The Monotone Convergence Theorem and a First Look at Infinite Series

Note 1

7f744b7eecb54041a6e188d2283abcff

A sequence (a_n) is {{c2::increasing}} if {{c1::$a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$.
}}

Note 2

cb73357863a14f808fcb79e9f2888e9d

A sequence (a_n) is {{c2::decreasing}} if {{c1::$a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.
}}

Note 3

428c29af1f87467cba4605f856da5dc0

A sequence (a_n) is {{c2::monotone}} if {{c1::it is either increasing or decreasing.}}

Note 4

f0effd26705b4fe2850675b4a8b69fa7

If a sequence is {{c3::monotone}} and {{c2::bounded,}} then {{c1::it converges.
}}

Note 5

f04966660a1d453499de164d33c3efd9

If a sequence is monotone and bounded, then it converges.

«{{c1::Monotone Convergence Theorem}}»

Note 6

fe52926982cd479399d0e77cf6fbb8ae

What is the key idea in the proof of the Monotone Convergence Theorem?

> The limit equals to $\sup \{a_n \mid n \in \mathbb{N}\}$

Note 7

b7b0d33916a74554bee0bb1e829b7a20

Let {{c3::(a_n) be a sequence.}} {{c2::An infinite series}} is {{c1::a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

}}

Note 8

024782c9319a441f91dfd2c8e8aac542

Let $\sum_{n=1}^{\infty} a_n$ be a series. We define the corresponding {{c2::sequence of partial sums}} by {{c1::

$$m \mapsto a_1 + a_2 + \cdots + a_m.$$

}}

Note 9

56563c7563df42c0a111a49ad4ae24ae

Let $\sum_{n=1}^{\infty} a_n$ be a series. {{c2::The sequence of partial sums}} is usually denoted {{c1::(s_m).}}

Note 10

dc59f9b31fff4dcb9113d42da885c946

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A}} if {{c1::the sequence of partial sums converges to A.}}

Note 11

356961ddcb85482c8155d43bd6d8061c

Let $\sum_{n=1}^{\infty} a_n$ be a series. If {{c2::$\sum_{n=1}^{\infty} a_n$ converges to A,}} we write

{{c1::

$$\sum_{n=1}^{\infty} a_n = A.$$

}}

Note 12

4819e0996d5d4eeb8ab8df01f58c8efe

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

> Yes.

Note 13

64c293a1a2f74541ba8e3ffa23fb54b2

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. What is the key idea in the proof?

> $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$.

Note 14

cd5ca73daf014641b49c5445adcd69b5

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

> No.

Note 15

184fe5e5e62b4c3f8a49c4ea6d26c240

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What is the key idea in the proof?

> Find a lower bound using powers of two.

Note 16

4608dd8499934012aadc1209fb34ec1e

{{c2::$\sum_{n=1}^{\infty} \frac{1}{n}$}} is called {{c1::the harmonic series.}}

Note 17

ccea4c33507e4d5f9387c996a8bb13ad

Let (a_n) be {{c5::a decreasing sequence}} and {{c4::$a_n \geq 0$.}} Then

$$\text{\{\{c2::} \sum_{n=1}^{\infty} a_n \text{ converges\}\}\{\{c3::} \iff \text{\}\}\{\{c1::} \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.\}\}}$$

«{{c6::Cauchy Condensation Test}}»

Note 18

88287ba71bd545459ba16b4e2ca5cb69

Let (a_n) be a decreasing sequence and $a_n \leq 0$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

What is the key idea in the proof?

> Group the element of a partial sum in chunks of size 2^m.

Note 19

7dfc9afff8a045caa6549458d3264c8d

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ {{c2::converges}} {{c3::if and only if}} {{c1::$p > 1$.}}

Note 20

666661971092437289591809 63a362d4

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. What is the key idea in the proof?

> The Cauchy Condensation Test and the convergence of geometric series.

Properties of Infinite Series

Note 1

51836e3c068e46888891ad60f449bd6

Let $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$. Under which condition does

$$\sum_{k=1}^{\infty} ca_k$$

converge?

> Always.

Note 2

548101004aba462b8e81b2c4f7cbd1b9

If $\sum_{k=1}^{\infty} a_k = A$ and $c \in \mathbf{R}$, then $\sum_{k=1}^{\infty} ca_k =$ {{c1::cA}}.

Note 3

30607fca749d4ea9814ec7460a102865

Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Under which condition does

$$\sum_{k=1}^{\infty} a_k + b_k$$

converge?

> Always.

Note 4

4f1064d2b18d4e889fa4e80010f532b1

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\sum_{k=1}^{\infty} a_k + b_k = \text{{{c1::}} } A + B. \text{ }}}$$

Note 5

6795efea2a204bfb90bf19f3ac01f60a

The series $\sum_{k=1}^{\infty} a_k$ {{c5::converges}} {{c4::if and only if,}} given {{c3::$\epsilon > 0$,}} there exists {{c2::an $N \in \mathbf{N}$}} such that whenever {{c2::$n > m \geq N$}} it follows that {{c1::

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

}}

Note 6

f83e35fa266b4b71ae674a5ae53196aa

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

«{{c1::Cauchy Criterion}}»

Note 7

255fd1a8d1ca40ddbe4706f396dcaad5

What is the key idea in the proof of the Cauchy Criterion for Series?

> Cauchy Criterion for the sequence of partial sums.

Note 8

2ccccd666d0d4025a48baaa6ac297e88

If the series $\sum_{k=1}^{\infty} a_k$ {{c2::converges,}} then {{c1::$(a_k) \to 0$.}}

Note 9

e553a27c1b0240b4a08a2d2e1291a1c5

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$. What is the key idea in the proof?

> Apply the Cauchy Criterion with $n = m + 1$.

Note 10

0314d6d2761e4bd1b24b1b858e9c5086

Assume (a_k) and (b_k) are sequences satisfying {{c3::$0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.}} If $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::converges,}} then $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::converges.}}

Note 11

03fddbcdb39340e0a421d24fe7298f2e

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty}$ {{c1::a_k}} {{c2::diverges,}} then $\sum_{k=1}^{\infty}$ {{c1::b_k}} {{c2::diverges.}}

Note 12

d6553b70220c4348a7c7692a58f91271

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

«{{c1::Comparison Test}}»

Note 13

7f40a1b03ff44e75af1465ca5e329e3e

What is the key idea in the proof of the Comparison Test for Series?

> Use the Cauchy Criterion explicitly.

Note 14

e02413e7068f47d28eab58d2542d2858

What series are considered in the Limit Comparison Test?

> Positive and one containing no zeros.

Note 15

cedc7eb1b2ac4f578caebcbaf4398f01

Which value is considered in the Limit Comparison Test?

> The limit of the ratio of corresponding terms.

Note 16

9ce4a06cfa6e42c7bae44e61649416d4

Which cases exist on the Limit Comparison Test?

> The limit is finite or is nonzero.

Note 17

bb6597f3ea41409da5895548c598ddae

What can we say from the Limit Comparison Test if the limit is finite?

> The denominator's series convergence implies that of the numerator.

Note 18

08f8caf25dd54ec8bd0b0a03a66d00f2

What can we say from the Limit Comparison Test if the limit is nonzero?

> The numerator's series convergence implies that of the denominator.

Note 19

8474f88f7b4140dabe637c96e7a5005d

What can we say from the Limit Comparison Test if the limit is finite and nonzero?

> The two series's convergences are equivalent.

Note 20

ca9aa1db61144f7e99c9c0ead13fed2f

What can we say from the Limit Comparison Test if the limit does not exist?

> Nothing.

Note 21

34848474b28a469dbb7bc1859e1ab612

What is the key idea in the proof of the Limit Comparison Test (finite limit)?

> The set of ratios is bounded above + the Comparison Test.

Note 22

6f66af55f5d042cb85559bf7718f0641

What is the key idea in the proof of the Limit Comparison Test (nonzero limit)?

> Swap the numerator and the denominator.

Note 23

1f9364c8930f4fedbfb3501d9a92ee2e

Statements about {{c2::convergence}} of sequences and series are immune to {{c1::changes in some finite number of initial terms.}}

Note 24

89c3e03f687b4c4aa41185f6c668d327

A series is called {{c2::geometric}} if it is of the form {{c1::

$$\sum_{k=0}^{\infty} ar^k.$$

}}

Note 25

4d18a586f7754236bac47a23a54ede43

The series $\sum_{k=0}^{\infty} ar^k$ {{c2::converges}} {{c3::if and only if}} {{c1::$|r| < 1$.}}

Note 26

f7ab1e58f37b4580a558de06c51dc6f7

Given $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \text{\{\{c1::}\frac{a}{1-r}.\text{\}\}}$$

Note 27

c409ec230f6741b796ea4ef3e8813d9c

Given $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. What is the key idea in the proof?

> Rewrite partial sums.

Note 28

28dc84fd3d384adea7a15102e07c644a

If {{c2::the series $\sum_{k=1}^{\infty} |a_k|$ converges,}} then {{c1::$\sum_{k=1}^{\infty} a_k$ converges.
}}

«{{c3::Absolute Convergence Test}}»

Note 29

fb10bc5e919347ffa66da221bf832aa3

What is the key idea in the proof of the Absolute Convergence Test?

> The Cauchy Criterion and the Triangle Inequality.

Note 30

998d23f7cbbb49ed885b7ef2f62bb629

Let (a_k) be {{c4::a decreasing sequence}} and {{c3::$(a_k) \to 0$.}} Then {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

}} {{c1::converges.}}

Note 31

df767d19abbf4031899b4a87577b2625

Let (a_k) be a decreasing sequence and $(a_k) \to 0$. Then

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges.

«{{c1::Alternating Series Test}}»

Note 32

5023b0a2f0ca4300bfa09b61e0ec0a9c

{{c1::An alternating series}} is a series of the form {{c2::

$$\sum_{k=0}^{\infty} (-1)^k a_k,$$

}} where {{c3::all $a_k > 0$.}}

Note 33

6fb766a68cd14aa395c223e4a0e95999

What is the key idea in the proof of the Alternating Series Test?

> The Cauchy criterion for the sequence of partial sums.

Note 34

9bfa24b4310b474db9705bceed02cc45

Which intervals are considered in the proof of the Alternating Series Test?

> Those formed by successive partial sums.

Note 35

a581365ace824e89ae7a397fe6d02f1d

In the proof of the Alternating Series Test, how to you choose $\Delta_{s_m, s_{m+1}}$, given $\epsilon > 0$?

> So that its length is less then ϵ.

Note 36

a77a5abf0f2a46e8af759deffbaeed9e

In the proof of the Alternating Series Test, what do you need to show about an interval $\Delta_{s_m, s_{m+1}}$?

> It contains all of the following partial sums.

Note 37

cb8249219a644a12b50a90701e47e548

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges absolutely,}} if {{c1::$\sum_{k=1}^{\infty} |a_k|$ converges.
}}

Note 38

c07bf73c30a04766803b1c0fae6b38d9

We say $\sum_{k=1}^{\infty} a_k$ {{c2::converges conditionally,}} if {{c1::it converges and does not converge absolutely.}}

Note 39

f54a6f91b89f42c7b548ace2e106608d

A series $\sum_{k=1}^{\infty} a_k$ is said to be {{c2::positive}} if {{c1::$a_k \geq 0$ for all $k \in \mathbf{N}$.}}

Note 40

c5acade4dde342f8b7ac4acec2278ac6

Any {{c2::positive}} convergent series must {{c1::converge absolutely.
}}

Note 41

e85b9eb09cfa4056b868f983703a571c

May a positive series diverge?

> Only to $+\infty$.

Note 42

b65eba46e51c438e933833ad313a4cf8

A {{c2::positive}} series converges {{c3::if and only if}} {{c1::the sequence of partial sums (s_n) is bounded.}}

Note 43

4ef68f3ca3544ea98fd3c54340c65ce5

Let $\sum_{k=1}^{\infty} a_k$ be a series and {{c3::$f : \mathbf{N} \to \mathbf{N}$ be 1–1 and onto.}} {{c2:: The series $\sum_{k=1}^{\infty} a_{f(k)}$}} is called {{c1::a rearrangement of $\sum_{k=1}^{\infty} a_k$.}}

Note 44

4071d910f5e6410cb2b01dfc73ae48da

If a series {{c2::converges absolutely,}} then {{c3::any rearrangement of this series}} {{c1::converges to the same limit.}}

Note 45

057430cb21934da7ac9bc037ba169eb5

If a series converges absolutely, then any rearrangement of this series converges to the same limit. What is the key idea in the proof?

> Substitute the original series' initial terms for the rearrangement's partial sum.

Note 46

d572332d7e36407ab1531e824f794b4b

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the original series' initial terms are substituted from the rearrangement's partial sum?

> So as to use the definition of convergence and the Cauchy Criterion for absolute convergence.

Note 47

574ee484bcf94971932baee731b90c95

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how many of the rearrangement's terms are taken for the partial sum?

▌ So as to contain the initial terms of the original sequence.

Note 48

c50d4f3043cb4ca38411c1b1dc20ae26

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::s_n}} to be {{c1::the original series' partial sum.}}

Note 49

2f9195ab94ee4143800fc5300d10d80f

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof we denote {{c2::t_n}} to be {{c1::the rearrangement' partial sum.}}

Note 50

1bacf92272b04fc98d69ac25f5fcdfe2

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, what do we show about $t_m - s_N$?

▌ $|t_m - s_N| < \varepsilon$

Note 51

6e8705bf5bd84118a85ac3eb8a1d5e28

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, why is it that $|t_m - s_N| < \varepsilon$?

▌ Due to the Cauchy Criterion.

Note 52

8ffac6aca55141b29861f55f5d1dd8fb

If a series converges absolutely, then any rearrangement of this series converges to the same limit. In the proof, how do you show $|t_m - A| < \varepsilon$?

> $|t_m - s_N + s_N - A|$ and the triangle inequality.

Note 53

b4e0eacc15f64559b6c255552fe3aadf

What series are considered in the Ratio Test?

> Strictly positive.

Note 54

dcfddd94a3304571a442fff1f7009cb8

Which value is considered in the Ratio Test?

> The limit of successive ratios.

Note 55

d00eda65eafa4efabe918bfacc3ff819

Which term is placed to the numerator in the Ratio Test?

> The next one.

Note 56

605c64a7226c48eebe5ee34d51cd470b

When does the Ratio Test let us conclude something?

> When the ratios approach a value other than 1.

Note 57

a70e3ac68ab947fc8e389e85e5f54588

Which cases exists on the Ratio Test?

> Ratios converge to less than, or greater than, 1.

Note 58

de649e2ae5cc4b3b93aac925d3b37d4b

What do we conclude from the Ratio Test when the ratios converge to something less than 1?

> The series converges.

Note 59

3bcf7fb3ba4f4ace92b222a3c8af9174

What do we conclude from the Ratio Test when the ratios converge to something greater than 1?

> The series diverges.

Note 60

90519e5b985b4f97a25636a1473b500d

What do we conclude from the Ratio Test when the ratios converge to 1?

> Nothing.

Note 61

4bab403524b240cda38745c2324966c0

What do we conclude from the Ratio Test when the ratios do not converge?

> Nothing.

Note 62

1a0caf850c00432b93871e8c66f3397b

Give an example when the Ratio Test is inconclusive and the series diverges.

> The harmonic series.

Note 63

0c417f771ac54fa3ad89fb5d65d5f10d

Give an example when the Ratio Test is inconclusive and the series converges.

> $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 64

0a54c42a8bd74ba883e310f36f865ca6

What is the nominal name of the Ratio Test?

▌ The d'Alambert's Ratio Test.

Note 65

f1e24cc124f84cf3a6d14e77ee23368b

What is the first step in proving the Ratio Test?

▌ Split $r < 1, r > 1$.

Note 66

127428f8805043978b16164456c8acf5

What is the key idea in the proof of the Ratio Test ($r > 1$)?

▌ The terms are eventually increasing.

Note 67

535154065a884eb7bf3e87e8d4b400e5

What is the first key idea in the proof of the Ratio Test ($r < 1$)?

▌ For $r < r' < 1$ the ratios are eventually less than r'.

Note 68

5ac59226423b4b8fb84c087795e5ed6f

What is the second key idea in the proof of the Ratio Test ($r < 1$)?

▌ Find an upper bound using a geometric series.

Note 69

ce4c6aa5f15044a2a804f11a91d677b7

What sequences are considered in the Root Test?

▌ Strictly positive.

Note 70

02964fce0fcd409cab46d91942e3f1c2

What value is considered in the Root Test?

▌ The limit of $\sqrt[n]{a_n}$.

Note 71

06c9e889bae041afb32a8f2da431bbf9

Which cases exist on the Root Test?

> n-th roots approach something less than, or greater than, 1.

Note 72

562b1b6b74e24c73ad75d944ff17d581

When does the Root Test let us conclude something?

> When n-th roots approach something other than 1.

Note 73

687fe6a03e28430189cd57632f9bae0b

What do we conclude from the Root Test if the limit is less than 1?

> The series converges.

Note 74

dd2315fb062b4bdf93ebe5072fc0d308

What do we conclude from the Root Test if the limit is greater than 1?

> The series diverges.

Note 75

7701686caac7412aa1b3375ff77e5a9e

What do we conclude from the Root Test if the limit converges to 1?

> Nothing.

Note 76

6200b936d6144cafb8b74ff7d9271a9d

Give an example when the root test is inconclusive and the series diverges.

> The harmonic series.

Note 77

6cd4fabac91944db96449403d2288e0a

Give an example when the root test is inconclusive and the series converges.

▌ $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note 78

644281f3c2614e2499993a48daca8aac

What is the nominal name for the Root Test?

▌ Cauchy's Radical Test.

Note 79

7021924723f142d489dc64e27e06c40b

What is the first step in proving the Root Test?

▌ Split $r < 1$, $r > 1$.

Note 80

ae27724cb07240fbb243221a41bb7f82

What is the first key idea in the proof of the Root Test ($r < 1$)?

▌ For $r < r' < 1$ the roots are eventually less than r'.

Note 81

64f3efecadd94ca8ad1277cba95ded2e

What is the second key idea in the proof of the Root Test ($r < 1$)?

▌ Find an upper bound using a geometric series.

Note 82

e4b13d2a78bc4010ad92b3574943d982

What is the key idea in the proof of the Root Test ($r > 1$)?

▌ Elements are eventually greater than 1.