

# Uniform Convergence of a Sequence of Functions

## Note 1

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Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges pointwise on  $A$  to a function  $f$  if for all  $x \in A$

$$(f_n(x)) \xrightarrow{n \rightarrow \infty} f(x).$$

}

## Note 2

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Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges pointwise on  $A$  to  $f$ , we write

$$(f_n \rightarrow f) \quad \text{or} \quad \lim_{n \rightarrow \infty} f_n = f.$$

## Note 3

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Let  $f_n(x) = \frac{x^2 + nx}{n}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = (x).$$

## Note 4

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Let  $f_n(x) = x^n$ ,  $f_n : [0, 1] \rightarrow \mathbb{R}$ .

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

## Note 5

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Let  $(f_n)$  be a sequence of function on a set  $A$ . We say  $(f_n)$  converges uniformly on  $A$  to a function  $f$  if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \\ |f_n - f| < \epsilon.$$

}

## Note 6

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Let  $(f_n)$  be a sequence of function on a set  $A$ . If  $(f_n)$  converges uniformly on  $A$  to  $f$ , we write

$$f_n \rightrightarrows f.$$

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## Note 7

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What is the key distinction between the definitions of pointwise and uniform convergences of a sequence of functions?

■ The dependence of  $N$  on  $x$ .

## Note 8

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What is the visual behind the uniform convergence of a sequence of functions?

■ Eventually every  $f_n$  is completely contained in the  $\epsilon$ -strip.

## Note 9

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Which is stronger, uniform or pointwise convergence?

■ Uniform convergence is stronger.

## Note 10

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Uniform convergence implies pointwise convergence.

## Note 11

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Let  $(f_n)$  be a sequence of function on a set  $A$ .

$$(f_n \rightrightarrows f) \iff \sup_{n \rightarrow \infty} |f_n - f| \rightarrow 0.$$

(in terms of sup)

## Note 12

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Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $f_n \Rightarrow f$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

## Note 13

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Let  $(f_n)$  be a sequence of function on a set  $A$ . Then  $f_n \Rightarrow f$  if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall m, n \geq N \\ |f_n - f_m| < \varepsilon.$$

«[Cauchy Criterion]»

## Note 14

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What is the key idea in the proof of necessity of the Cauchy Criterion for uniform convergence?

■ Follows immediately from the definition.

## Note 15

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What is the key idea in the proof of sufficiency of the Cauchy Criterion for uniform convergence?

■ Define a candidate for the limit and prove by definition.

## Note 16

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you define a candidate for the limit?

■ Use the pointwise limit.

### Note 17

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, how do you know the pointwise limit exists?

■ Due to the Cauchy's Criterion for sequences.

### Note 18

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In the proof of sufficiency of the Cauchy Criterion for uniform convergence, we have  $f_n \rightarrow f$ . How do you show that  $f_n \rightrightarrows f$ ?

■ Take the limit of the inequality from the Cauchy Criterion.

### Note 19

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{c\}$  the convergence is uniform  $\}$  and  $\{c\}$  all  $f_n$  are continuous at  $c$ ,  $\}$  then  $\{c\}$   $f$  is continuous at  $c$ .  $\}$

### Note 20

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If the convergence is uniform and all  $f_n$  are continuous at  $c$ , then  $f$  is continuous at  $c$ .

« $\{c\}$ : Continuous Limit Theorem $\}$ »

### Note 21

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What is the key idea in the proof of the Continuous Limit Theorem for a series of functions?

■ Triple triangle inequality after adding and subtracting  $f_N$ .

### Note 22

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Let  $f_n \rightarrow f$  on a set  $A$  and  $c \in A$ . If  $\{c\}$  the convergence is uniform and all  $f_n$  are continuous at  $c$ ,  $\}$  then

$$\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \{c\} \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x) \}$$

### Note 23

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Let  $f_n \rightarrow f$  on a set  $A$ . If each  $f_n$  is continuous, but  $f$  is discontinuous, then  $\{c\}$  the convergence is not uniform.  $\}$