

# Sets

## Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking,  $\{\{c2::a \text{ set}\}\}$  is  $\{\{c1::a \text{ collection of objects.}\}\}$

## Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects.  $\{\{c2::\text{Those objects}\}\}$  are referred to as  $\{\{c1::\text{the elements of the set.}\}\}$

## Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set  $A$ , we write  $\{\{c2::x \in A\}\}$  if  $\{\{c1::x \text{ is an element of } A.\}\}$

## Note 4

b25d749749a64c5b90880253d9839da8

Given a set  $A$ , we write  $\{\{c2::x \notin A\}\}$  if  $\{\{c1::x \text{ is not an element of } A.\}\}$

## Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets  $A$  and  $B$ ,  $\{\{c1::\text{the union}\}\}$  is written  $\{\{c2::A \cup B.\}\}$

## Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets  $A$ , and  $B$ ,  $\{\{c2::\text{the union}\}\}$  is  $\{\{c3::\text{defined}\}\}$  by the rule

$$x \in \{\{c2::A \cup B\}\} \text{ provided that } \{\{c1::x \in A \text{ or } x \in B.\}\}$$

## Note 7

8ce7db157931494bbfb6eee706e15efc

Given two sets  $A$  and  $B$ ,  $\{\{c1::\text{the intersection}\}\}$  is written  $\{\{c2::A \cap B.\}\}$

## Note 8

6a277df52de2409a98e48429d69b6d05

Given two sets  $A$  and  $B$ ,  $\{\{c2::\text{the intersection}\}\}$  is  $\{\{c3::\text{defined}\}\}$  by the rule

$$x \in \{\{c2::A \cap B\}\} \text{ provided that } \{\{c1::x \in A \text{ and } x \in B.\}\}$$

## Note 9

684951afc378458aa7bd27e67cdc499b

The set of natural numbers is denoted  $\mathbf{N}$ .

## Note 10

49d36a026d4b4678ab86fb6103571cce

$$\mathbf{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

## Note 11

797c81e5adb543e1a5d4cc67e64c5e09

The set of integers is denoted  $\mathbf{Z}$ .

## Note 12

d3c61bf891744c58b73cef543c6e100d

$$\mathbf{Z} \stackrel{\text{def}}{=} \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

## Note 13

57f085776972449f8bc14daf5cff6603

The set of rational numbers is denoted  $\mathbf{Q}$ .

## Note 14

f7e3370650134607853b41b2b1ecf54b

$$\mathbf{Q} \stackrel{\text{def}}{=} \left\{ \text{all fractions } \frac{p}{q} \text{ where } p, q \in \mathbf{Z} \text{ and } q \neq 0 \right\}.$$

## Note 15

faeac83cb5b740b6964551c85ad3e35b

The set of real numbers is denoted  $\mathbf{R}$ .

## Note 16

6e5da98964d645d09ad6989e85679c74

The empty set is the set that contains no elements.

## Note 17

206db0a0f3d042e49a9ca532e222201f

The empty set is denoted  $\emptyset$ .

## Note 18

2f0448d226db4b71b150acaed349a73b

Two sets  $A$  and  $B$  are said to be disjoint if  $A \cap B = \emptyset$ .

**Note 19**

e5d9d365e86640319ca5460ef8c4f05c

Given two sets  $A$  and  $B$ , we say  $\{\{c2::A \text{ is a subset of } B\}\}$  or  $\{\{c2::B \text{ contains } A\}\}$  if  $\{\{c1::\text{every element of } A \text{ is also an element of } B\}\}$

**Note 20**

c2bd27f1fc0d40e296dceef9c9789556

Given two sets  $A$  and  $B$ , the  $\{\{c3::\text{inclusion}\}\}$  relationship  $\{\{c2::A \subseteq B \text{ or } B \supseteq A\}\}$  is used to indicate that  $\{\{c1::A \text{ is a subset of } B\}\}$

**Note 21**

333e7c6716af48b7b9962ad803f0732f

Given two sets  $A$  and  $B$ ,  $\{\{c2::A = B\}\}$  means that  $\{\{c1::A \subseteq B \text{ and } B \subseteq A\}\}$

**Note 22**

74e93b42d46746dc9ec2b54f8366c435

Let  $A_1, A_2, A_3, \dots$  be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \dots$$

are all equivalent ways to indicate  $\{\{c1::\text{the set whose elements consist of any element that appears in at least on particular } A_n\}\}$

**Note 23**

69e4627a3e7149ef8be05479a2587b41

Let  $A_1, A_2, A_3, \dots$  be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \dots$$

are all equivalent ways to indicate  $\{\{c1::\text{the set whose elements consist of any element that appears in every } A_n\}\}$

**Note 24**

11a987e10fce4ceea69672f366597729

Given  $A \subseteq \mathbf{R}$ ,  $\{\{c2::\text{the complement of } A\}\}$  refers to  $\{\{c1::\text{the set of all elements of } \mathbf{R} \text{ not in } A\}\}$

**Note 25**

8b379552450b4672af82c17476c0ff13

Given  $A \subseteq \mathbf{R}$ ,  $\{\{c2::\text{the complement of } A\}\}$  is written  $\{\{c1::A^c\}\}$

## Note 26

a3459afa53264a7c82d9abd760a0c93e

Given  $A, B \subseteq \mathbf{R}$ ,

$$\{\{c2: (A \cap B)^c\}\} = \{\{c1: A^c \cup B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

## Note 27

c983927aa0304e51949e2f90a2ec2614

Given  $A, B \subseteq \mathbf{R}$ ,

$$\{\{c2: (A \cup B)^c\}\} = \{\{c1: A^c \cap B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

## Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

■ Demonstrate inclusion both ways.

# Functions

## Note 1

18930cfe4e445779bcec8a2fb53f23c

Given two sets  $A$  and  $B$ , a function from  $A$  to  $B$  is a rule or mapping that takes each element  $x \in A$  and associates with it a single element of  $B$ .

## Note 2

dfa898ef047e418fa8dfe9ce9582fd71

If  $f$  is a function from  $A$  to  $B$ , we write  $f : A \rightarrow B$ .

## Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let  $f : A \rightarrow B$ . Given  $x \in A$ , the expression  $f(x)$  is used to represent the element of  $B$  associated with  $x$  by  $f$ .

## Note 4

65568f366ca949888310668475dbe570

Let  $f : A \rightarrow B$ . The set  $A$  is called the domain of  $f$ .

## Note 5

7870a310786142fa938bcc843ca8e1ae

Let  $f : A \rightarrow B$ . The set  $\{f(x) \mid x \in A\}$  is called the range of  $f$ .

## Note 6

716c208c9ae849b89ec722aa17f20882

Given a function  $f$  and a subset  $A$  of its domain, the set

$$\{f(x) : x \in A\}$$

is called the range of  $f$  over the set  $A$ .

## Note 7

24aae21652754fcd1267ac61036a3ea

Given a function  $f$  and a subset  $A$  of its domain, the range of  $f$  over  $A$  is written  $f(A)$ .

### Note 8

6ed2fb1006634dcf81707a3c4d514857

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

■ Yes.

### Note 9

ee665e77ac9a45cf9a15d42549e6f382

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

■ No.

### Note 10

5d2e9d4e1e094e06b37bd87e2c9edff8

Given  $\{a, b \in \mathbf{R} \mid a \leq b\}$ , the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

is called a closed interval.

### Note 11

9f383a22fc724f8fa43af5cb65e0cd5a

Given  $a, b \in \mathbf{R}$  and  $a < b$ , the set

$$\{x \in \mathbf{R} : a < x < b\}$$

is called an open interval.

### Note 12

3143096eb895471bac4b2d5840d18758

Given  $a, b \in \mathbf{R}$  and  $a \leq b$ , the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

is written  $[a, b]$ .

### Note 13

604897f024bd4de78723fe8247290371

Given  $a, b \in \mathbf{R}$  and  $a < b$ , the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

is written  $(a, b)$ .

**Note 14**

a77dc72d26be45c185900ba7ff132b05

Let  $f(x) = x^2$ . Find two sets  $A$  and  $B$  for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

■ Singletons  $\{-1\}$  and  $\{1\}$ .

**Note 15**

6ed2fb1006634dcf81707a3c4d514857

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Then

$$\{f(A \cup B)\} = \{f(A) \cup f(B)\}.$$

**Note 16**

e088ae5ae1f24425a81dac09317978fd

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Then

$$\{f(A \cap B)\} \subseteq \{f(A) \cap f(B)\}.$$

**Note 17**

f951f5a5136248dcb413f59b3271d389

Given  $x \in \mathbf{R}$ , the absolute value of  $x$  is denoted  $|x|$ .

**Note 18**

624dda908fd64a1cadae2b61c1277c59

Given  $x \in \mathbf{R}$ ,

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

**Note 19**

0ab23d0afe1448e397cad330aea55883

Given  $a, b \in \mathbf{R}$ ,  $|ab| = |a| \cdot |b|$ .

**Note 20**

2b51f36fba524365b72001d318791436

Given  $a, b \in \mathbf{R}$ ,  $|a + b| \leq |a| + |b|$ .

«Triangle inequality»

### Note 21

4d6e77677e884f9c8ee877b9a32d48b5

Let  $f : A \rightarrow B$ . The function  $f$  is  $\{\{c2: \text{one-to-one}\}\}$  if  $\{\{c1::$

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

$\}\}$

### Note 22

56b2bf81daaf419ab1207c6693c981e6

Let  $f : A \rightarrow B$ . The function  $f$  is  $\{\{c2: \text{onto}\}\}$  if  $\{\{c1::$

$$\text{the range of } f \text{ equals } B.$$

$\}\}$

### Note 23

ccc8a358284a4b1f99f8e4336a2efdb9

Let  $\{\{c4:: f : D \rightarrow \mathbf{R}_*\}\}$  and  $\{\{c3:: B \subseteq \mathbf{R}_*\}\}$ . The set

$$\{x \in D : f(x) = B\}$$

$\}\}$  is called  $\{\{c1::$  the preimage of  $B$  under the function  $f$ . $\}$

### Note 24

b72f131ae6734bf694fd8f987bb2323d

Let  $f : D \rightarrow \mathbf{R}$  and  $A, B \subseteq \mathbf{R}$ . Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

■ Yes.

### Note 25

5b3116f568a34fe2be32f403d7d081d9

Let  $f : D \rightarrow \mathbf{R}$  and  $A, B \subseteq \mathbf{R}$ . Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

■ Yes.



# Logic and Proofs

## Note 1

a4d52b740f5b494696a5bdc956906cf2

Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem,  $\{\{c1: \text{those conditions}\} \text{ are known } \{\{c2: \text{as the theorem's hypotheses.}\}\}$

## Note 2

93f759e32dbf497cb30754e24c5b09f1

When in  $\{\{c3: \text{a proof by contradiction}\} \{\{c2: \text{the contradiction is with the theorem's hypothesis,}\}\}$  the proof is said to be  $\{\{c1: \text{contrapositive.}\}\}$

## Note 3

1f45350926704df98b0abdf205f4319c

Two real number  $a$  and  $b$  are  $\{\{c4: \text{equal}\} \{\{c3: \text{if and only if}\} \{\{c2: \text{for every real number } \epsilon > 0 \text{ it follows that } \{\{c1: } |a - b| < \epsilon.\}\}\}$

## Note 4

3ef90c9123e64df39ae9cd34271a7dcd

Two real number  $a$  and  $b$  are equal  $\iff$  for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ . What is the key idea in the proof?

■ By contradiction.

## Note 5

aab4bb967d814e87bd85608277093755

Let  $\{\{c3: S \subseteq \mathbf{N}.\}\}$  If  $\{\{c2: S \text{ contains } 1\}\}$  and  $\{\{c2: \text{whenever } S \text{ contains } n, \text{ it also contains } n + 1,\}\}$  then  $\{\{c1: S = \mathbf{N}.\}\}$

## Note 6

3dd92625856f408b9dc93fd36d82588d

Let  $S \subseteq \mathbf{N}$ . If  $S$  contains 1 and whenever  $S$  contains  $n$ , it also contains  $n + 1$ , then  $S = \mathbf{N}$ . This proposition is the fundamental principle behind  $\{\{c1: \text{induction.}\}\}$

### Note 7

40977a19a0d043c985df5676daa9f776

Does an induction argument imply the validity of the infinite case?

■ No, it doesn't.

### Note 8

91b673c484b442ec92dd47ad0ef95f6c

Do De Morgan's rules hold for an infinite collection of sets?

■ Yes, they do.

### Note 9

df9aa3b9e0c74da78d7e2a0a65276fcd

How De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

■ They are essentially the same.

# The Axiom of Completeness

## Note 1

d7df92f228f64fb28a9e353f0fcb3160

First,  $\mathbf{R}$  is  $\{\{c1: \text{an ordered field, which contains } \mathbf{Q} \text{ as a subfield.}\}\}$

## Note 2

6ac3816effb14ba682f20f91ae42bfd

What is the key distinction between  $\mathbf{R}$  and  $\mathbf{Q}$ ?

■ Axiom of Completeness.

## Note 3

7c2ddbcb52224d5cbad5c650d77e8a4f

$\{\{c1: \text{Every nonempty set of real numbers}\} \text{ that is } \{\{c2: \text{bounded above}\} \}$   
 $\}$  has  $\{\{c3: \text{a least upper bound.}\}\}$

« $\{\{c4: \text{Axiom of completeness}\}\}$ »

## Note 4

fddbb10e685c4ad49d1af25d241c03c0

Given a set  $A \subseteq \mathbf{R}$ ,  $\{\{c3: \text{a number } b \in \mathbf{R}\} \}$  such that  $\{\{c2: a \leq b \text{ for all } a \in A\} \}$  is called  $\{\{c1: \text{an upper bound for } A.\}\}$

## Note 5

1edcfd8354464c81ab51da0d4f2f2ca4

A set  $A \subseteq \mathbf{R}$  is  $\{\{c2: \text{bounded above}\} \}$  if  $\{\{c1: \text{there exists an upper bound for } A.\}\}$

## Note 6

c757fa0c676941b0a4abbccb3a67fb2a

Given a set  $A \subseteq \mathbf{R}$ ,  $\{\{c3: \text{a number } b \in \mathbf{R}\} \}$  such that  $\{\{c2: a \geq b \text{ for all } a \in A\} \}$  is called  $\{\{c1: \text{a lower bound for } A.\}\}$

## Note 7

3c9ba92f774e439dbcfb6c364a88f0ae

A set  $A \subseteq \mathbf{R}$  is  $\{\{c2: \text{bounded below}\} \}$  if  $\{\{c1: \text{there exists a lower bound for } A.\}\}$

## Note 8

9d2391299602497abd4fdfac14c71daa

Let  $A \subseteq \mathbf{R}$ . A real number  $s$  is the least upper bound for  $A$  if

- $s$  is an upper bound for  $A$ ;
- if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

## Note 9

5369939ee0f94abcaf65896355258f0d

The least upper bound of a set  $A \subseteq \mathbf{R}$  is also frequently called the supremum of  $A$ .

## Note 10

04884b60726641c6b8d7c2c3479f8b05

The least upper bound of a set  $A \subseteq \mathbf{R}$  is denoted  $\sup A$ .

## Note 11

afca84537fdd409e97254e6d36d736c3

Let  $A \subseteq \mathbf{R}$ . A real number  $s$  is the greatest lower bound for  $A$  if

- $s$  is a lower bound for  $A$ ;
- if  $b$  is any lower bound for  $A$ , then  $s \geq b$ .

## Note 12

41c9913ebc524f85be951737dc3e33e8

The greatest lower bound of a set  $A \subseteq \mathbf{R}$  is also frequently called the infimum of  $A$ .

## Note 13

7230c3d5f7ef4b62bc1fd6c5b94841f0

The greatest lower bound of a set  $A \subseteq \mathbf{R}$  is denoted  $\inf A$ .

## Note 14

51abcb89d7d486c9177cfc51b6e8721

Is it possible for a set  $A \subseteq \mathbf{R}$  to have multiple upper bounds?

■ Yes.

### Note 15

1c9d5ad3f35a47b0b12f27639fe4a409

Is it possible for a set  $A \subseteq \mathbf{R}$  to have multiple least upper bounds?

■ No.

### Note 16

8068979c7a6949fc9af88258008a9801

If  $s_1$  and  $s_2$  are both least upper bounds for a set  $A \subseteq \mathbf{R}$ , then

$\{\{c1::$

$$s_1 = s_2.$$

$\}\}$

### Note 17

466b264de27a44d3bd21221e39347d2e

What is the key idea in the proof of well-definedness of the least upper bound?

■  $s_1 \leq s_2$  and  $s_2 \leq s_1$ .

### Note 18

7100e899d7d44ffb89dbc0bac76ffb3f

Let  $A \subseteq \mathbf{R}$ .  $\{\{c4::$  A real number  $b\}$  is  $\{\{c3::$  a maximum of  $A\}\}$  if  $b$  is  $\{\{c2::$  an element of  $A\}\}$  and  $\{\{c1::$  an upper bound for  $A\}\}$

### Note 19

5795e83831c14208a2d2b3dac0e2b139

Let  $A \subseteq \mathbf{R}$ . A real number  $b$  is  $\{\{c3::$  a minimum of  $A\}\}$  if  $b$  is  $\{\{c2::$  an element of  $A\}\}$  and  $\{\{c1::$  a lower bound for  $A\}\}$

### Note 20

2ea41e2869754b64bdb6c221308f0c58

Let  $A \subseteq \mathbf{R}$  and  $\{\{c3:: c \in \mathbf{R}\}\}$ . Then  $\{\{c2:: c + A\}\} \stackrel{\text{def}}{=} \{\{c1:: \{c + a : a \in A\}\}\}$ .

## Note 21

f7518efec7b457d86040b99720ad110

Let  $A \subseteq \mathbf{R}$  be nonempty and bounded above, and let  $c \in \mathbf{R}$ . Then

$$\sup(c + A) = c + \sup A.$$

## Note 22

726f73a8cead495fa65f331e49a892ea

Let  $s \in \mathbf{R}$  be an upper bound for a set  $A \subseteq \mathbf{R}$ . Then  $s = \sup A$  if and only if, for every  $\epsilon > 0$ , there exists an element  $a$  in  $A$  satisfying  $s - \epsilon < a$ .

## Note 23

4161e1c933ba4349978c94d951259701

Let  $s \in \mathbf{R}$  be a lower bound for a set  $A \subseteq \mathbf{R}$ . Then  $s = \inf A$  if and only if, for every  $\epsilon > 0$ , there exists an element  $a$  in  $A$  satisfying  $s + \epsilon > a$ .

## Note 24

0f8f37e55f8e4046a19926f2955f843f

Let  $A \subseteq \mathbf{R}$ . How do  $\inf A$  and  $\sup A$  relate?

|

$$\inf A \leq \sup A.$$

## Note 25

882685715e2143a0b51a1e43390e1dbc

Every nonempty set of real numbers that is bounded below has a greatest lower bound.

## Note 26

87f1451906164b06b7ffe3cd51a2ec7f

Every nonempty set of real numbers that is bounded below has a greatest lower bound. What is the key idea in the proof?

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Infimum is the supremum for the set of lower bounds.