Sets

Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking, {{c2::a set}} is {{c1::a collection of objects.}}

Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects. {{c2::Those objects}} are referred to as {{c1::the elements of the set.}}

Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set A, we write {{c2::$x \in A$}} if {{c1::x is an element of A.}}

Note 4

b25d749749a64c5b90880253d9839da8

Given a set A, we write {{c2::$x \notin A$}} if {{c1::x is not an element of A.}}

Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets A and B, {{c1::the union}} is written {{c2::$A \cup B$.}}

Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets A, and B, {{c2::the union}} is {{c3::defined}} by the rule

$x \in$ {{c2::$A \cup B$}} provided that {{c1::$x \in A$ or $x \in B$.}}

Note 7

8ce7db157931494bbfb6eee706e15efc

Given two sets A and B, {{c1::the intersection}} is written {{c2::$A \cap B$.}}

Note 8

6a277df52de2409a98e48429d69b6d05

Given two sets A and B, {{c2::the intersection}} is {{c3::defined}} by the rule

$x \in$ {{c2::$A \cap B$}} provided that {{c1::$x \in A$ and $x \in B$.}}

Note 9

684951afc378458aa7bd27e67cdc499b

{{c2::The set of natural numbers}} is denoted {{c1::$\mathbf{N}$.}}

Note 10

49d36a026d4b4678ab86fb6103571cce

$$\{\{c2::\mathbf{N}\}\} \stackrel{\text{def}}{=} \left\{\{\{c1::1, 2, 3, \ldots\}\}\right\}.$$

Note 11

797c81e5adb543e1a5d4cc67e64c5e09

{{c2::The set of integers}} is denoted {{c1::$\mathbf{Z}$.}}

Note 12

d3c61bf891744c58b73cef543c6e100d

$$\{\{c2::\mathbf{Z}\}\} \stackrel{\text{def}}{=} \left\{\{\{c1:: \ldots, -2, -1, 0, 1, 2, \ldots\}\}\right\}.$$

Note 13

57f085776972449f8bc14daf5cff6603

{{c2::The set of rational numbers}} is denoted {{c1::$\mathbf{Q}$.}}

Note 14

f7e3370650134607853b41b2b1ecf54b

$$\{\{c3::\mathbf{Q}\}\} \stackrel{\text{def}}{=} \left\{\text{all } \{\{c2::\text{fractions } \frac{p}{q}\}\} \text{ where } \{\{c1::p, q \in \mathbf{Z} \text{ and } q \neq 0\}\}\right\}.$$

Note 15

faeac83cb5b740b6964551c85ad3e35b

{{c2::The set of real numbers}} is denoted {{c1::$\mathbf{R}$.}}

Note 16

6e5da98964d645d09ad6989e85679c74

{{c2::The empty}} set is {{c1::the set that contains no elements.}}

Note 17

206db0a0f3d042e49a9ca532e222201f

{{c2::The empty set}} is denoted {{c1::$\emptyset$.}}

Note 18

2f0448d226db4b71b150acaed349a73b

Two sets A and B are said to be {{c2::disjoint}} if {{c1::$A \cap B = \emptyset$.}}

Note 19

e5d9d365e86640319ca5460ef8c4f05c

Given two sets A and B, we say {{c2::A is a subset of B,}} or {{c2::B contains A}} if {{c1::every element of A is also an element of B.}}

Note 20

c2bd27f1fc0d40e296dceef9c9789556

Given two sets A and B, the {{c3::inclusion}} relationship {{c2::$A \subseteq B$ or $B \supseteq A$}} is used to indicate that {{c1::A is a subset of B.}}

Note 21

333e7c6716af48b7b9962ad803f0732f

Given two sets A and B, {{c2::$A = B$}} means that {{c1::$A \subseteq B$ and $B \subseteq A$.}}

Note 22

74e93b42d46746dc9ec2b54f8366c435

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in at least on particular A_n.}}

Note 23

69e4627a3e7149ef8be05479a2587b41

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in every A_n.}}

Note 24

11a987e10fce4ceea69672f366597729

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} refers to {{c1::the set of all elements of $\mathbf{R}$ not in A.}}

Note 25

8b379552450b4672af82c17476c0ff13

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} is written {{c1::A^c.}}

Note 26

a3459afa53264a7c82d9abd760a0c93e

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2::(A \cap B)^c\}\} = \{\{c1::A^c \cup B^c.\}\}$$

«{{c3::De Morgan's Law}}»

Note 27

c983927aa0304e51949e2f90a2ec2614

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2::(A \cup B)^c\}\} = \{\{c1::A^c \cap B^c.\}\}$$

«{{c3::De Morgan's Law}}»

Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

> Demonstrate inclusion both ways.

Functions

Note 1

18930cfe4e4445779bcec8a2fb53f23c

Given {{c3::two sets A and B,}} {{c2::a function from A to B}} is {{c1::a rule or mapping that takes each element $x \in A$ and associates with it a single element of B.}}

Note 2

dfa898ef047e418fa8dfe9ee9582fd71

{{c1::If f is a function from A to B,}} we write {{c2::$f : A \to B$.}}

Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let $f : A \to B$. Given {{c3::$x \in A$,}} the expression {{c2::$f(x)$}} is used to represent {{c1::the element of B associated with x by f.}}

Note 4

65568f366ca949888310668475dbe570

Let $f : A \to B$. {{c2::The set A}} is called {{c1::the domain of f.}}

Note 5

7870a310786142fa938bcc843ca8e1ae

Let $f : A \to B$. {{c2::The set $\{f(x) \mid x \in A\}$}} is called {{c1::the range of f.}}

Note 6

716c208c9ae849b89ec722aa17f20882

Given a function f and {{c3::a subset A of its domain,}} {{c2::the set

$$\{f(x) : x \in A\}$$

}} is called {{c1::the range of f over the set A.}}

Note 7

24aae21652754fcda1267ac61036a3ea

Given a function f and a subset A of its domain, {{c2::the range of f over A}} is written {{c1::$f(A)$.}}

Note 8

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

Yes.

Note 9

ee665e77ac9a45cf9a15d42549e6f382

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

No.

Note 10

5d2e9d4e1e094e06b37bd87e2c9edff8

Given {{c4::$a, b \in \mathbf{R}$}} and {{c3::$a \leq b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is called {{c1::a closed interval.}}

Note 11

9f383a22fc724f8fa43af5cb65e0cd5a

Given $a, b \in \mathbf{R}$ and {{c3::$a < b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is called {{c1::an open interval.}}

Note 12

3143096eb895471bac4b2d5840d18758

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is written {{c2::$[a, b]$.}}

Note 13

604897f024bd4de78723fe8247290371

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is written {{c2::(a, b).}}

Note 14

a77dc72d26be45c185900ba7ff132b05

Let $f(x) = x^2$. Find two sets A and B for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

> Singletons $\{-1\}$ and $\{1\}$.

Note 15

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Then

$$\text{\{\{c3::}f(A \cup B)\text{\}\}\{\{c1::} = \text{\}\}\{\{c2::}f(A) \cup f(B).\text{\}\}}$$

Note 16

e088ae5ae1f24425a81dac09317978fd

Let $f : D \to \mathbf{R},\ A, B \subseteq D$. Then

$$\text{\{\{c3::}f(A \cap B)\text{\}\}\{\{c1::} \subseteq \text{\}\}\{\{c2::}f(A) \cap f(B).\text{\}\}}$$

Note 17

f951f5a5136248dcb413f59b3271d389

Given $x \in \mathbf{R}$, {{c2::the absolute value of x}} is denoted {{c1::$|x|$.}}

Note 18

624dda908fd64a1cadae2b61c1277c59

Given $x \in \mathbf{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} \text{\{\{c1::}x,\text{\}\}} & \text{if } \text{\{\{c2::}x \geq 0\text{\}\}}, \\ \text{\{\{c1::}-x,\text{\}\}} & \text{if } \text{\{\{c2::}x < 0\text{\}\}}. \end{cases}$$

Note 19

0ab23d0afe1448e397cad330aea55883

Given $a, b \in \mathbf{R}$, $\quad |ab| = \text{\{\{c1::} |a| \cdot |b| \text{\}\}}.$

Note 20

2b51f36fba524365b72001d318791436

Given $a, b \in \mathbf{R}$, $\quad \text{\{\{c2::} |a + b| \text{\}\}\{\{c3::} \leq \text{\}\}\{\{c1::} |a| + |b| \text{\}\}}.$

«{{c4::Triangle inequality}}»

Note 21

4d6e77677e884f9c8ee877b9a32d48b5

Let $f : A \to B$. The function f is {{c2::one-to-one}} if {{c1::

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

}}

Note 22

8616006c8ab04495ad6996ba77fe1b74

{{c2::*One-to-one*}} is shortened as {{c1::1-1.}}

Note 23

56b2bf81daaf419ab1207c6693c981e6

Let $f : A \to B$. The function f is {{c2::onto}} if {{c1::

$$\text{the range of } f \text{ equals } B.$$

}}

Note 24

ccc8a358284a4b1f99f8e4336a2efdb9

Let {{c4::$f : D \to \mathbf{R}$}} and {{c3::$B \subseteq \mathbf{R}$.}} {{c2::The set

$$\{x \in D : f(x) = B\}$$

}} is called {{c1::the preimage of B}} {{c5::under}} the function f.

Note 25

b72f131ae6734bf694fd8f987bb2323d

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

▌ Yes.

Note 26

5b3116f568a34fe2be32f403d7d081d9

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

▌ Yes.

Logic and Proofs

Note 1

a4d52b740f5b494696a5bdc956906cf2

Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem, {{c1::those conditions}} are known {{c2::as the theorem's hypotheses.
}}

Note 2

93f759e32dbf497cb30754e24c5b09f1

When in {{c3::a proof by contradiction}} {{c2::the contradiction is with the theorem's hypothesis,}} the proof is said to be {{c1::contrapositive.
}}

Note 3

1f45350926704df98b0abdf205f4319c

Two real number a and b are {{c4::equal}} {{c3::if and only if}} {{c2::for every real number $\epsilon > 0$ it follows that}} {{c1::$|a - b| < \epsilon$.}}

Note 4

3ef90c9123e64df39ae9cd34271a7dcd

Two real number a and b are equal $\Longleftarrow$ for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$. What is the key idea in the proof?

> By contradiction.

Note 5

aab4bb967d814e87bd85608277093755

Let {{c3::$S \subseteq \mathbf{N}$.}} If {{c2::S contains 1}} and {{c2::whenever S contains n, it also contains $n + 1$,}} then {{c1::$S = \mathbf{N}$.}}

Note 6

3dd92625856f408b9dc93fd36d82588d

Let $S \subseteq \mathbf{N}$. If S contains 1 and whenever S contains n, it also contains $n + 1$, then $S = \mathbf{N}$. This proposition is the fundamental principle behind {{c1::induction.}}

Note 7

40977a19a0d043c985df5676daa9f776

Does an induction argument imply the validity of the infinite case?

> No, it doesn't.

Note 8

91b673c484b442ec92dd47ad0ef95f6c

Do De Morgan's rules hold for an infinite collection of sets?

> Yes, they do.

Note 9

df9aa3b9e0c74da78d7e2a0a65276fcd

How Do De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

> They are essentially the same.

The Axiom of Completeness

Note 1

d7df92f228f64fb28a9e353f0fcb3160

First, $\mathbf{R}$ is {{c1::an ordered field, which contains $\mathbf{Q}$ as a subfield.}}

Note 2

6ac3816effb14ba682f20f91ae42bfdf

What is the key distinction between $\mathbf{R}$ and $\mathbf{Q}$?

> The Axiom of Completeness.

Note 3

7c2ddbcb52224d5cbad5c650d77e8a4f

{{c1::Every nonempty set of real numbers}} that is {{c2::bounded above }} has {{c3::a least upper bound.}}

«{{c4::Axiom of completeness}}»

Note 4

fddbb10e685c4ad49d1af25d241c03c0

Given a set $A \subseteq \mathbf{R}$, {{c3::a number $b \in \mathbf{R}$}} such that {{c2::$a \leq b$ for all $a \in A$}} is called {{c1::an upper bound for A.}}

Note 5

1edcfd8354464c81ab51da0d4f2f2ca4

A set $A \subseteq \mathbf{R}$ is {{c2::bounded above}} if {{c1::there exists an upper bound for A.}}

Note 6

c757fa0c676941b0a4abbccb3a67fb2a

Given a set $A \subseteq \mathbf{R}$, {{c3::a number $b \in \mathbf{R}$}} such that {{c2::$a \geq b$ for all $a \in A$}} is called {{c1::a lower bound for A.}}

Note 7

3c9ba92f774e439dbcfb6c364a88f0ae

A set $A \subseteq \mathbf{R}$ is {{c2::bounded below}} if {{c1::there exists a lower bound for A.}}

Note 8

40f7ae4897174d37952c83f51894ab53

A set $A \subseteq \mathbf{R}$ is {{c2::bounded}} if {{c1::it is bounded above and below. }}

Note 9

9d2391299602497abd4fdfac14c71daa

Let $A \subseteq \mathbf{R}$. {{c4::A real number s}} is {{c3::the least upper bound for A}} if

- {{c2::s is an upper bound for A;}}
- {{c1::if b is any upper bound for A, then $s \leq b$.}}

Note 10

5369939ee0f94abcaf65896355258f0d

{{c2::The least upper bound}} for a set $A \subseteq \mathbf{R}$ is also frequently called {{c1::the supremum for A.}}

Note 11

04884b60726641c6b8d7c2c3479f8b05

{{c2::The least upper bound}} for a set $A \subseteq \mathbf{R}$ is denoted {{c1::sup A.}}

Note 12

afca84537fdd409e97254e6d36d736c3

Let $A \subseteq \mathbf{R}$. A real number s is {{c3::the greatest lower bound for A}} if

- {{c2::s is a lower bound for A;}}
- {{c1::if b is any lower bound for A, then $s \geq b$.}}

Note 13

41c9913ebc524f85be951737dc3e33e8

{{c2::The greatest lower bound}} for a set $A \subseteq \mathbf{R}$ is also frequently called {{c1::the infimum for A.}}

Note 14

7230c3d5f7ef4b62bc1fd6c5b94841f0

{{c2::The greatest lower bound}} for a set $A \subseteq \mathbf{R}$ is denoted {{c1::inf A.
}}

Note 15

51abcbb89d7d486c9177cfc51b6e8721

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple upper bounds?

▌ Yes.

Note 16

1c9d5ad3f35a47b0b12f27639fe4a409

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple least upper bounds?

▌ No.

Note 17

8068979c7a6949fc9af88258008a9801

If s_1 and s_2 are both least upper bounds for a set $A \subseteq \mathbf{R}$, then

{{c1::

$$s_1 = s_2.$$

}}

Note 18

466b264de27a44d3bd21221e39347d2e

What is the key idea in the proof of uniqueness of the least upper bound?

▌ $s_1 \leq s_2$ and $s_2 \leq s_1$.

Note 19

7100e899d7d44ffb89dbc0bac76ffb3f

Let $A \subseteq \mathbf{R}$. {{c4::A real number b}} is {{c3::a maximum of A}} if b is {{c2::an element of A}} and {{c1::an upper bound for A.}}

Note 20

5795e83831c14208a2d2b3dac0e2b139

Let $A \subseteq \mathbf{R}$. A real number b is {{c3::a minimum of A}} if b is {{c2::an element of A}} and {{c1::a lower bound for A.}}

Note 21

2ea41e2869754b64bdb6c221308f0c58

Let $A \subseteq \mathbf{R}$ and {{c3::$c \in \mathbf{R}$.}} Then {{c2::$c + A$}} $\stackrel{\text{def}}{=}$ {{c1:: $\{c + a : a \in A\}$ }}.

Note 22

f7518efeac7b457d86040b99720ad110

Let {{c2::$A \subseteq \mathbf{R}$ be nonempty and bounded above,}} and let {{c4::$c \in \mathbf{R}$.}} Then

$$\text{\{\{c3::} \sup(c + A)\text{\}\}} = \text{\{\{c1::} c + \sup A.\text{\}\}}$$

Note 23

726f73a8cead495fa65f331e49a892ea

Let $s \in \mathbf{R}$ be {{c5::an upper bound}} for a set $A \subseteq \mathbf{R}$. Then {{c3::$s = \sup A$}} {{c4::if and only if,}} {{c2::for every $\epsilon > 0$,}} {{c1::there exists an element a in A satisfying $s - \epsilon < a$.}}

Note 24

4161e1c933ba4349978c94d951259701

Let $s \in \mathbf{R}$ be {{c5::a lower bound}} for a set $A \subseteq \mathbf{R}$. Then {{c3::$s = \inf A$}} {{c4::if and only if,}} {{c2::for every $\epsilon > 0$,}} {{c1::there exists an element a in A satisfying $s + \epsilon > a$.}}

Note 25

0f8f37e55fbe4046a19926f2955f843f

Let $A \subseteq \mathbf{R}$ be nonempty and bounded. How do $\inf A$ and $\sup A$ relate?

> $$\inf A \leq \sup A.$$

Note 26

882685715e2143a0b51a1e43390e1dbc

{{c1::Every nonempty set of real numbers}} that is {{c2::bounded below}} has {{c3::a greatest lower bound.}}

Note 27

87f1451906164b06b7ffe3cd51a2ec7f

Every nonempty set of real numbers that is bounded below has a greatest lower bound. What is the key idea in the proof?

> Infimum is the supremum for the set of lower bounds.

Note 28

74b4cfb8b91d47b7afc1ae11a4b94ccb

Let $A_1, \ldots, A_n \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\{\{c2:: \sup\left(\bigcup_{k=1}^{n} A_k\right) \}\} = \{\{c1:: \max_k \sup A_k.\}\}$$

Note 29

c4f28c7f86554b8d83da1931799f4181

Let $A_1, A_2, \ldots$ be a collection of nonempty sets, each of which is bounded above. If {{c3:: $\bigcup_{k=1}^{\infty} A_k$ is bounded above,}} then

$$\{\{c2:: \sup\left(\bigcup_{k=1}^{\infty} A_k\right) \}\} = \{\{c1:: \sup_k \sup A_k.\}\}$$

Note 30

4c14ddc5fe394879915897bbb199442d

Let $A \subseteq \mathbf{R}$ and $c \in \mathbf{R}$. Then {{c2:: cA}} $\stackrel{\text{def}}{=}$ {{c1:: $\{c \cdot a : a \in A\}$ }}.

Note 31

8bdedbcb920f442787c9d475958a65dd

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If {{c2:: $c \geq 0$,}} it follows that

$$\sup(cA) = \{\{c1:: c \cdot \sup A.\}\}$$

Note 32

c96971d0b0eb40c39d1773c4f89a5588

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. If {{c2:: $c < 0$,}} it follows that

$$\sup(cA) = \{\{c1:: c \cdot \inf A.\}\}$$

Note 33

fded05f0fad74578a073f5a838a3a081

Let $A, B \subseteq \mathbf{R}$. Then {{c2:: $A + B$}} $\stackrel{\text{def}}{=}$ {{c1:: $\{a + b : a \in A \text{ and } b \in B\}$ }}.

Note 34

12d0a51ec08c4d2094ce3e4c6c8b506a

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\{\{c2{::}\ \sup(A+B)\}\} = \{\{c1{::}\ \sup A + \sup B.\}\}$$

Note 35

75698bb156aa40799fc85b1e2419efa2

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded above. Then

$$\sup(A+B) = \underbrace{\sup A}_{s} + \underbrace{\sup B}_{t}.$$

What is the key idea in the proof?

> For $\epsilon > 0$, choose $a > s - \frac{\epsilon}{2}$ and $b > t - \frac{\epsilon}{2}$.

Note 36

a6281cfff0a84b578d8cacdc6ea4779d

If {{c3::a is an upper bound for A}} and {{c2::$a \in A$,}} then {{c1::

$$a = \sup A.$$

}}

Note 37

eb0969a772e442dd8c3f57ed4f8ee1be

Let $A, B \subseteq \mathbf{R}$ and {{c3::$\sup A < \sup B$.}} Then there exists {{c2::$b \in B$ }} that is {{c1::an upper bound for A.}}

Note 38

6b667686c9644d8b9849c735110dac20

If A and B are {{c3::nonempty, disjoint sets with $A \cup B = \mathbf{R}$}} and {{c2::$a < b$ for all $a \in A$ and $b \in B$,}} then {{c1::there exists $c \in \mathbf{R}$ that is an upper bound for A and a lower bound for B.}}

«{{c4::Cut Property}}»

Note 39

545cb11592164c31badc3f21a1e29981

What is the key idea in the proof of the Cut Property?

Use the Axiom of Completeness.

Note 40

39aa54de461b426fbe225601c0663097

The Cut Property implies {{c1::the Axiom of Completeness.}}

Note 41

3a64720500f14d66a66401dd3f133a10

The Cut Property implies the Axiom of Completeness. What is the key idea in the proof?

Consider the set of the upper bounds and its complement.

Note 42

70244652872f4c1fb020d95cfaf88365

Let $A, B \subseteq \mathbf{R}$ be nonempty, bounded above, and satisfy $A \subseteq B$. How do $\sup A$ and $\sup B$ relate?

$$\sup A \leqslant \sup B.$$

Note 43

12dc792f7f78436ea2156c1cc15355de

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded, and let {{c2::$\sup A < \inf B$.}} Then {{c3::there exists a $c \in \mathbf{R}$}} satisfying {{c1::

$$a < c < b$$

for all $a \in A$ and $b \in B$.}}

Note 44

b21502823c8b4f59b97c454a58895487

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded, and let $\sup A < \inf B$. Then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$. What is the key idea in the proof?

Let $c = \frac{1}{2}(\sup A + \inf B)$.

Note 45

12e16f85a81743be8fd6073089decbea

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded. If {{c3::there exists a $c \in \mathbf{R}$}} satisfying {{c2::$a \leq c \leq b$ for all $a \in A$ and $b \in B$,}} then {{c1::

$$\sup A \leq \inf B.$$

}}

Note 46

7bf4ff436d8047b382ad38a9fdbc1c88

Let $A, B \subseteq \mathbf{R}$ be nonempty and bounded. If there exists a $c \in \mathbf{R}$ satisfying $a \leq c \leq b$ for all $a \in A$ and $b \in B$, then

$$\sup A \leq \inf B.$$

What is the key idea in the proof?

> c is an upper bound for A and a lower bound for B.

Consequences of Completeness

Note 1

379096944c2c44f89a8f8438cb155346

Let $a, b \in \mathbf{R}$ and $a \leqslant b$. Then, a is called the {{c1::left-hand endpoint }} of $[a, b]$.

Note 2

8f63313b30ce4a789cfc42d207c26d84

Let $a, b \in \mathbf{R}$ and $a \leqslant b$. Then, b is called the {{c1::right-hand endpoint }} of $[a, b]$.

Note 3

c92f2518ce984069b8bee160141746f6

Let $I_1, I_2, I_3, \ldots$ be {{c2::a collection of closed intervals}} such that {{c1::

$$I_n \supseteq I_{n+1} \quad \forall n \in \mathbf{N}.$$

}} Then, {{c3::$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.}}

«{{c4::Nested Interval Property}}»

Note 4

2d3a37b00fb54f299e2cc7a467f070c2

What is the key idea in the proof of the Nested Interval Property?

> Consider the least upper bound for the set of left-hand endpoints of the intervals.

Note 5

ffceba6cf93e418388528a047155adbe

Given {{c3::any number $x \in \mathbf{R}$,}} {{c2::there exists an $n \in \mathbf{N}$}} satisfying {{c1::

$$n > x.$$

}}

«{{c4::Archimedean Property}}»

Note 6

94598450d3494fed8e15214bc0f27e63

Given {{c3::any real number $y > 0$,}} {{c2::there exists an $n \in \mathbf{N}$}} satisfying {{c1::

$$\frac{1}{n} < y.$$

}}

Note 7

ec854594e00246c7a0bcb81abd56999d

What is the key idea in the proof of the Archimedean Property?

▌ Show that $\mathbf{N}$ is not bounded above.

Note 8

1cc2452b17744237a7e50a513b7a3d43

What is the key idea in proving that $\mathbf{N}$ is not bounded above?

▌ By contradiction + the Axiom of Completeness.

Note 9

f038c4a25eff4227b348bee12cf2d5a6

What makes a contradiction in proving that $\mathbf{N}$ is not bounded above?

▌ Choose $n > \sup \mathbf{N} - 1$.

Note 10

e625796365ee4200bdf4797f7b2648f2

Given $a \in \mathbf{R}$, {{c1::the smallest integer $\geq a$}} is written {{c2::$\lceil a \rceil$.}}

Note 11

e0aee99c343c4404877954f75d91407b

Given $a \in \mathbf{R}$, {{c1::the greatest integer $\leq a$}} is written {{c2::$\lfloor a \rfloor$.}}

Note 12

7d5c2fcf277a492f98b95ef314b9feff

For every a in $\mathbf{R}$, $\lfloor a \rfloor + 1${{c1:: $>$ }}{{c2::a}}.

Note 13

0e4bf87c94b44a01981cec995bc97cdd

For every a in $\mathbf{R}$, $\lfloor a \rfloor + 1 > a$. What is the key idea in the proof?

> Any number greater than $\lfloor a \rfloor$ is not $\leq a$ by definition of $\lfloor a \rfloor$.

Note 14

453c930dc7f1405fbd9ae6bd7546b396

For every {{c3::two real numbers a and b with $a < b$,}} {{c2::there exists a rational number r}} satisfying {{c1::

$$a < r < b.$$

}}

«{{c4::Density of $\mathbf{Q}$ in $\mathbf{R}$}}»

Note 15

ee8f91b90eed4a04b7649cce6ed643ed

What is the key idea in the proof of the density of $\mathbf{Q}$ in $\mathbf{R}$?

> Choose $1/n < b - a$ and consider $\dfrac{\lfloor na \rfloor + 1}{n}$.

Note 16

b1cab1949d9b406295cc27a4506d8848

A set B is {{c2::dense in $\mathbf{R}$}} if {{c1::an element of B can be found between any two real numbers $a < b$.}}

Note 17

3d1287674bbd40aba451b188049186b7

{{c2::The set of irrational number}} is denoted {{c1::$\mathbf{I}$.}}

Note 18

06f50f67b6ad4f5fb6e27378c5e15d8c

Is $\mathbf{N}$ dense in $\mathbf{R}$?

> No.

Note 19

433f404cebfb4e6086fca5eead6644ec

Is $\mathbf{Z}$ dense in $\mathbf{R}$?

▌ No.

Note 20

0d63ae8648dd4d2bae2004b138a20321

Is $\mathbf{Q}$ dense in $\mathbf{R}$?

▌ Yes.

Note 21

16c7710136254657b348bacbd4a6c2e6

Is $\mathbf{I}$ dense in $\mathbf{R}$?

▌ Yes.

Note 22

bd63e13dc03f4b1fa02450e1aee4d306

What is the key idea in the proof of the existence of square roots in $\mathbf{R}$?

$$\sqrt{x} = \sup\left\{t \in \mathbf{R} : t^2 < x\right\}.$$

Note 23

515e4a4736ae4a039f574ba78bf824fe

Given a real number $x \geq 0$, how do you show that

$$\sqrt{x} = \underbrace{\sup\left\{t \in \mathbf{R} : t^2 < x\right\}}_{\alpha}?$$

▌ Eliminate the possibilities that $\alpha^2 < x$ and $\alpha^2 > x$.

Note 24

e9bcf04b9032477ebf9e359f1b05b57d

Let $x \geq 0$ be a real number, and let $\alpha = \sup\left\{t \in \mathbf{R} : t^2 < x\right\}$. How do you eliminate the possibility that $\alpha^2 < x$?

▌ By contradiction, and show that α is not an upper bound.

Note 25

5d03bd0a824d4fb1ad88448d8d530518

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 < x$, how do you show that α is not an upper bound?

> Choose n such that $(\alpha + 1/n)^2 < x$.

Note 26

97cde662927848ed91f93ff91c7881e2

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 < x$, how do you chose n such that

$$\left(\alpha + \frac{1}{n}\right)^2 < x?$$

> Expand $(\alpha + 1/n)^2$ and notice that $\frac{1}{n^2} < \frac{1}{n}$.

Note 27

58056112e69c4c63ab3084525e5ba3b8

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. How do you eliminate the possibility that $\alpha^2 > x$?

> By contradiction, and show that α is not the **least** upper bound.

Note 28

80676249b5f94de9a51427ddf550a858

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 > x$, how do you show that α is not the **least** upper bound?

> Choose n such that $(\alpha - 1/n)^2 > x$.

Note 29

a2e8ec2f07fe45a089d1a36fa2db2076

Let $x \geq 0$ be a real number, and let $\alpha = \sup \{t \in \mathbf{R} : t^2 < x\}$. Assuming $\alpha^2 > x$, how do you chose n such that

$$\left(\alpha - \frac{1}{n}\right)^2 > x?$$

Expand $(\alpha - 1/n)^2$ and notice that $\frac{1}{n^2} > 0$.

Note 30

cb083651e17f4dfba2f203500a1daf6e

Let $\circ$ be a binary operation. A set B is {{c2::closed under $\circ$}} if {{c1::

$$a \circ b \in B \text{ for all } a, b \in B.$$

}}

Note 31

e08de199070b49108df207003392f74e

Let $a, b \in \mathbf{Q}$. Then $a + b \in$ {{c1::$\mathbf{Q}$}}.

Note 32

ae29e77e8a234750872588ce22aae036

Let $a, b \in \mathbf{Q}$. Then $ab \in$ {{c1::$\mathbf{Q}$}}.

Note 33

1e97c0352ebe4d9b8b9e662a38870a56

Let $a \in \mathbf{Q}$ and $t \in \mathbf{I}$. Then $a + t \in$ {{c1::$\mathbf{I}$}}.

Note 34

6089a02bb3424a79a0985a8e1464ef82

Let $a \in \mathbf{Q}$ and $t \in \mathbf{I}$. Then $at \in$ {{c1::$\mathbf{I}$}} as long as {{c2::$a \neq 0$.}}

Note 35

b2ed7798fa124138b24e4859423d7570

Let $a, b \in \mathbf{I}$. Then $a + b \in$ {{c1::$\mathbf{R}$}}.

Note 36

597d2e4ebaac4a96a0f2560b964b3da8

Let $a, b \in \mathbf{I}$. Then $ab \in$ {{c1::$\mathbf{R}$}}.

Note 37

6bc9bfd548bf4ef19cc45bf424a1c298

$$\bigcap_{n=1}^{\infty} [0, 1/n] = \text{{{c1::}} \{0\} .\text{}}}$$

Note 38

a2af66a8fd504a25b8f480c66232ea80

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \text{{{c1::}}\emptyset.\text{}}}$$

Note 39

eba83589c14749909068b51944e1133d

What is the key idea in the proof of the density of $\mathbf{I} \in \mathbf{R}$?

> Choose a rational $r \in (a - \sqrt{2}, b - \sqrt{2})$.

Note 40

6bf6abe0c1cd45bc9c6359e3e324f764

Given $a \in \mathbf{R}$, {{c2::the set

$$\{x \in \mathbf{R} : x \geq a\}$$

}} is called {{c1::an unbounded closed interval.}}

Note 41

60b55d16eafb45caab0e247d61efc17f

Given $a \in \mathbf{R}$, {{c2::the closed unbounded interval

$$\{x \in \mathbf{R} : x \geq a\}$$

}} is written {{c1::$[a, \infty)$.}}

Note 42

bc30145749df40c9bd720035fe0eb326

Let $I_1, I_2, \ldots$ be a sequence of closed bounded intervals with the property that

$\bigcap_{n=1}^{N} I_n${{c2:: $\neq \emptyset$}} for all $N \in \mathbf{N}$.

Then {{c1::$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.}}

Note 43

6992c08bbdbe4f57b2378e33cda019cd

Let $I_1, I_2, \ldots$ be a sequence of closed bounded intervals with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. What is the key idea in the proof?

> The sequence of $\bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ is a sequence of nested closed intervals.

Cardinality

Note 1

c5e9507eadd3494386ca568a39626c59

{{c2::The set A has the same cardinality as B}} if {{c1::there exists $f : A \to B$ that is 1-1 and onto.}}

Note 2

d4ab5f6d316748bb89d719f88ef325f1

Given two sets A and B, if {{c2::the set A has the same cardinality as B,}} we write {{c1::

$$A \sim B.$$

}}

Note 3

a1e546a198024d36a46b0b9ae054781b

A set A is {{c2::countable}} if {{c1::$\mathbf{N} \sim A$.}}

Note 4

1a81ca2429304e448ffcfa1af2800cf8

A set A is {{c2::uncountable}} if {{c1::it is infinite and not countable.}}

Note 5

a12fca23278d404d8d0f1264ba393ba8

Is $\mathbf{Q}$ countable?

> Yes.

Note 6

b534268ef8724eeeab96b7b79ad59c76

Is $\mathbf{R}$ countable?

> No.

Note 7

96665a3169ac4f4e9f3993be9e62a2d5

What is the key idea in the proof that $\mathbf{Q}$ is countable?

> Consider sets of fractions $\frac{p}{q}$ that are in lowest terms and satisfy
>
> $$|p| + |q| = n.$$

Note 8

2fa0660a601245579bf9a542518c4539

What is the key idea in the proof that $\mathbf{R}$ is uncountable?

> By contradiction + the Nested Interval Property.

Note 9

5deb81ae460e414694bc20774379bff3

How do you choose a sequence of nested closed intervals in the proof that $\mathbf{R}$ is uncountable? (The proof is by contradiction.)

> So that $x_{n+1} \notin I_{n+1}$.

Note 10

332b116bbc7b4a3c89d2bff540caf379

Is $\mathbf{I}$ countable?

> No.

Note 11

c16044c879634950a78626257511ee0f

{{c2::The union}} of two {{c3::countable}} sets {{c1::must be countable.}}

Note 12

e07afd17349641668e6f14c926635d3e

What is the key idea in the proof that $\mathbf{I}$ is uncountable?

> By contradiction and $\mathbf{R} = \mathbf{I} \cup \mathbf{Q}$.

Note 13

5bda3078241a4fe3bc1a1ac24d06bf5a

If {{c3::$A \subseteq B$}} and {{c2::B is countable,}} then {{c1::A is either countable or finite.}}

Note 14

6b552d0da71143549a35108597074df9

If $A \subseteq B$ and B is countable, then A is either countable or finite. What is the key idea in the proof?

> Arrange the elements of A in the order of their appearance in an enumeration of B.

Note 15

4f079706552a417c953a9ff8a89459fe

The union of a finite number of countable sets {{c1::must be countable.
}}

Note 16

f797a0a54e9046f780dec06fa2b08745

The union of an infinite number of countable sets {{c1::may not be countable.}}

Note 17

b9c328e78c044cb28eb9a6f178e009f7

The union of a finite number of countable sets must be countable. What is the key idea in the proof?

> By induction.

Note 18

21d32436609f4b9a923d48d13626bb7a

If A_n is a {{c2::countable}} set for each $n \in \mathbf{N}$, then {{c1::$\bigcup_{n=1}^{\infty} A_n$ is countable.}}

Note 19

94a84d1a14eb4212bfd68660babfc2c0

If A_n is a countable set for each $n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable. What is the key idea in the proof?

> Build the table and go over the diagonals.

Note 20

a8c32e05cddd4b4ea275c4872897bc8a

The relation of having the same cardinality is an {{c1::equivalence }} relation.

Note 21

1199795a91f4427e9a2b0821fcb04386

Any {{c3::infinite}} collection of {{c2::disjoint}} open intervals is {{c1::countable.}}

Note 22

291875161b8b4b41a8a6277d8cc24fad

Any infinite collection of disjoint open intervals is countable. What is the key idea in the proof?

> Identify every interval with a unique rational number.

Note 23

1a8331589b1940868a3bc1b0c91c4cf7

Any infinite collection of disjoint open intervals is countable. How do you identify intervals with rational numbers in the proof?

> Use the density of $\mathbf{Q}$ in $\mathbf{R}$.

Note 24

7683d3a14a97407ca3d749db1809a6b1

Any infinite collection of disjoint open intervals is countable. How does identifying every interval with a unique rational number proves the statement?

> The set of intervals has the same cardinality as an infinite subset of $\mathbf{Q}$, which is countable.

Note 25

292de4d7dca94aea8e23bafa6bc7e700

{{c1::

$$\{(x, y) : 0 < x, y < 1\}$$

}} is the set of points in the open unit square.

Note 26

73b05dd1d5e34d799f0d2d1223af7dbe

Let S be the set of points in the open unit square. Does

$$(0, 1) \sim S \quad ?$$

> Yes.

Note 27

89725d2670c840a69014380fd35f6f19

Let S be the set of points in the open unit square. What is the key idea in the proof that $(0, 1) \sim S$?

> Build 1-1 functions both ways.

Note 28

cc4e464c35924ac9b4f152a33012290b

Let S be the set of points in the open unit square. How do you build a 1-1 function $S \to (0, 1)$?

> Use the decimal expansion.

Note 29

b42d868ad9924603b6c27adbf4a0b382

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Then B must be {{c1::finite of countable.}}

Note 30

f57af6a7156c40fd9beefd791dc79aa3

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Then B must be finite of countable. What is the key idea in the proof?

> Intersections of B with $[1, \infty), [\frac{1}{2}, 1), [\frac{1}{4}, \frac{1}{2}), \ldots$ are all finite.

Note 31

96721d17c934420cb09f7fd795f9cfb8

A real number is {{c2::algebraic}} if {{c1::it is the root of a polynomial with integer coefficients.}}

Note 32

031dfba524594e20b08feff2b4dd08c7

A real number is {{c2::transcendental}} if {{c1::it is not algebraic.}}

Note 33

9e23c9868009403e93de695a10f8b396

Is $\sqrt{2}$ algebraic?

Yes.

Note 34

074ea363fa01453f96c59af510b45830

Is $\sqrt{2} + \sqrt{3}$ transcendental?

No.

Note 35

406e0379942d4ed488153e4c4ab4d655

Is the set of all algebraic numbers countable?

Yes.

Note 36

a48cbd4ab0a147efa5a36753de2419b3

The set of all algebraic numbers is countable. What is the key idea in the proof?

The set of all polynomials with integer coefficients is countable and every polynomial has a finite number of roots.

Note 37

ea08d8a514cc459c8dd180fd5086229b

Is the set of all transcendental numbers countable?

No.

Note 38

90f25d91d9824c6db8f11a98c0c0f924

The set of all transcendental numbers is uncountable. What is the key idea in the proof?

By contradiction + the set of algebraic numbers is countable.

Note 39

e05fa107509842c5b1ff2d19ba496322

Given two sets A and B,

$$\{\{c2::A \setminus B\}\} \stackrel{\text{def}}{=} \{\{c1:: \{x \in A : x \notin B\} .\}\}$$

Note 40

18655983a854412cbc80d2f31ec48f52

Let $f : D \to \mathbf{R}$ and D be nonempty. Then

$$\{\{c4::f(A \cap B)\}\} = \{\{c2::f(A) \cap f(B),\}\}$$

for every $A, B \subseteq D$, {{c3::if and only if}} {{c1::f is 1-1.}}

Note 41

e510aa0dce6245bfbb94b3ca7dbaa3b0

If there exist {{c2::functions $X \to Y$ and $Y \to X$}} that {{c1::are both 1-1,}} then {{c4::$X \sim Y$.}}

«{{c3::Schröder-Bernstein Theorem}}»

Note 42

1e32f5e583ee4dd4893be0b2beac0fd4

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. What is the strategy in the proof?

> Partition X and Y into pairs of disjoint subsets, so that the functions are onto on those sets, respectively.

Note 43

b51f8de70e8b4de3a10ed05eecc7db9e

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we define

$$\begin{aligned} A_1 &= \{\{c1::X \setminus g(Y),\}\} \\ A_{n+1} &= \{\{c2::g(f(A_n)).\}\} \end{aligned}$$

Note 44

a76880dee9b3458980e3bee5893cfcb8

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, what do we need to show about the sets $A_1, A_2, \ldots$?

> They are pairwise disjoint.

Note 45

d8ae68d1896d46deaa2128dee4caac77

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, what do we need to show about the sets $f(A_1), f(A_2), \ldots$?

> They are pairwise disjoint.

Note 46

6bfe498654ad41d5b222e6f1cdd75436

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, how do you show that the sets $A_1, A_2, \ldots$ are pairwise disjoint?

> By contradiction + use that f and g are 1-1.

Note 47

613766c151ef44118bef1f3a8fb86b0b

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. What are the first sets in the disjoint partitions, formed in the proof?

> $$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$
>
> $$(A \subseteq X, \quad B \subseteq Y)$$

Note 48

590fcbe354a54b2592824c025c02c701

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we have defined

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$

How do you show that g maps $Y \setminus A$ onto $X \setminus B$?

> Show that $g(Y \setminus B) = X \setminus A$ by the definition of set equality.

Note 49

0fdcbb1f320743f3bef0b9ea1fc12dc4

If there exist functions $f : X \to Y$ and $g : Y \to X$ that are both 1-1, then $X \sim Y$. In the proof, we have defined

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} f(A_n).$$

What's the "tricky" part in showing that $g(Y \setminus B) = X \setminus A$?

> "Shift" A_n's in $Y \setminus A$ by definition of A_n.

Note 50

272e809745ef4b90b8e96a218f48ca4e

How can you visualise the proof of the Schröder-Bernstein Theorem?

> Consider the chains, formed by repeated altering applications of f and g. For elements in a chain, starting in X, use f. For any other use g^{-1}.