Uniform Convergence of a Sequence of Functions

Note 1

1bf1a79b9eba47cf852e1a9c7468c5f7

Let (f_n) be {{c1::a sequence of function on a set A.}} We say {{c3::(f_n) converges pointwise on A to a function f}} if {{c2::for all $x \in A$

$$\Big(f_n(x)\Big) \xrightarrow[n\to\infty]{} f(x).$$

}}

Note 2

f11dc20a5619424cafc97ab1b4d64b5f

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges pointwise on A to f,}} we write

$$\text{\{\{c1::}f_n \to f\text{\}\}} \quad \text{or} \quad \text{\{\{c1::}\lim_{n\to\infty} f_n = f.\text{\}\}}$$

Note 3

6f3f051b9e0741dcbd85037d47c4fd19

Let $f_n(x) = \frac{x^2+nx}{n}$.

$$\lim_{n\to\infty} f_n(x) = \text{\{\{c1::}x.\text{\}\}}$$

Note 4

3c7731c6b70a4c28972a5ea2e88a1e5f

Let $f_n(x) = x^n,\ f_n : [0,1] \to \mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = \text{\{\{c1::}\begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x = 1. \end{cases}\text{\}\}}$$

Note 5

7218c9c8b0f04d4887dc2345da75c6c6

Let (f_n) be a sequence of function on a set A. We say {{c2::(f_n) converges uniformly on A to a function f}} if {{c1::

$$\forall \epsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall n \ge N$$
$$|f_n - f| < \epsilon.$$

}}

Note 6

c80f9e6c9feb486fb69c66c740b4fa7b

Let (f_n) be a sequence of function on a set A. If {{c2::(f_n) converges uniformly on A to f,}} we write {{c1::

$$f_n \rightrightarrows f.$$

}}

Note 7

77ef924775b2453cb303b726f3081917

What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

> The dependence of N on x.

Note 8

42d2e1017eac4382878c195aa5a4c54d

What is the visual behind the uniform convergence of a sequence of functions?

> Eventually every f_n is completely contained in the ϵ-strip.

Note 9

0c853e2f4ed04acf9dae0b00c1a751f3

Which is stronger, uniform or pointwise convergence?

> Uniform convergence is stronger.

Note 10

ed7804cf8d4d48d5b0efb426d130fb52

Uniform convergence implies {{c1::pointwise convergence.}}

Note 11

c9b4c187b4d54a78a9500289aa5899d0

Let (f_n) be a sequence of function on a set A.

{{c2::$f_n \rightrightarrows f$}} {{c3:: $\iff$ }} {{c1:: $\sup |f_n - f| \xrightarrow[n\to\infty]{} 0.$}}

(in terms of sup)

Note 12

1b59f18d7ccb47829cf7b7ea7576318c

Let (f_n) be a sequence of function on a set A. {{c3::Then (f_n) converges uniformly on A}} {{c4::if and only if}}

$$\{\{c1::\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N\}\}$$
$$\{\{c2::|f_n - f_m| < \varepsilon.\}\}$$

Note 13

2b9e4671775a43e9aa4a6b4d581b1658

Let (f_n) be a sequence of function on a set A. Then $f_n \rightrightarrows f$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N$$
$$|f_n - f_m| < \varepsilon.$$

«{{c1::Cauchy Criterion}}»

Note 14

3fa98b94397f4cc2b2d766dd41934f67

What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

> Follows immediately from the definition.

Note 15

f2d15c9af98f4b82956e48ed7df71fc9

What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

> Define a candidate for the limit and prove by definition.

Note 16

1525b27207e74da186a95d7656e895da

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

> Use the pointwise limit.

Note 17

0177dd65112f46c799884e104b39ef76

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

> Due to the Cauchy Criterion for sequences.

Note 18

a1ccae80d31b4f38a4fc876e1ffe4ae7

In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have $f_n \to f$. How do you show that $f_n \rightrightarrows f$?

> Take the limit of the inequality from the Cauchy Criterion.

Note 19

baab958475694fc08316e2031a57fa58

Let $f_n \to f$ on a set A and $c \in A$. If {{c3::the convergence is uniform }} and {{c2::all f_n are continuous at c,}} then {{c1::f is continuous at c.}}

Note 20

a026cf3ddb2f4d5b9a94b36b2bc20ef9

Let $f_n \to f$ on a set A and $c \in A$. If the convergence is uniform and all f_n are continuous at c, then f is continuous at c.

«{{c1::Continuous Limit Theorem}}»

Note 21

5fd08fca82504ff0af82d320da351ff7

What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

> Triple triangle inequality after adding and subtracting f_N.

Note 22

06425162bee447479d3a4f5c71c9cf2a

Let $f_n \to f$ on a set A and $c \in A$. If {{c1::the convergence is uniform and all f_n are continuous at c,}} then

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \text{\{\{c2::} \lim_{n \to \infty} \lim_{x \to c} f_n(x). \text{\}\}}$$

Note 23

05371b5a401f4756bc04fc154476e2c4

Let $f_n \to f$ on a set A. If each f_n is continuous, but f is discontinuous, then {{c1::the convergence is not uniform.}}

Note 24

a5ee2f3836bd4545afde8c2d7ecda40e

Give an example of a sequence of functions $f_n \to f$ such that

- each f_n is continuous almost everywhere; and
- f is nowhere continuous.

> Step-by-step construction of the Dirichlet's function.

Note 25

81c5e1a2081241d1973bb2cacde92627

Assume $f_n \to f$ on a set A and each f_n is uniformly continuous. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is uniformly continuous.}}

Note 26

f819f1c60074468ba1e718298059ade4

Assume $f_n \to f$ on a set A and each f_n is bounded. If {{c2::$f_n \rightrightarrows f$,}} then {{c1::f is bounded.}}

Note 27

b1fded6e729d40ba99a9d087781866dd

Assume $f_n \to f$ on a set A and each f_n has a finite number of discontinuities. If $f_n \rightrightarrows f$, then {{c1::f has at most a countable number of discontinuities.}}

Note 28

a010908ba95d473ea734442288757314

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If {{c2::f is discontinuous at c,}} then {{c1::all f_n are eventually discontinuous at c.}}

Note 29

5ca0ebc56cc947d1bc6a5ed00cd1617b

Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If f is discontinuous at c, then all f_n are eventually discontinuous at c. What is the key idea in the proof?

▌ By contradiction + choose a subsequence continuous at c.

Note 30

4c8d50b955be4fa0a3ba792c5699174f

Let f be {{c2::continuous}} on all of $\mathbf{R}$. Then $f(x + \frac{1}{n})$ {{c1::converges to f.}}

Note 31

59f59d25a40a4e72afdd62a2dd24bd13

Let f be {{c2::uniformly continuous}} on all of $\mathbf{R}$. Then $f(x + \frac{1}{n})$ {{c1::converges uniformly to f.}}

Uniform Convergence and Differentiation

Note 1

37f46dbb09f54423a835e842d402ee19

What sequence is considered in the Differentiable Limit Theorem?

> A sequence of differentiable functions that converges pointwise on a closed interval.

Note 2

19574e41800e43678628e78581f801cc

When applying the Differentiable Limit Theorem, is it necessary for the limit to be differentiable?

> No, this is one of the implications.

Note 3

5ef400e26d2541e589faa672492059bf

When do we conclude something form the Differentiable Limit Theorem?

> When the derivatives converge uniformly.

Note 4

f7da48c586d2457baad72d900c07defd

What do we conclude from The Differentiable Limit Theorem?

> The limit f is differentiable and $f' = \lim f_n'$.

Note 5

61acf9aeed834980a9dbaa77746b89e0

Let $f_n \to f$ on $[a, b]$ and each f_n is differentiable. What do we know about f if $f_n' \to g$?

> Nothing special.

Note 6

63a1ccb4818a4cd281f9b4d9513500a0

Let $f_n \to f$ on $[a, b]$ and each f_n is differentiable. What do we know about f if $f_n' \rightrightarrows g$?

> f is differentiable and $f' = g$.

Note 7

a720c08c553f46a0b0423c46f4c19a2e

What is the key idea in the proof of the Differentiable Limit Theorem?

> Rewrite the limit's derivative by definition.

Note 8

31222913007d4ceda945e1a21642c876

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right| ?$$

> Expand it using the triple triangle inequality involving f_N.

Note 9

d239aa3eedd346a69139a5a8b1d94ce7

In the proof of the Differentiable Limit Theorem, how do you choose N?

> By the Cauchy Criterion for $f_n' \rightrightarrows g$.

Note 10

70bbcff5bceb49c7b0abb25a8ab9be35

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| f_N'(x) - g(x) \right| ?$$

> Take the limit of the inequality from the Cauchy Criterion.

Note 11

ee6b7ee23d0a48a8a32afe978be50a7d

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left| \frac{f_N(x+h) - f_N(x)}{h} - f_N'(x) \right| ?$$

▌ Pick δ by the definition of differentiability of f_N.

Note 12

cdb10b03a9254c5abfe796106c1d3e9b

In the proof of the Differentiable Limit Theorem, how do you find an upper bound for

$$\left|\frac{f(x+h)-f(x)}{h}-\frac{f_N(x+h)-f_N(x)}{h}\right|?$$

▌ The Mean Value Theorem for $f_N - f_m$ and make $m \to \infty$.

Note 13

b4b2753226ff4d839269bbf795c02301

Let (f_n) be {{c4::a sequence of differentiable functions on $[a, b]$}} and {{c3::(f'_n) converge uniformly.}} If {{c2::$\lim f_n(x_0)$ exists for some x_0,}} then {{c1::(f_n) converges uniformly.}}

Note 14

8c542d7e30524e129805ce26973b0925

How can we weaken the hypothesis of the Differentiable Limit Theorem?

▌ (f_n) converges at a single point.

Series of Functions

Note 1

b2a393ededc84241bb594417273dea7e

Let (f_n) be {{c3::a sequence of functions on a set A.}} {{c2::A functional series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} f_n(x).$$

}}

Note 2

6291bcd4e0274102bfe4090eebac24ef

Let (f_n) be a sequence of functions on a set A. We say $\sum_n f_n(x)$ {{c2::converges pointwise on A to a function $f(x)$}} if {{c1::the sequence of partial sums converges pointwise on A to f.}}

Note 3

084d4603478b4dc48c0d1837ff30dfd8

Let (f_n) be a sequence of functions on a set A. If {{c2::$\sum_n f_n(x)$ converges pointwise to $f(x)$,}} we write {{c1::

$$f(x) = \sum_n f_n(x).$$

}}

Note 4

2922cd6ac8ff42fabe5bc630fa320169

Let (f_n) be a sequence of functions on a set A. We say $\sum f_n(x)$ {{c2::converges uniformly on A to a function $f(x)$}} if {{c1::the sequence of partial sums converges uniformly on A to f.}}

Note 5

2b28ab51bc7f45ca934cc405e7de388f

Let $\sum_n f_n(x)$ be a functional series. {{c1::A series

$$\sum_{n=k+1}^{\infty} f_n(x) \quad \text{for } k \in \mathbf{N},$$

}} is called {{c2::a tail of $\sum_n f_n(x)$.}}

Note 6

d633b0c9c968402aba5285afb115d682

A series $\sum_n f_n(x)$ {{c2::converges pointwise}} {{c3::only if}} {{c1::its tail converges pointwise to 0.}}

(in terms of the tail)

Note 7

16325daa37b14ddebc3939e1d2ea063b

A series $\sum_n f_n(x)$ {{c2::converges uniformly}} {{c3::only if}} {{c1::its tail converges uniformly to 0.}}

(in terms of the tail)

Note 8

891381b2ecd44c2cb160d114479f0b20

A series $\sum_n f_n(x)$ {{c2::converges pointwise}} {{c3::only if}} {{c1::$f_n \to 0$.
}}

Note 9

767a398cce7c40b781b0c39db5f9b9ac

A series $\sum_n f_n(x)$ {{c2::converges uniformly}} {{c3::only if}} {{c1::$f_n \rightrightarrows 0$.
}}

Note 10

c0a25e35d11c4560a26e2e463a31f725

What series is considered in the Term-by-term Continuity Theorem?

> A series of continuous functions.

Note 11

55e76f7381cf476bb7c32155d099bf7c

When do we conclude something from the Term-by-term Continuity Theorem?

> When the functional series converges uniformly.

Note 12

84af86f380cf48048b8e6b2c91e25d6c

What do we conclude from the Term-by-term Continuity Theorem when the series only converges pointwise?

> Nothing.

Note 13

a2c89255016b4abebcc0733f8178fdef

What do we conclude from the Term-by-term Continuity Theorem?

> The series' sum is continuous.

Note 14

9a06615f719646bb8e4bde3a605344f5

What series is considered in the Term-by-term Differentiability Theorem?

> A series of differentiable functions that converges pointwise on a closed interval.

Note 15

fa6705a7ca6141eeb7056368500bbdb0

When do we conclude something from the Term-by-term Differentiability Theorem?

> The derivatives' series converge uniformly.

Note 16

50a4a0c1c82c4129a14c9af763976811

What do we conclude form the Term-by-term Differentiability Theorem?

> $\sum f_n$ is differentiable and $(\sum f_n)' = \sum f'_n$.

Note 17

296676411bf5475eacdde73dc1c2b008

What series is considered in the Weierstrass M-Test?

> A series of bounded functions.

Note 18

5c393b177b724cf69790bafcf0ff7b23

When do we conclude something from the Weierstrass M-Test?

> When the series of “absolute” bounds converges.

Note 19

f964f11937374d53be121d3893daeef6

Which bounds are considered in the Weierstrass M-Test?

> The sequence of the functions’ “absolute” upper bounds.

Note 20

c48d5e20f5d24ca58a0c3bd71ab7b256

What do we conclude from the Weierstrass M-Test?

> The functional series converges uniformly.

Note 21

2f9827fda17c4670b0d2bd4728303ae4

What is the key idea in the proof of the Weierstrass M-Test?

> It follows from the Cauchy Criterion.

Note 22

f0d47c16fb4f4ab888dbaa2d8d17ef7a

What is the second implication of the Weierstrass M-Test?

> The series converges absolutely.

Note 23

5803d1bfa65b4180921e6b1443015177

Why does the Weierstrass M-Test implies absolute convergence?

> Absolute values have the same upper bounds.

Power Series

Note 1

575572e782e64317ba8228d5791138da

What is a power series (intuitively)?

> An infinite polynomial.

Note 2

3cd19400150446d68e6df4a87977e765

{{c2::A power series}} is {{c1::a series of the form

$$\sum_{n=1}^{\infty} a_n x^n.$$

}}

Note 3

59c245eadd1f4c7c84641a4a81a6cf9c

A power series is {{c2::a generalisation}} of {{c1::a polynomial.}}

Note 4

034c6da627e9416d94fe7048441924c4

If $\sum a_n x^n$ {{c2::converges at some point $x_0 \in \mathbf{R}$}} then {{c1::it converges absolutely for any x satisfying $|x| < |x_0|$.}}

Note 5

fed41f842cd54bb1b712f694b52659f9

If $\sum a_n x^n$ converges at some point $x_0 \in \mathbf{R}$ then it converges absolutely for any x satisfying $|x| < |x_0|$. What is the key idea in the proof?

> Make a geometric series by factoring out $\left|\frac{x}{x_0}\right|^n$.

Note 6

59b428fc86c24ff8aff670ff3a284435

If $\sum a_n x^n$ converges at some point $x_0 \in \mathbf{R}$ then it converges absolutely for any x satisfying $|x| < |x_0|$. In the proof, how do you turn $\sum |a_n x_0^n| \left|\frac{x}{x_0}\right|^n$ into a geometric series?

> $(a_n x^n)$ is bounded + the Comparison Test.

Note 7

573b21be0d10467d913040dfe4d493bb

Which form may be taken by the set of points for which $\sum a_n x^n$ converges?

> An interval centered around 0.

Note 8

ecdb3ab6a5bd4e23bcd67794066ab7c9

The set of points for which $\sum a_n x^n$ converges is always an interval centered around 0. What is the key idea in the proof?

> Use the "Interior Convergence" theorem.

Note 9

21ae4818657c4e16b4ef4b2585bc3c18

How is the set of points for which $\sum a_n x^n$ converges called?

> The interval of convergence.

Note 10

a11ccad617764a25a049ee310707b122

{{c2::The radius of convergence}} of $\sum a_n x^n$ is {{c1::the half length of its interval of convergence.}}

Note 11

e3918912ac244859ab2293dcbac39594

How does $\sum a_n x^n$ behave at the endpoints of its interval of convergence?

> Who knows...