Forcing and the independence of the continuum hypothesis

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Abstract

In these notes, intended as the plaintext part of the Flypitch project, we give a complete account of the independence of the continuum hypothesis from ZFC, with special attention paid to comparing the different approaches: generic sets, Boolean-valued models, and double-negation sheaves.

1 Preliminaries

1.1 First-order logic

1.1.1 General logical symbols

Definition 1.1. We reserve the following general logical symbols:

$\neg$	not
$\vee$	or
$\wedge$	and
$\forall$	for all
$\exists$	exists
$=$	equals
$(,)$	parentheses
$(x_i)_{i:\mathbb{N}}, (y_i)_{i:\mathbb{N}}, (z_i)_{i:\mathbb{N}}$	variables

1.1.2 First-order languages

Definition 1.2. A (first-order, one-sorted) **language** $\mathcal{L}$ comprises the following data:

(i) A collection of **constant symbols** $\mathsf{Const}(\mathcal{L})$,

(ii) a collection of **relation symbols** $\mathsf{Rel}(\mathcal{L})$,

(iii) a collection of **function symbols** $\mathsf{Funct}(\mathcal{L})$, and

(iv) an assignment of each symbol $S : \mathsf{Const}(\mathcal{L}) \cup \mathsf{Rel}(\mathcal{L}) \cup \mathsf{Funct}(\mathcal{L})$ to a natural number $\text{arity}(S) : \mathbb{N}$.

Whenever we interpret a language on some carrier A, we mean for constants c to be interpreted as elements of $A^{\text{arity}(c)}$, relations R to be interpreted as subsets of $A^{\text{arity}(R)}$, and for function symbols to be interpreted as functions $A^{\text{arity}(f)} \to A$.

Example 1.3. • The language of groups comprises a 1-ary constant symbol for the identity and a 2-ary function for group multiplication.

- The language of rings comprises constant symbols 0 and 1 and 2-ary functions for addition and multiplication.
- The language of set theory comprises just one 2-ary relation $\in$.

1.1.3 Terms, formulas, and sentences

Definition 1.4. A **term** is a string of symbols defined by structural induction as follows:

1. Any variable v is a term.
2. Any constant c is a term.
3. If $t_1, \ldots, t_n$ are terms of arities $a_1, \ldots, a_n$, then $(t_1, \ldots, t_n)$ is a term of arity $a_1 + \cdots + a_n$.
4. If t is a term and f is a function symbol with matching arities, then ft is a term.

Whenever we interpret our language on a carrier A, we mean for terms to be interpreted as functions into A which we can construct by composing existing constants (constant functions), basic functions (i.e. the interpretations of the function symbols), and variables (identity).

Definition 1.5. A **formula** is defined by structural induction as follows:

1. If t_1 and t_2 are terms of the same arity, $t_1 = t_2$ is a formula.
2. If t is a term and R is a relation symbol, and t and R have the same arity, then Rt is a formula.
3. If φ is a formula, $\neg\varphi$ is a formula.
4. If φ and ψ are formulas, then $\varphi \vee \psi$ is a formula.
5. If φ and ψ are formulas, then $\varphi \wedge \psi$ is a formula.
6. If φ is a formula containing a variable v, then $\exists v\varphi$ is a formula.
7. If φ is a formula containing a variable v, then $\forall v\varphi v$ is a formula.

Definition 1.6. Let φ be a formula containing the variables $x_1, \ldots, x_n$. We say that the variable x_k is **free** if x_k is not contained in a subformula of the form $\exists x_k\psi$ or $\forall x_k\psi$.

x_k is **bound** if it is not free.

Definition 1.7. A formula is a **sentence** (or **statement**) if it contains no free variables.

We write $\mathsf{Formulas}(\mathcal{L})$ for all the first-order formulas of $\mathcal{L}$, and we write $\mathsf{Sentences}(\mathcal{L})$ for all the first-order sentences of $\mathcal{L}$.

By convention, we always include sentences called true and false.

1.1.4 Predicate calculus and provability

Throughout this section, we fix a language $\mathcal{L}$.

Definition 1.8. A **propositional function** is a function $f : \mathsf{Prop}^k \to \mathsf{Prop}$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. The constant functions to true and false are propositional functions.

2. Each projection $(P_1, \ldots, P_k) \mapsto P_j$ is a propositional function.

3. If f and g are propositional functions, so are

$$\neg f, f \wedge g, f \vee g, f \to g, \text{ and } f \leftrightarrow g,$$

where the operations above are carried out pointwise in Prop.

f is a **tautology** if $\vdash \forall \vec{p} : \mathsf{Prop}^k, f\vec{p} \leftrightarrow \mathsf{true}$.

Definition 1.9. A **propositional combination** is a function $f : \mathsf{Sentences}(\mathcal{L})^k \to \mathsf{Sentences}(\mathcal{L})^k$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. Each projection $(B_1, \ldots, B_k) \mapsto B_j$ is a propositional combination.

2. If f and g are propositional combinations, so are

$$\neg f, f \wedge g, f \vee g, f \to g, \text{ and } f \leftrightarrow g,$$

where the operations are carried out pointwise in $\mathsf{Sentences}(\mathcal{L})$.

By sending projections to projections and symbols $(\neg, \wedge, \vee, \to, \leftrightarrow)$ to the corresponding operations on Prop, every propositional combination $f : \mathsf{Sentences}(\mathcal{L})^k \to \mathsf{Sentences}(\mathcal{L})$ can be realized as a propositional function $\mathbf{r}(f) : \mathsf{Prop}^k \to \mathsf{Prop}$.

Definition 1.10. The **predicate calculus** comprises the following rules for deducing sentences from other sentences. We call deducible sentences **valid**, and write $\vdash_{\mathcal{L}} \varphi$ to mean that the $\mathcal{L}$-sentence φ is valid (and to disambiguate from $\vdash$, which when used unadorned means "provable in the metatheory).

(a) (Rule of the propositional calculus) if f is a propositional combination taking k arguments such that $\mathbf{r}(f)$ is a tautology, then for any k sentences $A_1, \ldots, A_k$, the value of the propositional combination $f(\varphi_1, \ldots, \varphi_k)$ is a valid sentence.

(b) (Rule of modus ponens) If A and $A \to B$ are valid, then B is valid.

(c) (Rules of equality)

(i) $\forall x, x = x$, $\forall x \forall y, x = y \wedge y = x$, and $\forall x \forall y \forall z, x = y \wedge y = z \to x = z$ are all valid.

(ii) Let $\varphi(x)$ be a formula whose only free variable is x. Then

$$\forall x \forall y, (x = y) \to (\varphi(x) \to \varphi(y))$$

is valid.

(d) (Change of variable) If A is a sentence and A' represents A with all instances of a variable x switched to y, then $A \leftrightarrow A'$ is valid.

(e) (Rule of specialization "$\forall$-elimination") Let c be any constant symbol, and let $\varphi(x)$ be a formula whose only free variable is x. Then $(\forall x \varphi(x)) \to \varphi(c)$ is valid.

(f) ("$\neg$-introduction") If $\neg A \leftrightarrow (A \to \mathsf{false})$ is valid.

(g) (Generalization of constants "$\forall$-introduction") Let B be a sentence which does not contain the constant c or the variable x. Let $\varphi(x)$ be some formula such that $\varphi(c) \to B$ is valid. Then $\exists x \varphi(x) \to B$ is also valid.[1]

[1] In particular, using the next rule, if $\neg\varphi(c) \to \mathsf{false}$ is valid, so is $\exists x \neg\varphi(x) \to \mathsf{false}$, so is $\neg\exists x \neg\varphi(x)$, and therefore so is $\forall x \varphi(x)$.

(h) (de Morgan laws) Let $\varphi(x)$ have x as its only free variable. Let B be a sentence which does not contain x. Then the following are valid statements:

$$(\neg(\forall x \varphi(x))) \leftrightarrow (\exists x \neg \varphi(x))$$

$$((\forall x \varphi(x)) \wedge B) \leftrightarrow ((\forall x(\varphi(x) \wedge B)$$

$$((\exists x \varphi(x)) \wedge B) \leftrightarrow ((\exists x(\varphi(x) \wedge B)$$

Definition 1.11. Let S be a collection of sentences.

(i) We say that A is provable from S if there exist finitely many $B_1, \ldots, B_n : S$ such that $(B_1 \wedge \cdots \wedge B_n) \to A$ is valid.

(ii) We say that S is consistent if false is not valid.

Remark 1.12. One may wonder why we work with a type of formulas and not with a collection of Props directly. The problem with this is that everything needs to be typed, and so to reason about a predicate (say "$\in$") using Prop, we need some carrier type A such that $\in : A \to A \to \mathsf{Prop}$, so that e.g. $\in$ satisfies the axioms of set theory. But then what does it mean for some other type B to have an interpretation of $\in$ and the axioms it satisfies? There then needs to be a separate predicate $\in_B : B \to B \to \mathsf{Prop}$ satisfying the same *kind* of Props as $\in : A \to A \to \mathsf{Prop}$. We could proceed to define a typeclass of such $(B, \in_B)$, and we would then be working with models of set theory, but we would lack a way to reason syntactically about the axioms themselves.

1.2 Models and satisfiability

For the remainder of this section we fix a language $\mathcal{L}$.

Definition 1.13. An **$\mathcal{L}$-theory** is a collection of sentences from $\mathsf{Sentences}(\mathcal{L})$.

Definition 1.14. An **$\mathcal{L}$-structure** comprises the following data:

(i) A carrier type A,

(ii) an assignment of every $c : \mathsf{Const}(\mathcal{L})$ to a $c^A : A^{\text{arity}(c)}$,

(iii) an assignment of every $R : \mathsf{Rel}(\mathcal{L})$ to a subtype $R^A : A^{\text{arity}(R)} \to \mathsf{Prop}$,

(iv) an assignment of every $f : \mathsf{Funct}(\mathcal{L})$ to a function $f^A : A^{\text{arity}(f)} \to A$.

Definition 1.15. Let A be an $\mathcal{L}$-structure. Using the data of A being an $\mathcal{L}$-structure, we can inductively assign to every term t (of arity k and containing n free variables) a **realization $\mathbf{r}(t)$** : $A^n \to A^k$, as follows:

1. If $t = v$ for a variable v, $\mathbf{r}(t) = \text{id}_A = \lambda v, v$.
2. If $t = c$ for a constant symbol c, $\mathbf{r}(t) = A^0 \xrightarrow{c^A} A$.
3. If $t = (t_1, \ldots, t_m)$, then $\mathbf{r}(t) = \mathbf{r}(t_1) \times \cdots \times \mathbf{r}(t_m)$.
4. If $t = f(t_0)$ for some function symbol f, then $\mathbf{r}(t) = f^A \circ \mathbf{r}(t_0)$.

Definition 1.16. Let A be an $\mathcal{L}$-structure. Using the data of A being an $\mathcal{L}$-structure, we can inductively assign to every formula $\varphi(x_1, \ldots, x_n)$ (where $x_1, \ldots, x_n$ exhaust the free variables of φ) a **realization $\mathbf{r}(\varphi)$** : $A^n \to \mathsf{Prop}$, as follows:

1. If φ is of the form $t_1 = t_2$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ (where symbolic equality is realized as true equality).
2. If φ is of the form $R(t)$, $\mathbf{r}(R(t))$ is $R^A(\mathbf{r}(t))$.
3. If φ is of the form $\neg\psi$, then $\mathbf{r}(\varphi)$ is $\neg\mathbf{r}(\psi)$.
4. If φ is of the form $\psi \vee \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \vee \mathbf{r}(\psi')$.
5. If φ is of the form $\psi \wedge \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \wedge \mathbf{r}(\psi')$.
6. If φ is of the form $\exists v\psi$, then $\mathbf{r}(\varphi)$ is $\exists v\mathbf{r}(\varphi)$.
7. If φ is of the form $\forall v\psi$, then $\mathbf{r}(\varphi)$ is $\forall v\mathbf{r}(\varphi)$.

In particular, each sentence φ is sent to a Prop $\mathbf{r}(\varphi)$.

Definition 1.17. Let A be an $\mathcal{L}$-structure, and let φ be a sentence. We say that A **satisfies** φ, written

$$A \models \varphi,$$

if $\vdash \mathbf{r}(\varphi)$.

Definition 1.18. Let T be an $\mathcal{L}$-theory, and let A be an $\mathcal{L}$-structure. We say that A is a **model** of T if for every sentence $\varphi : T$, $A \models \varphi$.

Example 1.19. (Line graph) The **language of graphs** $\mathcal{L}_{\mathsf{Graph}}$ comprises a single 2-ary relation symbol E.

The **theory of graphs** Graph comprises the sentence $\forall x \forall y\, (E(x, y) \leftrightarrow E(y, x))$.

The natural numbers $\mathbb{N}$ can be viewed as a model of Graph as follows. We realize E as the set

$$(y = \operatorname{succ} x) \vee (x = \operatorname{succ} y) : \mathbb{N} \to \mathbb{N} \to \mathsf{Prop}$$

which is clearly symmetric.

Example 1.20. Let $\mathbf{Mod}(\mathsf{Graph})$ be the collection of graphs.[2] A **graph property** is a map $P : \mathbf{Mod}(\mathsf{Graph}) \to \mathsf{Prop}$ such that whenever $G \simeq G'$, $P(G) \leftrightarrow P(G')$. We say that G *satisfies* P if $P(g) \leftrightarrow \mathsf{true}$. A graph property is additionally said to be *monotone* if whenever $G \subseteq G'$ is a subgraph, then $P(G') \to P(G)$.

For example, the property of being a complete graph is not monotone, while the property of being cycle-free is.

The **graph evasiveness conjecture** says that for every monotone graph property P and every $n : \mathbb{N}$, one needs to ask $\binom{n}{2}$ questions of the form "is there an edge between v and w" to determine if an arbitrary graph on n vertices satisfies P.

Example 1.21. (Peano arithmetic)

The **language of Peano arithmetic** $\mathcal{L}_{\mathsf{PA}}$ comprises:

1. A 1-ary constant 0.
2. Three function symbols succ, $+$, $\times$.

[2] Warning: this is "large", so lives in the next universe up: one can interpret a trivial edge relation on *every* type in the current universe.

The **theory of Peano arithmetic** PA comprises:

1. $\forall x, s(x) \neq 0$
2. $\forall x \forall y, (s(x) = s(y)) \to x = y$
3. $\forall x, x + 0 = x$
4. $\forall x \forall y, x + s(y) = s(x + y)$
5. $\forall x, x \times 0 = 0$
6. $\forall x \forall y, x \times S(y) = (x \times y) + x$

Schema: For every $\mathcal{L}_{\mathsf{PA}}$-formula $\varphi(x)$ with one free variable x,

$$\big(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\operatorname{succ} x))\big) \to \forall x \varphi(x).$$

The **standard model** of PA is $\mathbb{N}$ with 0 realized as $0 : \mathbb{N}$, succ realized as $\operatorname{succ} : \mathbb{N} \to \mathbb{N}$, $+$ realized as $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, and $\times$ realized as $\times : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$.

By recursing on the inductive type of valid sentences and replacing every rule of the propositional calculus with the corresponding deduction rule for Prop, we can construct for every valid $\mathcal{L}$-sentence φ a proof that $\mathbf{r}(\varphi) \leftrightarrow \mathsf{true}$.

That is the soundness theorem. (In what follows, taking ψ to be true yields the assertion in the previous paragraph.)

Theorem 1.22. *(Soundness theorem) For every $\mathcal{L}$-structure and any sentences $\varphi, \psi : \mathsf{Sentences}(\mathcal{L})$,*

$$\vdash_{\mathcal{L}} \varphi \to \psi \quad \Longrightarrow \quad \vdash \mathbf{r}(\varphi) \to \mathbf{r}(\psi).$$

This happens regardless of which $\mathcal{L}$-structure is doing the realizing. When the $\mathcal{L}$-structure itself is a model of a theory T, then whenever $T \vdash_{\mathcal{L}} \psi$, then since there is some sentence $\varphi : T$ such that $\vdash_{\mathcal{L}} \varphi \to \psi$, Prop's modus ponens tells us that the model satisfies ψ also.

Example 1.23. For example, suppose we're working in the language of graphs expanded with two 1-ary constants a and b, and we know that there is some model M such that M satisfies the sole axiom that E is symmetric. We can show

$$\vdash (\forall x \forall y, \mathbf{r}(E)(x, y) \leftrightarrow \mathbf{r}(E)(y, x)) \to \mathbf{r}(E)(\mathbf{r}(a), \mathbf{r}(b)) \leftrightarrow \mathbf{r}(E)(\mathbf{r}(b), \mathbf{r}(a))$$

because we already know the antecedent and can apply Prop's $\forall$-elimination.

The converse of Theorem 1.22 is false. There may be some things which are incidentally true about the model which are not universally valid.

Example 1.24. Working again in the language of graphs, consider a complete graph on n vertices. Call this model M. M happens to satisfy the $\mathcal{L}$-sentence

$$\big(\forall x \forall y, E(x, y) \leftrightarrow E(y, x)\big) \to \big(\forall x \forall y \forall z, E(x, y) \wedge E(y, z) \to E(x, z)\big),$$

but this is not a valid $\mathcal{L}$-sentence. (Indeed, if it were, then the soundness theorem would imply that *every* graph has a transitive edge relation, which is not true.)

It will turn out that we can do the next best thing. If we rule out this kind of exception by requiring that $M \models \varphi$ *for every $\mathcal{L}$-structure* M (resp. every model M of T), then it follows that $\vdash_{\mathcal{L}} \varphi$ (resp. $T \vdash_{\mathcal{L}} \varphi$). This is the completeness theorem.

1.3 The completeness theorem

In this section, our goal will be to prove the **completeness theorem**:

Theorem 1.25. *Let T be an $\mathcal{L}$-theory. T is consistent if and only if there exists a model of T.*

First we will prove that if there exists a model M of T, then T is consistent.

Proof. We will show the contrapositive: if T is inconsistent, then there does not exist a model M of T.

Indeed, suppose that T is inconsistent. Suppose there is a model M. Then by the soundness theorem, $M \models \mathsf{false}$. By definition, this means that

$$\vdash \mathsf{false},$$

so we have shown that

$$\vdash (T \text{ inconsistent}) \wedge (\text{there exists a model } M \text{ of } T) \to \mathsf{false}$$

which is equivalent to

$$\vdash (T \text{ not inconsistent}) \vee (T \text{ does not have a model}),$$

which is equivalent to

$$\vdash T \text{ inconsistent } \to T \text{ does not have a model.}$$

Taking the contrapositive, we conclude that if T has a model, then T is consistent. $\square$

It then remains to show that if T is consistent, T has a model. We will use the Henkin construction.

1.3.1 The Henkin construction

Definition 1.26. Let T be an $\mathcal{L}$-theory. We say that T is a **Henkin theory** if, for every formula $\varphi(x)$, there is a constant $c : \mathsf{Const}(\mathcal{L})$ such that $T \vdash_{\mathcal{L}} (\exists x \varphi(x)) \to \varphi(c)$.

Example 1.27. Let $\mathcal{L}_{\text{field}}$ be the language of fields, which we define to be $\{0, 1, +, \times, (-)^{-1}\}$ (the usual language of rings augmented with an inversion operation), and let T be the usual axiomatization of a field of characteristic zero. T is not a Henkin theory, for there is no constant c such that e.g. $c = (1+1)^{-1}$.

Example 1.28. Let $\mathcal{L}_{\mathsf{PA}}$ be the language of Peano arithmetic (see Example 1.21). Let T be the collection of all $\mathcal{L}_{\mathsf{PA}}$-sentences ψ such that $N \models \psi$. Then T certainly contains the sentence $\exists x \forall y, x \cdot y = y$. However, $1 = \text{succ } 0$ is not a constant in the language, but rather a term. So T is not a Henkin theory.

However, if we *expand* $\mathcal{L}_{\mathsf{PA}}$ to a language $\mathcal{L}'$ with a constant symbol c_n for every natural number n, and if we let T' be the collection all $\mathcal{L}'$-sentences ψ such that $\mathbb{N}$ (viewed in the natural way as a model of $\mathcal{L}'$) satisfies ψ, then T' *is* a Henkin theory.

Proposition 1.29. *Let T be an $\mathcal{L}$-theory. If T is consistent, then there exists a language $\mathcal{L}'$ extending L and an $\mathcal{L}'$-theory T' extending T viewed as an $\mathcal{L}$'-theory, such that T' is a Henkin theory.*

Furthermore, if T is consistent, then T' is consistent.

Proof. Put $\mathcal{L}_0 \overset{\text{df}}{=} \mathcal{L}$ and $T_0 \overset{\text{df}}{=} T$. We define a chain of languages $\mathcal{L}_i$ and for each i we define an $\mathcal{L}_i$-theory T_i as follows: given $\mathcal{L}_n$ and $\mathcal{T}_n$, let $\mathcal{L}_{n+1}$ be the language obtained by adding a constant $c_{\varphi,x}$ where φ ranges over all $\mathcal{L}_n$-formulas and x ranges over the free variables of φ.

Having defined $\mathcal{L}_{n+1}$, we now define T_{n+1} to be

$$T_n \cup \{\exists x \varphi(x) \to \varphi(c_{\varphi,x})\}_{\varphi,x}$$

where above we have adjoined a sentence saying that the newly-adjoined constant $c_{\varphi,x}$ behaves as expected.

We put

$$T' \overset{\text{df}}{=} \bigcup_{n\,:\,\mathbb{N}} T_n.$$

By construction, T' is a Henkin theory.

It remains to show that if T is consistent, so is T'. If $T \vdash_{\mathcal{L}} \psi$, then from the finiteness of proofs, we must have that $T_n \vdash_{\mathcal{L}} \psi$ for some n. So, to show T' is consistent, it suffices to show that for each n, T_n is consistent.

We induct on n. The base case $T = T_0$ is by assumption. For the induction step, we must show that if T_n is consistent, then T_{n+1} is consistent.

Suppose towards the contrapositive that T_{n+1} is inconsistent. Since T_{n+1} is obtained by adjoining formulas of the form $\exists x \varphi(x) \to \varphi(c)$, there must be finitely many such formulas $\psi_1, \ldots, \psi_m : T_{n+1} \backslash T_n$ of this form, along with finitely many formulas $\rho_1, \ldots, \rho_n$ from T_n, such that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_m \to \mathsf{false}.$$

By material implication, we get that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \neg\psi_m,$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \neg(\exists x \varphi_m(x) \to \varphi_m(c_m)),$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to (\exists x \varphi_m(x)) \wedge \neg\varphi_m(c_m)),$$

and since c_m does not occur in the premise of the implication, we have that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to (\exists x \varphi_m(x)) \wedge \forall x \neg\varphi_m(x))$$

and therefore

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \mathsf{false}.$$

We conclude that

$$\rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1}$$

is inconsistent. Repeating this argument, we eliminate all the ψ_i and conclude that $\rho_1 \wedge \cdots \wedge \rho_n$ is inconsistent, and therefore that T is inconsistent. $\square$

Definition 1.30. To any Henkin $\mathcal{L}$-theory T, we can associate a canonical structure (a "term model") term(T) built from the closed terms (i.e. those not containing any variables).

First, we take the collection A of all closed $\mathcal{L}$-terms. We define a relation $E : A \to A \to \mathsf{Prop}$, with the convention that $\vdash E\ \ a_1\ \ a_2 \leftrightarrow \mathsf{true}$ if and only if $T \vdash_{\mathcal{L}} a_1 = a_2$. By the rules about equality that we have stipulated as part of the predicate calculus, E is an equivalence relation.

We put $\widetilde{A} \overset{\mathrm{df}}{=} A/E$. This will be the underlying type of the model.

For a constant $c : \mathsf{Const}(\mathcal{L})$, we put $c^{\widetilde{A}} \overset{\mathrm{df}}{=} c/E$ (c belongs to $\mathcal{L}_0$, and so is a closed term of $\mathcal{L}'$).

For a relation symbol $R : \mathsf{Rel}(\mathcal{L})$, we define $R^{\widetilde{A}} : \widetilde{A}^{\mathrm{arity}(R)} \to \mathsf{Prop}$ by $R^{\widetilde{A}}(a_1/E, \ldots, a_n/E) \leftrightarrow T' \vdash_{\mathcal{L}} R(a_1, \ldots, a_n)$.

For a function symbol $f : \mathsf{Funct}(\mathcal{L})$, we define $f^{\widetilde{A}} : \widetilde{A}^{\mathrm{arity}(f)} \to \widetilde{A}$ by

$$\lambda a_1/E \ \ \ldots \ \ a_n/E, f(a_1, \ldots, a_n)/E.$$

This completes the definition of term(T).

By the soundness theorem, if T is inconsistent, then term(T) cannot be a model of T. But, under suitable assumptions, the inverse is true.

Definition 1.31. An $\mathcal{L}$-theory T is **complete** if for every $\mathcal{L}$-sentence ψ,

$$\vdash (T \vdash_{\mathcal{L}} \psi) \vee (T \vdash_{\mathcal{L}} \neg\psi)\,.$$

Remark 1.32. Excluded middle in Prop implies that for any $\mathcal{L}$-structure M and every $\mathcal{L}$-sentence ψ,

$$\vdash (M \models \psi) \vee (M \models \neg\psi),$$

and therefore that the $\mathcal{L}$-theory of an $\mathcal{L}$-structure (i.e. the collection of all sentences true in the structure) is complete.

By invoking the axiom of choice, we can extend any consistent theory to a complete consistent theory. We will prove this.

Proposition 1.33. *Let T be a consistent $\mathcal{L}$-theory. There exists a complete, consistent$\mathcal{L}$-theory T' which contains T.*

To prove this, we will use Zorn's lemma. To start the argument, we prove the following lemma.

Lemma 1.34. *Suppose T is consistent. Let φ be an $\mathcal{L}$-sentence. Then $T \cup \{\varphi\}$ is consistent or $T \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. Then there exist sentences σ and ρ from T such that

$$\vdash_{\mathcal{L}} (\sigma \wedge \varphi) \to \mathsf{false} \quad \text{and} \quad \vdash_{\mathcal{L}} (\rho \wedge \neg\varphi) \to \mathsf{false}.$$

By $\neg$-introduction, we get

$$\vdash_{\mathcal{L}} \neg\,(\sigma \wedge \varphi) \quad \text{and} \quad \vdash_{\mathcal{L}} \neg\,(\rho \wedge \neg\varphi)$$

and by $\wedge$-introduction, we get

$$\vdash_{\mathcal{L}} (\neg(\sigma \wedge \varphi)) \wedge (\neg(\rho \wedge \neg\varphi))\,.$$

Since the finitary de Morgan laws are tautologies in the sense of 1.8, it follows that

$$\vdash_{\mathcal{L}} \neg\left(\sigma \vee \varphi \vee \rho \vee \neg\varphi\right).$$

Since the metatheory satisfies the law of the excluded middle, we have that the law of the excluded middle for $\mathcal{L}$-formulas is a tautology in the sense of 1.8. Therefore,

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \rho) \Leftrightarrow \vdash_{\mathcal{L}} \neg\sigma \wedge \neg\rho,$$

so by $\wedge$-elimination, $\vdash_{\mathcal{L}} \neg\sigma$ and $\vdash_{\mathcal{L}} \neg\rho$, so T is inconsistent. □

Proof of 1.33. Consider the poset of proper consistent extensions of T. If T is not complete, then the previous lemma shows that this poset is nonempty.

Now we show that we can take the union of a chain in this poset and obtain an upper bound on that chain.

Indeed, let $(T_i)_{i \in I}$ be a chain in this poset, and let T_∞ be its union. This is clearly a theory which contains all the theories in the chain (and also T). We need to show that it is consistent. Indeed, if it were inconsistent, then by the finiteness of proofs, there exists some T_n such that $T_n \vdash_{\mathcal{L}} \mathsf{false}$.

This now fits the hypotheses of Zorn's lemma, which gives us a maximal consistent extension of T' of T. If T' were not complete, then the previous lemma shows that we can extend it. □

Theorem 1.35. *Let T be a complete Henkin $\mathcal{L}$-theory. If T is consistent, then* $\mathrm{term}(T)$ *is a model of T.*

Proof. We will show that for every $\psi : \mathsf{Sentences}(\mathcal{L})$,

$$T \vdash_{\mathcal{L}} \psi \iff \mathrm{term}(T) \models \psi.$$

We will do this by a structural induction on formulas. In the base case, we have atomic sentences.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $a_1 = a_2$ where a_1 and a_2 are closed terms, then since $T \vdash_{\mathcal{L}} a_1 = a_2$, then $\vdash a_1^{\widetilde{A}} = a_2^{\widetilde{A}}$ (in $\widetilde{A}$), so $\mathrm{term}(T) \models \psi$.

  Conversely, if $\mathrm{term}(T) \models \psi$, then $\vdash a_1^{\widetilde{A}} = a_2^{\widetilde{A}}$, so by definition of the equivalence relation we used to define $\widetilde{A}$, $T \vdash_{\mathcal{L}} a_1 = a_2$.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $R(a_1, \ldots, a_n)$ where R is a relation symbol and $a_1, \ldots, a_n$ are closed terms, then since $T \vdash_{\mathcal{L}} R(a_1, \ldots, a_n)$, we have that $\vdash R^{\widetilde{A}}(a_1^{\widetilde{A}}, \ldots, a_n^{\widetilde{A}})$.

  Conversely, if $\mathrm{term}(T) \models \psi$, then $\vdash R^{\widetilde{A}}(a_1^{\widetilde{A}}, \ldots, a_n^{\widetilde{A}})$, so by definition of how we interpreted $\mathcal{L}$ onto $\widetilde{A}$, $T \vdash_{\mathcal{L}} R(a_1, \ldots, a_n)$.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $\varphi_1 \wedge \varphi_2$, then by $\wedge$-elimination in $\mathsf{Sentences}(\mathcal{L})$,

  $$\vdash (T \vdash_{\mathcal{L}} \psi) \to (T \vdash_{\mathcal{L}} \varphi_1) \wedge (T \vdash_{\mathcal{L}} \varphi_2).$$

  By the induction hypothesis, $\mathrm{term}(T) \models \varphi_1$ and $\mathrm{term}(T) \models \varphi_2$, so by $\wedge$-introduction in Prop, $\mathrm{term}(T) \models \varphi_1 \wedge \varphi_2$.

  Conversely, if $\mathrm{term}(T) \models \varphi_1 \wedge \varphi_2$, then by $\wedge$-elimination in Prop, $\mathrm{term}(T) \models \varphi_1$ and $\mathrm{term}(T) \models \varphi_2$. By the induction hypothesis, $T \vdash_{\mathcal{L}} \varphi_1$ and $T \vdash_{\mathcal{L}} \varphi_2$, so by $\wedge$-introduction in $\mathsf{Sentences}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi_1 \wedge \varphi_2$.

- Suppose $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $\neg\varphi$. The induction hypothesis says that $T \vdash_{\mathcal{L}} \varphi$ if and only if $\mathrm{term}(T) \models \varphi$. Since T is consistent, $T \ \vdash_{\mathcal{L}} \varphi$. Therefore, by the induction hypothesis, $\mathrm{term}(T) \not\models \varphi$. By the law of the excluded middle, $\mathrm{term}(T) \models \neg\varphi$.

  Conversely, suppose that $\mathrm{term}(T) \models \neg\varphi$. Then $\mathrm{term}(T) \not\models \varphi$, so by the induction hypothesis, $T \ \vdash_{\mathcal{L}} \varphi$. Since T was complete, $T \vdash_{\mathcal{L}} \varphi$.

  We omit the cases for $\wedge$ and $\rightarrow$, which are entirely analogous.

  We conclude that whenever ψ is quantifier-free, $T \vdash_{\mathcal{L}} \psi$ if and only if $\mathrm{term}(T) \models \psi$.

  To complete the proof, we must take care of quantifiers.

- Suppose that $T \vdash_{\mathcal{L}} \exists x \varphi(x)$, where $\varphi(x)$ satisfies the induction hypothesis that if we substitute a closed term c for x, $\varphi(c)$ is a sentence such that $T \vdash_{\mathcal{L}} \varphi(c)$ if and only if $\mathrm{term}(T) \models \varphi(c)$.

  Then, since T is a Henkin theory, there exists some c such that

  $$T \vdash_{\mathcal{L}} \varphi(c).$$

  By the induction hypothesis, we have that

  $$\mathrm{term}(T) \models \varphi(c),$$

  and therefore by $\exists$-introduction in Prop, we conclude that

  $$\mathrm{term}(T) \models \exists x \varphi(x).$$

  Conversely, suppose that $\mathrm{term}(T) \models \exists x \varphi(x)$. By $\exists$-elimination in Prop, there exists some $a/E : \widetilde{A}$ such that $\vdash \mathbf{r}(\varphi)(a/E)$, which is equivalent to $\mathrm{term}(T) \models \varphi(a)$. By the induction hypothesis, $T \vdash_{\mathcal{L}} \varphi(a)$, and by $\exists$-introduction in $\mathsf{Sentences}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \exists x \varphi(x)$.

- Similarly, suppose that $T \vdash_{\mathcal{L}} \forall x \varphi(x)$, where $\varphi(x)$ satisfies the induction hypothesis that if we substitute a closed term c for x, $\varphi(c)$ is a sentence such that $T \vdash_{\mathcal{L}} \varphi(c)$ if and only if $\mathrm{term}(T) \models \varphi(c)$.

  Then by $\forall$-elimination in $\mathsf{Sentences}(\mathcal{L})$, we have that for every constant $c : \mathsf{Const}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi(c)$. By the induction hypothesis, $\mathrm{term}(T) \models \varphi(c)$. Since the interpretations of c exhaust $\mathrm{term}(T)$, we conclude by $\forall$-introduction in Prop that $\mathrm{term}(T) \models \forall x \varphi(x)$.

  Conversely, suppose that $\mathrm{term}(T) \models \forall x \varphi(x)$. By $\forall$-elimination in Prop, for every $a/E \in \widetilde{A}$, $\vdash \mathbf{r}(\varphi)(a/E)$, which is equivalent to $\mathrm{term}(T) \models \varphi(a)$. By the induction hypothesis, for every $c : \mathsf{Const}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi(c)$.

  Since T is consistent, for every c, T does not prove $\neg\varphi(c)$, so $\mathrm{term}(T) \models \neg\varphi(c) \rightarrow \mathsf{false}$, and therefore $T \vdash \neg\varphi(c) \rightarrow \mathsf{false}$. Specializing to a c which does not occur in φ, we get from $\forall$-introduction (g) in $\mathsf{Sentences}(\mathcal{L})$ that $T \vdash_{\mathcal{L}} \exists x \neg\varphi(x) \rightarrow \mathsf{false}$. By the de Morgan laws, $T \vdash_{\mathcal{L}} \forall x \neg\neg\varphi(x)$. Since double-negation elimination is a propositional tautology, we conclude that $T \vdash_{\mathcal{L}} \forall x \varphi(x)$.

$\square$

Corollary 1.36. *Let T be a consistent $\mathcal{L}$-theory. Then T has a model.*

Proof. By 1.29, extend T to a Henkin theory T'. By 1.33, extend T' to a complete theory T''.

T'' is again Henkin: for any formula $\varphi(x)$, there already exists a c such that $T' \vdash_{L'} \exists x\varphi(x) \leftrightarrow \varphi(x)$, and T'' contains all the sentences of T'.

By 1.35, $\mathrm{term}(T'')$ is a model of T''. Since T'' contains T, $\mathrm{term}(T'')$ is also a model of T. ☐

This completes the proof of the completeness theorem.

1.4 The Löwenheim-Skolem theorem

TODO

2 ZFC

The language $\mathcal{L}_{\mathsf{ZFC}}$ of set theory comprises just one 2-ary relation $\in$. Now we give the definition of the $\mathcal{L}_{\mathsf{ZFC}}$-theory ZFC.

Definition 2.1. ZFC is defined to be the collection of following axioms and axiom schemas:

Extensionality

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

This says that every set is determined by its elements.

Notation 2.2. In what follows, we write "$x \subseteq y$" to abbreviate the formal statement $\forall z \in x, z \in y$.

Empty set

$$\exists x \forall y (\neg y \in x).$$

Viewing a model of ZFC as a directed tree, this says that every model has a least ("root") element.

Pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

This axiom says that we can form unordered pairs.

Notation 2.3. We denote z as above by $\{x, y\}$, adopt the convention that $\{x\} \stackrel{\text{df}}{=} \{x, x\}$, and we implement ordered pairs with *Kuratowski ordered pairs*, viz. $(x, y) \stackrel{\text{df}}{=} \{\{x\}, \{x, y\}\}$.

Now that we have defined ordered pairs, we can define functions (internal to ZFC):

Definition 2.4. A **function** is a set f of ordered pairs such that $(x, y) \wedge (x, z) \rightarrow y = z$.

Union

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists t (z \in t \wedge t \in x)).$$

This says that the y above is the union of all the members of x. Applying 2.1, we conclude that given sets x and y, there exists z such that $z = x \cup y$.

Definition 2.5. Let x be a set. We denote the **successor** of x to be the set $\succ x \stackrel{\text{df}}{=} x \cup \{x\}$.

Infinity

$$\exists x \left(\varnothing \in x \wedge \forall y (y \in x \rightarrow \succ y \in x) \right).$$

Replacement Let $\varphi(x, y, t_1, \ldots, t_k)$ be an $\mathcal{L}_{\mathsf{ZFC}}$-formula with at least two free variables. For each such formula,

$$\forall t_1 \ldots \forall t_k (\forall x \exists! y \varphi(x, y, t_1, \ldots, t_k) \to \forall u \exists v \forall r (r \in v \leftrightarrow \exists s (s \in u \wedge \varphi(s, r, t_1, \ldots, t_k)))).$$

is an axiom of ZFC.

This axiom says that if for fixed terms $t_1, \ldots, t_k$, $\varphi(x, y, t_1, \ldots, t_k)$ is the graph of a function sending x to y, then for each set u, the image of u under this function is again a set.

Note that the quantifiers above can range over the entire model of ZFC.

Powerset

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x).$$

Choice Let $y : x \to z$ abbreviate the $\mathcal{L}_{\mathsf{ZFC}}$-formula which says that y is a function from x to z.

$$\forall y \forall y \forall z, y : x \to z \wedge (y \neq \varnothing) \to \left(\exists f (f : x \to (\bigcup z) \wedge \forall a \in x, f(a) \in y(x)) \right).$$

More clearly, this says that for every x-indexed family of sets z, there exists a section to the projection $\bigcup z \twoheadrightarrow x$.

Regularity

$$\forall x \exists y (x = \varnothing \vee (y \in x \wedge \forall z (z \in x \to \neg z \in y)))$$

This asserts that every set contains an element which is minimal with respect to $\in$.

2.1 Ordinal numbers

Definition 2.6. We say that y is a (binary) **relation** on x if y is a set of ordered pairs from X.

Definition 2.7. We say that a relation $<$ on x is a **well-ordering** if:

1. $$\forall a \forall b, a = b \vee a < b \vee b < a.$$

2. $$\forall a \forall b \forall c, a < b \wedge b < c \to a < c.$$

3. $$\forall s \subseteq x, s \neq \varnothing \to \exists a (a \in x \wedge \forall b (b \in s \to \neg a < b)).$$

Definition 2.8. A set x is called **transitive** if $y \in x, z \in y \to z \in x$.

Definition 2.9. We say that a set α is an **ordinal** if it is well-ordered by the membership relation $\in$ and it is transitive. We abbreviate this assertion by $\mathrm{On}\,\alpha$.

2.2 Cardinal numbers

3 Boolean-valued models

3.1 Boolean algebras

Definition 3.1. A **preorder** B is a type B equipped with relations $\leqslant$ and $<$ satisfying the following properties:

1. $\forall a : B, a \leqslant a$
2. $\forall a, b, c : B, a \leqslant b \to b \leqslant c \to a \leqslant c$
3. $\forall a, b : B, a \leqslant b \wedge \neg b \leqslant a$
4. $\forall a, b : B, a < b \leftrightarrow (a \leqslant b \wedge \neg b \leqslant a)$

Definition 3.2. A **partial order** B is a preorder such that the $\leqslant$ relation is antisymmetric:

$$\forall a, b : B, a \leqslant b \to b \leqslant a \to a = b.$$

Definition 3.3. A **join-semilattice** B is a partial order with binary sup operation $\sqcup$ which satisfies the following properties:

1. $\forall a, b : B, a \leqslant a \sqcup b$
2. $\forall a, b : B, b \leqslant a \sqcup b$
3. $\forall a, b, c : B, a\ leqc \to b \leqslant c \to a \sqcup b \leqslant c.$

Definition 3.4. A **meet-semilattice** B is a partial order with a, binary infimum operation $\sqcap$ which satisfies the following properties:

1. $\forall a, b : B, a \sqcap b \leqslant a$
2. $\forall a, b : B, a \sqcap b \leqslant b$
3. $\forall a, b, c : B, a \leqslant b \to a \leqslant c \to a \leqslant b \sqcap c.$

Definition 3.5. A **lattice** B is a join-semilattice which is also a meet-semilattice.

Definition 3.6. A **distributive lattice** B is a lattice which satisfies the following property:[3]

$$\forall x, y, z : B, (x \sqcup y) \sqcap (x \sqcup z) \leqslant x \sqcup (y \sqcap z).$$

Definition 3.7. A lattice B has a **bottom element** $\bot$ if for every $a : B, \bot \leqslant a$,

Definition 3.8. A lattice B has a **top element** $\top$ if for every $a : B, a \leqslant \top$.

Definition 3.9. A **bounded lattice** is a lattice with a top and bottom element.

Definition 3.10. A **bounded distributive lattice** is a distributive lattice which is a, bounded.

Definition 3.11. A **Boolean algebra** B is a, bounded distributive lattice such that:

1. For every $a : B$, there exists an element $\neg a : B$ which satisfies the following properties:
   (a) $\forall x : B, x \sqcap \neg x = \bot$
   (b) $\forall x : B, x \sqcup \neg x = \top$
2. We additionally specify a **complementation operator** $\lambda x, y, x - y : B \to B$ which satisfies the property:
$$\forall x, y : B, x - y = x \sqcap \neg y.$$

[3]From the `mathlib` docstring: a distributive lattice can be defined to satisfy any of four equivalent distribution properties (of sup over inf or inf over sup, on the left or right). A classic example of a distributive lattice is the lattice of subsets of a set, and in fact this example is generic in the sense that every distributive lattice is realizable as a sublattice of a powerset lattice.

Definition 3.12. For convenience, we accumulate the previous definitions into a complete axiomatization of a boolean algebra. A boolean algebra is a type B with a specification of binary ordering relations $\leqslant$, $<$, a binary sup operation $\sqcup$, a binary inf operation $\sqcap$, top and bottom elements $\top$, $\bot$, a unary negation operator $\neg$, and a binary subtraction operator $-$, satisfying the following properties:

1. $\forall (a : B), a \leqslant a$
2. $(ab, c_1 : B), a \leqslant b \to b \leqslant c_1 \to a \leqslant c_1$
3. $(\forall (a, b : B), a < b \iff a \leqslant b \land \neg b \leqslant a)$
4. $(a, b : B), a \leqslant b \to b \leqslant a \to a = b$
5. $\forall (a, b : B), a \leqslant a \sqcup b$
6. $\forall (a, b : B), b \leqslant a \sqcup b$
7. $\forall (a, b, c_1 : B), a \leqslant c_1 \to b \leqslant c_1 \to a \sqcup b \leqslant c_1$
8. $\forall (a, b : B), a \sqcap b \leqslant a$
9. $\forall (a, b : B), a \sqcap b \leqslant b$
10. $\forall (a, b, c_1 : B), a \leqslant b \to a \leqslant c_1 \to a \leqslant b \sqcap c_1$
11. $\forall (x, y, z : B), (x \sqcup y) \sqcap (x \sqcup z) \leqslant x \sqcup y \sqcap z$
12. $\forall (a : B), a \leqslant \top$
13. $\forall (a : B), \bot \leqslant a$
14. $\forall (x : B), x \sqcap -x = \bot$
15. $\forall (x : B), x \sqcup -x = \top$
16. $\forall (x, y : B), x - y = x \sqcap -y$

4 Generic sets: Cohen's original proof

5 Sheaves and filterquotients