The Monotone Convergence Theorem and a First Look at Infinite Series

Note 1

A sequence (a_n) is {{c2::increasing}} if {{c1::$a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.

}}

Note 2

A sequence (a_n) is {{c2::decreasing}} if {{c1::$a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.

}}

Note 3

A sequence (a_n) is {{c2::monotone}} if {{c1::it is either increasing or decreasing.}}

Note 4

If a sequence is {{c3::monotone}} and {{c2::bounded,}} then {{c1::it converges.

}}

Note 5

If a sequence is monotone and bounded, then it converges.

«{{c1::Monotone Convergence Theorem}}»

Note 6

What is the key idea in the proof of the Monotone Convergence Theorem?

▌ The limit equals to $\sup \{a_n \mid n \in \mathbb{N}\}$

Note 7

Let {{c3::(a_n) be a sequence.}} {{c2::An infinite series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

}}

Note 8

Let $\sum_{n=1}^{\infty} a_n$ be a series. We define the corresponding sequence of partial sums by

$$m \mapsto a_1 + a_2 + \cdots + a_m.$$

Note 9

Let $\sum_{n=1}^{\infty} a_n$ be a series. The sequence of partial sums is usually denoted (s_m).

Note 10

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that $\sum_{n=1}^{\infty} a_n$ converges to A if the sequence of partial sums converges to A.

Note 11

Let $\sum_{n=1}^{\infty} a_n$ be a series. If $\sum_{n=1}^{\infty} a_n$ converges to A, we write

$$\sum_{n=1}^{\infty} a_n = A.$$

Note 12

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

❚ Yes.

Note 13

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. What is the key idea in the proof?

❚ $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$.

Note 14

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

▌ No.

Note 15

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What is the key idea in the proof?

▌ Find a lower bound using powers of two.

Note 16

{{c2::$\sum_{n=1}^{\infty} \frac{1}{n}$}} is called {{c1::the harmonic series.}}

Note 17

Let (a_n) be {{c5::a decreasing sequence}} and {{c4::$a_n \geq 0$.}} Then

$$\{\{c2::\sum_{n=1}^{\infty} a_n \text{ converges}\}\}\{\{c3:: \iff \}\}\{\{c1:: \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}\}\}$$

«{{c6::Cauchy Condensation Test}}»

Note 18

Let (a_n) be a decreasing sequence and $a_n \leq 0$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

What is the key idea in the proof?

▌ Group the element of a partial sum in chunks of size 2^m.

Note 19

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ {{c2::converges}} {{c3::if and only if}} {{c1::$p > 1$.}}

Note 20

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. What is the key idea in the proof?

The Cauchy Condensation Test and the convergence of geometric series.

Properties of Infinite Series

Note 1

Let $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$. Under which condition does

$$\sum_{k=1}^{\infty} ca_k$$

converge?

Always.

Note 2

If $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$, then $\sum_{k=1}^{\infty} ca_k = cA$.

Note 3

Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Under which condition does

$$\sum_{k=1}^{\infty} a_k + b_k$$

converge?

Always.

Note 4

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\sum_{k=1}^{\infty} a_k + b_k = A + B.$$

Note 5

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

Note 6

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

«{{c1::Cauchy Criterion}}»

Note 7

What is the key idea in the proof of the Cauchy Criterion for Series?

❚ Cauchy Criterion for the sequence of partial sums.

Note 8

If the series $\sum_{k=1}^{\infty} a_k$ {{c2::converges,}} then {{c1::$(a_k) \to 0$.}}

Note 9

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. What is the key idea in the proof?

❚ Apply the Cauchy Criterion with $n = m + 1$.

Note 10

Assume (a_k) and (b_k) are sequences satisfying {{c3::$0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.}} If $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::converges,}} then $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::converges.}}

Note 11

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::diverges,}} then $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::diverges.}}

Note 12

Assume (a_k) and (b_k) are sequences satisfying $0 \le a_k \le b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«{{c1::Comparison Test}}»

Note 13

What is the key idea in the proof of the Comparison Test for Series?

▌ Use the Cauchy Criterion explicitly.

Note 14

What series are considered in the Limit Comparison Test?

▌ Positive and one containing no zeros.

Note 15

Which value is considered in the Limit Comparison Test?

▌ The limit of the ratio of corresponding terms.

Note 16

Which cases exist on the Limit Comparison Test?

▌ The limit is finite or is nonzero.

Note 17

What can we say from the Limit Comparison Test if the limit is finite?

▌ The denominator's series convergence implies that of the numerator.

Note 18

What can we say from the Limit Comparison Test if the limit is nonzero?

> The numerator's series convergence implies that of the denominator.

Note 19

What can we say from the Limit Comparison Test if the limit is finite and nonzero?

> The two series's convergences are equivalent.

Note 20

What can we say from the Limit Comparison Test if the limit does not exist?

> Nothing.

Note 21

What is the key idea in the proof of the Limit Comparison Test (finite limit)?

> The set of ratios is bounded above + the Comparison Test.

Note 22

What is the key idea in the proof of the Limit Comparison Test (nonzero limit)?

> Swap the numerator and the denominator.

Note 23

Statements about {{c2::convergence}} of sequences and series are immune to {{c1::changes in some finite number of initial terms.}}

Note 24

A series is called geometric if it is of the form

$$\sum_{k=0}^{\infty} ar^k.$$

Note 25

The series $\sum_{k=0}^{\infty} ar^k$ converges if and only if $|r| < 1.$

Note 26

Given $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Note 27

Given $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. What is the key idea in the proof?

▌ Rewrite partial sums.

Note 28

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«Absolute Convergence Test»

Note 29

What is the key idea in the proof of the Absolute Convergence Test?

▌ The Cauchy Criterion and the Triangle Inequality.

Note 30

Let (a_k) be {{c4::a decreasing sequence}} and {{c3::$(a_k) \to 0$.}} Then {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

}} {{c1::converges.}}

Note 31

Let (a_k) be a decreasing sequence and $(a_k) \to 0$. Then

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges.

«{{c1::Alternating Series Test}}»

Note 32

{{c1::An alternating series}} is a series of the form {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k,$$

}} where {{c3::all $a_k > 0$.}}

Note 33

What is the key idea in the proof of the Alternating Series Test?

▌ The Cauchy criterion for the sequence of partial sums.

Note 34

Which intervals are considered in the proof of the Alternating Series Test?

| Those formed by successive partial sums.

Note 35

In the proof of the Alternating Series Test, how to you choose $\Delta_{s_m, s_{m+1}}$, given $\epsilon > 0$?

| So that its length is less then ϵ.

Note 36

In the proof of the Alternating Series Test, what do you need to show about an interval $\Delta_{s_m, s_{m+1}}$?

| It contains all of the following partial sums.

Note 37

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges absolutely,}} if {{c1::$\sum_{k=1}^{\infty} |a_k|$ converges.}}

Note 38

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges conditionally,}} if {{c1::it converges and does not converge absolutely.}}

Note 39

A series $\sum_{k=1}^{\infty} a_k$ is said to be {{c2::positive}} if {{c1::$a_k \geq 0$ for all $k \in \mathbf{N}$.}}

Note 40

Any {{c2::positive}} convergent series must {{c1::converge absolutely.}}

Note 41

May a positive series diverge?

> Only to $+\infty$.

Note 42

A positive series converges if and only if the sequence of partial sums (s_n) is bounded.

Note 43

Let $\sum_{k=1}^{\infty} a_k$ be a series and $f : \mathbf{N} \to \mathbf{N}$ be 1–1 and onto. The series $\sum_{k=1}^{\infty} a_{f(k)}$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Note 44

If a series converges absolutely, then any rearrangement of this series converges to the same limit.

Note 45

If a series converges absolutely, then any rearrangement of this series converges to the same limit. What is the key idea in the proof?

> Substitute the original series' initial terms for the rearrangement's partial sum.

Note 46

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the original series' initial terms are substituted from the rearrangement's partial sum?

> So as to use the definition of convergence and the Cauchy Criterion for absolute convergence.

Note 47

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the rearrangement's terms are taken for the partial sum?

▌ So as to contain the initial terms of the original sequence.

Note 48

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote s_n to be the original series' partial sum.

Note 49

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote t_n to be the rearrangement' partial sum.

Note 50

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, what do we show about $t_m - s_N$?

▌ $|t_m - s_N| < \varepsilon$

Note 51

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, why is it that $|t_m - s_N| < \varepsilon$?

▌ Due to the Cauchy Criterion.

Note 52

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how do you show $|t_m - A| < \varepsilon$?

▌ $|t_m - s_N + s_N - A|$ and the triangle inequality.

Note 53

What series are considered in the Ratio Test?

▌ Strictly positive.

Note 54

Which value is considered in the Ratio Test?

▌ The limit of successive ratios.

Note 55

Which term is placed to the numerator in the Ratio Test?

▌ The next one.

Note 56

When does the Ratio Test let us conclude something?

▌ When the ratios approach a value less than or greater then 1.

Note 57

Which cases exists on the Ratio Test?

▌ Ratios converge to less than, or greater than, 1.

Note 58

What do we conclude from the Ratio Test when the ratios converge to something less than 1?

The series converges.

Note 59

What do we conclude from the Ratio Test when the ratios converge to something greater than 1?

The series diverges.

Note 60

What do we conclude from the Ratio Test when the ratios converge to 1?

Nothing.

Note 61

What do we conclude from the Ratio Test when the ratios do not converge?

Nothing.

Note 62

Give an example when the Ratio Test is inconclusive and the series diverges.

The harmonic series.

Note 63

Give an example when the Ratio Test is inconclusive and the series converges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 64

What is the nominal name of the Ratio Test?

▌ The d'Alambert's Ratio Test.

Note 65

What is the first step in proving the Ratio Test?

▌ Split $r < 1, r > 1$.

Note 66

What is the key idea in the proof of the Ratio Test $(r > 1)$?

▌ The terms are eventually increasing.

Note 67

What is the first key idea in the proof of the Ratio Test $(r < 1)$?

▌ For $r < r' < 1$ ratios are eventually less than r'.

Note 68

What is the second key idea in the proof of the Ratio Test $(r < 1)$?

▌ Find an upper bound with a geometric series.