Forcing seminar roadmap

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Abstract

These are my notes for the first, organizational meeting of the reading seminar on forcing and the independence of the continuum hypothesis, held Thursday August 30 at 1PM in Thackeray 427.

Meeting time and place: Thursdays 2PM in Thackeray 427.

Seminar website: `https://www.pitt.edu/~jmh288/forcing-seminar/forcing-seminar.html`

1 Introduction

Forcing is a powerful technique which allows one to create new models of set theory where things like the continuum hypothesis, axiom of choice, and existence of non-measurable subsets of the reals are false. Its discovery earned Cohen a Fields Medal, and his proof of the independence of CH is something which everyone should see at least once.

This reading seminar will assume no background of its participants, and we will start from scratch.

Of course, we will all aim to learn something about how to prove that CH is independent of ZFC.

Ultimately, the goal of this seminar is to produce a set of extremely detailed notes for use as the human-readable part of the Flypitch project. I encourage participants to record their contributions to the seminar in the notes, which will be hosted at:

`https://www.github.com/flypitch/flypitch-notes/`.

Please let me know if you want to contribute and would like help with using `git` or GitHub.

2 Roadmap

We will aim for the seminar to end the same week that undergraduate classes end. So, starting from next Thursday 6 September, we have until Thursday 6 December, which gives us 91 days, or exactly 13 weeks.

Cohen's proof requires little background besides the basics of first-order logic and set theory. So, we will first cover the basics of first-order logic and set theory. In particular, we need to see the proof of Gödel's completeness theorem, and Gödel's proof that CH holds in the constructible universe L. Then we can start the on the independence of CH. So the seminar will be split into three parts:

1. Prerequisites from first-order logic
2. Prerequisites from set theory
3. The independence of CH from ZFC.

If we remain on schedule, there will be time left over to look at alternate proofs via Boolean-valued models and topos theory. Folklore has it that all these proofs are really "the same", so it would be interesting to see why.

The main texts for the reading seminar will be Cohen's monograph [4] and his original papers [2] and [3].

For additional background in logic, I recommend Manin's *A course in mathematical logic for mathematicians* [6] and Tent and Ziegler's *A course in model theory* [8]. Manin's book contains an account of the Boolean-valued model approach, but we should also look at Dana Scott's original paper [7]. The topos theoretic proof is due to Lawvere and Tierney, and a complete exposition is given in Tierney's monograph [9].

Now we will subdivide these three parts into specific topics for each week, and tentatively assign topics to the participants. The structure of this outline is roughly mirrored by the main text [4].

3 Responsibilities and schedule

Prerequisites from first-order logic

1. Syntax: first-order languages, terms, formulas, sentences, proof systems
2. Semantics: models of theories, satisfiability, Löwenheim-Skolem
3. Gödel's completeness theorem.

Prerequisites from set theory

1. ZFC, von Neumann's cumulative hierarchy, ordinals and cardinals
2. Inner models and the constructible universe L
3. CH holds in the constructible universe L.

The independence of CH from ZFC

1. *The independence of the continuum hypothesis, I*
2. *The independence of the continuum hypothesis, II*
3. (maybe) Proof via Boolean-valued models

4. (maybe) Topos-theoretic proof via filterquotients and double-negation sheaves.

Suggested texts

[1] J. Barwise, *Handbook of mathematical logic*, vol. 90, Elsevier, 1982.

[2] P. J. Cohen, *The independence of the continuum hypothesis*, Proceedings of the National Academy of Sciences, 50 (1964), pp. 1143–1148.

[3] ——, *The independence of the continuum hypothesis, ii*, Proceedings of the National Academy of Sciences, 51 (1964), pp. 105–110.

[4] ——, *Set theory and the continuum hypothesis*, Courier Corporation, 2008.

[5] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer New York, 1994.

[6] Y. I. Manin, *A course in mathematical logic for mathematicians*, vol. 53, Springer Science & Business Media, 2009.

[7] D. Scott, *A proof of the independence of the continuum hypothesis*, Mathematical Systems Theory, 1 (1967), pp. 89–111.

[8] K. Tent and M. Ziegler, *A course in model theory*, vol. 40, Cambridge University Press, 2012.

[9] M. Tierney, *Sheaf theory and the continuum hypothesis*, (1972), pp. 13–42.