

Sets

Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking, $\{\{c2::a \text{ set}\}\}$ is $\{\{c1::a \text{ collection of objects.}\}\}$

Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects. $\{\{c2::\text{Those objects}\}$ are referred to as $\{\{c1::\text{the elements of the set.}\}\}$

Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set A , we write $\{\{c2::x \in A\}\}$ if $\{\{c1::x \text{ is an element of } A.\}\}$

Note 4

b25d749749a64c5b90880253d9839da8

Given a set A , we write $\{\{c2::x \notin A\}\}$ if $\{\{c1::x \text{ is not an element of } A.\}\}$

Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets A and B , $\{\{c1::\text{the union}\}\}$ is written $\{\{c2::A \cup B.\}\}$

Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets A , and B , $\{\{c2::\text{the union}\}\}$ is $\{\{c3::\text{defined}\}\}$ by the rule

$$x \in \{\{c2::A \cup B\}\} \text{ provided that } \{\{c1::x \in A \text{ or } x \in B.\}\}$$

Note 7

8ce7db157931494bbfb6eee706e15efc

Given two sets A and B , $\{\{c1::\text{the intersection}\}\}$ is written $\{\{c2::A \cap B.\}\}$

Note 8

6a277df52de2409a98e48429d69b6d05

Given two sets A and B , $\{\{c2::\text{the intersection}\}\}$ is $\{\{c3::\text{defined}\}\}$ by the rule

$$x \in \{\{c2::A \cap B\}\} \text{ provided that } \{\{c1::x \in A \text{ and } x \in B.\}\}$$

Note 9

684951afc378458aa7bd27e67cdc499b

The set of natural numbers is denoted \mathbf{N} .

Note 10

49d36a026d4b4678ab86fb6103571cce

$$\mathbf{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

Note 11

797c81e5adb543e1a5d4cc67e64c5e09

The set of integers is denoted \mathbf{Z} .

Note 12

d3c61bf891744c58b73cef543c6e100d

$$\mathbf{Z} \stackrel{\text{def}}{=} \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Note 13

57f085776972449f8bc14daf5cff6603

The set of rational numbers is denoted \mathbf{Q} .

Note 14

f7e3370650134607853b41b2b1ecf54b

$$\mathbf{Q} \stackrel{\text{def}}{=} \left\{ \text{all fractions } \frac{p}{q} \text{ where } p, q \in \mathbf{Z} \text{ and } q \neq 0 \right\}.$$

Note 15

faeac83cb5b740b6964551c85ad3e35b

The set of real numbers is denoted \mathbf{R} .

Note 16

6e5da98964d645d09ad6989e85679c74

The empty set is the set that contains no elements.

Note 17

206db0a0f3d042e49a9ca532e222201f

The empty set is denoted \emptyset .

Note 18

2f0448d226db4b71b150acaed349a73b

Two sets A and B are said to be disjoint if $A \cap B = \emptyset$.

Note 19

e5d9d365e86640319ca5460ef8c4f05c

Given two sets A and B , we say $\{\{c2::A \text{ is a subset of } B\}\}$ or $\{\{c2::B \text{ contains } A\}\}$ if $\{\{c1::\text{every element of } A \text{ is also an element of } B\}\}$

Note 20

c2bd27f1fc0d40e296dceef9c9789556

Given two sets A and B , the $\{\{c3::\text{inclusion}\}\}$ relationship $\{\{c2::A \subseteq B \text{ or } B \supseteq A\}\}$ is used to indicate that $\{\{c1::A \text{ is a subset of } B\}\}$

Note 21

333e7c6716af48b7b9962ad803f0732f

Given two sets A and B , $\{\{c2::A = B\}\}$ means that $\{\{c1::A \subseteq B \text{ and } B \subseteq A\}\}$

Note 22

74e93b42d46746dc9ec2b54f8366c435

Let A_1, A_2, A_3, \dots be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \dots$$

are all equivalent ways to indicate $\{\{c1::\text{the set whose elements consist of any element that appears in at least on particular } A_n\}\}$

Note 23

69e4627a3e7149ef8be05479a2587b41

Let A_1, A_2, A_3, \dots be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \dots$$

are all equivalent ways to indicate $\{\{c1::\text{the set whose elements consist of any element that appears in every } A_n\}\}$

Note 24

11a987e10fce4ceea69672f366597729

Given $A \subseteq \mathbf{R}$, $\{\{c2::\text{the complement of } A\}\}$ refers to $\{\{c1::\text{the set of all elements of } \mathbf{R} \text{ not in } A\}\}$

Note 25

8b379552450b4672af82c17476c0ff13

Given $A \subseteq \mathbf{R}$, $\{\{c2::\text{the complement of } A\}\}$ is written $\{\{c1::A^c\}\}$

Note 26

a3459afa53264a7c82d9abd760a0c93e

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2: (A \cap B)^c\}\} = \{\{c1: A^c \cup B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

Note 27

c983927aa0304e51949e2f90a2ec2614

Given $A, B \subseteq \mathbf{R}$,

$$\{\{c2: (A \cup B)^c\}\} = \{\{c1: A^c \cap B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

■ Demonstrate inclusion both ways.

Functions

Note 1

18930cfe4e445779bcec8a2fb53f23c

Given two sets A and B , a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B .

Note 2

dfa898ef047e418fa8dfe9ce9582fd71

If f is a function from A to B , we write $f : A \rightarrow B$.

Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let $f : A \rightarrow B$. Given $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f .

Note 4

65568f366ca949888310668475dbe570

Let $f : A \rightarrow B$. The set A is called the domain of f .

Note 5

7870a310786142fa938bcc843ca8e1ae

Let $f : A \rightarrow B$. The set $\{f(x) \mid x \in A\}$ is called the range of f .

Note 6

716c208c9ae849b89ec722aa17f20882

Given a function f and a subset A of its domain, the set

$$\{f(x) : x \in A\}$$

is called the range of f over the set A .

Note 7

24aae21652754fcd1267ac61036a3ea

Given a function f and a subset A of its domain, the range of f over A is written $f(A)$.

Note 8

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \rightarrow \mathbf{R}$, $A, B \subseteq D$. Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

■ Yes.

Note 9

ee665e77ac9a45cf9a15d42549e6f382

Let $f : D \rightarrow \mathbf{R}$, $A, B \subseteq D$. Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

■ No.

Note 10

5d2e9d4e1e094e06b37bd87e2c9edff8

Given $\{\{c4::a, b \in \mathbf{R}\}\}$ and $\{\{c3::a \leq b\}\}$, $\{\{c2::$ the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

$\}\}$ is called $\{\{c1::$ a closed interval. $\}\}$

Note 11

9f383a22fc724f8fa43af5cb65e0cd5a

Given $a, b \in \mathbf{R}$ and $\{\{c3::a < b\}\}$, $\{\{c2::$ the set

$$\{x \in \mathbf{R} : a < x < b\}$$

$\}\}$ is called $\{\{c1::$ an open interval. $\}\}$

Note 12

3143096eb895471bac4b2d5840d18758

Given $a, b \in \mathbf{R}$ and $a \leq b$, $\{\{c1::$ the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

$\}\}$ is written $\{\{c2::[a, b].\}\}$

Note 13

604897f024bd4de78723fe8247290371

Given $a, b \in \mathbf{R}$ and $a \leq b$, $\{\{c1::$ the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

$\}\}$ is written $\{\{c2::(a, b).\}\}$

Note 14

a77dc72d26be45c185900ba7ff132b05

Let $f(x) = x^2$. Find two sets A and B for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

■ Singletons $\{-1\}$ and $\{1\}$.

Note 15

6ed2fb1006634dcf81707a3c4d514857

Let $f : D \rightarrow \mathbf{R}$, $A, B \subseteq D$. Then

$$f(A \cup B) = f(A) \cup f(B).$$

Note 16

e088ae5ae1f24425a81dac09317978fd

Let $f : D \rightarrow \mathbf{R}$, $A, B \subseteq D$. Then

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

Note 17

f951f5a5136248dcb413f59b3271d389

Given $x \in \mathbf{R}$, the absolute value of x is denoted $|x|$.

Note 18

624dda908fd64a1cadae2b61c1277c59

Given $x \in \mathbf{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note 19

0ab23d0afe1448e397cad330aea55883

Given $a, b \in \mathbf{R}$, $|ab| = |a| \cdot |b|$.

Note 20

2b51f36fba524365b72001d318791436

Given $a, b \in \mathbf{R}$, $|a + b| \leq |a| + |b|$.

«Triangle inequality»

Note 21

4d6e77677e884f9c8ee877b9a32d48b5

Let $f : A \rightarrow B$. The function f is $\{\{c2: \text{one-to-one}\}\}$ if $\{\{c1::$

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

$\}\}$

Note 22

56b2bf81daaf419ab1207c6693c981e6

Let $f : A \rightarrow B$. The function f is $\{\{c2: \text{onto}\}\}$ if $\{\{c1::$

$$\text{the range of } f \text{ equals } B.$$

$\}\}$

Note 23

ccc8a358284a4b1f99f8e4336a2efdb9

Let $\{\{c4:: f : D \rightarrow \mathbf{R}_*\}\}$ and $\{\{c3:: B \subseteq \mathbf{R}_*\}\}$ $\{\{c2::$ The set

$$\{x \in D : f(x) = B\}$$

$\}\}$ is called $\{\{c1::$ the preimage of B under the function f . $\}$

Note 24

b72f131ae6734bf694fd8f987bb2323d

Let $f : D \rightarrow \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

■ Yes.

Note 25

5b3116f568a34fe2be32f403d7d081d9

Let $f : D \rightarrow \mathbf{R}$ and $A, B \subseteq \mathbf{R}$. Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

■ Yes.

Logic and Proofs

Note 1

a4d52b740f5b494696a5bdc956906cf2

Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem, $\{\{c1: \text{those conditions}\} \text{ are known } \{\{c2: \text{as the theorem's hypotheses.}\}\}$

Note 2

93f759e32dbf497cb30754e24c5b09f1

When in $\{\{c3: \text{a proof by contradiction}\} \{\{c2: \text{the contradiction is with the theorem's hypothesis,}\}\}$ the proof is said to be $\{\{c1: \text{contrapositive.}\}\}$

Note 3

1f45350926704df98b0abdf205f4319c

Two real number a and b are $\{\{c4: \text{equal}\} \{\{c3: \text{if and only if}\} \{\{c2: \text{for every real number } \epsilon > 0 \text{ it follows that } \{\{c1: |a - b| < \epsilon.\}\}\}\}$

Note 4

3ef90c9123e64df39ae9cd34271a7dcd

Two real number a and b are equal \iff for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$. What is the key idea in the proof?

■ By contradiction.

Note 5

aab4bb967d814e87bd85608277093755

Let $\{\{c3: S \subseteq \mathbf{N}.\}\}$ If $\{\{c2: S \text{ contains } 1\}\}$ and $\{\{c2: \text{whenever } S \text{ contains } n, \text{ it also contains } n + 1,\}\}$ then $\{\{c1: S = \mathbf{N}.\}\}$

Note 6

3dd92625856f408b9dc93fd36d82588d

Let $S \subseteq \mathbf{N}$. If S contains 1 and whenever S contains n , it also contains $n + 1$, then $S = \mathbf{N}$. This proposition is the fundamental principle behind $\{\{c1: \text{induction.}\}\}$

Note 7

40977a19a0d043c985df5676daa9f776

Does an induction argument imply the validity of the infinite case?

■ No, it doesn't.

Note 8

91b673c484b442ec92dd47ad0ef95f6c

Do De Morgan's rules hold for an infinite collection of sets?

■ Yes, they do.

Note 9

df9aa3b9e0c74da78d7e2a0a65276fcd

How De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

■ They are essentially the same.

The Axiom of Completeness

Note 1

d7df92f228f64fb28a9e353f0fcb3160

First, \mathbf{R} is $\{\{c1: \text{an ordered field, which contains } \mathbf{Q} \text{ as a subfield.}\}\}$

Note 2

6ac3816effb14ba682f20f91ae42bfd

What is the key distinction between \mathbf{R} and \mathbf{Q} ?

■ Axiom of Completeness.

Note 3

7c2ddbbcb52224d5cbad5c650d77e8a4f

$\{\{c1: \text{Every nonempty set of real numbers}\} \text{ that is } \{\{c2: \text{bounded above}\} \}$
 $\}$ has $\{\{c3: \text{a least upper bound.}\}\}$

« $\{\{c4: \text{Axiom of completeness}\}\}$ »

Note 4

fd dbb10e685c4ad49d1af25d241c03c0

Given a set $A \subseteq \mathbf{R}$, $\{\{c3: \text{a number } b \in \mathbf{R}\} \}$ such that $\{\{c2: a \leq b \text{ for all } a \in A\} \}$ is called $\{\{c1: \text{an upper bound for } A.\}\}$

Note 5

1edcfd8354464c81ab51da0d4f2f2ca4

A set $A \subseteq \mathbf{R}$ is $\{\{c2: \text{bounded above}\} \}$ if $\{\{c1: \text{there exists an upper bound for } A.\}\}$

Note 6

c757fa0c676941b0a4abbccb3a67fb2a

Given a set $A \subseteq \mathbf{R}$, $\{\{c3: \text{a number } b \in \mathbf{R}\} \}$ such that $\{\{c2: a \geq b \text{ for all } a \in A\} \}$ is called $\{\{c1: \text{a lower bound for } A.\}\}$

Note 7

3c9ba92f774e439dbcfb6c364a88f0ae

A set $A \subseteq \mathbf{R}$ is $\{\{c2: \text{bounded below}\} \}$ if $\{\{c1: \text{there exists a lower bound for } A.\}\}$

Note 8

9d2391299602497abd4fdfac14c71daa

Let $A \subseteq \mathbf{R}$. A real number s is the least upper bound for A if

- s is an upper bound for A ;
- if b is any upper bound for A , then $s \leq b$.

Note 9

5369939ee0f94abcaf65896355258f0d

The least upper bound of a set $A \subseteq \mathbf{R}$ is also frequently called the supremum of A .

Note 10

04884b60726641c6b8d7c2c3479f8b05

The least upper bound of a set $A \subseteq \mathbf{R}$ is denoted $\sup A$.

Note 11

afca84537fdd409e97254e6d36d736c3

Let $A \subseteq \mathbf{R}$. A real number s is the greatest lower bound for A if

- s is a lower bound for A ;
- if b is any upper bound for A , then $s \geq b$.

Note 12

41c9913ebc524f85be951737dc3e33e8

The greatest lower bound of a set $A \subseteq \mathbf{R}$ is also frequently called the infimum of A .

Note 13

7230c3d5f7ef4b62bc1fd6c5b94841f0

The greatest lower bound of a set $A \subseteq \mathbf{R}$ is denoted $\inf A$.

Note 14

51abcb89d7d486c9177cfc51b6e8721

Is it possible for a set $A \subseteq \mathbf{R}$ to have multiple upper bounds?

■ Yes.

Note 15

1c9d5ad3f35a47b0b12f27639fe4a409

Is it possible for a set $A \subseteq \mathbf{R}$ for have multiple least upper bounds?

■ No.

Note 16

8068979c7a6949fc9af88258008a9801

If s_1 and s_2 are both least upper bounds for a set $A \subseteq \mathbf{R}$, then

$\{\{c1::$

$$s_1 = s_2.$$

$\}\}$

Note 17

466b264de27a44d3bd21221e39347d2e

What is the key idea in the proof of well-definedness of the least upper bound?

■ $s_1 \leq s_2$ and $s_2 \leq s_1$.

Note 18

7100e899d7d44ffb89dbc0bac76ffb3f

Let $A \subseteq \mathbf{R}$. $\{\{c4::$ A real number $b\}\}$ is $\{\{c3::$ a maximum of $A\}\}$ if b is $\{\{c2::$ an element of $A\}\}$ and $\{\{c1::$ an upper bound for $A\}\}$

Note 19

5795e83831c14208a2d2b3dac0e2b139

Let $A \subseteq \mathbf{R}$. A real number b is $\{\{c3::$ a minimum of $A\}\}$ if b is $\{\{c2::$ an element of $A\}\}$ and $\{\{c1::$ a lower bound for $A\}\}$

Note 20

2ea41e2869754b64bdb6c221308f0c58

Let $A \subseteq \mathbf{R}$ and $\{\{c3::c \in \mathbf{R}\}\}$. Then $\{\{c2::c + A\}\} \stackrel{\text{def}}{=} \{\{c1:: \{c + a : a \in A\} \}\}$.

Note 21

f7518efac7b457d86040b99720ad110

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. Then

$$\sup(c + A) = c + \sup A.$$

Note 22

726f73a8cead495fa65f331e49a892ea

Let $s \in \mathbf{R}$ be an upper bound for a set $A \subseteq \mathbf{R}$. Then $s = \sup A$ if and only if, for every $\epsilon > 0$, there exists an element a in A satisfying $s - \epsilon < a$.

Note 23

4161e1c933ba4349978c94d951259701

Let $s \in \mathbf{R}$ be a lower bound for a set $A \subseteq \mathbf{R}$. Then $s = \inf A$ if and only if, for every $\epsilon > 0$, there exists an element a in A satisfying $s + \epsilon > a$.