Sets

Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking, {{c2::a set}} is {{c1::a collection of objects.}}

Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects. {{c2::Those objects}} are referred to as {{c1::the elements of the set.}}

Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set A, we write {{c2::$x \in A$}} if {{c1::x is an element of A.}}

Note 4

b25d749749a64c5b90880253d9839da8

Given a set A, we write {{c2::$x \notin A$}} if {{c1::x is not an element of A.}}

Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets A and B, {{c1::the union}} is written {{c2::$A \cup B$.}}

Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets A, and B, {{c2::the union}} is {{c3::defined}} by the rule

{{c1::$x \in A \cup B$ provided that $x \in A$ or $x \in B$.}}

Note 7

8ce7db157931494bbfb6eee706e15efc

Given two sets A and B, {{c1::the intersection}} is written {{c2::$A \cap B$.}}

Note 8

6a277df52de2409a98e48429d69b6d05

Given two sets A and B, {{c2::the intersection}} is {{c3::defined}} by the rule

{{c1::$x \in A \cap B$ provided that $x \in A$ and $x \in B$.}}

Note 9

684951afc378458aa7bd27e67cdc499b

{{c2::The set of natural numbers}} is denoted {{c1::$\mathbf{N}$.}}

Note 10

49d36a026d4b4678ab86fb6103571cce

$${{c2::\mathbf{N}}} \overset{\text{def}}{=} \{{{c1::1, 2, 3, \dots}}\}.$$

Note 11

797c81e5adb543e1a5d4cc67e64c5e09

{{c2::The set of integers}} is denoted {{c1::$\mathbf{Z}$.}}

Note 12

d3c61bf891744c58b73cef543c6e100d

$${{c2::\mathbf{Z}}} \overset{\text{def}}{=} \{{{c1:: \dots, -2, -1, 0, 1, 2, \dots}}\}.$$

Note 13

57f085776972449f8bc14daf5cff6603

{{c2::The set of rational numbers}} is denoted {{c1::$\mathbf{Q}$.}}

Note 14

f7e3370650134607853b41b2b1ecf54b

$${{c3::\mathbf{Q}}} \overset{\text{def}}{=} \left\{\text{all } {{c2::\text{fractions } \frac{p}{q}}} \text{ where } {{c1::p, q \in \mathbf{Z} \text{ and } q \neq 0}}\right\}.$$

Note 15

faeac83cb5b740b6964551c85ad3e35b

{{c2::The set of real numbers}} is denoted {{c1::$\mathbf{R}$.}}

Note 16

6e5da98964d645d09ad6989e85679c74

{{c2::The empty}} set is {{c1::the set that contains no elements.}}

Note 17

206db0a0f3d042e49a9ca532e222201f

{{c2::The empty set}} is denoted {{c1::$\emptyset$.}}

Note 18

2f0448d226db4b71b150acaed349a73b

Two sets A and B are said to be {{c2::disjoint}} if {{c1::$A \cap B = \emptyset$.}}

Note 19

e5d9d365e86640319ca5460ef8c4f05c

Given two sets A and B, we say {{c2::A is a subset of B,}} or {{c2::B contains A}} if {{c1::every element of A is also an element of B.}}

Note 20

c2bd27f1fc0d40e296dceef9c9789556

Given two sets A and B, the {{c3::inclusion}} relationship {{c2::$A \subseteq B$ or $B \supseteq A$}} is used to indicate that {{c1::A is a subset of B.}}

Note 21

333e7c6716af48b7b9962ad803f0732f

Given two sets A and B, {{c2::$A = B$}} means that {{c1::$A \subseteq B$ and $B \subseteq A$.}}

Note 22

74e93b42d46746dc9ec2b54f8366c435

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in at least on particular A_n.}}

Note 23

69e4627a3e7149ef8be05479a2587b41

Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \cdots$$

are all equivalent ways to indicate {{c1::the set whose elements consist of any element that appears in every A_n.}}

Note 24

11a987e10fce4ceea69672f366597729

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} refers to {{c1::the set of all elements of $\mathbf{R}$ not in A.}}

Note 25

8b379552450b4672af82c17476c0ff13

Given $A \subseteq \mathbf{R}$, {{c2::the complement of A}} is written {{c1::A^c.}}

Note 26

a3459afa53264a7c82d9abd760a0c93e

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2::(A \cap B)^c\}\} = \{\{c1::A^c \cup B^c.\}\}$$

«{{c3::De Morgan's Law}}»

Note 27

c983927aa0304e51949e2f90a2ec2614

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2::(A \cup B)^c\}\} = \{\{c1::A^c \cap B^c.\}\}$$

«{{c3::De Morgan's Law}}»

Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

> Demonstrate inclusion both ways.

Functions

Note 1

18930cfe4e4445779bcec8a2fb53f23c

Given {{c3::two sets A and B,}} {{c2::a function from A to B}} is {{c1::a rule or mapping that takes each element $x \in A$ and associates with it a single element of B.}}

Note 2

dfa898ef047e418fa8dfe9ee9582fd71

{{c1::If f is a function from A to B,}} we write {{c2::$f : A \to B$.}}

Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let $f : A \to B$. Given {{c3::$x \in A$,}} the expression {{c2::$f(x)$}} is used to represent {{c1::the element of B associated with x by f.}}

Note 4

65568f366ca949888310668475dbe570

Let $f : A \to B$. {{c2::The set A}} is called {{c1::the domain of f.}}

Note 5

7870a310786142fa938bcc843ca8e1ae

Let $f : A \to B$. {{c2::The set $\{f(x) \mid x \in A\}$}} is called {{c1::the range of f.}}

Note 6

716c208c9ae849b89ec722aa17f20882

Given a function f and {{c3::a subset A of its domain,}} {{c2::the set

$$\{f(x) : x \in A\}$$

}} is called {{c1::the range of f over the set A.}}

Note 7

24aae21652754fcda1267ac61036a3ea

Given a function f and a subset A of its domain, {{c2::the range of f over A}} is written {{c2::$f(A)$.}}

Note 8

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R},\ \ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

▌ Yes.

Note 9

ee665e77ac9a45cf9a15d42549e6f382

Let $f : D \to \mathbf{R},\ \ A, B \subseteq D$. Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

▌ No.

Note 10

5d2e9d4e1e094e06b37bd87e2c9edff8

Given {{c4::$a, b \in \mathbf{R}$}} and {{c3::$a \leq b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is called {{c1::a closed interval.}}

Note 11

9f383a22fc724f8fa43af5cb65e0cd5a

Given $a, b \in \mathbf{R}$ and {{c3::$a < b$}}, {{c2::the set

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is called {{c1::an open interval.}}

Note 12

3143096eb895471bac4b2d5840d18758

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

}} is written {{c2::$[a, b]$.}}

Note 13

604897f024bd4de78723fe8247290371

Given $a, b \in \mathbf{R}$ and $a \leq b$, {{c1::the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

}} is written {{c2::(a, b).}}

Note 14

a77dc72d26be45c185900ba7ff132b05

Let $f(x) = x^2$. Find two sets A and B for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

> $[-1, 0]$ and $[0, 1]$.

Note 15

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Then

{{c3::$f(A \cup B)$}}{{c1:: $=$ }}{{c2::$f(A) \cup f(B)$.}}

Note 16

e088ae5ae1f24425a81dac09317978fd

Let $f : D \to \mathbf{R}$, $A, B \subseteq D$. Then

{{c3::$f(A \cap B)$}}{{c1:: $\subseteq$ }}{{c2::$f(A) \cap f(B)$.}}

Note 17

f951f5a5136248dcb413f59b3271d389

Given $x \in \mathbf{R}$, {{c2::the absolute value of x}} is denoted {{c1::$|x|$.}}

Note 18

624dda908fd64a1cadae2b61c1277c59

Given $x \in \mathbf{R}$,

$$|x| \overset{\text{def}}{=} \begin{cases} \text{\{\{c1::}x,\text{\}\}} & \text{if \{\{c2::}x \geq 0\text{\}\}}, \\ \text{\{\{c1::}-x,\text{\}\}} & \text{if \{\{c2::}x < 0\text{\}\}}. \end{cases}$$

Note 19

0ab23d0afe1448e397cad330aea55883

Given $a, b \in \mathbf{R}$, $|ab| =$ {{c1:: $|a| \cdot |b|$ }}.

Note 20

2b51f36fba524365b72001d318791436

Given $a, b \in \mathbf{R}$, {{c2:: $|a + b|$ }}{{c3:: $\leq$ }}{{c1:: $|a| + |b|$ }}.

«{{c4::Triangle inequality}}»

Note 21

4d6e77677e884f9c8ee877b9a32d48b5

Let $f : A \to B$. The function f is {{c2::one-to-one}} if {{c1::

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

}}

Note 22

56b2bf81daaf419ab1207c6693c981e6

Let $f : A \to B$. The function f is {{c2::onto}} if {{c1::

the range of f equals B.

}}

Note 23

ccc8a358284a4b1f99f8e4336a2efdb9

Let {{c4::$f : D \to \mathbf{R}$}} and {{c3::$B \subseteq \mathbf{R}$.}} {{c2::The set

$$\{x \in D : f(x) = B\}$$

}} is called {{c1::the preimage of B under the function f.}}

Note 24

b72f131ae6734bf694fd8f987bb2323d

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

▌ Yes.

Note 25

5b3116f568a34fe2be32f403d7d081d9

Let $f : D \to \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

▌ Yes.

Logic and Proofs

Note 1

93f759e32dbf497cb30754e24c5b09f1

When in {{c3::a proof by contradiction}} {{c2::the contradiction is with the theorem's hypothesis,}} the proof is said to be {{c1::contrapositive.
}}

Note 2

1f45350926704df98b0abdf205f4319c

Two real number a and b are {{c4::equal}} {{c3::if and only if}} {{c2::for every real number $\epsilon > 0$ it follows that}} {{c1::$|a - b| < \epsilon$.}}

Note 3

3ef90c9123e64df39ae9cd34271a7dcd

Two real number a and b are equal $\impliedby$ for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$. What is the key idea in the proof?

> By contradiction.

Note 4

aab4bb967d814e87bd85608277093755

Let {{c3::$S \subseteq \mathbf{N}$.}} If {{c2::S contains 1}} and {{c2::whenever S contains n, it also contains $n + 1$,}} then {{c1::$S = \mathbf{N}$.}}

Note 5

3dd92625856f408b9dc93fd36d82588d

Let $S \subseteq \mathbf{N}$. If S contains 1 and whenever S contains n, it also contains $n + 1$, then $S = \mathbf{N}$. This proposition is the fundamental principle behind {{c1::induction.}}

Note 6

40977a19a0d043c985df5676daa9f776

Does an induction argument imply the validity of the infinite case?

> No, it doesn't.

Note 7

91b673c484b442ec92dd47ad0ef95f6c

Do De Morgan's rules hold for an infinite collection of sets?

> Yes, they do.

Note 8

df9aa3b9e0c74da78d7e2a0a65276fcd

How De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

> They are essentially the same.