Definition and Examples

Note 1

9080791fc8754b0bb88c381c10acbdfc

Let G be a group. If {{c2::H is a subgroup of G}} we shall write {{c1::

$$H \leq G.$$

}}

Note 2

66e7f23728af4c9d8839d172e59d716a

Let G be a group and $H \leq G$. We shall denote the operation for H by {{c1::the same symbol as the operation for G.}}

Note 3

e76ada2ee6da4b5fb71966e9f7ce3ded

Let G be a group. If {{c2::$H \leq G$ and $H \neq G$}} we shall write {{c1::$H < G$.}}

Note 4

1d28c11c52c84bd0b639505598bb1dce

If H is a subgroup of G then any equation in the subgroup H may also be viewed as {{c1::an equation in the group G.}}

Note 5

8f5b7659618844608231416 45b5ea08b

Let G be a group and $H \leq G$. What is the identity of H?

> The identity of G.

Note 6

7c122a5400f64eba9a76438c1ff296ee

Let G be a group and $H \leq G$. The identity of H is the identity of G. What is the key idea in the proof?

> The identity is unique and it is the identity of G.

Note 7

83cba804764b43e2baf282ffee513694

Let G be a group. What is the minimal subgroup of G?

> The singleton $\{1\}$.

Note 8

d40df9f46a6b43ecb3fea9b5b37e5b1c

Let G be a group. What is the element that any subgroup of G must contain?

> The identity of G.

Note 9

bc95ad6358814213a6fff2eb0cfd544b

Let G be a group and $H \leq G$. What is the inverse of an element x in H?

> The inverse of x in G.

Note 10

be9f1756cf3449e8a6718069fd4aedf5

Let G be a group and $H \leq G$. Why is the notation x^{-1} unambiguous?

> The inverse in H is the same as the inverse in G.

Note 11

8aabd93df8a5437eb3e50c3e0d438381

Let G be a group. {{c2::The subgroup $\{1\}$ of G}} is called {{c1::the trivial subgroup.}}

Note 12

eb859714e1f34f4db3dc35755f562945

Let G be a group. {{c2::The trivial subgroup}} is denoted by {{c1::1.}}

Note 13

74ab3ceb9a36420fb53dc2b3a22eb5f2

Which subgroups does any group have?

> The trivial subgroup and the group itself.

Note 14

5683ff4198a74e9d988f501c925d85ad

If H is a subgroup of G and K is a subgroup of H, then {{c1::K is a subgroup of G.}}

Note 15

50a07cbb14aa4bed8866efcbedb0be4d

Which object is considered in the Subgroup Criterion?

> Any subset of a group.

Note 16

840038893a3642a18c3e43c4e89aed17

What are the conditions of the Subgroup Criterion?

> The subset is nonempty and closed under $(x, y) \mapsto x \cdot y^{-1}$.

Note 17

71291d04ca2941fca2fc08759d8fd302

What is the special case considered in the Subgroup Criterion?

> The subset is finite.

Note 18

a1e69be09e78402d989b3805b3dfc54f

What are the conditions of the Subgroup Criterion for a finite subset?

> The subset is nonempty and closed under the operation.

Note 19

5bcd55a73e184bcd9bcc32f1ee47da2e

What is the key idea in the proof of the Subgroup Criterion for a finite subset?

> Any element's inverse is it's n-th power.

Note 20

0e1ccaae016c4900ac96b733fb9e1764

Why is the set of 2-cycles in S_n not a subgroup of S_n?

> It does not contain the identity.

Note 21

587390d0450f4681a66bcbc8c0d5889c

Why is the set of reflection in D_{2n} not a subgroup of D_{2n}?

▌ It does not contain the identity.

Note 22

fc87d2283cb546708502ce325e326258

Why is the set of reflection in D_{2n} together with 1 not a subgroup of D_{2n}?

▌ Two distinct reflections induce a rotation.

Note 23

24b90e714649459ba38e6b40f07f6b2a

Is $\{1, r^2, s, sr^2\}$ a subgroup of D_8?

▌ Yes.

Note 24

eac99978715a4ec894b296f8e1ee52f3

Is $\{1, r, s, sr\}$ a subgroup of D_8?

▌ No.

Note 25

64ea968bdce94647b6fb2c351a60f2a2

Is $\{1, r^2, sr, sr^3\}$ a subgroup of D_8?

▌ Yes.

Note 26

678b87f890ac4d8da5be6a78cb619358

Is $\{1, r, r^2\}$ a subgroup of D_8?

▌ No.

Note 27

e036f3cc7667461b98e50e94ff3a8c80

Is $\{1, r, r^2, r^3\}$ a subgroup of D_8?

▌ Yes.

Note 28

209944ca7a524af3be44b398de974c2d

Give an example of a group and its infinite subset that is closed under the operations, but is not a subgroup of the original group.

> Positive integers under addition.

Note 29

547363a46106478187c20c5cbb868461

For what groups is the notion of the torsion subgroup introduced?

> For abelian groups.

Note 30

d29b9ffdb46c4c909fbfb2a438abb0a0

What is the torsion subgroup of an abelian group?

> The set of all the elements of a finite order.

Note 31

b2a854579339471d8ae41776f1661f29

Let G be an abelian group. What is the name of the set

$$\{g \in G : |g| < \infty\}\,?$$

> The torsion subgroup of G.

Note 32

2a685e6476b94b9eac539a17441574ef

Why is the notion of the torsion subgroup introduced only for abelian groups?

> For non-abelian groups the set is not guaranteed to form a subgroup.

Note 33

22a771a961c3498f88a030fabf778797

Give an example of a non-abelian group, who's "torsion subgroup" is not actually a subgroup.

> $GL_3(\mathbb{R})$

Note 34

f4edb9436c094103b0b9b82019185296

Give an example of two elements a, b in $GL_3(\mathbb{R})$ such that

$$|a|\,,|b| < \infty \quad \text{and} \quad |ab| = \infty.$$

$$
\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Note 35

f2c41a74f8a04bb892b471915e533055

What is the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$?

The set of elements who's first component is 0.

Note 36

4df6b5997d30483fb469565c89630322

When is the union of two subgroups also a subgroup?

If and only if one of the subgroups is a subset of the other.

Note 37

b523dcaec101461e902f34191451e112

When is the union of an infinite number of subgroups also a subgroup?

It depends.

Note 38

60129b39ceab4468915a6d2237915c1a

Let H and K be subgroups of G and $H \subseteq K$. What do we know about $H \cup K$?

It is a subgroup of G.

Note 39

791301f78ecf4800a13e3a0299c57028

Let H and K be subgroups of G. If $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$. What is the key idea in the proof?

By contradiction.

Note 40

cc8decdf60194667b3b27ff0941c9fc0

What is the special linear group?

▌ The set of square matrices who's determinant is 1.

Note 41

1e420dd97e1942b3b7bc70d71fc0953e

{{c2::The special linear group of $n \times n$ matrices over a field F}} is denoted {{c1::$SL_n(F)$.}}

Note 42

941aeeb281da4b009ffdc95864eddb3b

When is the intersection of two subgroups also a subgroup?

▌ Always.

Note 43

887cf7600d994fcd9662e35fc9719c62

When is the intersection of an infinite number of subgroups also a subgroup?

▌ Always.

Note 44

3bdd7a0f0e044c6b9c3c1811d4478f10

Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of G. Then {{c1::$\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G.}}

Centralizers and Normalizers, Stabilizers and Kernels

Note 1

93de251693e74655a5752529379e7081

For what do we define centralizers in groups?

> For nonempty subsets of the group.

Note 2

b46233e067ea4c24b38af57081ef1db3

Let G be a group and A be a nonempty subset of G. {{c1::The set

$$\{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$$

}} is called {{c2::the centralizer of A in G.}}

Note 3

3c2adb104b55494a8a248b4e6cf72980

Let G be a group and A be a nonempty subset of G. {{c2::The centralizer of A in G}} is denoted {{c1::

$$C_G(A) \, .$$

}}

Note 4

aeea9d02d1a8429ab94927313c1e2194

How can centralizers be redefined in terms of commutativity?

> As the set of all the elements that commute with every element of the subset.

Note 5

ae3968a709ba423b91c84596e63977c7

How do we call the set of elements of a group G that commute with every element of a given subset A of G?

> The centralizer of A in G.

Note 6

588fd51b4281485c87a74faa9ddbf8f5

Let G be a group and A be a nonempty subset of G. The centralizer of A in G forms {{c1::a subgroup of G.}}

Note 7

18e60ffefbe647a6aa9b7a9feeb58ef1

Let G be a group and A be a nonempty subset of G. When is the centralizer of A in G a subgroup of G?

▌ Always.

Note 8

23eee6bafc20447987eaab729108324e

Let G be a group and A be a nonempty subset of G. In the special case when $A = \{a\}$ we shall write {{c1::simply $C_G(a)$}} instead of {{c2::$C_G(\{a\})$.}}

Note 9

92e1f52031224232bf8ac69f4014862c

Let G be a group and $a \in G$. Then

$$\text{\{\{c1::}\langle a \rangle\text{\}\}} \subseteq C_G(a) \, .$$

Note 10

41ad6753d3624f379c6b0e82c31be987

Let G be a group and $a \in G$. Then

$$C_G(a^{-1}) = \text{\{\{c1::}C_G(a).\text{\}\}}$$

Note 11

656f69d38b7e4f41882f7feda5410dde

$$C_{Q_8}(i) = \text{\{\{c1::} \{1, -1, i, -i\} \ .\text{\}\}}$$

Note 12

1fd69e94ef324e30a0054ea4860105e4

$$C_{Q_8}(1) = \text{\{\{c1::}Q_8 \, .\text{\}\}}$$

Note 13

1ec04d8a609e442690da0ee9332a9647

For what do we define centers in groups?

▌ For the group itself.

Note 14

936431cf3df24996965ce022800fa1bc

Let G be a group. {{c2::The set of elements of G commuting with all elements of G}} is called {{c1::the center of G.}}

Note 15

4fdf7ab9640d453a9eb90b77b45f35b2

Let G be a group. {{c2::The center of G}} is denoted {{c1::$Z(G)$.}}

Note 16

b7d3c0377db64825b9428611799a938c

Let G be a group. The center of G forms {{c1::a subgroup of G.}}

Note 17

3c3e0bd81b194850bbeed0d6688646ea

Let G be a group. When is the center of G a subgroup of G?

▌ Always.

Note 18

10b56b05fab34d6a91d494a9c515f2a4

Let G be a group. {{c2::The center of G}} is the centralizer of {{c1::G in G.}}

Note 19

4c598c0713ef4d649fe3629dfcd8a0c7

For what do we define normalizers in groups?

▌ For nonempty subsets.

Note 20

ea05e39de520479892867fd132778337

Let G be a group and A be a nonempty subset of G. {{c2::The set

$$\{g \in G \mid gAg^{-1} = A\}$$

}} is called {{c1::the normalizer of A in G.}}

Note 21

c4b4c4424c6549c6b9afccb2945d17ee

Let G be a group and A be a nonempty subset of G. {{c2::The normalizer of A in G}} is denoted {{c1::

$$N_G(A)\,.$$

}}

Note 22

f831383807f34538b90b130d417dfb95

Let G be a group and A be a nonempty subset of G. The normalizer of A in G forms {{c1::a subgroup of G.}}

Note 23

b536e255fed54b02a1036b9baf6a7dc6

Let G be a group and A be a nonempty subset of G. When is the normalizer of A in G a subgroup of G?

> Always.

Note 24

2f5272d0d29344f79fee3fe3bc9cd61b

Let G be a group. How do $N_G(A)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G?

> $C_G(A) \leq N_G(A)$.

Note 25

18e1e07cc3434babbe772c427d74c950

Let G be a group. How do $Z(G)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G?

> $Z(G) \leq C_G(A)$.

Note 26

3a6dffba1da649fdb21855ef88b93490

How do $Z(G)$ and $N_G(A)$ relate for an arbitrary nonempty subset A of G?

> $Z(G) \leq N_G(A)$.

Note 27

730b6cdac2414954adf98bd2792c58c2

What is the smallest possible centralizer in a group?

> The center of the group.

Note 28

902c795be20e428bb4c2b4872658d5a8

Let G be a group. Then

$$C_G(\text{\{\{c2::}G\text{\}\}}) = \text{\{\{c1::}Z(G).\text{\}\}}$$

Note 29

552be3f0d86c4b46930159c7dc731d54

What is the largest possible centralizer in a group?

> The group itself.

Note 30

eaee66528de84c1f8c9fad762b3a6447

Let G be a group. Then

$$C_G(\text{\{\{c2::}1\text{\}\}}) = \text{\{\{c1::}G.\text{\}\}}$$

Note 31

b2cf1d78fcb14ecf8264f750afafece2

Let G be group. Then $Z(G) = G$ {{c2::if and only if}} {{c1::G is abelian.
}}

Note 32

90ce2f162b9d4f76b662613ffba40ced

Let G be an abelian group and A be a nonempty subset of G. Then

$$C_G(A) = \text{\{\{c1::}G\ .\text{\}\}}$$

Note 33

cfaa88a96e604867bcd19b10b115de3e

Let G be an abelian group and A be a nonempty subset of G. Then

$$N_G(A) = \text{\{\{c1::}G\text{.\}\}}$$

Note 34

98570a89b2904b749fe9cc594271e851

$$C_{D_8}\left(\left\{1, r, r^2, r^3\right\}\right) = \text{\{\{c1::}\left\{1, r, r^2, r^3\right\}\text{.\}\}}$$

Note 35

b19776816d5e4309b70ee8d73e045e52

$$N_{D_8}\left(\left\{1, r, r^2, r^3\right\}\right) = \text{\{\{c1::}D_8\text{.\}\}}$$

Note 36

d850d1653ce94d3780dcbb94b6080430

How do you show that $N_{D_8}\left(\left\{1, r, r^2, r^3\right\}\right) = D_8$?

> r and s must be included; as a subgroup the normalizer must be closed under multiplication.

Note 37

fadcb63215d9471db6ae7ebe8e76ee63

$$Z(D_6) = \text{\{\{c1::}\{1\}\text{.\}\}}$$

Note 38

46db60ed1736470f9534921d4505ea62

$$Z(D_8) = \text{\{\{c1::}\left\{1, r^2\right\}\text{.\}\}}$$

Note 39

2ce99da32df545fe9ec31ee8e0206c7f

$$C_{S_3}(\{1,\ (1\ 2)\}) = \text{\{\{c1::}\{1,\ (1\ 2)\}\text{.\}\}}$$

Note 40

2e782a1f12a6406db1161635406da5c7

How do you show that

$$C_{S_3}(\{1,\ (1\ 2)\}) = \{1,\ (1\ 2)\}\ ?$$

> $\{1,\ (1\ 2)\}$ is a subset + Lagrange's Theorem.

Note 41

d87a8e5254e643f1adf49c5fafb28600

$$N_{S_3}(\{1,\ (1\ 2)\}) = \{\{c1:: \{1,\ (1\ 2)\}\ .\}\}$$

Note 42

bd9c2331e5514956b74f445ad054106c

Let G be a group and $a \in G$. Then

$$N_G(\{1, a\}) = \{\{c1:: C_G(a).\}\}$$

Note 43

aed1e65dd5c9446a9de9fc7b16d42025

$$Z(S_3) = \{\{c1:: \{1\}\ .\}\}$$

Note 44

9b743f8b30c84ac99b22a6eb0f967267

Let G be a group acting on a set S and $s \in S$. {{c2::The stabilizer of s in G}} is denoted {{c1::

$$G_s\ .$$

}}

Note 45

6bc0f129d62348a487c114d0b7ad69f8

Let D_8 act on the set of four vertices of a square. What is the stabilizer of a vertex a in D_8?

> The identity and the reflection around the line of symmetry passing through a.

Note 46

041877bf3b974f4f8ac78eea7bb331f7

What is the kernel of the action of D_8 on the set of four vertices of a square?

> The identity subgroup.

Note 47

74e999e194f9439ea1a595397d377d68

Let G be a group and A be a nonempty subset of G. Then {{c2:: $N_G(A)$}} is {{c1::the stabilizer of A in G under the action of G on $\mathcal{P}(G)$ by left conjugation.}}

(in terms of group actions)

Note 48

39f5a3ad85424edc82a2480cee84987a

Let G be a group and A be a nonempty subset of G. Then {{c2:: $C_G(A)$}} is {{c1::the kernel of the action of $N_G(A)$ on A by left conjugation.}}

(in terms of group actions)

Note 49

ee92b4ef2d764cf8b83c56d984146ff4

Let G be a group. The {{c2::$Z(G)$}} is {{c1::the kernel of the action of G on G by left conjugation.}}

(in terms of group actions)

Note 50

7b91acc9b3d04a08b282faf013ef934c

Let G be a group.

$$C_G(Z(G)) = \text{\{\{c1::}G.\text{\}\}}$$

Note 51

0448d1d230f340db94d0400519d257f1

Let G be a group.

$$N_G(Z(G)) = \text{\{\{c1::}G.\text{\}\}}$$

Note 52

0d151553fc4c40c2ae19df1526b809da

Let G be a group and $A \subseteq B \subseteq G$. How do $C_G(A)$ and $C_G(B)$ relate?

> $$C_G(B) \leq C_G(A)\,.$$

Note 53

4b8d660e7831423b8ef7390c37250669

Let H be a subgroup of a group G. How do H and $N_G(H)$ relate?

> $H \leq N_G(H)$.

Note 54

10807a9a0243405280068a805327be8d

Let H be a subgroup of a group G. When is $H \leq N_G(H)$?

> Always.

Note 55

0b7463931b70417a883cb59616a67289

Let H be a subgroup of a group G. How do H and $C_G(H)$ relate?

> $H \leq C_G(H)$ if and only if H is abelian.

Note 56

524342fd119d4b8d9de7bd9f0bc8a088

Let H be a subgroup of a group G. When is $H \leq C_G(H)$?

> If and only if H is abelian.

Note 57

4f05768df1a340b2bf7df531f1eb9204

$Z(D_{2n}) =$ {{c1:: $\{1\}$ }} if {{c2::n is odd.}}

Note 58

d2260a07988940f09d3dd994d36cb649

$Z(D_{2n}) =$ {{c1:: $\left\{1, r^{n/2}\right\}$ }} if {{c2::n is even.}}

Note 59

7105261f89ed4022834274a4e5b47338

Let $G = S_n$ act naturally on $\{1, \ldots, n\}$. How many elements is there in G_i?

> $$(n-1)!$$

Note 60

081a1b44fb3b4eed9ad3be65055fb7c5

Let G be a group and {{c2::$H \leq G$, $A \subseteq G$.}}

$$N_H(A) \overset{\text{def}}{=} \text{{{c1::}} } \left\{h \in H \mid hAh^{-1} = A\right\}. \text{}}}$$

Note 61

b180b608cfd746638fcf52f5ae09c272

Let G be a group and $H \leq G$, $A \subseteq G$. Then

$$N_H(A) = \text{{{c1::}}} N_G(A) \cap H . \text{}}}$$

Note 62

1badf74997fd4d518ae14c54fa45cabf

Let G be a group and $H \leq G$, $A \subseteq G$. When is $N_H(A) \leq G$?

> Always.

Note 63

637554e55b3346398a3d6df04ca7f8e3

Let G be a group and $H \leq G$, $A \subseteq G$. Why is $N_H(A) \leq G$?

> It is the intersection of $N_G(A)$ and H.

Note 64

8193718985e042f495f4dddc01aff6d4

Let H a subgroup of order 2 in G. Then

$$N_G(H) = \text{{{c1::}}} C_G(H). \text{}}}$$

Note 65

eaab0949c29d4cc3bed1f53eabb0a5e2

Let $G = S_4$ act on $\mathbb{Z}[x_1, \ldots, x_4]$ by permuting the indices of the variables. Then

$$G_{x_1 x_2} \cong \text{\{\{c1::}(\mathbb{Z}/2\mathbb{Z})^2 \text{ .\}\}}$$

Note 66

85974b74f0df4ea6bcfbce29747e16c3

Let $G = S_4$ act on $\mathbb{Z}[x_1, \ldots, x_4]$ by permuting the indices of the variables. Then

$$G_{x_1 x_2 + x_3 x_4} \cong \text{\{\{c1::}D_8 \text{ .\}\}}$$

Note 67

84985d2f2f604141a40a8911d547b2b9

Let F be a field. What is the center of $H(F)$?

> The set of matrices $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $a \in F$.

Note 68

3266b5e63a934d0e829f448523636d0f

Let F be a field and $a, b \in F$. Then

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{\{\{c1::} \begin{bmatrix} 1 & 0 & a+b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ .\}\}}$$

Note 69

341f87c936e24ebbb62108347d1af6fe

Let F be a field. Then $Z(H(F)) \cong$ {{c1::$(F, +)$.}}