The Monotone Convergence Theorem and a First Look at Infinite Series

Note 1

A sequence (a_n) is {{c2::increasing}} if {{c1::$a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.}}

Note 2

A sequence (a_n) is {{c2::decreasing}} if {{c1::$a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.}}

Note 3

A sequence (a_n) is {{c2::monotone}} if {{c1::it is either increasing or decreasing.}}

Note 4

If a sequence is {{c3::monotone}} and {{c2::bounded,}} then {{c1::it converges.}}

Note 5

If a sequence is monotone and bounded, then it converges.

«{{c1::Monotone Convergence Theorem}}»

Note 6

What is the key idea in the proof of the Monotone Convergence Theorem?

▌ The limit equals to $\sup \{a_n \mid n \in \mathbb{N}\}$

Note 7

Let {{c3::(a_n) be a sequence.}} {{c2::An infinite series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

}}

Note 8

Let $\sum_{n=1}^{\infty} a_n$ be a series. We define the corresponding {{c2: sequence of partial sums}} by {{c1::

$$m \mapsto a_1 + a_2 + \cdots + a_m.$$

}}

Note 9

Let $\sum_{n=1}^{\infty} a_n$ be a series. {{c2:: The sequence of partial sums}} is usually denoted {{c1:: (s_m).}}

Note 10

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that {{c2:: $\sum_{n=1}^{\infty} a_n$ converges to A}} if {{c1:: the sequence of partial sums converges to A.}}

Note 11

Let $\sum_{n=1}^{\infty} a_n$ be a series. If {{c2:: $\sum_{n=1}^{\infty} a_n$ converges to A,}} we write

{{c1::

$$\sum_{n=1}^{\infty} a_n = A.$$

}}

Note 12

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

▌ Yes.

Note 13

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. What is the key idea in the proof?

▌ $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$.

Note 14

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

No.

Note 15

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What is the key idea in the proof?

Find a lower bound using powers of two.

Note 16

{{c2::$\sum_{n=1}^{\infty} \frac{1}{n}$}} is called {{c1::the harmonic series.}}

Note 17

{{c2::The harmonic series' partial sums}} are called {{c1::harmonic numbers.}}

Note 18

{{c2::The n-th harmonic number}} is denoted {{c1::H_n.}}

Note 19

What is the harmonic numbers' growth rate?

Logarithmic.

Note 20

How is H_n approximated with $\ln n$?

$\ln n + \text{a constant} + \delta_n, \quad \text{where } (\delta_n) \to 0.$

Note 21

What is the name of the constant term from the approximation of H_n with $\ln n$?

The Euler-Mascheroni constant.

Note 22

What is the value of the Euler-Mascheroni constant?

$$\lim_{n \to \infty} (H_n - \ln n).$$

Note 23

What is value of $\lim(H_n - \ln n)$?

The Euler-Mascheroni constant.

Note 24

The Euler-Mascheroni constant is usually denoted {{c1::γ.}}

Note 25

Let (a_n) be {{c5::a decreasing sequence}} and {{c4::$a_n \geq 0$.}} Then

$$\text{\{\{c2::}} \sum_{n=1}^{\infty} a_n \text{ converges}\text{\}\}\{\{c3::}} \iff \text{\}\}\{\{c1::}} \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}\text{\}\}}$$

«{{c6::Cauchy Condensation Test}}»

Note 26

Let (a_n) be a decreasing sequence and $a_n \leq 0$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

What is the key idea in the proof?

Group the element of a partial sum in chunks of size 2^m.

Note 27

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ {{c2::converges}} {{c3::if and only if}} {{c1::$p > 1$.}}

Note 28

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. What is the key idea in the proof?

> The Cauchy Condensation Test and the convergence of geometric series.

Properties of Infinite Series

Note 1

Let $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$. Under which condition does

$$\sum_{k=1}^{\infty} ca_k$$

converge?

❚ Always.

Note 2

If $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$, then $\sum_{k=1}^{\infty} ca_k = $ {{c1::cA}}.

Note 3

Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Under which condition does

$$\sum_{k=1}^{\infty} a_k + b_k$$

converge?

❚ Always.

Note 4

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\sum_{k=1}^{\infty} a_k + b_k = \text{\{\{c1::}A + B.\text{\}\}}$$

Note 5

If $\sum_{k=1}^{\infty} a_k$ {{c3::converges}} and $\sum_{k=1}^{\infty} b_k$ {{c3::diverges,}} then

$$\text{\{\{c2::}\sum_{k=1}^{\infty} a_k + b_k\text{\}\}}\text{\{\{c1::}\text{ diverges.}\text{\}\}}$$

Note 6

If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then

$$\sum_{k=1}^{\infty} a_k + b_k \text{ diverges.}$$

What is the key idea in the proof?

> By contradiction and $\sum b_k$ converges.

Note 7

The tail of a {{c2::convergent}} series {{c1::tends to 0.}}

Note 8

The series $\sum_{k=1}^{\infty} a_k$ {{c5::converges}} {{c4::if and only if,}} given {{c3:: $\epsilon > 0$,}} there exists {{c2::an $N \in \mathbf{N}$}} such that whenever {{c2:: $n > m \geq N$ }} it follows that {{c1::

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

}}

Note 9

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

«{{c1::Cauchy Criterion}}»

Note 10

What is the key idea in the proof of the Cauchy Criterion for Series?

> Cauchy Criterion for the sequence of partial sums.

Note 11

If the series $\sum_{k=1}^{\infty} a_k$ {{c2::converges,}} then {{c1::$(a_k) \to 0$.}}

Note 12

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. What is the key idea in the proof?

▌ Apply the Cauchy Criterion with $n = m + 1$.

Note 13

Assume (a_k) and (b_k) are sequences satisfying {{c3::$0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.}} If $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::converges,}} then $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::converges.}}

Note 14

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::diverges,}} then $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2:: diverges.}}

Note 15

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«{{c1::Comparison Test}}»

Note 16

What is the key idea in the proof of the Comparison Test for Series?

▌ Use the Cauchy Criterion explicitly.

Note 17

What series are considered in the Limit Comparison Test?

Positive and one containing no zeros.

Note 18

Which value is considered in the Limit Comparison Test?

The limit of the ratio of corresponding terms.

Note 19

Which cases exist on the Limit Comparison Test?

The limit is finite or is nonzero.

Note 20

What can we say from the Limit Comparison Test if the limit is finite?

The denominator's series convergence implies that of the numerator.

Note 21

What can we say from the Limit Comparison Test if the limit is nonzero?

The numerator's series convergence implies that of the denominator.

Note 22

What can we say from the Limit Comparison Test if the limit is finite and nonzero?

The two series's convergences are equivalent.

Note 23

What can we say from the Limit Comparison Test if the limit does not exist?

| Nothing.

Note 24

What is the key idea in the proof of the Limit Comparison Test (finite limit)?

| The set of ratios is bounded above + the Comparison Test.

Note 25

What is the key idea in the proof of the Limit Comparison Test (nonzero limit)?

| Swap the numerator and the denominator.

Note 26

Statements about {{c2::convergence}} of sequences and series are immune to {{c1::changes in some finite number of initial terms.}}

Note 27

A series is called {{c2::geometric}} if it is of the form {{c1::

$$\sum_{k=0}^{\infty} ar^k.$$

}}

Note 28

The series $\sum_{k=0}^{\infty} ar^k$ {{c2::converges}} {{c3::if and only if}} {{c1::$|r| < 1.$}}

Note 29

Given $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = {{c1::\frac{a}{1-r}.}}$$

Note 30

Given $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. What is the key idea in the proof?

❚ Rewrite the partial sums.

Note 31

If {{c2: the series $\sum_{k=1}^{\infty} |a_k|$ converges,}} then {{c1: $\sum_{k=1}^{\infty} a_k$ converges.}}

«{{c3: Absolute Convergence Test}}»

Note 32

What is the key idea in the proof of the Absolute Convergence Test?

❚ The Cauchy Criterion and the Triangle Inequality.

Note 33

Let (a_k) be a sequence. If {{c2: (a_k) is decreasing and approaches A,}} we say {{c1: (a_k) decreases to A.}}

Note 34

Let (a_k) be a sequence. If {{c2: (a_k) decreases to A,}} we write {{c1:

$$(a_k) \searrow A.$$

}}

Note 35

Let (a_k) be a sequence. If {{c2: (a_k) is increasing and approaches A,}} we say {{c1: (a_k) increases to A.}}

Note 36

Let (a_k) be a sequence. If {{c2::(a_k) increases to A,}} we write {{c1::

$$(a_k) \nearrow A.$$

}}

Note 37

Let (a_k) be {{c3::a sequence decreasing to zero.}} Then {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

}} {{c1::converges.}}

Note 38

Let (a_k) be a sequence decreasing to zero. Then

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges.

«{{c1::Alternating Series Test}}»

Note 39

What is the nominal name of the Alternating Series Test?

▌ Leibniz's Test.

Note 40

{{c1::An alternating series}} is a series of the form {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k,$$

}} where {{c3::all $a_k > 0$.}}

Note 41

What is the key idea in the proof of the Alternating Series Test?

❚ The Cauchy criterion for the sequence of partial sums.

Note 42

Which intervals are considered in the proof of the Alternating Series Test?

❚ Those formed by successive partial sums.

Note 43

In the proof of the Alternating Series Test, how to you choose $\Delta_{s_m, s_{m+1}}$, given $\epsilon > 0$?

❚ So that its length is less then ϵ.

Note 44

In the proof of the Alternating Series Test, what do you need to show about an interval $\Delta_{s_m, s_{m+1}}$?

❚ It contains all of the following partial sums.

Note 45

Does the alternating harmonic series converge?

❚ Yes.

Note 46

Why does the alternating harmonic series converge?

❚ Due to the Alternating Series Test.

Note 47

What does the alternating harmonic series converges to?

▍ $\ln 2$.

Note 48

$\sum \frac{(-1)^{n-1}}{n} = \ln 2$. What is the key idea in the proof?

▍ Use the logarithmic approximation of harmonic numbers.

Note 49

$\sum \frac{(-1)^{n-1}}{n} = \ln 2$. In the proof, how do you transform the partial sums as to use the logarithmic approximation of harmonic numbers?

▍ Add and subtract negative terms as to make them positive.

Note 50

We say $\sum_{k=1}^{\infty} a_k$ {{c2: converges absolutely,}} if {{c1: $\sum_{k=1}^{\infty} |a_k|$ converges.
}}

Note 51

We say $\sum_{k=1}^{\infty} a_k$ {{c2: converges conditionally,}} if {{c1: it converges and does not converge absolutely.}}

Note 52

A series $\sum_{k=1}^{\infty} a_k$ is said to be {{c2: positive}} if {{c1: $a_k \geq 0$ for all $k \in \mathbf{N}$.}}

Note 53

Any {{c2: positive}} convergent series must {{c1: converge absolutely.
}}

Note 54

May a positive series diverge?

▍ Only to $+\infty$.

Note 55

A {{c2::positive}} series converges {{c3::if and only if}} {{c1::the sequence of partial sums (s_n) is bounded.}}

Note 56

Let $\sum_{k=1}^{\infty} a_k$ be a series and {{c3::$f : \mathbf{N} \to \mathbf{N}$ be 1–1 and onto.}} {{c2:: The series $\sum_{k=1}^{\infty} a_{f(k)}$}} is called {{c1::a rearrangement of $\sum_{k=1}^{\infty} a_k$. }}

Note 57

If a series {{c2::converges absolutely,}} then {{c3::any rearrangement of this series}} {{c1::converges to the same limit.}}

Note 58

If a series converges absolutely, then any rearrangement of this series converges to the same limit. What is the key idea in the proof?

> Substitute the original series' initial terms for the rearrangement's partial sum.

Note 59

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the original series' initial terms are substituted from the rearrangement's partial sum?

> So as to use the definition of convergence and the Cauchy Criterion for absolute convergence.

Note 60

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the rearrangement's terms are taken for the partial sum?

> So as to contain the initial terms of the original sequence.

Note 61

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::s_n}} to be {{c1::the original series' partial sum.}}

Note 62

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::t_n}} to be {{c1::the rearrangement' partial sum.}}

Note 63

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, what do we show about $t_m - s_N$?

> $|t_m - s_N| < \varepsilon$

Note 64

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, why is it that $|t_m - s_N| < \varepsilon$?

> Due to the Cauchy Criterion.

Note 65

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how do you show $|t_m - A| < \varepsilon$?

> $|t_m - s_N + s_N - A|$ and the triangle inequality.

Note 66

What series are considered in the Ratio Test?

▌ Strictly positive.

Note 67

What value is considered in the Ratio Test?

▌ The limit of successive ratios.

Note 68

Which term is placed to the numerator in the Ratio Test?

▌ The next one.

Note 69

When does the Ratio Test let us conclude something?

▌ When the ratios approach a value other than 1.

Note 70

Which cases exists on the Ratio Test?

▌ Ratios converge to less than, or greater than, 1.

Note 71

What do we conclude from the Ratio Test when the ratios converge to something less than 1?

▌ The series converges.

Note 72

What do we conclude from the Ratio Test when the ratios converge to something greater than 1?

The series diverges.

Note 73

What do we conclude from the Ratio Test when the ratios converge to 1?

Nothing.

Note 74

What do we conclude from the Ratio Test when the ratios do not converge?

Nothing.

Note 75

Give an example when the Ratio Test is inconclusive and the series diverges.

The harmonic series.

Note 76

Give an example when the Ratio Test is inconclusive and the series converges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 77

What is the nominal name of the Ratio Test?

The d'Alambert's Ratio Test.

Note 78

What is the first step in proving the Ratio Test?

| Split $r < 1$, $r > 1$.

Note 79

What is the key idea in the proof of the Ratio Test ($r > 1$)?

| The terms are eventually increasing.

Note 80

What is the first key idea in the proof of the Ratio Test ($r < 1$)?

| For $r < r' < 1$ the ratios are eventually less than r'.

Note 81

What is the second key idea in the proof of the Ratio Test ($r < 1$)?

| Find an upper bound using a geometric series.

Note 82

What series are considered in the Root Test?

| Positive.

Note 83

What value is considered in the Root Test?

| The limit of $\sqrt[n]{a_n}$.

Note 84

Which cases exist on the Root Test?

| n-th roots approach something less than, or greater than, 1.

Note 85

When does the Root Test let us conclude something?

When n-th roots approach something other than 1.

Note 86

What do we conclude from the Root Test if the limit is less than 1?

The series converges.

Note 87

What do we conclude from the Root Test if the limit is greater than 1?

The series diverges.

Note 88

What do we conclude from the Root Test if the limit converges to 1?

Nothing.

Note 89

Give an example when the root test is inconclusive and the series diverges.

The harmonic series.

Note 90

Give an example when the root test is inconclusive and the series converges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 91

What is the nominal name for the Root Test?

| Cauchy's Radical Test.

Note 92

What is the first step in proving the Root Test?

| Split $r < 1, r > 1$.

Note 93

What is the first key idea in the proof of the Root Test ($r < 1$)?

| For $r < r' < 1$ the roots are eventually less than r'.

Note 94

What is the second key idea in the proof of the Root Test ($r < 1$)?

| Find an upper bound using a geometric series.

Note 95

What is the key idea in the proof of the Root Test ($r > 1$)?

| The elements are eventually greater than 1.

Note 96

What sequences are considered in the Summation-by-Parts formula?

| Arbitrary.

Note 97

What is the initial expression in the Summation-by-Parts formula?

$$\sum_{j=n}^{m} x_j y_j.$$

Note 98

Which terms are there in the transformed expression in the Summation-by-Parts formula?

❚ Two "free" terms and a sum.

Note 99

What is the first free term of the transformed expression in the Summation-by-Parts formula?

❚ The final partial sum times the next element.

Note 100

What is the second free term of the transformed expression in the Summation-by-Parts formula?

❚ Subtracting the partial sum preceding the range multiplied by the starting element.

Note 101

What is the sum term of the transformed expression in the Summation-by-Parts formula?

❚ The sum of partial sums multiplied by the successive differences.

Note 102

What is the order of successive differences in the sum term of the transformed expression in the Summation-by-Parts formula?

❚ The current minus the next.

Note 103

What is the range of summation in the sum term of the transformed expression in the Summation-by-Parts formula?

Same as the original.

Note 104

What is the value of the zeroth partial sum in the Summation-by-Parts formula?

Zero.

Note 105

What is the nominal name of the Summation-by-Parts formula?

The Abel Transformation.

Note 106

What is the key idea in the proof of the Summation-by-Parts formula?

Rewrite the sequence's values as the differences of successive partial sums.

Note 107

What series is considered in the Abel's Test?

A series formed by two sequence's products.

Note 108

What sequences are considered in the Abel's Test?

One, whose series converges, and one monotone and bounded.

Note 109

What do we conclude from the Abel's Test?

▎ The series of products converges.

Note 110

When can we conclude something from the Abel's Test?

▎ Whenever the hypothesis is satisfied.

Note 111

What is the first step in proving the Abel's Test?

▎ With no loss of generality, the sequence is decreasing.

Note 112

What is the key idea in the proof of the Abel's Test?

▎ Summation-by-Parts + the definition of convergence.

Note 113

To which sums is Summation-by-Parts applied in the proof of the Abel's Test?

▎ The products' series' partial sums.

Note 114

In the proof of the Abel's Test, how do you show that the partial sums converge?

▎ Both addends converge (after applying Summation-by-Parts).

Note 115

In the proof of the Abel's Test after applying Summation-by-Parts, how do you show that the "free" terms converge?

▌ It follows from the hypothesis.

Note 116

In the proof of the Abel's Test after applying Summation-by-Parts, how do you show that the sums converge?

▌ The Comparison Test for absolute convergence.

Note 117

To which series is the Comparison Test applied in the proof of the Abel's Test?

▌ The one generated after applying Summation-by-Parts.

Note 118

In the proof of the Abel's Test, where from do you get an upper bound when applying the Comparison Test?

▌ The partial sums converge and, thus, are bounded.

Note 119

What series is considered in the Dirichlet's Test?

▌ A series formed by two sequence's product.

Note 120

What sequences are considered in the Dirichlet's Test?

▌ One with bounded partials sums and one decreasing to zero.

Note 121

What do we conclude from the Dirichlet's Test?

▌ The product's series converges.

Note 122
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When can we conclude something from the Dirichlet's Test?

▌ Whenever the hypothesis is satisfied.

Note 123
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What is the key idea in the proof of the Dirichlet's Test?

▌ Summation-by-Parts + the definition of convergence.

Note 124
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To which sums is Summation-by-Parts applied in the proof of the Dirichlet's Test?

▌ The products' series' partial sum.

Note 125
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The Alternating Series Test can be derived as a special case of {{c1:the Dirichlet's Test.}}