f$, then {{c1::f has at most a countable number of discontinuities.}}

Note 28

a010908ba95d473ea734442288757314

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If {{c2::f is discontinuous at c,}} then {{c1::all f_n are eventually discontinuous at c.}}

Note 29

5ca0ebc56cc947d1bc6a5ed00cd1617b

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If f is discontinuous at c, then all f_n are eventually discontinuous at c. What is the key idea in the proof?

▌ By contradiction + choose a subsequence continuous at c.

Note 30

4c8d50b955be4fa0a3ba792c5699174f

Let f be {{c2::continuous}} on all of $\mathbf{R}$. Then $f(x+\frac{1}{n})$ {{c1::converges to f.}}

Note 31

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Let f be {{c2::uniformly continuous}} on all of $\mathbf{R}$. Then $f(x+\frac{1}{n})$ {{c1::converges uniformly to f.}}