

Definition and Examples

Note 1

9080791fc8754b0bb88c381c10acbdffc

Let G be a group. If H is a subgroup of G we shall write

$$H \leq G.$$

}

Note 2

66e7f23728af4c9d8839d172e59d716a

Let G be a group and $H \leq G$. We shall denote the operation for H by the same symbol as the operation for G .

Note 3

e76ada2ee6da4b5fb71966e9f7ce3ded

Let G be a group. If $H \leq G$ and $H \neq G$ we shall write $H < G$.

Note 4

1d28c11c52c84bd0b639505598bb1dce

If H is a subgroup of G then any equation in the subgroup H may also be viewed as an equation in the group G .

Note 5

8f5b765961884460823141645b5ea08b

Let G be a group and $H \leq G$. What is the identity of H ?

■ The identity of G .

Note 6

7c122a5400f64eba9a76438c1ff296ee

Let G be a group and $H \leq G$. The identity of H is the identity of G . What is the key idea in the proof?

■ The identity is unique and it is the identity of G .

Note 7

83cba804764b43e2baf282ffec513694

Let G be a group. What is the minimal subgroup of G ?

■ The singleton $\{1\}$.

Note 8

d40df9f46a6b43ecb3fea9b5b37e5b1c

Let G be a group. What is the element that any subgroup of G must contain?

■ The identity of G .

Note 9

bc95ad6358814213a6fff2eb0cfd544b

Let G be a group and $H \leq G$. What is the inverse of an element x in H ?

■ The inverse of x in G .

Note 10

be9f1756cf3449e8a6718069fd4aedf5

Let G be a group and $H \leq G$. Why is the notation x^{-1} unambiguous?

■ The inverse in H is the same as the inverse in G .

Note 11

8aab93df8a5437eb3e50c3e0d438381

Let G be a group. The subgroup $\{1\}$ of G is called the trivial subgroup.

Note 12

eb859714e1f34f4db3dc35755f562945

Let G be a group. The trivial subgroup is denoted by $\{1\}$.

Note 13

74ab3ceb9a36420fb53dc2b3a22eb5f2

Which subgroups does any group have?

■ The trivial subgroup and the group itself.

Note 14

5683ff4198a74e9d988f501e925d85ad

If H is a subgroup of G and K is a subgroup of H , then K is a subgroup of G .

Note 15

50a07cbb14aa4bed8866efcbdb0be4d

Which object is considered in the Subgroup Criterion?

- Any subset of a group.

Note 16

840038893a3642a18c3e43c4e89aed17

What are the conditions of the Subgroup Criterion?

- The subset is nonempty and closed under $(x, y) \mapsto x \cdot y^{-1}$.

Note 17

71291d04ca2941fca2fc08759d8fd302

What is the special case considered in the Subgroup Criterion?

- The subset is finite.

Note 18

a1e69be09e78402d989b3805b3dfc54f

What are the conditions of the Subgroup Criterion for a finite subset?

- The subset is nonempty and closed under the operation.

Note 19

5bcd55a73e184bcd9bcc32f1ee47da2e

What is the key idea in the proof of the Subgroup Criterion for a finite subset?

- Any element's inverse is its n -th power.

Note 20

0e1ccaae016c4900ac96b733fb9e1764

Why is the set of 2-cycles in S_n not a subgroup of S_n ?

- It does not contain the identity.

Note 21

587390d0450f4681a66bcb8c0d5889c

Why is the set of reflection in D_{2n} not a subgroup of D_{2n} ?

■ It does not contain the identity.

Note 22

fc87d2283cb546708502ce325e326258

Why is the set of reflection in D_{2n} together with 1 not a subgroup of D_{2n} ?

■ Two distinct reflections induce a rotation.

Note 23

24b90e714649459ba38e6b40f07f6b2a

Is $\{1, r^2, s, sr^2\}$ a subgroup of D_8 ?

■ Yes.

Note 24

cac99978715a4ec894b296f8e1ce52f3

Is $\{1, r, s, sr\}$ a subgroup of D_8 ?

■ No.

Note 25

64ea968bdce94647b6fb2c351a60f2a2

Is $\{1, r^2, sr, sr^3\}$ a subgroup of D_8 ?

■ Yes.

Note 26

678b87f890ac4d8da5be6a78cb619358

Is $\{1, r, r^2\}$ a subgroup of D_8 ?

■ No.

Note 27

e036f3cc7667461b98e50e94ff3a8c80

Is $\{1, r, r^2, r^3\}$ a subgroup of D_8 ?

■ Yes.

Note 28

209944ca7a524af3be44b398de974c2d

Give an example of a group and its infinite subset that is closed under the operations, but is not a subgroup of the original group.

■ Positive integers under addition.

Note 29

547363a46106478187c20c5cbb868461

For what groups is the notion of the torsion subgroup introduced?

■ For abelian groups.

Note 30

d29b9ffdb46c4c909fbfb2a438abb0a0

What is the torsion subgroup of an abelian group?

■ The set of all the elements of a finite order.

Note 31

b2a854579339471d8ae41776f1661f29

Let G be an abelian group. What is the name of the set

$$\{g \in G : |g| < \infty\} ?$$

■ The torsion subgroup of G .

Note 32

2a685e6476b94b9eac539a17441574ef

Why is the notion of the torsion subgroup introduced only for abelian groups?

■ For non-abelian groups the set is not guaranteed to form a subgroup.

Note 33

22a771a961c3498f88a030fabf778797

Give an example of a non-abelian group, who's "torsion subgroup" is not actually a subgroup.

■ $GL_3(\mathbb{R})$

Note 34

f4edb9436c094103b0b9b82019185296

Give an example of two elements a, b in $GL_3(\mathbb{R})$ such that

$$|a|, |b| < \infty \quad \text{and} \quad |ab| = \infty.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note 35

f2c41a74f8a04bb892b471915e533055

What is the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$?

■ The set of elements whose first component is 0.

Note 36

4df6b5997d30483fb469565c89630322

When is the union of two subgroups also a subgroup?

■ If and only if one of the subgroups is a subset of the other.

Note 37

b5f23dcaec101461e902f34191451e112

When is the union of an infinite number of subgroups also a subgroup?

■ It depends.

Note 38

60129b39ceab4468915a6d2237915c1a

Let H and K be subgroups of G and $H \subseteq K$. What do we know about $H \cup K$?

■ It is a subgroup of G .

Note 39

791301f78ecf4800a13e3a0299c57028

Let H and K be subgroups of G . If $H \cup K$ is a subgroup of G , then $H \subseteq K$ or $K \subseteq H$. What is the key idea in the proof?

■ By contradiction.

Note 40

cc8decdf60194667b3b27ff0941c9fc0

What is the special linear group?

■ The set of square matrices whose determinant is 1.

Note 41

1e420dd97e1942b3b7bc70d71fc0953e

The special linear group of $n \times n$ matrices over a field F is denoted $SL_n(F)$.

Note 42

941aeeb281da4b009ffdc95864eddb3b

When is the intersection of two subgroups also a subgroup?

■ Always.

Note 43

887cf7600d994fed9662e35fc9719c62

When is the intersection of an infinite number of subgroups also a subgroup?

■ Always.

Note 44

3bdd7a0f0e044c6b9c3c1811d4478f10

Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of G . Then $\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G .

Centralizers and Normalizers, Stabilizers and Kernels

Note 1

93de251693e74655a5752529379e7081

For what do we define centralizers in groups?

■ For nonempty subsets of the group.

Note 2

b46233e067ea4c24b38af57081ef1db3

Let G be a group and A be a nonempty subset of G . The set

$$\{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$$

is called the centralizer of A in G .

Note 3

3c2adb104b55494a8a248b4e6cf72980

Let G be a group and A be a nonempty subset of G . The centralizer of A in G is denoted

$$C_G(A).$$

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Note 4

aeaa9d02d1a8429ab94927313c1e2194

How can centralizers be redefined in terms of commutativity?

■ As the set of all the elements that commute with every element of the subset.

Note 5

ae3968a709ba423b91c84596e63977c7

How do we call the set of elements of a group G that commute with every element of a given subset A of G ?

■ The centralizer of A in G .

Note 6

588fd51b4281485c87a74faa9ddb8f5

Let G be a group and A be a nonempty subset of G . The centralizer of A in G forms a subgroup of G .

Note 7

18e60ffefbe647a6aa9b7a9feeb58ef1

Let G be a group and A be a nonempty subset of G . When is the centralizer of A in G a subgroup of G ?

■ Always.

Note 8

23eee6bafc20447987eaab729108324e

Let G be a group and A be a nonempty subset of G . In the special case when $A = \{a\}$ we shall write $\{\{c1::\text{simply } C_G(a)\}\}$ instead of $\{\{c2:: C_G(\{a\}).\}\}$

Note 9

92e1f52031224232bf8ac69f4014862c

Let G be a group and $a \in G$. Then

$$\{\{c1::\langle a \rangle\}\} \subseteq C_G(a) .$$

Note 10

41ad6753d3624f379c6b0e82c31be987

Let G be a group and $a \in G$. Then

$$C_G(a^{-1}) = \{\{c1:: C_G(a).\}\}$$

Note 11

656f69d38b7e4f41882f7feda5410dde

$$C_{Q_8}(i) = \{\{c1:: \{1, -1, i, -i\} .\}\}$$

Note 12

1fd69e94ef324e30a0054ea4860105e4

$$C_{Q_8}(1) = \{\{c1:: Q_8 .\}\}$$

Note 13

1ec04d8a609e442690da0ee9332a9647

For what do we define centers in groups?

■ For the group itself.

Note 14

936431cf3df24996965ce022800fa1bc

Let G be a group. The set of elements of G commuting with all elements of G is called the center of G .

Note 15

4fdf7ab9640d453a9eb90b77b45f35b2

Let G be a group. The center of G is denoted $Z(G)$.

Note 16

b7d3c0377db64825b9428611799a938c

Let G be a group. The center of G forms a subgroup of G .

Note 17

3c3e0bd81b194850bbee0d6688646ea

Let G be a group. When is the center of G a subgroup of G ?

■ Always.

Note 18

10b56b05fab34d6a91d494a9c515f2a4

Let G be a group. The center of G is the centralizer of G in G .

Note 19

4c598c0713ef4d649fe3629dfcd8a0c7

For what do we define normalizers in groups?

■ For nonempty subsets.

Note 20

ea05e39de520479892867fd132778337

Let G be a group and A be a nonempty subset of G . The set

$$\{g \in G \mid gAg^{-1} = A\}$$

is called the normalizer of A in G .

Note 21

c4b4c4424c6549c6b9afccb2945d17ee

Let G be a group and A be a nonempty subset of G . The normalizer of A in G is denoted

$$N_G(A).$$

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Note 22

f831383807f34538b90b130d417dfb95

Let G be a group and A be a nonempty subset of G . The normalizer of A in G forms a subgroup of G .

Note 23

b536e255fed54b02a1036b9baf6a7dc6

Let G be a group and A be a nonempty subset of G . When is the normalizer of A in G a subgroup of G ?

Always.

Note 24

2f5272d0d29344f79fee3fe3bc9cd61b

Let G be a group. How do $N_G(A)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G ?

$C_G(A) \leq N_G(A)$.

Note 25

18e1e07cc3434babbe772c427d74c950

Let G be a group. How do $Z(G)$ and $C_G(A)$ relate for an arbitrary nonempty subset A of G ?

$Z(G) \leq C_G(A)$.

Note 26

3a6dffba1da649fdb21855ef88b93490

How do $Z(G)$ and $N_G(A)$ relate for an arbitrary nonempty subset A of G ?

$$\mathbf{I} \quad Z(G) \leq N_G(A).$$

Note 27

730b6cdac2414954adf98bd2792c58c2

What is the smallest possible centralizer in a group?

\mathbf{I} The center of the group.

Note 28

902c795be20e428bb4c2b4872658d5a8

Let G be a group. Then

$$C_G(\langle G \rangle) = \langle Z(G) \rangle$$

Note 29

552be3f0d86c4b46930159c7dc731d54

What is the largest possible centralizer in a group?

\mathbf{I} The group itself.

Note 30

eaece66528de84c1f8c9fad762b3a6447

Let G be a group. Then

$$C_G(\langle 1 \rangle) = \langle G \rangle$$

Note 31

b2cf1d78fcb14ecf8264f750afafece2

Let G be group. Then $Z(G) = G$ if and only if G is abelian.

Note 32

90ce2f162b9d4f76b662613ffb40ced

Let G be an abelian group and A be a nonempty subset of G . Then

$$C_G(A) = \langle G \rangle$$

Note 33

cfaa88a96e604867bcd19b10b115de3e

Let G be an abelian group and A be a nonempty subset of G .
Then

$$N_G(A) = \langle G \rangle$$

Note 34

98570a89b2904b749fe9cc594271e851

$$C_{D_8}(\{1, r, r^2, r^3\}) = \langle \{1, r, r^2, r^3\} \rangle$$

Note 35

b19776816d5e4309b70ee8d73e045e52

$$N_{D_8}(\{1, r, r^2, r^3\}) = \langle D_8 \rangle$$

Note 36

d850d1653ce94d3780dcb94b6080430

How do you show that $N_{D_8}(\{1, r, r^2, r^3\}) = D_8$?

| r and s must be included; as a subgroup the normalizer must be closed under multiplication.

Note 37

fadcb63215d9471db6ae7ebe8e76ee63

$$Z(D_6) = \langle \{1\} \rangle$$

Note 38

46db60ed1736470f9534921d4505ea62

$$Z(D_8) = \langle \{1, r^2\} \rangle$$

Note 39

2ce99da32df545fe9ec31ee8e0206c7f

$$C_{S_3}(\{1, (1\ 2)\}) = \langle \{1, (1\ 2)\} \rangle$$

Note 40

2e782a1f12a6406db1161635406da5c7

How do you show that

$$C_{S_3}(\{1, (1\ 2)\}) = \{1, (1\ 2)\}?$$

■ $\{1, (1\ 2)\}$ is a subset + Lagrange's Theorem.

Note 41

d87a8e5254e643f1adf49c5fafb28600

$$N_{S_3}(\{1, (1\ 2)\}) = \{1, (1\ 2)\}.$$

Note 42

bd9c2331e5514956b74f445ad054106c

Let G be a group and $a \in G$. Then

$$N_G(\{1, a\}) = C_G(a).$$

Note 43

aed1e65dd5c9446a9de9fc7b16d42025

$$Z(S_3) = \{1\}.$$

Note 44

9b743f8b30c84ac99b22a6eb0f967267

Let G be a group acting on a set S and $s \in S$. The stabilizer of s in G is denoted

$$G_s.$$

Note 45

6bc0f129d62348a487c114d0b7ad69f8

Let D_8 act on the set of four vertices of a square. What is the stabilizer of a vertex a in D_8 ?

■ The identity and the reflection around the line of symmetry passing through a .

Note 46

041877bf3b974f4f8ac78eea7bb331f7

What is the kernel of the action of D_8 on the set of four vertices of a square?

■ The identity subgroup.

Note 47

74e999e194f9439ea1a595397d377d68

Let G be a group and A be a nonempty subset of G . Then $\langle N_G(A) \rangle$ is the stabilizer of A in G under the action of G on $\mathcal{P}(G)$ by left conjugation.

(in terms of group actions)

Note 48

39f5a3ad85424edc82a2480cee84987a

Let G be a group and A be a nonempty subset of G . Then $\langle C_G(A) \rangle$ is the kernel of the action of $N_G(A)$ on A by left conjugation.

(in terms of group actions)

Note 49

ee92b4ef2d764cf8b83c56d984146ff4

Let G be a group. The $\langle Z(G) \rangle$ is the kernel of the action of G on G by left conjugation.

(in terms of group actions)

Note 50

7b91acc9b3d04a08b282faf013ef934c

Let G be a group.

$$C_G(Z(G)) = \langle G \rangle$$

Note 51

0448d1d230f340db94d0400519d257f1

Let G be a group.

$$N_G(Z(G)) = \langle G \rangle$$

Note 52

0d151553fc4c40c2ae19df1526b809da

Let G be a group and $A \subseteq B \subseteq G$. How do $C_G(A)$ and $C_G(B)$ relate?

|

$$C_G(B) \leq C_G(A).$$