Basic Axioms and Examples

Note 1

cca4f1927b2c4eeaa3123dbcf0680bc0

Given a set G, {{c2::a binary operation $\star$ on G}} is {{c1::a function

$$\star : G \times G \to G.$$

}}

Note 2

7732d25ebb1e40dd9696c1c921803c17

Given a binary operation $\star$ on a set G, for any $a, b \in G$ we shall write {{c2::$a \star b$}} for {{c1::$\star(a, b)$.}}

Note 3

4fc60827250f4af4ab6a669ac7632568

A binary operation $\star$ on a set G is {{c2::associative}} if {{c1::for all $a, b, c \in G$ we have

$$a \star (b \star c) = (a \star b) \star c.$$

}}

Note 4

192d8d86f22349cabcd9f4229fc45290

If $\star$ is a binary operation on a set G we say elements a and b of G {{c1::commute}} if {{c2::

$$a \star b = b \star a.$$

}}

Note 5

e5cbf512d6a54c91950c65450a07a501

A binary operation $\star$ on a set G is {{c2::commutative}} if {{c1::for all $a, b \in G$ we have

$$a \star b = b \star a.$$

}}

Note 6

36b096eebd7f4264ab071a5fa4eefe13

Suppose that $\star$ is a binary operation on a set G and $H \subseteq G$. If {{c2::the restriction of $\star$ to H is a binary operation on H,}} then H is said to be {{c1::closed under $\star$.}}

Note 7

644b1cd8fa014885ad295ae5c089e5a7

{{c3::A group}} is {{c2::an ordered pair $(G, \star)$}} where {{c1::G is a set and $\star$ is a binary operation on G}} satisfying {{c4::the group axioms.}}

Note 8

5de4e717b4814adf8aed4f8d9a93322c

How many axiom are there in the definition of a group $(G, \star)$?

> Three.

Note 9

2de690f5008a4b8c8691e36308e44295

What is the first axiom from the definition of a group $(G, \star)$?

> $\star$ is associative.

Note 10

4fcc137e66a048459cc73d6735e4ccea

Given a binary operation $\star$ on a set G, {{c3::an element $e \in G$}} is called {{c2::an identity of G}} if {{c1::for all $a \in G$ we have

$$a \star e = e \star a = a.$$

}}

Note 11

a3cd125f152f432082757242096a76ef

What is the second axiom from the definition of a group $(G, \star)$?

> There exists an identity of G.

Note 12

5d438f0c3fb24b1a97507e81f868846e

Given a binary operation $\star$ on a set G and $a \in G$, {{c3::an element $\tilde{a} \in G$}} is called {{c2::an inverse of a}} if {{c1::

$$a \star \tilde{a} = \tilde{a} \star a = e.$$

}}

Note 13

d840b7b910d740f3bea231c74feba51c

Given a binary operation $\star$ on a set G and $a \in G$, {{c2::an inverse of a}} is usually denoted {{c1::a^{-1}.}}

Note 14

c4c56a11c6f746b3ae287ee386b4e12b

What is the third axiom from the definition of a group $(G, \star)$?

> For all $a \in G$ there exists a^{-1}.

Note 15

be05e23d350d4f49a65602b65045f888

A group $(G, \star)$ is called {{c2::abelian}} if {{c1::$\star$ is commutative.}}

Note 16

978f23382d594a28a3de168b7f661c30

We shall say G is {{c2::a group under $\star$}} if {{c1::$(G, \star)$ is a group.}}

Note 17

497f01593d7f4ffabb546b455788b354

We shall say a set G is {{c2::a group}} if {{c1::G is a group under an operation that is clear from the context.}}

Note 18

61ea2504ca474fe4aae902eb1965576c

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are {{c2::groups}} under {{c1::$+$.}}

Note 19

84b6a231d3934ab3b4f63226549a9589

$\mathbb{Q} - \{0\}$, $\mathbb{R} - \{0\}$, $\mathbb{C} - \{0\}$ are {{c2::groups}} under {{c1::$\times$.}}

Note 20

3051cd354f5040e2bdf0809e005635ed

$\mathbb{Q}^+, \mathbb{R}^+$ are {{c2::groups}} under {{c1::$\times$.}}

Note 21

21f924e833cd4e0bbae5f4588dff47b5

Is $\mathbb{Z} - \{0\}$ a group under $\times$?

> No. (There is no inverse.)

Note 22

3e94ca73ca344269bb98d94a22204fd9

If $(A, \star)$ and $(B, \diamond)$ are {{c4::groups,}} then the group {{c2::$A \times B$,}} whose operation is {{c1::defined componentwise:

$$(a, b)(c, d) = (a \star c, b \diamond d),$$

}} is called {{c3::the direct product of the two groups.}}

Note 23

e23d8e577b3948af9b0cadd5df7c9141

If $(G, \star)$ is a group, then {{c2::the identity of G}} is {{c1::unique.}}

Note 24

5b5391986e9b49ea9c5f9f73813e9594

If $(G, \star)$ is a group, then the identity of G is unique. What is the key idea in the proof?

> Consider the product of two arbitrary identities.

Note 25

0989a259fae446c48bb0f6c40394efd0

If $(G, \star)$ is a group, then for every $a \in G$, {{c2::a^{-1}}} is {{c1::uniquely determined.}}

Note 26

f0b0a651592c466ba8067beb3b1570b8

If $(G, \star)$ is a group, then for every $a \in G$, a^{-1} is uniquely determined. What is the key idea in the proof?

> Multiply an inverse on the right by $a \star a^{-1}$.

Note 27

4a6a6806d8874839bb7956d76e384333

If $(G, \star)$ is a group and $a \in G$, then

$$(a^{-1})^{-1} = \text{{{c1::}} a. \text{}}}$$

Note 28

9ab0e972d6a24baea99f1577ebf03423

If $(G, \star)$ is a group and $a, b \in G$, then

$$\{\{c2::(a \star b)^{-1}\}\} = \{\{c1::(b^{-1}) \star (a^{-1}).\}\}$$

Note 29

69b3db6e70ad4629aa55a855b8df8096

If $(G, \star)$ is a group and $a_1, \ldots, a_n \in G$, then the value of

$$a_1 \star \cdots \star a_n$$

is {{c2::independent}} of {{c1::how the expression is bracketed.}}

«{{c3::The generalized associative law}}»

Note 30

05cc8fd523084650adb46704dde222a7

What is the key idea in the proof of the generalized associative law for a group $(G, \star)$?

> By induction.