EX. 1

Si considerino due variabili aleatorie uniformi, X con alfabeto $\{0,1,2\}$ e Y, con alfabeto $\{-1,0,1\}$, indipendenti.

1. Scrivere, in forma tabellare, la PMF congiunta delle due variabili aleatorie X e Y.
2. Calcolare il coefficiente di correlazione tra la variabile aleatoria $Z = X + Y$ e la variabile aleatoria X.

$X \sim U(a,b)$ con $\mathcal{A}_X = \{0,1,2\}$ ed $Y \sim U(a,b)$ con $\mathcal{A}_Y = \{-1,0,1\}$ indip.

PMF Congiunta?

$p_{XY} : (x,y) \in \mathcal{A}_X \times \mathcal{A}_Y \to p_{XY}(x,y) \to P(\{X=x\} \cap \{Y=y\})$

$\to$ ovvero $\sum_{i \in \mathcal{A}_X} \sum_{j \in \mathcal{A}_Y} x_i y_i \cdot P(p_X \cdot p_Y)$

$\mathcal{A}_X$	$\mathcal{A}_Y$	p_X	p_Y	p_{XY}
0	-1	1/3	1/3	1/9
0	0	1/3	1/3	1/9
0	1	⋮	⋮	⋮
1	-1			
1	0			
1	1			
2	-1			
2	0	⋮	⋮	⋮
2	1	1/3	1/3	1/9

$\Rightarrow p_{XY} = 1/9 \quad \forall\, x,y \in \mathcal{A}_X \times \mathcal{A}_Y$

Q2: Coefficiente di correlazione tra Z e X

dove $Z = X + Y$

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \, \sigma_Y} \Rightarrow \begin{array}{l} C_{XY} = ? \\ \sigma_X, \sigma_Y = ? \end{array}$$

$E[X] = \sum_{k \in \mathcal{A}_X} x_i \cdot p_X(i) = \frac{1}{3}(0+1+2) = 1 \quad \mu_X$

$\Rightarrow \overline{X^2} = \sum_{k \in \mathcal{A}_X} x_i^2 \cdot p_X(i) = \frac{1}{3}(0+1+4) = \frac{5}{3} \quad \overline{X^2}$

$E[Y] = \frac{1}{3} \cdot (-1+0+1) = 0 \quad \mu_Y$

$\Rightarrow \overline{Y^2} = \frac{1}{3}(1+0+1) = \frac{2}{3} \quad \overline{Y^2}$

- $\sigma_X^2 = E[(X-\mu_X)^2] = \overline{X^2} - \mu_X^2 = \frac{5}{3} - 1 = \frac{2}{3} \quad \sigma_X^2$
- $\sigma_Y^2 = \frac{2}{3} \quad \sigma_Y^2$

$E[Z] = E[X+Y] = \mu_X + \mu_Y = 1 \quad \mu_Z$

$\sigma_Z^2 = E[(Z-\mu_Z)^2] = \overline{Z^2} - \mu_Z^2 \to \overline{Z^2} = E[Z^2] = E[(X-Y)^2] = \overline{X^2} + \overline{Y^2} - 2XY$

indipendenti $\Rightarrow E[XY] = \mu_X \mu_Y = 0$

$\Rightarrow \sigma_Z^2 = \overline{X^2} + \overline{Y^2} = \frac{5}{3} + \frac{2}{3} = \frac{7}{3} \quad \sigma_Z^2$

$\Rightarrow C_{XZ} = E[(X-\mu_X)(Z-\mu_Z)] = E[XZ] - \mu_X\mu_Z - \cancel{\mu_X\mu_Z} + \cancel{\mu_X\mu_Z} = E[XZ] - \mu_X\mu_Z$

$E[XZ] = E[X(X+Y)] = \overline{X^2} + \underset{0}{\cancel{E[XY]}} = \frac{5}{3}$

$C_{XZ} = \frac{5}{3} - (1 \cdot 1) = \frac{5}{3} - 1 = \frac{2}{3}$

$$\Rightarrow \rho_{XY} = \frac{\frac{2}{3}}{\sqrt{\frac{7}{3}} \cdot \sqrt{\frac{2}{3}}} \cong 0.53$$