

# Forcing and the independence of the continuum hypothesis

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## **Abstract**

In these notes, intended as the plaintext part of the Flypitch project, we give a complete account of the independence of the continuum hypothesis from ZFC, with special attention paid to comparing the different approaches: generic sets, Boolean-valued models, and double-negation sheaves.

# 1 Preliminaries

## 1.1 First-order logic

### 1.1.1 General logical symbols

**Definition 1.1.** We reserve the following general logical symbols:

$\neg$	not
$\vee$	or
$\wedge$	and
$\forall$	for all
$\exists$	exists
$=$	equals
$(, )$	parentheses
$(x_i)_{i:\mathbb{N}}, (y_i)_{i:\mathbb{N}}, (z_i)_{i:\mathbb{N}}$	variables

### 1.1.2 First-order languages

**Definition 1.2.** A (first-order, one-sorted) **language**  $\mathcal{L}$  comprises the following data:

- (i) A collection of **constant symbols**  $\text{Const}(\mathcal{L})$ ,
- (ii) a collection of **relation symbols**  $\text{Rel}(\mathcal{L})$ ,
- (iii) a collection of **function symbols**  $\text{Funct}(\mathcal{L})$ , and
- (iv) an assignment of each symbol  $S : \text{Const}(\mathcal{L}) \cup \text{Rel}(\mathcal{L}) \cup \text{Funct}(\mathcal{L})$  to a natural number  $\text{arity}(S) : \mathbb{N}$ .

Whenever we interpret a language on some carrier  $A$ , we mean for constants  $c$  to be interpreted as elements of  $A^{\text{arity}(c)}$ , relations  $R$  to be interpreted as subsets of  $A^{\text{arity}(R)}$ , and for function symbols to be interpreted as functions  $A^{\text{arity}(f)} \rightarrow A$ .

**Example 1.3.** • The language of groups comprises a 1-ary constant symbol for the identity and a 2-ary function for group multiplication.

- The language of rings comprises constant symbols 0 and 1 and 2-ary functions for addition and multiplication.
- The language of set theory comprises just one 2-ary relation  $\in$ .

### 1.1.3 Terms, formulas, and sentences

**Definition 1.4.** A **term** is a string of symbols defined by structural induction as follows:

1. Any variable  $v$  is a term.
2. Any constant  $c$  is a term.
3. If  $t_1, \dots, t_n$  are terms of arities  $a_1, \dots, a_n$ , then  $(t_1, \dots, t_n)$  is a term of arity  $a_1 + \dots + a_n$ .
4. If  $t$  is a term and  $f$  is a function symbol with matching arities, then  $ft$  is a term.

Whenever we interpret our language on a carrier  $A$ , we mean for terms to be interpreted as functions into  $A$  which we can construct by composing existing constants (constant functions), basic functions (i.e. the interpretations of the function symbols), and variables (identity).

**Definition 1.5.** A **formula** is defined by structural induction as follows:

1. If  $t_1$  and  $t_2$  are terms of the same arity,  $t_1 = t_2$  is a formula.
2. If  $t$  is a term and  $R$  is a relation symbol, and  $t$  and  $R$  have the same arity, then  $Rt$  is a formula.
3. If  $\varphi$  is a formula,  $\neg\varphi$  is a formula.
4. If  $\varphi$  and  $\psi$  are formulas, then  $\varphi \vee \psi$  is a formula.
5. If  $\varphi$  and  $\psi$  are formulas, then  $\varphi \wedge \psi$  is a formula.
6. If  $\varphi$  is a formula containing a variable  $v$ , then  $\exists v\varphi$  is a formula.
7. If  $\varphi$  is a formula containing a variable  $v$ , then  $\forall v\varphi$  is a formula.

**Definition 1.6.** Let  $\varphi$  be a formula containing the variables  $x_1, \dots, x_n$ . We say that the variable  $x_k$  is **free** if  $x_k$  is not contained in a subformula of the form  $\exists x_k\psi$  or  $\forall x_k\psi$ .

$x_k$  is **bound** if it is not free.

**Definition 1.7.** A formula is a **sentence** (or **statement**) if it contains no free variables.

We write  $\text{Formulas}(\mathcal{L})$  for all the first-order formulas of  $\mathcal{L}$ , and we write  $\text{Sentences}(\mathcal{L})$  for all the first-order sentences of  $\mathcal{L}$ .

By convention, we always include sentences called **true** and **false**.

#### 1.1.4 Predicate calculus and provability

Throughout this section, we fix a language  $\mathcal{L}$ .

**Definition 1.8.** A **propositional function** is a function  $f : \text{Prop}^k \rightarrow \text{Prop}$ , for some  $1 < k : \mathbb{N}$  which we define inductively as follows:

1. The constant functions to **true** and **false** are propositional functions.
2. Each projection  $(P_1, \dots, P_k) \mapsto P_j$  is a propositional function.
3. If  $f$  and  $g$  are propositional functions, so are

$$\neg f, f \wedge g, f \vee g, f \rightarrow g, \text{ and } f \leftrightarrow g,$$

where the operations above are carried out pointwise in  $\text{Prop}$ .

$f$  is a **tautology** if  $\vdash \forall \vec{p} : \text{Prop}^k, f\vec{p} \leftrightarrow \text{true}$ .

**Definition 1.9.** A **propositional combination** is a function  $f : \text{Sentences}(\mathcal{L})^k \rightarrow \text{Sentences}(\mathcal{L})^k$ , for some  $1 < k : \mathbb{N}$  which we define inductively as follows:

1. Each projection  $(B_1, \dots, B_k) \mapsto B_j$  is a propositional combination.

2. If  $f$  and  $g$  are propositional combinations, so are

$$\neg f, f \wedge g, f \vee g, f \rightarrow g, \text{ and } f \leftrightarrow g,$$

where the operations are carried out pointwise in  $\text{Sentences}(\mathcal{L})$ .

By sending projections to projections and symbols  $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$  to the corresponding operations on  $\text{Prop}$ , every propositional combination  $f : \text{Sentences}(\mathcal{L})^k \rightarrow \text{Sentences}(\mathcal{L})$  can be realized as a propositional function  $\mathbf{r}(f) : \text{Prop}^k \rightarrow \text{Prop}$ .

**Definition 1.10.** The **predicate calculus** comprises the following rules for deducing sentences from other sentences. We call deducible sentences **valid**, and write  $\vdash_{\mathcal{L}} \varphi$  to mean that the  $\mathcal{L}$ -sentence  $\varphi$  is valid (and to disambiguate from  $\vdash$ , which when used unadorned means “provable in the metatheory”).

- (a) (Rule of the propositional calculus) if  $f$  is a propositional combination taking  $k$  arguments such that  $\mathbf{r}(f)$  is a tautology, then for any  $k$  sentences  $A_1, \dots, A_k$ , the value of the propositional combination  $f(\varphi_1, \dots, \varphi_k)$  is a valid sentence.
- (b) (Rule of modus ponens) If  $A$  and  $A \rightarrow B$  are valid, then  $B$  is valid.
- (c) (Rules of equality)
  - (i)  $\forall x, x = x$ ,  $\forall x \forall y, x = y \wedge y = x$ , and  $\forall x \forall y \forall z, x = y \wedge y = z \rightarrow x = z$  are all valid.
  - (ii) Let  $\varphi(x)$  be a formula whose only free variable is  $x$ . Then

$$\forall x \forall y, (x = y) \rightarrow (\varphi(x) \rightarrow \varphi(y))$$

is valid.

- (d) (Change of variable) If  $A$  is a sentence and  $A'$  represents  $A$  with all instances of a variable  $x$  switched to  $y$ , then  $A \leftrightarrow A'$  is valid.
- (e) (Rule of specialization “ $\forall$ -elimination”) Let  $c$  be any constant symbol, and let  $\varphi(x)$  be a formula whose only free variable is  $x$ . Then  $(\forall x \varphi(x)) \rightarrow \varphi(c)$  is valid.
- (f) (“ $\neg$ -introduction”) If  $\neg A \leftrightarrow (A \rightarrow \text{false})$  is valid.
- (g) (Generalization of constants “ $\forall$ -introduction”) Let  $B$  be a sentence which does not contain the constant  $c$  or the variable  $x$ . Let  $\varphi(x)$  be some formula such that  $\varphi(c) \rightarrow B$  is valid. Then  $\exists x \varphi(x) \rightarrow B$  is also valid.<sup>1</sup>
- (h) (de Morgan laws) Let  $\varphi(x)$  have  $x$  as its only free variable. Let  $B$  be a sentence which does not contain  $x$ . Then the following are valid statements:

$$(\neg(\forall x \varphi(x))) \leftrightarrow (\exists x \neg \varphi(x))$$

$$((\forall x \varphi(x)) \wedge B) \leftrightarrow ((\forall x (\varphi(x) \wedge B))$$

$$((\exists x \varphi(x)) \wedge B) \leftrightarrow ((\exists x (\varphi(x) \wedge B))$$

**Definition 1.11.** Let  $S$  be a collection of sentences.

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<sup>1</sup>In particular, using the next rule, if  $\neg \varphi(c) \rightarrow \text{false}$  is valid, so is  $\exists x \neg \varphi(x) \rightarrow \text{false}$ , so is  $\neg \exists x \neg \varphi(x)$ , and therefore so is  $\forall x \varphi(x)$ .

- (i) We say that  $A$  is provable from  $S$  if there exist finitely many  $B_1, \dots, B_n : S$  such that  $(B_1 \wedge \dots \wedge B_n) \rightarrow A$  is valid.
- (ii) We say that  $S$  is consistent if **false** is not valid.

**Remark 1.12.** One may wonder why we work with a type of formulas and not with a collection of Props directly. The problem with this is that everything needs to be typed, and so to reason about a predicate (say “ $\in$ ”) using **Prop**, we need some carrier type  $A$  such that  $\in : A \rightarrow A \rightarrow \mathbf{Prop}$ , so that e.g.  $\in$  satisfies the axioms of set theory. But then what does it mean for some other type  $B$  to have an interpretation of  $\in$  and the axioms it satisfies? There then needs to be a separate predicate  $\in_B : B \rightarrow B \rightarrow \mathbf{Prop}$  satisfying the same *kind* of Props as  $\in : A \rightarrow A \rightarrow \mathbf{Prop}$ . We could proceed to define a typeclass of such  $(B, \in_B)$ , and we would then be working with models of set theory, but we would lack a way to reason syntactically about the axioms themselves.

## 1.2 Models and satisfiability

For the remainder of this section we fix a language  $\mathcal{L}$ .

**Definition 1.13.** An  $\mathcal{L}$ -**theory** is a collection of sentences from  $\mathbf{Sentences}(\mathcal{L})$ .

**Definition 1.14.** An  $\mathcal{L}$ -**structure** comprises the following data:

- (i) A carrier type  $A$ ,
- (ii) an assignment of every  $c : \mathbf{Const}(\mathcal{L})$  to a  $c^A : A^{\mathbf{arity}(c)}$ ,
- (iii) an assignment of every  $R : \mathbf{Rel}(\mathcal{L})$  to a subtype  $R^A : A^{\mathbf{arity}(R)} \rightarrow \mathbf{Prop}$ ,
- (iv) an assignment of every  $f : \mathbf{Func}(\mathcal{L})$  to a function  $f^A : A^{\mathbf{arity}(f)} \rightarrow A$ .

**Definition 1.15.** Let  $A$  be an  $\mathcal{L}$ -structure. Using the data of  $A$  being an  $\mathcal{L}$ -structure, we can inductively assign to every term  $t$  (of arity  $k$  and containing  $n$  free variables) a **realization**  $\mathbf{r}(t) : A^n \rightarrow A^k$ , as follows:

1. If  $t = v$  for a variable  $v$ ,  $\mathbf{r}(t) = \text{id}_A = \lambda v. v$ .
2. If  $t = c$  for a constant symbol  $c$ ,  $\mathbf{r}(t) = A^0 \xrightarrow{c^A} A$ .
3. If  $t = (t_1, \dots, t_m)$ , then  $\mathbf{r}(t) = \mathbf{r}(t_1) \times \dots \times \mathbf{r}(t_m)$ .
4. If  $t = f(t_0)$  for some function symbol  $f$ , then  $\mathbf{r}(t) = f^A \circ \mathbf{r}(t_0)$ .

**Definition 1.16.** Let  $A$  be an  $\mathcal{L}$ -structure. Using the data of  $A$  being an  $\mathcal{L}$ -structure, we can inductively assign to every formula  $\varphi(x_1, \dots, x_n)$  (where  $x_1, \dots, x_n$  exhaust the free variables of  $\varphi$ ) a **realization**  $\mathbf{r}(\varphi) : A^n \rightarrow \mathbf{Prop}$ , as follows:

1. If  $\varphi$  is of the form  $t_1 = t_2$ , then  $\mathbf{r}(\varphi)$  is  $\mathbf{r}(t_1) = \mathbf{r}(t_2)$  (where symbolic equality is realized as true equality).
2. If  $\varphi$  is of the form  $R(t)$ ,  $\mathbf{r}(\varphi)$  is  $R^A(\mathbf{r}(t))$ .
3. If  $\varphi$  is of the form  $\neg\psi$ , then  $\mathbf{r}(\varphi)$  is  $\neg\mathbf{r}(\psi)$ .
4. If  $\varphi$  is of the form  $\psi \vee \psi'$ , then  $\mathbf{r}(\varphi)$  is  $\mathbf{r}(\psi) \vee \mathbf{r}(\psi')$ .

5. If  $\varphi$  is of the form  $\psi \wedge \psi'$ , then  $\mathbf{r}(\varphi)$  is  $\mathbf{r}(\psi) \wedge \mathbf{r}(\psi')$ .
6. If  $\varphi$  is of the form  $\exists v\psi$ , then  $\mathbf{r}(\varphi)$  is  $\exists v\mathbf{r}(\psi)$ .
7. If  $\varphi$  is of the form  $\forall v\psi$ , then  $\mathbf{r}(\varphi)$  is  $\forall v\mathbf{r}(\psi)$ .

In particular, each sentence  $\varphi$  is sent to a  $\mathbf{Prop} \mathbf{r}(\varphi)$ .

**Definition 1.17.** Let  $A$  be an  $\mathcal{L}$ -structure, and let  $\varphi$  be a sentence. We say that  $A$  **satisfies**  $\varphi$ , written

$$A \models \varphi,$$

if  $\vdash \mathbf{r}(\varphi)$ .

**Definition 1.18.** Let  $T$  be an  $\mathcal{L}$ -theory, and let  $A$  be an  $\mathcal{L}$ -structure. We say that  $A$  is a **model** of  $T$  if for every sentence  $\varphi : T$ ,  $A \models \varphi$ .

**Example 1.19.** (Line graph) The **language of graphs**  $\mathcal{L}_{\mathbf{Graph}}$  comprises a single 2-ary relation symbol  $E$ .

The **theory of graphs**  $\mathbf{Graph}$  comprises the sentence  $\forall x\forall y (E(x, y) \leftrightarrow E(y, x))$ .

The natural numbers  $\mathbb{N}$  can be viewed as a model of  $\mathbf{Graph}$  as follows. We realize  $E$  as the set

$$(y = \text{succ } x) \vee (x = \text{succ } y) : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbf{Prop}$$

which is clearly symmetric.

**Example 1.20.** Let  $\mathbf{Mod}(\mathbf{Graph})$  be the collection of graphs.<sup>2</sup> A **graph property** is a map  $P : \mathbf{Mod}(\mathbf{Graph}) \rightarrow \mathbf{Prop}$  such that whenever  $G \simeq G'$ ,  $P(G) \leftrightarrow P(G')$ . We say that  $G$  *satisfies*  $P$  if  $P(g) \leftrightarrow \text{true}$ . A graph property is additionally said to be *monotone* if whenever  $G \subseteq G'$  is a subgraph, then  $P(G') \rightarrow P(G)$ .

For example, the property of being a complete graph is not monotone, while the property of being cycle-free is.

The **graph evasiveness conjecture** says that for every monotone graph property  $P$  and every  $n : \mathbb{N}$ , one needs to ask  $\binom{n}{2}$  questions of the form “is there an edge between  $v$  and  $w$ ” to determine if an arbitrary graph on  $n$  vertices satisfies  $P$ .

**Example 1.21.** (Peano arithmetic)

The **language of Peano arithmetic**  $\mathcal{L}_{\mathbf{PA}}$  comprises:

1. A 1-ary constant 0.
2. Three function symbols  $\text{succ}$ ,  $+$ ,  $\times$ .

The **theory of Peano arithmetic**  $\mathbf{PA}$  comprises:

1.  $\forall x, s(x) \neq 0$
2.  $\forall x\forall y, (s(x) = s(y)) \rightarrow x = y$

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<sup>2</sup>Warning: this is “large”, so lives in the next universe up: one can interpret a trivial edge relation on *every* type in the current universe.

3.  $\forall x, x + 0 = x$
4.  $\forall x \forall y, x + s(y) = s(x + y)$
5.  $\forall x, x \times 0 = 0$
6.  $\forall x \forall y, x \times S(y) = (x \times y) + x$

Schema: For every  $\mathcal{L}_{\text{PA}}$ -formula  $\varphi(x)$  with one free variable  $x$ ,

$$(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(\text{succ } x))) \rightarrow \forall x \varphi(x).$$

The **standard model** of PA is  $\mathbb{N}$  with 0 realized as  $0 : \mathbb{N}$ , succ realized as  $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$ , + realized as  $+ : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ , and  $\times$  realized as  $\times : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ .

By recursing on the inductive type of valid sentences and replacing every rule of the propositional calculus with the corresponding deduction rule for **Prop**, we can construct for every valid  $\mathcal{L}$ -sentence  $\varphi$  a proof that  $\mathbf{r}(\varphi) \leftrightarrow \text{true}$ .

That is the soundness theorem. (In what follows, taking  $\psi$  to be **true** yields the assertion in the previous paragraph.)

**Theorem 1.22.** (*Soundness theorem*) For every  $\mathcal{L}$ -structure and any sentences  $\varphi, \psi : \text{Sentences}(\mathcal{L})$ ,

$$\vdash_{\mathcal{L}} \varphi \rightarrow \psi \quad \Longrightarrow \quad \vdash \mathbf{r}(\varphi) \rightarrow \mathbf{r}(\psi).$$

This happens regardless of which  $\mathcal{L}$ -structure is doing the realizing. When the  $\mathcal{L}$ -structure itself is a model of a theory  $T$ , then whenever  $T \vdash_{\mathcal{L}} \psi$ , then since there is some sentence  $\varphi : T$  such that  $\vdash_{\mathcal{L}} \varphi \rightarrow \psi$ , **Prop**'s modus ponens tells us that the model satisfies  $\psi$  also.

**Example 1.23.** For example, suppose we're working in the language of graphs expanded with two 1-ary constants  $a$  and  $b$ , and we know that there is some model  $M$  such that  $M$  satisfies the sole axiom that  $E$  is symmetric. We can show

$$\vdash (\forall x \forall y, \mathbf{r}(E)(x, y) \leftrightarrow \mathbf{r}(E)(y, x)) \rightarrow \mathbf{r}(E)(\mathbf{r}(a), \mathbf{r}(b)) \leftrightarrow \mathbf{r}(E)(\mathbf{r}(b), \mathbf{r}(a))$$

because we already know the antecedent and can apply **Prop**'s  $\forall$ -elimination.

The converse of Theorem 1.22 is false. There may be some things which are incidentally true about the model which are not universally valid.

**Example 1.24.** Working again in the language of graphs, consider a complete graph on  $n$  vertices. Call this model  $M$ .  $M$  happens to satisfy the  $\mathcal{L}$ -sentence

$$(\forall x \forall y, E(x, y) \leftrightarrow E(y, x)) \rightarrow (\forall x \forall y \forall z, E(x, y) \wedge E(y, z) \rightarrow E(x, z)),$$

but this is not a valid  $\mathcal{L}$ -sentence. (Indeed, if it were, then the soundness theorem would imply that *every* graph has a transitive edge relation, which is not true.)

It will turn out that we can do the next best thing. If we rule out this kind of exception by requiring that  $M \models \varphi$  for *every*  $\mathcal{L}$ -structure  $M$  (resp. every model  $M$  of  $T$ ), then it follows that  $\vdash_{\mathcal{L}} \varphi$  (resp.  $T \vdash_{\mathcal{L}} \varphi$ ). This is the completeness theorem.

### 1.3 The completeness theorem

In this section, our goal will be to prove the **completeness theorem**:

**Theorem 1.25.** *Let  $T$  be an  $\mathcal{L}$ -theory.  $T$  is consistent if and only if there exists a model of  $T$ .*

First we will prove that if there exists a model  $M$  of  $T$ , then  $T$  is consistent.

*Proof.* We will show the contrapositive: if  $T$  is inconsistent, then there does not exist a model  $M$  of  $T$ .

Indeed, suppose that  $T$  is inconsistent. Suppose there is a model  $M$ . Then by the soundness theorem,  $M \models \text{false}$ . By definition, this means that

$$\vdash \text{false},$$

so we have shown that

$$\vdash (T \text{ inconsistent}) \wedge (\text{there exists a model } M \text{ of } T) \rightarrow \text{false}$$

which is equivalent to

$$\vdash (T \text{ not inconsistent}) \vee (T \text{ does not have a model}),$$

which is equivalent to

$$\vdash T \text{ inconsistent} \rightarrow T \text{ does not have a model}.$$

Taking the contrapositive, we conclude that if  $T$  has a model, then  $T$  is consistent. □

It then remains to show that if  $T$  is consistent,  $T$  has a model. We will use the Henkin construction.

#### 1.3.1 The Henkin construction

**Definition 1.26.** Let  $T$  be an  $\mathcal{L}$ -theory. We say that  $T$  is a **Henkin theory** if, for every formula  $\varphi(x)$ , there is a constant  $c : \text{Const}(\mathcal{L})$  such that  $T \vdash_{\mathcal{L}} (\exists x \varphi(x)) \rightarrow \varphi(c)$ .

**Example 1.27.** Let  $\mathcal{L}_{\text{field}}$  be the language of fields, which we define to be  $\{0, 1, +, \times, (-)^{-1}\}$  (the usual language of rings augmented with an inversion operation), and let  $T$  be the usual axiomatization of a field of characteristic zero.  $T$  is not a Henkin theory, for there is no constant  $c$  such that e.g.  $c = (1 + 1)^{-1}$ .

**Example 1.28.** Let  $\mathcal{L}_{\text{PA}}$  be the language of Peano arithmetic (see Example 1.21). Let  $T$  be the collection of all  $\mathcal{L}_{\text{PA}}$ -sentences  $\psi$  such that  $\mathbb{N} \models \psi$ . Then  $T$  certainly contains the sentence  $\exists x \forall y, x \cdot y = y$ . However,  $1 = \text{succ } 0$  is not a constant in the language, but rather a term. So  $T$  is not a Henkin theory.

However, if we *expand*  $\mathcal{L}_{\text{PA}}$  to a language  $\mathcal{L}'$  with a constant symbol  $c_n$  for every natural number  $n$ , and if we let  $T'$  be the collection all  $\mathcal{L}'$ -sentences  $\psi$  such that  $\mathbb{N}$  (viewed in the natural way as a model of  $\mathcal{L}'$ ) satisfies  $\psi$ , then  $T'$  is a Henkin theory.



**Proposition 1.29.** *Let  $T$  be an  $\mathcal{L}$ -theory. If  $T$  is consistent, then there exists a language  $\mathcal{L}'$  extending  $\mathcal{L}$  and an  $\mathcal{L}'$ -theory  $T'$  extending  $T$  viewed as an  $\mathcal{L}'$ -theory, such that  $T'$  is a Henkin theory.*

*Furthermore, if  $T$  is consistent, then  $T'$  is consistent.*

*Proof.* Put  $\mathcal{L}_0 \stackrel{\text{df}}{=} \mathcal{L}$  and  $T_0 \stackrel{\text{df}}{=} T$ . We define a chain of languages  $\mathcal{L}_i$  and for each  $i$  we define an  $\mathcal{L}_i$ -theory  $T_i$  as follows: given  $\mathcal{L}_n$  and  $T_n$ , let  $\mathcal{L}_{n+1}$  be the language obtained by adding a constant  $c_{\varphi,x}$  where  $\varphi$  ranges over all  $\mathcal{L}_n$ -formulas and  $x$  ranges over the free variables of  $\varphi$ .

Having defined  $\mathcal{L}_{n+1}$ , we now define  $T_{n+1}$  to be

$$T_n \cup \{\exists x \varphi(x) \rightarrow \varphi(c_{\varphi,x})\}_{\varphi,x}$$

where above we have adjoined a sentence saying that the newly-adjoined constant  $c_{\varphi,x}$  behaves as expected.

We put

$$T' \stackrel{\text{df}}{=} \bigcup_{n:\mathbb{N}} T_n.$$

By construction,  $T'$  is a Henkin theory.

It remains to show that if  $T$  is consistent, so is  $T'$ . If  $T \vdash_{\mathcal{L}} \psi$ , then from the finiteness of proofs, we must have that  $T_n \vdash_{\mathcal{L}} \psi$  for some  $n$ . So, to show  $T'$  is consistent, it suffices to show that for each  $n$ ,  $T_n$  is consistent.

We induct on  $n$ . The base case  $T = T_0$  is by assumption. For the induction step, we must show that if  $T_n$  is consistent, then  $T_{n+1}$  is consistent.

Suppose towards the contrapositive that  $T_{n+1}$  is inconsistent. Since  $T_{n+1}$  is obtained by adjoining formulas of the form  $\exists x \varphi(x) \rightarrow \varphi(c)$ , there must be finitely many such formulas  $\psi_1, \dots, \psi_m : T_{n+1} \setminus T_n$  of this form, along with finitely many formulas  $\rho_1, \dots, \rho_n$  from  $T_n$ , such that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_m \rightarrow \text{false}.$$

By material implication, we get that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_{m-1} \rightarrow \neg \psi_m,$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_{m-1} \rightarrow \neg(\exists x \varphi_m(x) \rightarrow \varphi_m(c_m)),$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_{m-1} \rightarrow (\exists x \varphi_m(x)) \wedge \neg \varphi_m(c_m),$$

and since  $c_m$  does not occur in the premise of the implication, we have that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_{m-1} \rightarrow (\exists x \varphi_m(x)) \wedge \forall x \neg \varphi_m(x)$$

and therefore

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \dots \wedge \rho_n \wedge \psi_1 \wedge \dots \wedge \psi_{m-1} \rightarrow \text{false}.$$

We conclude that

$$\rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1}$$

is inconsistent. Repeating this argument, we eliminate all the  $\psi_i$  and conclude that  $\rho_1 \wedge \cdots \wedge \rho_n$  is inconsistent, and therefore that  $T$  is inconsistent.  $\square$

**Definition 1.30.** To any Henkin  $\mathcal{L}$ -theory  $T$ , we can associate a canonical structure (a “term model”)  $\text{term}(T)$  built from the closed terms (i.e. those not containing any variables).

First, we take the collection  $A$  of all closed  $\mathcal{L}$ -terms. We define a relation  $E : A \rightarrow A \rightarrow \text{Prop}$ , with the convention that  $\vdash E \ a_1 \ a_2 \leftrightarrow \text{true}$  if and only if  $T \vdash_{\mathcal{L}} a_1 = a_2$ . By the rules about equality that we have stipulated as part of the predicate calculus,  $E$  is an equivalence relation.

We put  $\tilde{A} \stackrel{\text{df}}{=} A/E$ . This will be the underlying type of the model.

For a constant  $c : \text{Const}(\mathcal{L})$ , we put  $c^{\tilde{A}} \stackrel{\text{df}}{=} c/E$  ( $c$  belongs to  $\mathcal{L}_0$ , and so is a closed term of  $\mathcal{L}'$ ).

For a relation symbol  $R : \text{Rel}(\mathcal{L})$ , we define  $R^{\tilde{A}} : \tilde{A}^{\text{arity}(R)} \rightarrow \text{Prop}$  by  $R^{\tilde{A}}(a_1/E, \dots, a_n/E) \leftrightarrow T' \vdash_{\mathcal{L}} R(a_1, \dots, a_n)$ .

For a function symbol  $f : \text{Funct}(\mathcal{L})$ , we define  $f^{\tilde{A}} : \tilde{A}^{\text{arity}(f)} \rightarrow \tilde{A}$  by

$$\lambda a_1/E \ \dots \ a_n/E, f(a_1, \dots, a_n)/E.$$

This completes the definition of  $\text{term}(T)$ .

By the soundness theorem, if  $T$  is inconsistent, then  $\text{term}(T)$  cannot be a model of  $T$ . But, under suitable assumptions, the inverse is true.

**Definition 1.31.** An  $\mathcal{L}$ -theory  $T$  is **complete** if for every  $\mathcal{L}$ -sentence  $\psi$ ,

$$\vdash (T \vdash_{\mathcal{L}} \psi) \vee (T \vdash_{\mathcal{L}} \neg \psi).$$

**Remark 1.32.** Excluded middle in **Prop** implies that for any  $\mathcal{L}$ -structure  $M$  and every  $\mathcal{L}$ -sentence  $\psi$ ,

$$\vdash (M \models \psi) \vee (M \models \neg \psi),$$

and therefore that the  $\mathcal{L}$ -theory of an  $\mathcal{L}$ -structure (i.e. the collection of all sentences true in the structure) is complete.

By invoking the axiom of choice, we can extend any consistent theory to a complete consistent theory. We will prove this.

**Proposition 1.33.** *Let  $T$  be a consistent  $\mathcal{L}$ -theory. There exists a complete, consistent  $\mathcal{L}$ -theory  $T'$  which contains  $T$ .*

To prove this, we will use Zorn’s lemma. To start the argument, we prove the following lemma.

**Lemma 1.34.** *Suppose  $T$  is consistent. Let  $\varphi$  be an  $\mathcal{L}$ -sentence. Then  $T \cup \{\varphi\}$  is consistent or  $T \cup \{\neg \varphi\}$  is consistent.*

*Proof.* Suppose that both  $T \cup \{\varphi\}$  and  $T \cup \{\neg\varphi\}$  are inconsistent. Then there exist sentences  $\sigma$  and  $\rho$  from  $T$  such that

$$\vdash_{\mathcal{L}}(\sigma \wedge \varphi) \rightarrow \text{false} \quad \text{and} \quad \vdash_{\mathcal{L}}(\rho \wedge \neg\varphi) \rightarrow \text{false}.$$

By  $\neg$ -introduction, we get

$$\vdash_{\mathcal{L}} \neg(\sigma \wedge \varphi) \quad \text{and} \quad \vdash_{\mathcal{L}} \neg(\rho \wedge \neg\varphi)$$

and by  $\wedge$ -introduction, we get

$$\vdash_{\mathcal{L}} (\neg(\sigma \wedge \varphi)) \wedge (\neg(\rho \wedge \neg\varphi)).$$

Since the finitary de Morgan laws are tautologies in the sense of 1.8, it follows that

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \varphi \vee \rho \vee \neg\varphi).$$

Since the metatheory satisfies the law of the excluded middle, we have that the law of the excluded middle for  $\mathcal{L}$ -formulas is a tautology in the sense of 1.8. Therefore,

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \rho) \Leftrightarrow \vdash_{\mathcal{L}} \neg\sigma \wedge \neg\rho,$$

so by  $\wedge$ -elimination,  $\vdash_{\mathcal{L}} \neg\sigma$  and  $\vdash_{\mathcal{L}} \neg\rho$ , so  $T$  is inconsistent.  $\square$

*Proof of 1.33.* Consider the poset of proper consistent extensions of  $T$ . If  $T$  is not complete, then the previous lemma shows that this poset is nonempty.

Now we show that we can take the union of a chain in this poset and obtain an upper bound on that chain.

Indeed, let  $(T_i)_{i \in I}$  be a chain in this poset, and let  $T_\infty$  be its union. This is clearly a theory which contains all the theories in the chain (and also  $T$ ). We need to show that it is consistent. Indeed, if it were inconsistent, then by the finiteness of proofs, there exists some  $T_n$  such that  $T_n \vdash_{\mathcal{L}} \text{false}$ .

This now fits the hypotheses of Zorn's lemma, which gives us a maximal consistent extension of  $T'$  of  $T$ . If  $T'$  were not complete, then the previous lemma shows that we can extend it.  $\square$

**Theorem 1.35.** *Let  $T$  be a complete Henkin  $\mathcal{L}$ -theory. If  $T$  is consistent, then  $\text{term}(T)$  is a model of  $T$ .*

*Proof.* We will show that for every  $\psi : \text{Sentences}(\mathcal{L})$ ,

$$T \vdash_{\mathcal{L}} \psi \iff \text{term}(T) \models \psi.$$

We will do this by a structural induction on formulas. In the base case, we have atomic sentences.

- If  $T \vdash_{\mathcal{L}} \psi$  and  $\psi$  is of the form  $a_1 = a_2$  where  $a_1$  and  $a_2$  are closed terms, then since  $T \vdash_{\mathcal{L}} a_1 = a_2$ , then  $\vdash a_1^{\tilde{A}} = a_2^{\tilde{A}}$  (in  $\tilde{A}$ ), so  $\text{term}(T) \models \psi$ .

Conversely, if  $\text{term}(T) \models \psi$ , then  $\vdash a_1^{\tilde{A}} = a_2^{\tilde{A}}$ , so by definition of the equivalence relation we used to define  $\tilde{A}$ ,  $T \vdash_{\mathcal{L}} a_1 = a_2$ .

- If  $T \vdash_{\mathcal{L}} \psi$  and  $\psi$  is of the form  $R(a_1, \dots, a_n)$  where  $R$  is a relation symbol and  $a_1, \dots, a_n$  are closed terms, then since  $T \vdash_{\mathcal{L}} R(a_1, \dots, a_n)$ , we have that  $\vdash R^{\tilde{A}}(a_1^{\tilde{A}}, \dots, a_n^{\tilde{A}})$ .

Conversely, if  $\text{term}(T) \models \psi$ , then  $\vdash R^{\tilde{A}}(a_1^{\tilde{A}}, \dots, a_n^{\tilde{A}})$ , so by definition of how we interpreted  $\mathcal{L}$  onto  $\tilde{A}$ ,  $T \vdash_{\mathcal{L}} R(a_1, \dots, a_n)$ .

- If  $T \vdash_{\mathcal{L}} \psi$  and  $\psi$  is of the form  $\varphi_1 \wedge \varphi_2$ , then by  $\wedge$ -elimination in **Sentences**( $\mathcal{L}$ ),

$$\vdash (T \vdash_{\mathcal{L}} \psi) \rightarrow (T \vdash_{\mathcal{L}} \varphi_1) \wedge (T \vdash_{\mathcal{L}} \varphi_2).$$

By the induction hypothesis,  $\text{term}(T) \models \varphi_1$  and  $\text{term}(T) \models \varphi_2$ , so by  $\wedge$ -introduction in **Prop**,  $\text{term}(T) \models \varphi_1 \wedge \varphi_2$ .

Conversely, if  $\text{term}(T) \models \varphi_1 \wedge \varphi_2$ , then by  $\wedge$ -elimination in **Prop**,  $\text{term}(T) \models \varphi_1$  and  $\text{term}(T) \models \varphi_2$ . By the induction hypothesis,  $T \vdash_{\mathcal{L}} \varphi_1$  and  $T \vdash_{\mathcal{L}} \varphi_2$ , so by  $\wedge$ -introduction in **Sentences**( $\mathcal{L}$ ),  $T \vdash_{\mathcal{L}} \varphi_1 \wedge \varphi_2$ .

- Suppose  $T \vdash_{\mathcal{L}} \psi$  and  $\psi$  is of the form  $\neg\varphi$ . The induction hypothesis says that  $T \vdash_{\mathcal{L}} \varphi$  if and only if  $\text{term}(T) \models \varphi$ . Since  $T$  is consistent,  $T \not\vdash_{\mathcal{L}} \varphi$ . Therefore, by the induction hypothesis,  $\text{term}(T) \not\models \varphi$ . By the law of the excluded middle,  $\text{term}(T) \models \neg\varphi$ .

Conversely, suppose that  $\text{term}(T) \models \neg\varphi$ . Then  $\text{term}(T) \not\models \varphi$ , so by the induction hypothesis,  $T$  does not prove  $\varphi$ . Since  $T$  was complete,  $T \vdash_{\mathcal{L}} \neg\varphi$ .

We omit the cases for  $\wedge$  and  $\rightarrow$ , which are entirely analogous.

We conclude that whenever  $\psi$  is quantifier-free,  $T \vdash_{\mathcal{L}} \psi$  if and only if  $\text{term}(T) \models \psi$ .

To complete the proof, we must take care of quantifiers.

- Suppose that  $T \vdash_{\mathcal{L}} \exists x\varphi(x)$ , where  $\varphi(x)$  satisfies the induction hypothesis that if we substitute a closed term  $c$  for  $x$ ,  $\varphi(c)$  is a sentence such that  $T \vdash_{\mathcal{L}} \varphi(c)$  if and only if  $\text{term}(T) \models \varphi(c)$ .

Then, since  $T$  is a Henkin theory, there exists some  $c$  such that

$$T \vdash_{\mathcal{L}} \varphi(c).$$

By the induction hypothesis, we have that

$$\text{term}(T) \models \varphi(c),$$

and therefore by  $\exists$ -introduction in **Prop**, we conclude that

$$\text{term}(T) \models \exists x\varphi(x).$$

Conversely, suppose that  $\text{term}(T) \models \exists x\varphi(x)$ . By  $\exists$ -elimination in **Prop**, there exists some  $a/E : \tilde{A}$  such that  $\vdash \mathbf{r}(\varphi)(a/E)$ , which is equivalent to  $\text{term}(T) \models \varphi(a)$ . By the induction hypothesis,  $T \vdash_{\mathcal{L}} \varphi(a)$ , and by  $\exists$ -introduction in **Sentences**( $\mathcal{L}$ ),  $T \vdash_{\mathcal{L}} \exists x\varphi(x)$ .

- Similarly, suppose that  $T \vdash_{\mathcal{L}} \forall x\varphi(x)$ , where  $\varphi(x)$  satisfies the induction hypothesis that if we substitute a closed term  $c$  for  $x$ ,  $\varphi(c)$  is a sentence such that  $T \vdash_{\mathcal{L}} \varphi(c)$  if and only if  $\text{term}(T) \models \varphi(c)$ .

Then by  $\forall$ -elimination in **Sentences**( $\mathcal{L}$ ), we have that for every constant  $c : \text{Const}(\mathcal{L})$ ,  $T \vdash_{\mathcal{L}} \varphi(c)$ . By the induction hypothesis,  $\text{term}(T) \models \varphi(c)$ . Since the interpretations of  $c$  exhaust  $\text{term}(T)$ , we conclude by  $\forall$ -introduction in **Prop** that  $\text{term}(T) \models \forall x\varphi(x)$ .

Conversely, suppose that  $\text{term}(T) \models \forall x \varphi(x)$ . By  $\forall$ -elimination in **Prop**, for every  $a/E \in \tilde{A}$ ,  $\vdash \mathbf{r}(\varphi)(a/E)$ , which is equivalent to  $\text{term}(T) \models \varphi(a)$ . By the induction hypothesis, for every  $c : \text{Const}(\mathcal{L})$ ,  $T \vdash_{\mathcal{L}} \varphi(c)$ .

Suppose towards a contradiction that  $T$  does not prove  $\forall x \varphi(x)$ . Since  $T$  was complete,  $T$  proves  $\exists x \neg \varphi(x)$ . Since we have already proved the cases for  $\exists$  and  $\neg$ , we conclude that  $\text{term}(T) \models \exists x \neg \varphi(x)$ , and by the axiom of choice we can find a witness  $c \in \text{term}(T)$  such that  $\text{term}(T) \models \neg \varphi(c)$ . This contradicts the conclusion of the previous paragraph.

□

**Corollary 1.36.** *Let  $T$  be a consistent  $\mathcal{L}$ -theory. Then  $T$  has a model.*

*Proof.* By 1.29, extend  $T$  to a Henkin theory  $T'$ . By 1.33, extend  $T'$  to a complete theory  $T''$ .

$T''$  is again Henkin: for any formula  $\varphi(x)$ , there already exists a  $c$  such that  $T' \vdash_{L'} \exists x \varphi(x) \leftrightarrow \varphi(c)$ , and  $T''$  contains all the sentences of  $T'$ .

By 1.35,  $\text{term}(T'')$  is a model of  $T''$ . Since  $T''$  contains  $T$ ,  $\text{term}(T'')$  is also a model of  $T$ .

□

This completes the proof of the completeness theorem.

## 1.4 The Löwenheim-Skolem theorem

TODO

## 2 ZFC

The language  $\mathcal{L}_{\text{ZFC}}$  of set theory comprises just one 2-ary relation  $\in$ . Now we give the definition of the  $\mathcal{L}_{\text{ZFC}}$ -theory ZFC.

**Definition 2.1.** ZFC is defined to be the collection of following axioms and axiom schemas:

### Extensionality

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

This says that every set is determined by its elements.

**Notation 2.2.** In what follows, we write “ $x \subseteq y$ ” to abbreviate the formal statement  $\forall z \in x, z \in y$ .

### Empty set

$$\exists x \forall y (\neg y \in x).$$

Viewing a model of ZFC as a directed tree, this says that every model has a least (“root”) element.

### Pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

This axiom says that we can form unordered pairs.

**Notation 2.3.** We denote  $z$  as above by  $\{x, y\}$ , adopt the convention that  $\{x\} \stackrel{\text{df}}{=} \{x, x\}$ , and we implement ordered pairs with *Kuratowski ordered pairs*, viz.  $(x, y) \stackrel{\text{df}}{=} \{\{x\}, \{x, y\}\}$ .

Now that we have defined ordered pairs, we can define functions (internal to ZFC):

**Definition 2.4.** A **function** is a set  $f$  of ordered pairs such that  $(x, y) \wedge (x, z) \rightarrow y = z$ .

### Union

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists t (z \in t \wedge t \in x)).$$

This says that the  $y$  above is the union of all the members of  $x$ . Applying 2.1, we conclude that given sets  $x$  and  $y$ , there exists  $z$  such that  $z = x \cup y$ .

**Definition 2.5.** Let  $x$  be a set. We denote the **successor** of  $x$  to be the set  $\succ x \stackrel{\text{df}}{=} x \cup \{x\}$ .

### Infinity

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow \succ y \in x)).$$

**Replacement** Let  $\varphi(x, y, t_1, \dots, t_k)$  be an  $\mathcal{L}_{\text{ZFC}}$ -formula with at least two free variables. For each such formula,

$$\forall t_1 \dots \forall t_k (\forall x \exists! y \varphi(x, y, t_1, \dots, t_k) \rightarrow \forall u \exists v \forall r (r \in v \leftrightarrow \exists s (s \in u \wedge \varphi(s, r, t_1, \dots, t_k)))).$$

is an axiom of ZFC.

This axiom says that if for fixed terms  $t_1, \dots, t_k$ ,  $\varphi(x, y, t_1, \dots, t_k)$  is the graph of a function sending  $x$  to  $y$ , then for each set  $u$ , the image of  $u$  under this function is again a set.

Note that the quantifiers above can range over the entire model of ZFC.

## Powerset

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x).$$

**Choice** Let  $y : x \rightarrow z$  abbreviate the  $\mathcal{L}_{\text{ZFC}}$ -formula which says that  $y$  is a function from  $x$  to  $z$ .

$$\forall y \forall y \forall z, y : x \rightarrow z \wedge (y \neq \emptyset) \rightarrow \left( \exists f (f : x \rightarrow \bigcup z) \wedge \forall a \in x, f(a) \in y(x) \right).$$

More clearly, this says that for every  $x$ -indexed family of sets  $z$ , there exists a section to the projection  $\bigcup z \rightarrow x$ .

## Regularity

$$\forall x \exists y (x = \emptyset \vee (y \in x \wedge \forall z (z \in x \rightarrow \neg z \in y)))$$

This asserts that every set contains an element which is minimal with respect to  $\in$ .

## 2.1 Ordinal numbers

**Definition 2.6.** We say that  $y$  is a (binary) **relation** on  $x$  if  $y$  is a set of ordered pairs from  $X$ .

**Definition 2.7.** We say that a relation  $<$  on  $x$  is a **well-ordering** if:

1.

$$\forall a \forall b, a = b \vee a < b \vee b < a.$$

2.

$$\forall a \forall b \forall c, a < b \wedge b < c \rightarrow a < c.$$

3.

$$\forall s \subseteq x, s \neq \emptyset \rightarrow \exists a (a \in x \wedge \forall b (b \in s \rightarrow \neg a < b)).$$

**Definition 2.8.** A set  $x$  is called **transitive** if  $y \in x, z \in y \rightarrow z \in x$ .

**Definition 2.9.** We say that a set  $\alpha$  is an **ordinal** if it is well-ordered by the membership relation  $\in$  and it is transitive. We abbreviate this assertion by  $\text{On } \alpha$ .

## 2.2 Cardinal numbers

## 3 Boolean-valued models

### 3.1 Boolean algebras

**Definition 3.1.** A **preorder**  $B$  is a type  $B$  equipped with relations  $\leq$  and  $<$  satisfying the following properties:

1.  $\forall a : B, a \leq a$
2.  $\forall a, b, c : B, a \leq b \rightarrow b \leq c \rightarrow a \leq c$
3.  $\forall a, b : B, a \leq b \wedge \neg b \leq a$
4.  $\forall a, b : B, a < b \leftrightarrow (a \leq b \wedge \neg b \leq a)$

**Definition 3.2.** A **partial order**  $B$  is a preorder such that the  $\leq$  relation is antisymmetric:

$$\forall a, b : B, a \leq b \rightarrow b \leq a \rightarrow a = b.$$

**Definition 3.3.** A **join-semilattice**  $B$  is a partial order with binary sup operation  $\sqcup$  which satisfies the following properties:

1.  $\forall a, b : B, a \leq a \sqcup b$
2.  $\forall a, b : B, b \leq a \sqcup b$
3.  $\forall a, b, c : B, a \leq c \rightarrow b \leq c \rightarrow a \sqcup b \leq c.$

**Definition 3.4.** A **meet-semilattice**  $B$  is a partial order with a binary infimum operation  $\sqcap$  which satisfies the following properties:

1.  $\forall a, b : B, a \sqcap b \leq a$
2.  $\forall a, b : B, a \sqcap b \leq b$
3.  $\forall a, b, c : B, a \leq b \rightarrow a \leq c \rightarrow a \leq b \sqcap c.$

**Definition 3.5.** A **lattice**  $B$  is a join-semilattice which is also a meet-semilattice.

**Definition 3.6.** A **distributive lattice**  $B$  is a lattice which satisfies the following property:<sup>3</sup>

$$\forall x, y, z : B, (x \sqcup y) \sqcap (x \sqcup z) \leq x \sqcup (y \sqcap z).$$

**Definition 3.7.** A lattice  $B$  has a **bottom element**  $\perp$  if for every  $a : B, \perp \leq a$ ,

**Definition 3.8.** A lattice  $B$  has a **top element**  $\top$  if for every  $a : B, a \leq \top$ .

**Definition 3.9.** A **bounded lattice** is a lattice with a top and bottom element.

**Definition 3.10.** A **bounded distributive lattice** is a distributive lattice which is bounded.

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<sup>3</sup>From the `mathlib` docstring: a distributive lattice can be defined to satisfy any of four equivalent distribution properties (of sup over inf or inf over sup, on the left or right). A classic example of a distributive lattice is the lattice of subsets of a set, and in fact this example is generic in the sense that every distributive lattice is realizable as a sublattice of a powerset lattice.



**Definition 3.11.** A **Boolean algebra**  $B$  is a bounded distributive lattice such that:

1. For every  $a : B$ , there exists an element  $\neg a : B$  which satisfies the following properties:
  - (a)  $\forall x : B, x \sqcap \neg x = \perp$
  - (b)  $\forall x : B, x \sqcup \neg x = \top$
2. We additionally specify a **complementation operator**  $\lambda x, y, x - y : B \rightarrow B$  which satisfies the property:

$$\forall x, y : B, x - y = x \sqcap \neg y.$$

**Definition 3.12.** For convenience, we accumulate the previous definitions into a complete axiomatization of a boolean algebra. A boolean algebra is a type  $B$  with a specification of binary ordering relations  $\leq, <$ , a binary sup operation  $\sqcup$ , a binary inf operation  $\sqcap$ , top and bottom elements  $\top, \perp$ , a unary negation operator  $\neg$ , and a binary subtraction operator  $-$ , satisfying the following properties:

1.  $\forall (a : B), a \leq a$
2.  $(ab, c_1 : B), a \leq b \rightarrow b \leq c_1 \rightarrow a \leq c_1$
3.  $(\forall (a, b : B), a < b \iff a \leq b \wedge \neg b \leq a)$
4.  $(a, b : B), a \leq b \rightarrow b \leq a \rightarrow a = b$
5.  $\forall (a, b : B), a \leq a \sqcup b$
6.  $\forall (a, b : B), b \leq a \sqcup b$
7.  $\forall (a, b, c_1 : B), a \leq c_1 \rightarrow b \leq c_1 \rightarrow a \sqcup b \leq c_1$
8.  $\forall (a, b : B), a \sqcap b \leq a$
9.  $\forall (a, b : B), a \sqcap b \leq b$
10.  $\forall (a, b, c_1 : B), a \leq b \rightarrow a \leq c_1 \rightarrow a \leq b \sqcap c_1$
11.  $\forall (x, y, z : B), (x \sqcup y) \sqcap (x \sqcup z) \leq x \sqcup y \sqcap z$
12.  $\forall (a : B), a \leq \top$
13.  $\forall (a : B), \perp \leq a$
14.  $\forall (x : B), x \sqcap \neg x = \perp$
15.  $\forall (x : B), x \sqcup \neg x = \top$
16.  $\forall (x, y : B), x - y = x \sqcap \neg y$

**Definition 3.13.** A **complete lattice**  $B$  is a bounded lattice which has operations  $\text{Sup}, \text{Inf} : \text{set } B \rightarrow B$ ,

**Definition 3.14.** A **complete distributive lattice**  $B$  is a complete lattice which additionally satisfies the following properties:

1.  $\forall a : B, s : \text{set } B, (\bigcap_{b \in s}, a \sqcup b \leq a \sqcup \text{Inf } s)$
2.  $\forall a : B, s : \text{set } B, a \sqcap \text{Sup } s \leq (\bigcup_{b \in s}, a \sqcap b).$

**Definition 3.15.** A **complete boolean algebra**  $B$  is a boolean algebra which is also a complete distributive lattice.

### 3.2 Boolean-valued models

Fix  $\mathcal{L}$  a first-order language and  $T$  an  $\mathcal{L}$ -theory. Fix  $\mathbb{B}$  a boolean algebra.

**Definition 3.16.** A  $\mathbb{B}$ -valued  $\mathcal{L}$ -structure is the following data:

- (i) A carrier type  $A$ ,
- (ii) an assignment of every  $c : \text{Const}(\mathcal{L})$  to a  $c^A : A^{\text{arity}(c)}$ ,
- (iii) an assignment of every  $R : \text{Rel}(\mathcal{L})$  to a  $\mathbb{B}$ -valued map  $R^A : A^{\text{arity}(R)} \rightarrow \mathbb{B}$ ; in particular an assignment of a binary  $\mathbb{B}$ -valued map for the equality symbol, and
- (iv) an assignment of every  $f : \text{Func}(\mathcal{L})$  to a function  $f^A : A^{\text{arity}(f)} \rightarrow A$ .

If  $\phi(\vec{x})$  is a formula, we write  $[[\phi(\vec{x})]]^A$  to mean  $\phi(\vec{x})$  viewed as a  $\mathbb{B}$ -valued function (taking as many arguments as it has free variables). The previous data must satisfy the following properties:

1. For every  $a : A$ ,  $[[a = a]]^A = 1$ .
2. For every  $a, b : A$ ,  $[[a = b]]^A = [[b = a]]^A$ .
3. For every  $a, b, c : A$ ,  $[[a = b]]^A \cap [[b = c]]^A \leq [[a = c]]^A$ .
4. For every  $n$ -ary function symbol  $R$ , and for all  $n$ -tuples  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$ ,

$$\left( \bigcap_{i=1}^n [[a_i = b_i]]^A \right) \cap [[R(a_1, \dots, a_n)]]^A \leq [[R(b_1, \dots, b_n)]]^A$$

5. For every  $n$ -ary function symbol  $f$ , for every  $a, b : A$  and every  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  in  $A^n$ , the following three properties hold:

$$\left( \bigcap_{i=1}^n [[a_i = b_i]]^A \right) \cap [[f(a_1, \dots, a_n) = a]]^A \leq [[f(b_1, \dots, b_n) = b]]^A,$$

$$\bigsqcup_{a:A} [[f(a_1, \dots, a_n) = a]]^A, \text{ and}$$

$$[[f(a_1, \dots, a_n) = a]]^A \cap [[f(a_1, \dots, a_n) = b]]^A \leq [[a = b]]^A.$$

### 3.3 An example of a Boolean-valued model (experimental)

A preset is a type-indexed family of presets.

This means that there is an associated indicator function on the “disjoint union” of all types, and it is 1 if and only if its argument is inside the indexing type.

Let  $\mathbb{B}$  be a complete Boolean algebra. We construct a  $\mathbb{B}$ -valued model analogous to the standard model  $\mathbb{W}$  as follows:

A  **$\mathbb{B}$ -valued preset** is an  $\alpha$ -indexed family of  $\mathbb{B}$ -valued presets  $\times \mathbb{B}$  (i.e. a type-indexed family of presets with boolean truth values for membership).

Such objects are called  $\mathbb{B}$ -names. A  $\mathbb{B}$ -name  $u$  is specified by the following data:

1. An indexing type  $\alpha$ .
2. An indexing function  $A : \alpha \rightarrow \text{bSet } \mathbb{B}$ .
3. A truth-value function  $B : \alpha \rightarrow \mathbb{B}$ .

## 4 The maximum principle

**Definition 4.1.** Let  $\mathbb{B}$  be a bounded lattice. An **antichain** in  $\mathbb{B}$  is a subset  $A \subseteq \mathbb{B}$  satisfying the following property:

$$\forall x \in S, \forall y \in S, (x \neq y \rightarrow x \sqcap y = \perp).$$

**Definition 4.2.** Let  $\mathbb{B}$  be a bounded lattice. An **indexed antichain** is the data of a type  $I$  and a map  $A : I \rightarrow \mathbb{B}$  satisfying the following property:

$$\forall i, j : I, (i \neq j \rightarrow A(i) \sqcap A(j) = \perp).$$

**Definition 4.3.** Let  $I$  be a type, and let  $A : I \rightarrow \mathbb{B}$  be an indexed antichain. Let  $u : I \rightarrow V^{\mathbb{B}}$  be an  $I$ -indexed family of  $\mathbb{B}$ -valued sets. The **mixture** of  $u$  with respect to  $A$  is the following  $\mathbb{B}$ -valued set, which we will specify by the data  $\langle \alpha_\mu, A_\mu, B_\mu \rangle$ :

- $\alpha_\mu$  is defined to be the  $\Sigma$ -type

$$\alpha_\mu \stackrel{\text{df}}{=} \Sigma_{(i:I)} \alpha_{u(i)}.$$

- $A_\mu$  is defined to be the map

$$(A_\mu : \alpha_\mu \rightarrow V^{\mathbb{B}}) \stackrel{\text{df}}{=} \lambda \langle i, x \rangle, A_{u(i)}(x).$$

- $B_\mu$  is defined to be the map

$$(B_\mu : \alpha_\mu \rightarrow \mathbb{B}) \stackrel{\text{df}}{=} \lambda \langle i, x \rangle, \bigsqcup (j : \iota), A(j) \sqcap (A_{u(i)} x) \in^{\mathbb{B}} u(j).$$

We focus on a special case of mixtures when  $I$  is a type with only two elements.

**Definition 4.4.** Let  $a_1, a_2 : \mathbb{B}$ , and let  $u_1, u_2 \in V^{\mathbb{B}}$ . The **two-term mixture** of  $u_1$  and  $u_2$  is the following  $\mathbb{B}$ -valued set, given by the data  $\langle \alpha_\mu, A_\mu, B_\mu \rangle$ :

- $\alpha_\mu \stackrel{\text{df}}{=} \alpha_{u_1} \oplus \alpha_{u_2}$ .
- $(A_\mu : \alpha_\mu \rightarrow V^{\mathbb{B}}) \stackrel{\text{df}}{=} \lambda a, A_{u_1} a$  (if  $a : \alpha_{u_1}$ ), and  $\lambda a, A_{u_2} a$  otherwise.
- $(B_\mu : \alpha_\mu \rightarrow \mathbb{B}) \stackrel{\text{df}}{=} \lambda (a : \alpha_{u_1} \oplus \alpha_{u_2}), (a_1 \sqcap (A_\mu(a) \in^{\mathbb{B}} u_1)) \sqcup (a_2 \sqcap (A_\mu(a) \in^{\mathbb{B}} u_2))$

We write  $a_1 \cdot u_1 + a_2 \cdot u_2$  for the two-term mixture of  $u_1$  and  $u_2$  with respect to  $a_1$  and  $a_2$ .

#### 4.0.1 Corollaries of the maximum principle

**Lemma 4.5.** *Let  $X$  be a  $\mathbb{B}$ -valued set, and let  $u_1$  and  $u_2$  be  $\mathbb{B}$ -valued sets such that  $u_1 \in^{\mathbb{B}} X = \top$  and  $u_2 \in^{\mathbb{B}} X = \top$ . Let  $a_1$  and  $a_2$  be an antichain such that  $a_1 \sqcup a_2 = \top$ .*

*Then  $a_1 \cdot u_1 + a_2 \cdot u_2 \in^{\mathbb{B}} X = \top$ .*

*Proof.* Let  $U = a_1 \cdot u_1 + a_2 \cdot u_2$ . We must show that

$$\top \leq U \in^{\mathbb{B}} X.$$

By the mixing lemma, we have that  $a_1 \leq U =^{\mathbb{B}} u_1$  and that  $a_2 \leq U =^{\mathbb{B}} u_2$ . Therefore,  $a_1 \sqcup a_2 = \top \leq U =^{\mathbb{B}} u_1 \sqcup U =^{\mathbb{B}} u_2$ .

Conjuncting this with the facts that  $T \leq u_1 \in^{\mathbb{B}} X$  and  $T \leq u_2 \in^{\mathbb{B}} X$  and then eliminating the previous Boolean-valued disjunction, we see that it suffices to prove that

$$(U =^{\mathbb{B}} u_1) \cap (u_1 \in^{\mathbb{B}} X) \leq U \in^{\mathbb{B}} X \text{ and } (U =^{\mathbb{B}} u_1) \cap (u_2 \in^{\mathbb{B}} X) \leq U \in^{\mathbb{B}} X,$$

and both of these obligations are precisely the left  $=^{\mathbb{B}}$ -extensionality of  $\in^{\mathbb{B}}$ .  $\square$

**Lemma 4.6.** *Let  $X$  be a  $\mathbb{B}$ -valued set, and let  $u_1$  and  $u_2$  be  $\mathbb{B}$ -valued sets. Let  $a_1$  and  $a_2$  be an antichain such that  $a_1 \sqcup a_2 = \top$ , such that  $a_1 = u_2 \subseteq^{\mathbb{B}} u_1$ . Then  $\top \leq u_2 \subseteq^{\mathbb{B}} a_1 \cdot u_1 + a_2 \cdot u_2$ .*

*Proof.* Let  $U$  be the two-term mixture  $a_1 \cdot u_1 + a_2 \cdot u_2$ . We need to show that

$$\top \leq \bigcap_{w:V^{\mathbb{B}}} w \in^{\mathbb{B}} u_2 \Rightarrow w \in^{\mathbb{B}} U,$$

i.e. that for every  $w : V^{\mathbb{B}}$ ,

$$w \in^{\mathbb{B}} u_2 \leq w \in^{\mathbb{B}} U.$$

Unfolding the definition of  $\in^{\mathbb{B}}$ , we restate our goal as:

$$\bigsqcup_{i_y:\alpha_{u_2}} B_{u_2}(i_y) \cap w =^{\mathbb{B}} A_{u_2}(i_y) \leq \bigsqcup_{i_z:\alpha_U} B_U(i_z) \cap w =^{\mathbb{B}} A_U(i_z).$$

Eliminating the existential quantifier on the left, we fix an arbitrary  $i : \alpha_{u_2}$  and now must show that:

$$B_{u_2}(i) \cap w =^{\mathbb{B}} A_{u_2}(i) \leq \bigsqcup_{i_z:\alpha_U} B_U(i_z) \cap w =^{\mathbb{B}} A_U(i_z).$$

Instantiating the existential quantifier on the right with  $i$ , it now suffices to show:

$$B_{u_2}(i) \cap w =^{\mathbb{B}} A_{u_2}(i) \leq B_U(i) \cap w =^{\mathbb{B}} A_{u_2}(i),$$

and cancelling like terms, it suffices to show that

$$(B_{u_2}(i) \leq B_U(i)) \iff \left( B_{u_2} \leq \left( a_1 \cap A_U(i) \in^{\mathbb{B}} u_1 \right) \sqcup \left( a_2 \cap A_U(i) \in^{\mathbb{B}} u_2 \right) \right).$$

Rewriting the right hand side with de Morgan's law and splitting into cases, we see that it now suffices to show:

$$1. B_{u_2}(i) \leq (a_1 \sqcup a_2) = \top.$$

2.  $B_{u_2}(i) \leq (a_1 \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_2).$
3.  $B_{u_2}(i) \leq (a_2 \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_1).$
4.  $B_{u_2}(i) \leq (A_{u_2}(i) \in^{\mathbb{B}} u_2).$

All the cases except (3) follow immediately from previous results.

For (3), note that we have by assumption that  $a_2 = (\neg(u_2 \subseteq^{\mathbb{B}} u_1)) \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_1$ , so now our goal is to show

$$B_{u_2}(i) \leq (u_2 \subseteq^{\mathbb{B}} u_1) \Rightarrow A_{u_2}(i) \in^{\mathbb{B}} u_1 \iff \left( (u_2 \subseteq^{\mathbb{B}} u_1) \cap B_{u_2}(i) \leq A_{u_2}(i) \in^{\mathbb{B}} u_1 \right).$$

Instantiating the universal quantifier on the left hand side with  $A_{u_2}(i)$ , it now suffices to show that

$$A_{u_2}(i) \in^{\mathbb{B}} u_2 \Rightarrow A_{u_2}(i) \in^{\mathbb{B}} u_1 \cap B_{u_2}(i) \leq A_{u_2}(i) \in^{\mathbb{B}} u_1.$$

Since  $B_{u_2}(i) \leq A_{u_2}(i) \in^{\mathbb{B}} u_2$ , we are finished after applying  $\leq$ -transitivity and Boolean-valued implication elimination.  $\square$

**Lemma 4.7.** *Let  $\phi : V^{\mathbb{B}} \rightarrow \mathbb{B}$  be an  $=^{\mathbb{B}}$ -extensional function, such that  $[[\exists x \phi(x)]] = \top$ .*

- (i) *For every  $\mathbb{B}$ -valued set  $v$ , there exists a  $\mathbb{B}$ -valued set  $u$  such that  $\phi(u) = 1$  and  $\phi(v) = [[u = v]]$ .*
- (ii) *If  $\psi : V^{\mathbb{B}} \rightarrow \mathbb{B}$  is another  $=^{\mathbb{B}}$ -extensional function, such that for every  $u$ ,  $\phi(u) = \top$  implies  $\psi(u) = \top$ , then*

$$\left( \bigcap_{(x:V^{\mathbb{B}})} \phi(x) \rightarrow \psi(x) \right) = \top.$$

*Proof.* (i) Since  $[[\exists x \phi(x)]] = \top$ , we obtain, using the maximum principle, a  $w$  such that  $\phi(w) = \top$ .

Put  $b = \phi(v)$ , and let  $u$  be the two-term mixture  $u = b \cdot v + (\neg b) \cdot w$ . We make the following observations:

- (a)  $b \leq u =^{\mathbb{B}} v$  by the mixing lemma.
- (b)  $b \leq \phi(v)$  by reflexivity.
- (c)  $\neg b \leq u =^{\mathbb{B}} w$  by the mixing lemma.
- (d)  $\neg b \leq \phi(w)$  because everything is bounded by  $\top$ .

Putting these observations together, we conclude that

$$\top = b \sqcup \neg b \leq [[u = v \cap \phi(v)]] \sqcup [[u = w \cap \phi(w)]] \leq [[\phi(u)]].$$

It remains to show that  $\phi(v) = [[u = v]]$ . Since  $\phi$  is  $=^{\mathbb{B}}$ -extensional, we have the inequality

$$[[u = v]] = [[u = v]] \cap [[\phi(u)]] \leq [[\phi(v)]].$$

Conversely, by definition of  $u$ ,

$$\phi(v) = b \leq [[u = v]].$$

- (ii) Let  $v \in V^{\mathbb{B}}$ . We may use part (1) to choose  $u \in V^{\mathbb{B}}$  such that  $\phi(u) = \top$ , and such that  $\phi(v) = u =^{\mathbb{B}} v$ .

Our goal is then to show that  $\phi(v) \leq \psi(v)$ . So, we calculate

$$\phi(v) \leq \phi(v) \cap \top = \phi(v) \cap \psi(u) = (u =^{\mathbb{B}} v) \cap \psi(u) \leq \psi(v),$$

since  $\psi$  was  $=^{\mathbb{B}}$ -extensional.

□

## 5 The fundamental theorem of forcing for Boolean-valued models

The aim of this section is to prove the *fundamental theorem of forcing* (for Boolean-valued models), which states that for any complete Boolean algebra  $B$ ,  $V^B$  satisfies all the ZFC axioms.

### 5.1 The axiom of choice

Let  $x$  be a  $\mathbb{B}$ -set. By the maximum principle, there exists a  $\mathbb{B}$ -set  $u_x$  such that  $[[u_x \in x]] = \bigsqcup_{(y:V^B)} [[y \in x]]$ . By the axiom of choice in the metatheory, we can then define a *global choice function*  $\text{choice} : \text{bSet } \mathbb{B} \rightarrow \text{bSet } \mathbb{B}$ .

Naïvely, this is not a choice function from the point of view of  $V^{\mathbb{B}}$ , as it may not be  $=^{\mathbb{B}}$ -extensional.

However, by restricting  $\text{choice}$  to a  $\mathbb{B}$ -set, we can fulfill the requirements of (AC').

**Definition 5.1.** The *axiom of choice* is the sentence (AC):

$$\forall u, \exists f, [\text{Fun}(f) \wedge (\text{dom}(f) = u) \wedge (\forall x, x \in u \rightarrow (x \neq \emptyset \rightarrow f(x) \in x))].$$

It is provably equivalent over ZF to the following sentence<sup>4</sup>:

**Definition 5.2.** (AC'):

$$\forall x, \exists y, \forall z, z \in x \rightarrow (z \neq \emptyset \rightarrow \exists! w, (w \in z \rightarrow \exists v, (v \in y \rightarrow (z \in v \wedge w \in v))))).$$

**Theorem 5.3.** The sentence AC' has truth value  $\top$  in  $V^B$ .

*Proof.* Translated into Boolean truth values, we see that

$$\begin{aligned} [[\text{AC}']] &= \bigsqcap_{(x:V^B)} \bigsqcup_{(y:V^B)} \bigsqcap_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \emptyset]] \Rightarrow \perp) \\ &\Rightarrow (\bigsqcup_{(w:V^B)} [[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]). \end{aligned}$$

It then suffices to show that for every  $x : V^B$ ,

$$\top \leq \bigsqcup_{(y:V^B)} \bigsqcap_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \emptyset]] \Rightarrow \perp)$$

<sup>4</sup>See <http://us.metamath.org/mpeuni/ac3.html>

$$\Rightarrow (\bigsqcup_{(w:V^B)}![[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]).$$

We will finish the proof by constructing a  $y$  with which we instantiate the (Boolean-valued) existential quantifier.

Before proceeding with the construction, we indicate how the argument would proceed after instantiating the  $y$ , which will expose the properties that such a  $y$  must satisfy.

So, fix a  $y$ . Instantiating the quantifier, we now have to show

$$\begin{aligned} \top &\leq \bigsqcup_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \emptyset]] \Rightarrow \perp) \\ &\Rightarrow (\bigsqcup_{(w:V^B)}![[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]). \end{aligned}$$

Introducing the  $z$ , we then have to show that for any  $z : V^B$ ,

$$\begin{aligned} \top &\leq [[z \in x]] \Rightarrow ([[z = \emptyset]] \Rightarrow \perp) \\ &\Rightarrow (\bigsqcup_{(w:V^B)}![[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]). \end{aligned}$$

Introducing the implications (using, say, the deduction theorem), it suffices to show that

$$[[z \in x]] \Rightarrow ([[z = \emptyset]] \Rightarrow \perp) \leq (\bigsqcup_{(w:V^B)}![[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]).$$

It is easy to see that for any  $z : V^B$ ,

$$[[z = \emptyset]] = \bigsqcap_{w:V^B} [[w \in z]] \Rightarrow \perp.$$

Let  $u_z$  be the witness produced by the maximum principle such that

$$[[u_z \in z]] = \bigsqcup_{(w:V^B)} [[w \in z]].$$

Therefore, it suffices to show that

$$[[z \in x]] \cap [[u_z \in z]] \leq (\bigsqcup_{(w:V^B)}![[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \cap [[w \in v]]).$$

Instantiating the quantifier on the right-hand side with  $u_z$  and unfolding the definition of the Boolean-valued version of the bounded “exists unique” quantifier, it now suffices to show that

$$[[z \in x]] \cap [[u_z \in z]] \leq \bigsqcap_{(w':V^B)} [[w' \in z]] \Rightarrow \left( \bigsqcup_{(v:V^B)} [[v \in y]] \cap [[z \in v]] \cap [[w' \in v]] \right) \Rightarrow [[w' = u_z]].$$

Now we describe how to construct the necessary  $y$ .

**TODO(jesse): Fix this to include injectivization and finish the proof.**

□

### 5.1.1 Zorn's lemma

The axiom of choice is equivalent to the following version of Zorn's lemma, which states that any set whose members are ordered by  $\subseteq$  and is closed under unions of chains contains a maximal element.

**Definition 5.4.** (ZL):

$$\begin{aligned} \forall x, (\forall y, (y \subseteq x \wedge \forall w_1 \in y, \forall w_2 \in y, w_1 \subseteq w_2 \vee w_2 \subseteq w_1) \implies \\ (\bigcup y) \in x) \implies \exists m \in x, \forall z \in x, m \subseteq z \implies m = x \end{aligned}$$

The aim of this section is to prove the following theorem:

**Theorem 5.5.** *Zorn's lemma has truth-value  $\top$  in  $V^B$ .*

To do this, we introduce the notion of a *core* of a  $\mathbb{B}$ -valued set.

**Definition 5.6.** Let  $u$  be a  $\mathbb{B}$ -valued set. Let  $S$  be a set of  $\mathbb{B}$ -valued sets. We say that  $S$  is a **core** for  $u$  if the following properties hold:

- (i) For every  $v \in S$ ,  $[[x \in u]] = 1$ , and
- (ii) For each  $y \in V^B$  such that  $[[y \in u]] = 1$ , there exists a *unique*  $v_y \in S$  such that  $[[y = v_y]] = 1$ .

**Lemma 5.7.** *Let  $u$  be a  $\mathbb{B}$ -valued set. There exists a set of  $\mathbb{B}$ -valued sets  $S$  which is a core for  $u$ .*

*Proof.* Fix  $x : V^B$ . We define the set

$$f_x \stackrel{\text{df}}{=} \{(a, u_B(a) \cap u_A(a) =^{\mathbb{B}} x) \mid a \in u_\alpha\}.$$

That is,  $f_x$  is the graph-relation of the map

$$\lambda a : u_\alpha, u_B(a) \cap u_A(a) =^{\mathbb{B}} x.$$

As such,  $f_x : \text{set}(u_\alpha \times \beta)$ . Consider the map

$$(\lambda x : V^B, f_x) : V^B \rightarrow \text{set}(u_\alpha \times \beta).$$

Since  $\text{set}(u_\alpha \times \beta)$  is a small type, we may perform the same type of smallness argument as we did for the maximum principle, and select lifts of every fiber, obtaining a set  $S' : \text{set } V^B$  which satisfies the following property: for every  $x$ , there exists a  $y$  in  $S'$  such that  $f_x = f_y$ .

There is an equivalence relation  $\sim$  on  $V^B$ , defined by  $x_1 \sim x_2 \iff (x_1 =^{\mathbb{B}} x_2 = \top)$ . This restricts to an equivalence relation on any subset of  $V^B$ . In particular, we consider the restriction of  $\sim$  to the subset of  $S'' \stackrel{\text{df}}{=} \{x \in S' \mid x \in^{\mathbb{B}} u = \top\}$ . We define  $S$  to be a transversal of the  $\sim$ -equivalence classes of  $S''$ .

It remains to verify that  $S$  is a core for  $u$ . Since  $S \subseteq S''$  and  $S''$  by definition is made up of  $\mathbb{B}$ -valued sets  $x$  such that  $x \in^{\mathbb{B}} u = \top$ , item (i) of the definition of a core is satisfied.

To verify item (ii), let  $y : V^B$  such that  $y \in^{\mathbb{B}} u = \top$ . By construction of  $S'$ , there exists a  $y' \in S'$  such that  $f_{y'} = f_y$ . Since the supremum of (the values of)  $f_y$  is  $\top$ , the supremum of (the values of)



$f_{y'}$  is  $\top$ , and therefore  $y' \in^{\mathbb{B}} u = \top$ . By construction of  $S''$ , there exists a unique  $y'' \in S''$  such that  $y'' =^{\mathbb{B}} y' = \top$ .

We claim that  $y =^{\mathbb{B}} y'' = \top$ . To prove this claim, by the transitivity of Boolean-valued equality, it suffices to show that  $y =^{\mathbb{B}} y' = \top$ . We calculate:

$$\top = \top \sqcap \top = \left( \bigsqcup_{(a:u_\alpha)} y =^{\mathbb{B}} u_A(\alpha) \right) \sqcap \left( \bigsqcup_{(a:u_\alpha)} u_A(\alpha) =^{\mathbb{B}} y' \right) \leq (y =^{\mathbb{B}} u_A(\alpha)) \sqcap (u_A(\alpha) =^{\mathbb{B}} y') \leq y =^{\mathbb{B}} y'.$$

To complete the proof, we must show that  $y''$  is unique for  $y$ . So, let  $y'''$  be an element of  $S''$  such that  $y''' =^{\mathbb{B}} y = \top$ . To show that  $y''' = y''$ , it suffices to show that  $y''' =^{\mathbb{B}} y' = \top$ , because then  $y'''$  will belong to the  $\sim$ -class of  $y'$  and by construction of  $S''$ ,  $y'''$  will have to be equal to  $y''$ .

However, we have already seen that  $y =^{\mathbb{B}} y' = \top$ , so this follows from the transitivity of  $=^{\mathbb{B}}$ . □

**Remark 5.8.** If  $u$  is nonempty with truth-value  $\top$ , then the maximum principle ensures that any core of  $u$  is nonempty.

**Lemma 5.9.** *Let  $u$  be a  $\mathbb{B}$ -valued set such that  $[[u \neq \emptyset]] = \top$ . Let  $S$  be a core for  $u$ . Then for any  $B$ -valued set  $x$ , there exists a  $y \in S$  such that  $[[x = y]] = [[x \in u]]$ .*

*Proof.* Up to an equality of terms in  $\mathbb{B}$ , this is exactly 4.7. □

*Proof of 5.4.* Let  $X$  be a  $\mathbb{B}$ -valued set such that  $(X, \subseteq)$  is a nonempty inductive partially ordered set. To prove Zorn's lemma, we must show that  $X$  has a maximal element.

Let  $S$  be a core for  $X$ . We define the relation  $\leq: S \rightarrow S \rightarrow \mathbf{Prop}$  as follows:

$$\lambda s_1 s_2, [[s_1 \subseteq s_2]] = \top.$$

It is easy to check that  $\leq$  is a partial order on  $S$ . To apply the Zorn's lemma available in the metatheory, we must show that  $\leq$  is inductive, i.e. that every chain has an upper bound.

So, fix a chain  $C$  of  $S$ . We can turn  $C$  into a  $\mathbb{B}$ -valued set  $\hat{C}$  by declaring the truth-value of all elements to be  $\top$ .

We claim that if  $C$  was a chain in  $S$ , then  $V^{\mathbb{B}} \models^{\mathbb{B}} \hat{C}$  “is a chain in  $X$ .”

*Proof of claim.* Formally, “ $\hat{C}$  is a chain” is the sentence

$$\forall u_1 \in \hat{C}, \forall u_2 \in \hat{C}, u_1 \subseteq u_2 \vee u_2 \subseteq u_1,$$

and so we must show that

$$\top \leq \prod_{u_1 \in C} \prod_{u_2 \in C} u_1 \subseteq^{\mathbb{B}} u_2 \vee u_2 \subseteq^{\mathbb{B}} u_1,$$

or equivalently,

$$\forall u_1 \in C, \forall u_2 \in C, \top \leq u_1 \subseteq^{\mathbb{B}} u_2 \vee u_2 \subseteq^B u_1.$$

So, fix  $u_1$  and  $u_2$  in  $C$ . Since  $C$  was a chain in  $S$ , either  $[[u_1 \subseteq u_2]] = \top$  or  $[[u_2 \subseteq u_1]] = \top$ .

Without loss of generality, suppose that  $[[u_1 \subseteq u_2]] = \top$ , as the other case is entirely symmetric. Then  $\top \leq u_1 \subseteq^{\mathbb{B}} u_2$ , and by the transitivity of  $\leq$ , it suffices to show that

$$u_1 \subseteq^{\mathbb{B}} u_2 \leq u_1 \subseteq^{\mathbb{B}} u_2 \sqcup u_2 \subseteq^{\mathbb{B}} u_1,$$

which immediately follows from the defining property of the binary sup.

That  $\hat{C}$  is a chain in  $X$  means that

$$\top \leq \bigcap_{u \in C} u \in^{\mathbb{B}} X.$$

But, each  $u$  belongs to  $C$ , which is a subset of a core  $S$  for  $X$ , so  $u \in^{\mathbb{B}} X = \top$ .  $\square$

Now we claim that for any chain  $C$  in  $S$ , there exists a  $u : V^{\mathbb{B}}$  such that  $V^{\mathbb{B}} \models^{\mathbb{B}} \text{“} u \text{ is an upper bound for } \hat{C} \text{ in } X \text{”}$ .

*Proof of claim.* We will use  $u = \bigcup \hat{C}$ . We need to show two things:

- (i)  $\top \leq u \in^{\mathbb{B}} X$ , and
- (ii)  $\top \leq \bigcap_{w \in C} w \subseteq^{\mathbb{B}} u$ .

Item (ii) is an immediate consequence of the specification of the  $\mathbb{B}$ -valued unionset operation.

For item (i), we have by assumption that  $X$  is inductive, so that

$$\begin{aligned} T &\leq \bigcap_{(Y:V^{\mathbb{B}})} \left( \left( Y \subseteq^{\mathbb{B}} X \cap \left( \bigcap_{w_1 \in Y} \bigcap_{w_2 \in Y} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right) \Rightarrow (\bigcup Y) \in^{\mathbb{B}} X \right) \\ &\leq \left( u \subseteq^{\mathbb{B}} X \cap \left( \bigcap_{w_1 \in u} \bigcap_{w_2 \in u} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right) \Rightarrow (\bigcup Y) \in^{\mathbb{B}} X \end{aligned}$$

Therefore, it suffices to show that

$$T \leq \left( \hat{C} \subseteq^{\mathbb{B}} X \cap \left( \bigcap_{w_1 \in \hat{C}} \bigcap_{w_2 \in \hat{C}} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right),$$

and so it suffices to show:

- 1.  $T \leq \hat{C} \subseteq^{\mathbb{B}} X$ , and
- 2.  $T \leq \left( \bigcap_{w_1 \in \hat{C}} \bigcap_{w_2 \in \hat{C}} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right),$

This is exactly the conclusion of the previous claim.  $\square$

In particular,  $[[u \in x]] = \top$ , so by virtue of  $S$  being a core, there exists a  $w \in S$  such that  $[[w = u]] = \top$ . Then  $w$  is an upper bound for  $C$  in  $S$ : for any other  $w' \in S$ ,  $[[w' \in \hat{C}]] = 1$ , so  $[[w' \subseteq w]] = \top$ , and therefore  $w' \leq w$ .

So  $S$  is inductive and we may apply Zorn's lemma. Therefore,  $S$  has a maximal element  $c$ . Since  $S$  is a core,  $[[c \in X]] = \top$ .

To finish the proof, it now suffices to show that  $c$  is a maximal element of  $X$ . That is, we must show that

$$V^{\mathbb{B}} \models^{\mathbb{B}} \forall x \in X, c \subseteq x \rightarrow x = c.$$

That is, we must show that

$$\top \leq \bigcap_{(x \in X)} c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

Introducing the universal quantifier, it suffices to show that for any  $x \in V^{\mathbb{B}}$ ,

$$\top \leq x \in^{\mathbb{B}} X \Rightarrow c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c,$$

and introducing the implication, it suffices to show that

$$x \in^{\mathbb{B}} X \leq c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

By 5.9, we can find a  $y$  such that  $x \in^{\mathbb{B}} X = [[x = y]]$ , so we rewrite the above inequality to

$$[[x = y]] \leq c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

Introducing another implication, we have

$$[[x = y]] \cap c \subseteq^{\mathbb{B}} x \leq [[x = c]].$$

since  $\subseteq^{\mathbb{B}}$  is  $=^{\mathbb{B}}$ -extensional, by applying the transitivity of  $\leq$ , it now suffices to show that

$$[[x = y]] \cap (c \subseteq^{\mathbb{B}} y) \leq [[x = c]].$$

If we can show that  $c \subseteq^{\mathbb{B}} y \leq [[y = c]]$ , then we are done, by applying transitivity of  $\leq$  and then by applying the transitivity of  $=^{\mathbb{B}}$ .

Let  $a : \mathbb{B}$  be  $c \subseteq^{\mathbb{B}} y$ . We let  $v$  be the two-term mixture  $a \cdot y + (\neg a) \cdot c$ .

We claim that  $v \in^{\mathbb{B}} X = \top$ .

*Proof of claim.* This follows directly from 4.5. □

By virtue of  $S$  being a core, there exists a  $z \in S$  such that  $[[v = z]] = \top$ .

We claim that  $[[c \subseteq v]] = \top$ .

*Proof of claim.* This follows directly from 4.6. □

Since  $[[v = z]]$ , it follows that  $[[c \subseteq z]] = \top$ , and therefore

$$c \leq z$$

in  $S$ , and since  $c$  is  $\leq$ -maximal,  $c = z$ , so  $c =^{\mathbb{B}} z = \top$ .

Now we calculate:

$$\begin{aligned} [[c \subseteq y]] &= a \leq [[y = v]] \\ &\leq [[y = v]] \cap [[v = z]] \\ &\leq [[y = z]] = [[y = c]], \end{aligned}$$

and the proof is complete. □

- 6 Generic sets: Cohen's original proof
- 7 Sheaves and filterquotients