Uniform Convergence of a Sequence of Functions

Note 1

Let (f_n) be {{c1::a sequence of function on a set A.}} We say {{c3::(f_n) converges pointwise on A to a function f}} if {{c2::for all $x \in A$

$$\left(f_n(x) \right) \xrightarrow[n \to \infty]{} f(x).$$

}}

Note 2

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges pointwise on A to f,}} we write

$$\{\{c1::f_n \to f\}\} \quad \text{or} \quad \{\{c1:: \lim_{n \to \infty} f_n = f.\}\}$$

Note 3

Let $f_n(x) = \frac{x^2 + nx}{n}$.

$$\lim_{n \to \infty} f_n(x) = \{\{c1::x.\}\}$$

Note 4

Let $f_n(x) = x^n, \; f_n : [0,1] \to \mathbb{R}$.

$$\lim_{n \to \infty} f_n(x) = \left\{ \{\{c1:: \begin{array}{ll} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{array} \}\} \right.$$

Note 5

Let (f_n) be a sequence of function on a set A. We say {{c2::(f_n) converges uniformly on A to a function f}} if {{c1::

$$\forall \epsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall n \geq N$$
$$|f_n - f| < \epsilon.$$

}}

Note 6

Let (f_n) be a sequence of function on a set A. If (f_n) converges uniformly on A to f, we write

$$f_n \rightrightarrows f.$$

Note 7

What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

▌ The dependence of N on x.

Note 8

What is the visual behind the uniform convergence of a sequence of functions?

▌ Eventually every f_n is completely contained in the ϵ-strip.

Note 9

Which is stronger, uniform or pointwise convergence?

▌ Uniform convergence is stronger.

Note 10

Uniform convergence implies pointwise convergence.

Note 11

Let (f_n) be a sequence of function on a set A.

$$f_n \rightrightarrows f \iff \sup |f_n - f| \xrightarrow[n \to \infty]{} 0.$$

(in terms of sup)

Note 12

Let (f_n) be a sequence of function on a set A. {{c3::Then (f_n) converges uniformly on A}} {{c4::if and only if}}

$$\{\{c1::\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N\}\}$$
$$\{\{c2::|f_n - f_m| < \varepsilon.\}\}$$

Note 13

Let (f_n) be a sequence of function on a set A. Then $f_n \rightrightarrows f$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N$$
$$|f_n - f_m| < \varepsilon.$$

«{{c1::Cauchy Criterion}}»

Note 14

What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

▌ Follows immediately from the definition.

Note 15

What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

▌ Define a candidate for the limit and prove by definition.

Note 16

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

▌ Use the pointwise limit.

Note 17

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

▌ Due to the Cauchy Criterion for sequences.

Note 18

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have $f_n \to f$. How do you show that $f_n \rightrightarrows f$?

▌ Take the limit of the inequality from the Cauchy Criterion.

Note 19

Let $f_n \to f$ on a set A and $c \in A$. If {{c3:: the convergence is uniform }} and {{c2:: all f_n are continuous at c,}} then {{c1:: f is continuous at c.}}

Note 20

Let $f_n \to f$ on a set A and $c \in A$. If the convergence is uniform and all f_n are continuous at c, then f is continuous at c.

«{{c1::Continuous Limit Theorem}}»

Note 21

What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

▌ Triple triangle inequality after adding and subtracting f_N.

Note 22

Let $f_n \to f$ on a set A and $c \in A$. If {{c1:: the convergence is uniform and all f_n are continuous at c,}} then

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = {}_{\{\{c2::}} \lim_{n \to \infty} \lim_{x \to c} f_n(x).{}_{\}\}}$$

Note 23

Let $f_n \to f$ on a set A. If each f_n is continuous, but f is discontinuous, then {{c1::the convergence is not uniform.}}

Note 24

Give an example of a sequence of functions $f_n \to f$ such that

- each f_n is continuous almost everywhere; and

- f is nowhere continuous.

▍ Step-by-step construction of the Dirichlet's function.

Note 25

Assume $f_n \to f$ on a set A and each f_n is uniformly continuous. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is uniformly continuous.}}

Note 26

Assume $f_n \to f$ on a set A and each f_n is bounded. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is bounded.}}

Note 27

Assume $f_n \to f$ on a set A and each f_n has a finite number of discontinuities. If $f_n \rightrightarrows f$, then {{c1::f has at most a countable number of discontinuities.}}

Note 28

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If {{c2::f is discontinuous at c,}} then {{c1::all f_n are eventually discontinuous at c.}}

Note 29

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If f is discontinuous at c, then all f_n are eventually discontinuous at c. What is the key idea in the proof?

By contradiction + choose a subsequence continuous at c.