Forcing and the independence of the continuum hypothesis

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Abstract

In these notes, intended as the plaintext part of the Flypitch project, we give a complete account of the independence of the continuum hypothesis from ZFC, with special attention paid to comparing the different approaches: generic sets, Boolean-valued models, and double-negation sheaves.

1 Preliminaries

1.1 First-order logic

1.1.1 General logical symbols

Definition 1.1. We reserve the following general logical symbols:

$\neg$	not
$\vee$	or
$\wedge$	and
$\forall$	for all
$\exists$	exists
$=$	equals
$(,)$	parentheses
$(x_i)_{i:\mathbb{N}}, (y_i)_{i:\mathbb{N}}, (z_i)_{i:\mathbb{N}}$	variables

1.1.2 First-order languages

Definition 1.2. A (first-order, one-sorted) **language** $\mathcal{L}$ comprises the following data:

(i) A collection of **constant symbols** $\mathsf{Const}(\mathcal{L})$,

(ii) a collection of **relation symbols** $\mathsf{Rel}(\mathcal{L})$,

(iii) a collection of **function symbols** $\mathsf{Funct}(\mathcal{L})$, and

(iv) an assignment of each symbol $S : \mathsf{Const}(\mathcal{L}) \cup \mathsf{Rel}(\mathcal{L}) \cup \mathsf{Funct}(\mathcal{L})$ to a natural number $\text{arity}(S) : \mathbb{N}$.

Whenever we interpret a language on some carrier A, we mean for constants c to be interpreted as elements of $A^{\text{arity}(c)}$, relations R to be interpreted as subsets of $A^{\text{arity}(R)}$, and for function symbols to be interpreted as functions $A^{\text{arity}(f)} \to A$.

Example 1.3.
- The language of groups comprises a 1-ary constant symbol for the identity and a 2-ary function for group multiplication.
- The language of rings comprises constant symbols 0 and 1 and 2-ary functions for addition and multiplication.
- The language of set theory comprises just one 2-ary relation $\in$.

1.1.3 Terms, formulas, and sentences

Definition 1.4. A **term** is a string of symbols defined by structural induction as follows:

1. Any variable v is a term.
2. Any constant c is a term.
3. If $t_1, \dots, t_n$ are terms of arities $a_1, \dots, a_n$, then $(t_1, \dots, t_n)$ is a term of arity $a_1 + \cdots + a_n$.
4. If t is a term and f is a function symbol with matching arities, then ft is a term.

Whenever we interpret our language on a carrier A, we mean for terms to be interpreted as functions into A which we can construct by composing existing constants (constant functions), basic functions (i.e. the interpretations of the function symbols), and variables (identity).

Definition 1.5. A **formula** is defined by structural induction as follows:

1. If t_1 and t_2 are terms of the same arity, $t_1 = t_2$ is a formula.
2. If t is a term and R is a relation symbol, and t and R have the same arity, then Rt is a formula.
3. If φ is a formula, $\neg\varphi$ is a formula.
4. If φ and ψ are formulas, then $\varphi \vee \psi$ is a formula.
5. If φ and ψ are formulas, then $\varphi \wedge \psi$ is a formula.
6. If φ is a formula containing a variable v, then $\exists v\varphi$ is a formula.
7. If φ is a formula containing a variable v, then $\forall v\varphi v$ is a formula.

Definition 1.6. Let φ be a formula containing the variables $x_1, \ldots, x_n$. We say that the variable x_k is **free** if x_k is not contained in a subformula of the form $\exists x_k\psi$ or $\forall x_k\psi$.

x_k is **bound** if it is not free.

Definition 1.7. A formula is a **sentence** (or **statement**) if it contains no free variables.

We write $\mathsf{Formulas}(\mathcal{L})$ for all the first-order formulas of $\mathcal{L}$, and we write $\mathsf{Sentences}(\mathcal{L})$ for all the first-order sentences of $\mathcal{L}$.

By convention, we always include sentences called true and false.

1.1.4 Predicate calculus and provability

Throughout this section, we fix a language $\mathcal{L}$.

Definition 1.8. A **propositional function** is a function $f : \mathsf{Prop}^k \to \mathsf{Prop}$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. The constant functions to true and false are propositional functions.
2. Each projection $(P_1, \ldots, P_k) \mapsto P_j$ is a propositional function.
3. If f and g are propositional functions, so are

$$\neg f, f \wedge g, f \vee g, f \to g, \text{ and } f \leftrightarrow g,$$

   where the operations above are carried out pointwise in Prop.

f is a **tautology** if $\vdash \forall \vec{p} : \mathsf{Prop}^k, f\vec{p} \leftrightarrow \mathsf{true}$.

Definition 1.9. A **propositional combination** is a function $f : \mathsf{Sentences}(\mathcal{L})^k \to \mathsf{Sentences}(\mathcal{L})^k$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. Each projection $(B_1, \ldots, B_k) \mapsto B_j$ is a propositional combination.

2. If f and g are propositional combinations, so are

$$\neg f, f \wedge g, f \vee g, f \to g, \text{ and } f \leftrightarrow g,$$

where the operations are carried out pointwise in $\mathsf{Sentences}(\mathcal{L})$.

By sending projections to projections and symbols $(\neg, \wedge, \vee, \to, \leftrightarrow)$ to the corresponding operations on Prop, every propositional combination $f : \mathsf{Sentences}(\mathcal{L})^k \to \mathsf{Sentences}(\mathcal{L})$ can be realized as a propositional function $\mathbf{r}(f) : \mathsf{Prop}^k \to \mathsf{Prop}$.

Definition 1.10. The **predicate calculus** comprises the following rules for deducing sentences from other sentences. We call deducible sentences **valid**, and write $\vdash_{\mathcal{L}} \varphi$ to mean that the $\mathcal{L}$-sentence φ is valid (and to disambiguate from $\vdash$, which when used unadorned means "provable in the metatheory).

(a) (Rule of the propositional calculus) if f is a propositional combination taking k arguments such that $\mathbf{r}(f)$ is a tautology, then for any k sentences $A_1, \ldots, A_k$, the value of the propositional combination $f(\varphi_1, \ldots, \varphi_k)$ is a valid sentence.

(b) (Rule of modus ponens) If A and $A \to B$ are valid, then B is valid.

(c) (Rules of equality)

  (i) $\forall x, x = x$, $\forall x \forall y, x = y \wedge y = x$, and $\forall x \forall y \forall z, x = y \wedge y = z \to x = z$ are all valid.

  (ii) Let $\varphi(x)$ be a formula whose only free variable is x. Then

$$\forall x \forall y, (x = y) \to (\varphi(x) \to \varphi(y))$$

  is valid.

(d) (Change of variable) If A is a sentence and A' represents A with all instances of a variable x switched to y, then $A \leftrightarrow A'$ is valid.

(e) (Rule of specialization "$\forall$-elimination") Let c be any constant symbol, and let $\varphi(x)$ be a formula whose only free variable is x. Then $(\forall x \varphi(x)) \to \varphi(c)$ is valid.

(f) ("$\neg$-introduction") If $\neg A \leftrightarrow (A \to \mathsf{false})$ is valid.

(g) (Generalization of constants "$\forall$-introduction") Let B be a sentence which does not contain the constant c or the variable x. Let $\varphi(x)$ be some formula such that $\varphi(c) \to B$ is valid. Then $\exists x \varphi(x) \to B$ is also valid.[1]

(h) (de Morgan laws) Let $\varphi(x)$ have x as its only free variable. Let B be a sentence which does not contain x. Then the following are valid statements:

$$(\neg(\forall x \varphi(x))) \leftrightarrow (\exists x \neg \varphi(x))$$

$$((\forall x \varphi(x)) \wedge B) \leftrightarrow ((\forall x (\varphi(x) \wedge B)$$

$$((\exists x \varphi(x)) \wedge B) \leftrightarrow ((\exists x (\varphi(x) \wedge B)$$

Definition 1.11. Let S be a collection of sentences.

[1] In particular, using the next rule, if $\neg\varphi(c) \to \mathsf{false}$ is valid, so is $\exists x \neg \varphi(x) \to \mathsf{false}$, so is $\neg \exists x \neg \varphi(x)$, and therefore so is $\forall x \varphi(x)$.

(i) We say that A is provable from S if there exist finitely many $B_1, \ldots, B_n : S$ such that $(B_1 \wedge \cdots \wedge B_n) \to A$ is valid.

(ii) We say that S is consistent if false is not valid.

Remark 1.12. One may wonder why we work with a type of formulas and not with a collection of Props directly. The problem with this is that everything needs to be typed, and so to reason about a predicate (say "$\in$") using Prop, we need some carrier type A such that $\in : A \to A \to \mathsf{Prop}$, so that e.g. $\in$ satisfies the axioms of set theory. But then what does it mean for some other type B to have an interpretation of $\in$ and the axioms it satisfies? There then needs to be a separate predicate $\in_B : B \to B \to \mathsf{Prop}$ satisfying the same *kind* of Props as $\in : A \to A \to \mathsf{Prop}$. We could proceed to define a typeclass of such $(B, \in_B)$, and we would then be working with models of set theory, but we would lack a way to reason syntactically about the axioms themselves.

1.2 Models and satisfiability

For the remainder of this section we fix a language $\mathcal{L}$.

Definition 1.13. An **$\mathcal{L}$-theory** is a collection of sentences from $\mathsf{Sentences}(\mathcal{L})$.

Definition 1.14. An **$\mathcal{L}$-structure** comprises the following data:

(i) A carrier type A,

(ii) an assignment of every $c : \mathsf{Const}(\mathcal{L})$ to a $c^A : A^{\mathrm{arity}(c)}$,

(iii) an assignment of every $R : \mathsf{Rel}(\mathcal{L})$ to a subtype $R^A : A^{\mathrm{arity}(R)} \to \mathsf{Prop}$,

(iv) an assignment of every $f : \mathsf{Funct}(\mathcal{L})$ to a function $f^A : A^{\mathrm{arity}(f)} \to A$.

Definition 1.15. Let A be an $\mathcal{L}$-structure. Using the data of A being an $\mathcal{L}$-structure, we can inductively assign to every term t (of arity k and containing n free variables) a **realization** $\mathbf{r}(t) : A^n \to A^k$, as follows:

1. If $t = v$ for a variable v, $\mathbf{r}(t) = \mathrm{id}_A = \lambda v, v$.
2. If $t = c$ for a constant symbol c, $\mathbf{r}(t) = A^0 \xrightarrow{c^A} A$.
3. If $t = (t_1, \ldots, t_m)$, then $\mathbf{r}(t) = \mathbf{r}(t_1) \times \cdots \times \mathbf{r}(t_m)$.
4. If $t = f(t_0)$ for some function symbol f, then $\mathbf{r}(t) = f^A \circ \mathbf{r}(t_0)$.

Definition 1.16. Let A be an $\mathcal{L}$-structure. Using the data of A being an $\mathcal{L}$-structure, we can inductively assign to every formula $\varphi(x_1, \ldots, x_n)$ (where $x_1, \ldots, x_n$ exhaust the free variables of φ) a **realization** $\mathbf{r}(\varphi) : A^n \to \mathsf{Prop}$, as follows:

1. If φ is of the form $t_1 = t_2$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ (where symbolic equality is realized as true equality).
2. If φ is of the form $R(t)$, $\mathbf{r}(R(t))$ is $R^A(\mathbf{r}(t))$.
3. If φ is of the form $\neg\psi$, then $\mathbf{r}(\varphi)$ is $\neg\mathbf{r}(\psi)$.
4. If φ is of the form $\psi \vee \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \vee \mathbf{r}(\psi')$.

5. If φ is of the form $\psi \wedge \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \wedge \mathbf{r}(\psi')$.
6. If φ is of the form $\exists v\psi$, then $\mathbf{r}(\varphi)$ is $\exists v\mathbf{r}(\varphi)$.
7. If φ is of the form $\forall v\psi$, then $\mathbf{r}(\varphi)$ is $\forall v\mathbf{r}(\varphi)$.

In particular, each sentence φ is sent to a Prop $\mathbf{r}(\varphi)$.

Definition 1.17. Let A be an $\mathcal{L}$-structure, and let φ be a sentence. We say that A **satisfies** φ, written

$$A \models \varphi,$$

if $\vdash \mathbf{r}(\varphi)$.

Definition 1.18. Let T be an $\mathcal{L}$-theory, and let A be an $\mathcal{L}$-structure. We say that A is a **model** of T if for every sentence $\varphi : T$, $A \models \varphi$.

Example 1.19. (Line graph) The **language of graphs** $\mathcal{L}_{\mathsf{Graph}}$ comprises a single 2-ary relation symbol E.

The **theory of graphs** Graph comprises the sentence $\forall x \forall y\, (E(x,y) \leftrightarrow E(y,x))$.

The natural numbers $\mathbb{N}$ can be viewed as a model of Graph as follows. We realize E as the set

$$(y = \operatorname{succ} x) \vee (x = \operatorname{succ} y) : \mathbb{N} \to \mathbb{N} \to \mathsf{Prop}$$

which is clearly symmetric.

Example 1.20. Let $\mathbf{Mod}(\mathsf{Graph})$ be the collection of graphs.[2] A **graph property** is a map $P : \mathbf{Mod}(\mathsf{Graph}) \to \mathsf{Prop}$ such that whenever $G \simeq G'$, $P(G) \leftrightarrow P(G')$. We say that G *satisfies* P if $P(g) \leftrightarrow \mathsf{true}$. A graph property is additionally said to be *monotone* if whenever $G \subseteq G'$ is a subgraph, then $P(G') \to P(G)$.

For example, the property of being a complete graph is not monotone, while the property of being cycle-free is.

The **graph evasiveness conjecture** says that for every monotone graph property P and every $n : \mathbb{N}$, one needs to ask $\binom{n}{2}$ questions of the form "is there an edge between v and w" to determine if an arbitrary graph on n vertices satisfies P.

Example 1.21. (Peano arithmetic)

The **language of Peano arithmetic** $\mathcal{L}_{\mathsf{PA}}$ comprises:

1. A 1-ary constant 0.
2. Three function symbols $\operatorname{succ}, +, \times$.

The **theory of Peano arithmetic** PA comprises:

1. $\forall x, s(x) \neq 0$
2. $\forall x \forall y, (s(x) = s(y)) \to x = y$

[2] Warning: this is "large", so lives in the next universe up: one can interpret a trivial edge relation on *every* type in the current universe.

3. $\forall x, x + 0 = x$
4. $\forall x \forall y, x + s(y) = s(x + y)$
5. $\forall x, x \times 0 = 0$
6. $\forall x \forall y, x \times S(y) = (x \times y) + x$

Schema: For every $\mathcal{L}_{\mathsf{PA}}$-formula $\varphi(x)$ with one free variable x,

$$\big(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\operatorname{succ} x))\big) \to \forall x \varphi(x).$$

The **standard model** of PA is $\mathbb{N}$ with 0 realized as $0 : \mathbb{N}$, succ realized as $\operatorname{succ} : \mathbb{N} \to \mathbb{N}$, $+$ realized as $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, and $\times$ realized as $\times : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$.

By recursing on the inductive type of valid sentences and replacing every rule of the propositional calculus with the corresponding deduction rule for Prop, we can construct for every valid $\mathcal{L}$-sentence φ a proof that $\mathbf{r}(\varphi) \leftrightarrow \mathsf{true}$.

That is the soundness theorem. (In what follows, taking ψ to be true yields the assertion in the previous paragraph.)

Theorem 1.22. *(Soundness theorem) For every $\mathcal{L}$-structure and any sentences $\varphi, \psi : \mathsf{Sentences}(\mathcal{L})$,*

$$\vdash_{\mathcal{L}} \varphi \to \psi \quad \Longrightarrow \quad \vdash \mathbf{r}(\varphi) \to \mathbf{r}(\psi).$$

This happens regardless of which $\mathcal{L}$-structure is doing the realizing. When the $\mathcal{L}$-structure itself is a model of a theory T, then whenever $T \vdash_{\mathcal{L}} \psi$, then since there is some sentence $\varphi : T$ such that $\vdash_{\mathcal{L}} \varphi \to \psi$, Prop's modus ponens tells us that the model satisfies ψ also.

Example 1.23. For example, suppose we're working in the language of graphs expanded with two 1-ary constants a and b, and we know that there is some model M such that M satisfies the sole axiom that E is symmetric. We can show

$$\vdash (\forall x \forall y, \mathbf{r}(E)(x, y) \leftrightarrow \mathbf{r}(E)(y, x)) \to \mathbf{r}(E)(\mathbf{r}(a), \mathbf{r}(b)) \leftrightarrow \mathbf{r}(E)(\mathbf{r}(b), \mathbf{r}(a))$$

because we already know the antecedent and can apply Prop's $\forall$-elimination.

The converse of Theorem 1.22 is false. There may be some things which are incidentally true about the model which are not universally valid.

Example 1.24. Working again in the language of graphs, consider a complete graph on n vertices. Call this model M. M happens to satisfy the $\mathcal{L}$-sentence

$$\big(\forall x \forall y, E(x, y) \leftrightarrow E(y, x)\big) \to \big(\forall x \forall y \forall z, E(x, y) \wedge E(y, z) \to E(x, z)\big),$$

but this is not a valid $\mathcal{L}$-sentence. (Indeed, if it were, then the soundness theorem would imply that *every* graph has a transitive edge relation, which is not true.)

It will turn out that we can do the next best thing. If we rule out this kind of exception by requiring that $M \models \varphi$ *for every $\mathcal{L}$-structure M* (resp. every model M of T), then it follows that $\vdash_{\mathcal{L}} \varphi$ (resp. $T \vdash_{\mathcal{L}} \varphi$). This is the completeness theorem.

1.3 The completeness theorem

In this section, our goal will be to prove the **completeness theorem**:

Theorem 1.25. *Let T be an $\mathcal{L}$-theory. T is consistent if and only if there exists a model of T.*

First we will prove that if there exists a model M of T, then T is consistent.

Proof. We will show the contrapositive: if T is inconsistent, then there does not exist a model M of T.

Indeed, suppose that T is inconsistent. Suppose there is a model M. Then by the soundness theorem, $M \models \mathsf{false}$. By definition, this means that

$$\vdash \mathsf{false},$$

so we have shown that

$$\vdash (T \text{ inconsistent}) \wedge (\text{there exists a model } M \text{ of } T) \to \mathsf{false}$$

which is equivalent to

$$\vdash (T \text{ not inconsistent}) \vee (T \text{ does not have a model}),$$

which is equivalent to

$$\vdash T \text{ inconsistent } \to T \text{ does not have a model.}$$

Taking the contrapositive, we conclude that if T has a model, then T is consistent. $\square$

It then remains to show that if T is consistent, T has a model. We will use the Henkin construction.

1.3.1 The Henkin construction

Definition 1.26. Let T be an $\mathcal{L}$-theory. We say that T is a **Henkin theory** if, for every formula $\varphi(x)$, there is a constant $c : \mathsf{Const}(\mathcal{L})$ such that $T \vdash_{\mathcal{L}} (\exists x \varphi(x)) \to \varphi(c)$.

Example 1.27. Let $\mathcal{L}_{\text{field}}$ be the language of fields, which we define to be $\{0, 1, +, \times, (-)^{-1}\}$ (the usual language of rings augmented with an inversion operation), and let T be the usual axiomatization of a field of characteristic zero. T is not a Henkin theory, for there is no constant c such that e.g. $c = (1+1)^{-1}$.

Example 1.28. Let $\mathcal{L}_{\mathsf{PA}}$ be the language of Peano arithmetic (see Example 1.21). Let T be the collection of all $\mathcal{L}_{\mathsf{PA}}$-sentences ψ such that $N \models \psi$. Then T certainly contains the sentence $\exists x \forall y, x \cdot y = y$. However, $1 = \text{succ } 0$ is not a constant in the language, but rather a term. So T is not a Henkin theory.

However, if we *expand* $\mathcal{L}_{\mathsf{PA}}$ to a language $\mathcal{L}'$ with a constant symbol c_n for every natural number n, and if we let T' be the collection all $\mathcal{L}'$-sentences ψ such that $\mathbb{N}$ (viewed in the natural way as a model of $\mathcal{L}'$) satisfies ψ, then T' *is* a Henkin theory.

Proposition 1.29. *Let T be an $\mathcal{L}$-theory. If T is consistent, then there exists a language $\mathcal{L}'$ extending L and an $\mathcal{L}'$-theory T' extending T viewed as an $\mathcal{L}$'-theory, such that T' is a Henkin theory.*

Furthermore, if T is consistent, then T' is consistent.

Proof. Put $\mathcal{L}_0 \stackrel{\mathrm{df}}{=} \mathcal{L}$ and $T_0 \stackrel{\mathrm{df}}{=} T$. We define a chain of languages $\mathcal{L}_i$ and for each i we define an $\mathcal{L}_i$-theory T_i as follows: given $\mathcal{L}_n$ and $\mathcal{T}_n$, let $\mathcal{L}_{n+1}$ be the language obtained by adding a constant $c_{\varphi,x}$ where φ ranges over all $\mathcal{L}_n$-formulas and x ranges over the free variables of φ.

Having defined $\mathcal{L}_{n+1}$, we now define T_{n+1} to be

$$T_n \cup \{\exists x\varphi(x) \to \varphi(c_{\varphi,x})\}_{\varphi,x}$$

where above we have adjoined a sentence saying that the newly-adjoined constant $c_{\varphi,x}$ behaves as expected.

We put

$$T' \stackrel{\mathrm{df}}{=} \bigcup_{n\,:\,\mathbb{N}} T_n.$$

By construction, T' is a Henkin theory.

It remains to show that if T is consistent, so is T'. If $T \vdash_{\mathcal{L}} \psi$, then from the finiteness of proofs, we must have that $T_n \vdash_{\mathcal{L}} \psi$ for some n. So, to show T' is consistent, it suffices to show that for each n, T_n is consistent.

We induct on n. The base case $T = T_0$ is by assumption. For the induction step, we must show that if T_n is consistent, then T_{n+1} is consistent.

Suppose towards the contrapositive that T_{n+1} is inconsistent. Since T_{n+1} is obtained by adjoining formulas of the form $\exists x\varphi(x) \to \varphi(c)$, there must be finitely many such formulas $\psi_1, \ldots, \psi_m : T_{n+1} \backslash T_n$ of this form, along with finitely many formulas $\rho_1, \ldots, \rho_n$ from T_n, such that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_m \to \mathsf{false}.$$

By material implication, we get that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \neg\psi_m,$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \neg(\exists x\varphi_m(x) \to \varphi_m(c_m)),$$

which is equivalent to

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to (\exists x\varphi_m(x)) \wedge \neg\varphi_m(c_m)),$$

and since c_m does not occur in the premise of the implication, we have that

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to (\exists x\varphi_m(x)) \wedge \forall x\neg\varphi_m(x))$$

and therefore

$$\vdash_{\mathcal{L}_{n+1}} \rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1} \to \mathsf{false}.$$

We conclude that

$$\rho_1 \wedge \cdots \wedge \rho_n \wedge \psi_1 \wedge \cdots \wedge \psi_{m-1}$$

is inconsistent. Repeating this argument, we eliminate all the ψ_i and conclude that $\rho_1 \wedge \cdots \wedge \rho_n$ is inconsistent, and therefore that T is inconsistent. □

Definition 1.30. To any Henkin $\mathcal{L}$-theory T, we can associate a canonical structure (a "term model") term(T) built from the closed terms (i.e. those not containing any variables).

First, we take the collection A of all closed $\mathcal{L}$-terms. We define a relation $E : A \to A \to \mathsf{Prop}$, with the convention that $\vdash E\ a_1\ a_2 \leftrightarrow \mathsf{true}$ if and only if $T \vdash_{\mathcal{L}} a_1 = a_2$. By the rules about equality that we have stipulated as part of the predicate calculus, E is an equivalence relation.

We put $\widetilde{A} \stackrel{\text{df}}{=} A/E$. This will be the underlying type of the model.

For a constant $c : \mathsf{Const}(\mathcal{L})$, we put $c^{\widetilde{A}} \stackrel{\text{df}}{=} c/E$ (c belongs to $\mathcal{L}_0$, and so is a closed term of $\mathcal{L}'$).

For a relation symbol $R : \mathsf{Rel}(\mathcal{L})$, we define $R^{\widetilde{A}} : \widetilde{A}^{\text{arity}(R)} \to \mathsf{Prop}$ by $R^{\widetilde{A}}(a_1/E, \ldots, a_n/E) \leftrightarrow T' \vdash_{\mathcal{L}} R(a_1, \ldots, a_n)$.

For a function symbol $f : \mathsf{Funct}(\mathcal{L})$, we define $f^{\widetilde{A}} : \widetilde{A}^{\text{arity}(f)} \to \widetilde{A}$ by

$$\lambda a_1/E \quad \ldots \quad a_n/E, f(a_1, \ldots, a_n)/E.$$

This completes the definition of term(T).

By the soundness theorem, if T is inconsistent, then term(T) cannot be a model of T. But, under suitable assumptions, the inverse is true.

Definition 1.31. An $\mathcal{L}$-theory T is **complete** if for every $\mathcal{L}$-sentence ψ,

$$\vdash (T \vdash_{\mathcal{L}} \psi) \vee (T \vdash_{\mathcal{L}} \neg\psi)\,.$$

Remark 1.32. Excluded middle in Prop implies that for any $\mathcal{L}$-structure M and every $\mathcal{L}$-sentence ψ,

$$\vdash (M \models \psi) \vee (M \models \neg\psi),$$

and therefore that the $\mathcal{L}$-theory of an $\mathcal{L}$-structure (i.e. the collection of all sentences true in the structure) is complete.

By invoking the axiom of choice, we can extend any consistent theory to a complete consistent theory. We will prove this.

Proposition 1.33. *Let T be a consistent $\mathcal{L}$-theory.uuuu There exists a complete, consistent $\mathcal{L}$-theory T' which contains T.*

To prove this, we will use Zorn's lemma. To start the argument, we prove the following lemma.

Lemma 1.34. *Suppose T is consistent. Let φ be an $\mathcal{L}$-sentence. Then $T \cup \{\varphi\}$ is consistent or $T \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. Then there exist sentences σ and ρ from T such that

$$\vdash_{\mathcal{L}} (\sigma \wedge \varphi) \to \mathsf{false} \quad \text{and} \quad \vdash_{\mathcal{L}} (\rho \wedge \neg\varphi) \to \mathsf{false}.$$

By $\neg$-introduction, we get

$$\vdash_{\mathcal{L}} \neg\,(\sigma \wedge \varphi) \quad \text{and} \quad \vdash_{\mathcal{L}} \neg\,(\rho \wedge \neg\varphi)$$

and by $\wedge$-introduction, we get

$$\vdash_{\mathcal{L}} (\neg(\sigma \wedge \varphi)) \wedge (\neg(\rho \wedge \neg\varphi))\,.$$

Since the finitary de Morgan laws are tautologies in the sense of 1.8, it follows that

$$\vdash_{\mathcal{L}} \neg\,(\sigma \vee \varphi \vee \rho \vee \neg\varphi)\,.$$

Since the metatheory satisfies the law of the excluded middle, we have that the law of the excluded middle for $\mathcal{L}$-formulas is a tautology in the sense of 1.8. Therefore,

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \rho) \Leftrightarrow \vdash_{\mathcal{L}} \neg\sigma \wedge \neg\rho,$$

so by $\wedge$-elimination, $\vdash_{\mathcal{L}} \neg\sigma$ and $\vdash_{\mathcal{L}} \neg\rho$, so T is inconsistent. □

Proof of 1.33. Consider the poset of proper consistent extensions of T. If T is not complete, then the previous lemma shows that this poset is nonempty.

Now we show that we can take the union of a chain in this poset and obtain an upper bound on that chain.

Indeed, let $(T_i)_{i \in I}$ be a chain in this poset, and let T_∞ be its union. This is clearly a theory which contains all the theories in the chain (and also T). We need to show that it is consistent. Indeed, if it were inconsistent, then by the finiteness of proofs, there exists some T_n such that $T_n \vdash_{\mathcal{L}} \mathsf{false}$.

This now fits the hypotheses of Zorn's lemma, which gives us a maximal consistent extension of T' of T. If T' were not complete, then the previous lemma shows that we can extend it. □

Theorem 1.35. *Let T be a complete Henkin $\mathcal{L}$-theory. If T is consistent, then* $\mathrm{term}(T)$ *is a model of T.*

Proof. We will show that for every $\psi : \mathsf{Sentences}(\mathcal{L})$,

$$T \vdash_{\mathcal{L}} \psi \iff \mathrm{term}(T) \models \psi.$$

We will do this by a structural induction on formulas. In the base case, we have atomic sentences.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $a_1 = a_2$ where a_1 and a_2 are closed terms, then since $T \vdash_{\mathcal{L}} a_1 = a_2$, then $\vdash a_1^{\widetilde{A}} = a_2^{\widetilde{A}}$ (in $\widetilde{A}$), so $\mathrm{term}(T) \models \psi$.

  Conversely, if $\mathrm{term}(T) \models \psi$, then $\vdash a_1^{\widetilde{A}} = a_2^{\widetilde{A}}$, so by definition of the equivalence relation we used to define $\widetilde{A}$, $T \vdash_{\mathcal{L}} a_1 = a_2$.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $R(a_1, \dots, a_n)$ where R is a relation symbol and $a_1, \dots, a_n$ are closed terms, then since $T \vdash_{\mathcal{L}} R(a_1, \dots, a_n)$, we have that $\vdash R^{\widetilde{A}}(a_1^{\widetilde{A}}, \dots, a_n^{\widetilde{A}})$.

  Conversely, if $\mathrm{term}(T) \models \psi$, then $\vdash R^{\widetilde{A}}(a_1^{\widetilde{A}}, \dots, a_n^{\widetilde{A}})$, so by definition of how we interpreted $\mathcal{L}$ onto $\widetilde{A}$, $T \vdash_{\mathcal{L}} R(a_1, \dots, a_n)$.

- If $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $\varphi_1 \wedge \varphi_2$, then by $\wedge$-elimination in $\mathsf{Sentences}(\mathcal{L})$,

$$\vdash (T \vdash_{\mathcal{L}} \psi) \to (T \vdash_{\mathcal{L}} \varphi_1) \wedge (T \vdash_{\mathcal{L}} \varphi_2).$$

  By the induction hypothesis, $\mathrm{term}(T) \models \varphi_1$ and $\mathrm{term}(T) \models \varphi_2$, so by $\wedge$-introduction in Prop, $\mathrm{term}(T) \models \varphi_1 \wedge \varphi_2$.

  Conversely, if $\mathrm{term}(T) \models \varphi_1 \wedge \varphi_2$, then by $\wedge$-elimination in Prop, $\mathrm{term}(T) \models \varphi_1$ and $\mathrm{term}(T) \models \varphi_2$. By the induction hypothesis, $T \vdash_{\mathcal{L}} \varphi_1$ and $T \vdash_{\mathcal{L}} \varphi_2$, so by $\wedge$-introduction in $\mathsf{Sentences}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi_1 \wedge \varphi_2$.

- Suppose $T \vdash_{\mathcal{L}} \psi$ and ψ is of the form $\neg\varphi$. The induction hypothesis says that $T \vdash_{\mathcal{L}} \varphi$ if and only if $\mathrm{term}(T) \models \varphi$. Since T is consistent, $T \ \vdash_{\mathcal{L}} \varphi$. Therefore, by the induction hypothesis, $\mathrm{term}(T) \not\models \varphi$. By the law of the excluded middle, $\mathrm{term}(T) \models \neg\varphi$.

  Conversely, suppose that $\mathrm{term}(T) \models \neg\varphi$. Then $\mathrm{term}(T) \not\models \varphi$, so by the induction hypothesis, T does not prove φ. Since T was complete, $T \vdash_{\mathcal{L}} \varphi$.

  We omit the cases for $\wedge$ and $\to$, which are entirely analogous.

  We conclude that whenever ψ is quantifier-free, $T \vdash_{\mathcal{L}} \psi$ if and only if $\mathrm{term}(T) \models \psi$.

  To complete the proof, we must take care of quantifiers.

- Suppose that $T \vdash_{\mathcal{L}} \exists x \varphi(x)$, where $\varphi(x)$ satisfies the induction hypothesis that if we substitute a closed term c for x, $\varphi(c)$ is a sentence such that $T \vdash_{\mathcal{L}} \varphi(c)$ if and only if $\mathrm{term}(T) \models \varphi(c)$.

  Then, since T is a Henkin theory, there exists some c such that

$$T \vdash_{\mathcal{L}} \varphi(c).$$

  By the induction hypothesis, we have that

$$\mathrm{term}(T) \models \varphi(c),$$

  and therefore by $\exists$-introduction in Prop, we conclude that

$$\mathrm{term}(T) \models \exists x \varphi(x).$$

  Conversely, suppose that $\mathrm{term}(T) \models \exists x \varphi(x)$. By $\exists$-elimination in Prop, there exists some $a/E : \widetilde{A}$ such that $\vdash \mathbf{r}(\varphi)(a/E)$, which is equivalent to $\mathrm{term}(T) \models \varphi(a)$. By the induction hypothesis, $T \vdash_{\mathcal{L}} \varphi(a)$, and by $\exists$-introduction in $\mathsf{Sentences}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \exists x \varphi(x)$.

- Similarly, suppose that $T \vdash_{\mathcal{L}} \forall x \varphi(x)$, where $\varphi(x)$ satisfies the induction hypothesis that if we substitute a closed term c for x, $\varphi(c)$ is a sentence such that $T \vdash_{\mathcal{L}} \varphi(c)$ if and only if $\mathrm{term}(T) \models \varphi(c)$.

  Then by $\forall$-elimination in $\mathsf{Sentences}(\mathcal{L})$, we have that for every constant $c : \mathsf{Const}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi(c)$. By the induction hypothesis, $\mathrm{term}(T) \models \varphi(c)$. Since the interpretations of c exhaust $\mathrm{term}(T)$, we conclude by $\forall$-introduction in Prop that $\mathrm{term}(T) \models \forall x \varphi(x)$.

Conversely, suppose that $\mathrm{term}(T) \models \forall x \varphi(x)$. By $\forall$-elimination in Prop, for every $a/E \in \widetilde{A}$, $\vdash \mathbf{r}(\varphi)(a/E)$, which is equivalent to $\mathrm{term}(T) \models \varphi(a)$. By the induction hypothesis, for every $c : \mathsf{Const}(\mathcal{L})$, $T \vdash_{\mathcal{L}} \varphi(c)$.

Suppose towards a contradiction that T does not prove $\forall x \varphi(x)$. Since T was complete, T proves $\exists x \neg \varphi(x)$. Since we have already proved the cases for $\exists and \neg$, we conclude that $\mathrm{term}(T) \models \exists x \neg \varphi(x)$, and by the axiom of choice we can find a witness $c \in \mathrm{term}(T)$ such that $\mathrm{term}(T) \models \neg \varphi(c)$. This contradicts the conclusion of the previous paragraph.

□

Corollary 1.36. *Let T be a consistent $\mathcal{L}$-theory. Then T has a model.*

Proof. By 1.29, extend T to a Henkin theory T'. By 1.33, extend T' to a complete theory T''.

T'' is again Henkin: for any formula $\varphi(x)$, there already exists a c such that $T' \vdash_{L'} \exists x \varphi(x) \leftrightarrow \varphi(x)$, and T'' contains all the sentences of T'.

By 1.35, $\mathrm{term}(T'')$ is a model of T''. Since T'' contains T, $\mathrm{term}(T'')$ is also a model of T. □

This completes the proof of the completeness theorem.

1.4 The Löwenheim-Skolem theorem

TODO

2 ZFC

The language $\mathcal{L}_{\mathsf{ZFC}}$ of set theory comprises just one 2-ary relation $\in$. Now we give the definition of the $\mathcal{L}_{\mathsf{ZFC}}$-theory ZFC.

Definition 2.1. ZFC is defined to be the collection of following axioms and axiom schemas:

Extensionality

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y).$$

This says that every set is determined by its elements.

Notation 2.2. In what follows, we write "$x \subseteq y$" to abbreviate the formal statement $\forall z \in x, z \in y$.

Empty set

$$\exists x \forall y (\neg y \in x).$$

Viewing a model of ZFC as a directed tree, this says that every model has a least ("root") element.

Pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

This axiom says that we can form unordered pairs.

Notation 2.3. We denote z as above by $\{x, y\}$, adopt the convention that $\{x\} \stackrel{\text{df}}{=} \{x, x\}$, and we implement ordered pairs with *Kuratowski ordered pairs*, viz. $(x, y) \stackrel{\text{df}}{=} \{\{x\}, \{x, y\}\}$.

Now that we have defined ordered pairs, we can define functions (internal to ZFC):

Definition 2.4. A **function** is a set f of ordered pairs such that $(x, y) \wedge (x, z) \to y = z$.

Union

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists t (z \in t \wedge t \in x)).$$

This says that the y above is the union of all the members of x. Applying 2.1, we conclude that given sets x and y, there exists z such that $z = x \cup y$.

Definition 2.5. Let x be a set. We denote the **successor** of x to be the set $\succ x \stackrel{\text{df}}{=} x \cup \{x\}$.

Infinity

$$\exists x \left(\varnothing \in x \wedge \forall y (y \in x \to \succ y \in x) \right).$$

Replacement Let $\varphi(x, y, t_1, \ldots, t_k)$ be an $\mathcal{L}_{\mathsf{ZFC}}$-formula with at least two free variables. For each such formula,

$$\forall t_1 \ldots \forall t_k (\forall x \exists! y \varphi(x, y, t_1, \ldots, t_k) \to \forall u \exists v \forall r (r \in v \leftrightarrow \exists s (s \in u \wedge \varphi(s, r, t_1, \ldots, t_k)))).$$

is an axiom of ZFC.

This axiom says that if for fixed terms $t_1, \ldots, t_k$, $\varphi(x, y, t_1, \ldots, t_k)$ is the graph of a function sending x to y, then for each set u, the image of u under this function is again a set.

Note that the quantifiers above can range over the entire model of ZFC.

Powerset

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x).$$

Choice Let $y : x \to z$ abbreviate the $\mathcal{L}_{\mathsf{ZFC}}$-formula which says that y is a function from x to z.

$$\forall y \forall y \forall z, y : x \to z \wedge (y \neq \varnothing) \to \left(\exists f (f : x \to (\bigcup z) \wedge \forall a \in x, f(a) \in y(x)) \right).$$

More clearly, this says that for every x-indexed family of sets z, there exists a section to the projection $\bigcup z \twoheadrightarrow x$.

Regularity

$$\forall x \exists y (x = \varnothing \vee (y \in x \wedge \forall z (z \in x \to \neg z \in y)))$$

This asserts that every set contains an element which is minimal with respect to $\in$.

2.1 Ordinal numbers

Definition 2.6. We say that y is a (binary) **relation** on x if y is a set of ordered pairs from X.

Definition 2.7. We say that a relation $<$ on x is a **well-ordering** if:

1. $$\forall a \forall b, a = b \vee a < b \vee b < a.$$

2. $$\forall a \forall b \forall c, a < b \wedge b < c \to a < c.$$

3. $$\forall s \subseteq x, s \neq \varnothing \to \exists a (a \in x \wedge \forall b (b \in s \to \neg a < b)).$$

Definition 2.8. A set x is called **transitive** if $y \in x, z \in y \to z \in x$.

Definition 2.9. We say that a set α is an **ordinal** if it is well-ordered by the membership relation $\in$ and it is transitive. We abbreviate this assertion by $\operatorname{Ord} \alpha$.

2.2 Cardinal numbers

Definition 2.10. A set x is a **cardinal number** if it is an ordinal number satisfying the following extra property: for every $y \in x$, there exists no bijection between y and x.

3 Boolean-valued models

3.1 Boolean algebras

Definition 3.1. A **preorder** B is a type B equipped with relations $\leqslant$ and $<$ satisfying the following properties:

1. $\forall a : B, a \leqslant a$
2. $\forall a, b, c : B, a \leqslant b \to b \leqslant c \to a \leqslant c$
3. $\forall a, b : B, a \leqslant b \wedge \neg b \leqslant a$
4. $\forall a, b : B, a < b \leftrightarrow (a \leqslant b \wedge \neg b \leqslant a)$

Definition 3.2. A **partial order** B is a preorder such that the $\leqslant$ relation is antisymmetric:

$$\forall a, b : B, a \leqslant b \to b \leqslant a \to a = b.$$

Definition 3.3. A **join-semilattice** B is a partial order with binary sup operation $\sqcup$ which satisfies the following properties:

1. $\forall a, b : B, a \leqslant a \sqcup b$
2. $\forall a, b : B, b \leqslant a \sqcup b$
3. $\forall a, b, c : B, a \ leqc \to b \leqslant c \to a \sqcup b \leqslant c.$

Definition 3.4. A **meet-semilattice** B is a partial order with a binary infimum operation $\sqcap$ which satisfies the following properties:

1. $\forall a, b : B, a \sqcap b \leqslant a$
2. $\forall a, b : B, a \sqcap b \leqslant b$
3. $\forall a, b, c : B, a \leqslant b \to a \leqslant c \to a \leqslant b \sqcap c.$

Definition 3.5. A **lattice** B is a join-semilattice which is also a meet-semilattice.

Definition 3.6. A **distributive lattice** B is a lattice which satisfies the following property:[3]

$$\forall x, y, z : B, (x \sqcup y) \sqcap (x \sqcup z) \leqslant x \sqcup (y \sqcap z).$$

Definition 3.7. A lattice B has a **bottom element** $\bot$ if for every $a : B, \bot \leqslant a$,

Definition 3.8. A lattice B has a **top element** $\top$ if for every $a : B, a \leqslant \top$.

Definition 3.9. A **bounded lattice** is a lattice with a top and bottom element.

Definition 3.10. A **bounded distributive lattice** is a distributive lattice which is bounded.

[3] From the `mathlib` docstring: a distributive lattice can be defined to satisfy any of four equivalent distribution properties (of sup over inf or inf over sup, on the left or right). A classic example of a distributive lattice is the lattice of subsets of a set, and in fact this example is generic in the sense that every distributive lattice is realizable as a sublattice of a powerset lattice.

Definition 3.11. A **Boolean algebra** B is a bounded distributive lattice such that:

1. For every $a : B$, there exists an element $\neg a : B$ which satisfies the following properties:
   (a) $\forall x : B, x \sqcap \neg x = \bot$
   (b) $\forall x : B, x \sqcup \neg x = \top$
2. We additionally specify a **complementation operator** $\lambda x, y, x - y : B \to B$ which satisfies the property:
$$\forall x, y : B, x - y = x \sqcap \neg y.$$

Definition 3.12. For convenience, we accumulate the previous definitions into a complete axiomatization of a boolean algebra. A boolean algebra is a type B with a specification of binary ordering relations $\leqslant$, $<$, a binary sup operation $\sqcup$, a binary inf operation $\sqcap$, top and bottom elements $\top$, $\bot$, a unary negation operator $\neg$, and a binary subtraction operator $-$, satisfying the following properties:

1. $\forall (a : B), a \leqslant a$
2. $(ab, c_1 : B), a \leqslant b \to b \leqslant c_1 \to a \leqslant c_1$
3. $(\forall (a, b : B), a < b \iff a \leqslant b \wedge \neg b \leqslant a)$
4. $(a, b : B), a \leqslant b \to b \leqslant a \to a = b$
5. $\forall (a, b : B), a \leqslant a \sqcup b$
6. $\forall (a, b : B), b \leqslant a \sqcup b$
7. $\forall (a, b, c_1 : B), a \leqslant c_1 \to b \leqslant c_1 \to a \sqcup b \leqslant c_1$
8. $\forall (a, b : B), a \sqcap b \leqslant a$
9. $\forall (a, b : B), a \sqcap b \leqslant b$
10. $\forall (a, b, c_1 : B), a \leqslant b \to a \leqslant c_1 \to a \leqslant b \sqcap c_1$
11. $\forall (x, y, z : B), (x \sqcup y) \sqcap (x \sqcup z) \leqslant x \sqcup y \sqcap z$
12. $\forall (a : B), a \leqslant \top$
13. $\forall (a : B), \bot \leqslant a$
14. $\forall (x : B), x \sqcap -x = \bot$
15. $\forall (x : B), x \sqcup -x = \top$
16. $\forall (x, y : B), x - y = x \sqcap -y$

Definition 3.13. A **complete lattice** B is a bounded lattice which has operations $\operatorname{Sup}, \operatorname{Inf} : \operatorname{set} B \to B$,

Definition 3.14. A **complete distributive lattice** B is a complete lattice which additionally satisfies the following properties:

1. $\forall a : B, s : \operatorname{set} B, (\bigsqcap_{b \in s}, a \sqcup b \leqslant a \sqcup \operatorname{Inf} s)$
2. $\forall a : B, s : \operatorname{set} B, a \sqcap \operatorname{Sup} s \leqslant (\bigsqcup_{b \in s}, a \sqcap b)$.

Definition 3.15. A **complete boolean algebra** B is a boolean algebra which is also a complete distributive lattice.

3.2 Boolean-valued models

Fix $\mathcal{L}$ a first-order language and T an $\mathcal{L}$-theory. Fix $\mathbb{B}$ a boolean algebra.

Definition 3.16. A **$\mathbb{B}$-valued $\mathcal{L}$-structure** is the following data:

(i) A carrier type A,

(ii) an assignment of every $c : \mathsf{Const}(\mathcal{L})$ to a $c^A : A^{\text{arity}(c)}$,

(iii) an assignment of every $R : \mathsf{Rel}(\mathcal{L})$ to a $\mathbb{B}$-valued map $R^A : A^{\text{arity}(R)} \to \mathbb{B}$; in particular an assignment of a binary $\mathbb{B}$-valued map for the equality symbol, and

(iv) an assignment of every $f : \mathsf{Funct}(\mathcal{L})$ to a function $f^A : A^{\text{arity}(f)} \to A$.

If $\phi(\vec{x})$ is a formula, we write $[[\phi(\vec{x})]]^A$ to mean $\phi(\vec{x})$ viewed as a $\mathbb{B}$-valued function (taking as many arguments as it has free variables). The previous data must satisfy the following properties:

1. For every $a : A$, $[[a = a]]^A = 1$.
2. For every $a, b : A$, $[[a = b]]^A = [[b = a]]^A$.
3. For every $a, b, c : A$, $[[a = b]]^A \sqcap [[b = c]]^A \leqslant [[a = c]]^A$.
4. For every n-ary function symbol R, and for all n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$,

$$\left(\prod_{i=1}^{n} [[a_i = b_i]]^A\right) \sqcap [[R(a_1, \ldots, a_n)]]^A \leqslant [[R(b_1, \ldots, b_n)]]^A$$

5. For every n-ary function symbol f, for every $a, b : A$ and every $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ in A^n, the followng three properties hold:

$$\left(\prod_{i=1}^{n} [[a_i = b_i]]^A\right) \sqcap [[f(a_1, \ldots, a_N) = a]]^A \leqslant [[f(b_1, \ldots, b_n) = b]]^A,$$

$$\bigsqcup_{a:A} [[f(a_1, \ldots, a_n) = a]]^A, \text{ and}$$

$$[[f(a_1, \ldots, a_n) = a]]^A \sqcap [[f(a_1, \ldots, a_n) = b]]^A \leqslant [[a = b]]^A.$$

3.3 An example of a Boolean-valued model (experimental)

A preset is a type-indexed family of presets.

This means that there is an associated indicator function on the "disjoint union" of all types, and it is 1 if and only if its argument is inside the indexing type.

Let $\mathbb{B}$ be a complete Boolean algebra. We construct a $\mathbb{B}$-valued model analogous to the standard model $\mathbb{W}$ as follows:

A **$\mathbb{B}$-valued preset** is an α-indexed family of $\mathbb{B}$-valued presets $\times$ $\mathbb{B}$ (i.e. a type-indexed family of presets with boolean truth values for membership).

Such objects are called $\mathbb{B}$-names. A $\mathbb{B}$-name u is specified by the following data:

1. An indexing type α.
2. An indexing function $A : \alpha \to \mathrm{bSet}\,\mathbb{B}$.
3. A truth-value function $B : \alpha \to \mathbb{B}$.

4 The maximum principle

Definition 4.1. Let $\mathbb{B}$ be a bounded lattice. An **antichain** in $\mathbb{B}$ is a subset $A \subseteq \mathbb{B}$ satisfying the following property:
$$\forall x \in S, \forall y \in S, (x \neq y \to x \sqcap y = \bot)\,.$$

Definition 4.2. Let $\mathbb{B}$ be a bounded lattice. An **indexed antichain** is the data of a type I and a map $\mathsf{A} : I \to \mathbb{B}$ satisfying the following property:
$$\forall i, j : I, (i \neq j \to \mathsf{A}(i) \sqcap \mathsf{A}(j) = \bot)\,.$$

Definition 4.3. Let I be a type, and let $\mathsf{A} : I \to \mathbb{B}$ be an indexed antichain. Let $u : I \to V^{\mathbb{B}}$ be an I-indexed family of $\mathbb{B}$-valued sets. The **mixture** of u with respect to A is the following $\mathbb{B}$-valued set, which we will specify by the data $\langle \alpha_\mu, A_\mu, B_\mu \rangle$:

- α_μ is defined to be the Σ-type
$$\alpha_\mu \overset{\text{df}}{=} \Sigma_{(i:I)} \alpha_{u(i)}.$$
- A_μ is defined to be the map
$$\left(A_\mu : \alpha_\mu \to V^{\mathbb{B}}\right) \overset{\text{df}}{=} \lambda \langle i, x \rangle, A_{u(i)}(x).$$
- B_μ is defined to be the map
$$(B_\mu : \alpha_\mu \to \mathbb{B}) \overset{\text{df}}{=} \lambda \langle i, x \rangle, \bigsqcup (j : \iota), \mathsf{A}(j) \sqcap (A_{u(i)} x) \in^{\mathbb{B}} u(j).$$

We focus on a special case of mixtures when I is a type with only two elements.

Definition 4.4. Let $a_1, a_2 : \mathbb{B}$, and let $u_1, u_2 \in V^{\mathbb{B}}$. The **two-term mixture** of u_1 and u_2 is the following $\mathbb{B}$-valued set, given by the data $\langle \alpha_\mu, A_\mu, B_\mu \rangle$:

- $\alpha_\mu \overset{\text{df}}{=} \alpha_{u_1} \oplus \alpha_{u_2}$.
- $\left(A_\mu : \alpha_\mu \to V^{\mathbb{B}}\right) \overset{\text{df}}{=} \lambda a, A_{u_1} a$ (if $a : \alpha_{u_1}$), and $\lambda a, A_{u_2} a$ otherwise.
- $(B_\mu : \alpha_\mu \to \mathbb{B}) \overset{\text{df}}{=} \lambda (a : \alpha_{u_1} \oplus \alpha_{u_2}), \left(a_1 \sqcap (A_\mu(a) \in^{\mathbb{B}} u_1)\right) \sqcup \left(a_2 \sqcap (A_\mu(a) \in^{\mathbb{B}} u_2\right)$

We write $a_1 \cdot u_1 + a_2 \cdot u_2$ for the two-term mixture of u_1 and u_2 with respect to a_1 and a_2.

4.0.1 Corollaries of the maximum principle

Lemma 4.5. *Let X be a $\mathbb{B}$-valued set, and let u_1 and u_2 be $\mathbb{B}$-valued sets such that $u_1 \in^{\mathbb{B}} X = \top$ and $u_2 \in^{\mathbb{B}} X = \top$. Let a_1 and a_2 be an antichain such that $a_1 \sqcup a_2 = \top$.*

Then $a_1 \cdot u_1 + a_2 \cdot u_2 \in^{\mathbb{B}} X = \top$.

Proof. Let $U = a_1 \cdot u_1 + a_2 \cdot u_2$. We must show that

$$\top \leqslant U \in^{\mathbb{B}} X.$$

By the mixing lemma, we have that $a_1 \leqslant U =^{\mathbb{B}} u_1$ and that $a_2 \leqslant U =^{\mathbb{B}} u_2$. Therefore, $a_1 \sqcup a_2 = \top \leqslant U =^{\mathbb{B}} u_1 \sqcup U =^{\mathbb{B}} u_2$.

Conjuncting this with the facts that $T \leqslant u_1 \in^{\mathbb{B}} X$ and $T \leqslant u_2 \in^{\mathbb{B}} X$ and then eliminating the previous Boolean-valued disjunction, we see that it suffices to prove that

$$(U =^{\mathbb{B}} u_1) \sqcap (u_1 \in^{\mathbb{B}} X) \leqslant U \in^{\mathbb{B}} X \text{ and } (U =^{\mathbb{B}} u_1) \sqcap (u_2 \in^{\mathbb{B}} X) \leqslant U \in^{\mathbb{B}} X,$$

and both of these obligations are precisely the left $=^{\mathbb{B}}$-extensionality of $\in^{\mathbb{B}}$. □

Lemma 4.6. *Let X be a $\mathbb{B}$-valued set, and let u_1 and u_2 be $\mathbb{B}$-valued sets. Let a_1 and a_2 be an antichain such that $a_1 \sqcup a_2 = \top$, such that $a_1 = u_2 \subseteq^{\mathbb{B}} u_1$. Then $\top \leqslant u_2 \subseteq^{\mathbb{B}} a_1 \cdot u_1 + a_2 \cdot u_2$.*

Proof. Let U be the two-term mixture $a_1 \cdot u_1 + a_2 \cdot u_2$. We need to show that

$$\top \leqslant \prod_{w : V^{\mathbb{B}}} w \in^{\mathbb{B}} u_2 \Rightarrow w \in^{\mathbb{B}} U,$$

i.e. that for every $w : V^{\mathbb{B}}$,

$$w \in^{\mathbb{B}} u_2 \leqslant w \in^{\mathbb{B}} U.$$

Unfolding the definition of $\in^{\mathbb{B}}$, we restate our goal as:

$$\bigsqcup_{i_y : \alpha_{u_2}} B_{u_2}(i_y) \sqcap w =^{\mathbb{B}} A_{u_2}(i_y) \leqslant \bigsqcup_{i_z : \alpha_U} B_U(i_z) \sqcap w =^{\mathbb{B}} A_U(i_z).$$

Eliminating the existential quantifier on the left, we fix an arbitrary $i : \alpha_{u_2}$ and now must show that:

$$B_{u_2}(i) \sqcap w =^{\mathbb{B}} A_{u_2}(i) \leqslant \bigsqcup_{i_z : \alpha_U} B_U(i_z) \sqcap w =^{\mathbb{B}} A_U(i_z).$$

Instantiating the existential quantifier on the right with i, it now suffices to show:

$$B_{u_2}(i) \sqcap w =^{\mathbb{B}} A_{u_2}(i) \leqslant B_U(i) \sqcap w =^{\mathbb{B}} A_{u_2}(i),$$

and cancelling like terms, it suffices to show that

$$(B_{u_2}(i) \leqslant B_U(i)) \iff \left(B_{u_2} \leqslant \left(a_1 \sqcap A_U(i) \in^{\mathbb{B}} u_1\right) \sqcup \left(a_2 \sqcap A_U(i) \in^{\mathbb{B}} u_2\right)\right).$$

Rewriting the right hand side with de Morgan's law and splitting into cases, we see that it now suffices to show:

1. $B_{u_2}(i) \leqslant (a_1 \sqcup a_2) = \top$.

2. $B_{u_2}(i) \leqslant (a_1 \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_2)$.
3. $B_{u_2}(i) \leqslant (a_2 \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_1)$.
4. $B_{u_2}(i) \leqslant (A_{u_2}(i) \in^{\mathbb{B}} u_2)$.

All the cases except (3) follow immediately from previous results.

For (3), note that we have by assumption that $a_2 = (\neg(u_2 \subseteq^{\mathbb{B}} u_1)) \sqcup A_{u_2}(i) \in^{\mathbb{B}} u_1$, so now our goal is to show

$$B_{u_2}(i) \leqslant (u_2 \subseteq^{\mathbb{B}} u_1) \Rightarrow A_{u_2}(i) \in^{\mathbb{B}} u_1 \iff \Big((u_2 \subseteq^{\mathbb{B}} u_1) \sqcap B_{u_2}(i) \leqslant A_{u_2}(i) \in^{\mathbb{B}} u_1\Big).$$

Instantiating the universal quantifier on the left hand side with $A_{u_2}(i)$, it now suffices to show that

$$A_{u_2}(i) \in^{\mathbb{B}} u_2 \Rightarrow A_{u_2}(i) \in^{\mathbb{B}} u_1 \sqcap B_{u_2}(i) \leqslant A_{u_2}(i) \in^{\mathbb{B}} u_1.$$

Since $B_{u_2}(i) \leqslant A_{u_2}(i) \in^{\mathbb{B}} u_2$, we are finished after applying $\leqslant$-transitivity and Boolean-valued implication elimination. □

Lemma 4.7. *Let $\phi : V^{\mathbb{B}} \to \mathbb{B}$ be an $=^{\mathbb{B}}$-extensional function, such that $[[\exists x \phi(x)]] = \top$.*

(i) *For every $\mathbb{B}$-valued set v, there exists a $\mathbb{B}$-valued set u such that $\phi(u) = 1$ and $\phi(v) = [[u = v]]$.*

(ii) *If $\psi : V^{\mathbb{B}} \to \mathbb{B}$ is another $=^{\mathbb{B}}$-extensional function, such that for every u, $\phi(u) = \top$ implies $\psi(u) = \top$, then*

$$\left(\prod_{(x:V^{\mathbb{B}})} \phi(x) \to \psi(x)\right) = \top.$$

Proof. **(i)** Since $[[\exists x \phi(x)]] = \top$, we obtain, using the maximum principle, a w such that $\phi(w) = \top$.

Put $b = \phi(v)$, and let u be the two-term mixture $u = b \cdot v + (\neg b) \cdot w$. We make the following observations:

(a) $b \leqslant u =^{\mathbb{B}} v$ by the mixing lemma.

(b) $b \leqslant \phi(v)$ by reflexivity.

(c) $\neg b \leqslant u =^{\mathbb{B}} w$ by the mixing lemma.

(d) $\neg b \leqslant \phi(w)$ because everything is bounded by $\top$.

Putting these observations together, we conclude that

$$\top = b \sqcup \neg b \leqslant [[u = v \sqcap \phi(v)]] \sqcup [[u = w \sqcap \phi(w)]] \leqslant [[\phi(u)]].$$

It remains to show that $\phi(v) = [[u = v]]$. Since ϕ is $=^{\mathbb{B}}$-extensional, we have the inequality

$$[[u = v]] = [[u = v]] \sqcap [[\phi(u)]] \leqslant [[\phi(v)]].$$

Conversely, by definition of u,

$$\phi(v) = b \leqslant [[u = v]].$$

(ii) Let $v \in V^{\mathbb{B}}$. We may use part (1) to choose $u \in V^{\mathbb{B}}$ such that $\phi(u) = \top$, and such that $\phi(v) = u =^{\mathbb{B}} v$.

Our goal is then to show that $\phi(v) \leqslant \psi(v)$. So, we calculate

$$\phi(v) \leqslant \phi(v) \sqcap \top = \phi(v) \sqcap \psi(u) = (u =^{\mathbb{B}} v) \sqcap \psi(u) \leqslant \psi(v),$$

since ψ was $=^{\mathbb{B}}$-extensional.

□

5 The fundamental theorem of forcing for Boolean-valued models

The aim of this section is to prove the *fundamental theorem of forcing* (for Boolean-valued models), which states that for any complete Boolean algebra B, V^B satisfies all the ZFC axioms.

5.1 Powersets in $V^{\mathbb{B}}$

Definition 5.1. Let $u : V^{\mathbb{B}}$ be a $\mathbb{B}$-valued set. Let $\chi : \alpha_u \to \mathbb{B}$ be a function. We associate to χ a $\mathbb{B}$-valued set $\widetilde{\chi}$, which we think of as being the subset of u induced by χ. We define $\widetilde{\chi}$ by specifying the data $\langle \alpha, A, B \rangle$:

- $\alpha \stackrel{\text{df}}{=} \alpha_u$,
- $A \stackrel{\text{df}}{=} A_u$,
- $B \stackrel{\text{df}}{=} \chi$.

Definition 5.2. Let u be a $\mathbb{B}$-valued set. We define the **powerset** $\mathcal{P}(u) : V^{\mathbb{B}} := \langle \alpha, A, B \rangle$ as follows:

- $\alpha \stackrel{\text{df}}{=} \alpha_u \to \mathbb{B}$,
- $A : \alpha \to V^{\mathbb{B}} \stackrel{\text{df}}{=} \lambda \chi : \alpha_u \to \mathbb{B}, \widetilde{\chi}$,
- $B : \alpha \to \mathbb{B} \stackrel{\text{df}}{=} \lambda \chi : \alpha_u \to \mathbb{B}, \widetilde{\chi} \subseteq^{\mathbb{B}} u$.

5.2 The axiom of choice

Let x be a $\mathbb{B}$-set. By the maximum principle, there exists a $\mathbb{B}$-set u_x such that $[[u_x \in x]] = \bigsqcup_{(y:V^B)} [[y \in x]]$. By the axiom of choice in the metatheory, we can then define a *global choice function* $\mathsf{choice} : \mathrm{bSet}\,\mathbb{B} \to \mathrm{bSet}\,\mathbb{B}$.

Naïvely, this is not a choice function from the point of view of $V^{\mathbb{B}}$, as it may not be $=^{\mathbb{B}}$-extensional.

However, by restricting choice to a $\mathbb{B}$-set, we can fulfill the requirements of (AC').

Definition 5.3. The *axiom of choice* is the sentence (AC):

$$\forall u, \exists f, [\mathrm{Fun}(f) \wedge (\mathrm{dom}(f) = u) \wedge (\forall x, x \in u \to (x \neq \varnothing \to f(x) \in x))] .$$

It is provably equivalent over ZF to the following sentence[4]:

[4]See http://us.metamath.org/mpeuni/ac3.html

Definition 5.4. (AC'):

$$\forall x, \exists y, \forall z, z \in x \to (z \neq \varnothing \to \exists! w, (w \in z \to \exists v, (v \in y \to (z \in v \wedge w \in v))))\,.$$

Theorem 5.5. *The sentence* AC' *has truth value* $\top$ *in* V^B*.*

Proof. Translated into Boolean truth values, we see that

$$[[\mathsf{AC}']] = \bigsqcap_{(x:V^B)} \bigsqcup_{(y:V^B)} \bigsqcap_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \varnothing]] \Rightarrow \bot)$$

$$\Rightarrow (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

It then suffices to show that for every $x : V^B$,

$$\top \leqslant \bigsqcup_{(y:V^B)} \bigsqcap_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \varnothing]] \Rightarrow \bot)$$

$$\Rightarrow (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

We will finish the proof by constructing a y with which we instantiate the (Boolean-valued) existential quantifier.

Before proceeding with the construction, we indicate how the argument would proceed after instantiating the y, which will expose the properties that such a y must satisfy.

So, fix a y. Instantiating the quantifier, we now have to show

$$\top \leqslant \bigsqcap_{(z:V^B)} [[z \in x]] \Rightarrow ([[z = \varnothing]] \Rightarrow \bot)$$

$$\Rightarrow (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

Introducing the z, we then have to show that for any $z : V^B$,

$$\top \leqslant [[z \in x]] \Rightarrow ([[z = \varnothing]] \Rightarrow \bot)$$

$$\Rightarrow (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

Introducing the implications (using, say, the deduction theorem), it suffices to show that

$$[[z \in x]] \Rightarrow ([[z = \varnothing]] \Rightarrow \bot) \leqslant (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

It is easy to see that for any $z : V^B$,

$$[[z = \varnothing]] = \bigsqcap_{w:V^B} [[w \in z]] \Rightarrow \bot.$$

Let u_z be the witness produced by the maximum principle such that

$$[[u_z \in z]] = \bigsqcup_{(w:V^B)} [[w \in z]].$$

Therefore, it suffices to show that

$$[[z \in x]] \sqcap [[u_z \in z]] \leqslant (\bigsqcup!_{(w:V^B)}[[w \in z]] \Rightarrow \bigsqcup_{(v:V^B)} [[v \in y]] \Rightarrow [[z \in v]] \sqcap [[w \in v]].$$

Instantiating the quantifier on the right-hand side with u_z and unfolding the definition of the Boolean-valued version of the bounded "exists unique" quantifier, it now suffices to show that

$$[[z \in x]] \sqcap [[u_z \in z]] \leqslant \bigsqcap_{(w':V^B)} [[w' \in z]] \Rightarrow \left(\bigsqcup_{(v:V^B)} [[v \in y]] \sqcap [[z \in v]] \sqcap [[w' \in v]] \right) \Rightarrow [[w' = u_z]].$$

Now we describe how to construct the necessary y.

TODO(jesse): Fix this to include injectivization and finish the proof.

□

5.2.1 Zorn's lemma

The axiom of choice is equivalent to the following version of Zorn's lemma, which states that any set whose members are ordered by $\subseteq$ and is closed under unions of chains contains a maximal element.

Definition 5.6. (ZL):

$$\forall x, (\forall y, (y \subseteq x \wedge \forall w_1 \in y, \forall w_2 \in y, w_1 \subseteq w_2 \vee w_2 \subseteq w_1) \implies$$

$$(\bigcup y) \in x) \implies \exists m \in x, \forall z \in x, m \subseteq x \implies m = x$$

The aim of this section is to prove the following theorem:

Theorem 5.7. *Zorn's lemma has truth-value* $\top$ *in* V^B.

To do this, we introduce the notion of a *core* of a $\mathbb{B}$-valued set.

Definition 5.8. Let u be a $\mathbb{B}$-valued set. Let S be a set of $\mathbb{B}$-valued sets. We say that S is a **core** for u if the following properties hold:

(i) For every $v \in S$, $[[x \in u]] = 1$, and

(ii) For each $y \in V^B$ such that $[[y \in u]] = 1$, there exists a *unique* $v_y \in S$ such that $[[y = v_y]] = 1$.

Lemma 5.9. *Let u be a $\mathbb{B}$-valued set. There exists a set of $\mathbb{B}$-valued sets S which is a core for u.*

Proof. Fix $x : V^{\mathbb{B}}$. We define the set

$$f_x \stackrel{\text{df}}{=} \{(a, u_B(a) \sqcap u_A(a) =^{\mathbb{B}} x) \,|\, a \in u_\alpha\}.$$

That is, f_x is the graph-relation of the map

$$\lambda a : u_\alpha, u_B(a) \sqcap u_A(a) =^{\mathbb{B}} x.$$

As such, $f_x : \mathrm{set}(u_\alpha \times \beta)$. Consider the map

$$(\lambda x : V^{\mathbb{B}}, f_x) : V^{\mathbb{B}} \to \mathrm{set}(u_\alpha \times \beta).$$

Since $\mathrm{set}(u_\alpha \times \beta)$ is a small type, we may perform the same type of smallness argument as we did for the maximum principle, and select lifts of every fiber, obtaining a set $S' : \mathrm{set}\, V^{\mathbb{B}}$ which satisfies the following property: for every x, there exists a y in S' such that $f_x = f_y$.

There is an equivalence relation $\sim$ on V^B, defined by $x_1 \sim x_2 \iff \left(x_1 =^{\mathbb{B}} x_2 = \top\right)$. This restricts to an equivalence relation on any subset of $V^{\mathbb{B}}$. In particular, we consider the restriction of $\sim$ to the subset of $S'' \stackrel{\text{df}}{=} \{x \in S' \,|\, x \in^{\mathbb{B}} u = \top\}$. We define S to be a transversal of the $\sim$-equivalence classes of S''.

It remains to verify that S is a core for u. Since $S \subseteq S''$ and S'' by definition is made up of $\mathbb{B}$-valued sets x such that $x \in^{\mathbb{B}} u = \top$, item **(i)** of the definition of a core is satisfied.

To verify item **(ii)**, let $y : V^{\mathbb{B}}$ such that $y \in^{\mathbb{B}} u = \top$. By construction of S', there exists a $y' \in S'$ such that $f_{y'} = f_y$. Since the supremum of (the values of) f_y is $\top$, the supremum of (the values of) $f_{y'}$ is $\top$, and therefore $y' \in^{\mathbb{B}} u = \top$. By construction of S'', there exists a unique $y'' \in S''$ such that $y'' =^{\mathbb{B}} y' = \top$.

We claim that $y =^{\mathbb{B}} y'' = \top$. To prove this claim, by the transitivity of Boolean-valued equality, it suffices to show that $y =^{\mathbb{B}} y' = \top$.

Let a be an arbitrary element of α_u. By the transitivity of $=^{\mathbb{B}}$,

$$y =^{\mathbb{B}} A_u a \sqcap A_u a =^{\mathbb{B}} y' \leqslant y =^{\mathbb{B}} y'.$$

This is equivalent to

$$\left(\bigsqcup_{(a:\alpha_u)} y =^{\mathbb{B}} A_u a \sqcap A_u a =^{\mathbb{B}} y' \right) \leqslant y =^{\mathbb{B}} y',$$

and it is easy to check that this implies

$$\left(\bigsqcup_{(a:\alpha_u)} B_u a \sqcap y =^{\mathbb{B}} A_u a \sqcap A_u a =^{\mathbb{B}} y' \sqcap B_u a \right) \leqslant y =^{\mathbb{B}} y'$$

Since $f_y = f_{y'}$, the left-hand side of this can be rewritten to

$$\left(\bigsqcup_{(a:\alpha_u)} B_u a \sqcap y =^{\mathbb{B}} A_u a \sqcap B_u a \sqcap y =^{\mathbb{B}} A_u a \right) \leqslant y =^{\mathbb{B}} y',$$

and it follows that $\top \leqslant y \in^{\mathbb{B}} u = \bigsqcup B_u a \sqcap y =^{\mathbb{B}} A_u a \leqslant y =^{\mathbb{B}} y',$, as required.

To complete the proof, we must show that y'' is unique for y. So, let y''' be an element of S'' such that $y''' =^{\mathbb{B}} y = \top$. To show that $y''' = y''$, it suffices to show that $y''' =^{\mathbb{B}} y' = \top$, because then y''' will belong to the $\sim$-class of y' and by construction of S'', y''' will have to be equal to y''.

However, we have already seen that $y =^{\mathbb{B}} y' = \top$, so this follows from the transitivity of $=^{\mathbb{B}}$.

$\square$

Remark 5.10. If u is nonempty with truth-value $\top$, then the maximum principle ensures that any core of u is nonempty.

Lemma 5.11. *Let u be a $\mathbb{B}$-valued set such that $[[u \neq \varnothing]] = \top$. Let S be a core for u. Then for any B-valued set x, there exists a $y \in S$ such that $[[x = y]] = [[x \in u]]$.*

Proof. Up to an equality of terms in $\mathbb{B}$, this is exactly 4.7. $\square$

Proof of 5.6. Let X be a $\mathbb{B}$-valued set such that $(X, \subseteq)$ is a nonempty inductive partially ordered set. To prove Zorn's lemma, we must show that X has a maximal element.

Let S be a core for X. We define the relation $\leqslant : S \to S \to \mathsf{Prop}$ as follows:

$$\lambda s_1 s_2, [[s_1 \subseteq s_2]] = \top.$$

It is easy to check that $\leqslant$ is a partial order on S. To apply the Zorn's lemma available in the metatheory, we must show that $\leqslant$ is inductive, i.e. that every chain has an upper bound.

So, fix a chain C of S. We can turn C into a $\mathbb{B}$-valued set $\widehat{C}$ by declaring the truth-value of all elements to be $\top$.

We claim that if C was a chain in S, then $V^{\mathbb{B}} \models^{\mathbb{B}} \widehat{C}$ "is a chain in X."

Proof of claim. Formally, "$\widehat{C}$ is a chain" is the sentence

$$\forall u_1 \in \widehat{C}, \forall u_2 \in \widehat{C}, u_1 \subseteq u_2 \vee u_2 \subseteq u_1,$$

and so we must show that

$$\top \leqslant \prod_{u_1 \in C} \prod_{u_2 \in C} u_1 \subseteq^{\mathbb{B}} u_2 \vee u_2 \subseteq^{\mathbb{B}} u_1,$$

or equivalently,

$$\forall u_1 \in C, \forall u_2 \in C, T \leqslant u_1 \subseteq^{\mathbb{B}} u_2 \vee u_2 \subseteq^{B} u_1.$$

So, fix u_1 and u_2 in C. Since C was a chain in S, either $[[u_1 \subseteq u_2]] = \top$ or $[[u_2 \subseteq u_1]] = \top$.

Without loss of generality, suppose that $[[u_1 \subseteq u_2]] = \top$, as the other case is entirely symmetric. Then $\top \leqslant u_1 \subseteq^{\mathbb{B}} u_2$, and by the transitivity of $\leqslant$, it suffices to show that

$$u_1 \subseteq^{\mathbb{B}} u_2 \leqslant u_1 \subseteq^{\mathbb{B}} u_2 \sqcup u_2 \subseteq^{\mathbb{B}} u_1,$$

which immediately follows from the defining property of the binary sup.

That $\widehat{C}$ is a chain in X means that

$$\top \leqslant \prod_{u \in C} u \in^{\mathbb{B}} X.$$

But, each u belongs to C, which is a subset of a core S for X, so $u \in^{\mathbb{B}} X = \top$. $\square$

Now we claim that for any chain C in S, there exists a $u : V^{\mathbb{B}}$ such that $V^{\mathbb{B}} \models^{\mathbb{B}}$ "u is an upper bound for $\widehat{C}$ in X".

Proof of claim. We will use $u = \bigcup \widehat{C}$. We need to to show two things:

(i) $\top \leqslant u \in^{\mathbb{B}} X$, and

(ii) $\top \leqslant \prod_{w \in C} w \subseteq^{\mathbb{B}} u$.

Item **(ii)** is an immediate consequence of the specification of the $\mathbb{B}$-valued unionset operation.

For item **(i)**, we have by assumption that X is inductive, so that

$$T \leqslant \prod_{(Y:V^{\mathbb{B}})} \left(\left(Y \subseteq^{\mathbb{B}} X \sqcap \left(\prod_{w_1 \in Y} \prod_{w_2 \in Y} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right) \Rightarrow (\bigcup Y) \in^{\mathbb{B}} X \right)$$

$$\leqslant \left(u \subseteq^{\mathbb{B}} X \sqcap \left(\prod_{w_1 \in u} \prod_{w_2 \in u} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right) \Rightarrow (\bigcup Y) \in^{\mathbb{B}} X$$

Therefore, it suffices to show that

$$T \leqslant \left(\widehat{C} \subseteq^{\mathbb{B}} X \sqcap \left(\prod_{w_1 \in \widehat{C}} \prod_{w_2 \in \widehat{C}} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right) \right),$$

and so it suffices to show:

1. $T \leqslant \widehat{C} \subseteq^{\mathbb{B}} X$, and
2. $T \leqslant \left(\prod_{w_1 \in \widehat{C}} \prod_{w_2 \in \widehat{C}} w_1 \subseteq^{\mathbb{B}} w_2 \sqcup w_2 \subseteq^{\mathbb{B}} w_1 \right)$,

This is exactly the conclusion of the previous claim. □

In particular, $[[u \in x]] = \top$, so by virtue of S being a core, there exists a $w \in S$ such that $[[w = u]] = \top$. Then w is an upper bound for C in S: for any other $w' \in S$, $[[w' \in \widehat{C}]] = 1$, so $[[w' \subseteq w]] = \top$, and therefore $w' \leqslant w$.

So S is inductive and we may apply Zorn's lemma. Therefore, S has a maximal element c. Since S is a core, $[[c \in X]] = \top$.

To finish the proof, it now suffices to show that c is a maximal element of X. That is, we must show that

$$V^{\mathbb{B}} \models^{\mathbb{B}} \forall x \in X, c \subseteq x \rightarrow x = c.$$

That is, we must show that

$$\top \leqslant \prod_{(x \in X)} c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

Introducing the universal quantifier, it suffices to show that for any $x \in V^{\mathbb{B}}$,

$$\top \leqslant x \in^{\mathbb{B}} X \Rightarrow c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c,$$

and introducing the implication, it suffices to show that

$$x \in^{\mathbb{B}} X \leqslant c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

By 5.11, we can find a y such that $x \in^{\mathbb{B}} X = [[x = y]]$, so we rewrite the above inequality to

$$[[x = y]] \leqslant c \subseteq^{\mathbb{B}} x \Rightarrow x =^{\mathbb{B}} c.$$

Introducing another implication, we have

$$[[x = y]] \sqcap c \subseteq^{\mathbb{B}} x \leqslant [[x = c]].$$

since $\subseteq^{\mathbb{B}}$ is $=^{\mathbb{B}}$-extensional, by applying the transitivity of $\leqslant$, it now suffices to show that

$$[[x = y]] \sqcap (c \subseteq^{\mathbb{B}} y) \leqslant [[x = c]].$$

If we can show that $c \subseteq^{\mathbb{B}} y \leqslant [[y = c]]$, then we are done, by applying transitivity of $\leqslant$ and then by applying the transitivity of $=^{\mathbb{B}}$.

Let $a : \mathbb{B}$ be $c \subseteq^{\mathbb{B}} y$. We let v be the two-term mixture $a \cdot y + (\neg a) \cdot c$.

We claim that $v \in^{\mathbb{B}} X = \top$.

Proof of claim. This follows directly from 4.5. □

By virtue of S being a core, there exists a $z \in S$ such that $[[v = z]] = \top$.

We claim that $[[c \subseteq v]] = \top$.

Proof of claim. This follows directly from 4.6. □

Since $[[v = z]]$, it follows that $[[c \subseteq z]] = \top$, and therefore

$$c \leqslant z$$

in S, and since c is $\leqslant$-maximal, $c = z$, so $c =^{\mathbb{B}} z = \top$.

Now we calculate:

$$\begin{aligned}
[[c \subseteq y]] = a &\leqslant [[y = v]] \\
&\leqslant [[y = v]] \sqcap [[v = z]] \\
&\leqslant [[y = z]] = [[y = c]],
\end{aligned}$$

and the proof is complete. □

5.3 Ordinals in $V^{\mathbb{B}}$

Lemma 5.12. *Let $u \in V^{\mathbb{B}}$. Then*

$$\mathrm{Ord}^{\mathbb{B}}(u) = \bigsqcup_{\alpha:\mathrm{ORD}} u =^{\mathbb{B}} \check{\alpha}.$$

Proof. Since the property of being an ordinal is preserved by taking check-names, for an arbitrary ordinal α,

$$u =^{\mathbb{B}} \check{\alpha} = u =^{\mathbb{B}} \check{\alpha} \sqcap \top = u =^{\mathbb{B}} \check{\alpha} \sqcap \mathrm{Ord}^{\mathbb{B}}(\check{\alpha}) \leqslant \mathrm{Ord}^{\mathbb{B}}(u),$$

since the interpretation of any first-order formula is $=^{\mathbb{B}}$-extensional. So, the inequality

$$\bigsqcup_{\alpha:\mathrm{ORD}} u =^{\mathbb{B}} \check{\alpha} \leqslant \mathrm{Ord}^{\mathbb{B}}(u)$$

follows.

To show the other direction, we first note that since $\eta \neq \xi \implies \check{\eta} =^{\mathbb{B}} \check{\xi} = \bot$, for any $x : \alpha_u$, the map $\xi \mapsto A_u(x) =^{\mathbb{B}} \check{\xi}$ is injective when restricted to $D_X \overset{\text{df}}{=} \{\xi : A_u(x) =^{\mathbb{B}} \check{\xi} > \bot\}$ (as a function into $\mathbb{B}$).

Since $\mathbb{B}$ is small, so is D_X. Put $D = \bigcup_{x:\alpha_u} D_x$.

If α_0 is an ordinal greater than every ordinal in D, then for every $x : a_u$,

$$\check{\alpha_0} =^{\mathbb{B}} A_u(x) = 0,$$

so

$$\check{\alpha_0} \in^{\mathbb{B}} u = 0.$$

Since the ordinals are well-ordered and satisfy a trichotomy with respect to $\in$, it follows that

$$\mathrm{Ord}^{\mathbb{B}}(u) \leqslant u \in^{\mathbb{B}} \check{\alpha_0} \sqcup u =^{\mathbb{B}} \check{\alpha_0} \sqcup \check{\alpha_0} \in^{\mathbb{B}} u.$$

Therefore, $\mathrm{Ord}^{\mathbb{B}}(u) \leqslant u \in^{\mathbb{B}} \check{\alpha_0} \sqcup u =^{\mathbb{B}} \check{\alpha_0}$.

In the first case, $u \in^{\mathbb{B}} \check{\alpha_0}$, and is therefore equal to some member of α_0, which must be the check-name of an ordinal, and therefore an ordinal.

In the second case, $u =^{\mathbb{B}} \check{\alpha_0}$ obviously implies that there exists an ordinal whose check-name is equal to u. □

5.4 The consistency of $\neg\mathsf{CH}$

By the Boolean-valued soundness theorem, if ZFC proves CH, then CH has truth-value $\top$ in every Boolean-valued model of ZFC. Therefore, to show $\neg(\mathsf{ZFC} \vdash \mathsf{CH})$, it suffices to show that there exists a model of ZFC where CH has truth-value not equal to $\top$.

Throughout the rest of this section, $\mathcal{S}$ will denote the type of Prop-valued functions on $\omega_2 \times \omega$.

We will show that CH is not true in $V^{\mathbb{B}}$, where $\mathbb{B}$ is defined to be the complete boolean algebra of regular opens of $2^{\mathcal{S}}$. Our goal will be to prove:

Theorem 5.13. *CH is not true in $V^{\mathbb{B}}$. Specifically, $V^{\mathbb{B}} \models \aleph_2 \leqslant |2^{\omega}|$.*

Before proceeding with the proof of 5.13, we set up some prerequisite definitions and lemmas.

Lemma 5.14. *$\mathbb{B}$ has the countable chain condition.*

Proof. TODO(floris) □

Lemma 5.15. *$\check{\aleph}_1$ is a cardinal.*

Proof. It suffices to show that for every $f \in V^{\mathbb{B}}$ and $\eta \in \aleph_1$ (in $V^{\mathbb{B}}$) that

$$a(f,\eta) = [[f \text{ is a function whose domain is } \eta \text{ and whose range is } \aleph_1]] = \perp.$$

So, suppose towards a contradiction that there exists some $f \in V^{\mathbb{B}}$ and $\eta \in \aleph_1$ such that $a(f,\eta) > 0$. Since η is an ordinal, we have that η is equal to some $\check{\beta}$.

We claim that $\beta < \aleph_1$. For otherwise, $\aleph_1 \subseteq \beta$, so then

$$\check{\aleph}_1 \subseteq \check{\beta} = \eta,$$

and therefore $\eta \in \eta$, a contradiction.

So, we may assume that $\beta < \aleph_1$. Therefore, β is countable.

We have that

$$a \leqslant \prod_{\eta < \aleph_1} \bigsqcup_{\xi < \beta} [[f(\widetilde{xi}) = \check{\xi}]] \sqcap a$$

(This restriction of quantifiers can be proved using a similar argument as above.)

Then for each $\eta < \aleph_1$, there exists a least $\xi_\eta < \beta$ such that

$$[[f(\check{\xi_\eta}) = \widetilde{\eta}]] \sqcap a \neq 0.$$

Since the map $\eta \mapsto \xi_\eta$ is a map from the uncountable $\aleph_1$ to the countable β, there must exist a $\gamma < \beta$ such that the fiber of this map over γ is uncountable.

It is then easy to check that the set

$$\{[[f(\check{\gamma}) = \check{\eta}]] \sqcap a : \eta \in X\}$$

is an uncountable antichain in B, which contradicts the fact that B had the CCC. $\square$

Lemma 5.16. *In* $V^{\mathbb{B}}$. $\aleph_{\check{1}} \subseteq \check{\aleph}_1$.

Proof. By the universal property of the cardinal successor operation (this is using the fact that $\check{\aleph}_1$ is actually a cardinal), it suffices to show that $\aleph_{\check{0}} \subsetneq \check{\aleph}_1$. We have that $\aleph_{\check{0}} = \check{\aleph}_0$, so it suffices to show that $\check{\aleph}_0 \subsetneq \check{\aleph}_1$. Since the check-name of an ordinal is an ordinal, this is equivalent to showing that $\check{\aleph}_0 \in \check{\aleph}_1$. This follows from the facts that $\aleph_0 \in \aleph_1$ and taking check-names preserves membership. $\square$

Lemma 5.17. *Let u be a $\mathbb{B}$-valued set, and let $F : \alpha_u \to V^{\mathbb{B}}$ be a function which is $=^{\mathbb{B}}$-extensional, i.e. for every i and j in α_u, $A_u(i) =^{\mathbb{B}} A_u(j) \leqslant F(i) =^{\mathbb{B}} F(j)$. Then $V^{\mathbb{B}}$ thinks the set of all pairs $(A_u(i), F(i))$ is a function from u to $\langle \alpha_u, F, \lambda i, \top \rangle$.*

Lemma 5.18. *Let u be a $\mathbb{B}$-valued set, and let $F : \alpha_u \to V^{\mathbb{B}}$ be a function as in 5.17. If, for every i and j in α_u, if $i \neq j$ then $F(i) =^{\mathbb{B}} F(j) \leqslant \perp$, then $V^{\mathbb{B}}$ thinks $\widetilde{F}$ is injective.*

Definition 5.19. Let $(a,b) : A \times B$. The **principal open** associated to (a,b) is the collection of all subsets S of $A \times B$ such that $(a,b) \in S$. We denote it by $\mathbf{P}_{(a,b)}$. It is a regular open in the product topology on $\mathrm{set}(A \times B)$.

Definition 5.20. For each $\nu < \omega_2$, we define a $\mathbb{B}$-valued set u_ν (which we will think of as a new subset of ω).

Let $\chi_\nu : \alpha_\omega \to \mathbb{B}$ be the function defined by $\lambda n : \mathbb{N}, \mathbf{P}_{(\nu,n)}$. We define

$$u_\nu \stackrel{\text{df}}{=} \widetilde{\chi_\nu}$$

(see 5.1 for a definition of the notation $\chi \mapsto \widetilde{\chi}$). Each u_ν is called a **Cohen real**.

This lemma says that $V^{\mathbb{B}}$ thinks that every Cohen real is a subset of ω:

Lemma 5.21. $u_\nu \subseteq^{\mathbb{B}} \omega = \top$.

Proof. By definition, $\subseteq^{\mathbb{B}}$ unfolds to

$$u_\nu \subseteq^{\mathbb{B}} \omega = \bigsqcap_{k:\alpha_{u_\nu}} B_{u_\nu}(k) \Rightarrow A_{u_\nu}(k) \in^{\mathbb{B}} \omega,$$

and since it suffices to show that

$$\top \leqslant \bigsqcap_{k:\alpha_{u_\nu}} B_{u_\nu}(k) \Rightarrow A_{u_\nu}(k) \in^{\mathbb{B}} \omega,$$

then this is equivalent to showing that for every $k : \mathbb{N}$,

$$\top \leqslant B_{u_\nu}(k) \Rightarrow A_{u_\nu}(k) \in^{\mathbb{B}} \omega$$

However, by the definition of $\widetilde{\chi}$, $A_{u_\nu}(k)$ is actually just A_ω, so this becomes

$$\top \leqslant B_{u_\nu}(k) \Rightarrow A_\omega(k) \in^{\mathbb{B}} \omega,$$

and since what $V^{\mathbb{B}}$ thinks is ω is actually $\check{\omega}$, this is equivalent to

$$(\top \leqslant B_{u_\nu}(k) \Rightarrow \top) \iff B_{u_\nu}(k) \leqslant \top,$$

which is always true. $\square$

This lemma says that $V^{\mathbb{B}}$ thinks that every Cohen real is *distinct*:

Lemma 5.22. *Let μ and ν belong to* $\operatorname{set}(\operatorname{set}(\omega))$*. If $\mu \neq \nu$, then $u_\mu =^{\mathbb{B}} u_\nu = \bot$.*

Proof. Suppose towards a contradiction that there exist μ and ν which are distinct such that $u_\mu =^{\mathbb{B}} u_\nu > \bot$. Since the Cohen poset P is a basis for the Boolean algebra of regular opens, there exists a $p \in P$ such that $p \Vdash u_\mu = \mu_\nu$. Viewing p as a partial function with a finite domain, we may choose an $n : \mathbb{N}$ such that for every $\xi : \omega_2$, (ξ, n) is not in $\operatorname{dom}(p)$.

We now a define an extension p' of p, so that $p \leqslant p$. Put

$$p' \stackrel{\text{df}}{=} p \cup \{((\mu, n), 1)\} \cup \{(\nu, n), 0)\},$$

That is, we define p' to be equal to p everywhere except at (μ, n) and (ν, n). on which we define the values of p' to be 1 and 0, respectively. (This extension is well-defined since n was chosen to not occur in the domain of p.)

Then $p' \Vdash \check{n} \in u_\mu \check{n} \notin u_\nu$. It follows that $p' \Vdash u_\mu \neq u_\nu$. However, since $p' \leqslant p$, p' forces every statement that p forces, so

$$p' \Vdash u_\mu = u_\nu,$$

a contradiction. □

It remains to construct a map from $\omega_{\check{2}}$ to $\mathcal{P}(\omega)$ which $V^{\mathbb{B}}$ believes is injective. We do this by internalizing to $V^{\mathbb{B}}$ the map $\nu \mapsto u_\nu$. In 5.17 we saw how a function out of the domain of a $\mathbb{B}$-valued set could be internalized into a function in $V^{\mathbb{B}}$. Applying the same construction here, we obtain

$$f : V^{\mathbb{B}} := \langle \alpha, A, B \rangle,$$

where:

- $\alpha \overset{\text{df}}{=} \alpha_{\check{\omega_2}}$
- $A \overset{\text{df}}{=} F$
- $B \overset{\text{df}}{=} \lambda_-, \top$

Since $V^{\mathbb{B}}$ believes the values of the function are all distinct, it follows from 5.18 that the function is injective.

By 5.21, $V^{\mathbb{B}}$ thinks this is an injective function into $\mathcal{P}(\omega)$.

Since $\mathbb{B}$ has the countable chain condition, by 5.16 we have that $\aleph_1$ is a subset of $\check{\aleph}_1$. Since taking check-names preserves subsets, $\check{\aleph}_1$ is a subset of $\check{\aleph}_2$. Therefore, $V^{\mathbb{B}}$ thinks there is an injective function from $\aleph_1$ into $\mathcal{P}(\omega)$: the continuum hypothesis is false in $V^{\mathbb{B}}$.

6 Generic sets: Cohen's original proof

7 Sheaves and filterquotients