

Uniform Convergence of a Sequence of Functions

Note 1

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Let (f_n) be a sequence of function on a set A . We say (f_n) converges pointwise on A to a function f if for all $x \in A$

$$\left(f_n(x)\right)_{n \rightarrow \infty} \longrightarrow f(x).$$

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Note 2

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Let (f_n) be a sequence of function on a set A . If (f_n) converges pointwise on A to f , we write

$$(f_n \rightarrow f) \quad \text{or} \quad \lim_{n \rightarrow \infty} f_n = f.$$

Note 3

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Let $f_n(x) = \frac{x^2 + nx}{n}$.

$$\lim_{n \rightarrow \infty} f_n(x) = x.$$

Note 4

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Let $f_n(x) = x^n$, $f_n : [0, 1] \rightarrow \mathbb{R}$.

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

Note 5

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Let (f_n) be a sequence of function on a set A . We say (f_n) converges uniformly on A to a function f if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \\ |f_n - f| < \epsilon.$$

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Note 6

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Let (f_n) be a sequence of function on a set A . If (f_n) converges uniformly on A to f , we write

$$f_n \rightrightarrows f.$$

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Note 7

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What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

■ The dependence of N on x .

Note 8

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What is the visual behind the uniform convergence of a sequence of functions?

■ Eventually every f_n is completely contained in the ϵ -strip.

Note 9

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Which is stronger, uniform or pointwise convergence?

■ Uniform convergence is stronger.

Note 10

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Uniform convergence implies pointwise convergence.

Note 11

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Let (f_n) be a sequence of function on a set A .

$$(f_n \rightrightarrows f) \iff \sup_{n \rightarrow \infty} |f_n - f| \rightarrow 0.$$

(in terms of sup)

Note 12

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Let (f_n) be a sequence of function on a set A . Then (f_n) converges uniformly on A if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

Note 13

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Let (f_n) be a sequence of function on a set A . Then $f_n \Rightarrow f$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

«[Cauchy Criterion]»

Note 14

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What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

■ Follows immediately from the definition.

Note 15

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What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

■ Define a candidate for the limit and prove by definition.

Note 16

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

■ Use the pointwise limit.

Note 17

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do we know the pointwise limit exists?

■ Due to the Cauchy Criterion for sequences.

Note 18

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have $f_n \rightarrow f$. How do you show that $f_n \rightrightarrows f$?

■ Take the limit of the inequality from the Cauchy Criterion.

Note 19

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Let $f_n \rightarrow f$ on a set A and $c \in A$. If $\{\{c\}::\text{the convergence is uniform}\}$ and $\{\{c\}::\text{all } f_n \text{ are continuous at } c,\}$ then $\{\{c\}::f \text{ is continuous at } c,\}$

Note 20

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Let $f_n \rightarrow f$ on a set A and $c \in A$. If the convergence is uniform and all f_n are continuous at c , then f is continuous at c .

« $\{\{c\}::\text{Continuous Limit Theorem}\}$ »

Note 21

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What is the key idea in the proof of the Continuous Limit Theorem for a sequence of functions?

■ Triple triangle inequality after adding and subtracting f_N .

Note 22

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Let $f_n \rightarrow f$ on a set A and $c \in A$. If $\{\{c\}::\text{the convergence is uniform}\}$ and all f_n are continuous at c , then

$$\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \{\{c\}:: \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x).\}$$

Note 23

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Let $f_n \rightarrow f$ on a set A . If each f_n is continuous, but f is discontinuous, then $\{\{c1:: \text{the convergence is not uniform.}\}$

Note 24

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Give an example of a sequence of functions $f_n \rightarrow f$ such that

- each f_n is continuous almost everywhere; and
- f is nowhere continuous.

■ Step-by-step construction of the Dirichlet's function.

Note 25

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Assume $f_n \rightarrow f$ on a set A and each f_n is uniformly continuous. If $\{\{c2:: f_n \rightrightarrows f,\}\}$ then $\{\{c1:: f \text{ is uniformly continuous.}\}\}$

Note 26

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Assume $f_n \rightarrow f$ on a set A and each f_n is bounded. If $\{\{c2:: f_n \rightrightarrows f,\}\}$ then $\{\{c1:: f \text{ is bounded.}\}\}$

Note 27

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Assume $f_n \rightarrow f$ on a set A and each f_n has a finite number of discontinuities. If $f_n \rightrightarrows f$, then $\{\{c1:: f \text{ has at most a countable number of discontinuities.}\}\}$

Note 28

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Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If $\{\{c2:: f \text{ is discontinuous at } c,\}\}$ then $\{\{c1:: \text{all } f_n \text{ are eventually discontinuous at } c.\}\}$

Note 29

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Assume $f_n \rightrightarrows f$ on a set A and $c \in A$. If f is discontinuous at c , then all f_n are eventually discontinuous at c . What is the key idea in the proof?

■ By contradiction + choose a subsequence continuous at c .