Introduction to “Tsunami” Model

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Goals for this presentation

- Some motivation for studying partial differential equations (PDEs)
- Flux-form for hyperbolic PDEs
- An overview of the shallow water equations
- Finite-volume methods which conserve physical quantities
- Riemann solvers
- A quick overview of the Tsunami code

Motivations for PDEs

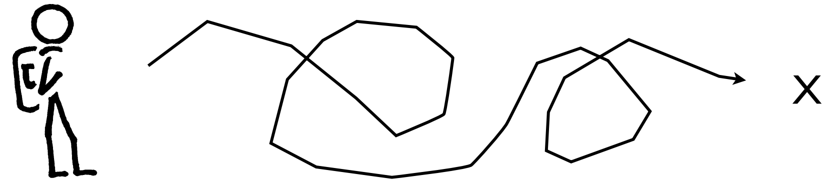
Πάντα ῥεῖ καὶ οὐδὲν μένει

(Everything flows, nothing stands still)

Heraclitus (535 - 475 B.C.E.)

How reality unfolds

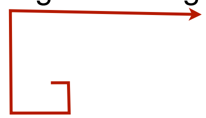
If we begin at an initial starting point and know how our position *changes* at every moment, then, *in principle,* we know our path for all time...



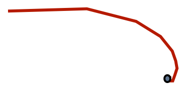
In Mathematics this is referred to as an *initial value problem*

Examples

At 9:00, I am at the certain point in the Sahara desert and start walking at 1 step per second eastward. After ten steps I turn 90 degrees clockwise (i.e. to the right). I then double the number of steps in each interval before turning 90 degrees again and again. Where am I at 13:00? at 19:00? at *t?*



Same starting point, but at each instant, my speed decreases such that after ten seconds I'm going half as fast. After each ten-step interval I veer right by the constant angle of 15 degrees. Where do I end up?



From discrete to continuous

Of course, in Mathematics, discrete changes are described through differences, or in the parlances “deltas”. The theory of the transition from discrete to continuous systems is the subject of *Calculus*.

$$\delta \Rightarrow \partial$$

Hyperbolic PDE: General Definition

Most generally: consider order- κ PDE defined on $\Re^m$, and a smooth *manifold* $S \subset \Re^m$ of dimension m-1, and normal $n(P) \perp S \text{ at } P$

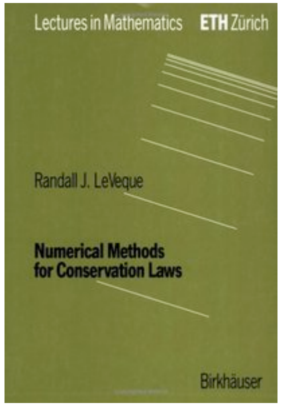
Cauchy Problem: find the solution u which satifies,

$$\begin{aligned} u(x) &= f_0(x) & \forall x \in S \\ \frac{\partial^k u(x)}{\partial n^k} &= f_k(x) & k = 1, \ldots \kappa - 1 \end{aligned}$$

A partial differential equation is hyperbolic at a point P provided that the Cauchy Problem is uniquely solvable in a neighborhood of P for any initial data given on a non-characteristic hypersurface passing through P

Hyperbolic PDE: Features

- Solutions to hyperbolic PDEs are "wave-like": disturbances propagate along characteristics with finite speed
- The distinguishes them from parabolic/ elliptical PDEs were a perturbation in initial or boundary values are *felt instantly* in the entire domain
- Note: the PDE is only hyperbolic w.r.t. a specific point; the same PDE might also have parabolic regimes in parts of the domain
- Often express conservation laws

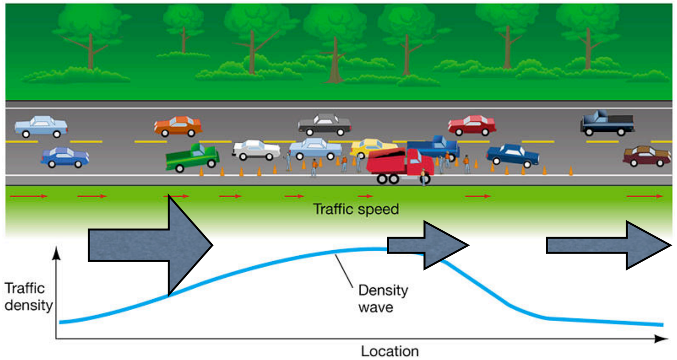


ETH Eidgenössische Technische Hochschule Zürich Swiss Federal Institute of Technology Zurich

CSCS Swiss National Supercomputing Centre

A ‘simple’ example: traffic

- Traffic slows down at a constriction, car flux is reduced
- The density of cars behind the constriction increases



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Cars in / s. - cars out /s. = # cars in constriction / s.

Net flux of cars = net increase in density

Cars are neither created nor destroyed

<u>Traffic Shockwave</u>

Continuity Equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho)$$

ρ	density
$\mathbf{v}$	velocity
∇	$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$$\frac{\partial}{\partial t}\iiint_D \rho dV = -\iiint_D \nabla \cdot (\mathbf{v}\rho) dV \qquad = -\oint_{\partial D} \mathbf{n} \cdot \mathbf{v}\rho dA$$ (Gauss)

The change in mass in D is equal to the flux of mass through its boundary

Flux form of a PDE

- General form for many hyperbolic PDEs
- Also known as *conservative* form

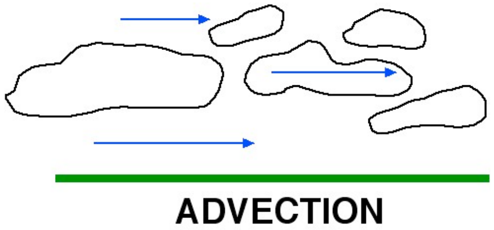
$$\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{x},t) + \nabla \cdot \mathbf{F}(\mathbf{\Psi}(\mathbf{x},t)) = 0$$

Example: continuity equation (mass is conserved)

$$\begin{aligned} \Psi &= \rho \\ \mathbf{F} &= \{v_x\rho, v_y\rho, v_z\rho\} = \mathbf{v}\Psi \end{aligned}$$

Advection (or "Transport")

- Definition: transport of a substance (or some conserved quantity) by a fluid
- In atmosphere: substance travels with wind velocity



Linear Advection

Scalar one-dimensional: $$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

Scalar multi-dimensional: $$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$

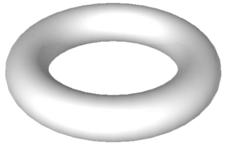
Vector-valued 1D: $$\frac{\partial}{\partial t}\mathbf{\Phi}(x,t) + \mathbf{A}\frac{\partial}{\partial x}\mathbf{\Phi}(x,t) = 0$$

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi}_x = 0$$

Why linear? Assume solutions ϕ_1, ϕ_2
then the following is also a solution: $\alpha\phi_1 + \beta\phi_2$

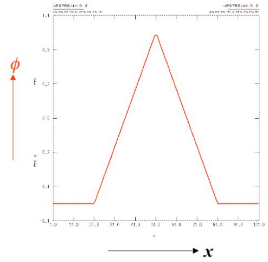
(Take a few minutes)

Analytical Solution

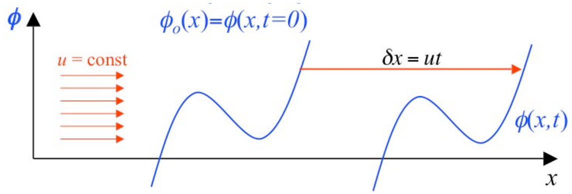


Assume a "1D" torus:

Initial Condition: $\phi_0(x)$



Analytic Solution:



Linear Advection: Flux Formulation?

Relationship:
$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{u}\phi) = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi + \phi \nabla \cdot \mathbf{u}$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{u}\phi) = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$

Flux form:
$$\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{x},t) + \nabla \cdot \mathbf{F}(\mathbf{\Psi}(\mathbf{x},t)) \qquad \begin{array}{lcl} \Psi & = & \phi \\ \mathbf{F} & = & \mathbf{u}\Psi \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}$$

Translation: in a divergence-free velocity field (incompressible medium), density is merely advected

Example: vector-valued advection

Consider the eigen-decomposition of **A**:

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi}_x = 0 \quad \Rightarrow \quad \mathbf{\Phi}_t + \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\mathbf{\Phi}_x = 0$$

Eigenvalues/vectors: $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ $\mathbf{R} = [\mathbf{r_1}, \ldots, \mathbf{r_m}]$

$$\mathbf{R}^{-1}\mathbf{\Phi}_t + \mathbf{\Lambda}\mathbf{R}^{-1}\mathbf{\Phi}_x = 0$$

Characteristic variables: $\mathbf{\Psi} = \mathbf{R}^{-1}\mathbf{\Phi}$

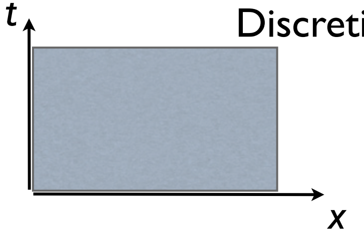
$$\mathbf{\Psi}_t + \mathbf{\Lambda}\mathbf{\Psi}_x = 0 \Longrightarrow (\psi_p)_t + \lambda_p(\psi_p)_x = 0, \quad \text{for } p = 1, \ldots m$$

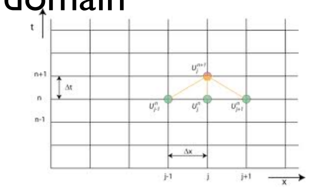
Solutions: $\psi_p(x,t) = \psi_p(x - \lambda_p t, 0)$

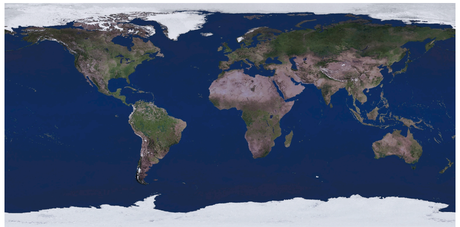
$$\mathbf{\Phi}(x,t) = \sum_{p=1}^{m} \psi_p(x,t)\mathbf{r}_p$$

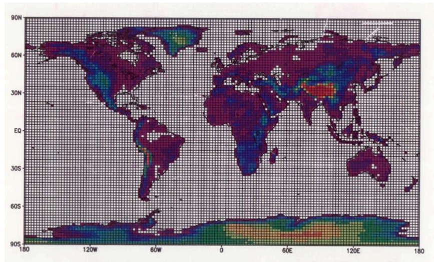
Numerical solution

Discretization of domain









Finite difference approach

Taylor expansion is your best friend

$$\phi_i^n \approx \phi(x,t)$$

$$\phi_{i+1}^n \approx \phi(x+\Delta x,t) = \phi(x,t) + \Delta x \frac{\partial \phi}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \phi}{\partial x^2} + O((\Delta x)^3)$$

$$\phi_{i-1}^n \approx \phi(x-\Delta x,t) = \phi(x,t) - \Delta x \frac{\partial \phi}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \phi}{\partial x^2} + O((\Delta x)^3)$$

$$\phi_i^{n+1} \approx \phi(x,t+\Delta t) = \phi(x,t) + \Delta t \frac{\partial \phi}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 \phi}{\partial t^2} + O((\Delta t)^3)$$

Simple methods

Zeroeth order time stepping scheme: $\phi_i^{n+1} = \phi_i^n$

First order approximations:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{\partial \phi}{\partial t} + O(\Delta t) \qquad \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} = \frac{\partial \phi}{\partial x} + O(\Delta x)$$

Resulting *explicit* scheme (*first order in space and time*)

$$\phi_i^{n+1} = \phi_i^n - \underbrace{\frac{u\Delta t}{\Delta x}}_{\alpha}(\phi_i^n - \phi_{i-1}^n)$$

"Upwind scheme"

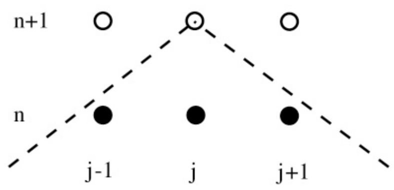
MATLAB: ideal software to test schemes

Courant-Friedrichs-Levy

The CFL condition is a necessary condition for *stability*

Information travels along *characteristics*. In this case straight lines with slope

$$\alpha = \frac{u\Delta t}{\Delta x}$$



Non-linear advection

Also known as the *inviscid Burgers' equation*

Advective formulation $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Homework: assume solutions ϕ_1, ϕ_2
is $\alpha\phi_1 + \beta\phi_2$ a solution?

Is there a flux formulation?

$$\frac{\partial u}{\partial t} + \frac{\partial\left(\frac{1}{2}u^2\right)}{\partial x} = 0$$

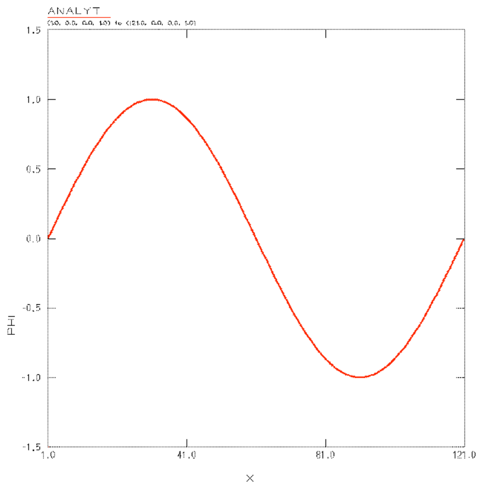
$$\begin{aligned} \Psi &= u \\ F &= \left(\frac{\Psi^2}{2}\right) \end{aligned}$$

Nonlinear advection

Illustrates development of shocks

A diffusive term avoids non-physical behavior ("Burgers' equation")

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$



Conservative Methods

Upstream scheme
for non-lin. advection

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left[(u_i^n)^2 - (u_{i-1}^n)^2\right]$$

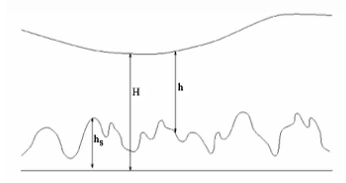
Consider discrete equivalent of spatial integral of Ψ

$$\int_{-L/2}^{L/2} u dx \approx \sum_{i=1}^{M} u_j \Delta x \qquad \sum_{i=1}^{M} u_j^{n+1} = \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{2\Delta x}\left[\sum_{i=1}^{M}(u_i^n)^2 - \sum_{i=1}^{M}(u_{i-1}^n)^2\right]$$

$$\sum_{i=1}^{M}(u_i^n)^2 = \sum_{j=1}^{M}(u_{i-1}^n)^2 \implies \sum_{i=1}^{M} u_j^{n+1} = \sum_{i=1}^{M} u_i^n$$

Shallow water equations

The shallow water equations describe propagation of a vertical column of fluid.



The whole column moves (generally!) only in shallow water (a bay, lagoon, bathtub, or a very thin atmosphere).

Not to be confused with deep water where there are rotations in the column:

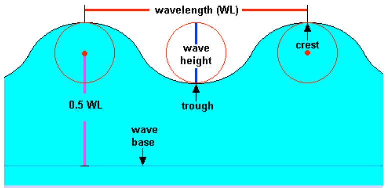


Figure 1. Depiction of deep water waves (water deeper than 0.5 WL).

Shallow water equations

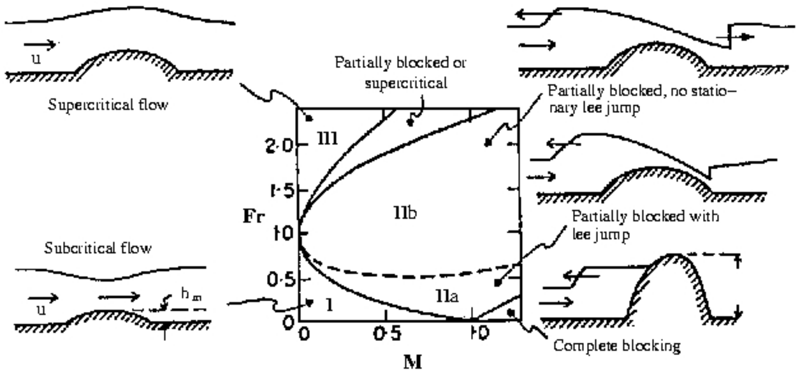
In the shallow water system, mass and momentum (1,2, or 3 dimensions) are conserved.

1-D: $$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix} h \\ hv \end{pmatrix}}_{\Psi} + \frac{\partial}{\partial x}\underbrace{\begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}}_{\mathbf{F}(\Psi)} = \mathbf{0}$$

2-D planar: $$\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{x},t) + \nabla \cdot \mathbf{F}(\mathbf{\Psi}(\mathbf{x},t)) = 0$$

$$\Psi = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} \psi_2 & \psi_3 \\ \psi_2^2/\psi_1 + g\psi_1^2/2 & \psi_2\psi_3/\psi_1 \\ \psi_2\psi_3/\psi_1 & \psi_3^2/\psi_1 + g\psi_1^2/2 \end{pmatrix}$$

Shallow water flow situations



Speed (celerity) of shallow water wave: $c = \sqrt{gh}$

Dimensionless Froude number: $Fr = |u|/c$

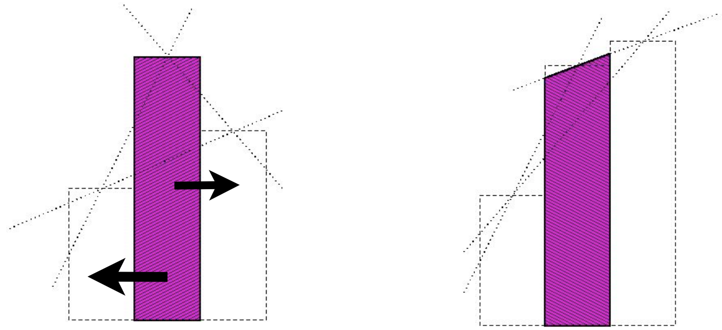
Ratio of base height to free surface at rest: $M = h_s/H_{rest}$

Shallow water simulations

- Propagation of perturbation in bathtub
- Tsunami simulation

Finite Volume Methods

Consider the value ϕ_i^n as the mean value of the function within a cell i

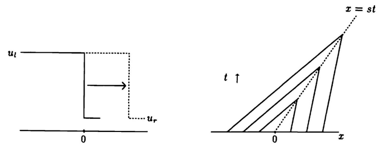


Fluxes are passed between cells, mean values updated

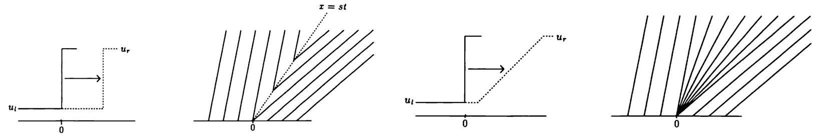
Shock propagation

Consider the propagation speed s of a discontinuity:

Case 1:



Case 2:



Rangine-Hugoniot jump conditions:

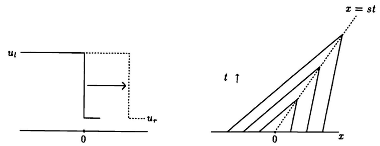
$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi}_x = 0 \implies \mathbf{A}(\mathbf{\Phi}_l - \mathbf{\Phi}_r) = s(\mathbf{\Phi}_l - \mathbf{\Phi}_r)$$

$$\mathbf{\Phi}_t + \mathbf{f}(\mathbf{\Phi})_x = 0 \implies \mathbf{f}(\mathbf{\Phi}_l - \mathbf{\Phi}_r) = s(\mathbf{\Phi}_l - \mathbf{\Phi}_r)$$

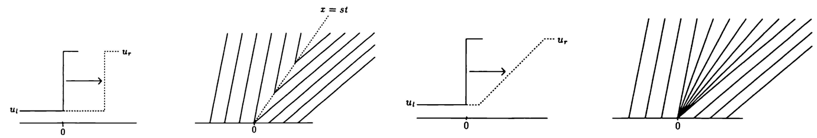
Riemann solvers

Consider the propagation speed s of a discontinuity:

Case 1:



Case 2:



Rangine-Hugoniot jump conditions:

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$$\mathbf{\Phi}_t + \mathbf{f}(\mathbf{\Phi})_x = 0 \quad \Longrightarrow \quad \mathbf{f}(\mathbf{\Phi}_l - \mathbf{\Phi}_r) = s(\mathbf{\Phi}_l - \mathbf{\Phi}_r)$$

Riemann problem

Linear formulation: $\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi}_x = 0 \quad \mathbf{\Phi}(x,0) = \begin{cases} \mathbf{\Phi_l} & x > 0 \\ \mathbf{\Phi_r} & x < 0 \end{cases}$

Strict hyperbolicity (eigenvalues real, distinct): $\lambda_1 < \lambda_2 < \cdots < \lambda_m$

$$\mathbf{\Phi}_l = \sum_{p=1}^{m} \alpha_p \mathbf{r}_p \quad \mathbf{\Phi}_r = \sum_{p=1}^{m} \beta_p \mathbf{r}_p$$

Formulate with characteristic variables: $\mathbf{\Psi} = \mathbf{R}^{-1}\mathbf{\Phi}$

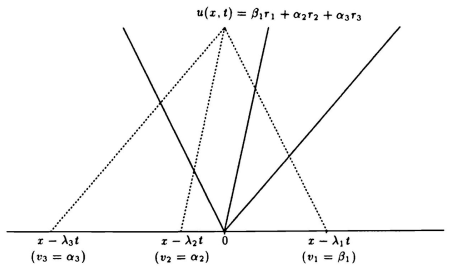
$$\psi_p(x,0) = \begin{cases} \alpha_\mathbf{p} & x < 0 \\ \beta_\mathbf{p} & x > 0 \end{cases} \quad \psi_p(x,0) = \begin{cases} \alpha_\mathbf{p} & x - \lambda_p t < 0 \\ \beta_\mathbf{p} & x - \lambda_p t > 0 \end{cases}$$

Solution depends on downstream 'regime':

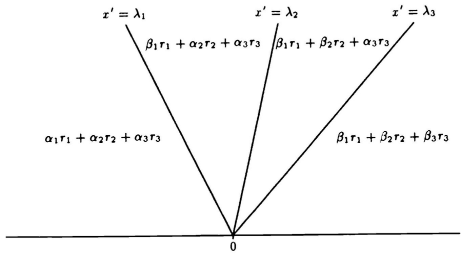
$$\mathbf{\Phi}(x,t) = \sum_{p=1}^{P(x,t)} \beta_p \mathbf{r}_p + \sum_{p=P(x,t)+1}^{m} \alpha_p \mathbf{r}_p \qquad P(x,t) = \{\max(p) \mid x - \lambda_p t > 0\}$$

Riemann solution

Riemann construction (m=3)

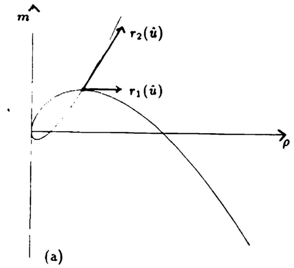


Solutions:



What have we learned

- There is a explicit solution to the one-dimensional linear scalar and vector-valued advection equation
- Instead of considering infinitely many points, consider finite number of cells, each with constant value
- Propagation of discontinuities between cells can be calculated explicitly by solving the Riemann problem



That's nice, but

- Interesting problems are 2-D, 3-D or n-D
- They are not usually linear

Riemann-based Finite Volume Numerical Method

Numerical method is based on multiple approximations

- Non-linear problems *linearized* for a sufficiently small time step:

  $\mathbf{\Phi}_t + \mathbf{F}(\mathbf{\Phi})_x = 0$ Jacobian: $\mathbf{A}(\mathbf{\Phi}) = \mathbf{F}'(\mathbf{\Phi})$

  Quasilinear form: $\mathbf{\Phi}_t + \mathbf{A}(\mathbf{\Phi})\mathbf{\Phi}_x = 0$

- From every Riemann solution, derive a *flux* for a given time step; these fluxes are added/subtracted from cells, resulting in new cells, each with a mean value, and thus new discontinuities
- For multi-dimensional problems: use the Riemann solver to calculate fluxes in x, y, etc. Update cells in X, in Y, or a clever combination (this can cause *numerical artifacts*)

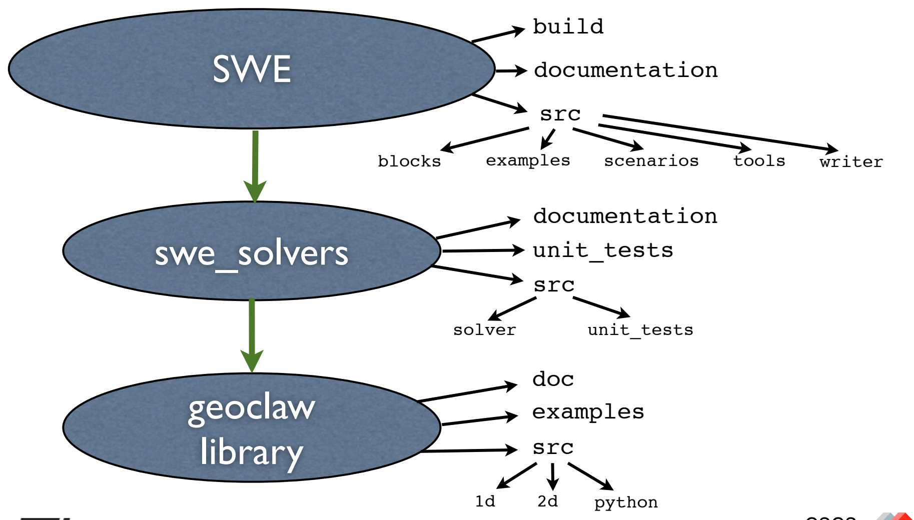
Tsunami Model (SWE)

Solves the 2D shallow water equation $\dfrac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{x},t) + \nabla \cdot \mathbf{F}(\mathbf{\Psi}(\mathbf{x},t)) = 0$

$$\Psi = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} \psi_2 & \psi_3 \\ \psi_2^2/\psi_1 + g\psi_1^2/2 & \psi_2\psi_3/\psi_1 \\ \psi_2\psi_3/\psi_1 & \psi_3^2/\psi_1 + g\psi_1^2/2 \end{pmatrix}$$

- Forms the Jacobian: $\mathbf{A}(\mathbf{\Phi}) = \mathbf{F}'(\mathbf{\Phi})$
- Defines domain into square cells, initialized with mean values
- At each time step:
  - ✳ Evaluates Jacobian to get characteristic tangents, eigenvalues
  - ✳ Solves Riemann problem at each cell face in X and Y
  - ✳ Determines cell-cell fluxes in X and Y by averaging solution
  - ✳ Updates cell values with these fluxes

Tsunami code structure



Thank you for your attention!