Finite Elements

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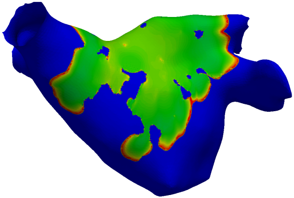
Why FEM



- Can solve PDEs on complicated domains.
- Have flexibility to increase order of accuracy and match the numerics to the physics.
- FEM has an elegant mathematical formulation that lends itself to mathematical analysis and flexible code implementation.

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This course

- In this course we blend theoretical analysis with computer implementation.
- 14 lectures + 3 tutorials (Dr Cotter) - assessed by examination (50%).
- 15 computer labs (Dr Ham) - assessed by coursework (50%)
- Dr Cotter Office Hour (Huxley 755): 5-6pm Mondays.

Introduction

Definition (Poisson's equation in the unit square)

Let $\Omega = [0,1] \times [0,1]$ be the unit square. For a given function f, we seek u such that

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u := -\nabla^2 u = f,$$

$$u(0,y) = u(1,y) = 0,\ \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0. \tag{1}$$

Triangulation

The finite element space that we shall use is defined on a triangulation.

Definition

Triangulation Let Ω be a polygonal subdomain of $\mathbb{R}^2$. A triangulation $\mathcal{T}$ of Ω is a set of triangles $\{K_i\}_{i=1}^N$, such that:

1. $\operatorname{int} K_i \cap K_j = \emptyset, \quad i \neq j$, where int denotes the interior of a set.
2. $\cup K_i = \bar{\Omega}$, the closure of Ω.
3. No vertex of the interior of any triangle is located in the interior of any other triangle in $\mathcal{T}$.

P1

Definition (P1 finite element space)

Let $\mathcal{T}$ be a triangulation of Ω. Then the P1 finite element space is a space V_h containing all functions v such that

1. $v \in C^0(\Omega)$,
2. $v|_{K_i}$ is a linear function for each $K_i \in \mathcal{T}$.

We also define the following subspace,

$$\mathring{V}_h = \{v \in V_h : v(0, y) = v(1, y) = 0\}. \tag{2}$$

Definition (L^2 norm)

For a real-valued function f on a domain Ω, with Lebesgue integral

$$\int_\Omega f(x)\,\mathrm{d}x, \tag{3}$$

we define the L^2 norm of f,

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\,\mathrm{d}x\right)^{1/2}. \tag{4}$$

Definition (L^2)

We define $L^2(\Omega)$ as the set of functions

$$L^2(\Omega) = \{f : \|f\|_{L^2}(\Omega) < \infty\}, \tag{5}$$

and identify two functions f and g if $\|f - g\|_{L^2(\Omega)} = 0$.

Example

Consider the two functions f and g defined on $\Omega = [0,1] \times [0,1]$ with

$$f(x,y) = \begin{cases} 1 & x \geq 0.5, \\ 0 & x < 0.5, \end{cases}, g(x,y) = \begin{cases} 1 & x > 0.5, \\ 0 & x \leq 0.5, \end{cases}. \quad (6)$$

Since f and g only differ on the line $x = 0.5$ which has zero area, then $\|f - g\|_{L^2(\Omega)} = 0$, and so $f \equiv g$ in L^2.

Definition (Finite element derivative)

The finite element partial derivative $\frac{\partial^{FE}}{\partial x_i}u$ of u is defined in $L^2(\Omega)$ such that restricted to K_i, we have

$$\frac{\partial^{FE} u}{\partial x_i}\Big|_{K_i} = \frac{\partial u}{\partial x_i}. \tag{7}$$

Exercise

Let V_h be a P1 finite element space for a triangulation $\mathcal{T}$ of Ω. For all $u \in V_h$, show that the definition above uniquely defines $\frac{\partial^{FE} u}{\partial x_i}$ in $L^2(\Omega)$.

Exercise

Let $u \in C^1(\Omega)$. Show that the finite element partial derivative and the usual derivative are equal in $L^2(\Omega)$.

Remark

In view of this, in this section we will consider all derivatives to be finite element derivatives. In later sections we shall consider an even more general definition of the derivative which contains both of these definitions.

Towards the finite element discretisation

Proceeding with the finite element discretisation, we assume that we have a solution u to Equation (1) that is smooth (i.e. $u \in C^1$). We take $v \in \mathring{V}_h$, multiply by Equation (1), and integrate over the domain. Integration by parts in each triangle then gives

$$\sum_i \left(\int_{K_i} \nabla v \cdot \nabla u \, \mathrm{d}x - \int_{\partial K_i} v n \cdot \nabla u \, \mathrm{d}S \right) = \int_\Omega v f \, \mathrm{d}x, \qquad (8)$$

where n is the unit outward pointing normal to K_i.

Next, we consider each interior edge f in the triangulation, formed as the intersection between two neighbouring triangles $K_i \cap K_j$. If $i > j$, then we label the K_i side of f with a $+$, and the K_j side with a $-$. Then, denoting Γ as the union of all such interior edges, we can rewrite our equation as

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x - \int_\Gamma v n^+ \cdot \nabla u + v n^- \cdot \nabla u \,\mathrm{d}S - \int_{\partial\Omega} v n \cdot \nabla u \,\mathrm{d}S = \int_\Omega v f \,\mathrm{d}x, \tag{9}$$

where $n^\pm$ is the unit normal to f pointing from the $\pm$ side into the $\mp$ side. Since $n^- = -n^+$, the interior edge integrals vanish.

Further, on the boundary, either v vanishes (at $x = 0$ and $x = 1$) or $n \cdot \nabla u$ vanishes (at $y = 0$ and $y = 1$), and we obtain

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}x = \int_\Omega v f \, \mathrm{d}x. \tag{10}$$

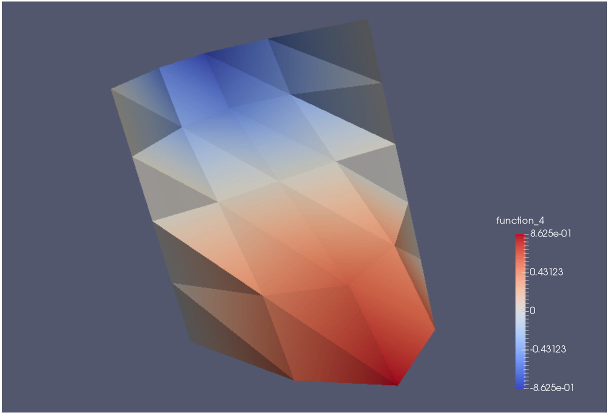
The finite element approximation is then defined by requiring that this equation holds for all $v \in \mathring{V}_h$ and when we restrict $u \in \mathring{V}_h$.

Definition (Finite element approximation)

The finite element approximation $u_h \in \mathring{V}_h$ to the solution u of Poisson's equation is defined by

$$\int_\Omega \nabla v \cdot \nabla u_h \, \mathrm{d}x = \int_\Omega v f \, \mathrm{d}x, \quad \forall v \in \mathring{V}_h. \tag{11}$$

Some plots



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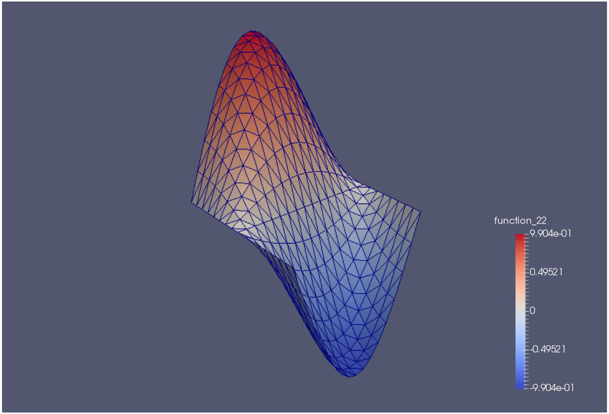
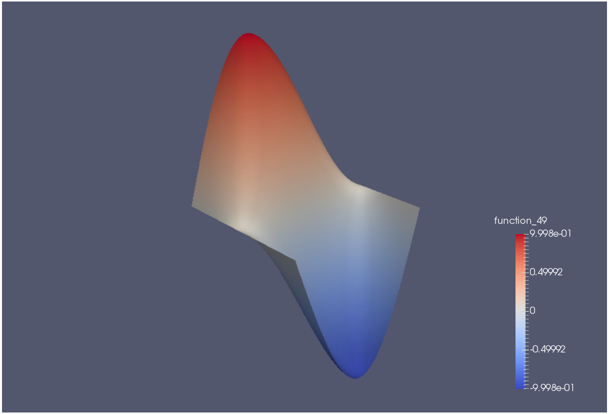
Some plots



Some plots



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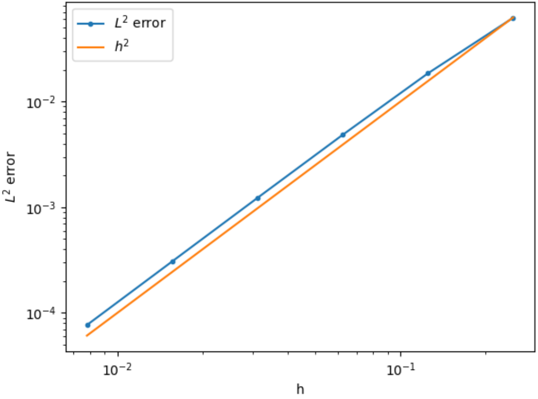
Convergence of error



Several questions arise at this point.

1. Is u_h unique?
2. What is the size of the error $u - u_h$?
3. Does this error go to zero as the mesh is refined?
4. For what types of functions f can these questions be answered?
5. What other kinds of finite element spaces are there?
6. How do we extend this approach to other PDEs?
7. How can we calculate u_h using a computer?

We shall aim to address these questions, at least partially, through the rest of this course. For now, we concentrate on the final question.

In this course we shall mostly concentrate on finite element methods for elliptic PDEs, of which Poisson's equation is an example, using continuous finite element spaces, of which $P1$ is an example. The design, analysis and implementation of finite methods for PDEs is a huge field of current research, and includes parabolic and elliptic PDEs and other PDEs from elasticity, fluid dynamics, electromagnetism, mathematical biology, mathematical finance, astrophysics and cosmology, etc. This course is intended as a starting point to introduce the general concepts that can be applied in all of these areas.

Definition (Nodal basis for $P1$)

Let $\{z_i\}_{i=1}^M$ indicate the vertices in the triangulation $\mathcal{T}$. For each vertex z_i, we define a basis function $\phi_i \in V_h$ by

$$\phi_i(z_j) = \delta_{ij} := \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{12}$$

We can define a similar basis for $\mathring{V}_h$ by removing the basis functions ϕ_i corresponding to vertices z_i on the Dirichlet boundaries $x = 0$ and $x = 1$.

If we expand u_h and v in the basis for $\mathring{V}_h$,

$$u_h(x) = \sum_i u_i \phi_i(x), \quad v(x) = \sum_i v_i \phi_i(x), \tag{13}$$

into Equation (11), then we obtain

$$\sum_i v_i \left(\sum_j \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}x v_j - \int_\Omega \phi_i f \,\mathrm{d}x \right) = 0. \tag{14}$$

Since this equation must hold for all $v \in \mathring{V}_h$, then it must hold for all basis coefficients v_i, and we obtain the matrix-vector system

$$K\mathrm{u} = \mathrm{f}, \tag{15}$$

where

$$K_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}x, \tag{16}$$

$$\mathrm{u} = \left(u_1, u_2, \ldots, u_{\bar{M}}\right)^T, \tag{17}$$

$$\mathrm{f} = \left(f_1, f_2, \ldots, f_{\bar{M}}\right)^T, \quad f_i = \int_\Omega \phi_i f \,\mathrm{d}x. \tag{18}$$

The system is square, but we do not currently know that K is invertible. This is equivalent to the finite element approximation having a unique solution u_h, which we shall establish in later sections.

Putting solvability aside for the moment, the goal of the implementation sections of this course is to explain how to efficiently form K and f, and solve this system. For now we note a few following aspects that suggest that this might be possible. First, the matrix K and vector f can be written as sums over elements,

$$K_{ij} = \sum_{K \in \mathcal{T}} \int_K \nabla \phi_i \cdot \nabla \phi_j \,\mathrm{d}x, \quad f_i = \sum_{K \in \mathcal{T}} \int_K \phi_i f \,\mathrm{d}x. \qquad (19)$$

For each entry in the sum for K_{ij}, the integrand is composed entirely of polynomials (actually constants in this particular case, but we shall shortly consider finite element spaces using polynomials of higher degree).

- This motivates our starting point in exposing the computer implementation, namely the integration of polynomials over triangles using quadrature rules. T
- This will also motivate an efficient way to construct derivatives of polynomials evaluated at quadrature points.
- Further, we shall shortly develop an interpolation operator $\mathcal{I}$ such that $\mathcal{I}_f \in V_h$. If we replace f by $\mathcal{I}_f$ in the approximations above, then the evaluation of f_i can also be performed via quadrature rules.

- Further, the matrix K is very sparse, since in most triangles, both ϕ_i and ϕ_j are zero. Any efficient implementation must make use of this and avoid computing integrals that return zero.
- This motivates the concept of global assembly, the process of looping over elements, computing only the contributions to K that are non-zero from that element.
- Finally, the sparsity of K means that the system should be solved using numerical linear algebra algorithms that can exploit this sparsity.