

# Uniform Convergence of a Sequence of Functions

## Note 1

1bf1a79b9eba47cf852e1a9c7468c5f7

Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges pointwise on  $A$  to a function  $f$  if for all  $x \in A$

$$(f_n(x)) \xrightarrow{n \rightarrow \infty} f(x).$$

}

## Note 2

f11dc20a5619424cafc97ab1b4d64b5f

Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges pointwise on  $A$  to  $f$ , we write

$$(f_n \rightarrow f) \quad \text{or} \quad \lim_{n \rightarrow \infty} f_n = f.$$

## Note 3

6f3f051b9e0741debd85037d47c4fd19

Let  $f_n(x) = \frac{x^2 + nx}{n}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = (x).$$

## Note 4

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Let  $f_n(x) = x^n$ ,  $f_n : [0, 1] \rightarrow \mathbb{R}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

## Note 5

7218c9c8b0f04d4887dc2345da75c6c6

Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges uniformly on  $A$  to a function  $f$  if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \\ |f_n - f| < \epsilon.$$

}

## Note 6

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Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges uniformly on  $A$  to  $f$ , we write

$$f_n \rightrightarrows f.$$

}}

## Note 7

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What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

■ The dependence of  $N$  on  $x$ .

## Note 8

42d2e1017eac4382878c195aa5a4c54d

What is the visual behind the uniform convergence of a sequence of functions?

■ Eventually every  $f_n$  is completely contained in the  $\epsilon$ -strip.

## Note 9

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Which is stronger, uniform or pointwise convergence?

■ Uniform convergence is stronger.

## Note 10

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Uniform convergence implies pointwise convergence.

## Note 11

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Let  $(f_n)$  be a sequence of function on a set  $A$ .

$$(f_n \rightrightarrows f) \iff \sup_{n \rightarrow \infty} |f_n - f| \rightarrow 0.$$

(in terms of sup)

### Note 12

1b59f18d7ccb47829cf7b7ea7576318c

Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $(f_n)$  converges uniformly on  $A$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

### Note 13

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Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $f_n \Rightarrow f$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

«[Cauchy Criterion]»

### Note 14

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What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

■ Follows immediately from the definition.

### Note 15

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What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

■ Define a candidate for the limit and prove by definition.

### Note 16

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

■ Use the pointwise limit.

### Note 17

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

■ Due to the Cauchy Criterion for sequences.

### Note 18

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have  $f_n \rightarrow f$ . How do you show that  $f_n \rightrightarrows f$ ?

■ Take the limit of the inequality from the Cauchy Criterion.

### Note 19

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{\{c\}::\text{the convergence is uniform}\}$  and  $\{\{c\}::\text{all } f_n \text{ are continuous at } c,\}$  then  $\{\{c\}::f \text{ is continuous at } c,\}$

### Note 20

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If the convergence is uniform and all  $f_n$  are continuous at  $c$ , then  $f$  is continuous at  $c$ .

« $\{\{c\}::\text{Continuous Limit Theorem}\}$ »

### Note 21

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What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

■ Triple triangle inequality after adding and subtracting  $f_N$ .

### Note 22

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{\{c\}::\text{the convergence is uniform}\}$  and all  $f_n$  are continuous at  $c$ , then

$$\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \{\{c\}:: \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x).\}$$

### Note 23

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Let  $f_n \rightarrow f$  on a set  $A$ . If each  $f_n$  is continuous, but  $f$  is discontinuous, then  $\{\{c1:: \text{the convergence is not uniform.}\}$

### Note 24

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Give an example of a sequence of functions  $f_n \rightarrow f$  such that

- each  $f_n$  is continuous almost everywhere; and
- $f$  is nowhere continuous.

■ Step-by-step construction of the Dirichlet's function.

### Note 25

81c5e1a2081241d1973bb2cacde92627

Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  is uniformly continuous. If  $\{\{c2:: f_n \rightrightarrows f,\}\}$  then  $\{\{c1:: f \text{ is uniformly continuous.}\}\}$

### Note 26

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Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  is bounded. If  $\{\{c2:: f_n \rightrightarrows f,\}\}$  then  $\{\{c1:: f \text{ is bounded.}\}\}$

### Note 27

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Assume  $f_n \rightarrow f$  on a set  $A$  and each  $f_n$  has a finite number of discontinuities. If  $f_n \rightrightarrows f$ , then  $\{\{c1:: f \text{ has at most a countable number of discontinuities.}\}\}$

### Note 28

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Assume  $f_n \rightrightarrows f$  on a set  $A$  and  $c \in A$ . If  $\{\{c2:: f \text{ is discontinuous at } c,\}\}$  then  $\{\{c1:: \text{all } f_n \text{ are eventually discontinuous at } c.\}\}$

### Note 29

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Assume  $f_n \rightrightarrows f$  on a set  $A$  and  $c \in A$ . If  $f$  is discontinuous at  $c$ , then all  $f_n$  are eventually discontinuous at  $c$ . What is the key idea in the proof?

■ By contradiction + choose a subsequence continuous at  $c$ .

### Note 30

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Let  $f$  be  $\{\{c2: \text{continuous}\}\}$  on all of  $\mathbf{R}$ . Then  $f(x + \frac{1}{n})$   $\{\{c1: \text{converges to } f.\}\}$

### Note 31

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Let  $f$  be  $\{\{c2: \text{uniformly continuous}\}\}$  on all of  $\mathbf{R}$ . Then  $f(x + \frac{1}{n})$   $\{\{c1: \text{converges uniformly to } f.\}\}$

### Note 32

4973de785b1848b3b54d17231e4b30ae

Which algebraic operations preserve uniform convergence?

■ Scalar multiplication, addition and taking absolute value.

### Note 33

7631431e512c41e6ae5297fc6ceac974

Let  $(f_n)$  and  $(g_n)$  be uniformly convergent. Does  $(f_n + g_n)$  necessarily converge uniformly?

■ Yes.

### Note 34

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Let  $(f_n)$  and  $(g_n)$  be uniformly convergent. Does  $(f_n g_n)$  necessarily converge uniformly?

■ No.

### Note 35

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Give an example of two sequences of functions that converge uniformly, but whose product only converges pointwise.

■  $x$  and  $\frac{1}{n}$  on all of  $\mathbf{R}$ .

### Note 36

fec44b8a25b24523a0bb9d452feb5b7d

Give a “more visual” example showing that uniform convergence is not always preserved under multiplication.

|  $x^2$  and  $\frac{(\sin x)^n}{n}$ .

### Note 37

6076795b810c44ce93275f5095e37919

Let  $(f_n)$  and  $(g_n)$  be uniformly convergent. If both sequences are uniformly bounded, then  $(f_n g_n)$  converge uniformly.

### Note 38

287b3fa1a8654b3ea17d85983cd0a1b6

What does it mean for a sequence of functions to be equicontinuous at a point  $x_0$ ?

| Each function is continuous and the choice of  $\delta$  does not depend on  $n$ .

### Note 39

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A sequence of functions  $(f_n)$  is called equicontinuous at  $x_0$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$|f_n(x_0) - f_n(x)| < \epsilon$$

for all  $n$  and  $|x - x_0| < \delta$ .

### Note 40

9e1ef57494fd424a90d01b53c1c23050

On which values does  $\delta$  depend on, in the definition of equicontinuity?

| It depends on  $x_0$  and  $\epsilon$ .

### Note 41

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What does it mean for a sequence of functions to be uniformly equicontinuous?

Each function is uniformly continuous at the choice of  $\delta$  does not depend on  $n$ .

### Note 42

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A sequence of functions  $(f_n)$  is called uniformly equicontinuous if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$|f_n(x) - f_n(y)| < \epsilon$$

for all  $n$  and  $|x - y| < \delta$ .

### Note 43

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On which values does  $\delta$  depend on, in the definition of uniform equicontinuity?

Only on  $\epsilon$ .

### Note 44

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The notions of equicontinuity and uniform equicontinuity are equivalent if the sequence of functions is defined on a closed interval.

### Note 45

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Give an example of a sequence of functions  $(f_n)$  such that

- each  $f_n$  is uniformly continuous and
- $(f_n)$  is **not** uniformly equicontinuous.

$(x^n)$  on  $[0, 1]$ .



# Uniform Convergence and Differentiation

## Note 1

37f46dbb09f54423a835e842d402ee19

What sequence is considered in the Differentiable Limit Theorem?

■ A sequence of differentiable functions that converges point-wise on a closed interval.

## Note 2

19574e41800e43678628e78581f801cc

When applying the Differentiable Limit Theorem, is it necessary for the limit to be differentiable?

■ No, this is one of the implications.

## Note 3

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When do we conclude something from the Differentiable Limit Theorem?

■ When the derivatives converge uniformly.

## Note 4

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What do we conclude from The Differentiable Limit Theorem?

■ The limit  $f$  is differentiable and  $f' = \lim f'_n$ .

## Note 5

61acf9aecd834980a9dbaa77746b89e0

Let  $f_n \rightarrow f$  on  $[a, b]$  and each  $f_n$  is differentiable. What do we know about  $f$  if  $f'_n \rightarrow g$ ?

■ Nothing special.

## Note 6

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Let  $f_n \rightarrow f$  on  $[a, b]$  and each  $f_n$  is differentiable. What do we know about  $f$  if  $f'_n \rightrightarrows g$ ?

■  $f$  is differentiable and  $f' = g$ .

### Note 7

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What is the key idea in the proof of the Differentiable Limit Theorem?

■ Rewrite the limit's derivative by definition.

### Note 8

31222913007d4ceda945e1a21642c876

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right|?$$

■ Expand it using the triple triangle inequality involving  $f_N$ .

### Note 9

d3239aa3eedd346a69139a5a8b1d94ce7

In the proof of the Differentiable Limit Theorem, how do you choose  $N$ ?

■ By the Cauchy Criterion for  $f'_n \Rightarrow g$ .

### Note 10

70bbcff5bceb49c7b0abb25a8ab9be35

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$|f'_N(x) - g(x)|?$$

■ Take the limit of the inequality from the Cauchy Criterion.

### Note 11

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In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f_N(x+h) - f_N(x)}{h} - f'_N(x) \right|?$$

■ Pick  $\delta$  by the definition of differentiability of  $f_N$ .

## Note 12

cdb10b03a9254c5abfe796106c1d3e9b

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - \frac{f_N(x+h) - f_N(x)}{h} \right|?$$

■ The Mean Value Theorem for  $f_N - f_m$  and make  $m \rightarrow \infty$ .

## Note 13

b4b2753226ff4d839269bbf795c02301

Let  $(f_n)$  be a sequence of differentiable functions on  $[a, b]$  and  $(f'_n)$  converge uniformly. If  $\lim_{n \rightarrow \infty} f_n(x_0)$  exists for some  $x_0$ , then  $(f_n)$  converges uniformly.

## Note 14

8c542d7e30524e129805ce26973b0925

How can we weaken the hypothesis of the Differentiable Limit Theorem?

■  $(f_n)$  converges at a single point.

# Series of Functions

## Note 1

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . A functional series is a formal expression of the form

$$\sum_{n=1}^{\infty} f_n(x).$$

}}

## Note 2

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Let  $(f_n)$  be a sequence of functions on a set  $A$ . We say  $\sum_n f_n(x)$  converges pointwise on  $A$  to a function  $f(x)$  if the sequence of partial sums converges pointwise on  $A$  to  $f$ .

## Note 3

084d4603478b4dc48c0d1837ff30dfd8

Let  $(f_n)$  be a sequence of functions on a set  $A$ . If  $\sum_n f_n(x)$  converges pointwise to  $f(x)$ , we write

$$f(x) = \sum_n f_n(x).$$

}}

## Note 4

2922cd6ac8ff42fab5bc630fa320169

Let  $(f_n)$  be a sequence of functions on a set  $A$ . We say  $\sum f_n(x)$  converges uniformly on  $A$  to a function  $f(x)$  if the sequence of partial sums converges uniformly on  $A$  to  $f$ .

## Note 5

2b28ab51bc7f45ca934cc405e7de388f

Let  $\sum_n f_n(x)$  be a functional series. A series

$$\sum_{n=k+1}^{\infty} f_n(x) \quad \text{for } k \in \mathbb{N},$$

is called a tail of  $\sum_n f_n(x)$ .

## Note 6

d633b0c9c968402aba5285afb115d682

A series  $\sum_n f_n(x)$   $\{\{c2::\text{converges pointwise}\} \{\{c3::\text{only if}\} \{\{c1::\text{its tail converges pointwise to 0.}\}\}$

(in terms of the tail)

## Note 7

16325daa37b14ddebc3939e1d2ea063b

A series  $\sum_n f_n(x)$   $\{\{c2::\text{converges uniformly}\} \{\{c3::\text{only if}\} \{\{c1::\text{its tail converges uniformly to 0.}\}\}$

(in terms of the tail)

## Note 8

891381b2ecd44c2cb160d114479f0b20

A series  $\sum_n f_n(x)$   $\{\{c2::\text{converges pointwise}\} \{\{c3::\text{only if}\} \{\{c1::f_n \rightarrow 0.\}\}\}$

## Note 9

767a398cce7c40b781b0c39db5f9b9ac

A series  $\sum_n f_n(x)$   $\{\{c2::\text{converges uniformly}\} \{\{c3::\text{only if}\} \{\{c1::f_n \Rightarrow 0.\}\}\}$

## Note 10

c0a25e35d11c4560a26e2e463a31f725

What series is considered in the Term-by-term Continuity Theorem?

■ A series of continuous functions.

## Note 11

55e76f7381cf476bb7c32155d099bf7c

When do we conclude something from the Term-by-term Continuity Theorem?

■ When the functional series converges uniformly.

## Note 12

84af86f380cf48048b8e6b2c91e25d6c

What do we conclude from the Term-by-term Continuity Theorem when the series only converges pointwise?

■ Nothing.

### Note 13

a2c89255016b4abebcc0733f8178fdef

What do we conclude from the Term-by-term Continuity Theorem?

■ The series' sum is continuous.

### Note 14

9a06615f719646bb8e4bde3a605344f5

What series is considered in the Term-by-term Differentiability Theorem?

■ A series of differentiable functions that converges pointwise on a closed interval.

### Note 15

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When do we conclude something from the Term-by-term Differentiability Theorem?

■ The derivatives' series converge uniformly.

### Note 16

50a4a0c1c82c4129a14c9af763976811

What do we conclude from the Term-by-term Differentiability Theorem?

■  $\sum f_n$  is differentiable and  $(\sum f_n)' = \sum f_n'$ .

### Note 17

296676411bf5475eacdde73dc1c2b008

What series is considered in the Weierstrass M-Test?

■ A series of bounded functions.

### Note 18

5c393b177b724cf69790bafc0ff7b23

When do we conclude something from the Weierstrass M-Test?

- When the series of “absolute” bounds converges.

### Note 19

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Which bounds are considered in the Weierstrass M-Test?

- The sequence of the functions’ “absolute” upper bounds.

### Note 20

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What do we conclude from the Weierstrass M-Test?

- The functional series converges uniformly.

### Note 21

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What is the key idea in the proof of the Weierstrass M-Test?

- It follows from the Cauchy Criterion.

### Note 22

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What is the second implication of the Weierstrass M-Test?

- The series converges absolutely.

### Note 23

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Why does the Weierstrass M-Test implies absolute convergence?

- Absolute values have the same upper bounds.

# Power Series

## Note 1

575572e782e64317ba8228d5791138da

What is a power series (intuitively)?

■ An infinite polynomial.

## Note 2

3cd19400150446d68e6df4a87977e765

A power series is a series of the form

$$\sum_{n=1}^{\infty} a_n x^n.$$

}}

## Note 3

59c245cadd1f4c7c84641a4a81a6cf9c

A power series is a generalisation of a polynomial.

## Note 4

b9f30748736c4d1f92c345b4946d1e1a

From now on, we shall assume that any power series is centered at zero, unless otherwise agreed.

## Note 5

034c6da627e9416d94fe7048441924c4

If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| < |x_0|$ .

## Note 6

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What do we know about  $\sum a_n x^n$  if it converges at some  $x_0$ ?

■ It converges absolutely withing the open interval.

## Note 7

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If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| < |x_0|$ . What is the key idea in the proof?



Make a geometric series by factoring out  $\left|\frac{x}{x_0}\right|^n$ .

### Note 8

59b428fc86c24ff8aff670ff3a284435

If  $\sum a_n x^n$  converges at some point  $x_0 \in \mathbf{R}$  then it converges absolutely for any  $x$  satisfying  $|x| < |x_0|$ . In the proof, how do you turn  $\sum |a_n x_0^n| \left|\frac{x}{x_0}\right|^n$  into a geometric series?

$(a_n x^n)$  is bounded + the Comparison Test.

### Note 9

573b21be0d10467d913040dfe4d493bb

Which form may be taken by the set of points for which  $\sum a_n x^n$  converges?

An interval centered around 0.

### Note 10

ecdb3ab6a5bd4e23bcd67794066ab7c9

The set of points for which  $\sum a_n x^n$  converges is always an interval centered around 0. What is the key idea in the proof?

Use the “Interior Convergence” theorem.

### Note 11

21ae4818657c4e16b4ef4b2585bc3c18

How is the set of points for which  $\sum a_n x^n$  converges called?

The interval of convergence.

### Note 12

cc247e245b4d47ce8e408ff25ad39c6d

Every power series converges absolutely within the interior of its interval of convergence.)

### Note 13

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Every power series converges absolutely within the interior of its interval of convergence. What is the key idea in the proof?

■ Follows from the “Interior Convergence” theorem.

### Note 14

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[[c2: The radius of convergence]] of  $\sum a_n x^n$  is [[c1: the half length of its interval of convergence.]]

### Note 15

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How does  $\sum a_n x^n$  behave at the endpoints of its interval of convergence?

■ Who knows. . .

### Note 16

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What are the simplest methods for calculating the radius of convergence of a power series?

■ Using either the Root Test or the Ratio Test.

### Note 17

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How can you use the Root Test to calculate the radius of convergence of a power series?

■ Take the inverse of the coefficients’ roots’ limit.

### Note 18

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How can you use the Ratio Test to calculate the radius of convergence of a power series?

■ Take the inverse of the coefficients’ ratios’ limit.

### Note 19

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Let  $R$  be the radius of convergence of  $\sum a_n x^n$ . Then

$$R = \left( \limsup \sqrt[n]{|a_n|} \right)^{-1}.$$

«[[c2: Cauchy–Hadamard Theorem]]»

## Note 20

197b7547ea3349ce827335925cf42930

In the Cauchy-Hadamard Theorem, what happens when

$$\limsup \sqrt[n]{|a_n|} = 0?$$

■ The radius is infinite.

## Note 21

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In the Cauchy-Hadamard Theorem, what happens when

$$\limsup \sqrt[n]{|a_n|} = \infty?$$

■ The radius equals to 0.

## Note 22

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What is the key idea in the proof of the Cauchy-Hadamard Theorem?

■ The Root Test.

## Note 23

9223bf629d1a492a892f77c69e4d1cad

What does it mean for a power series to be centered at  $a \neq 0$ ?

■ It is expressed in terms of  $(x - a)$ .

## Note 24

b41b3e3920ae4372a12438b11d262544

Let  $\sum a_n(x - a)^n$  be a power series. Then  $\{\{c2::the\ value\ a\}\}$  is called  $\{\{c1::the\ center\ of\ the\ series.\}\}$

## Note 25

36acf2e7094146dd8a30193845ea7928

Any power series centered at  $a \neq 0$  may be turned into  $\{\{c2::a\}$  series centered at 0 $\}$  by  $\{\{c1::substituting\}$

$$\bar{x} = x - a.$$

}}

## Note 26

872483e057f74ba7adddfbd543a41a8

If  $\sum a_n x^n$  converges absolutely at a point  $x_0$ , then it converges uniformly on  $[-c, c]$ , where  $c = |x_0|$ .

## Note 27

8bfec12c2af34918aa416ea9071592ca

What do we know about  $\sum a_n x^n$  if it converges absolutely at some  $x_0$ ?

■ It converges uniformly on the closed interval.

## Note 28

091856fb98e84114b9b21203616d0e36

If  $\sum a_n x^n$  converges absolutely at a point  $x_0$ , then it converges uniformly on  $[-c, c]$ , where  $c = |x_0|$ . What is the key idea in the proof?

■ The Weierstrass M-Test.

## Note 29

f8f6226e01f74c2f99f8a59913c6b044

If  $\sum a_n x^n$  converges absolutely at a point  $x_0$ , then it converges uniformly on  $[-c, c]$ , where  $c = |x_0|$ . What is used as the sequence of upper bounds in the proof?

■ The values at  $x_0$ .

## Note 30

f8f6226e01f74c2f99f8a59913c6b044

Let  $\sum a_n x^n$  converge absolutely at a point  $x_0$ . Then not only it converges uniformly at  $[-c, c]$ , where  $c = |x_0|$ , but also it converges absolutely at  $[-c, c]$ .

## Note 31

23b23a23d1eb4887a5bb4f0edc237a7d

Let  $R$  be the radius of convergence of  $\sum a_n x^n$ . If  $\sum a_n x^n$  converges absolutely at  $x = R$ , then it converges uniformly on  $[-R, R]$ .

### Note 32

95017b4b22bf4a64a943cc3064449625

Let  $R$  be the radius of convergence of  $\sum a_n x^n$ . Then for any  $r \in [0, R)$ , the series  $\sum a_n x^n$  converges uniformly on  $[-r, r]$ .

### Note 33

57e2d62cb3ce40e48ee6290824fbafeb

Let  $R$  be the radius of convergence of  $\sum a_n x^n$ . Then for any  $r \in [0, R)$ , the series  $\sum a_n x^n$  converges uniformly on  $[-r, r]$ . What is the key idea in the proof?

■ The series converges absolutely at  $x = r$ .

### Note 34

6752618ba2a441c0925b861d87f54ec5

A family of functions is called uniformly bounded if all of the functions can be bounded by the same constant.

### Note 35

6762cdc6398d434b8757d1770c21a186

How is the Abel's test modified for functional series?

■ The series converges uniformly and the monotone sequence is uniformly bounded.

### Note 36

48bf5bd4803c4bbd914aae81b730ee5d

What do we conclude from the Abel's test when it is applied to functional series?

■ The product's series converges uniformly.

### Note 37

4a5eae4d10b5402bb82a217882420fe1

What is the key idea in the proof of the Abel's test for functional series?

■ Identical to numerical series.

### Note 38

8d72659c94aa40eaba69706c0a5d9b08

How is the Dirichlet's test modified for functional series?

- The partial sums are uniformly bounded and the monotone sequence converges uniformly (to 0).

### Note 39

0b66eddc70b42f3bc716d6bfbac7683

What do we conclude from the Dirichlet's test when it is applied to functional series?

- The product's series converges uniformly.

### Note 40

da9daf8da106446d9e2cb20b78b1df0c

What is the key idea in the proof of the Dirichlet's test for functional series?

- Identical to numerical series.

### Note 41

6d76ab5f2fcd4fed9bcd7b3956adf74

What object is considered in the Abel's Theorem?

- Any power series.

### Note 42

1ff48bc8962a4da0b548a6ac2ee95cd5

When do we conclude something from the Abel's Theorem?

- When the power series converges at an end-point of its interval of convergence.

### Note 43

d2056277e2184080838572a8a28bf953

What do we conclude from the Abel's Theorem?

The power series converges uniformly at  $[0, R]$  or  $[-R, 0]$ , respectively.

#### Note 44

d9f5a15d2a074638a3befb9b4ae99500

Why in the Abel's Theorem we only consider a half of the interval  $[-R, R]$ ?

To keep our attention on a single end-point.

#### Note 45

cf9f508a5aca4e4b911bd777d82e2e69

Give an example of a power series that converges at  $x = R$ , but diverges at  $x = -R$ .

Alternating  $\frac{x^n}{n}$ .

#### Note 46

7f4749f737ab47a7874b1cfe9761d482

What is the key idea in the proof of the Abel's Theorem?

Factor out  $\left(\frac{x}{R}\right)^n$  and apply the Abel's Test.

#### Note 47

261655c9501e48aa8c75ba9fa5bd13cb

When applying the Abel's Test in the proof of the Abel's Theorem, which sequence is the “sequentially” uniformly convergent one?

$(a_n R^n)$  view as a functional series.

#### Note 48

08cb33e043454d9eb9e1ad2e379b2228

When applying the Abel's Test in the proof of the Abel's Theorem, which sequence is the monotone, uniformly bounded one?

The factored  $\left(\frac{x}{R}\right)^n$ .

### Note 49

1d25928aac204afd8878944bd82b08ed

How can the theorems related to the uniform convergence of a power series be summarised?

■ The converge is uniform on any compact set within its interval of convergence.

### Note 50

815678a33a2347fcafec7a196cebc701

Any power series  $\{\{c_1\}$  converges uniformly $\}$  on any  $\{\{c_2\}$  compact $\}$  subset of its interval of convergence.

### Note 51

63eed7cd06074230b34d1df61cbe8f5b

Any power series converges uniformly on any compact subset of its interval of convergence. What is the key idea in the proof?

■ The series converges uniformly on the closed interval, that wraps the compact subset.

### Note 52

14710a3a80de483c8122e797da679c72

Let  $\sum a_n x^n$  be a power series. What is the interval of convergence of

$$\sum n a_n x^{n-1} ?$$

■ An interval with the same radius.

### Note 53

29c970bf736841719d13b9711ea4fb75

How does the differentiated power series behave at the end-points of the original interval of convergence?

■ How knows. . .

### Note 54

a855331e13c74a9bb8d2ed157378136a

Give an example of a power series, that converges at one of end-points of its interval of convergence, but who's differentiated series diverges at the same very point.



■ The alternating  $\frac{x^n}{n}$  at  $x = 1$ .

### Note 55

d3327533574d4f658356b768d5e931dc

Let  $\sum a_n x^n$  be a power series. Then  $\sum n a_n x^{n-1}$  has the same radius of convergence. What is the key idea in the proof?

■ Use the Cauchy-Hadamard Theorem.

### Note 56

0ceca97e50034485eadc1030687ebd4

Let  $(a_n)$  and  $(b_n)$  be sequences. Then

$$\limsup a_n b_n = \{\{c1::[Who\ knows\ \dots]\}$$

### Note 57

d54a087cff04470bb2a7e141f206b7a1

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\{\{c2::\lim a_n > 0,\}$  then

$$\limsup a_n b_n = \{\{c1::\lim a_n \cdot \limsup b_n.\}$$

### Note 58

8ed2ade8fae24fde9eedbbe63b8da6a

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\{\{c2::\lim a_n < 0,\}$  then

$$\limsup a_n b_n = \{\{c1::\lim a_n \cdot \liminf b_n.\}$$

### Note 59

d907711a2c0148c885d62302233039cf

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\lim a_n > 0$ , then

$$\limsup a_n b_n = \lim a_n \cdot \limsup b_n.$$

What is the key idea in the proof?

■ Show that it is the largest subsequential limit.

### Note 60

109e7bda00f849b59010ded779de8cd6

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\lim a_n > 0$ , then

$$\limsup a_n b_n = \lim a_n \cdot \limsup b_n.$$

In the proof, how do you apply the product rule to

$$\lim a_{k_j} b_{k_j} ?$$

■ Choose a convergent subsequence of  $(b_{k_j})$ .

### Note 61

408cb435462e45868a152c7eaf22cf60

Any power series is  $\{\{c2::\text{continuous}\}\}$  on  $\{\{c1::\text{its interval of convergence.}\}\}$

### Note 62

750b19a7e31d4dfb84f1e450d92925ae

Any power series is continuous on its interval of convergence. What is the key idea in the proof?

■ Consider a compact subset, that contains a given point.

### Note 63

6658d21f879a4693930642304e0a975c

Any power series is  $\{\{c2::\text{infinitely differentiable}\}\}$  on  $\{\{c1::\text{the interior of its interval of convergence.}\}\}$

### Note 64

c6b5c255a9f94434a2b258c5379db034

Any power series is differentiable on the interior of its interval of convergence. What is the key idea in the proof?

■ Consider a compact subset, that contains a given point.

### Note 65

21c8e2fa0e6e4ba2944f6926e9428b0b

How do you find the derivative of a power series?

■ Differentiate term-by-term.

### Note 66

a2913d0326f04c109c3bf15f49aa239e

Is it possible for a power series to be differentiable at an end-point of its interval of convergence?

■ Yes.

### Note 67

84c4cd40364d4c39af2f53c8cd3886bd

Give an example of a power series that is differentiable at an end-point of its interval of convergence.

■ Alternating  $\frac{x^n}{n^2}$  at  $x = 1$ .

### Note 68

c1da1dd7fa1a439f813932d1ff197f10

Let  $\sum a_n x^n$  converge at an end-point  $x_0 \neq 0$  of its interval of convergence. When is differentiable at this point?

■ Whenever the differentiated series converges at  $x_0$ .

### Note 69

5f268586b8f94f03bdcd24ccc44c984c

What is the interval of convergence of

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots?$$

■  $(-1, 1]$ .

### Note 70

00d179b62ee64669a1c21d4ce1302819

For what values of  $x$  is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

differentiable?

■  $-1 < x < 1$ .

### Note 71

8630c8226f464037a5fa5b0989071505

Give an example of a power series that converges conditionally at both  $x = -1$  and  $x = 1$ .

■  $\frac{1}{n}x^n$  with the sign alternating every two terms.

### Note 72

c392909466044067aa165dcb760bfa64

How do you find the antiderivative of a power series?

■ Antidifferentiate term-by-term.

### Note 73

c37db23d6c9d4e238f6be62803b940ae

Let  $\sum a_n x^n$  be a power series. On which set does

$$\sum \frac{a_n}{n+1} x^{n+1}$$

converge?

■ The original interval of convergence, perhaps including some extra end-points.

### Note 74

1fbab8666cfe4c99a76c37833048a2fc

Let  $\sum a_n x^n$  be a power series. What is the radius of convergence of

$$\sum \frac{a_n}{n+1} x^{n+1} ?$$

■ Same as the original.

### Note 75

b3fa1fa7f12a448e89b81223dd868bf7

Let  $\sum a_n x^n$  be a power series. How does

$$\sum \frac{a_n}{n+1} x^{n+1}$$

behave at the end-points of its interval of convergence?

■ It must converge if the original series does so.

### Note 76

26f4c030950e4d3c85009bc6ced85569

Let  $\sum a_n x^n$  be a power series. Then

$$\sum \frac{a_n}{n+1} x^{n+1}$$

converges at the original interval of convergence. What is the key idea in the proof?

■ The Abel's Test for numerical series.

### Note 77

77211252be1f4def99a19cdec10f109d

On which set a power series is guaranteed to have an antiderivative?

■ The entire interval of convergence.

### Note 78

bcde90c525294c3db8bcaaf6299e84fc

How do you find the value of

$$\sum \frac{n}{2^n} ?$$

■ It is almost the derivative of  $\sum x^n$  at  $x = \frac{1}{2}$ .

### Note 79

b5059ae5d9994c48a996c35e8510f9dc

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

### Note 80

7b06e0dcbf124495b1eb712e00b053bd

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2}.$$

### Note 81

6ff3656419b34e41bfe8b0a4f65deac8

What two series are considered in the theorem about uniqueness of power series representations?

Two power series that are equal on a neighbourhood of their center.

### Note 82

31ec7ac8c7f24b1db2ceb913c8a0bbfb

In which sense are power series representations unique (in the corresponding theorem)?

Any two that are equal must have the same coefficients.

### Note 83

998be995286641e295809a60dce45d7f

Give an example of two power series such that  $\sum a_n x^n = \sum b_n x^n$  around zero, but  $a_n \neq b_n$  for some  $n$ .

This is impossible.

### Note 84

68af15fb6c1f44198ab2ac1268165c0a

If two power series converge to the same function on a neighborhood of their center, they must have the same coefficients.

### Note 85

1021fba1eac945cab927d72d99ca82f

If two power series converge to the same function on a neighborhood of their center, they must have the same coefficients. What is the key idea in the proof?

Consider their derivatives at 0.

### Note 86

44d53e9cbf1b49e3b2ce3910c91ad0cd

Let  $f(x) = \sum a_n x^n$  converge on  $(-R, R)$ . How many such power series satisfy  $f'(x) = f(x)$  and  $f(0) = 1$ ?

Only one.

**Note 87**

67db9a3dab9843f197d07e488b4e27c0

Let  $f(x) = \sum a_n x^n$  converge on  $(-R, R)$ . If  $f' = f$  and  $f(0) = 1$  then

$$a_n = \frac{1}{n!}$$

**Note 88**

8f20edffa46d42369821b7fcbded533c

Let  $f(x) = \sum a_n x^n$  converge on  $(-R, R)$ . If  $f' = f$  and  $f(0) = 1$  then  $a_n = \frac{1}{n!}$ . What is the key idea in the proof?

■ Use the uniqueness of power series representations.

**Note 89**

21136fe33d9e42959f29e35d7bf271f6

What is notably special about the series  $f(x) = \sum \frac{1}{n!} x^n$ ?

■  $f' = f$ .

# Taylor Series

## Note 1

0914695ea5d547e89e22d1d7ab5601c6

What is the power series representation of  $\arctan(x)$ ?

■  $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

## Note 2

ac27df93ef7844dd92ff6795d7a234c3

How do you find the power series representation of  $\arctan(x)$  (assuming no knowledge about its derivatives)?

■ Deduce from the power series expansion of  $\frac{1}{1-x}$ .

## Note 3

785d121c13ad44fa886120397df7525d

How do you produce the power series representation of  $\arctan(x)$  from the power series expansion of  $\frac{1}{1-x}$ ?

■ Substitute  $-x^2$  and take the antiderivative.

## Note 4

39fae994dab84daa9432a0de86cf210a

What is the assumption of the Taylor's Formula?

■ A function has a valid power series representation.

## Note 5

a0f51af429a6460aa52fb799c41d0f17

In the hypothesis of the Taylor's Formula, on which set must the power series representation be valid?

■ A nontrivial interval centered at zero.

## Note 6

9146ee4f363a40dda0907c2520bbf327

What is the statement of the Taylor's Formula?



■ The  $n$ -th coefficient is  $\frac{f^{(n)}(0)}{n!}$ .

### Note 7

c0172988148c43abab348375810bc483

What is the key idea in the proof of the Taylor's Formula?

■ Calculate the  $n$ -th derivatives of the power series.

### Note 8

be1c0cd1dc67489292f987020b36a626

Let  $f$  be infinitely differentiable at a point  $a$ . Then the series

$$\sum \frac{f^{(n)}(a)}{n!} (x - a)^n$$

is called the Taylor series of  $f$  centered at  $a$ .

### Note 9

4b80359f67c848879bb3267507874f83

For which functions is the Taylor series defined?

■ Infinitely differentiable at the center point.

### Note 10

0cb2b10acf66420d8ae9921589cd0c6f

Let  $f$  be infinitely differentiable at a point  $a$ . Then the coefficients  $\frac{f^{(n)}(a)}{n!}$  are called the Taylor coefficients.

### Note 11

34c2ad6abed54d1e9c13f827f9b22afd

A Taylor series centered at zero is also called a Maclaurin series.

### Note 12

5d307a1691c34348a72199f607ac5036

Under which condition does an infinitely differentiable function has a valid Taylor series representation?

■ The sequence of derivatives is uniformly bounded.

### Note 13

6249d19d4a914f009f213ecd73e31b8b

In the sufficient condition of the validity of the Taylor series representation, on which set must the derivatives be uniformly bounded?

- The interior of the interval of convergence.

### Note 14

9df76917e6de4ddea16d1c992f22e0eb

An infinitely differentiable function has a valid Taylor series representation, if its sequence of derivatives is uniformly bounded. What is the key idea in the proof?

- Follows from the Lagrange's Remainder Theorem.

### Note 15

f5641a38607b4ef09fb94eb42636cd0c

Is there anything special about zero, when speaking of the Taylor series?

- Nothing but the notational simplicity.

### Note 16

a559dfe7fa134c90ada27d41069f61ce

On which set of points is the Taylor series of  $f$  always guaranteed to converge to  $f$ ?

- Only at the center point.

### Note 17

dbd63641408d44a783796f4d22f094fd

Give a counterexample against the validity of the Taylor series representation.

- $e^{-\frac{1}{x^2}}$  extended to be continuous at 0.

### Note 18

0efd72deacc64762a35d64e0f4430920

What is notable about  $e^{-\frac{1}{x^2}}$ ?

■ It is *extremely* flat at the origin.

### Note 19

70ed29e038784ac188c2ee5466b96528

Why doesn't  $e^{-\frac{1}{x}}$  work out as a counterexample against the validity of the Taylor series representation?

■ It tends to infinity as  $x \rightarrow 0^-$ .