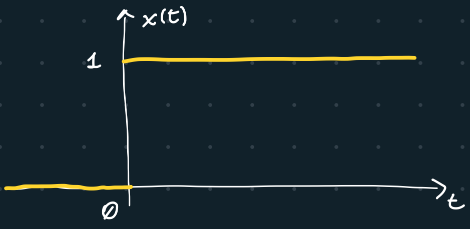
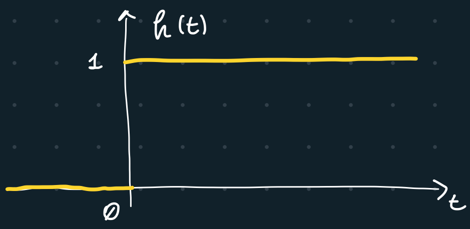




Abbiamo $x(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

e $h(t) = u(t)$

$y(t) = x(t) * h(t)$

Metodo 1.

a) Sostituire $t = \tau$

$\Rightarrow \quad x(t) \to x(\tau) \qquad h(t) \to h(\tau)$

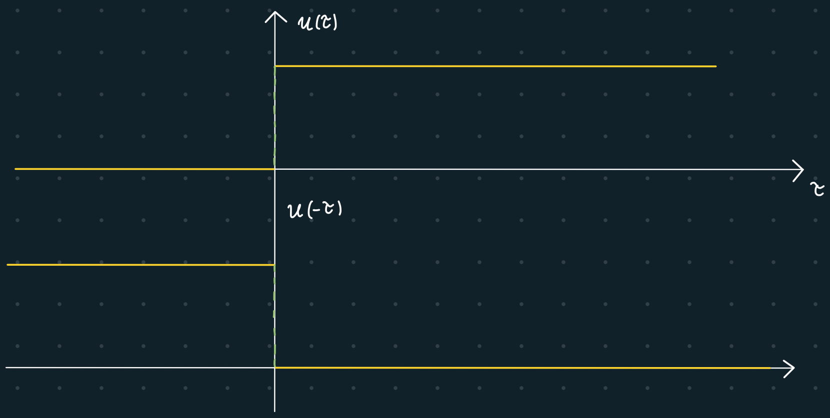
b) Flip di $h(\tau) \Rightarrow \quad h(\tau) \to h(-\tau)$

c) Operazione di time Shifting : $h(t-\tau) \to h[-(\tau - t)]$

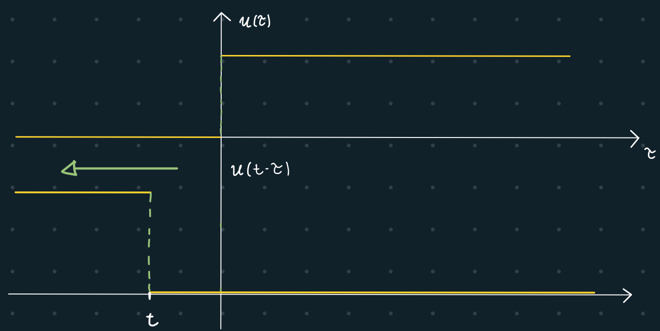
d) $x(\tau) \cdot h(t-\tau)$

e) $\displaystyle\int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau$

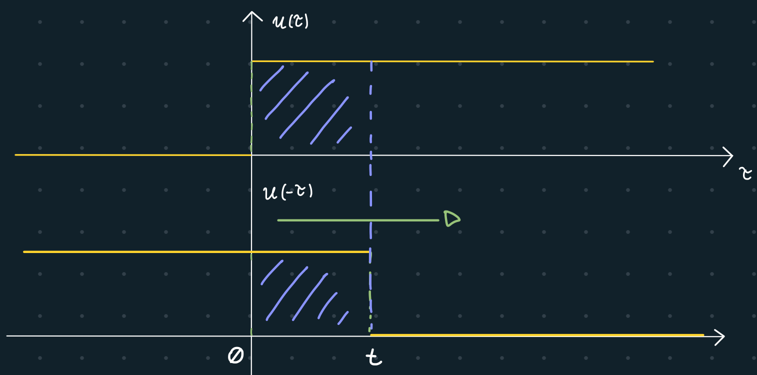
loop



← Step b : Sostituzione + Flip



Caso 1: $t < 0 \to y(t) = \emptyset$



Caso 2: $t > 0$

$y(t) = \text{conv}(x(t), h(t))$

$\displaystyle \Rightarrow \int_0^t u(\tau) \cdot u(t-\tau)\, d\tau = \int_0^t d\tau = t$

$\Rightarrow y(t) = \begin{cases} t & \text{per } t \geq 0 \\ 0 & \text{per } t < 0 \end{cases} = \tau_{amp}(t) = \tau(t)$

Metodo 2:

moltiplicazione

$$y(t) = x(t) * h(t) \rightarrow y(t) = u(t) * u(t) \Longleftrightarrow Y(f) = \mathcal{F}(u(t)) \cdot \mathcal{F}(u(t))$$

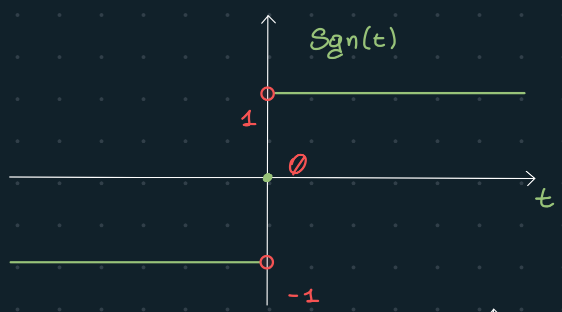
$$\mathcal{F}(u(t)) = \int_{-\infty}^{+\infty} u(t) \cdot e^{-j2\pi f t}\, dt = \int_{0}^{+\infty} e^{-j2\pi f t}\, dt = -\frac{1}{j2\pi f} \cdot e^{-j2\pi f t}\Big|_{0}^{+\infty} = -\frac{1}{j2\pi f}\left[+\infty - 1\right]$$

Non Integrabile!

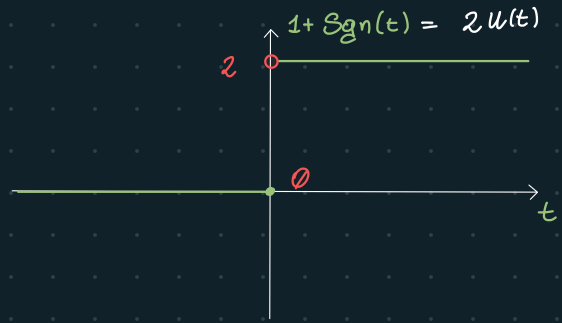
Infatti $\int_{-\infty}^{+\infty} |u(t)|\, dt = \int_{-\infty}^{+\infty} u(t)\, dt = +\infty$ non ASSOLUTAMENTE integrabile.

→ Per calcolare la $\mathcal{F}(u(t))$ dobbiamo rendere $u(t)$ convergente

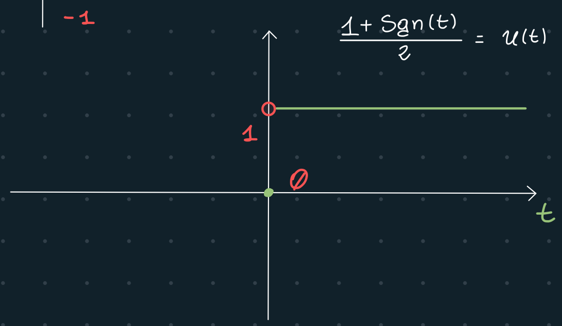
→ Sappiamo che $\operatorname{sgn}(t) \Longleftrightarrow \frac{2j}{\omega}$ → Se riusciamo a scrivere $u(t)$ in termini di $\operatorname{sgn}(t)$ possiamo Trasformare!



Amplitude Shifting



Amplitude Shift $\frac{1+\operatorname{sgn}(t)}{2}$



$$\Rightarrow u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

⇒ A questo punto possiamo Trasformare:

$$\mathcal{F}(u(t)) = \underbrace{\mathcal{F}\left(\tfrac{1}{2}\right)}_{2\pi \cdot \frac{1}{2} \cdot \delta(\omega)} + \mathcal{F}\left(\tfrac{1}{2}\operatorname{sgn}(t)\right) = \cancel{2}\pi \cdot \frac{1}{\cancel{2}}\delta(\omega) + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{j\omega} = \boxed{\pi\delta(\omega) + \frac{1}{j\omega}}$$

$\mathcal{F}(u(t))$

$$\Rightarrow u(t) \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$Y(\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\cdot\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \pi^2\delta^2(\omega) + \frac{2\pi\delta(\omega)}{j\omega} + \frac{1}{j^2\omega^2}$$

$Y(f)$

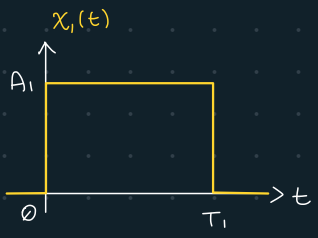
$$y(t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi^2\delta^2(\omega)\, d\omega}_{\frac{1}{2}} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \cdot \frac{\delta(\omega)}{j\omega} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{j^2\omega^2}\, d\omega$$

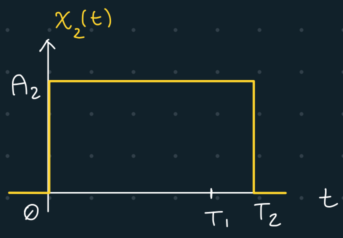
$$= \frac{1}{2} + \int_{-\infty}^{+\infty} \delta(\omega) \cdot \frac{1}{j\omega}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{j^2\omega^2}\, d\omega$$

Tips & Tricks *per la convoluzione*

- La conv di due segnali rettangolari di DURATA UGUALE sarà Triangolare
- La conv di due segnali rettangolari di DURATA DIVERSA sarà Trapezoidale

Caso 1: Same len



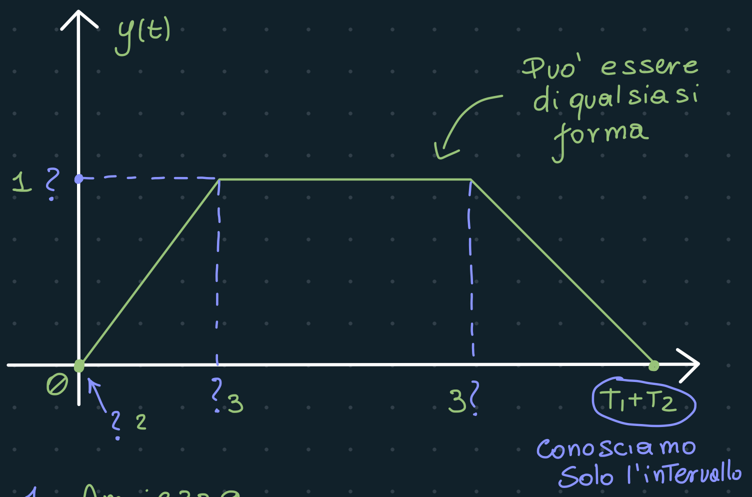


$T_1 < T_2$

→ Sia $y(t)$ il segnale risultato;

→ $x_1(t)$ con $0 \leq t \leq T_1$
$\quad x_2(t)$ con $0 \leq t \leq T_2$ con $T_2 > T_1$

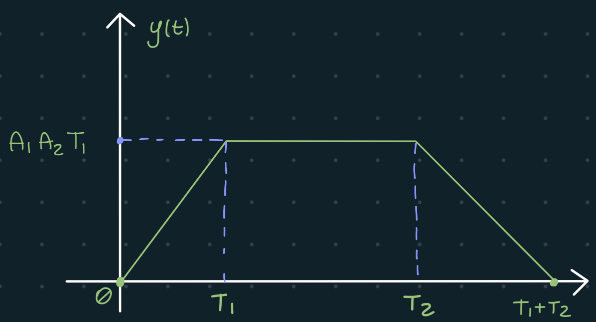
→ $x_1(t) * x_2(t) = y(t)$ con $0 \leq t \leq T_1 + T_2$



1. Ampiezza
   Dati $A_0 x_1(t) * A_1 x_2(t) = A_0 A_1 T_1 \, Conv(x_1 x_2)$

2. Punto di partenza
   E' Dato dalla proprietà 10 di Durata della conv

3. Intervallo costante
   Sappiamo che il primo punto è T_1
   ⇒ $2°$ punto $= \cancel{T_1} + T_2 - \cancel{T_1} = T_2$

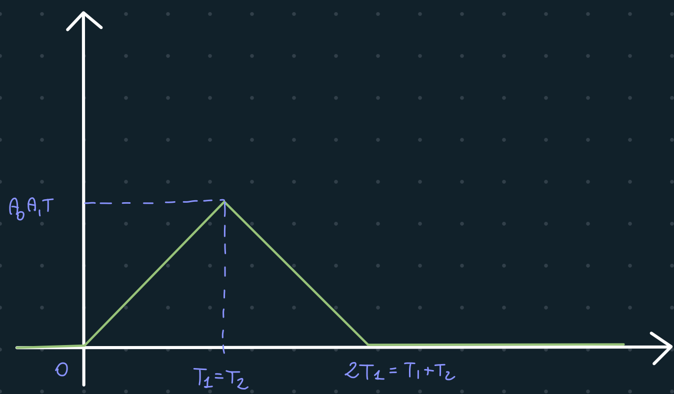


Caso 2: Diff. len.

→ Cosa ci aspettiamo che accada?

Se $T_1 = T_2$

avremo un Trapezio dove T_1 e T_2 coincidono → Non è più un trapezio:



Problema 2:

$x(t+5) * \delta(t-7) = ?$

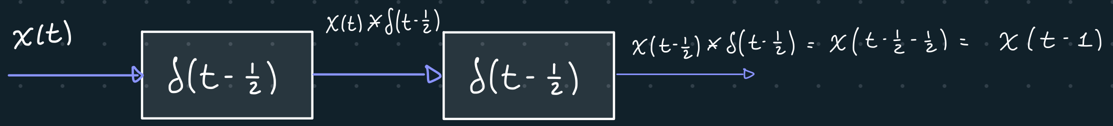
Proprietà 4 della Conv

$\rightarrow x(t) * \delta(t-k) = x(t-k)$

$\Rightarrow x(t+5-7) = x(t-2)$ Ans.

Prob 3:

$h(t) = \delta(t-0.5)$ Se due sistemi del genere vengono posti in cascata, Quale sarà l'impulse resp del sys risultante?



Se $y(t) = x(t-1)$ e $x(t)$ $\Rightarrow$ $x(t) * h_0(t-k) = x(t-1)$

$\Rightarrow h_0(t) = \delta(t-1)$ Ans.

MA Possiamo usare la proprietà: II) Associativa

$\rightarrow (x_1(t) * x_2(t)) * x_3(t) = x_1(t) * (x_2(t) * x_3(t))$

$\rightarrow$ In Termini del nostro problema: $(x(t) * h(t)) * h(t) = x(t) * (h(t) * h(t))$

$\Rightarrow h_{eq}(t) = \delta(t-\frac{1}{2}) * \delta(t-\frac{1}{2}) = \delta(t-\frac{1}{2}-\frac{1}{2}) = \delta(t-1)$ Ans.

Prob Bonus:

Un Sys LTI ha R.I. $h(t-\tau)$; che succede se $x(t) = x(t-\tau)$?

$x(t-\tau) * h(t-\tau) = y[t-(\tau+\tau)] = y(t-2\tau)$ Ans.

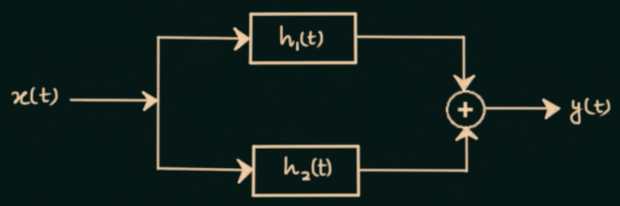
Prop Time Inv.

$x(t-T_0) * x(t-T_1) = y[t-(T_0+T_1)]$

come se fosse una messa in evidenza

Prob 4:

Consider the parallel combination of two LTI systems shown in the figure.



The impulse responses of the systems are

$h_1(t) = 2\delta(t+2) - 3\delta(t+1)$

$h_2(t) = \delta(t-2)$

If the input x(t) is a unit step signal, then the energy of y(t) is ______

Se $x(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

e $h_1(t) = 2\delta(t+2) - 3\delta(t+1)$

$h_2(t) = \delta(t-2)$

$\mathcal{E}_y = ?$

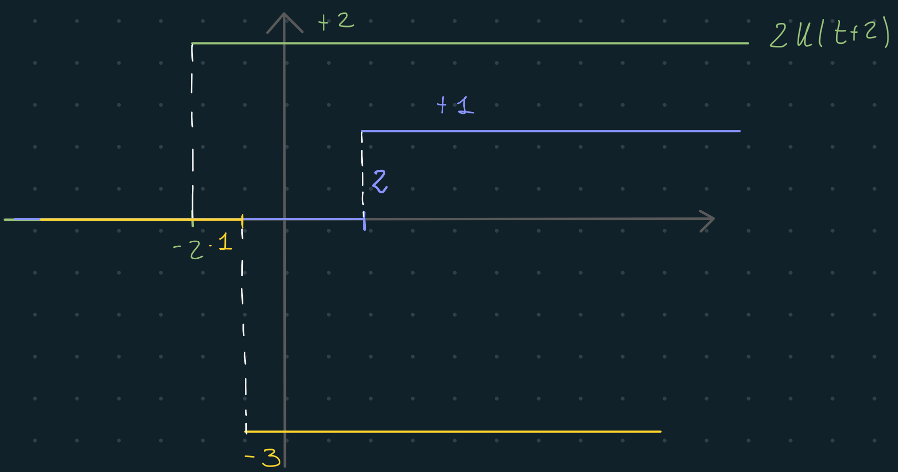
Sys: $\big(x(t) * h_1(t)\big) + \big(x(t) * h_2(t)\big) = y(t)$

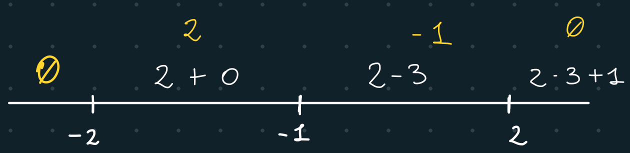
$\rightarrow$ Prop 3 Distributiva $\rightarrow$ Eq Sys: $x(t) * \big(h_1(t) + h_2(t)\big) = y(t)$

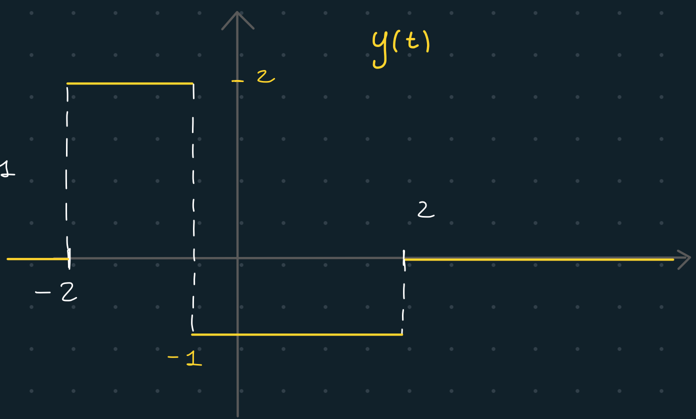
$\Rightarrow h_e(t) = h_1(t) + h_2(t) = 2\delta(t+2) - 3\delta(t+1) + \delta(t-2)$

$\rightarrow y(t) = 2u(t) * \delta(t+2) - 3u(t) * \delta(t+1) + u(t) * \delta(t-2)$

$= 2u(t+2) - 3u(t-1) + u(t-2)$



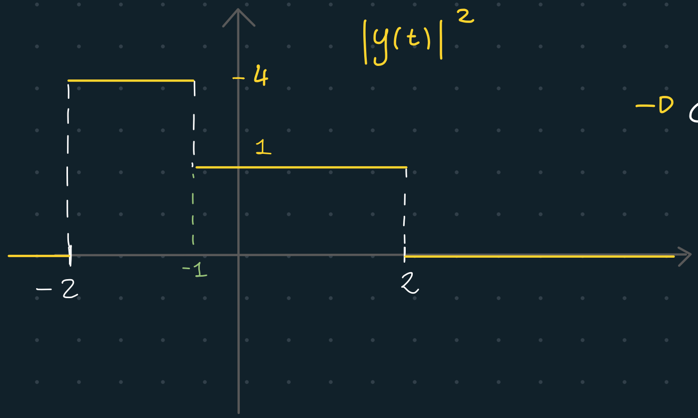




$\rightarrow \mathcal{E}_y = ?$

$$\mathcal{E}_y = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

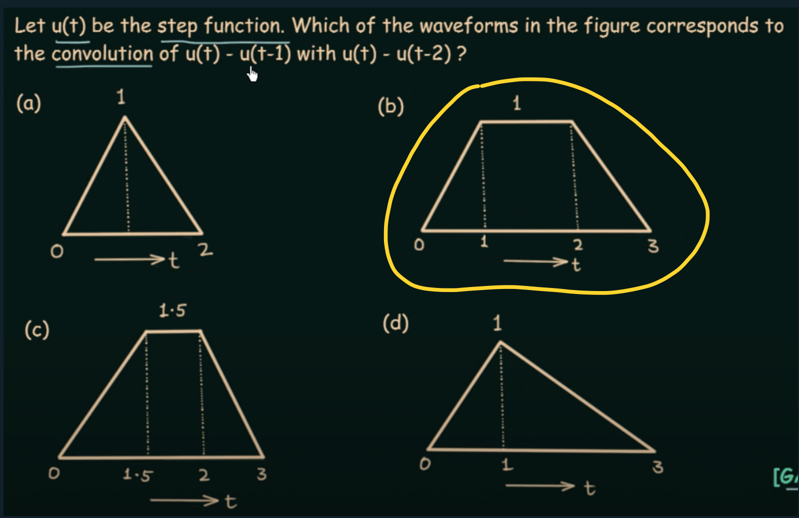
$$|y(t)|^2 = \begin{cases} 0 & \text{per } t < -2 \cup t > 2 \\ 2^2 & \text{per } -2 \leq t \leq -1 \\ (-1)^2 & \text{per } -1 \leq t \leq 2 \end{cases}$$



$$\rightarrow \mathcal{E}_y = \cancel{\int_{-\infty}^{-2} 0 \, dt} + 4\int_{-2}^{-1} dt + \int_{-1}^{2} dt + \cancel{\int_{2}^{+\infty} 0 \, dt}$$

$$= 4 \left[t \right]_{-2}^{-1} + \left[t \right]_{-1}^{2} = 4(-1+2) + 2+1 = 4+3 = 7$$

Prob 5:



$$\underbrace{\left(u(t) - u(t-1)\right)}_{\text{Rect 1}} * \underbrace{\left(u(t) - u(t-2)\right)}_{\text{Rect 2}}$$

$\Rightarrow$ · $\text{Rect}_1 = \Pi\left(\frac{t - 1}{1}\right)^{T_1}$

· $\text{Rect}_2 = \Pi\left(\frac{t - 2}{2}\right)^{T_2}$

$t \neq 0$ per

$x_1(t) \rightarrow \quad 0 < t < 1$

$x_2(t) \rightarrow \quad 0 < t < 2$

- Starting point

  $y(t) \neq 0$ per $0 < t < 3$

- Ampiezza Max

  $\rightarrow A_0 = A_1 \, A_2 \cdot T_1 = 1$

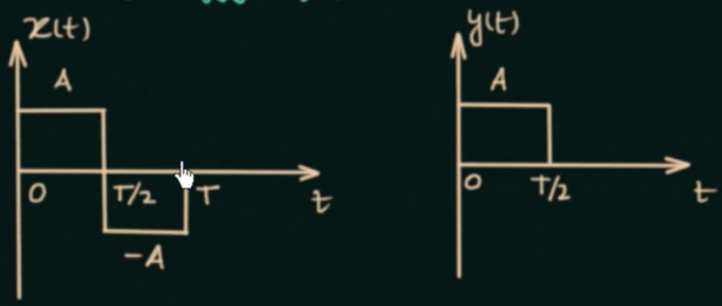
- Il segnale è costante da a...

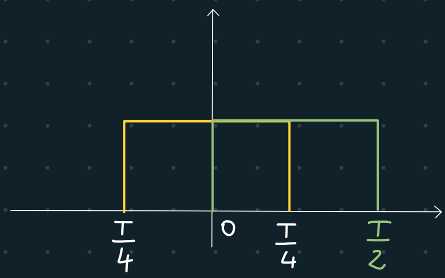
  $P_1 = T_1 = 1 \qquad P_2 = T_2 + T_1 - T_1 = T_2 = 2$

$\Rightarrow$ Il segnale $y(t)$ e' il (b)

Prob 6:

Question: Find $z(t) = x(t) * Y(t)$





$$\frac{T}{2} + \frac{T}{4} = \frac{2T+T}{4} = \frac{3}{4}T$$

• Decomponiamo il segnale $x(t)$

$$x(t) = Rect_1 + Rect_2 \quad \begin{cases} Rect_1 = \Pi\left(\dfrac{t - \frac{T}{4}}{\frac{T}{2}}\right) \\ Rect_2 = -\Pi\left(\dfrac{t - \frac{3}{4}T}{\frac{T}{2}}\right) \end{cases} \quad x(t) = \Pi\left(\frac{4t - T}{\underset{T_0}{2T}}\right) - \Pi\left(\frac{4t-3T}{\underset{T_1}{2T}}\right) \qquad T_0 = T_1$$

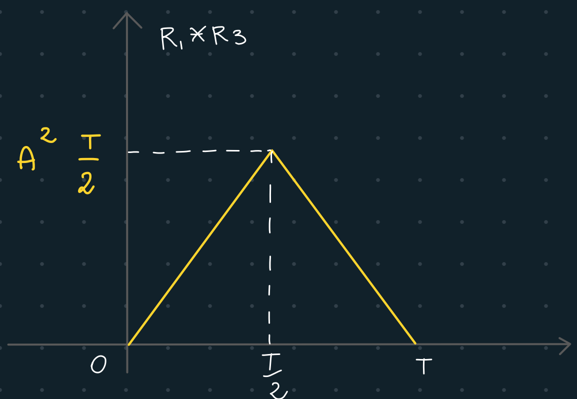
$$\Rightarrow y(t) = \underbrace{\left[\Pi\left(\frac{4t-T}{2T}\right) * \Pi\left(\frac{4t-T}{2T}\right)\right]}_{\text{Triangolo}}^{(a)} - \underbrace{\left[\Pi\left(\frac{4t-3T}{2T}\right) * \Pi\left(\frac{4t-T}{2T}\right)\right]}_{\text{TRIANGOLO}}^{(b)}$$

a) $Rect_1 \neq 0 \rightarrow 0 \leq t \leq \frac{T}{2}$ $\quad \Rightarrow Conv(R_1, R_2) \neq 0 \rightarrow \underline{0 \leq t \leq T}$

$Rect_3 \neq 0 \rightarrow 0 \leq t \leq \frac{T}{2}$

$\rightarrow T_0 = T_1 \Rightarrow$ Max in $t = T_0 = T_1 = \frac{T}{2}$

$\rightarrow$ Ampl $= A_0 \, A_1 \, T_0 = \frac{T}{2} A^2$



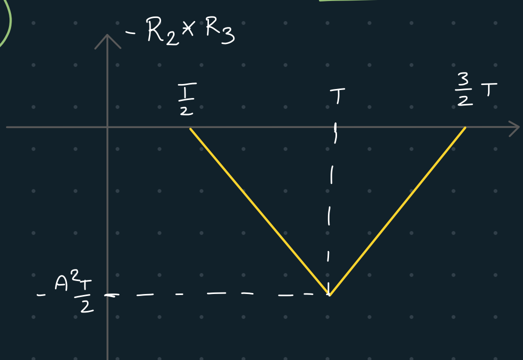
b) $Rect_2 \neq 0 \rightarrow \frac{T}{2} \leq t \leq T$

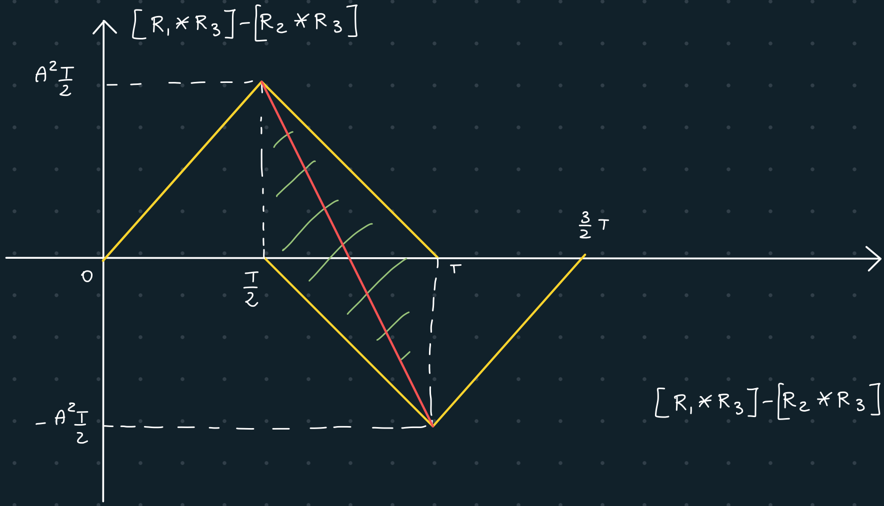
$Rect_3 \neq 0 \rightarrow 0 \leq t \leq \frac{T}{2}$

$\Rightarrow Conv(R_2, R_3) \neq 0 \rightarrow \underline{\frac{T}{2} < t < \frac{3}{2}T}$

• Ampl : $A^2 \frac{T}{2}$

• Max = $\frac{3}{2}T - \frac{1}{2}T = T$ Max





$$[R_1 * R_3]-[R_2 * R_3] = \begin{cases} t & 0 \leq t \leq \frac{T}{2} \\ [-t-T]+\left[-t-\frac{T}{2}\right] & \frac{T}{2} < t < T \\ t-\frac{3}{2}T & T < t < \frac{3}{2}T \\ 0 & \text{otw} \end{cases}$$

- $-t-T-t-\frac{T}{2} = -2t-\frac{3}{2}T = -\left(\frac{4t+3T}{2}\right)$

$$\Rightarrow y(t) = \begin{cases} t & 0 \leq t \leq \frac{T}{2} \\ -\left(2t+\frac{3}{2}T\right) & \frac{T}{2} \leq t \leq T \\ t-\frac{3}{2}T & T \leq t \leq \frac{3}{2}T \\ 0 & \text{Altrimenti} \end{cases}$$