

Forcing and the independence of the continuum hypothesis

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Abstract

In these notes, intended as the plaintext part of the Flypitch project, we give a complete account of the independence of the continuum hypothesis from ZFC, with special attention paid to comparing the different approaches: generic sets, Boolean-valued models, and double-negation sheaves.

1 Preliminaries

1.1 First-order logic

1.1.1 General logical symbols

Definition 1.1. We reserve the following general logical symbols:

\neg	not
\vee	or
\wedge	and
\forall	for all
\exists	exists
$=$	equals
$(,)$	parentheses
$(x_i)_{i:\mathbb{N}}, (y_i)_{i:\mathbb{N}}, (z_i)_{i:\mathbb{N}}$	variables

1.1.2 First-order languages

Definition 1.2. A (first-order, one-sorted) **language** \mathcal{L} comprises the following data:

- (i) A collection of **constant symbols** $\text{Const}(\mathcal{L})$,
- (ii) a collection of **relation symbols** $\text{Rel}(\mathcal{L})$,
- (iii) a collection of **function symbols** $\text{Funct}(\mathcal{L})$, and
- (iv) an assignment of each symbol $S : \text{Const}(\mathcal{L}) \cup \text{Rel}(\mathcal{L}) \cup \text{Funct}(\mathcal{L})$ to a natural number $\text{arity}(S) : \mathbb{N}$.

Whenever we interpret a language on some carrier A , we mean for constants c to be interpreted as elements of $A^{\text{arity}(c)}$, relations R to be interpreted as subsets of $A^{\text{arity}(R)}$, and for function symbols to be interpreted as functions $A^{\text{arity}(f)} \rightarrow A$.

Example 1.3. • The language of groups comprises a 1-ary constant symbol for the identity and a 2-ary function for group multiplication.

- The language of rings comprises constant symbols 0 and 1 and 2-ary functions for addition and multiplication.
- The language of set theory comprises just one 2-ary relation \in .

1.1.3 Terms, formulas, and sentences

Definition 1.4. A **term** is a string of symbols defined by structural induction as follows:

1. Any variable v is a term.
2. Any constant c is a term.
3. If t_1, \dots, t_n are terms of arities a_1, \dots, a_n , then (t_1, \dots, t_n) is a term of arity $a_1 + \dots + a_n$.
4. If t is a term and f is a function symbol with matching arities, then ft is a term.

Whenever we interpret our language on a carrier A , we mean for terms to be interpreted as functions into A which we can construct by composing existing constants (constant functions), basic functions (i.e. the interpretations of the function symbols), and variables (identity).

Definition 1.5. A **formula** is defined by structural induction as follows:

1. If t_1 and t_2 are terms of the same arity, $t_1 = t_2$ is a formula.
2. If t is a term and R is a relation symbol, and t and R have the same arity, then Rt is a formula.
3. If φ is a formula, $\neg\varphi$ is a formula.
4. If φ and ψ are formulas, then $\varphi \vee \psi$ is a formula.
5. If φ and ψ are formulas, then $\varphi \wedge \psi$ is a formula.
6. If φ is a formula containing a variable v , then $\exists v\varphi$ is a formula.
7. If φ is a formula containing a variable v , then $\forall v\varphi$ is a formula.

Definition 1.6. Let φ be a formula containing the variables x_1, \dots, x_n . We say that the variable x_k is **free** if x_k is not contained in a subformula of the form $\exists x_k\psi$ or $\forall x_k\psi$.

x_k is **bound** if it is not free.

Definition 1.7. A formula is a **sentence** (or **statement**) if it contains no free variables.

We write $\text{Formulas}(\mathcal{L})$ for all the first-order formulas of \mathcal{L} , and we write $\text{Sentences}(\mathcal{L})$ for all the first-order sentences of \mathcal{L} .

By convention, we always include sentences called **true** and **false**.

1.1.4 Predicate calculus and provability

Throughout this section, we fix a language \mathcal{L} .

Definition 1.8. A **propositional function** is a function $f : \text{Prop}^k \rightarrow \text{Prop}$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. Each projection $(P_1, \dots, P_k) \mapsto P_j$ is a propositional function.
2. If f and g are propositional functions, so are

$$\neg f, f \wedge g, f \vee g, f \rightarrow g, \text{ and } f \leftrightarrow g,$$

where the operations above are carried out pointwise in Prop .

f is a **tautology** if $\vdash \forall \vec{p} : \text{Prop}^k, f\vec{p} \leftrightarrow \text{true}$.

Definition 1.9. A **propositional combination** is a function $f : \text{Sentences}(\mathcal{L})^k \rightarrow \text{Sentences}(\mathcal{L})^k$, for some $1 < k : \mathbb{N}$ which we define inductively as follows:

1. Each projection $(B_1, \dots, B_k) \mapsto B_j$ is a propositional combination.
2. If f and g are propositional combinations, so are

$$\neg f, f \wedge g, f \vee g, f \rightarrow g, \text{ and } f \leftrightarrow g,$$

where the operations are carried out pointwise in $\text{Sentences}(\mathcal{L})$.

By sending projections to projections and symbols $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$ to the corresponding operations on Prop , every propositional combination $f : \text{Sentences}(\mathcal{L})^k \rightarrow \text{Sentences}(\mathcal{L})^k$ can be realized as a propositional function $\mathbf{r}(f) : \text{Prop}^k \rightarrow \text{Prop}$.

Definition 1.10. The **predicate calculus** comprises the following rules for deducing sentences from other sentences. We call deducible sentences **valid**, and write $\vdash_{\mathcal{L}} \varphi$ to mean that the \mathcal{L} -sentence φ is valid (and to disambiguate from \vdash , which when used unadorned means “provable in the metatheory”).

- (a) (Rule of the propositional calculus) if f is a propositional combination taking k arguments such that $\mathbf{r}(f)$ is a tautology, then for any k sentences A_1, \dots, A_k , the value of the propositional combination $f(\varphi_1, \dots, \varphi_k)$ is a valid sentence.
- (b) (Rule of modus ponens) If A and $A \rightarrow B$ are valid, then B is valid.
- (c) (Rules of equality)

(i) $\forall x, x = x$, $\forall x \forall y, x = y \wedge y = x$, and $\forall x \forall y \forall z, x = y \wedge y = z \rightarrow x = z$ are all valid.

(ii) Let $\varphi(x)$ be a formula whose only free variable is x . Then

$$\forall x \forall y, (x = y) \rightarrow (\varphi(x) \rightarrow \varphi(y))$$

is valid.

- (d) (Change of variable) If A is a sentence and A' represents A with all instances of a variable x switched to y , then $A \leftrightarrow A'$ is valid.
- (e) (Rule of specialization “ \forall -elimination”) Let c be any constant symbol, and let $\varphi(x)$ be a formula whose only free variable is x . Then $(\forall x \varphi(x)) \rightarrow \varphi(c)$ is valid.
- (f) $A \leftrightarrow (A \rightarrow \text{true})$ is valid.
- (g) (“ \neg -introduction”) If $\neg A \leftrightarrow (A \rightarrow \text{false})$ is valid.

- (h) (Generalization of constants “ \forall -introduction”) Let B be a sentence which does not contain the constant c or the variable x . Let $\varphi(x)$ be some formula such that $\varphi(x) \rightarrow B$ is valid. Then $\exists x\varphi(x) \rightarrow B$ is also valid.¹
- (i) (de Morgan laws) Let $\varphi(x)$ have x as its only free variable. Let B be a sentence which does not contain x . Then the following are valid statements:

$$(\neg(\forall x\varphi(x))) \leftrightarrow (\exists x\neg\varphi(x))$$

$$((\forall x\varphi(x)) \wedge B) \leftrightarrow ((\forall x(\varphi(x) \wedge B))$$

$$((\exists x\varphi(x)) \wedge B) \leftrightarrow ((\exists x(\varphi(x) \wedge B))$$

Definition 1.11. Let S be a collection of sentences.

- (i) We say that A is provable from S if there exist finitely many $B_1, \dots, B_n : S$ such that $(B_1 \wedge \dots \wedge B_n) \rightarrow A$ is valid.
- (ii) We say that S is consistent if **false** is not valid.

1.2 Models and satisfiability

For the remainder of this section we fix a language \mathcal{L} .

Definition 1.12. An \mathcal{L} -theory is a collection of sentences from $\text{Sentences}(\mathcal{L})$.

Definition 1.13. An \mathcal{L} -structure comprises the following data:

- (i) A carrier type A ,
- (ii) an assignment of every $c : \text{Const}(\mathcal{L})$ to a $c^A : A^{\text{arity}(c)}$,
- (iii) an assignment of every $R : \text{Rel}(\mathcal{L})$ to a subtype $R^A : A^{\text{arity}(R)} \rightarrow \text{Prop}$,
- (iv) an assignment of every $f : \text{Funct}(\mathcal{L})$ to a function $f^A : A^{\text{arity}(f)} \rightarrow A$.

Definition 1.14. Let A be an \mathcal{L} -structure. Using the data of A being an \mathcal{L} -structure, we can inductively assign to every term t (of arity k and containing n free variables) a **realization** $\mathbf{r}(t) : A^n \rightarrow A^k$, as follows:

1. If $t = v$ for a variable v , $\mathbf{r}(t) = \text{id}_A = \lambda v. v$.
2. If $t = c$ for a constant symbol c , $\mathbf{r}(t) = A^0 \xrightarrow{c^A} A$.
3. If $t = (t_1, \dots, t_m)$, then $\mathbf{r}(t) = \mathbf{r}(t_1) \times \dots \times \mathbf{r}(t_m)$.
4. If $t = f(t_0)$ for some function symbol f , then $\mathbf{r}(t) = f^A \circ \mathbf{r}(t_0)$.

Definition 1.15. Let A be an \mathcal{L} -structure. Using the data of A being an \mathcal{L} -structure, we can inductively assign to every formula $\varphi(x_1, \dots, x_n)$ (where x_1, \dots, x_n exhaust the free variables of φ) a **realization** $\mathbf{r}(\varphi) : A^n \rightarrow \text{Prop}$, as follows:

1. If φ is of the form $t_1 = t_2$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ (where symbolic equality is realized as true equality).

¹In particular, using the next rule, if $\neg\varphi(c) \rightarrow \text{false}$ is valid, so is $\exists x\neg\varphi(x) \rightarrow \text{false}$, so is $\neg\exists x\neg\varphi(x)$, and therefore so is $\forall x\varphi(x)$.

2. If φ is of the form $R(t)$, $\mathbf{r}(R(t))$ is $R^A(\mathbf{r}(t))$.
3. If φ is of the form $\neg\psi$, then $\mathbf{r}(\varphi)$ is $\neg\mathbf{r}(\psi)$.
4. If φ is of the form $\psi \vee \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \vee \mathbf{r}(\psi')$.
5. If φ is of the form $\psi \wedge \psi'$, then $\mathbf{r}(\varphi)$ is $\mathbf{r}(\psi) \wedge \mathbf{r}(\psi')$.
6. If φ is of the form $\exists v\psi$, then $\mathbf{r}(\varphi)$ is $\exists v\mathbf{r}(\psi)$.
7. If φ is of the form $\forall v\psi$, then $\mathbf{r}(\varphi)$ is $\forall v\mathbf{r}(\psi)$.

In particular, each sentence φ is sent to a $\mathbf{Prop} \mathbf{r}(\varphi)$.

Definition 1.16. Let A be an \mathcal{L} -structure, and let φ be a sentence. We say that A **satisfies** φ , written

$$A \models \varphi,$$

if $\vdash \mathbf{r}(\varphi)$.

Definition 1.17. Let T be an \mathcal{L} -theory, and let A be an \mathcal{L} -structure. We say that A is a **model** of T if for every sentence $\varphi : T$, $A \models \varphi$.

Example 1.18. (Line graph) The **language of graphs** $\mathcal{L}_{\text{Graph}}$ comprises a single 2-ary relation symbol E .

The **theory of graphs** \mathbf{Graph} comprises the sentence $\forall x \forall y (E(x, y) \leftrightarrow E(y, x))$.

The natural numbers \mathbb{N} can be viewed as a model of \mathbf{Graph} as follows. We realize E as the set

$$(y = \text{succ } x) \vee (x = \text{succ } y) : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbf{Prop}$$

which is clearly symmetric.

Example 1.19. (Peano arithmetic)

The **language of Peano arithmetic** \mathcal{L}_{PA} comprises:

1. A 1-ary constant 0.
2. Three function symbols $\text{succ}, +, \times$.

The **theory of Peano arithmetic** \mathbf{PA} comprises:

1. $\forall x, s(x) \neq 0$
2. $\forall x \forall y, (s(x) = s(y)) \rightarrow x = y$
3. $\forall x, x + 0 = x$
4. $\forall x \forall y, x + s(y) = s(x + y)$
5. $\forall x, x \times 0 = 0$
6. $\forall x \forall y, x \times S(y) = (x \times y) + x$

Schema: For every \mathcal{L}_{PA} -formula $\varphi(x)$ with one free variable x ,

$$(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(\text{succ } x))) \rightarrow \forall x \varphi(x).$$

The **standard model** of PA is \mathbb{N} with 0 realized as $0 : \mathbb{N}$, succ realized as $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$, + realized as $+$: $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$, and \times realized as \times : $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$.

By recursing on the inductive type of valid sentences and replacing every rule of the propositional calculus with the corresponding deduction rule for **Prop**, we can construct for every valid \mathcal{L} -sentence φ a proof that $\mathbf{r}(\varphi) \leftrightarrow \mathbf{true}$.

That is the soundness theorem. (In what follows, taking ψ to be **true** yields the assertion in the previous paragraph.)

Theorem 1.20. (*Soundness theorem*) *For every \mathcal{L} -structure and any sentences $\varphi, \psi : \text{Sentences}(\mathcal{L})$,*

$$\vdash_{\mathcal{L}} \varphi \rightarrow \psi \quad \Longrightarrow \quad \vdash \mathbf{r}(\varphi) \rightarrow \mathbf{r}(\psi).$$

This happens regardless of which \mathcal{L} -structure is doing the realizing. When the \mathcal{L} -structure itself is a model of a theory T , then whenever $T \vdash_{\mathcal{L}} \psi$, then since there is some sentence $\varphi : T$ such that $\vdash_{\mathcal{L}} \varphi \rightarrow \psi$, **Prop**'s modus ponens tells us that the model satisfies ψ also.

Example 1.21. For example, suppose we're working in the language of graphs expanded with two 1-ary constants a and b , and we know that there is some model M such that M satisfies the sole axiom that E is symmetric. We can show

$$\vdash (\forall x \forall y, \mathbf{r}(E)(x, y) \leftrightarrow \mathbf{r}(E)(y, x)) \rightarrow \mathbf{r}(E)(\mathbf{r}(a), \mathbf{r}(b)) \leftrightarrow \mathbf{r}(E)(\mathbf{r}(b), \mathbf{r}(a))$$

because we already know the antecedent and can apply **Prop**'s \forall -elimination.

The converse of Theorem 1.20 is false. There may be some things which are incidentally true about the model which are not universally valid.

Example 1.22. Working again in the language of graphs, consider a complete graph on n vertices. Call this model M . M happens to satisfy the \mathcal{L} -sentence

$$(\forall x \forall y, E(x, y) \leftrightarrow E(y, x)) \rightarrow (\forall x \forall y \forall z, E(x, y) \wedge E(y, z) \rightarrow E(x, z)),$$

but this is not a valid \mathcal{L} -sentence. (Indeed, if it were, then the soundness theorem would imply that *every* graph has a transitive edge relation, which is not true.)

It will turn out that we can do the next best thing. If we rule out this kind of exception by requiring that $M \models \varphi$ for every \mathcal{L} -structure M (resp. every model M of T), then it follows that $\vdash_{\mathcal{L}} \varphi$ (resp. $T \vdash_{\mathcal{L}} \varphi$). This is the completeness theorem.

1.3 The completeness theorem

In this section, our goal will be to prove the **completeness theorem**:

Theorem 1.23. *Let T be an \mathcal{L} -theory. T is consistent if and only if there exists a model of T .*

First we will prove that if there exists a model M of T , then T is consistent.

Proof. Suppose T is inconsistent. If T has a model M , then by the soundness theorem $M \models \mathbf{false}$. By definition, this means that in **Prop**, we have proved that $\mathbf{true} \leftrightarrow \mathbf{false}$, a contradiction. \square

It then remains to show that if T is consistent, T has a model. We will first prove Theorem 1.23 in the case where the language \mathcal{L} of T is *relational*, i.e. has no function symbols. Later, we will show that to every theory \mathcal{L} -theory T we can associate a relational language \mathcal{L}_{rel} and an \mathcal{L}_{rel} -theory T_{rel} by replacing function symbols with their graph relations. Then we will show that if T is consistent, so is T_{rel} , and that every model of T_{rel} gives rise to a model of T , which will give the full completeness theorem.

Before proceeding, we prove a lemma, valid for any consistent theory in any language.

Lemma 1.24. *Suppose T is consistent. Let φ be an \mathcal{L} -sentence. Then $T \cup \{\varphi\}$ is consistent or $T \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. Then there exist sentences σ and ρ from T such that

$$\vdash_{\mathcal{L}}(\sigma \wedge \varphi) \rightarrow \text{false} \quad \text{and} \quad \vdash_{\mathcal{L}}(\rho \wedge \neg\varphi) \rightarrow \text{false}.$$

By \neg -introduction, we get

$$\vdash_{\mathcal{L}} \neg(\sigma \wedge \varphi) \quad \text{and} \quad \vdash_{\mathcal{L}} \neg(\rho \wedge \neg\varphi)$$

and by \wedge -introduction, we get

$$\vdash_{\mathcal{L}} (\neg(\sigma \wedge \varphi)) \wedge (\neg(\rho \wedge \neg\varphi)).$$

Since the finitary de Morgan laws are tautologies in the sense of 1.8, it follows that

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \varphi \vee \rho \vee \neg\varphi).$$

Since the metatheory satisfies the law of the excluded middle, we have that the law of the excluded middle for \mathcal{L} -formulas is a tautology in the sense of 1.8. Therefore,

$$\vdash_{\mathcal{L}} \neg(\sigma \vee \rho) \Leftrightarrow \vdash_{\mathcal{L}} \neg\sigma \wedge \neg\rho,$$

so by \wedge -elimination, $\vdash_{\mathcal{L}} \neg\sigma$ and $\vdash_{\mathcal{L}} \neg\rho$, so T is inconsistent. \square

Theorem 1.25. *Suppose that S is a relational theory, containing no quantifiers, and which is consistent. Then S has a model.*

Proof. We start by choosing a well-ordering of S , which induces a well-ordering of the constant and relation symbols which appear in S . In turn, this induces a lexicographic ordering on all sentences of the form $c_i = c_j$ and $R_\beta(c_1, \dots, c_n)$ where c_i, c_j and R_β are constant and relation symbols occurring in S . Collect these sentences into a single well-ordered set (F_α) .

Now, we inductively decide whether the F_α should be true or false consistent with S . We put $G_0 \stackrel{\text{df}}{=} F_0$ if $S \cup \{F_0\}$ is consistent; otherwise we put $G_0 \stackrel{\text{df}}{=} \neg F_0$. Similarly, for $\beta > 0$ we put $G_\beta \stackrel{\text{df}}{=} F_\beta$ if $S \cup \{G_\alpha \mid \alpha < \beta\} \cup \{F_\beta\}$ is consistent, and we put $G_\beta \stackrel{\text{df}}{=} \neg F_\beta$ otherwise.

From the previous lemma, at each stage β of this construction, $S_\beta \stackrel{\text{df}}{=} S \cup \{G_\alpha \mid \alpha < \beta\}$ is consistent. Since any inconsistency is derivable from finitely many other sentences, the union

$$H \stackrel{\text{df}}{=} \bigcup_{\beta} S_\beta$$

is consistent.

Now, there is a natural equivalence relation on the collection \mathcal{C} of all constant symbols which occur in H , given by

$$c \sim_{\mathcal{C}} c' \iff c = c' : H.$$

Since \mathcal{C} is well-ordered, we may pick the least element of each $\sim_{\mathcal{C}}$ -class, and collect them as \mathcal{C}' . We will make \mathcal{C}' into a model of H . First, we realize every constant symbol c as the chosen least representative of its $\sim_{\mathcal{C}}$ -class.

For every (n -ary) relation symbol R_β , we realize R_β by putting

$$R_\beta^{\mathcal{C}}(c_1, \dots, c_n) \leftrightarrow (R(c_1, \dots, c_n) : H).$$

This is a model of H : for any atomic or negated-atomic sentence σ which is consistent with S , σ must arise as some G_α and therefore by construction is satisfied by M . Since S is quantifier-free, every sentence splits into a disjunction of conjunctions of atomic or negated-atomic sentences, so by induction every quantifier-free sentence consistent with S is again satisfied by M . \square

Theorem 1.26. *Now suppose that S is a relational theory, which possibly contains quantifiers, and is consistent. Then S has a model.*

Proof. \square

Proposition 1.27. *Let T be an \mathcal{L} -theory, and let T_{rel} be the associated \mathcal{L}_{rel} -theory obtained by replacing function symbols with their graphs. Then any model $M_{\text{rel}} \models T_{\text{rel}}$ can be viewed as a model $M \models T$.*

Sketch. For every function symbol f of \mathcal{L} , we interpret f as the function specified by the graph relation Γ_f in \mathcal{L}_{rel} , which was axiomatized in T_{rel} to be the graph of a function. This gives an \mathcal{L} -structure M . Since $M \models T_{\text{rel}}$ and every sentence of T_{rel} is either a modified version of a sentence in T or asserts that a new relation is a graph of a function, $M \models T$. \square

Proposition 1.28. *Let T be an \mathcal{L} -theory. If T is consistent, then T_{rel} is consistent.*

Sketch. Suppose towards the contrapositive that T_{rel} is inconsistent. Then there is a proof from T_{rel} of false. It suffices to show that replacing the graphs Γ_f by the functions f induces a deduction-preserving map from the valid \mathcal{L}_{rel} -sentences to the valid \mathcal{L} -sentences, for then we will have a proof from T of false. This can be done by induction and a case-by-case analysis of the rules of deduction. \square

1.4 The Henkin construction

In this section, we present an alternate proof of the completeness theorem, due to Henkin, by building “term models”.

TODO(jesse)

1.5 The Löwenheim-Skolem theorem

TODO(kody)

- 2 ZFC
- 3 Generic sets: Cohen's original proof
- 4 Boolean-valued models
- 5 Sheaves and filterquotients