
Le3 29

Derivazione nel dominio della frequenza (proprietà)

$$Y(f) = \frac{d}{df} X(f) = \frac{d}{df} \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t} \, dt = \int_{-\infty}^{+\infty} x(t) \cdot \frac{d}{df} e^{-j2\pi f t} \, dt$$

$$\hookrightarrow -j2\pi t \, e^{-j2\pi f t}$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot (-j2\pi t) \, e^{-j2\pi f t} \, dt = +j2\pi \int_{-\infty}^{+\infty} x(t) \cdot (-t) \cdot e^{-j2\pi f t} \, dt$$

$$S(t) = -t\,x(t)$$

$$= j2\pi \int_{-\infty}^{+\infty} S(t) \cdot e^{-j2\pi f t} \, dt$$

$$\mathcal{F}\left(-t \cdot x(t)\right)$$

$$\rightarrow j2\pi \cdot \left(-t \cdot x(t)\right) = \frac{d}{df} X(f) \rightarrow -t\,x(t) = \frac{d X(f)}{j2\pi \cdot df}$$

$$\rightarrow t \cdot x(t) = -\frac{1}{j} \cdot \frac{d X(f)}{2\pi \, df} \Rightarrow \boxed{t\,x(t) = \frac{j\,d X(f)}{2\pi \, df} = \frac{j}{2\pi} \cdot \frac{d}{df} X(f)}$$

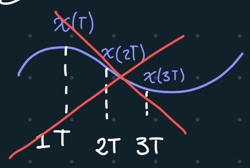
$$\hookrightarrow -\frac{1}{j} \cdot \frac{j}{j} = -\frac{j}{j^2} = -\frac{j}{-1} = j$$

Replicazione e campionamento

Non periodico!

Tempo Continuo possiamo usare come segnale generatore un qualsiasi segnale

$$\tilde{x}(t) = rep_T\big[\underbrace{x(t)}_{\text{S. Generatore}}\big] = \sum_{K=-\infty}^{+\infty} x(t - K\underset{\text{Periodo}}{T})$$



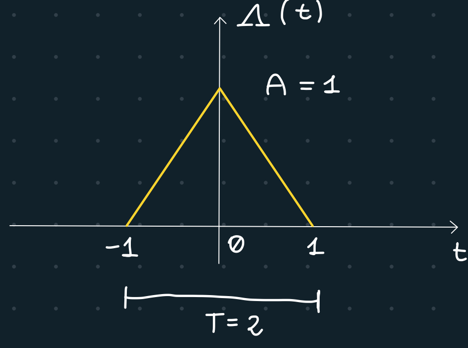
$\cancel{rep[x(t)] = [x(t), x(2T), x(KT)]}$ sbagliato

Tempo Discreto

$$\tilde{x}(n) = rep_N\big[x(n)\big] = \sum_{K=-\infty}^{+\infty} x(n - K\overset{\text{Periodo}}{N})$$

Esempio di Replicazione a tempo continuo

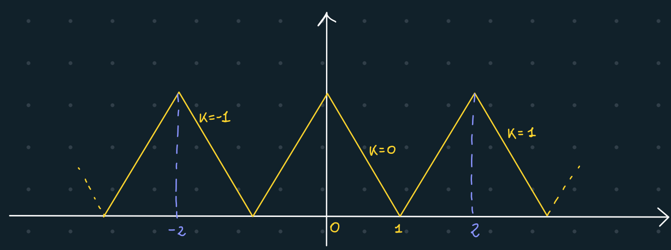
$$x(t) = \Lambda\left(\frac{t}{T}\right) = \Lambda(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & otw \end{cases}$$



$$\tilde{S}(t) = rep_T\big[x(t)\big] = \sum_{K=-\infty}^{+\infty} \Lambda(t - TK)$$

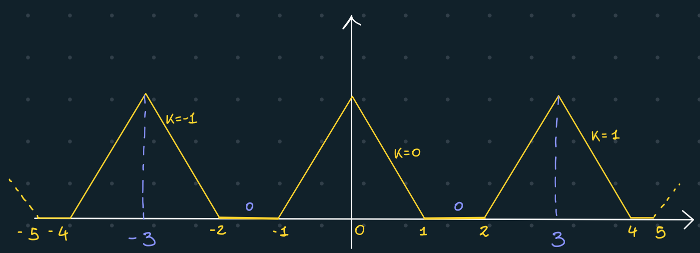
Scegliamo $T = 2 \geq$ Periodo della finestra

$K=0 \to \Lambda(t)$
$K=1 \to \Lambda(t-2)$
$K=2 \to \Lambda(t-4)$ } $\to$ Shift Right $T_0 > 0$
$K=-1 \to \Lambda(t+2)$
$K=-2 \to \Lambda(t+4)$ } $\to$ Shift Left $T_0 < 0$

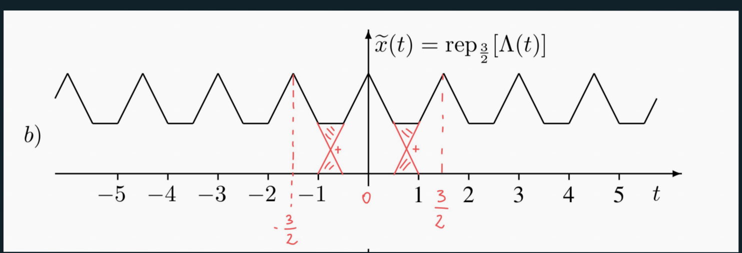


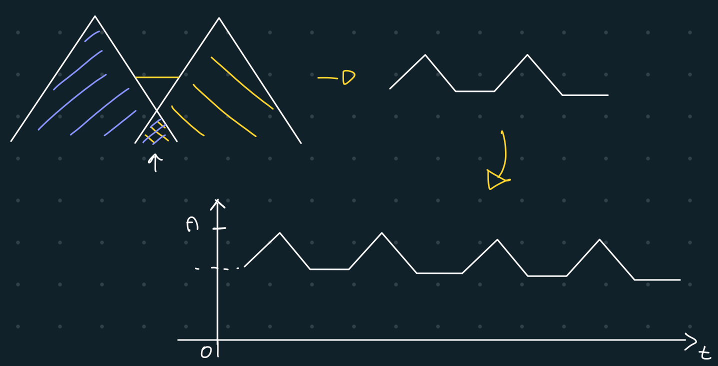
Scegliamo $T = 3 \geq$ Periodo della finestra

$K=0 \to \Lambda(t)$
$K=1 \to \Lambda(t-3)$
$K=2 \to \Lambda(t-6)$
$K=-1 \to \Lambda(t+3)$
$K=-2 \to \Lambda(t+6)$



Scegliamo $T = \frac{3}{2} \leq$ Periodo della finestra





Quanto vale la trasformata di Fourier del Segnale replicato?

$$\tilde{x}(t) = rep_T[x(t)] = \sum_{K=-\infty}^{+\infty} x(t-KT)$$

1) Ci riscriviamo $\quad x(t) * \delta(t-KT) = x(t-KT) \cdot \underbrace{\delta(t-KT)}_{1}$

$$\rightarrow \tilde{x}(t) = \sum_{K=-\infty}^{+\infty} x(t) * \delta(t-KT)$$

2) Sfruttiamo la proprieta' Distributiva

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$

$$\rightarrow x(t) * \underbrace{\sum_{K=-\infty}^{+\infty} \delta(t-KT)}_{\tilde{\delta}(t)}$$

Treno Campionatore

3) Treno campionatore in frequenza

$$\tilde{\delta}_T(t) = \sum_{K=-\infty}^{+\infty} \delta(t-K) \rightleftharpoons \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta\left(f - \frac{m}{T}\right) = \tilde{\delta}_T(f)$$

Treno Campionatore in frequenza

4) Proprieta' della Trasformata Ad una conv. nel t corrisponde una mul nella freq:

$$x(t) * y(t) \rightleftharpoons X(f) \cdot Y(f)$$

$$\rightarrow \tilde{x}(t) = x(t) * \sum_{K=-\infty}^{+\infty} \rightleftharpoons X(f) \cdot \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta\left(f - \frac{m}{T}\right) = \tilde{X}(f)$$

$\rightarrow$ Portiamo tutti i membri "dentro" $\rightarrow \displaystyle\sum_{m=-\infty}^{+\infty} \frac{1}{T} X(f) \cdot \delta\left(f - \frac{m}{T}\right)$

$\underbrace{\text{Prop Delta} \rightarrow X\left(\frac{m}{T}\right) \cdot \delta\left(f - \frac{m}{T}\right)}_{\text{Proprieta' del campionamento della Delta}}$

5) Morale della favola

- $\underbrace{\tilde{x}(t) = x(t) * \sum_{K=-\infty}^{+\infty} x(t-KT)}_{\text{Replicazione nel Tempo}} \rightleftharpoons \underbrace{\sum_{m=-\infty}^{+\infty} \frac{1}{T} \cdot X\left(\frac{m}{T}\right) \cdot \delta\left(f - \frac{m}{T}\right)}_{\text{campionamento in frequenza}} = \tilde{X}(f)$ Tempo continuo

- $\tilde{x}(n) = x(n) * \displaystyle\sum_{m=-\infty}^{+\infty} x(n-mT) \rightleftharpoons \sum_{m=-\infty}^{+\infty} \frac{1}{N} \cdot X\left(\frac{m}{N}\right) \cdot \delta\left(f - \frac{m}{N}\right)$ Tempo Discreto

Periodo di riproduzione

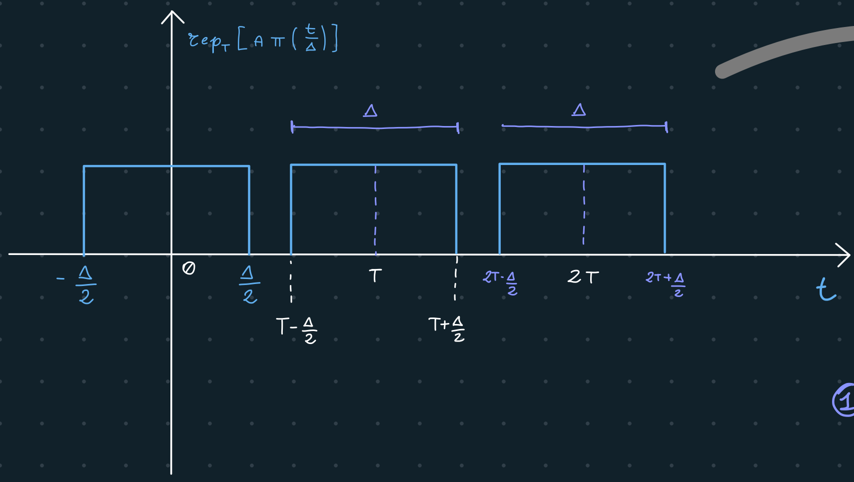
Periodo di Campionamento

$$\boxed{\text{Replicazione} \rightleftharpoons \text{Campionamento}}$$

Tempo $\quad$ Frequenza

Esempio: Treno di impulsi rettangolari

$$\tilde{x}(t) = rep_T\left[A\,\Pi\left(\frac{t}{\Delta}\right)\right], \quad T > \Delta$$



Dominio della freq?

1) Proprieta' rep/comp.

$$\rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} x(t-kT)$$

periodo di rep

$$\Updownarrow$$

$$① \rightarrow \tilde{X}(f) = \sum_{m=-\infty}^{+\infty} \frac{1}{T}\, X\left(\frac{m}{T}\right) \cdot \delta\left(f - \frac{m}{T}\right)$$

?

$f = \frac{m}{T}$

$$A\cdot\Pi\left(\frac{t}{T}\right) \overset{F.T.}{\rightleftharpoons} A\cdot T\,\mathrm{Sinc}(fT) \longrightarrow A\cdot\Pi\left(\frac{t}{\Delta}\right) \rightleftharpoons A\cdot\Delta\,\mathrm{Sinc}(f\Delta)$$

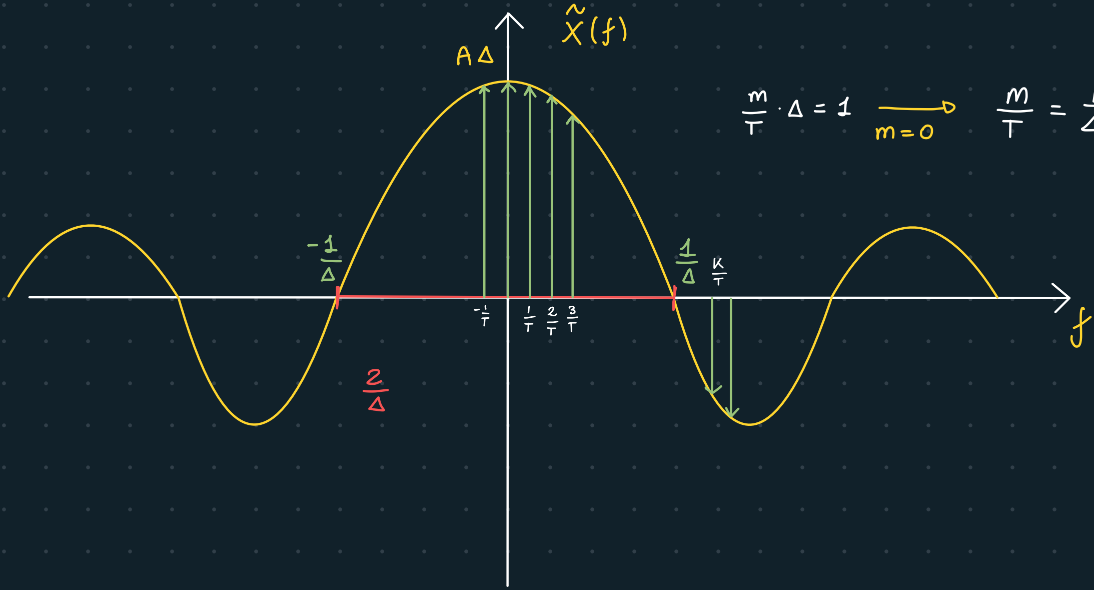
→ Sostituiamo nella ①

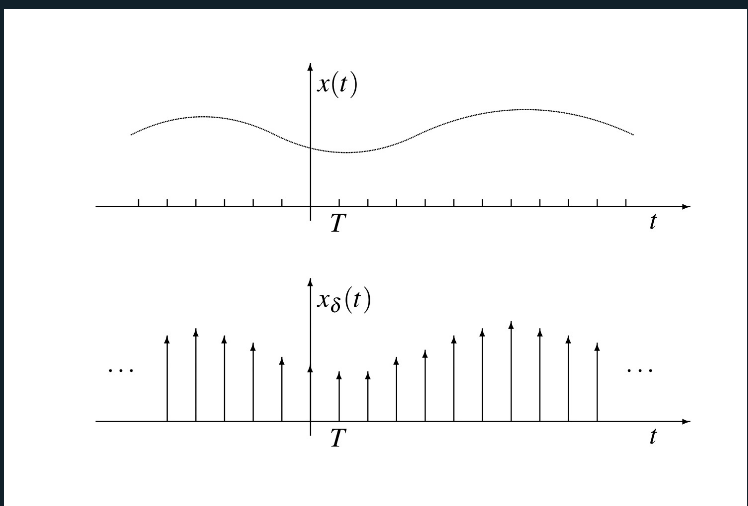
$$\tilde{X}(f) = \sum_{m=-\infty}^{+\infty} \underbrace{\frac{1}{T}}_{\text{campionato con freq } \frac{1}{T}} \underbrace{A\Delta\,\mathrm{Sinc}\left(\frac{m}{T}\cdot\Delta\right)}_{\text{Segnale sinc}} \cdot \underbrace{\delta\left(f-\frac{m}{T}\right)}_{\text{prop. del campionamento}}$$

Ans

Graficchiamo

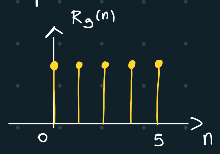
$$\frac{m}{T}\cdot\Delta = 1 \xrightarrow[m=0]{} \frac{m}{T} = \frac{1}{\Delta}$$





Esempio: Rep. Seq Rett

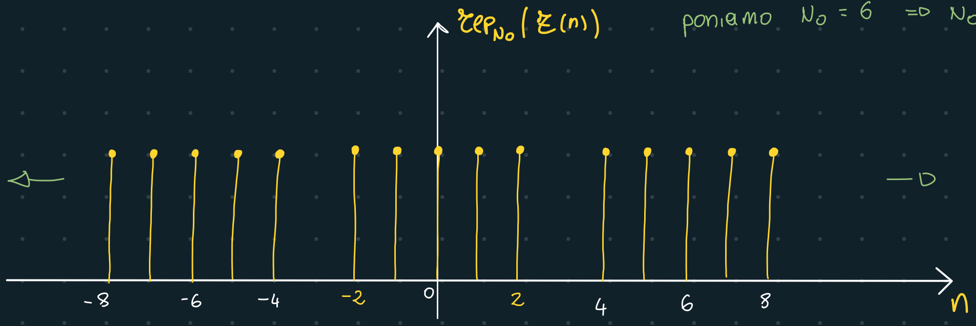
$R_N(n) = R_5(n)$



$\rightarrow$ Traccia: $\tilde{z}(n) = rep_{N_0}[z(n)] = rep_{N_0}[R_5(n+2)]$ -2 !

question $\rightarrow$ $\tilde{Z}(\nu) = ?$

poniamo $N_0 = 6 \Rightarrow N_0 > T_R$



0) $R_N(n) \rightleftharpoons e^{-j\pi\nu(N-1)} \cdot \dfrac{\sin(\pi\nu N)}{\sin(\pi\nu)}$

1) $R_5(n) \rightleftharpoons e^{-j\pi\nu 4} \cdot \dfrac{\sin(5\pi\nu)}{\sin(\pi\nu)}$

2) $x(n-T_0) \rightleftharpoons \tilde{X}(f) \cdot e^{-j2\pi\nu T_0} \xrightarrow{T_0 = -2} R_5(n+2) = e^{-j\pi\nu 4} \cdot \dfrac{\sin(5\pi\nu)}{\sin(\pi\nu)} \cdot e^{-j2\pi\nu(-2)}$

$\rightarrow R_5(n+2) \overset{F.T.}{\rightleftharpoons} \cancel{e^{-j\pi\nu 4}} \cdot \dfrac{\sin(5\pi\nu)}{\sin(\pi\nu)} \cdot \cancel{e^{j\pi\nu 4}}$

$\rightarrow R_5(n+2) \rightleftharpoons \dfrac{\sin(5\pi\nu)}{\sin(\pi\nu)} = \tilde{X}(f)$

Trasformata della rep: $rep_{N_0}[x(t)] \rightleftharpoons \displaystyle\sum_{m=-\infty}^{+\infty} \frac{1}{T} \cdot X\left(\frac{m}{T}\right) \cdot \delta\left(\nu - \frac{m}{T}\right)$

Sostituire

$\rightarrow rep_{N_0}[R_5(n+2)] \rightleftharpoons \displaystyle\sum_{m=-\infty}^{+\infty} \frac{1}{N_0} \frac{\sin\left(5\pi \frac{m}{N_0}\right)}{\sin\left(\pi \frac{m}{N_0}\right)} \cdot \delta\left(\nu - \frac{m}{N_0}\right)$

una sorta di sinc

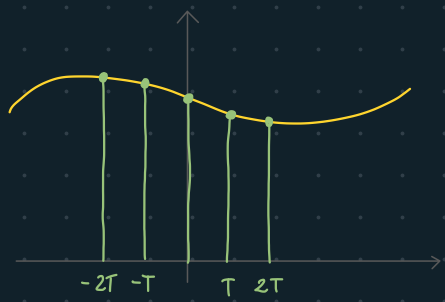
$\rightarrow$ Dimostrazione Matlab a 1:15

Esempio inverso: Segnale campionato $\rightarrow$ Replicazione

Tempo Continuo

$$x_\delta(t) = \sum_{K=-\infty}^{+\infty} x(KT) \cdot \delta(t-KT)$$

Periodo



$$x(KT)\cdot\delta(t-KT) \leftarrow x(t)\cdot\delta(t-KT)$$

$$\rightarrow x_\delta(t) = \sum_{K=-\infty}^{+\infty} x(t)\, \delta(t-KT)$$

Const

$$\rightarrow x_\delta(t) = x(t) \underbrace{\sum_{K=-\infty}^{+\infty} \delta(t-KT)}_{\tilde{\delta}_T(t)} \rightarrow x_\delta(t) = x(t)\cdot \tilde{\delta}_T(t)$$

$$\rightarrow x(t)\cdot y(t) \rightleftharpoons X(f) * Y(f) \longrightarrow X_\delta(f) = X(f) * \sum_{m=-\infty}^{+\infty} \frac{1}{T}\cdot\delta\left(f-\frac{m}{T}\right)$$

$$\rightarrow X_\delta(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X(f)*\delta\left(f-\frac{m}{T}\right) \rightarrow X_\delta(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(f-\frac{m}{T}\right) \underbrace{\delta\left(f-\frac{m}{T}\right)}_{1}$$

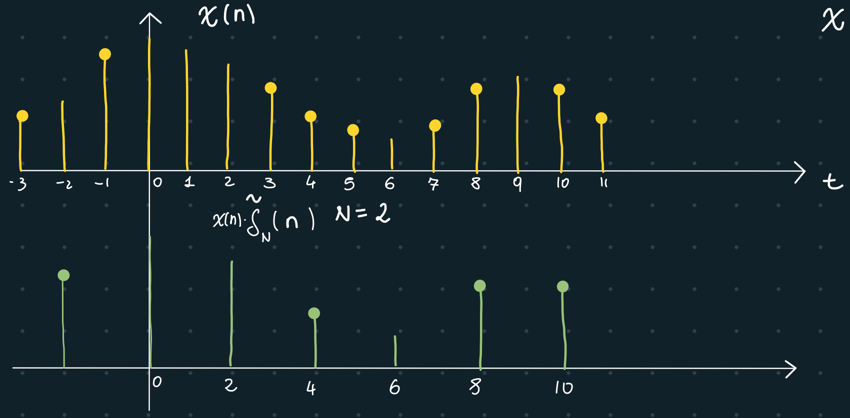
Prop del time shifting della Delta

$$\rightarrow X_\delta(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(f-\frac{m}{T}\right)$$

Riproduzione

Morale della favola: Campionamento (Tempo) $\Longleftrightarrow$ Riproduzione (frequenza)

Esempio inverso: S. Campionato $\Rightarrow$ Riproduzione — Tempo Discreto



$$x_\delta(n) = \sum_{k=-\infty}^{+\infty} x(kN)\, \delta(n-kN)$$

$$x_\delta(n) = x(n) \cdot \tilde{\delta}_N(n)$$

Tempo

$$x_\delta(n) \rightleftharpoons X(\nu) * \frac{1}{N}\sum_{m=-\infty}^{+\infty} \delta\left(\nu - \frac{k}{N}\right)$$

tempo

$$\Rightarrow x_\delta(n) \rightleftharpoons \frac{1}{N}\sum_{m=-\infty}^{+\infty} X\left(\nu - \frac{k}{N}\right)$$

freq

Campionamento — Riproduzione

Proprietà $X(\nu) * \delta\left(\nu - \frac{k}{N}\right) \Rightarrow X\left(\nu - \frac{k}{N}\right)$

Esercizio:

$$x(t) = \text{Sinc}^2(2t) * \text{rep}_{4\pi}\left[e^{-|t|}\right] \xrightarrow{F.T} X(t) = ?$$

1) $\underline{x(t) * y(t)} \rightleftharpoons \underline{x(t) \cdot y(t)}$

2) $\text{Sinc}^2(2t) = \text{Sinc}(2t) \cdot \text{Sinc}(2t) \rightleftharpoons \mathcal{F}(\text{sinc}(2t)) * \mathcal{F}(\text{sinc}(2t))$

$\to$ $\text{Sinc}(t) \rightleftharpoons \Pi(f)$ $\to$ $\text{Sinc}(2t)$ Sfruttiamo la prop $x(Bt) \rightleftharpoons \frac{1}{|B|} \cdot X\left(\frac{f}{B}\right)$

$\Rightarrow$ $\underline{\text{Sinc}(2t) \rightleftharpoons \frac{1}{2}\Pi\left(\frac{f}{2}\right)}$

Concludiamo il punto 2: $\text{Sinc}^2(2t) \rightleftharpoons \frac{1}{2}\Pi\left(\frac{f}{2}\right) * \frac{1}{2}\Pi\left(\frac{f}{2}\right)$ $T=2$

$\to$ Sappiamo • $\Pi\left(\frac{t}{T}\right) * \Pi\left(\frac{t}{T}\right) = T\Lambda\left(\frac{t}{T}\right)$

• $A \cdot x(t) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$

$\to$ $\text{Sinc}^2(2t) \rightleftharpoons \frac{1}{2} \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}\Lambda\left(\frac{f}{2}\right) = \frac{1}{2}\Lambda\left(\frac{f}{2}\right)$ $\mathcal{F}(\text{sinc}^2(2t))$

3) $\text{rep}_{4\pi}\left[e^{-|t|}\right] \rightleftharpoons$ Campionamento

$\to$ $e^{-|t|} \rightleftharpoons \dfrac{2a}{a^2+(2\pi f)^2} = \dfrac{2}{1+(2\pi f)^2}$

$f = \dfrac{m}{4\pi}$ $\to$ $\dfrac{2}{1+\left(\cancel{2\pi}\,\dfrac{m}{\cancel{4\pi}_{2}}\right)^2} = \dfrac{2}{1+\left(\dfrac{m}{2}\right)^2} = \dfrac{2}{1+\dfrac{m^2}{4}}$

• $T = 4\pi$

$$\text{rep}_T[x(t)] \rightleftharpoons \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(\frac{m}{T}\right) \cdot \delta\left(f - \frac{m}{T}\right)$$

$\to$ $\text{rep}_{4\pi}\left[e^{-|t|}\right] \rightleftharpoons \dfrac{1}{4\pi}\displaystyle\sum_{m=-\infty}^{+\infty} \dfrac{2}{1+\dfrac{m^2}{4}} \cdot \delta\left(f - \dfrac{m}{4\pi}\right)$ $\mathcal{F}\left(e^{-|t|}\right)$

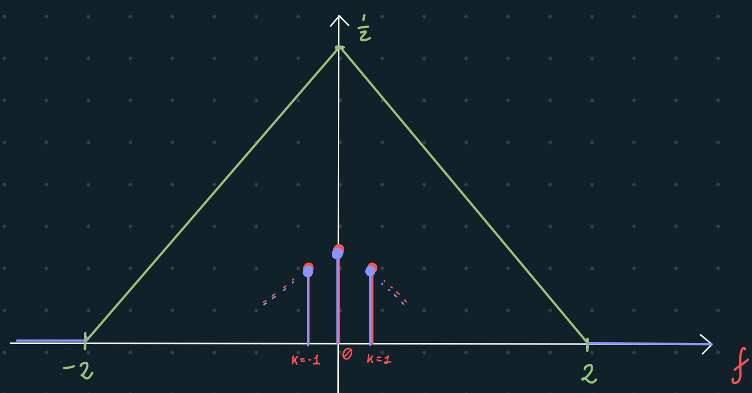
4) Morale Della favola:

Applico il punto 1: $X(f) = \dfrac{1}{2}\Lambda\left(\dfrac{f}{2}\right) \dfrac{1}{4\pi}\displaystyle\sum_{m=-\infty}^{+\infty} \dfrac{2}{1+\dfrac{m^2}{4}} \cdot \delta\left(f - \dfrac{m}{4\pi}\right)$

Durata

Ampiezza Segnale Campionamento

5) Come graficarlo?



$X(f)$

$\frac{1}{2}\Lambda\left(\frac{f}{2}\right)$

Campionamento

$K=0 \to \dfrac{1}{\cancel{4}_{2}\pi} \dfrac{\cancel{2}}{1} \cdot = \dfrac{1}{2\pi}$

$K=1 \to \dfrac{1}{\cancel{4}_{2}\pi} \cdot \dfrac{2}{1+\frac{1}{4}} = \dfrac{1}{\frac{4+1}{4} \cdot 2\pi} = \dfrac{2\,\cancel{4}}{5\,\cancel{10}\pi}$