Definition and Examples

Note 1

9080791fc8754b0bb88c381c10acbdfc

Let G be a group. If {{c2::H is a subgroup of G}} we shall write {{c1::

$$H \leq G.$$

}}

Note 2

66e7f23728af4c9d8839d172e59d716a

Let G be a group and $H \leq G$. We shall denote the operation for H by {{c1::the same symbol as the operation for G.}}

Note 3

e76ada2ee6da4b5fb71966e9f7ce3ded

Let G be a group. If {{c2::$H \leq G$ and $H \neq G$}} we shall write {{c1::$H < G$.}}

Note 4

1d28c11c52c84bd0b639505598bb1dce

If H is a subgroup of G then any equation in the subgroup H may also be viewed as {{c1::an equation in the group G.}}

Note 5

8f5b765961884460823141645b5ea08b

Let G be a group and $H \leq G$. What is the identity of H?

> The identity of G.

Note 6

7c122a5400f64eba9a76438c1ff296ee

Let G be a group and $H \leq G$. The identity of H is the identity of G. What is the key idea in the proof?

> The identity is unique and it is the identity of G.

Note 7

83cba804764b43e2baf282ffee513694

Let G be a group. What is the minimal subgroup of G?

> The singleton $\{1\}$.

Note 8

d40df9f46a6b43ecb3fea9b5b37e5b1c

Let G be a group. What is the element that any subgroup of G must contain?

> The identity of G.

Note 9

bc95ad6358814213a6fff2eb0cfd544b

Let G be a group and $H \leq G$. What is the inverse of an element x in H?

> The inverse of x in G.

Note 10

be9f1756cf3449e8a6718069fd4aedf5

Let G be a group and $H \leq G$. Why is the notation x^{-1} unambiguous?

> The inverse in H is the same as the inverse in G.

Note 11

8aabd93df8a5437eb3e50c3e0d438381

Let G be a group. {{c2::The subgroup $\{1\}$ of G}} is called {{c1::the trivial subgroup.}}

Note 12

eb859714e1f34f4db3dc35755f562945

Let G be a group. {{c2::The trivial subgroup}} is denoted by {{c1::1.}}

Note 13

74ab3ceb9a36420fb53dc2b3a22eb5f2

Which subgroups does any group have?

> The trivial subgroup and the group itself.

Note 14

5683ff4198a74e9d988f501c925d85ad

If H is a subgroup of G and K is a subgroup of H, then {{c1::K is a subgroup of G.}}

Note 15

50a07cbb14aa4bed8866efcbedb0be4d

Which object is considered in the Subgroup Criterion?

> Any subset of a group.

Note 16

840038893a3642a18c3e43c4e89aed17

What are the conditions of the Subgroup Criterion?

> The subset is nonempty and closed under $(x, y) \mapsto x \cdot y^{-1}$.

Note 17

71291d04ca2941fca2fc08759d8fd302

What is the special case considered in the Subgroup Criterion?

> The subset is finite.

Note 18

a1e69be09e78402d989b3805b3dfc54f

What are the conditions of the Subgroup Criterion for a finite subset?

> The subset is nonempty and closed under the operation.

Note 19

5bcd55a73e184bcd9bcc32f1ee47da2e

What is the key idea in the proof of the Subgroup Criterion for a finite subset?

> Any element's inverse is it's n-th power.

Note 20

0e1ccaae016c4900ac96b733fb9e1764

Why is the set of 2-cycles in S_n not a subgroup of S_n?

> It does not contain the identity.

Note 21

587390d0450f4681a66bcbc8c0d5889c

Why is the set of reflection in D_{2n} not a subgroup of D_{2n}?

- It does not contain the identity.

Note 22

fc87d2283cb546708502ce325e326258

Why is the set of reflection in D_{2n} together with 1 not a subgroup of D_{2n}?

- Two distinct reflections induce a rotation.

Note 23

24b90e714649459ba38e6b40f07f6b2a

Is $\{1, r^2, s, sr^2\}$ a subgroup of D_8?

- Yes.

Note 24

eac99978715a4ec894b296f8e1ee52f3

Is $\{1, r, s, sr\}$ a subgroup of D_8?

- No.

Note 25

64ea968bdce94647b6fb2c351a60f2a2

Is $\{1, r^2, sr, sr^3\}$ a subgroup of D_8?

- Yes.

Note 26

678b87f890ac4d8da5be6a78cb619358

Is $\{1, r, r^2\}$ a subgroup of D_8?

- No.

Note 27

e036f3cc7667461b98e50e94ff3a8c80

Is $\{1, r, r^2, r^3\}$ a subgroup of D_8?

- Yes.

Note 28

209944ca7a524af3be44b398de974c2d

Give an example of a group and its infinite subset that is closed under the operations, but is not a subgroup of the original group.

Positive integers under addition.

Note 29

547363a46106478187c20c5cbb868461

For what groups is the notion of the torsion subgroup introduced?

For abelian groups.

Note 30

d29b9ffdb46c4c909fbfb2a438abb0a0

What is the torsion subgroup of an abelian group?

The set of all the elements of a finite order.

Note 31

b2a854579339471d8ae41776f1661f29

Let G be an abelian group. What is the name of the set

$$\{g \in G : |g| < \infty\}\,?$$

The torsion subgroup of G.

Note 32

2a685e6476b94b9eac539a17441574ef

Why is the notion of the torsion subgroup introduced only for abelian groups?

For non-abelian groups the set is not guaranteed to form a subgroup.

Note 33

22a771a961c3498f88a030fabf778797

Give an example of a non-abelian group, who's "torsion subgroup" is not actually a subgroup.

$GL_3(\mathbb{R})$

Note 34

f4edb9436c094103b0b9b82019185296

Give an example of two elements a, b in $GL_3(\mathbb{R})$ such that

$$|a|\,, |b| < \infty \quad \text{and} \quad |ab| = \infty.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note 35

f2c41a74f8a04bb892b471915e533055

What is the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$?

> The set of elements who's first component is 0.

Note 36

4df6b5997d30483fb469565c89630322

When is the union of two subgroups also a subgroup?

> If and only if one of the subgroups is a subset of the other.

Note 37

b523dcaec101461e902f34191451e112

When is the union of an infinite number of subgroups also a subgroup?

> It depends.

Note 38

60129b39ceab4468915a6d2237915c1a

Let H and K be subgroups of G and $H \subseteq K$. What do we know about $H \cup K$?

> It is a subgroup of G.

Note 39

791301f78ecf4800a13e3a0299c57028

Let H and K be subgroups of G. If $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$. What is the key idea in the proof?

> By contradiction.

Note 40

cc8decdf60194667b3b27ff0941c9fc0

What is the special linear group?

> The set of square matrices who's determinant is 1.

Note 41

1e420dd97e1942b3b7bc70d71fc0953e

{{c2::The special linear group of $n \times n$ matrices over a field F}} is denoted {{c1::$SL_n(F)$.}}

Note 42

941aeeb281da4b009ffdc95864eddb3b

When is the intersection of two subgroups also a subgroup?

> Always.

Note 43

887cf7600d994fcd9662e35fc9719c62

When is the intersection of an infinite number of subgroups also a subgroup?

> Always.

Note 44

3bdd7a0f0e044c6b9c3c1811d4478f10

Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of G. Then {{c1::$\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G.}}

Centralizers and Normalizers, Stabilizers and Kernels

Note 1

93de251693e74655a5752529379e7081

For what do we define centralizers in groups?

> For nonempty subsets of the group.

Note 2

b46233e067ea4c24b38af57081ef1db3

Let G be a group and A be a nonempty subset of G. {{c1::The set

$$\{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$$

}} is called {{c2::the centralizer of A in G.}}

Note 3

3c2adb104b55494a8a248b4e6cf72980

Let G be a group and A be a nonempty subset of G. {{c2::The centralizer of A in G}} is denoted {{c1::

$$C_G(A)\,.$$

}}

Note 4

aeea9d02d1a8429ab94927313c1e2194

How can centralizers be redefined in terms of commutativity?

> As the set of all the elements that commute with every element of the subset.

Note 5

588fd51b4281485c87a74faa9ddbf8f5

Let G be a group and A be a nonempty subset of G. The centralizer of A in G forms {{c1::a subgroup of G.}}

Note 6

18e60ffefbe647a6aa9b7a9feeb58ef1

Let G be a group and A be a nonempty subset of G. When is the centralizer of A in G a subgroup of G?

Always.

Note 7

23eee6bafc20447987eaab729108324e

Let G be a group and A be a nonempty subset of G. In the special case when $A = \{a\}$ we shall write {{c1::simply $C_G(a)$}} instead of {{c2::$C_G(\{a\})$.}}

Note 8

92e1f52031224232bf8ac69f4014862c

Let G be a group and $a \in G$. Then

$$\text{\{\{c1::}\langle a \rangle\text{\}\}} \subseteq C_G(a)\,.$$

Note 9

656f69d38b7e4f41882f7feda5410dde

$$C_{Q_8}(i) = \text{\{\{c1::}\ \{1, -1, i, -i\}\ .\text{\}\}}$$

Note 10

1fd69e94ef324e30a0054ea4860105e4

$$C_{Q_8}(1) = \text{\{\{c1::}Q_8\ .\text{\}\}}$$

Note 11

1ec04d8a609e442690da0ee9332a9647

For what do we define centers in groups?

For the group itself.

Note 12

936431cf3df24996965ce022800fa1bc

Let G be a group. {{c2::The set of elements of G commuting with all elements of G}} is called {{c1::the center of G.}}

Note 13

4fdf7ab9640d453a9eb90b77b45f35b2

Let G be a group. {{c2::The center of G}} is denoted {{c1::$Z(G)$.}}

Note 14

b7d3c0377db64825b9428611799a938c

Let G be a group. The center of G forms {{c1::a subgroup of G.}}

Note 15

3c3e0bd81b194850bbeed0d6688646ea

Let G be a group. When is the center of G a subgroup of G?

> Always.

Note 16

10b56b05fab34d6a91d494a9c515f2a4

Let G be a group. {{c2::The center of G}} is the centralizer of {{c1::G in G.}}

Note 17

4c598c0713ef4d649fe3629dfcd8a0c7

For what do we define normalizers in groups?

> For nonempty subsets.

Note 18

ea05e39de520479892867fd132778337

Let G be a group and A be a nonempty subset of G. {{c2::The set

$$\{g \in G \mid gAg^{-1} = A\}$$

}} is called {{c1::the normalizer of A in G.}}

Note 19

c4b4c4424c6549c6b9afccb2945d17ee

Let G be a group and A be a nonempty subset of G. {{c2::The normalizer of A in G}} is denoted {{c1::

$$N_G(A)\,.$$

}}

Note 20

f831383807f34538b90b130d417dfb95

Let G be a group and A be a nonempty subset of G. The normalizer of A in G forms {{c1::a subgroup of G.}}

Note 21

b536e255fed54b02a1036b9baf6a7dc6

Let G be a group and A be a nonempty subset of G. When is the normalizer of A in G a subgroup of G?

> Always.

Note 22

2f5272d0d29344f79fee3fe3bc9cd61b

Let G be a group. How do $N_G(A)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G?

> $C_G(A) \leq N_G(A)$.

Note 23

18e1e07cc3434babbe772c427d74c950

Let G be a group. How do $Z(G)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G?

> $Z(G) \leq C_G(A)$.

Note 24

3a6dffba1da649fdb21855ef88b93490

How do $Z(G)$ and $N_G(A)$ relate for an arbitrary nonempty subset A of G?

> $Z(G) \leq N_G(A)$.

Note 25

730b6cdac2414954adf98bd2792c58c2

What is the smallest possible centralizer in a group?

> The center of the group.

Note 26

902c795be20e428bb4c2b4872658d5a8

Let G be a group. Then

$$C_G(\text{\{\{c2::}G\text{\}\}}) = \text{\{\{c1::}Z(G).\text{\}\}}$$

Note 27

552be3f0d86c4b46930159c7dc731d54

What is the largest possible centralizer in a group?

> The group itself.

Note 28

eaee66528de84c1f8c9fad762b3a6447

Let G be a group. Then

$$C_G(\text{\{\{c2::}1\text{\}\}}) = \text{\{\{c1::}G.\text{\}\}}$$

Note 29

b2cf1d78fcb14ecf8264f750afafece2

Let G be group. Then $Z(G) = G$ {{c2::if and only if}} {{c1::G is abelian.
}}

Note 30

90ce2f162b9d4f76b662613ffba40ced

Let G be an abelian group and A be a nonempty subset of G. Then

$$C_G(A) = \text{\{\{c1::}G\ .\text{\}\}}$$

Note 31

cfaa88a96e604867bcd19b10b115de3e

Let G be an abelian group and A be a nonempty subset of G. Then

$$N_G(A) = \text{\{\{c1::}G\ .\text{\}\}}$$

Note 32

98570a89b2904b749fe9cc594271e851

$$C_{D_8}\left(\{1, r, r^2, r^3\}\right) = \text{\{\{c1::}\ \{1, r, r^2, r^3\}\ .\text{\}\}}$$

Note 33

b19776816d5e4309b70ee8d73e045e52

$$N_{D_8}\left(\{1, r, r^2, r^3\}\right) = \text{\{\{c1::}D_8\ .\text{\}\}}$$

Note 34

d850d1653ce94d3780dcbb94b6080430

How do you show that $N_{D_8}\left(\{1, r, r^2, r^3\}\right) = D_8$?

> r and s must be included; as a subgroup the normalizer must be closed under multiplication.

Note 35

fadcb63215d9471db6ae7ebe8e76ee63

$$Z(D_6) = \text{\{\{c1::} \{1\} \text{ .\}\}}$$

Note 36

46db60ed1736470f9534921d4505ea62

$$Z(D_8) = \text{\{\{c1::} \{1, r^2\} \text{ .\}\}}$$

Note 37

2ce99da32df545fe9ec31ee8e0206c7f

$$C_{S_3}(\{1,\ (1\ 2)\}) = \text{\{\{c1::} \{1,\ (1\ 2)\} \text{ .\}\}}$$

Note 38

2e782a1f12a6406db1161635406da5c7

How do you show that

$$C_{S_3}(\{1,\ (1\ 2)\}) = \{1,\ (1\ 2)\}\,?$$

> $\{1,\ (1\ 2)\}$ is a subset + Lagrange's Theorem.

Note 39

d87a8e5254e643f1adf49c5fafb28600

$$N_{S_3}(\{1,\ (1\ 2)\}) = \text{\{\{c1::} \{1,\ (1\ 2)\} \text{ .\}\}}$$

Note 40

bd9c2331e5514956b74f445ad054106c

Let G be a group and $a \in G$. Then

$$N_G(\{1, a\}) = \text{\{\{c1::} C_G(a). \text{\}\}}$$

Note 41

aed1e65dd5c9446a9de9fc7b16d42025

$$Z(S_3) = \text{\{\{c1::} \{1\} \text{ .\}\}}$$

Note 42

9b743f8b30c84ac99b22a6eb0f967267

Let G be a group acting on a set S and $s \in S$. {{c2::The stabilizer of s in G}} is denoted {{c1::

$$G_s\,.$$

}}

Note 43

6bc0f129d62348a487c114d0b7ad69f8

Let D_8 act on the set of four vertices of a square. What is the stabilizer of a vertex a in D_8?

> The identity and the reflection around the line of symmetry passing through a.

Note 44

041877bf3b974f4f8ac78eea7bb331f7

What is the kernel of the action of D_8 on the set of four vertices of a square?

> The identity subgroup.

Note 45

74e999e194f9439ea1a595397d377d68

Let G be a group and A be a nonempty subset of G. Then {{c2:: $N_G(A)$}} is {{c1::the stabilizer of A in G under the action of G on $\mathcal{P}(G)$ by left conjugation.}}

(in terms of group actions)

Note 46

39f5a3ad85424edc82a2480cee84987a

Let G be a group and A be a nonempty subset of G. Then {{c2:: $C_G(A)$}} is {{c1::the kernel of the action of $N_G(A)$ on A by left conjugation.}}

(in terms of group actions)

Note 47

ee92b4ef2d764cf8b83c56d984146ff4

Let G be a group. The {{c2::$Z(G)$}} is {{c1::the kernel of the action of G on G by left conjugation.}}

(in terms of group actions)