# Note 1

b79ffdaa29944c45ae643c2735ddfeee

The names of the authors of Abstract Algebra are

- {{c1::David S. Dummit,}}
- {{c2::Richard M. Foote.}}

# Basics

## Note 1

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{{c2::The Cartesian product}} of {{c3::two sets $A$ and $B$}} is {{c1::the collection

$$\{(a, b) \mid a \in A, b \in B\},$$

of ordered pairs of elements from $A$ and $B$.}}

## Note 2

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{{c2::The Cartesian product}} of two sets $A$ and $B$ is denoted {{c1:: $A \times B$.}}

## Note 3

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{{c2::$\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$}} denote {{c1::the positive (nonzero) elements in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.}}

## Note 4

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Given a function $f : A \to B$, {{c1::the set $B$}} is called the codomain of $f$.

## Note 5

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Given a function $f : A \to B$ and {{c3::a $b \in B$,}} {{c2::the preimage of $\{b\}$ under $f$}} is called {{c1::the fiber of $f$ over $b$.}}

## Note 6

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If $f : A \to B$ and $g : B \to C$, then the {{c1::composite map}}

$$g \circ f : A \to C$$

is defined by

$$(g \circ f)(a) = g(f(a)).$$

## Note 7

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A function $f : A \to B$ {{c3::has a left inverse}} if there is a function $g$ : {{c2::$B \to A$}}, such that {{c1::

$$g \circ f = id_A.$$

}}

## Note 8

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A function $f : A \to B$ {{c3::has a right inverse}} if there is a function $g$ : {{c2::$B \to A$}}, such that {{c1::

$$f \circ g = id_B.$$

}}

## Note 9

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A map $f$ is {{c1::injective}} if and only if $f$ has a {{c2::left}} inverse.

## Note 10

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A map $f$ is {{c1::surjective}} if and only if $f$ has a {{c2::right}} inverse.

## Note 11

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A {{c2::permutation}} of a set $A$ is {{c1::a bijection from $A$ to itself.}}

## Note 12

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If $A \subseteq B$ and $f : B \to C$, {{c2::the restriction of $f$ to $A$}} is denoted {{c1::$f|_A$.}}

## Note 13

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If $A \subseteq B$ and $g : A \to C$ and there is a function $f : B \to C$ such that {{c2::$f|_A = g$,}} we shall say $f$ is {{c1::an extension of $g$ to $B$.}}

## Note 14

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{{c2::A binary relation on a set $A$}} is {{c1::a subset $R$ of $A \times A$.}}

## Note 15

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Let $R$ be a binary relation on a set $A$. We write {{c2::$a \sim b$}} if {{c1::$(a, b) \in R$.}}

## Note 16

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A binary relation $R$ on $A$ is said to be {{c2::reflexive}} if {{c1::

$$a \sim a, \text{ for all } a \in A.$$

}}

## Note 17

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A binary relation $R$ on $A$ is said to be {{c2::symmetric}} if {{c1::

$$a \sim b \text{ implies } b \sim a \text{ for all } a, b \in A.$$

}}

## Note 18

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A binary relation $R$ on $A$ is said to be {{c2::transitive}} if {{c1::

$$a \sim b \text{ and } b \sim c \text{ implies } a \sim c \text{ for all } a, bc \in A.$$

}}

## Note 19

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A binary relation is {{c2::an equivalence relation}} if {{c1::it is reflexive, symmetric and transitive.}}

## Note 20

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If $\sim$ defines an {{c3::equivalence}} relation on $A$, then {{c2::the equivalence class}} of $a \in A$ is defined to be {{c1::

$$\{x \in A \mid x \sim a\}.$$

}}

## Note 21

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If $C$ is an equivalence class, {{c2::any element of $C$}} is called {{c1::a representative of the class $C$.}}

## Note 22

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{{c2::A partition of a set $A$}} is {{c3::any collection $\{A_i \mid i \in I\}$ of nonempty subsets of $A$}} such that {{c1::$A$ is the disjoint union of all $A_i$.}}

## Note 23

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If $\sim$ defines an equivalence relation on $A$ then {{c2::the set of equivalence classes of $\sim$}} form {{c1::a partition of $A$.}}

# Properties of the Integers

## Note 1

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Let $a, b \in \mathbb{Z}$. We write {{c2::$a \mid b$}} if {{c1::$a$ divides $b$.}}

## Note 2

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Let $a, b \in \mathbb{Z}$ with $a \neq 0$. We write {{c2::$a \nmid b$}} if {{c1::$a$ does not divide $b$.}}

## Note 3

533403fe830341a39cee216314b861e8

Let $a, b \in$ {{c3::$\mathbb{Z} - \{0\}$ }}. {{c2::The greatest common divisor of $a$ and $b$}} is denoted by {{c1::$(a, b)$.}}

## Note 4

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Let $a, b \in$ {{c3::$\mathbb{Z} - \{0\}$ }}. If {{c2::$(a, b) = 1$,}} we say that $a$ and $b$ are {{c1::relatively prime.}}

## Note 5

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If $a, b \in \mathbb{Z} - \{0\}$, then there exists unique $q, r \in \mathbb{Z}$ such that

$$a = qb + r \text{ and } 0 \leqslant r < |b|,$$

where $q$ is {{c1::the quotient}} and $r$ {{c1::the remainder.}}

«{{c2::Division Algorithm}}»

## Note 6

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If $a, b \in \mathbb{Z} - \{0\}$, then there exist $x, y \in$ {{c3::$\mathbb{Z}$}} such that

$$\{\{c2::(a, b)\}\} = \{\{c1::xa + yb.\}\}$$

## Note 7

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If $p$ is prime and $p \mid ab$, for some $a, b \in \mathbb{Z}$, then {{c1::

$$\text{either } p \mid a \text{ or } p \mid b.$$

}}

## Note 8

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{{c3::The Euler $\varphi$-function}} is defined as follows: for $n \in$ {{c2::$\mathbb{Z}^+$}} let $\varphi(n)$ be {{c1::the number of positive integers $a \leq n$ with $a$ relatively prime to $n$.}}

## Note 9

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Let $\varphi$ stand for the Euler $\varphi$-function. If $p$ is {{c3::prime}} and $a \geq 1$, then

$$\{\{c2::\varphi(p^a)\}\} = \{\{c1::p^a - p^{a-1}.\}\}$$

## Note 10

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Let $\varphi$ stand for the Euler $\varphi$-function. Then

$$\{\{c1::\varphi(ab) = \varphi(a)\varphi(b)\}\} \quad \text{if } \{\{c2::(a,b) = 1.\}\}$$

# The Integers Modulo n

## Note 1

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Let {{c3::$n \in \mathbb{Z}^+$,}} {{c4::$a, b \in \mathbb{Z}$.}} Then {{c2::$a$ is congruent to $b$ mod $n$}} if {{c1::

$$n \mid (b - a).$$

}}

## Note 2

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Let $n \in \mathbb{Z}^+$. If {{c2::$a$ is congruent to $b$ mod $n$,}} we write {{c1::

$$a \equiv b \pmod{n}.$$

}}

## Note 3

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Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$. {{c1::The equivalence class of $a$ with respect to congruence mod $n$}} is called {{c2::the congruence class or residue class of $a$ mod $n$.}}

## Note 4

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Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$. {{c2::The congruence class of $a$ mod $n$}} is denoted {{c1::$\bar{a}$.}}

## Note 5

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Let $n \in \mathbb{Z}^+$. There are {{c1::precisely $n$}} distinct equivalence classes mod $n$.

## Note 6

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Let $n \in \mathbb{Z}^+$. {{c1::The set of equivalence classes under the relation of congruence mod $n$}} is denoted by {{c2::

$$\mathbb{Z}/n\mathbb{Z}.$$

}}

## Note 7

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Let $n \in \mathbb{Z}^+$. The set $\mathbb{Z}/n\mathbb{Z}$ is called {{c1::the integers modulo $n$.}}

## Note 8

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Let $n \in \mathbb{Z}^+$. The process of {{c2::finding the equivalence class mod $n$ of some integer $a$}} is often referred to as {{c1::reducing $a$ mod $n$.
}}

## Note 9

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Let $n \in \mathbb{Z}^+$. {{c1::The smallest non-negative integer congruent to $a$ mod $n$}} is called {{c2::the least residue of $a$ mod $n$.}}

## Note 10

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Let $n \in \mathbb{Z}^+$ and $\bar{a}, \bar{b} \in \mathbb{Z}/n\mathbb{Z}$. Then

$$\text{\{\{c2::}\bar{a} + \bar{b}\text{\}\}} \stackrel{\text{def}}{=} \text{\{\{c1::}\overline{a+b}.\text{\}\}}$$

## Note 11

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Let $n \in \mathbb{Z}^+$ and $\bar{a}, \bar{b} \in \mathbb{Z}/n\mathbb{Z}$. Then

$$\text{\{\{c2::}\bar{a} \cdot \bar{b}\text{\}\}} \stackrel{\text{def}}{=} \text{\{\{c1::}\overline{ab}.\text{\}\}}$$

## Note 12

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Let $n \in \mathbb{Z}^+$. {{c2::$(\mathbb{Z}/n\mathbb{Z})^\times$}} $\stackrel{\text{def}}{=} \{\bar{a} \mid$ {{c1::there exists $\bar{c}$ with $\bar{a} \cdot \bar{c} = \bar{1}$}}$\}$.

## Note 13

866ff8df0c6d4a589ee1a49fb58fb15e

Let $n \in \mathbb{Z}^+$. Then

$$\text{\{\{c2::}(\mathbb{Z}/n\mathbb{Z})^\times\text{\}\}} = \{\bar{a} \mid \text{\{\{c1::}(a, n) = 1\text{\}\}}\}.$$

## Note 14

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Let $n \in \mathbb{Z}^+$. Then $(a, n) = 1$ implies $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. What is the key idea in the proof?

> Represent $(a, n)$ as a $\mathbb{Z}$-linear combination of $a$ and $n$.

## Note 15

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Let $n \in \mathbb{Z}^+$. Then $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ implies $(a, n) = 1$. What is the key idea in the proof?

> By contradiction and multiply $ac \equiv 1$ by $\frac{n}{(a,n)}$.

## Note 16

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Let $n \in \mathbb{Z}^+$. The number of elements in $(\mathbb{Z}/n\mathbb{Z})^\times$ is {{c1::$\varphi(n)$ where $\varphi$ denotes the Euler $\varphi$-function.}}