The Monotone Convergence Theorem and a First Look at Infinite Series

Note 1

7f744b7eecb54041a6e188d2283abcff

A sequence (a_n) is {{c2::increasing}} if {{c1::$a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.
}}

Note 2

cb73357863a14f808fcb79e9f2888e9d

A sequence (a_n) is {{c2::decreasing}} if {{c1::$a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.
}}

Note 3

428c29af1f87467cba4605f856da5dc0

A sequence (a_n) is {{c2::monotone}} if {{c1::it is either increasing or decreasing.}}

Note 4

f0effd26705b4fe2850675b4a8b69fa7

If a sequence is {{c3::monotone}} and {{c2::bounded,}} then {{c1::it converges.
}}

Note 5

f04966660a1d453499de164d33c3efd9

If a sequence is monotone and bounded, then it converges.

«{{c1::Monotone Convergence Theorem}}»

Note 6

fe52926982cd479399d0e77cf6fbb8ae

What is the key idea in the proof of the Monotone Convergence Theorem?

> The limit equals to $\sup \{a_n \mid n \in \mathbb{N}\}$

Note 7

b7b0d33916a74554bee0bb1e829b7a20

Let {{c3::(a_n) be a sequence.}} {{c2::An infinite series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

}}

Note 8

024782c9319a441f91dfd2c8e8aac542

Let $\sum_{n=1}^{\infty} a_n$ be a series. We define the corresponding {{c2::sequence of partial sums}} by {{c1::

$$m \mapsto a_1 + a_2 + \cdots + a_m.$$

}}

Note 9

56563c7563df42c0a111a49ad4ae24ae

Let $\sum_{n=1}^{\infty} a_n$ be a series. {{c2::The sequence of partial sums}} is usually denoted {{c1::(s_m).}}

Note 10

dc59f9b31fff4dcb9113d42da885c946

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A}} if {{c1::the sequence of partial sums converges to A.}}

Note 11

356961ddcb85482c8155d43bd6d8061c

Let $\sum_{n=1}^{\infty} a_n$ be a series. If {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A,}} we write {{c1::

$$\sum_{n=1}^{\infty} a_n = A.$$

}}

Note 12

4819e0996d5d4eeb8ab8df01f58c8efe

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

> Yes.

Note 13

64c293a1a2f74541ba8e3ffa23fb54b2

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. What is the key idea in the proof?

> $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$.

Note 14

cd5ca73daf014641b49c5445adcd69b5

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

▌ No.

Note 15

184fe5e5e62b4c3f8a49c4ea6d26c240

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What is the key idea in the proof?

▌ Find a lower bound using powers of two.

Note 16

4608dd8499934012aadc1209fb34ec1e

{{c2::$\sum_{n=1}^{\infty} \frac{1}{n}$}} is called {{c1::the harmonic series.}}

Note 17

c09166f03686451eabbc0fbfeff75b48

{{c2::The harmonic series' partial sums}} are called {{c1::harmonic numbers.}}

Note 18

967408ec06384fc5bcebcfe9d34754e3

{{c2::The n-th harmonic number}} is denoted {{c1::H_n.}}

Note 19

b41eb87209464924a744a8142f77f9fc

What is the harmonic numbers' growth rate?

▌ Logarithmic.

Note 20

758809447e7f453ea7b35e206473125c

How is H_n approximated with $\ln n$?

▌ $\ln n + \text{a constant} + \delta_n, \quad \text{where } (\delta_n) \to 0.$

Note 21

73dd17acc2ae4135a8b47403834cc4b6

What is the name of the constant term from the approximation of H_n with $\ln n$?

▌ The Euler-Mascheroni constant.

Note 22

867a8efe526b49388eea0f0a54e8ec20

What is the value of the Euler-Mascheroni constant?

$$\lim_{n\to\infty} (H_n - \ln n).$$

Note 23

8361e3e94e624c89b3279fd526ece194

What is value of $\lim(H_n - \ln n)$?

The Euler-Mascheroni constant.

Note 24

5af1c127b9d44469b37ca4390dbcc30f

The Euler-Mascheroni constant is usually denoted {{c1::γ.}}

Note 25

ccea4c33507e4d5f9387c996a8bb13ad

Let (a_n) be {{c4::a positive decreasing sequence.}} Then

$$\text{\{\{c2::} \sum_{n=1}^{\infty} a_n \text{ converges\}\}\{\{c3::} \iff \text{\}\}\{\{c1::} \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.\}\}}$$

«{{c6::Cauchy Condensation Test}}»

Note 26

88287ba71bd545459ba16b4e2ca5cb69

Let (a_n) be a decreasing sequence and $a_n \leq 0$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

What is the key idea in the proof?

Group the element of a partial sum in chunks of size 2^m.

Note 27

7dfc9afff8a045caa6549458d3264c8d

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ {{c2::converges}} {{c3::if and only if}} {{c1::$p > 1$.}}

Note 28

66666197109243728959180963a362d4

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. What is the key idea in the proof?

> The Cauchy Condensation Test and the convergence of geometric series.

Note 29

02c63840577e480888a118f988cd15f5

Let

$$x_1 = 3 \quad \text{and} \quad x_{n+1} = \frac{1}{4 - x_n}.$$

How do you prove that (x_n) converges?

> Use the Monotone Convergence Theorem.

Note 30

c2af7f6b381941e4a6225dc675a8b580

Let

$$x_1 = 3 \quad \text{and} \quad x_{n+1} = \frac{1}{4 - x_n}.$$

How do you proof that x_n is monotone?

> By induction.

Note 31

f3e64eda5bb1491ea87a57a0627b885c

Let

$$x_1 = 3 \quad \text{and} \quad x_{n+1} = \frac{1}{4 - x_n}.$$

Given that $\lim x_n$ exists, how do you find its value?

> Take the limit of the equality.

Note 32

c9b146dff86b4f93bb189517510996a0

What is the geometric mean of two real numbers x and y?

$$\sqrt{x \cdot y}$$

Note 33

5a27b7e61ce148dbbd2ed59ebe4e62af

What is the arithmetic mean of two real numbers x and y?

$$\frac{x + y}{2}$$

Note 34

f2819387c22a4fa0bb81aa43dc5713a1

How do the geometric mean and the arithmetic mean of two positive real number relate?

Geometric $\leq$ arithmetic.

Note 35

2139b7a22c4943d4a6a97cea13e2d4c7

What, vaguely, is the limit superior of a sequence?

The limit of consecutive supremums.

Note 36

a0aaa8a84d2e45d4ab724b0acab3f929

For which sequences is the limit superior always well-defined?

Bounded above.

Note 37

8f61f74e27fb429aa768dd1cd4d18e9f

Which set's supremum is taken in the definition of the limit superior of a sequence?

Of all the elements starting from the n-th.

Note 38

fedfa9f4b6af42368cedc5eca7373c93

Let (a_n) be a sequence. {{c2::The limit superior of (a_n)}} is denoted

{{c1::

$$\limsup a_n.$$

}}

Note 39

2139b7a22c4943d4a6a97cea13e2d4c7

What, vaguely, is the limit inferior of a sequence?

> The limit of consecutive infimums.

Note 40

72a567f6068242118de0a4d171564149

For which sequences is the limit inferior always well-defined?

> Bounded below.

Note 41

a392b9b00d884f018250ab615f16dcba

Let (a_n) be a sequence. {{c2::The limit inferior of (a_n)}} is denoted

{{c1::

$$\liminf a_n.$$

}}

Note 42

6d07e6f3644649ab814c1c25ac7f8ea7

Let (a_n) be a bounded sequence. How do $\liminf a_n$ and $\limsup a_n$ relate?

> $$\liminf a_n \leq \limsup a_n.$$

Note 43

2de9f038606f4764a5ff29923042646c

Let (a_n) be a bounded sequence.

{{c2::$(a_n) \to a$}} {{c3:: $\iff$ }} $\liminf a_n =$ {{c1::$\limsup a_n = a.$}}

Note 44

53b4f3cf3d6a41bfb0e016a3935e7891

Let (a_n) be {{c3::a sequence.}} {{c2::An infinite product}} is {{c1::a formal expression of the form

$$\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \cdots$$

}}

Note 45

df2c87f7d21745eca396cc7544135480

{{c2::A partial product}} of an infinite product is {{c1::the product of the first m terms.}}

Note 46

9d34fc764e4440759885868ead946718

{{c2::The sequence of partial products}} of an infinite product is usually denoted {{c1::(p_m).}}

Note 47

d3c3280660df404ba970bb6ce71933e7

We say that an infinite product {{c2::converges to A}} if {{c1::the corresponding sequence of partial products converges to A.}}

Note 48

707abc48979742b7b9a5a7c0646a03d7

Does $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ converge?

▌ No.

Note 49

d5d2d9e020c04d0aa59e356b7a11447c

$$\prod_{n=1}^{\infty}(1+a_n)\,, \quad \text{where {{c2::}}} a_n \geq 0, \text{ {{}}}$$

{{c3::converges}} if and only if {{c1::$\sum a_n$ converges.}}

Note 50

eb2761aa4e8e47008fbc04a66a05d627

Let (a_n) be a positive sequence.

$$\prod (1 + a_n) \text{ converges} \implies \sum a_n \text{ converges.}$$

What is the key idea in the proof?

▌ A partial product's expansion contains the partial sum.

Note 51

51003bd899f44cd3ae8c7ceef7a14dc5

Let (a_n) be a positive sequence.

$$\prod (1 + a_n) \text{ converges} \impliedby \sum a_n \text{ converges.}$$

What is the key idea in the proof?

▌ Give an upper bound for $1 + a_n$ with an exponential.

Note 52

000b398dbcc04190a753fdbee51f11f8

Given $x \geq 0$, provide an upper bound for $1 + x$ with an exponential.

▌ $1 + x \leq e^x$.

Note 53

523b13cecd6249faac6c5f8c39e0c77f

What is the visual representation behind the inequality

$$1 + x \leq e^x \quad ?$$

▌ $y = 1 + x$ is tangent to $y = e^x$, which is convex.

Properties of Infinite Series

Note 1

51836e3c068e46888891ad60f449bd6

Let $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$. Under which condition does

$$\sum_{k=1}^{\infty} ca_k$$

converge?

> Always.

Note 2

548101004aba462b8e81b2c4f7cbd1b9

If $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$, then $\sum_{k=1}^{\infty} ca_k =$ {{c1::cA}}.

Note 3

30607fca749d4ea9814ec7460a102865

Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Under which condition does

$$\sum_{k=1}^{\infty} a_k + b_k$$

converge?

> Always.

Note 4

4f1064d2b18d4e889fa4e80010f532b1

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\sum_{k=1}^{\infty} a_k + b_k = \text{{{c1::}} A + B. \text{}}}$$

Note 5

7c0abecdaf8e4bc19ba89cb4fe114bd6

If $\sum_{k=1}^{\infty} a_k$ {{c3::converges}} and $\sum_{k=1}^{\infty} b_k$ {{c3::diverges,}} then

$$\text{{{c2::}} \sum_{k=1}^{\infty} a_k + b_k \text{}}{{c1::} \text{ diverges.}\text{}}}$$

Note 6

b5cdf0512d8d457cb2a379569c3be2d3

If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then

$$\sum_{k=1}^{\infty} a_k + b_k \text{ diverges.}$$

What is the key idea in the proof?

> By contradiction and $\sum b_k$ converges.

Note 7

41d1b8798dd64b74a5b57efd33beaa27

The tail of a {{c2::convergent}} series {{c1::tends to 0.}}

Note 8

6795efea2a204bfb90bf19f3ac01f60a

The series $\sum_{k=1}^{\infty} a_k$ {{c5::converges}} {{c4::if and only if,}} given {{c3:: $\epsilon > 0$,}} there exists {{c2::an $N \in \mathbf{N}$}} such that whenever {{c2::$n > m \geq N$ }} it follows that {{c1::

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

}}

Note 9

f83e35fa266b4b71ae674a5ae53196aa

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

«{{c1::Cauchy Criterion}}»

Note 10

255fd1a8d1ca40ddbe4706f396dcaad5

What is the key idea in the proof of the Cauchy Criterion for Series?

> Cauchy Criterion for the sequence of partial sums.

Note 11

2ccccd666d0d4025a48baaa6ac297e88

If the series $\sum_{k=1}^{\infty} a_k$ {{c2::converges,}} then {{c1::$(a_k) \to 0$.}}

Note 12

e553a27c1b0240b4a08a2d2e1291a1c5

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. What is the key idea in the proof?

> Apply the Cauchy Criterion with $n = m + 1$.

Note 13

0314d6d2761e4bd1b24b1b858e9c5086

Assume (a_k) and (b_k) are sequences satisfying {{c3::$0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.}} If $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::converges,}} then $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::converges.}}

Note 14

03fddbcdb39340e0a421d24fe7298f2e

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::diverges,}} then $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::diverges.}}

Note 15

d6553b70220c4348a7c7692a58f91271

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«{{c1::Comparison Test}}»

Note 16

7f40a1b03ff44e75af1465ca5e329e3e

What is the key idea in the proof of the Comparison Test for Series?

> Use the Cauchy Criterion explicitly.

Note 17

e02413e7068f47d28eab58d2542d2858

What series are considered in the Limit Comparison Test?

> Positive and one containing no zeros.

Note 18

cedc7eb1b2ac4f578caebcbaf4398f01

Which value is considered in the Limit Comparison Test?

> The limit of the ratio of corresponding terms.

Note 19

9ce4a06cfa6e42c7bae44e61649416d4

Which cases exist on the Limit Comparison Test?

> The limit is finite or is nonzero.

Note 20

bb6597f3ea41409da5895548c598ddae

What can we say from the Limit Comparison Test if the limit is finite?

> The denominator’s series convergence implies that of the numerator.

Note 21

08f8caf25dd54ec8bd0b0a03a66d00f2

What can we say from the Limit Comparison Test if the limit is nonzero?

> The numerator’s series convergence implies that of the denominator.

Note 22

8474f88f7b4140dabe637c96e7a5005d

What can we say from the Limit Comparison Test if the limit is finite and nonzero?

> The two series’s convergences are equivalent.

Note 23

ca9aa1db61144f7e99c9c0ead13fed2f

What can we say from the Limit Comparison Test if the limit does not exist?

> Nothing.

Note 24

34848474b28a469dbb7bc1859e1ab612

What is the key idea in the proof of the Limit Comparison Test (finite limit)?

> The set of ratios is bounded above + the Comparison Test.

Note 25

6f66af55f5d042cb85559bf7718f0641

What is the key idea in the proof of the Limit Comparison Test (nonzero limit)?

> Swap the numerator and the denominator.

Note 26

1f9364c8930f4fedbfb3501d9a92ee2e

Statements about {{c2::convergence}} of sequences and series are immune to {{c1::changes in some finite number of initial terms.}}

Note 27

89c3e03f687b4c4aa41185f6c668d327

A series is called {{c2::geometric}} if it is of the form {{c1::

$$\sum_{k=0}^{\infty} ar^k.$$

}}

Note 28

4d18a586f7754236bac47a23a54ede43

The series $\sum_{k=0}^{\infty} ar^k$ {{c2::converges}} {{c3::if and only if}} {{c1::$|r| < 1$.}}

Note 29

f7ab1e58f37b4580a558de06c51dc6f7

Given $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \text{\{\{c1::}\frac{a}{1-r}.\text{\}\}}$$

Note 30

c409ec230f6741b796ea4ef3e8813d9c

Given $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. What is the key idea in the proof?

> Rewrite the partial sums.

Note 31

28dc84fd3d384adea7a15102e07c644a

If {{c2::the series $\sum_{k=1}^{\infty} |a_k|$ converges,}} then {{c1::$\sum_{k=1}^{\infty} a_k$ converges.
}}

«{{c3::Absolute Convergence Test}}»

Note 32

fb10bc5e919347ffa66da221bf832aa3

What is the key idea in the proof of the Absolute Convergence Test?

> The Cauchy Criterion and the Triangle Inequality.

Note 33

9e485bfc6cf2430e8c654c0404657fdf

Let (a_k) be a sequence. If {{c2::(a_k) is decreasing and approaches A,}} we say {{c1::(a_k) decreases to A.}}

Note 34

c25d4896df3146b68a046db8ad0db7b2

Let (a_k) be a sequence. If {{c2::(a_k) decreases to A,}} we write {{c1::

$$(a_k) \searrow A.$$

}}

Note 35

b3913aa4697f4849ae2b0a876b7412ab

Let (a_k) be a sequence. If {{c2::(a_k) is increasing and approaches A,}} we say {{c1::(a_k) increases to A.}}

Note 36

e24175a89ff848fa93c82f0fc0830dd9

Let (a_k) be a sequence. If {{c2::(a_k) increases to A,}} we write {{c1::

$$(a_k) \nearrow A.$$

}}

Note 37

998d23f7cbbb49ed885b7ef2f62bb629

Let (a_k) be {{c3::a sequence decreasing to zero.}} Then {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

}} {{c1::converges.}}

Note 38

df767d19abbf4031899b4a87577b2625

Let (a_k) be a sequence decreasing to zero. Then

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges.

«{{c1::Alternating Series Test}}»

Note 39

61711709fb284700a09065f04aedcf0d

What is the nominal name of the Alternating Series Test?

▌ Leibniz's Test.

Note 40

5023b0a2f0ca4300bfa09b61e0ec0a9c

{{c1::An alternating series}} is a series of the form {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k,$$

}} where {{c3::all $a_k > 0$.}}

Note 41

6fb766a68cd14aa395c223e4a0e95999

What is the key idea in the proof of the Alternating Series Test?

> The Cauchy criterion for the sequence of partial sums.

Note 42

9bfa24b4310b474db9705bceed02cc45

Which intervals are considered in the proof of the Alternating Series Test?

> Those formed by successive partial sums.

Note 43

a581365ace824e89ae7a397fe6d02f1d

In the proof of the Alternating Series Test, how to you choose $\Delta_{s_m, s_{m+1}}$, given $\epsilon > 0$?

> So that its length is less then ϵ.

Note 44

a77a5abf0f2a46e8af759deffbaeed9e

In the proof of the Alternating Series Test, what do you need to show about an interval $\Delta_{s_m, s_{m+1}}$?

> It contains all of the following partial sums.

Note 45

337566470e054b4cb38ea03a6a388ce0

Does the alternating harmonic series converge?

> Yes.

Note 46

ced51236176744dc901d6cd2463ed6fd

Why does the alternating harmonic series converge?

> Due to the Alternating Series Test.

Note 47

0e3cb5d839ba49f3aa704f2bfeffb052

What does the alternating harmonic series converge to?

▌ $\ln 2$.

Note 48

92cff082756243d9a9d2f060a0aec391

$\sum \frac{(-1)^{n-1}}{n} = \ln 2$. What is the key idea in the proof?

▌ Use the logarithmic approximation of harmonic numbers.

Note 49

be5d93836bcf452e9c9263d6206ce81b

$\sum \frac{(-1)^{n-1}}{n} = \ln 2$. In the proof, how do you transform the partial sums as to use the logarithmic approximation of H_n.

▌ Add and subtract negative terms as to make them positive.

Note 50

cb8249219a644a12b50a90701e47e548

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges absolutely,}} if {{c1::$\sum_{k=1}^{\infty} |a_k|$ converges.}}

Note 51

c07bf73c30a04766803b1c0fae6b38d9

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges conditionally,}} if {{c1::it converges and does not converge absolutely.}}

Note 52

f54a6f91b89f42c7b548ace2e106608d

A series $\sum_{k=1}^{\infty} a_k$ is said to be {{c2::positive}} if {{c1::$a_k \geq 0$ for all $k \in \mathbf{N}$.}}

Note 53

c5acade4dde342f8b7ac4acec2278ac6

Any {{c2::positive}} convergent series must {{c1::converge absolutely.}}

Note 54

e85b9eb09cfa4056b868f983703a571c

May a positive series diverge?

▌ Only to $+\infty$.

Note 55

b65eba46e51c438e933833ad313a4cf8

A {{c2::positive}} series converges {{c3::if and only if}} {{c1::the sequence of partial sums (s_n) is bounded.}}

Note 56

4ef68f3ca3544ea98fd3c54340c65ce5

Let $\sum_{k=1}^{\infty} a_k$ be a series and {{c3::$f : \mathbf{N} \to \mathbf{N}$ be 1–1 and onto.}} {{c2:: The series $\sum_{k=1}^{\infty} a_{f(k)}$}} is called {{c1::a rearrangement of $\sum_{k=1}^{\infty} a_k$. }}

Note 57

4071d910f5e6410cb2b01dfc73ae48da

If a series {{c2::converges absolutely,}} then {{c3::any rearrangement of this series}} {{c1::converges to the same limit.}}

Note 58

057430cb21934da7ac9bc037ba169eb5

If a series converges absolutely, then any rearrangement of this series converges to the same limit. What is the key idea in the proof?

> Subtract the original series' initial terms for the rearrangement's partial sum.

Note 59

d572332d7e36407ab1531e824f794b4b

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the original series' initial terms are subtracted from the rearrangement's partial sum?

> So as to use the definition of convergence and the Cauchy Criterion for absolute convergence.

Note 60

574ee484bcf94971932baee731b90c95

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the rearrangement's terms are taken for the partial sum?

So as to contain the initial terms of the original sequence.

Note 61

c50d4f3043cb4ca38411c1b1dc20ae26

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::s_n}} to be {{c1::the original series' partial sum.}}

Note 62

2f9195ab94ee4143800fc5300d10d80f

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::t_n}} to be {{c1::the rearrangement' partial sum.}}

Note 63

1bacf92272b04fc98d69ac25f5fcdfe2

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, what do we show about $t_m - s_N$?

$|t_m - s_N| < \varepsilon$

Note 64

6e8705bf5bd84118a85ac3eb8a1d5e28

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, why is it that $|t_m - s_N| < \varepsilon$?

Due to the Cauchy Criterion.

Note 65

8ffac6aca55141b29861f55f5d1dd8fb

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how do you show $|t_m - A| < \varepsilon$?

$|t_m - s_N + s_N - A|$ and the triangle inequality.

Note 66

d0ce809592604649888a354c618fd0ec

Are positive series immune to rearrangement?

> Yes.

Note 67

de28685020ea44d4998072ea240cb29c

Why are convergent positive series immune to rearrangement?

> They must converge absolutely.

Note 68

125d4c6fb0df43ac826a676d13ca67e8

Why are divergent positive series immune to rearrangement?

> Large enough partial sums contain the original initial terms.

Note 69

96c6d35aa4854b3781e1b3e4d59bfb49

What series is considered in the Riemann Series Theorem?

> Conditionally convergent.

Note 70

811acdda0a24480388e62f060d18d67e

What do we conclude from the Riemann Series Theorem?

> Rearrangements may converge to any chosen value.

Note 71

c7e493071a114013a18f4ff1b7bdf8a5

To which value can a conditionally convergent series' rearrangement converge (due to the Riemann Series Theorem)?

> Any real number, $\pm\infty$ or nothing (i.e. it may also diverge).

Note 72

4feb9c52558943278e61f143708d6d96

What do we conclude from the Riemann Series Theorem when the series is absolutely convergent?

▌ This is out of the theorem’s scope.

Note 73

2d7decf83860424eb1ccfd16074bce4d

What is the first step in proving the Riemann Series Theorem?

▌ Split for the limiting value being finite/infinite/nonexistent.

Note 74

5d58155034074cd9a49eea0ec8af064e

What is the key idea in the proof of the Riemann Series Theorem (finite limit)?

▌ Make the partial sums revolve around the given value.

Note 75

1f5a84357034420c8a97d48d8c110ddd

What is the algorithm for building the partial sums in the proof of the Riemann Series Theorem?

▌ Go up till you get above, then down to get below and repeat.

Note 76

a865d0048c1d4f3fb5d1c37bab47fcc6

In the proof of the Riemann Series Theorem, why can we make the partial sums revolve around the given value?

▌ Both “up” and “down” motions are unlimited.

Note 77

2f65a75387fa4c6c8a5f52993a2512a8

In the proof of the Riemann Series Theorem, why are both “up” and “down” motions unlimited?

▌ Due to convergence being conditional.

Note 78

0fc52132c4e748a3919527215e6a9bae

In the proof of the Riemann Series Theorem, why are the revolving partial sums approaching the given value?

▌ The terms must tend to zero.

Note 79

5b7ee13ee0d44167bca803673ded00bf

What is the key idea in the proof on the Riemann Series Theorem (infinite limit)?

▌ Go up two units, down one unit and repeat.

Note 80

d13a96547a804e139bfd1a25a5f4d303

What is the key idea in the proof on the Riemann Series Theorem (no limit)?

▌ Go up over 1, then down below 0 and repeat.

Note 81

817e11b891694f8bb974b530bda3015a

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

is {{c2::a rearrangement}} of {{c1::the harmonic series.}}

Note 82

3cb62d056cdf4830898fbdd672aef478

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \text{\{\{c1::}\frac{1}{2}\ln 2.\text{\}\}}$$

Note 83

a961182dd27140969e35373da31fdbc3

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln 2.$$

What is the key idea in the proof (intuitively)?

▌ Collapse $\frac{1}{k} - \frac{1}{2k}$.

Note 84

e36370a244d444edb06b2037f16d05b0

If $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges absolutely. Is this true?

▌ Yes, it is.

Note 85

64d44699e8e445089178e11a54560668

Assume $\sum a_n$ converges absolutely. What can we tell about $\sum a_n^2$?

▌ It converges absolutely.

Note 86

aa0da9a453cf405dbd207a83925a030c

Assume $\sum a_n$ converges absolutely. Then $\sum a_n^2$ converges absolutely. What is the key idea in the proof?

▌ Absolute values are eventually < 1 + the Comparison Test.

Note 87

9562860ae3544066a13fce0c8e105bff

If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges. Is this true?

▌ No, it's false.

Note 88

b518cd6950614ffba71bcc13155cdf31

If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges. Provide a counterexample.

▌ Alternating harmonic series and alternating $\frac{1}{\ln n}$.

Note 89

60feb468c7a54a7fb244e9e0c8b61c47

If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges. Is this true?

▌ Yes, it is.

Note 90

a3db2ba7fe3e4210a033c14756cda177

Assume $\sum a_n$ converges conditionally. What do we know about $\sum n^2 a_n$?

▌ It diverges.

Note 91

0d6253382e994acdbb84a82dcfa1152b

If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges. What is the key idea in the proof?

▌ By contradiction; $(n^2 a_n)$ is bounded.

Note 92

918203417e3c433499de22e1f1e71b37

If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges. In the proof (by contradiction), how do you show that $\sum |a_n|$ converges?

▌ $(n^2 a_n)$ is bounded; the Comparison Test.

Note 93

d67d12138d8741b2a9f636eaee48e7d4

If $\sum n^2 a_n$ converges, then $\sum a_n$ {{c1::converges absolutely.}}

Note 94

b4e0eacc15f64559b6c255552fe3aadf

What series are considered in the Ratio Test?

▌ Strictly positive.

Note 95

dcfddd94a3304571a442fff1f7009cb8

What value is considered in the Ratio Test?

▌ The limit of successive ratios.

Note 96

d00eda65eafa4efabe918bfacc3ff819

Which term is placed to the numerator in the Ratio Test?

▌ The next one.

Note 97

605c64a7226c48eebe5ee34d51cd470b

When does the Ratio Test let us conclude something?

> When the ratios approach a value other than 1.

Note 98

a70e3ac68ab947fc8e389e85e5f54588

Which cases exist on the Ratio Test?

> Ratios converge to less than, or greater than, 1.

Note 99

de649e2ae5cc4b3b93aac925d3b37d4b

What do we conclude from the Ratio Test when the ratios converge to something less than 1?

> The series converges.

Note 100

3bcf7fb3ba4f4ace92b222a3c8af9174

What do we conclude from the Ratio Test when the ratios converge to something greater than 1?

> The series diverges.

Note 101

90519e5b985b4f97a25636a1473b500d

What do we conclude from the Ratio Test when the ratios converge to 1?

> Nothing.

Note 102

4bab403524b240cda38745c2324966c0

What do we conclude from the Ratio Test when the ratios do not converge?

> Nothing.

Note 103

1a0caf850c00432b93871e8c66f3397b

Give an example when the Ratio Test is inconclusive and the series diverges.

▌ The harmonic series.

Note 104

0c417f771ac54fa3ad89fb5d65d5f10d

Give an example when the Ratio Test is inconclusive and the series converges.

▌ $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 105

0a54c42a8bd74ba883e310f36f865ca6

What is the nominal name of the Ratio Test?

▌ The d'Alambert's Ratio Test.

Note 106

f1e24cc124f84cf3a6d14e77ee23368b

What is the first step in proving the Ratio Test?

▌ Split $r < 1$, $r > 1$.

Note 107

127428f8805043978b16164456c8acf5

What is the key idea in the proof of the Ratio Test ($r > 1$)?

▌ The terms are eventually increasing.

Note 108

535154065a884eb7bf3e87e8d4b400e5

What is the first key idea in the proof of the Ratio Test ($r < 1$)?

▌ For $r < r' < 1$ the ratios are eventually less than r'.

Note 109

5ac59226423b4b8fb84c087795e5ed6f

What is the second key idea in the proof of the Ratio Test ($r < 1$)?

> Find an upper bound using a geometric series.

Note 110

ce4c6aa5f15044a2a804f11a91d677b7

What series are considered in the Root Test?

> Positive.

Note 111

02964fce0fcd409cab46d91942e3f1c2

What value is considered in the Root Test?

> The limit of $\sqrt[n]{a_n}$.

Note 112

06c9e889bae041afb32a8f2da431bbf9

Which cases exist on the Root Test?

> n-th roots approach something less than, or greater than, 1.

Note 113

562b1b6b74e24c73ad75d944ff17d581

When does the Root Test let us conclude something?

> When n-th roots approach something other than 1.

Note 114

687fe6a03e28430189cd57632f9bae0b

What do we conclude from the Root Test if the limit is less than 1?

> The series converges.

Note 115

dd2315fb062b4bdf93ebe5072fc0d308

What do we conclude from the Root Test if the limit is greater than 1?

> The series diverges.

Note 116

7701686caac7412aa1b3375ff77e5a9e

What do we conclude from the Root Test if the limit converges to 1?

> Nothing.

Note 117

6200b936d6144cafb8b74ff7d9271a9d

Give an example when the root test is inconclusive and the series diverges.

> The harmonic series.

Note 118

6cd4fabac91944db96449403d2288e0a

Give an example when the root test is inconclusive and the series converges.

> $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 119

644281f3c2614e2499993a48daca8aac

What is the nominal name for the Root Test?

> Cauchy's Radical Test.

Note 120

7021924723f142d489dc64e27e06c40b

What is the first step in proving the Root Test?

> Split $r < 1, r > 1$.

Note 121

ae27724cb07240fbb243221a41bb7f82

What is the first key idea in the proof of the Root Test ($r < 1$)?

> For $r < r' < 1$ the roots are eventually less than r'.

Note 122

64f3efecadd94ca8ad1277cba95ded2e

What is the second key idea in the proof of the Root Test ($r < 1$)?

> Find an upper bound using a geometric series.

Note 123

e4b13d2a78bc4010ad92b3574943d982

What is the key idea in the proof of the Root Test ($r > 1$)?

> The elements are eventually greater than 1.

Note 124

41062fe4246b41d4b8790bbf53f1bf3f

The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots$ certainly converges. Why?

> Partial products are monotone and bounded.

Note 125

0eacbc2fbf2f4db396ec33f55d598543

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots = \{\{c1::0.\}\}$$

Note 126

c417bd39783c42c081558d3730954f29

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots = 0.$$

What is the key idea in the proof?

> By contradiction + rewriting a partial product as a sum of successive differences reveals the harmonic series.

Note 127

ac6f35c13b28462188b13785fe0f255e

Find examples of two series $\sum a_n$ and $\sum b_n$ such that

- their terms are strictly positive and decreasing; and
- $\sum \min a_n, b_n$ converges.

> Take $\sum \frac{1}{2^n}$ and on every next block "rise" one of the sequences to make it sum to 1.

Note 128

391b719f11404d53959a2e258908f1d0

What sequences are considered in the Summation-by-Parts formula?

> Arbitrary.

Note 129

f1a472048eb0400cafd7a7d7b0e049cc

What is the initial expression in the Summation-by-Parts formula?

> $$\sum_{j=n}^{m} x_j y_j .$$

Note 130

1424da07cc0f4c7e9e792ba2daad165c

Which terms are there in the transformed expression in the Summation-by-Parts formula?

> Two "free" terms and a sum.

Note 131

eb188d69b3c74c42814da0030ab179ca

What is the first free term of the transformed expression in the Summation-by-Parts formula?

> The final partial sum times the next element.

Note 132

1af3ccefd5714f279390596beb66afdb

What is the second free term of the transformed expression in the Summation-by-Parts formula?

> Subtracting the partial sum preceding the range multiplied by the starting element.

Note 133

ed63990568ac41ff9e0d1b7535e91d62

What is the sum term of the transformed expression in the Summation-by-Parts formula?

> The sum of partial sums multiplied by the successive differences.

Note 134

e46ff0f795c84a08a97ab92916d689f7

What is the order of successive differences in the sum term of the transformed expression in the Summation-by-Parts formula?

> The current minus the next.

Note 135

67d6011a7aa7477da37cbb1ab2899cea

What is the range of summation in the sum term of the transformed expression in the Summation-by-Parts formula?

> Same as the original.

Note 136

84d1c74bb7fd496a90c6f85103bb2793

What is the value of the zeroth partial sum in the Summation-by-Parts formula?

> Zero.

Note 137

6153ba6cc000482694e8ffdcea302fd4

What is the nominal name of the Summation-by-Parts formula?

> The Abel Transformation.

Note 138

a637cd28783d4349916b7db04a7b8eef

What is the key idea in the proof of the Summation-by-Parts formula?

> Rewrite the sequence's values as the differences of successive partial sums.

Note 139

fbd64e1c82f047908a69529a47757a02

What, vaguely, is the statement of the Abel's Test?

> Convergent series times monotone bounded sequence is convergent.

Note 140

0f341db289494976a66d37a683abea82

What series is considered in the Abel's Test?

> A series formed by two sequence's products.

Note 141

7a5c1013788240b382ef972b1f7fd607

What sequences are considered in the Abel's Test?

> One, whose series converges, and one monotone and bounded.

Note 142

ddbb9e296a424788bf71b1e3b0a066a8

What do we conclude from the Abel's Test?

> The series of products converges.

Note 143

5e107977cb19443f9b7b7c162281129a

When can we conclude something from the Abel's Test?

> Whenever the hypothesis is satisfied.

Note 144

874fa5f12343413792c9ef518001baa2

What is the first step in proving the Abel's Test?

▌ With no loss of generality, the sequence is decreasing.

Note 145 338b5b7693534b4ea659c8f8f55b1583

What is the key idea in the proof of the Abel's Test?

▌ Summation-by-Parts + the definition of convergence.

Note 146 a09ddc5d281c4426acd43d366e76dc2c

To which sums is Summation-by-Parts applied in the proof of the Abel's Test?

▌ The products' series' partial sums.

Note 147 300bbcd9c31945c3bf02eeb30031651b

In the proof of the Abel's Test, how do you show that the partial sums converge?

▌ Both addends converge (after applying Summation-by-Parts).

Note 148 c53b15140c664228832eab2b80cc06c4

In the proof of the Abel's Test after applying Summation-by-Parts, how do you show that the "free" terms converge?

▌ It follows from the hypothesis.

Note 149 7438567b1b904753b18ef4713bb2def2

In the proof of the Abel's Test after applying Summation-by-Parts, how do you show that the sums converge?

▌ The Comparison Test for absolute convergence.

Note 150 418af51ce9214731ae729971dc8feff4

To which series is the Comparison Test applied in the proof of the Abel's Test?

> The one generated after applying Summation-by-Parts.

Note 151

35b449cb77ee4b26a4ea6e221660aece

In the proof of the Abel's Test, where from do you get an upper bound when applying the Comparison Test?

> The partial sums converge and, thus, are bounded.

Note 152

f87a7596fb944f9e8998cb889ea120e9

What, vaguely, is the statement of the Dirichlet's Test?

> "Bounded" series times infinitesimal sequence is convergent.

Note 153

576a36d9c5f047c59d57212e4781326f

What series is considered in the Dirichlet's Test?

> A series formed by two sequence's product.

Note 154

15db5e34f9784e7d900ceb16e32cc428

What sequences are considered in the Dirichlet's Test?

> One with bounded partials sums and one decreasing to zero.

Note 155

d4cc82820fb6425fa4c4839f216cb490

What do we conclude from the Dirichlet's Test?

> The product's series converges.

Note 156

f8624d3bb31347acba618aeb453083d1

When can we conclude something from the Dirichlet's Test?

▌ Whenever the hypothesis is satisfied.

Note 157

d4efe889ccf943438eb6d487589e7554

What is the key idea in the proof of the Dirichlet's Test?

▌ Summation-by-Parts + the definition of convergence.

Note 158

7b3d965ec2bb4498818d1b01c686ca76

To which sums is Summation-by-Parts applied in the proof of the Dirichlet's Test?

▌ The products' series' partial sums.

Note 159

b7c76052781e4eea809d4e1c5d892fec

The Alternating Series Test can be derived as a special case of {{c1::the Dirichlet's Test.}}

Note 160

f770236dcbdd457c902c4b7848051692

How do you prove that $\sum(-1)^{\lfloor \log_2 n \rfloor}\frac{1}{n}$ diverges?

▌ Use the contrapositive of the Cauchy criterion

Note 161

cf765ced874c41759ac91d2ef359a2ae

In the Abel's and Dirichlet's Tests {{c2::the non-monotone sequence }} is responsible for {{c1::the sign alternation.}}

Note 162

b33fdd1764a743b4a67ea1c97b9bc645

What's special about $\sum_n \sin n\alpha$?

▌ It's partial sums are bounded.

Note 163

84dfa5c69983414ea8a2bd7384b58069

The partial sums of $\sum_n \sin n\alpha$ are bounded. What is the key idea in the proof?

> Multiply a partial sum by $\sin\alpha$ so that almost everything cancels out.

Note 164

537743d91b994a9a98dd7819d5bd6120

The partial sums of $\sum_n \sin n\alpha$ are bounded. In the proof, why does almost everything cancel out after multiplying by $\sin\alpha$?

> Due to the formula for a product of two $\sin$s.