Sets

Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking, {{c2::a set}} is {{c1::a collection of objects.}}

Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects. {{c2::Those objects}} are referred to as {{c1::the elements of the set.}}

Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set A, we write {{c2::$x \in A$}} if {{c1::x is an element of A.}}

Note 4

b25d749749a64c5b90880253d9839da8

Given a set A, we write {{c2::$x \notin A$}} if {{c1::x is not an element of A.}}

Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets A and B, {{c1::the union}} is written {{c2::$A \cup B$.}}

Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets A, and B, {{c2::the union}} is {{c3::defined}} by the rule

$x \in$ {{c2::$A \cup B$}} provided that {{c1::$x \in A$ or $x \in B$.}}

Note 7

8ce7db157931494bbfb6eee706e15efc

Given two sets A and B, {{c1::the intersection}} is written {{c2::$A \cap B$.}}

Note 8

6a277df52de2409a98e48429d69b6d05

Given two sets A and B, {{c2::the intersection}} is {{c3::defined}} by the rule

$x \in$ {{c2::$A \cap B$}} provided that {{c1::$x \in A$ and $x \in B$.}}

Note 9

684951afc378458aa7bd27e67cdc499b

{{c2::The set of natural numbers}} is denoted {{c1::$\mathbf{N}$.}}

Note 10

49d36a026d4b4678ab86fb6103571cce

$$\text{\{\{c2::}\mathbf{N}\text{\}\}} \overset{\text{def}}{=} \{\text{\{\{c1::}1, 2, 3, \dots\text{\}\}}\}.$$

Note 11

797c81e5adb543e1a5d4cc67e64c5e09

{{c2::The set of integers}} is denoted {{c1::$\mathbf{Z}$.}}

Note 12

d3c61bf891744c58b73cef543c6e100d

$$\text{\{\{c2::}\mathbf{Z}\text{\}\}} \overset{\text{def}}{=} \{\text{\{\{c1::} \dots, -2, -1, 0, 1, 2, \dots\text{\}\}}\}.$$

Note 13

57f085776972449f8bc14daf5cff6603

{{c2::The set of rational numbers}} is denoted {{c1::$\mathbf{Q}$.}}

Note 14

f7e3370650134607853b41b2b1ecf54b

$$\text{\{\{c3::}\mathbf{Q}\text{\}\}} \overset{\text{def}}{=} \left\{\text{all \{\{c2::fractions } \frac{p}{q}\text{\}\} where \{\{c1::}p, q \in \mathbf{Z} \text{ and } q \neq 0\text{\}\}}\right\}.$$

Note 15

faeac83cb5b740b6964551c85ad3e35b

{{c2::The set of real numbers}} is denoted {{c1::$\mathbf{R}$.}}

Note 16

6e5da98964d645d09ad6989e85679c74

{{c2::The empty}} set is {{c1::the set that contains no elements.}}

Note 17

206db0a0f3d042e49a9ca532e222201f

{{c2::The empty set}} is denoted {{c1::$\emptyset$.}}

Note 18

2f0448d226db4b71b150acaed349a73b

Two sets A and B are said to be {{c2::disjoint}} if {{c1::$A \cap B = \emptyset$.}}

Note 19

e5d9d365e86640319ca5460ef8c4f05c

Given two sets A and B, we say {{c2::A is a subset of B,}} or {{c2::B contains A}} if {{c1::every element of A is also an element of B.}}

Note 20

c2bd27f1fc0d40e296dceef9c9789556

Given two sets A and B, the {{c3::inclusion}} relationship {{c2::$A \subseteq B$ or $B \supseteq A$}} is used to indicate that {{c1::A is a subset of B.}}

Note 21

333e7c6716af48b7b9962ad803f0732f

Given two sets A and B, {{c2::$A = B$}} means that {{c1::$A \subseteq B$ and $B \subseteq A$.}}

Note 22

74e93b42d46746dc9ec2b54f8366c435

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in at least on particular A_n.
}}

Note 23

69e4627a3e7149ef8be05479a2587b41

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in every A_n.}}

Note 24

11a987e10fce4ceea69672f366597729

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} refers to {{c1::the set of all elements of $\mathbf{R}$ not in A.}}

Note 25

8b379552450b4672af82c17476c0ff13

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} is written {{c1::A^c.}}

Note 26

a3459afa53264a7c82d9abd760a0c93e

Given $A, B \subseteq \mathbf{R}$,

$$\text{\{\{c2::}(A \cap B)^c\text{\}\}} = \text{\{\{c1::}A^c \cup B^c.\text{\}\}}$$

«{{c3::De Morgan's Law}}»

Note 27

c983927aa0304e51949e2f90a2ec2614

Given $A, B \subseteq \mathbf{R}$,

$$\text{\{\{c2::}(A \cup B)^c\text{\}\}} = \text{\{\{c1::}A^c \cap B^c.\text{\}\}}$$

«{{c3::De Morgan's Law}}»

Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

> Demonstrate inclusion both ways.

Functions

Note 1

18930cfe4e4445779bcec8a2fb53f23c

Given {{c3::two sets A and B,}} {{c2::a function from A to B}} is {{c1::a rule or mapping that takes each element $x \in A$ and associates with it a single element of B.}}

Note 2

dfa898ef047e418fa8dfe9ee9582fd71

{{c1::If f is a function from A to B,}} we write {{c2::$f : A \to B$.}}

Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let $f : A \to B$. Given {{c3::$x \in A$,}} the expression {{c2::$f(x)$}} is used to represent {{c1::the element of B associated with x by f.}}

Note 4

65568f366ca949888310668475dbe570

Let $f : A \to B$. {{c2::The set A}} is called {{c1::the domain of f.}}

Note 5

7870a310786142fa938bcc843ca8e1ae

Let $f : A \to B$. {{c2::The set $\{f(x) \mid x \in A\}$}} is called {{c1::the range of f.}}

Note 6

716c208c9ae849b89ec722aa17f20882

Given a function f and {{c3::a subset A of its domain,}} {{c2::the set

$$\{f(x) : x \in A\}$$

}} is called {{c1::the range of f over the set A.}}

Note 7

24aae21652754fcda1267ac61036a3ea

Given a function f and a subset A of its domain, {{c2::the range of f over A}} is written {{c1::$f(A)$.}}

Note 8

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R},\ \ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

▌ Yes.

Note 9

ee665e77ac9a45cf9a15d42549e6f382

Let $f : D \to \mathbf{R},\ \ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

▌ No.

Note 10

5d2e9d4e1e094e06b37bd87e2c9edff8

Given {{c4::$a, b \in \mathbf{R}$}} and {{c3::$a \leq b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is called {{c1::a closed interval.}}

Note 11

9f383a22fc724f8fa43af5cb65e0cd5a

Given $a, b \in \mathbf{R}$ and {{c3::$a < b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is called {{c1::an open interval.}}

Note 12

3143096eb895471bac4b2d5840d18758

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is written {{c2::$[a, b]$.}}

Note 13

604897f024bd4de78723fe8247290371

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is written {{c2::(a, b).}}

Note 14

a77dc72d26be45c185900ba7ff132b05

Let $f(x) = x^2$. Find two sets A and B for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

Singletons $\{-1\}$ and $\{1\}$.

Note 15

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Then

{{c3::$f(A \cup B)$}}{{c1:: $=$ }}{{c2::$f(A) \cup f(B)$.}}

Note 16

e088ae5ae1f24425a81dac09317978fd

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Then

{{c3::$f(A \cap B)$}}{{c1:: $\subseteq$ }}{{c2::$f(A) \cap f(B)$.}}

Note 17

f951f5a5136248dcb413f59b3271d389

Given $x \in \mathbf{R}$, {{c2::the absolute value of x}} is denoted {{c1::$|x|$.}}

Note 18

624dda908fd64a1cadae2b61c1277c59

Given $x \in \mathbf{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} \text{\{\{c1::}x,\text{\}\}} & \text{if } \text{\{\{c2::}x \geq 0\text{\}\}}, \\ \text{\{\{c1::}-x,\text{\}\}} & \text{if } \text{\{\{c2::}x < 0\text{\}\}}. \end{cases}$$

Note 19

0ab23d0afe1448e397cad330aea55883

Given $a, b \in \mathbf{R}$, $|ab| =$ {{c1:: $|a| \cdot |b|$ }}.

Note 20

2b51f36fba524365b72001d318791436

Given $a, b \in \mathbf{R}$, {{c2:: $|a+b|$ }}{{c3:: $\leq$ }}{{c1:: $|a| + |b|$ }}.

«{{c4::Triangle inequality}}»

Note 21

4d6e77677e884f9c8ee877b9a32d48b5

Let $f : A \to B$. The function f is {{c2::one-to-one}} if {{c1::

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

}}

Note 22

56b2bf81daaf419ab1207c6693c981e6

Let $f : A \to B$. The function f is {{c2::onto}} if {{c1::

the range of f equals B.

}}

Note 23

ccc8a358284a4b1f99f8e4336a2efdb9

Let {{c4::$f : D \to \mathbf{R}$}} and {{c3::$B \subseteq \mathbf{R}$.}} {{c2::The set

$$\{x \in D : f(x) = B\}$$

}} is called {{c1::the preimage of B under the function f.}}

Note 24

b72f131ae6734bf694fd8f987bb2323d

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

▌ Yes.

Note 25

5b3116f568a34fe2be32f403d7d081d9

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

▌ Yes.

Logic and Proofs

Note 1

a4d52b740f5b494696a5bdc956906cf2

Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem, {{c1::those conditions}} are known {{c2::as the theorem's hypotheses.}}

Note 2

93f759e32dbf497cb30754e24c5b09f1

When in {{c3::a proof by contradiction}} {{c2::the contradiction is with the theorem's hypothesis,}} the proof is said to be {{c1::contrapositive.}}

Note 3

1f45350926704df98b0abdf205f4319c

Two real number a and b are {{c4::equal}} {{c3::if and only if}} {{c2::for every real number $\epsilon > 0$ it follows that}} {{c1::$|a - b| < \epsilon$.}}

Note 4

3ef90c9123e64df39ae9cd34271a7dcd

Two real number a and b are equal $\Longleftarrow$ for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$. What is the key idea in the proof?

> By contradiction.

Note 5

aab4bb967d814e87bd85608277093755

Let {{c3::$S \subseteq \mathbf{N}$.}} If {{c2::S contains 1}} and {{c2::whenever S contains n, it also contains $n + 1$,}} then {{c1::$S = \mathbf{N}$.}}

Note 6

3dd92625856f408b9dc93fd36d82588d

Let $S \subseteq \mathbf{N}$. If S contains 1 and whenever S contains n, it also contains $n + 1$, then $S = \mathbf{N}$. This proposition is the fundamental principle behind {{c1::induction.}}

Note 7

40977a19a0d043c985df5676daa9f776

Does an induction argument imply the validity of the infinite case?

> No, it doesn't.

Note 8

91b673c484b442ec92dd47ad0ef95f6c

Do De Morgan's rules hold for an infinite collection of sets?

> Yes, they do.

Note 9

df9aa3b9e0c74da78d7e2a0a65276fcd

How De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

> They are essentially the same.

The Axiom of Completeness

Note 1

d7df92f228f64fb28a9e353f0fcb3160

First, $\mathbf{R}$ is {{c1::an ordered field, which contains $\mathbf{Q}$ as a subfield.}}

Note 2

6ac3816effb14ba682f20f91ae42bfdf

What is the key distinction between $\mathbf{R}$ and $\mathbf{Q}$?

> The Axiom of Completeness.

Note 3

7c2ddbcb52224d5cbad5c650d77e8a4f

{{c1::Every nonempty set of real numbers}} that is {{c2::bounded above}} has {{c3::a least upper bound.}}

«{{c4::Axiom of completeness}}»

Note 4

fddbb10e685c4ad49d1af25d241c03c0

Given a set $A \subseteq \mathbf{R}$, {{c3::a number $b \in \mathbf{R}$}} such that {{c2::$a \leq b$ for all $a \in A$}} is called {{c1::an upper bound for A.}}

Note 5

1edcfd8354464c81ab51da0d4f2f2ca4

A set $A \subseteq \mathbf{R}$ is {{c2::bounded above}} if {{c1::there exists an upper bound for A.}}

Note 6

c757fa0c676941b0a4abbccb3a67fb2a

Given a set $A \subseteq \mathbf{R}$, {{c3::a number $b \in \mathbf{R}$}} such that {{c2::$a \geq b$ for all $a \in A$}} is called {{c1::a lower bound for A.}}

Note 7

3c9ba92f774e439dbcfb6c364a88f0ae

A set $A \subseteq \mathbf{R}$ is {{c2::bounded below}} if {{c1::there exists a lower bound for A.}}

Note 8

40f7ae4897174d37952c83f51894ab53

A set $A \subseteq \mathbf{R}$ is {{c2::bounded}} if {{c1::it is bounded above and below.}}

Note 9

9d2391299602497abd4fdfac14c71daa

Let $A \subseteq \mathbf{R}$. {{c4::A real number s}} is {{c3::the least upper bound for A}} if

- {{c2::s is an upper bound for A;}}
- {{c1::if b is any upper bound for A, then $s \leq b$.}}

Note 10

5369939ee0f94abcaf65896355258f0d

{{c2::The least upper bound}} of a set $A \subseteq \mathbf{R}$ is also frequently called {{c1::the supremum of A.}}

Note 11

04884b60726641c6b8d7c2c3479f8b05

{{c2::The least upper bound}} of a set $A \subseteq \mathbf{R}$ is denoted {{c1::sup A.}}

Note 12

afca84537fdd409e97254e6d36d736c3

Let $A \subseteq \mathbf{R}$. A real number s is {{c3::the greatest lower bound for A}} if

- {{c2::s is a lower bound for A;}}
- {{c1::if b is any lower bound for A, then $s \geq b$.}}

Note 13

41c9913ebc524f85be951737dc3e33e8

{{c2::The greatest lower bound}} of a set $A \subseteq \mathbf{R}$ is also frequently called {{c1::the infimum of A.}}

Note 14

7230c3d5f7ef4b62bc1fd6c5b94841f0

{{c2::The greatest lower bound}} of a set $A \subseteq \mathbf{R}$ is denoted {{c1::inf A.
}}

Note 15

51abcbb89d7d486c9177cfc51b6e8721

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple upper bounds?

▌ Yes.

Note 16

1c9d5ad3f35a47b0b12f27639fe4a409

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple least upper bounds?

▌ No.

Note 17

8068979c7a6949fc9af88258008a9801

If s_1 and s_2 are both least upper bounds for a set $A \subseteq \mathbf{R}$, then

{{c1::

$$s_1 = s_2.$$

}}

Note 18

466b264de27a44d3bd21221e39347d2e

What is the key idea in the proof of uniqueness of the least upper bound?

▌ $s_1 \leqslant s_2$ and $s_2 \leqslant s_1$.

Note 19

7100e899d7d44ffb89dbc0bac76ffb3f

Let $A \subseteq \mathbf{R}$. {{c4::A real number b}} is {{c3::a maximum of A}} if b is {{c2::an element of A}} and {{c1::an upper bound for A.}}

Note 20

5795e83831c14208a2d2b3dac0e2b139

Let $A \subseteq \mathbf{R}$. A real number b is {{c3::a minimum of A}} if b is {{c2::an element of A}} and {{c1::a lower bound for A.}}

Note 21

2ea41e2869754b64bdb6c221308f0c58

Let $A \subseteq \mathbf{R}$ and {{c3::$c \in \mathbf{R}$.}} Then {{c2::$c + A$}} $\stackrel{\text{def}}{=}$ {{c1:: $\{c + a : a \in A\}$ }}.

Note 22

f7518efeac7b457d86040b99720ad110

Let {{c2:: $A \subseteq \mathbf{R}$ be nonempty and bounded above,}} and let {{c4:: $c \in \mathbf{R}$.}} Then

$$\{\{c3:: \sup(c + A)\}\} = \{\{c1:: c + \sup A.\}\}$$

Note 23

726f73a8cead495fa65f331e49a892ea

Let $s \in \mathbf{R}$ be {{c5::an upper bound}} for a set $A \subseteq \mathbf{R}$. Then {{c3:: $s = \sup A$}} {{c4::if and only if,}} {{c2::for every $\epsilon > 0$,}} {{c1::there exists an element a in A satisfying $s - \epsilon < a$.}}

Note 24

4161e1c933ba4349978c94d951259701

Let $s \in \mathbf{R}$ be {{c5::a lower bound}} for a set $A \subseteq \mathbf{R}$. Then {{c3:: $s = \inf A$}} {{c4::if and only if,}} {{c2::for every $\epsilon > 0$,}} {{c1::there exists an element a in A satisfying $s + \epsilon > a$.}}

Note 25

0f8f37e55fbe4046a19926f2955f843f

Let $A \subseteq \mathbf{R}$ be nonempty and bounded. How do $\inf A$ and $\sup A$ relate?

> $$\inf A \leq \sup A.$$

Note 26

882685715e2143a0b51a1e43390e1dbc

{{c1::Every nonempty set of real numbers}} that is {{c2::bounded below }} has {{c3::a greatest lower bound.}}

Note 27

87f1451906164b06b7ffe3cd51a2ec7f

Every nonempty set of real numbers that is bounded below has a greatest lower bound. What is the key idea in the proof?

> Infimum is the supremum for the set of lower bounds.

Note 28

74b4cfb8b91d47b7afc1ae11a4b94ccb

Let $A_1, \ldots, A_n \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\{\{c2:: \sup\left(\bigcup_{k=1}^{n} A_k\right) \}\} = \{\{c1:: \max_k \sup A_k .\}\}$$

Note 29

c4f28c7f86554b8d83da1931799f4181

Let $A_1, A_2, \ldots$ be a collection of nonempty sets, each of which is bounded above. If {{c3::$\bigcup_{k=1}^{\infty} A_k$ is bounded above,}} then

$$\{\{c2:: \sup\left(\bigcup_{k=1}^{\infty} A_k\right) \}\} = \{\{c1:: \sup_k \sup A_k .\}\}$$

Note 30

4c14ddc5fe394879915897bbb199442d

Let $A \subseteq \mathbf{R}$ and $c \in \mathbf{R}$. Then {{c2::cA}} $\stackrel{\text{def}}{=}$ {{c1:: $\{c \cdot a : a \in A\}$ }}.

Note 31

8bdedbcb920f442787c9d475958a65dd

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If {{c2::$c \geq 0$,}} it follows that

$$\sup(cA) = \{\{c1:: c \cdot \sup A.\}\}$$

Note 32

c96971d0b0eb40c39d1773c4f89a5588

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If {{c2::$c < 0$,}} it follows that

$$\sup(cA) = \{\{c1:: c \cdot \inf A.\}\}$$

Note 33

fded05f0fad74578a073f5a838a3a081

Let $A, B \subseteq \mathbf{R}$. Then {{c2::$A + B$}} $\stackrel{\text{def}}{=}$ {{c1:: $\{a + b : a \in A \text{ and } b \in B\}$ }}.

Note 34

12d0a51ec08c4d2094ce3e4c6c8b506a

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\{\{c2:: \sup(A+B)\}\} = \{\{c1:: \sup A + \sup B.\}\}$$

Note 35

75698bb156aa40799fc85b1e2419efa2

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\sup(A+B) = \underbrace{\sup A}_{s} + \underbrace{\sup B}_{t}.$$

What is the key idea in the proof?

> For $\epsilon > 0$, choose $a > s - \frac{\epsilon}{2}$ and $b > t - \frac{\epsilon}{2}$.

Note 36

a6281cfff0a84b578d8cacdc6ea4779d

If {{c3::a is an upper bound for A}} and {{c2::$a \in A$,}} then {{c1::

$$a = \sup A.$$

}}

Note 37

eb0969a772e442dd8c3f57ed4f8ee1be

Let $A, B \subseteq \mathbf{R}$ and {{c3::$\sup A < \sup B$.}} Then there exists {{c2::$b \in B$ }} that is {{c1::an upper bound for A.}}

Note 38

6b667686c9644d8b9849c735110dac20

If A and B are {{c3::nonempty, disjoint sets with $A \cup B = \mathbf{R}$}} and {{c2::$a < b$ for all $a \in A$ and $b \in B$,}} then {{c1::there exists $c \in \mathbf{R}$ that is an upper bound for A and a lower bound for B.}}

«{{c4::Cut Property}}»

Note 39

545cb11592164c31badc3f21a1e29981

What is the key idea in the proof of the Cut Property?

> Use the Axiom of Completeness.

Note 40

39aa54de461b426fbe225601c0663097

The Cut Property implies {{c1::the Axiom of Completeness.}}

Note 41

3a64720500f14d66a66401dd3f133a10

The Cut Property implies the Axiom of Completeness. What is the key idea in the proof?

> Consider the set of the upper bounds and its complement.