Uniform Convergence of a Sequence of Functions

Note 1

1bf1a79b9eba47cf852e1a9c7468c5f7

Let (f_n) be {{c1::a sequence of function on a set A.}} We say {{c3::(f_n) converges pointwise on A to a function f}} if {{c2::for all $x \in A$

$$\Big(f_n(x)\Big) \underset{n\to\infty}{\longrightarrow} f(x).$$

}}

Note 2

f11dc20a5619424cafc97ab1b4d64b5f

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges pointwise on A to f,}} we write

$$\{\{c1::f_n \to f\}\} \quad \text{or} \quad \{\{c1:: \lim_{n\to\infty} f_n = f.\}\}$$

Note 3

6f3f051b9e0741dcbd85037d47c4fd19

Let $f_n(x) = \frac{x^2+nx}{n}$.

$$\lim_{n\to\infty} f_n(x) = \{\{c1::x.\}\}$$

Note 4

3c7731c6b70a4c28972a5ea2e88a1e5f

Let $f_n(x) = x^n,\ f_n : [0,1] \to \mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = \{\{c1:: \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x = 1. \end{cases} \}\}$$

Note 5

7218c9c8b0f04d4887dc2345da75c6c6

Let (f_n) be a sequence of function on a set A. We say {{c2::(f_n) converges uniformly on A to a function f}} if {{c1::

$$\forall \epsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall n \ge N$$
$$|f_n - f| < \epsilon.$$

}}

Note 6

c80f9e6c9feb486fb69c66c740b4fa7b

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges uniformly on A to f,}} we write {{c1::

$$f_n \rightrightarrows f.$$

}}

Note 7

77ef924775b2453cb303b726f3081917

What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

> The dependence of N on x.

Note 8

42d2e1017eac4382878c195aa5a4c54d

What is the visual behind the uniform convergence of a sequence of functions?

> Eventually every f_n is completely contained in the ϵ-strip.

Note 9

0c853e2f4ed04acf9dae0b00c1a751f3

Which is stronger, uniform or pointwise convergence?

> Uniform convergence is stronger.

Note 10

ed7804cf8d4d48d5b0efb426d130fb52

Uniform convergence implies {{c1::pointwise convergence.}}

Note 11

c9b4c187b4d54a78a9500289aa5899d0

Let (f_n) be a sequence of function on a set A.

{{c2::$f_n \rightrightarrows f$}} {{c3:: $\Longleftrightarrow$ }} {{c1:: $\sup |f_n - f| \xrightarrow[n\to\infty]{} 0.$}}

(in terms of sup)

Note 12

1b59f18d7ccb47829cf7b7ea7576318c

Let (f_n) be a sequence of function on a set A. {{c3::Then (f_n) converges uniformly on A}} {{c4::if and only if}}

$$\text{\{\{c1::}\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N\text{\}\}}$$
$$\text{\{\{c2::}|f_n - f_m| < \varepsilon.\text{\}\}}$$

Note 13

2b9e4671775a43e9aa4a6b4d581b1658

Let (f_n) be a sequence of function on a set A. Then $f_n \rightrightarrows f$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N$$
$$|f_n - f_m| < \varepsilon.$$

«{{c1::Cauchy Criterion}}»

Note 14

3fa98b94397f4cc2b2d766dd41934f67

What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

> Follows immediately from the definition.

Note 15

f2d15c9af98f4b82956e48ed7df71fc9

What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

> Define a candidate for the limit and prove by definition.

Note 16

1525b27207e74da186a95d7656e895da

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

> Use the pointwise limit.

Note 17

0177dd65112f46c799884e104b39ef76

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

> Due to the Cauchy Criterion for sequences.

Note 18

a1ccae80d31b4f38a4fc876e1ffe4ae7

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have $f_n \to f$. How do you show that $f_n \rightrightarrows f$?

> Take the limit of the inequality from the Cauchy Criterion.

Note 19

baab958475694fc08316e2031a57fa58

Let $f_n \to f$ on a set A and $c \in A$. If {{c3::the convergence is uniform }} and {{c2::all f_n are continuous at c,}} then {{c1::f is continuous at c.}}

Note 20

a026cf3ddb2f4d5b9a94b36b2bc20ef9

Let $f_n \to f$ on a set A and $c \in A$. If the convergence is uniform and all f_n are continuous at c, then f is continuous at c.

«{{c1::Continuous Limit Theorem}}»

Note 21

5fd08fca82504ff0af82d320da351ff7

What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

> Triple triangle inequality after adding and subtracting f_N.

Note 22

06425162bee447479d3a4f5c71c9cf2a

Let $f_n \to f$ on a set A and $c \in A$. If {{c1::the convergence is uniform and all f_n are continuous at c,}} then

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \text{\{\{c2::}\lim_{n \to \infty} \lim_{x \to c} f_n(x).\text{\}\}}$$

Note 23

05371b5a401f4756bc04fc154476e2c4

Let $f_n \to f$ on a set A. If each f_n is continuous, but f is discontinuous, then {{c1::the convergence is not uniform.}}

Note 24

a5ee2f3836bd4545afde8c2d7ecda40e

Give an example of a sequence of functions $f_n \to f$ such that

- each f_n is continuous almost everywhere; and
- f is nowhere continuous.

> Step-by-step construction of the Dirichlet's function.

Note 25

81c5e1a2081241d1973bb2cacde92627

Assume $f_n \to f$ on a set A and each f_n is uniformly continuous. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is uniformly continuous.}}

Note 26

f819f1c60074468ba1e718298059ade4

Assume $f_n \to f$ on a set A and each f_n is bounded. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is bounded.}}

Note 27

b1fded6e729d40ba99a9d087781866dd

Assume $f_n \to f$ on a set A and each f_n has a finite number of discontinuities. If $f_n \rightrightarrows f$, then {{c1::f has at most a countable number of discontinuities.}}

Note 28

a010908ba95d473ea734442288757314

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If {{c2::f is discontinuous at c,}} then {{c1::all f_n are eventually discontinuous at c.}}

Note 29

5ca0ebc56cc947d1bc6a5ed00cd1617b

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If f is discontinuous at c, then all f_n are eventually discontinuous at c. What is the key idea in the proof?

By contradiction + choose a subsequence continuous at c.

Note 30

4c8d50b955be4fa0a3ba792c5699174f

Let f be {{c2::continuous}} on all of $\mathbf{R}$. Then $f(x+\frac{1}{n})$ {{c1::converges to f.}}

Note 31

59f59d25a40a4e72afdd62a2dd24bd13

Let f be {{c2::uniformly continuous}} on all of $\mathbf{R}$. Then $f(x+\frac{1}{n})$ {{c1::converges uniformly to f.}}

Uniform Convergence and Differentiation

Note 1

37f46dbb09f54423a835e842d402ee19

What sequence is considered in the Differentiable Limit Theorem?

> A sequence of differentiable functions that converges pointwise on a closed interval.

Note 2

19574e41800e43678628e78581f801cc

When applying the Differentiable Limit Theorem, is it necessary for the limit to be differentiable?

> No, this is one of the implications.

Note 3

5ef400e26d2541e589faa672492059bf

When do we conclude something form the Differentiable Limit Theorem?

> When the derivatives converge uniformly.

Note 4

f7da48c586d2457baad72d900c07defd

What do we conclude from The Differentiable Limit Theorem?

> The limit f is differentiable and $f' = \lim f'_n$.

Note 5

61acf9aeed834980a9dbaa77746b89e0

Let $f_n \to f$ on $[a, b]$ and each f_n is differentiable. What do we know about f if $f'_n \to g$?

> Nothing special.

Note 6

63a1ccb4818a4cd281f9b4d9513500a0

Let $f_n \to f$ on $[a, b]$ and each f_n is differentiable. What do we know about f if $f'_n \rightrightarrows g$?

> f is differentiable and $f' = g$.

Note 7

a720c08c553f46a0b0423c46f4c19a2e

What is the key idea in the proof of the Differentiable Limit Theorem?

> Rewrite the limit's derivative by definition.

Note 8

31222913007d4ceda945e1a21642c876

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right| ?$$

> Expand it using the triple triangle inequality involving f_N.

Note 9

d239aa3eedd346a69139a5a8b1d94ce7

In the proof of the Differentiable Limit Theorem, how do you choose N?

> By the Cauchy Criterion for $f_n' \rightrightarrows g$.

Note 10

70bbcff5bceb49c7b0abb25a8ab9be35

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| f_N'(x) - g(x) \right| ?$$

> Take the limit of the inequality from the Cauchy Criterion.

Note 11

ee6b7ee23d0a48a8a32afe978be50a7d

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f_N(x+h) - f_N(x)}{h} - f_N'(x) \right| ?$$

▌ Pick δ by the definition of differentiability of f_N.

Note 12

cdb10b03a9254c5abfe796106c1d3e9b

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left|\frac{f(x+h)-f(x)}{h}-\frac{f_N(x+h)-f_N(x)}{h}\right|?$$

▌ The Mean Value Theorem for $f_N - f_m$ and make $m \to \infty$.

Note 13

b4b2753226ff4d839269bbf795c02301

Let (f_n) be {{c4::a sequence of differentiable functions on $[a, b]$}} and {{c3::(f_n') converge uniformly.}} If {{c2::$\lim f_n(x_0)$ exists for some x_0,}} then {{c1::(f_n) converges uniformly.}}

Note 14

8c542d7e30524e129805ce26973b0925

How can we weaken the hypothesis of the Differentiable Limit Theorem?

▌ (f_n) converges at a single point.

Series of Functions

Note 1

b2a393ededc84241bb594417273dea7e

Let (f_n) be {{c3::a sequence of functions on a set A.}} {{c2::A functional series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} f_n(x).$$

}}

Note 2

6291bcd4e0274102bfe4090eebac24ef

Let (f_n) be a sequence of functions on a set A. We say $\sum_n f_n(x)$ {{c2::converges pointwise on A to a function $f(x)$}} if {{c1::the sequence of partial sums converges pointwise on A to f.}}

Note 3

084d4603478b4dc48c0d1837ff30dfd8

Let (f_n) be a sequence of functions on a set A. If {{c2::$\sum_n f_n(x)$ converges pointwise to $f(x)$,}} we write {{c1::

$$f(x) = \sum_n f_n(x).$$

}}

Note 4

2922cd6ac8ff42fabe5bc630fa320169

Let (f_n) be a sequence of functions on a set A. We say $\sum f_n(x)$ {{c2::converges uniformly on A to a function $f(x)$}} if {{c1::the sequence of partial sums converges uniformly on A to f.}}

Note 5

2b28ab51bc7f45ca934cc405e7de388f

Let $\sum_n f_n(x)$ be a functional series. {{c1::A series

$$\sum_{n=k+1}^{\infty} f_n(x) \quad \text{for } k \in \mathbf{N},$$

}} is called {{c2::a tail of $\sum_n f_n(x)$.}}

Note 6

d633b0c9c968402aba5285afb115d682

A series $\sum_n f_n(x)$ {{c2::converges pointwise}} {{c3::only if}} {{c1::its tail converges pointwise to 0.}}

(in terms of the tail)

Note 7

16325daa37b14ddebc3939e1d2ea063b

A series $\sum_n f_n(x)$ {{c2::converges uniformly}} {{c3::only if}} {{c1::its tail converges uniformly to 0.}}

(in terms of the tail)

Note 8

891381b2ecd44c2cb160d114479f0b20

A series $\sum_n f_n(x)$ {{c2::converges pointwise}} {{c3::only if}} {{c1::$f_n \to 0$.
}}

Note 9

767a398cce7c40b781b0c39db5f9b9ac

A series $\sum_n f_n(x)$ {{c2::converges uniformly}} {{c3::only if}} {{c1::$f_n \rightrightarrows 0$.
}}

Note 10

c0a25e35d11c4560a26e2e463a31f725

What series is considered in the Term-by-term Continuity Theorem?

> A series of continuous functions.

Note 11

55e76f7381cf476bb7c32155d099bf7c

When do we conclude something from the Term-by-term Continuity Theorem?

> When the functional series converges uniformly.

Note 12

84af86f380cf48048b8e6b2c91e25d6c

What do we conclude from the Term-by-term Continuity Theorem when the series only converges pointwise?

Nothing.

Note 13

a2c89255016b4abebcc0733f8178fdef

What do we conclude from the Term-by-term Continuity Theorem?

The series' sum is continuous.

Note 14

9a06615f719646bb8e4bde3a605344f5

What series is considered in the Term-by-term Differentiability Theorem?

A series of differentiable functions that converges pointwise on a closed interval.

Note 15

fa6705a7ca6141eeb7056368500bbdb0

When do we conclude something from the Term-by-term Differentiability Theorem?

The derivatives' series converge uniformly.

Note 16

50a4a0c1c82c4129a14c9af763976811

What do we conclude form the Term-by-term Differentiability Theorem?

$\sum f_n$ is differentiable and $(\sum f_n)' = \sum f_n'$.

Note 17

296676411bf5475eacdde73dc1c2b008

What series is considered in the Weierstrass M-Test?

A series of bounded functions.

Note 18

5c393b177b724cf69790bafcf0ff7b23

When do we conclude something from the Weierstrass M-Test?

▌ When the series of “absolute” bounds converges.

Note 19

f964f11937374d53be121d3893daeef6

Which bounds are considered in the Weierstrass M-Test?

▌ The sequence of the functions’ “absolute” upper bounds.

Note 20

c48d5e20f5d24ca58a0c3bd71ab7b256

What do we conclude from the Weierstrass M-Test?

▌ The functional series converges uniformly.

Note 21

2f9827fda17c4670b0d2bd4728303ae4

What is the key idea in the proof of the Weierstrass M-Test?

▌ It follows from the Cauchy Criterion.

Note 22

f0d47c16fb4f4ab888dbaa2d8d17ef7a

What is the second implication of the Weierstrass M-Test?

▌ The series converges absolutely.

Note 23

5803d1bfa65b4180921e6b1443015177

Why does the Weierstrass M-Test implies absolute convergence?

▌ Absolute values have the same upper bounds.

Power Series

Note 1
575572e782e64317ba8228d5791138da

What is a power series (intuitively)?

> An infinite polynomial.

Note 2
3cd19400150446d68e6df4a87977e765

{{c2::A power series}} is {{c1::a series of the form

$$\sum_{n=1}^{\infty} a_n x^n.$$

}}

Note 3
59c245eadd1f4c7c84641a4a81a6cf9c

A power series is {{c2::a generalisation}} of {{c1::a polynomial.}}

Note 4
034c6da627e9416d94fe7048441924c4

If $\sum a_n x^n$ {{c2::converges at some point $x_0 \in \mathbf{R}$}} then {{c1::it converges absolutely}} for any {{c3::x satisfying $|x| < |x_0|$.}}

Note 5
cf119f74fc394dc3a2d9d0c72dd70be5

What do we know about $\sum a_n x^n$ if it converges at some x_0?

> It converges absolutely withing the open interval.

Note 6
fed41f842cd54bb1b712f694b52659f9

If $\sum a_n x^n$ converges at some point $x_0 \in \mathbf{R}$ then it converges absolutely for any x satisfying $|x| < |x_0|$. What is the key idea in the proof?

> Make a geometric series by factoring out $\left|\frac{x}{x_0}\right|^n$.

Note 7

59b428fc86c24ff8aff670ff3a284435

If $\sum a_n x^n$ converges at some point $x_0 \in \mathbf{R}$ then it converges absolutely for any x satisfying $|x| < |x_0|$. In the proof, how do you turn $\sum |a_n x_0^n| \left|\frac{x}{x_0}\right|^n$ into a geometric series?

▌ $(a_n x^n)$ is bounded + the Comparison Test.

Note 8

573b21be0d10467d913040dfe4d493bb

Which form may be taken by the set of points for which $\sum a_n x^n$ converges?

▌ An interval centered around 0.

Note 9

ecdb3ab6a5bd4e23bcd67794066ab7c9

The set of points for which $\sum a_n x^n$ converges is always an interval centered around 0. What is the key idea in the proof?

▌ Use the "Interior Convergence" theorem.

Note 10

21ae4818657c4e16b4ef4b2585bc3c18

How is the set of points for which $\sum a_n x^n$ converges called?

▌ The interval of convergence.

Note 11

cc247e245b4d47ce8e408ff25ad39c6d

Every power series {{c1::converges absolutely}} withing {{c2::the interior of its interval of convergence.}}

Note 12

0fce527887bb4236b7813a76f877c418

Every power series converges absolutely withing the interior of its interval of convergence. What is the key idea in the proof?

Follows from the "Interior Convergence" theorem.

Note 13

a11ccad617764a25a049ee310707b122

{{c2::The radius of convergence}} of $\sum a_n x^n$ is {{c1::the half length of its interval of convergence.}}

Note 14

e3918912ac244859ab2293dcbac39594

How does $\sum a_n x^n$ behave at the endpoints of its interval of convergence?

Who knows...

Note 15

18c470fb2da44b60a1d569d93b89f643

What are the simplest methods for calculating the radius of convergence of a power series?

Using either the Root Test or the Ratio Test.

Note 16

b5c35bb7db58465a910f8283bf5f6196

How can you use the Root Test to calculate the radius of convergence of a power series?

Take the inverse of the coefficients' roots' limit.

Note 17

1badd0dc0e5c4500aa468131632c62b9

How can you use the Ratio Test to calculate the radius of convergence of a power series?

Take the inverse of the coefficients' ratios' limit.

Note 18

819016ab8f2c4bfc971839823a9fd8e0

Let R be the radius of convergence of $\sum a_n x^n$. Then

$$R = \text{\{\{c1::} \left(\limsup \sqrt[n]{|a_n|}\right)^{-1} . \text{\}\}}$$

«{{c2::Cauchy–Hadamard Theorem}}»

Note 19

197b7547ea3349ce827335925cf42930

In the Cauchy-Hadamard Theorem, what happens when

$$\limsup \sqrt[n]{|a_n|} = 0\,?$$

> The radius is infinite.

Note 20

e998f9fea2924dc8b9884bfb954bfedd

In the Cauchy-Hadamard Theorem, what happens when

$$\limsup \sqrt[n]{|a_n|} = \infty\,?$$

> The radius equals to 0.

Note 21

b32f1ebd412842729b113cf6836014e4

What is the key idea in the proof of the Cauchy-Hadamard Theorem?

> The Root Test.

Note 22

9223bf629d1a492a892f77c69e4d1cad

What does it mean for a power series to be centered at $a \neq 0$?

> It is expressed in terms of $(x - a)$.

Note 23

b41b3e3920ae4372a12438b11d262544

Let $\sum a_n(x-a)^n$ be a power series. Then {{c2::the value a}} is called {{c1::the center of the series.}}

Note 24

36acf2e7094146dd8a30193845ea7928

Any power series centered at $a \neq 0$ may be turned into {{c2::a series centered at 0}} by {{c1::substituting

$$\bar{x} = x - a.$$

}}

Note 25

872483e057f74ba7addfdfbd543a41a8

If $\sum a_n x^n$ {{c2::converges absolutely at a point x_0,}} then {{c1::it converges uniformly}} on {{c3::$[-c, c]$, where $c = |x_0|$.}}

Note 26

8bfee12c2af34918aa416ea9071592ca

What do we know about $\sum a_n x^n$ if it converges absolutely at some x_0?

> It converges uniformly on the closed interval.

Note 27

091856fb98e84114b9b21203616d0e36

If $\sum a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on $[-c, c]$, where $c = |x_0|$. What is the key idea in the proof?

> The Weierstrass M-Test.

Note 28

ffe00dc944e34558ba9caec75dbcf5cc

If $\sum a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on $[-c, c]$, where $c = |x_0|$. What is used as the sequence of upper bounds in the proof?

> The values at x_0.

Note 29

23b23a23d1eb4887a5bb4f0edc237a7d

Let R be the radius of convergence of $\sum a_n x^n$. If {{c2::$\sum a_n x^n$ converges absolutely at $x = R$,}} then {{c1::it converges uniformly on $[-R, R]$.}}

Note 30

95017b4b22bf4a64a943cc3064449625

Let R be the radius of convergence of $\sum a_n x^n$. Then for any $r \in$ {{c3::$[0, R)$}}, the series $\sum a_n x^n$ {{c2::converges uniformly}} on {{c1::$[-r, r]$.}}

Note 31

57e2d62cb3ce40e48ee6290824fbafeb

Let R be the radius of convergence of $\sum a_n x^n$. Then for any $r \in [0, R)$, the series $\sum a_n x^n$ converges uniformly on $[-r, r]$. What is the key idea in the proof?

The series converges absolutely at $x = r$.