Sets

Note 1

Intuitively speaking, {{c2::a set}} is {{c1::a collection of objects.}}

Note 2

Intuitively speaking, a set is a collection of objects. {{c2::Those objects}} are referred to as {{c1::the elements of the set.}}

Note 3

Given a set A, we write {{c2::$x \in A$}} if {{c1::x is an element of A.}}

Note 4

Given a set A, we write {{c2::$x \notin A$}} if {{c1::x is not an element of A.}}

Note 5

Given two sets A and B, {{c1::the union}} is written {{c2::$A \cup B$.}}

Note 6

Given two sets A, and B, {{c2::the union}} is {{c3::defined}} by the rule

$$x \in \text{{{c2::}} } A \cup B\text{}} \text{ provided that } \text{{{c1::}} } x \in A \text{ or } x \in B.\text{}}$$

Note 7

Given two sets A and B, {{c1::the intersection}} is written {{c2::$A \cap B$.}}

Note 8

Given two sets A and B, {{c2::the intersection}} is {{c3::defined}} by the rule

$$x \in \text{{{c2::}} } A \cap B\text{}} \text{ provided that } \text{{{c1::}} } x \in A \text{ and } x \in B.\text{}}$$

Note 9

The set of natural numbers is denoted $\mathbf{N}$.

Note 10

$$\mathbf{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

Note 11

The set of integers is denoted $\mathbf{Z}$.

Note 12

$$\mathbf{Z} \stackrel{\text{def}}{=} \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Note 13

The set of rational numbers is denoted $\mathbf{Q}$.

Note 14

$$\mathbf{Q} \stackrel{\text{def}}{=} \left\{ \text{all fractions } \frac{p}{q} \text{ where } p, q \in \mathbf{Z} \text{ and } q \neq 0 \right\}.$$

Note 15

The set of real numbers is denoted $\mathbf{R}$.

Note 16

The empty set is the set that contains no elements.

Note 17

The empty set is denoted $\emptyset$.

Note 18

Two sets A and B are said to be disjoint if $A \cap B = \emptyset$.

Note 19

Given two sets A and B, we say A is a subset of B, or B contains A if every element of A is also an element of B.

Note 20

Given two sets A and B, the inclusion relationship $A \subseteq B$ or $B \supseteq A$ is used to indicate that A is a subset of B.

Note 21

Given two sets A and B, $A = B$ means that $A \subseteq B$ and $B \subseteq A$.

Note 22

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate the set whose elements consist of any element that appears in at least on particular A_n.

Note 23

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \cdots$$

are all equivalent ways to indicate the set whose elements consist of any element that appears in every A_n.

Note 24

Given $A \subseteq \mathbf{R}$, the complement of A refers to the set of all elements of $\mathbf{R}$ not in A.

Note 25

Given $A \subseteq \mathbf{R}$, the complement of A is written A^c.

Note 26

Given $A, B \subseteq \mathbf{R}$,

$$_{\{\{c2::}}(A \cap B)^c {}_{\}\}} = {}_{\{\{c1::}}A^c \cup B^c._{\}\}}$$

«{{c3::De Morgan's Law}}»

Note 27

Given $A, B \subseteq \mathbf{R}$,

$$_{\{\{c2::}}(A \cup B)^c {}_{\}\}} = {}_{\{\{c1::}}A^c \cap B^c._{\}\}}$$

«{{c3::De Morgan's Law}}»

Note 28

What is the key idea in the proof of De Morgan's Laws?

❚ Demonstrate inclusion both ways.

Functions

Note 1

Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B.

Note 2

If f is a function from A to B, we write $f : A \rightarrow B$.

Note 3

Let $f : A \rightarrow B$. Given $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f.

Note 4

Let $f : A \rightarrow B$. The set A is called the domain of f.

Note 5

Let $f : A \rightarrow B$. The set $\{f(x) \mid x \in A\}$ is called the range of f.

Note 6

Given a function f and a subset A of its domain, the set

$$\{f(x) : x \in A\}$$

is called the range of f over the set A.

Note 7

Given a function f and a subset A of its domain, the range of f over A is written $f(A)$.

Note 8

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

❚ Yes.

Note 9

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

❚ No.

Note 10

Given $a, b \in \mathbf{R}$ and $a \leq b$, the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

is called a closed interval.

Note 11

Given $a, b \in \mathbf{R}$ and $a < b$, the set

$$\{x \in \mathbf{R} : a < x < b\}$$

is called an open interval.

Note 12

Given $a, b \in \mathbf{R}$ and $a \leq b$, the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

is written $[a, b]$.

Note 13

Given $a, b \in \mathbf{R}$ and $a \leq b$, the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

is written (a, b).

Note 14

Let $f(x) = x^2$. Find two sets A and B for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

▎ Singletons $\{-1\}$ and $\{1\}$.

Note 15

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Then

$$\{\{c3::f(A \cup B)\}\}\{\{c1:: = \}\}\{\{c2::f(A) \cup f(B).\}\}$$

Note 16

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Then

$$\{\{c3::f(A \cap B)\}\}\{\{c1:: \subseteq \}\}\{\{c2::f(A) \cap f(B).\}\}$$

Note 17

Given $x \in \mathbf{R}$, {{c2::the absolute value of x}} is denoted {{c1::$|x|$.}}

Note 18

Given $x \in \mathbf{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} \{\{c1::x, \}\} & \text{if } \{\{c2::x \geq 0\}\}, \\ \{\{c1:: - x, \}\} & \text{if } \{\{c2::x < 0\}\}. \end{cases}$$

Note 19

Given $a, b \in \mathbf{R}$, $|ab| = {}$ {{c1::$|a| \cdot |b|$ }}.

Note 20

Given $a, b \in \mathbf{R}$, {{c2::$|a + b|$ }}{{c3:: $\leq$ }}{{c1:: $|a| + |b|$ }}.

«{{c4::Triangle inequality}}»

Note 21

Let $f : A \to B$. The function f is one-to-one if

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

Note 22

One-to-one is shortened as 1-1.

Note 23

Let $f : A \to B$. The function f is onto if

$$\text{the range of } f \text{ equals } B.$$

Note 24

Let $f : D \to \mathbf{R}$ and $B \subseteq \mathbf{R}.$ The set

$$\{x \in D : f(x) = B\}$$

is called the preimage of B under the function f.

Note 25

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

❚ Yes.

Note 26

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

❚ Yes.

Logic and Proofs

Note 1

Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem, {{c1::those conditions}} are known {{c2::as the theorem's hypotheses. }}

Note 2

When in {{c3::a proof by contradiction}} {{c2::the contradiction is with the theorem's hypothesis,}} the proof is said to be {{c1::contrapositive.}}

Note 3

Two real number a and b are {{c4::equal}} {{c3::if and only if}} {{c2::for every real number $\epsilon > 0$ it follows that}} {{c1::$|a - b| < \epsilon$.}}

Note 4

Two real number a and b are equal $\impliedby$ for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$. What is the key idea in the proof?

▌ By contradiction.

Note 5

Let {{c3::$S \subseteq \mathbf{N}$.}} If {{c2::S contains 1}} and {{c2::whenever S contains n, it also contains $n + 1$,}} then {{c1::$S = \mathbf{N}$.}}

Note 6

Let $S \subseteq \mathbf{N}$. If S contains 1 and whenever S contains n, it also contains $n + 1$, then $S = \mathbf{N}$. This proposition is the fundamental principle behind {{c1::induction.}}

Note 7

Does an induction argument imply the validity of the infinite case?

> No, it doesn't.

Note 8

Do De Morgan's rules hold for an infinite collection of sets?

> Yes, they do.

Note 9

How Do De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

> They are essentially the same.

The Axiom of Completeness

Note 1

First, $\mathbf{R}$ is {{c1:an ordered field, which contains $\mathbf{Q}$ as a subfield.}}

Note 2

What is the key distinction between $\mathbf{R}$ and $\mathbf{Q}$?

▌ The Axiom of Completeness.

Note 3

{{c1:Every nonempty set of real numbers}} that is {{c2:bounded above}} has {{c3:a least upper bound.}}

«{{c4:Axiom of completeness}}»

Note 4

Given a set $A \subseteq \mathbf{R}$, {{c3:a number $b \in \mathbf{R}$}} such that {{c2:$a \leq b$ for all $a \in A$}} is called {{c1:an upper bound for A.}}

Note 5

A set $A \subseteq \mathbf{R}$ is {{c2:bounded above}} if {{c1:there exists an upper bound for A.}}

Note 6

Given a set $A \subseteq \mathbf{R}$, {{c3:a number $b \in \mathbf{R}$}} such that {{c2:$a \geq b$ for all $a \in A$}} is called {{c1:a lower bound for A.}}

Note 7

A set $A \subseteq \mathbf{R}$ is {{c2:bounded below}} if {{c1:there exists a lower bound for A.}}

Note 8

A set $A \subseteq \mathbf{R}$ is {{c2:bounded}} if {{c1:it is bounded above and below.}}

Note 9

Let $A \subseteq \mathbf{R}$. {{c4: A real number s}} is {{c3: the least upper bound for A}} if

- {{c2: s is an upper bound for A;}}

- {{c1: if b is any upper bound for A, then $s \leq b$.}}

Note 10

{{c2: The least upper bound}} for a set $A \subseteq \mathbf{R}$ is also frequently called {{c1: the supremum for A.}}

Note 11

{{c2: The least upper bound}} for a set $A \subseteq \mathbf{R}$ is denoted {{c1: sup A.}}

Note 12

Let $A \subseteq \mathbf{R}$. A real number s is {{c3: the greatest lower bound for A}} if

- {{c2: s is a lower bound for A;}}

- {{c1: if b is any lower bound for A, then $s \geq b$.}}

Note 13

{{c2: The greatest lower bound}} for a set $A \subseteq \mathbf{R}$ is also frequently called {{c1: the infimum for A.}}

Note 14

{{c2: The greatest lower bound}} for a set $A \subseteq \mathbf{R}$ is denoted {{c1: inf A.}}

Note 15

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple upper bounds?

| Yes.

Note 16

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple least upper bounds?

| No.

Note 17

If s_1 and s_2 are both least upper bounds for a set $A \subseteq \mathbf{R}$, then

{{c1::

$$s_1 = s_2.$$

}}

Note 18

What is the key idea in the proof of uniqueness of the least upper bound?

| $s_1 \leq s_2$ and $s_2 \leq s_1$.

Note 19

Let $A \subseteq \mathbf{R}$. {{c4::A real number b}} is {{c3::a maximum of A}} if b is {{c2::an element of A}} and {{c1::an upper bound for A.}}

Note 20

Let $A \subseteq \mathbf{R}$. A real number b is {{c3::a minimum of A}} if b is {{c2::an element of A}} and {{c1::a lower bound for A.}}

Note 21

Let $A \subseteq \mathbf{R}$ and {{c3::$c \in \mathbf{R}$.}} Then {{c2::$c + A$}} $\overset{\text{def}}{=}$ {{c1::$\{c + a : a \in A\}$}}.

Note 22

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. Then

$$\sup(c + A) = c + \sup A.$$

Note 23

Let $s \in \mathbf{R}$ be an upper bound for a set $A \subseteq \mathbf{R}$. Then $s = \sup A$ if and only if, for every $\epsilon > 0,$ there exists an element a in A satisfying $s - \epsilon < a.$

Note 24

Let $s \in \mathbf{R}$ be a lower bound for a set $A \subseteq \mathbf{R}$. Then $s = \inf A$ if and only if, for every $\epsilon > 0,$ there exists an element a in A satisfying $s + \epsilon > a.$

Note 25

Let $A \subseteq \mathbf{R}$ be nonempty and bounded. How do $\inf A$ and $\sup A$ relate?

$$\inf A \leq \sup A.$$

Note 26

Every nonempty set of real numbers that is bounded below has a greatest lower bound.

Note 27

Every nonempty set of real numbers that is bounded below has a greatest lower bound. What is the key idea in the proof?

Infimum is the supremum for the set of lower bounds.

Note 28

74b4cfb8b91d47b7afc1ae11a4b94ccb

Let $A_1, \ldots, A_n \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max_k \sup A_k.$$

Note 29

c4f28c7f86554b8d83da1931799f4181

Let $A_1, A_2, \ldots$ be a collection of nonempty sets, each of which is bounded above. If $\bigcup_{k=1}^{\infty} A_k$ is bounded above, then

$$\sup\left(\bigcup_{k=1}^{\infty} A_k\right) = \sup_k \sup A_k.$$

Note 30

4c14ddc5fe394879915897bbb199442d

Let $A \subseteq \mathbf{R}$ and $c \in \mathbf{R}$. Then $cA \overset{\text{def}}{=} \{c \cdot a : a \in A\}$.

Note 31

8bdedbcb920f442787c9d475958a65dd

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If $c \geq 0,$ it follows that

$$\sup(cA) = c \cdot \sup A.$$

Note 32

c96971d0b0eb40c39d1773c4f89a5588

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If $c < 0,$ it follows that

$$\sup(cA) = c \cdot \inf A.$$

Note 33

fded05f0fad74578a073f5a838a3a081

Let $A, B \subseteq \mathbf{R}$. Then $A + B \overset{\text{def}}{=} \{a + b : a \in A \text{ and } b \in B\}$.

Note 34

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\{\{c2:: \sup(A + B)\}\} = \{\{c1:: \sup A + \sup B.\}\}$$

Note 35

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\sup(A + B) = \underbrace{\sup A}_{s} + \underbrace{\sup B}_{t}.$$

What is the key idea in the proof?

▌ For $\epsilon > 0$, choose $a > s - \frac{\epsilon}{2}$ and $b > t - \frac{\epsilon}{2}$.

Note 36

If {{c3:: a is an upper bound for A}} and {{c2:: $a \in A$,}} then {{c1::

$$a = \sup A.$$

}}

Note 37

Let $A, B \subseteq \mathbf{R}$ and {{c3:: $\sup A < \sup B$.}} Then there exists {{c2:: $b \in B$ }} that is {{c1:: an upper bound for A.}}

Note 38

If A and B are {{c3:: nonempty, disjoint sets with $A \cup B = \mathbf{R}$}} and {{c2:: $a < b$ for all $a \in A$ and $b \in B$,}} then {{c1:: there exists $c \in \mathbf{R}$ that is an upper bound for A and a lower bound for B.}}

«{{c4::Cut Property}}»

Note 39

What is the key idea in the proof of the Cut Property?

| Use the Axiom of Completeness.

Note 40

The Cut Property implies {{c1: the Axiom of Completeness.}}

Note 41

The Cut Property implies the Axiom of Completeness. What is the key idea in the proof?

| Consider the set of the upper bounds and its complement.

Note 42

Let $A, B \subseteq \mathbf{R}$ be nonempty, bounded above, and satisfy $A \subseteq B$. How do $\sup A$ and $\sup B$ relate?

$$\sup A \leqslant \sup B.$$

Note 43

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded, and let {{c2: $\sup A < \inf B$.}} Then {{c3: there exists a $c \in \mathbf{R}$}} satisfying {{c1:

$$a < c < b$$

for all $a \in A$ and $b \in B$.}}

Note 44

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded, and let $\sup A < \inf B$. Then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$. What is the key idea in the proof?

| Let $c = \frac{1}{2}(\sup A + \inf B)$.

Note 45

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded. If {{c3: there exists a $c \in \mathbf{R}$}} satisfying {{c2: $a \leq c \leq b$ for all $a \in A$ and $b \in B$,}} then {{c1:

$$\sup A \leq \inf B.$$

}}

Note 46

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded. If there exists a $c \in \mathbf{R}$ satisfying $a \leq c \leq b$ for all $a \in A$ and $b \in B$, then

$$\sup A \leq \inf B.$$

What is the key idea in the proof?

▎ c is an upper bound for A and a lower bound for B.

Consequences of Completeness

Note 1

Let $a, b \in \mathbf{R}$ and $a \leqslant b$. Then, a is called the {{c1::left-hand endpoint }} of $[a, b]$.

Note 2

Let $a, b \in \mathbf{R}$ and $a \leqslant b$. Then, b is called the {{c1::right-hand endpoint }} of $[a, b]$.

Note 3

Let $I_1, I_2, I_3, \ldots$ be {{c2::a collection of closed intervals}} such that

{{c1::

$$I_n \supseteq I_{n+1} \quad \forall n \in \mathbf{N}.$$

}} Then, {{c3::$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.}}

«{{c4::Nested Interval Property}}»

Note 4

What is the key idea in the proof of the Nested Interval Property?

> Consider the least upper bound for the set of left-hand endpoints of the intervals.

Note 5

Given {{c3::any number $x \in \mathbf{R}$,}} {{c2::there exists an $n \in \mathbf{N}$}} satisfying

{{c1::

$$n > x.$$

}}

«{{c4::Archimedean Property}}»

Note 6

Given {{c3::any real number $y > 0$,}} {{c2::there exists an $n \in \mathbf{N}$}} satisfying {{c1::

$$\frac{1}{n} < y.$$

}}

Note 7

What is the key idea in the proof of the Archimedean Property?

▌ Show that $\mathbf{N}$ is not bounded above.

Note 8

What is the key idea in proving that $\mathbf{N}$ is not bounded above?

▌ By contradiction + the Axiom of Completeness.

Note 9

What makes a contradiction in proving that $\mathbf{N}$ is not bounded above?

▌ Choose $n > \sup \mathbf{N} - 1$.

Note 10

Given $a \in \mathbf{R}$, {{c1::the smallest integer $\geq a$}} is written {{c2::$\lceil a \rceil$.}}

Note 11

Given $a \in \mathbf{R}$, {{c1::the greatest integer $\leq a$}} is written {{c2::$\lfloor a \rfloor$.}}

Note 12

For every a in $\mathbf{R}$, $\lfloor a \rfloor + 1$ {{c1:: > }}{{c2::a}}.

Note 13

For every a in $\mathbf{R}$, $\lfloor a \rfloor + 1 > a$. What is the key idea in the proof?

> Any number greater than $\lfloor a \rfloor$ is not $\leq a$ by definition of $\lfloor a \rfloor$.

Note 14

For every {{c3::two real numbers a and b with $a < b$,}} {{c2::there exists a rational number r}} satisfying {{c1::

$$a < r < b.$$

}}

«{{c4::Density of $\mathbf{Q}$ in $\mathbf{R}$}}»

Note 15

What is the key idea in the proof of the density of $\mathbf{Q}$ in $\mathbf{R}$?

> Choose $1/n < b - a$ and consider $\dfrac{\lfloor na \rfloor + 1}{n}$.

Note 16

A set B is {{c2::dense in $\mathbf{R}$}} if {{c1::an element of B can be found between any two real numbers $a < b$.}}

Note 17

{{c2::The set of irrational number}} is denoted {{c1::$\mathbf{I}$.}}

Note 18

Is $\mathbf{N}$ dense in $\mathbf{R}$?

> No.

Note 19

Is $\mathbf{Z}$ dense in $\mathbf{R}$?

No.

Note 20

Is $\mathbf{Q}$ dense in $\mathbf{R}$?

Yes.

Note 21

Is $\mathbf{I}$ dense in $\mathbf{R}$?

Yes.

Note 22

What is the key idea in the proof of the existence of square roots in $\mathbf{R}$?

$$\sqrt{x} = \sup \left\{ t \in \mathbf{R} : t^2 < x \right\}.$$

Note 23

Given a real number $x \geq 0$, how do you show that

$$\sqrt{x} = \underbrace{\sup \left\{ t \in \mathbf{R} : t^2 < x \right\}}_{\alpha}?$$

Eliminate the possibilities that $\alpha^2 < x$ and $\alpha^2 > x$.

Note 24

Let $x \geq 0$ be a real number, and let $\alpha = \sup \left\{ t \in \mathbf{R} : t^2 < x \right\}$. How do you eliminate the possibility that $\alpha^2 < x$?

By contradiction, and show that α is not an upper bound.

Note 25

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 < x$, how do you show that α is not an upper bound?

| Choose n such that $(\alpha + 1/n)^2 < x$.

Note 26

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 < x$, how do you chose n such that

$$\left(\alpha + \frac{1}{n}\right)^2 < x?$$

| Expand $(\alpha + 1/n)^2$ and notice that $\frac{1}{n^2} < \frac{1}{n}$.

Note 27

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. How do you eliminate the possibility that $\alpha^2 > x$?

| By contradiction, and show that α is not the **least** upper bound.

Note 28

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 > x$, how do you show that α is not the **least** upper bound?

| Choose n such that $(\alpha - 1/n)^2 > x$.

Note 29

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 > x$, how do you chose n such that

$$\left(\alpha - \frac{1}{n}\right)^2 > x?$$

> Expand $(\alpha - 1/n)^2$ and notice that $\frac{1}{n^2} > 0$.

Note 30

Let $\circ$ be a binary operation. A set B is {{c2::closed under $\circ$}} if {{c1::

$$a \circ b \in B \text{ for all } a, b \in B.$$

}}

Note 31

Let $a, b \in \mathbf{Q}$. Then $a + b \in$ {{c1::$\mathbf{Q}$}}.

Note 32

Let $a, b \in \mathbf{Q}$. Then $ab \in$ {{c1::$\mathbf{Q}$}}.

Note 33

Let $a \in \mathbf{Q}$ and $t \in \mathbf{I}$. Then $a + t \in$ {{c1::$\mathbf{I}$}}.

Note 34

Let $a \in \mathbf{Q}$ and $t \in \mathbf{I}$. Then $at \in$ {{c1::$\mathbf{I}$}} as long as {{c2::$a \neq 0$.}}

Note 35

Let $a, b \in \mathbf{I}$. Then $a + b \in$ {{c1::$\mathbf{R}$}}.

Note 36

Let $a, b \in \mathbf{I}$. Then $ab \in$ {{c1::$\mathbf{R}$}}.

Note 37

$$\bigcap_{n=1}^{\infty} [0, 1/n] = \text{{{c1::} } \{0\} \text{ .}}$$

Note 38

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \text{{{c1::}} \emptyset.}$$

Note 39

What is the key idea in the proof of the density of $\mathbf{I} \in \mathbf{R}$?

> Choose a rational $r \in (a - \sqrt{2}, b - \sqrt{2})$.

Note 40

Given $a \in \mathbf{R}$, the set

$$\{x \in \mathbf{R} : x \geq a\}$$

is called an unbounded closed interval.

Note 41

Given $a \in \mathbf{R}$, the closed unbounded interval

$$\{x \in \mathbf{R} : x \geq a\}$$

is written $[a, \infty)$.

Note 42

Let $I_1, I_2, \ldots$ be a sequence of closed bounded intervals with the property that

$$\bigcap_{n=1}^{N} I_n \neq \emptyset \text{ for all } N \in \mathbf{N}.$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$

Note 43

Let $I_1, I_2, \ldots$ be a sequence of closed bounded intervals with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. What is the key idea in the proof?

> The sequence of $\bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ is a sequence of nested closed intervals.

Cardinality

Note 1

The set A has the same cardinality as B if there exists $f : A \to B$ that is 1-1 and onto.

Note 2

Given two sets A and B, if the set A has the same cardinality as B, we write

$$A \sim B.$$

Note 3

A set A is countable if $\mathbf{N} \sim A$.

Note 4

A set A is uncountable if it is infinite and not countable.

Note 5

Is $\mathbf{Q}$ countable?

❚ Yes.

Note 6

Is $\mathbf{R}$ countable?

❚ No.

Note 7

What is the key idea in the proof that $\mathbf{Q}$ is countable?

Consider sets of fractions $\frac{p}{q}$ that are in lowest terms and satisfy

$$|p| + |q| = n.$$

Note 8

What is the key idea in the proof that $\mathbf{R}$ is uncountable?

▌ By contradiction + the Nested Interval Property.

Note 9

How do you choose a sequence of nested closed intervals in the proof that $\mathbf{R}$ is uncountable? (The proof is by contradiction.)

▌ So that $x_{n+1} \notin I_{n+1}$.

Note 10

Is $\mathbf{I}$ countable?

▌ No.

Note 11

{{c2::The union}} of two {{c3::countable}} sets {{c1::must be countable.}}

Note 12

What is the key idea in the proof that $\mathbf{I}$ is uncountable?

▌ By contradiction and $\mathbf{R} = \mathbf{I} \cup \mathbf{Q}$.

Note 13

If {{c3::$A \subseteq B$}} and {{c2::B is countable,}} then {{c1::A is either countable or finite.}}

Note 14

If $A \subseteq B$ and B is countable, then A is either countable or finite. What is the key idea in the proof?

> Arrange the elements of A in the order of their appearance in an enumeration of B.

Note 15

4f079706552a417c953a9ff8a89459fe

The union of a finite number of countable sets {{c1::must be countable.
}}

Note 16

f797a0a54e9046f780dec06fa2b08745

The union of an infinite number of countable sets {{c1::may not be countable.}}

Note 17

b9c328e78c044cb28eb9a6f178e009f7

The union of a finite number of countable sets must be countable. What is the key idea in the proof?

> By induction.

Note 18

21d32436609f4b9a923d48d13626bb7a

If A_n is a {{c2::countable}} set for each $n \in \mathbf{N}$, then {{c1::$\bigcup_{n=1}^{\infty} A_n$ is countable.}}

Note 19

94a84d1a14eb4212bfd68660babfc2c0

If A_n is a countable set for each $n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable. What is the key idea in the proof?

> Build the table and go over the diagonals.

Note 20

a8c32e05cddd4b4ea275c4872897bc8a

The relation of having the same cardinality is an {{c1::equivalence}} relation.

Note 21

1199795a91f4427e9a2b0821fcb04386

Any {{c3::infinite}} collection of {{c2::disjoint}} open intervals is {{c1::countable.}}

Note 22

Any infinite collection of disjoint open intervals is countable. What is the key idea in the proof?

▌ Identify every interval with a unique rational number.

Note 23

Any infinite collection of disjoint open intervals is countable. How do you identify intervals with rational numbers in the proof?

▌ Use the density of $\mathbf{Q}$ in $\mathbf{R}$.

Note 24

Any infinite collection of disjoint open intervals is countable. How does identifying every interval with a unique rational number proves the statement?

▌ The set of intervals has the same cardinality as an infinite subset of $\mathbf{Q}$, which is countable.

Note 25

{{c1::

$$\{(x, y) : 0 < x, y < 1\}$$

}} is the set of points in the open unit square.

Note 26

Let S be the set of points in the open unit square. Does

$$(0, 1) \sim S \quad ?$$

▌ Yes.

Note 27

Let S be the set of points in the open unit square. What is the key idea in the proof that $(0, 1) \sim S$?

▌ Build 1-1 functions both ways.

Note 28

Let S be the set of points in the open unit square. How do you build a 1-1 function $S \to (0, 1)$?

▌ Use the decimal expansion.

Note 29

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Then B must be finite of countable.

Note 30

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Then B must be finite of countable. What is the key idea in the proof?

▌ Intersections of B with $[1, \infty), [\frac{1}{2}, 1), [\frac{1}{4}, \frac{1}{2}), \ldots$ are all finite.

Note 31

A real number is algebraic if it is the root of a polynomial with integer coefficients.

Note 32

A real number is transcendental if it is not algebraic.

Note 33

Is $\sqrt{2}$ algebraic?

▍ Yes.

Note 34

Is $\sqrt{2} + \sqrt{3}$ transcendental?

▍ No.

Note 35

Is the set of all algebraic numbers countable?

▍ Yes.

Note 36

The set of all algebraic numbers is countable. What is the key idea in the proof?

▌ The set of all polynomials with integer coefficients is countable and every polynomial has a finite number of roots.

Note 37

Is the set of all transcendental numbers countable?

▍ No.

Note 38

The set of all transcendental numbers is uncountable. What is the key idea in the proof?

▍ By contradiction + the set of algebraic numbers is countable.

Note 39

Given two sets A and B,

$$A \setminus B \overset{\text{def}}{=} \{x \in A : x \notin B\}.$$

Note 40

Let $f : D \to \mathbf{R}$ and D be nonempty. Then

$$f(A \cap B) = f(A) \cap f(B),$$

for every $A, B \subseteq D$, if and only if f is 1-1.

Note 41

If there exist functions $X \to Y$ and $Y \to X$ that are both 1-1, then $X \sim Y$.

«Schröder-Bernstein Theorem»

Note 42

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. What is the strategy in the proof?

Partition X and Y into pairs of disjoint subsets, so that the functions are onto on those sets, respectively.

Note 43

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we define

$$A_1 = X \setminus g(Y),$$
$$A_{n+1} = g(f(A_n)).$$

Note 44

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, what do we need to show about the sets $A_1, A_2, \ldots$?

They are pairwise disjoint.

Note 45

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, what do we need to show about the sets $f(A_1), f(A_2), \ldots$?

| They are pairwise disjoint.

Note 46

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, how do you show that the sets $A_1, A_2, \ldots$ are pairwise disjoint?

| By contradiction + use that f and g are 1-1.

Note 47

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. What are the first sets in the disjoint partitions, formed in the proof?

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$

$$(A \subseteq X, \quad B \subseteq Y)$$

Note 48

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we have defined

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$

How do you show that g maps $Y \setminus A$ onto $X \setminus B$?

| Show that $g(Y \setminus B) = X \setminus A$ by the definition of set equality.

Note 49

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we have defined

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$

What's the "tricky" part in showing that $g(Y \setminus B) = X \setminus A$?

| "Shift" A_n's in $Y \setminus A$ by definition of A_n.

Note 50

How can you visualise the proof of the Schröder-Bernstein Theorem?

| Consider the chains, formed by repeated altering applications of f and g. For elements in a chain, starting in X, use f. For any other use g^{-1}.

Cantor's Theorem

Note 1

Let S be the set consisting of all sequences of 0's and 1's. Is S countable or uncountable?

❚ It's uncountable.

Note 2

Let S be the set consisting of all sequences of 0's and 1's. Then S is uncountable. What is the key idea in the proof?

❚ Cantor's Diagonalization Method.

Note 3

Given a set A, {{c2::the power set}} refers to {{c1::the collection of all subsets of A.}}

Note 4

Given a set A, {{c2::its power set}} is denoted {{c1::$P(A)$.}}

Note 5

If A is a finite set with {{c2::n}} elements, then $P(A)$ has {{c1::2^n}} elements.

Note 6

Given {{c5::any set A,}} there {{c3::does not exist}} a function $f : $ {{c2::$A \to P(A)$}} that is {{c1::onto.}}

«{{c4::Cantor's Theorem}}»

Note 7

Given any set A, there does not exist a function $f : A \to P(A)$ that is onto. What is the key idea in the proof?

By contradiction + consider $\{x \in A : x \notin f(x)\}$.

Note 8

Given any set A, there does not exist a function $f : A \to P(A)$ that is onto. Let $B = \{x \in A : x \notin f(x)\}$. What leads to the contradiction in the proof?

If f is onto, $B = f(a)$, but then $a \in B$ and $a \notin B$ both lead to a contradiction.

Note 9

$$P(\mathbf{N}) \sim \text{\{\{c1: }\mathbf{R}.\text{\}\}}$$

Note 10

What is the key idea in the proof that $P(\mathbf{N}) \sim \mathbf{R}$?

$P(\mathbf{N}) \sim$ the set of sequences of 0's and 1's.

Note 11

Let S be the set consisting of all sequences of 0's and 1's. Then $S \sim [0, 1]$. What is the key idea in the proof?

Use the binary representation to build 1-1 functions both ways.

Note 12

Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable?

It's countable.

Note 13

The set of all functions from $\{0, 1\}$ to $\mathbf{N}$ is countable. What is the key idea in the proof?

▌ It's equivalent to the set of pairs of natural numbers.

Note 14
82a052351d304f8fb4fab2c9211f304d

Is the set of all function from $\mathbf{N}$ to $\{0, 1\}$ countable or uncountable?

▌ It's uncountable.

Note 15
bd563dce0b92425ba3aa43f5347f3b9c

The set of all function from $\mathbf{N}$ to $\{0, 1\}$ is uncountable. What is the key idea in the proof?

▌ It's equivalent to $P(\mathbf{N})$.

Note 16
a1ed65c929e64d3084657f55695dd3c6

Given a set B, {{c3::a subset $\mathcal{A}$ of $P(B)$}} is called {{c2::an antichain}} if {{c1::no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$.}}

Note 17
ec400dbf9991441a98efd6ca60656856

Does $P(\mathbf{N})$ contain an uncountable antichain?

▌ Yes.

Note 18
fc00f2238e1045d093490aadf4a5aa94

Provide an example of an uncountable antichain in $P(\mathbf{N})$.

▌
$$\left\{ \{2k - b_k : k \in \mathbf{N}\} : b_1, b_2, \ldots \in 0, 1 \right\}$$