

# Uniform Convergence of a Sequence of Functions

## Note 1

1bf1a79b9eba47cf852e1a9c7468c5f7

Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges pointwise on  $A$  to a function  $f$  if for all  $x \in A$

$$\left(f_n(x)\right)_{n \rightarrow \infty} \longrightarrow f(x).$$

}

## Note 2

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Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges pointwise on  $A$  to  $f$ , we write

$$(f_n \rightarrow f) \quad \text{or} \quad \lim_{n \rightarrow \infty} f_n = f.$$

## Note 3

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Let  $f_n(x) = \frac{x^2 + nx}{n}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = x.$$

## Note 4

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Let  $f_n(x) = x^n$ ,  $f_n : [0, 1] \rightarrow \mathbb{R}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

## Note 5

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Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges uniformly on  $A$  to a function  $f$  if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \\ |f_n - f| < \epsilon.$$

}

## Note 6

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Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges uniformly on  $A$  to  $f$ , we write

$$f_n \rightrightarrows f.$$

}}

## Note 7

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What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

■ The dependence of  $N$  on  $x$ .

## Note 8

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What is the visual behind the uniform convergence of a sequence of functions?

■ Eventually every  $f_n$  is completely contained in the  $\epsilon$ -strip.

## Note 9

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Which is stronger, uniform or pointwise convergence?

■ Uniform convergence is stronger.

## Note 10

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Uniform convergence implies pointwise convergence.

## Note 11

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Let  $(f_n)$  be a sequence of function on a set  $A$ .

$$(f_n \rightrightarrows f) \iff \sup_{n \rightarrow \infty} |f_n - f| \rightarrow 0.$$

(in terms of sup)

### Note 12

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Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $(f_n)$  converges uniformly on  $A$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

### Note 13

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Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $f_n \Rightarrow f$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

«[Cauchy Criterion]»

### Note 14

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What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

■ Follows immediately from the definition.

### Note 15

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What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

■ Define a candidate for the limit and prove by definition.

### Note 16

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

■ Use the pointwise limit.

### Note 17

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

■ Due to the Cauchy Criterion for sequences.

### Note 18

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have  $f_n \rightarrow f$ . How do you show that  $f_n \rightrightarrows f$ ?

■ Take the limit of the inequality from the Cauchy Criterion.

### Note 19

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{\{c\}::\text{the convergence is uniform}\}$  and  $\{\{c\}::\text{all } f_n \text{ are continuous at } c,\}$  then  $\{\{c\}::f \text{ is continuous at } c,\}$

### Note 20

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If the convergence is uniform and all  $f_n$  are continuous at  $c$ , then  $f$  is continuous at  $c$ .

« $\{\{c\}::\text{Continuous Limit Theorem}\}$ »

### Note 21

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What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

■ Triple triangle inequality after adding and subtracting  $f_N$ .

### Note 22

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{\{c\}::\text{the convergence is uniform}\}$  and all  $f_n$  are continuous at  $c$ , then

$$\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \{\{c\}:: \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x).\}$$

### Note 23

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Let  $f_n \rightarrow f$  on a set  $A$ . If each  $f_n$  is continuous, but  $f$  is discontinuous, then  $\{\{c1:: \text{the convergence is not uniform.}\}$

### Note 24

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Give an example of a sequence of functions  $f_n \rightarrow f$  such that

- each  $f_n$  is continuous almost everywhere; and
- $f$  is nowhere continuous.

■ Step-by-step construction of the Dirichlet's function.

### Note 25

81c5e1a2081241d1973bb2cacde92627

Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  is uniformly continuous. If  $\{\{c2:: f_n \rightrightarrows f,\}\}$  then  $\{\{c1:: f \text{ is uniformly continuous.}\}\}$

### Note 26

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Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  is bounded. If  $\{\{c2:: f_n \rightrightarrows f,\}\}$  then  $\{\{c1:: f \text{ is bounded.}\}\}$

### Note 27

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Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  has a finite number of discontinuities. If  $f_n \rightrightarrows f$ , then  $\{\{c1:: f \text{ has at most a countable number of discontinuities.}\}\}$

### Note 28

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Assume  $f_n \rightrightarrows f$  on a set  $A$  and  $c \in A$ . If  $\{\{c2:: f \text{ is discontinuous at } c,\}\}$  then  $\{\{c1:: \text{all } f_n \text{ are eventually discontinuous at } c.\}\}$

### Note 29

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Assume  $f_n \rightrightarrows f$  on a set  $A$  and  $c \in A$ . If  $f$  is discontinuous at  $c$ , then all  $f_n$  are eventually discontinuous at  $c$ . What is the key idea in the proof?

■ By contradiction + choose a subsequence continuous at  $c$ .

### Note 30

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Let  $f$  be  $\{\{c2: \text{continuous}\}\}$  on all of  $\mathbf{R}$ . Then  $f(x + \frac{1}{n})$   $\{\{c1: \text{converges to } f.\}\}$

### Note 31

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Let  $f$  be  $\{\{c2: \text{uniformly continuous}\}\}$  on all of  $\mathbf{R}$ . Then  $f(x + \frac{1}{n})$   $\{\{c1: \text{converges uniformly to } f.\}\}$

# Uniform Convergence and Differentiation

## Note 1

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What sequence is considered in the Differentiable Limit Theorem?

■ A sequence of differentiable functions that converges point-wise on a closed interval.

## Note 2

19574e41800e43678628e78581f801cc

When applying the Differentiable Limit Theorem, is it necessary for the limit to be differentiable?

■ No, this is one of the implications.

## Note 3

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When do we conclude something from the Differentiable Limit Theorem?

■ When the derivatives converges uniformly.

## Note 4

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What do we conclude from The Differentiable Limit Theorem?

■ The limit  $f$  is differentiable and  $f' = \lim f'_n$ .

## Note 5

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Let  $f_n \rightarrow f$  on  $[a, b]$  and each  $f_n$  is differentiable. What do we know about  $f$  if  $f'_n \rightarrow g$ ?

■ Nothing special.

## Note 6

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Let  $f_n \rightarrow f$  on  $[a, b]$  and each  $f_n$  is differentiable. What do we know about  $f$  if  $f'_n \rightrightarrows g$ ?

■  $f$  is differentiable and  $f' = g$ .

### Note 7

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What is the key idea in the proof of the Differentiable Limit Theorem?

■ Rewrite the limit's derivative by definition.

### Note 8

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In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right|?$$

■ Expand it using the triple triangle inequality involving  $f_N$ .

### Note 9

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In the proof of the Differentiable Limit Theorem, how do you choose  $N$ ?

■ By the Cauchy Criterion for  $f'_n \Rightarrow g$ .

### Note 10

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In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$|f'_N(x) - g(x)|?$$

■ Take the limit of the inequality from the Cauchy Criterion.

### Note 11

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In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f_N(x+h) - f_N(x)}{h} - f'_N(x) \right|?$$



■ Pick  $\delta$  by the definition of differentiability of  $f_N$ .

## Note 12

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In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - \frac{f_N(x+h) - f_N(x)}{h} \right|?$$

■ The Mean Value Theorem for  $f_N - f_m$  and make  $m \rightarrow \infty$ .

## Note 13

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Let  $(f_n)$  be a sequence of differentiable functions on  $[a, b]$  and  $(f'_n)$  converge uniformly. If  $\lim_{n \rightarrow \infty} f_n(x_0)$  exists for some  $x_0$ , then  $(f_n)$  converges uniformly.

## Note 14

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How can we weaken the hypothesis of the Differentiable Limit Theorem?

■  $(f_n)$  converges at a single point.

# Series of Functions

## Note 1

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . A functional series is a formal expression of the form

$$\sum_{n=1}^{\infty} f_n(x).$$

}}

## Note 2

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . We say  $\sum_n f_n(x)$  converges pointwise on  $A$  to a function  $f(x)$  if the sequence of partial sums converges pointwise on  $A$  to  $f$ .

## Note 3

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . If  $\sum_n f_n(x)$  converges pointwise to  $f(x)$ , we write

$$f(x) = \sum_n f_n(x).$$

}}

## Note 4

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . We say  $\sum f_n(x)$  converges uniformly on  $A$  to a function  $f(x)$  if the sequence of partial sums converges uniformly on  $A$  to  $f$ .

## Note 5

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Let  $\sum_n f_n(x)$  be a functional series. A series

$$\sum_{n=k+1}^{\infty} f_n(x) \quad \text{for } k \in \mathbb{N},$$

is called a tail of  $\sum_n f_n(x)$ .

## Note 6

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A series  $\sum_n f_n(x)$  converges pointwise if and only if its tail converges pointwise to 0.

(in terms of the tail)

## Note 7

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A series  $\sum_n f_n(x)$  converges uniformly if and only if its tail converges uniformly to 0.

(in terms of the tail)

## Note 8

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A series  $\sum_n f_n(x)$  converges pointwise only if  $f_n \rightarrow 0$ .

## Note 9

767a398cce7c40b781b0c39db5f9b9ac

A series  $\sum_n f_n(x)$  converges uniformly only if  $f_n \Rightarrow 0$ .

## Note 10

c0a25e35d11c4560a26e2e463a31f725

What series is considered in the Term-by-term Continuity Theorem?

■ A series of continuous functions.

## Note 11

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When do we conclude something from the Term-by-term Continuity Theorem?

■ When the functional series converges uniformly.

## Note 12

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What do we conclude from the Term-by-term Continuity Theorem when the series only converges pointwise?

■ Nothing.

### Note 13

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What do we conclude from the Term-by-term Continuity Theorem?

■ The series' sum is continuous.

### Note 14

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What series is considered in the Term-by-term Differentiability Theorem?

■ A series of differentiable functions that converges pointwise on a closed interval.

### Note 15

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When do we conclude something from the Term-by-term Differentiability Theorem?

■ The derivatives' series converge uniformly.

### Note 16

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What do we conclude from the Term-by-term Differentiability Theorem?

■  $\sum f_n$  is differentiable and  $(\sum f_n)' = \sum f_n'$ .

### Note 17

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What series is considered in the Weierstrass M-Test?

■ A series of bounded functions.

### Note 18

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When do we conclude something from the Weierstrass M-Test?

- When the series of “absolute” bounds converges.

### Note 19

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Which bounds are considered in the Weierstrass M-Test?

- The upper bound for the absolute value of  $f_n(x)$ .

### Note 20

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What do we conclude from the Weierstrass M-Test?

- The functional series converges.

### Note 21

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What is the key idea in the proof of the Weierstrass M-Test?

- A corollary of the Cauchy Criterion.

### Note 22

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What is the second implication of the Weierstrass M-Test?

- The series converges absolutely.

### Note 23

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Why does the Weierstrass M-Test implies absolute convergence?

- Absolute values have the same upper bounds.

# Power Series

## Note 1

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What is a power series (intuitively)?

■ An infinite polynomial.

## Note 2

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A power series is a series of the form

$$\sum_{n=1}^{\infty} a_n x^n.$$

}}

## Note 3

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A power series is a generalisation of a finite polynomial.

## Note 4

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If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| \leq |x_0|$ .

## Note 5

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If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| \leq |x_0|$ . What is the key idea in the proof?

■ Make a geometric series by factoring out  $\left| \frac{x}{x_0} \right|^n$ .

## Note 6

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If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| \leq |x_0|$ . In the proof, how do you turn  $\sum |a_n x_0^n| \left| \frac{x}{x_0} \right|^n$  into a geometric series?

■  $(a_n x^n)$  is bounded + the Comparison Test.

### Note 7

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Which form may be taken by the set of points for which  $\sum a_n x^n$  converges?

■ An interval centered around 0.

### Note 8

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The set of points for which  $\sum a_n x^n$  converges is always an interval centered around 0. What is the key idea in the proof?

■ Use the “Interior Convergence” theorem.

### Note 9

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How is the set of points for which  $\sum a_n x^n$  converges called?

■ The interval of convergence.

### Note 10

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[[c2: The radius of convergence]] of  $\sum a_n x^n$  is [[c1: the half of the length of its interval of convergence.]]

### Note 11

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How does  $\sum a_n x^n$  behave at the endpoints of its interval of convergence?

■ Who knows. . .