

# Sets

## Note 1

097312afe75d4a3d9eaa0c1f4c63748a

Intuitively speaking,  $\{\{c2::a \text{ set}\}\}$  is  $\{\{c1::a \text{ collection of objects.}\}\}$

## Note 2

85e21cf985524b80a8c00eb4608f34be

Intuitively speaking, a set is a collection of objects.  $\{\{c2::\text{Those objects}\}$  are referred to as  $\{\{c1::\text{the elements of the set.}\}\}$

## Note 3

12b96daebbc04070b74e2a6f74e5b268

Given a set  $A$ , we write  $\{\{c2::x \in A\}\}$  if  $\{\{c1::x \text{ is an element of } A.\}\}$

## Note 4

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Given a set  $A$ , we write  $\{\{c2::x \notin A\}\}$  if  $\{\{c1::x \text{ is not an element of } A.\}\}$

## Note 5

39565306ec4e40e18136e7eb88fc817a

Given two sets  $A$  and  $B$ ,  $\{\{c1::\text{the union}\}\}$  is written  $\{\{c2::A \cup B.\}\}$

## Note 6

73bf0eb1d16c4c5da368e326b4739d5b

Given two sets  $A$ , and  $B$ ,  $\{\{c2::\text{the union}\}\}$  is  $\{\{c3::\text{defined}\}\}$  by the rule

$$x \in \{\{c2::A \cup B\}\} \text{ provided that } \{\{c1::x \in A \text{ or } x \in B.\}\}$$

## Note 7

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Given two sets  $A$  and  $B$ ,  $\{\{c1::\text{the intersection}\}\}$  is written  $\{\{c2::A \cap B.\}\}$

## Note 8

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Given two sets  $A$  and  $B$ ,  $\{\{c2::\text{the intersection}\}\}$  is  $\{\{c3::\text{defined}\}\}$  by the rule

$$x \in \{\{c2::A \cap B\}\} \text{ provided that } \{\{c1::x \in A \text{ and } x \in B.\}\}$$

## Note 9

684951afc378458aa7bd27e67cdc499b

The set of natural numbers is denoted  $\mathbf{N}$ .

## Note 10

49d36a026d4b4678ab86fb6103571cce

$$\mathbf{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

## Note 11

797c81e5adb543e1a5d4cc67e64c5e09

The set of integers is denoted  $\mathbf{Z}$ .

## Note 12

d3c61bf891744c58b73cef543c6e100d

$$\mathbf{Z} \stackrel{\text{def}}{=} \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

## Note 13

57f085776972449f8bc14daf5cff6603

The set of rational numbers is denoted  $\mathbf{Q}$ .

## Note 14

f7e3370650134607853b41b2b1ecf54b

$$\mathbf{Q} \stackrel{\text{def}}{=} \left\{ \text{all fractions } \frac{p}{q} \text{ where } p, q \in \mathbf{Z} \text{ and } q \neq 0 \right\}.$$

## Note 15

faeac83cb5b740b6964551c85ad3e35b

The set of real numbers is denoted  $\mathbf{R}$ .

## Note 16

6e5da98964d645d09ad6989e85679c74

The empty set is the set that contains no elements.

## Note 17

206db0a0f3d042e49a9ca532e222201f

The empty set is denoted  $\emptyset$ .

## Note 18

2f0448d226db4b71b150acaed349a73b

Two sets  $A$  and  $B$  are said to be disjoint if  $A \cap B = \emptyset$ .

### Note 19

e5d9d365e86640319ca5460ef8c4f05c

Given two sets  $A$  and  $B$ , we say  $\{\{c2::A \text{ is a subset of } B,\}$  or  $\{\{c2::B \text{ contains } A,\}$  if  $\{\{c1::\text{every element of } A \text{ is also an element of } B.\}$

### Note 20

c2bd27f1fc0d40e296dceef9c9789556

Given two sets  $A$  and  $B$ , the  $\{\{c3::\text{inclusion}\}$  relationship  $\{\{c2::A \subseteq B \text{ or } B \supseteq A,\}$  is used to indicate that  $\{\{c1::A \text{ is a subset of } B.\}$

### Note 21

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Given two sets  $A$  and  $B$ ,  $\{\{c2::A = B,\}$  means that  $\{\{c1::A \subseteq B \text{ and } B \subseteq A.\}$

### Note 22

74e93b42d46746dc9ec2b54f8366c435

Let  $A_1, A_2, A_3, \dots$  be an infinite collection of sets. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \dots$$

are all equivalent ways to indicate  $\{\{c1::\text{the set whose elements consist of any element that appears in at least on particular } A_n.\}$

### Note 23

69e4627a3e7149ef8be05479a2587b41

Let  $A_1, A_2, A_3, \dots$  be an infinite collection of sets. Notationally,

$$\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \cap \dots$$

are all equivalent ways to indicate  $\{\{c1::\text{the set whose elements consist of any element that appears in every } A_n.\}$

### Note 24

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Given  $A \subseteq \mathbf{R}$ ,  $\{\{c2::\text{the complement of } A,\}$  refers to  $\{\{c1::\text{the set of all elements of } \mathbf{R} \text{ not in } A.\}$

### Note 25

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Given  $A \subseteq \mathbf{R}$ ,  $\{\{c2::\text{the complement of } A,\}$  is written  $\{\{c1::A^c.\}$

## Note 26

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Given  $A, B \subseteq \mathbf{R}$ ,

$$\{\{c2: (A \cap B)^c\}\} = \{\{c1: A^c \cup B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

## Note 27

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Given  $A, B \subseteq \mathbf{R}$ ,

$$\{\{c2: (A \cup B)^c\}\} = \{\{c1: A^c \cap B^c.\}\}$$

« $\{\{c3: \text{De Morgan's Law}\}\}$ »

## Note 28

09322548137b46529467f2946a4952d4

What is the key idea in the proof of De Morgan's Laws?

■ Demonstrate inclusion both ways.

# Functions

## Note 1

18930cfe4e445779bcec8a2fb53f23c

Given two sets  $A$  and  $B$ , a function from  $A$  to  $B$  is a rule or mapping that takes each element  $x \in A$  and associates with it a single element of  $B$ .

## Note 2

dfa898ef047e418fa8dfe9ce9582fd71

If  $f$  is a function from  $A$  to  $B$ , we write  $f : A \rightarrow B$ .

## Note 3

c2730dafa0fe4bf4bede66b7199b48b9

Let  $f : A \rightarrow B$ . Given  $x \in A$ , the expression  $f(x)$  is used to represent the element of  $B$  associated with  $x$  by  $f$ .

## Note 4

65568f366ca949888310668475dbe570

Let  $f : A \rightarrow B$ . The set  $A$  is called the domain of  $f$ .

## Note 5

7870a310786142fa938bcc843ca8e1ae

Let  $f : A \rightarrow B$ . The set  $\{f(x) \mid x \in A\}$  is called the range of  $f$ .

## Note 6

716c208c9ae849b89ec722aa17f20882

Given a function  $f$  and a subset  $A$  of its domain, the set

$$\{f(x) : x \in A\}$$

is called the range of  $f$  over the set  $A$ .

## Note 7

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Given a function  $f$  and a subset  $A$  of its domain, the range of  $f$  over  $A$  is written  $f(A)$ .

### Note 8

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Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Is it unconditionally true that

$$f(A \cup B) = f(A) \cup f(B)?$$

■ Yes.

### Note 9

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Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Is it unconditionally true that

$$f(A \cap B) = f(A) \cap f(B)?$$

■ No.

### Note 10

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Given  $\{\{c4::a, b \in \mathbf{R}\}\}$  and  $\{\{c3::a \leq b\}\}$ ,  $\{\{c2::$ the set

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

$\}\}$  is called  $\{\{c1::$ a closed interval. $\}\}$

### Note 11

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Given  $a, b \in \mathbf{R}$  and  $\{\{c3::a < b\}\}$ ,  $\{\{c2::$ the set

$$\{x \in \mathbf{R} : a < x < b\}$$

$\}\}$  is called  $\{\{c1::$ an open interval. $\}\}$

### Note 12

3143096eb895471bac4b2d5840d18758

Given  $a, b \in \mathbf{R}$  and  $a \leq b$ ,  $\{\{c1::$ the closed interval

$$\{x \in \mathbf{R} : a \leq x \leq b\}$$

$\}\}$  is written  $\{\{c2::[a, b].\}\}$

### Note 13

604897f024bd4de78723fe8247290371

Given  $a, b \in \mathbf{R}$  and  $a \leq b$ ,  $\{\{c1::$ the open interval

$$\{x \in \mathbf{R} : a < x < b\}$$

$\}\}$  is written  $\{\{c2::(a, b).\}\}$

**Note 14**

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Let  $f(x) = x^2$ . Find two sets  $A$  and  $B$  for which

$$f(A \cap B) \neq f(A) \cap f(B).$$

■  $[-1, 0]$  and  $[0, 1]$ .

**Note 15**

6ed2fb1006634dcf81707a3c4d514857

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Then

$$\{f(A \cup B)\} = \{f(A) \cup f(B)\}.$$

**Note 16**

e088ae5ae1f24425a81dac09317978fd

Let  $f : D \rightarrow \mathbf{R}$ ,  $A, B \subseteq D$ . Then

$$\{f(A \cap B)\} \subseteq \{f(A) \cap f(B)\}.$$

**Note 17**

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Given  $x \in \mathbf{R}$ , the absolute value of  $x$  is denoted  $|x|$ .

**Note 18**

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Given  $x \in \mathbf{R}$ ,

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

**Note 19**

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Given  $a, b \in \mathbf{R}$ ,  $|ab| = |a| \cdot |b|$ .

**Note 20**

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Given  $a, b \in \mathbf{R}$ ,  $|a + b| \leq |a| + |b|$ .

«Triangle inequality»

### Note 21

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Let  $f : A \rightarrow B$ . The function  $f$  is  $\{\{c2: \text{one-to-one}\}\}$  if  $\{\{c1::$

$$a_1 \neq a_2 \text{ in } A \text{ implies that } f(a_1) \neq f(a_2) \text{ in } B.$$

$\}\}$

### Note 22

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Let  $f : A \rightarrow B$ . The function  $f$  is  $\{\{c2: \text{onto}\}\}$  if  $\{\{c1::$

$$\text{the range of } f \text{ equals } B.$$

$\}\}$

### Note 23

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Let  $\{\{c4:: f : D \rightarrow \mathbf{R}_*\}\}$  and  $\{\{c3:: B \subseteq \mathbf{R}_*\}\}$ . The set

$$\{x \in D : f(x) = B\}$$

$\}\}$  is called  $\{\{c1::$  the preimage of  $B$  under the function  $f$ . $\}$

### Note 24

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Let  $f : D \rightarrow \mathbf{R}$  and  $A, B \subseteq \mathbf{R}$ . Is it unconditionally true that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

■ Yes.

### Note 25

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Let  $f : D \rightarrow \mathbf{R}$  and  $A, B \subseteq \mathbf{R}$ . Is it unconditionally true that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)?$$

■ Yes.



# Logic and Proofs

## Note 1

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Many mathematical theorems are conditional statements, whose proofs deduce conclusions from conditions. Given such a theorem,  $\{\{c1::\text{those conditions}\}\}$  are known  $\{\{c2::\text{as the theorem's hypotheses.}\}\}$

## Note 2

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When in  $\{\{c3::\text{a proof by contradiction}\}\}$   $\{\{c2::\text{the contradiction is with the theorem's hypothesis,}\}\}$  the proof is said to be  $\{\{c1::\text{contrapositive.}\}\}$

## Note 3

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Two real number  $a$  and  $b$  are  $\{\{c4::\text{equal}\}\}$   $\{\{c3::\text{if and only if}\}\}$   $\{\{c2::\text{for every real number } \epsilon > 0 \text{ it follows that}\}\}$   $\{\{c1::|a - b| < \epsilon.\}\}$

## Note 4

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Two real number  $a$  and  $b$  are equal  $\iff$  for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ . What is the key idea in the proof?

■ By contradiction.

## Note 5

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Let  $\{\{c3::S \subseteq \mathbf{N}.\}\}$  If  $\{\{c2::S \text{ contains } 1\}\}$  and  $\{\{c2::\text{whenever } S \text{ contains } n, \text{ it also contains } n + 1,\}\}\}$  then  $\{\{c1::S = \mathbf{N}.\}\}$

## Note 6

3dd92625856f408b9dc93fd36d82588d

Let  $S \subseteq \mathbf{N}$ . If  $S$  contains 1 and whenever  $S$  contains  $n$ , it also contains  $n + 1$ , then  $S = \mathbf{N}$ . This proposition is the fundamental principle behind  $\{\{c1::\text{induction.}\}\}$

### Note 7

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Does an induction argument imply the validity of the infinite case?

■ No, it doesn't.

### Note 8

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Do De Morgan's rules hold for an infinite collection of sets?

■ Yes, they do.

### Note 9

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How De Morgan's rules for an infinite collection of sets defer from that for a finite collection?

■ They are essentially the same.