

Advanced Models and Methods in Operations
Research
Dynamic programming

Florian Fontan

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Introduction

The partition problem and the subset sum problem

The knapsack problem

The single-night star observation scheduling problem

Dynamic programming as a tree search

Conclusion

- ▶ Former student of ORCO (2015–2016)
- ▶ PhD in Operations Research
- ▶ Engineer at Artelys: <https://www.artelys.com/>
 - ▶ Artelys is an independent company specialized in optimization, decision support and modeling
 - ▶ Design and implementation of optimization algorithms for industrial clients
 - ▶ Development for the NLP/MINLP solver Artelys Knitro
<https://www.artelys.com/solvers/knitro/>
- ▶ My GitHub: <https://github.com/fontanf/>

Organization

- ▶ 4 classes with me:
 - ▶ Dynamic programming
 - ▶ Heuristic tree search
 - ▶ Column generation heuristics
 - ▶ Project presentations
- ▶ 1.5 hours lecture / 1.5 hours practical training
- ▶ Goal of the classes: understanding the theory of the methods and being able to implement them to solve practical problems

Organization

- ▶ Evaluation:
 - ▶ Not in the final exam
 - ▶ Project by groups of 3 or 4
 - ▶ Implementation of the algorithms studied in the classes
 - ▶ First deadline with feedbacks
 - ▶ Second deadline with final grade
 - ▶ Please do not put your code on a public repository
 - ▶ You can keep it on a private repository. It can be valuable if you decide to apply for a company some day
- ▶ All materials (slides, projects...) are online
<https://github.com/fontanf/teaching>
- ▶ E-mail: dev@florian-fontan.fr

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Problem definition

Partition problem

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We consider a slightly more general variant:

Subset sum problem (decision version)

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Link between the partition problem and the subset sum problem?

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Link between the partition problem and the subset sum problem?

The partition problem is a subset sum problem with capacity

$$C = \frac{1}{2} \sum_{j=1}^n w_j.$$

Recursive function

For all $j = 0, \dots, n$, $c = 0, \dots, C$, let us define:

$$F(j, c) = \begin{cases} \text{True} & \text{if among items } 1, \dots, j, \text{ there exists} \\ & \text{a subset of items with total weight } c \\ \text{False} & \text{otherwise} \end{cases}$$

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What is the relation between the subset sum problem and F ?

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What is the relation between the subset sum problem and F ?

The subset sum problem is equivalent to determining the value of $F(n, C)$.

Computing $F(n, C)$

We compute $F(j, c)$ with the following recursive formula:

$$F(j, c) = \begin{cases} \text{True} & \text{if } j = 0 \text{ and } c = 0 \\ \text{False} & \text{if } j = 0 \text{ and } c \neq 0 \\ F(j-1, c) & \text{if } j \neq 0 \text{ and } c < w_j \\ F(j-1, c) & \text{otherwise} \\ \text{or } F(j-1, c - w_j) \end{cases}$$

Divide-and-conquer

We can now solve the subset sum problem with a divide-and-conquer algorithm:

```
function  $F(w, j, c)$   
  if  $j == 0$  then  
    if  $c == 0$  then  
      return True  
    else  
      return False  
  else if  $c < w_j$  then  
    return  $F(j - 1, c)$   
  else  
    return  $F(j - 1, c)$  or  $F(j - 1, c - w[j])$   
function subsetsum( $w, C$ )  
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 $O(2^n)$

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- ▶ Time complexity?
 $O(2^n)$
- ▶ Space complexity?
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Is there a way to improve the time complexity?

Divide-and-conquer example

Consider an instance I of the subset sum problem with

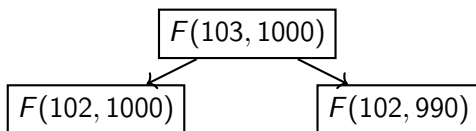
- ▶ $n = 103$
- ▶ $C = 1000$
- ▶ $w_{101} = 30, w_{102} = 20, w_{103} = 10.$

$$F(103, 1000)$$

Divide-and-conquer example

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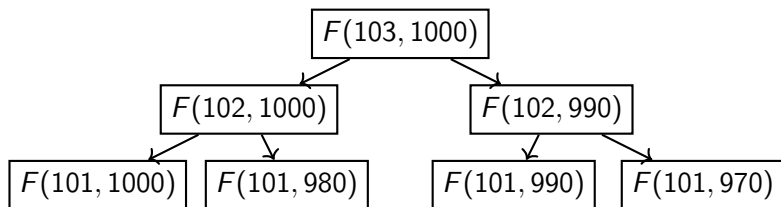
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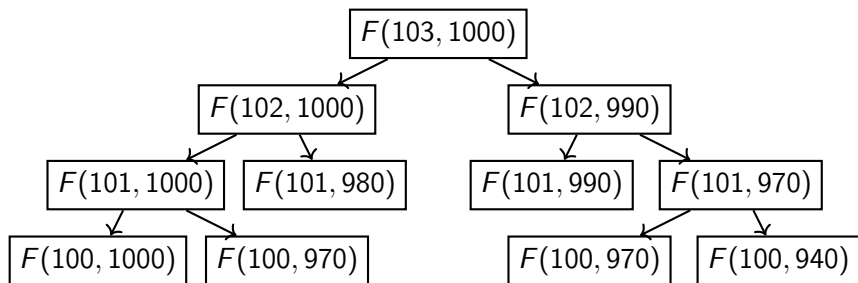
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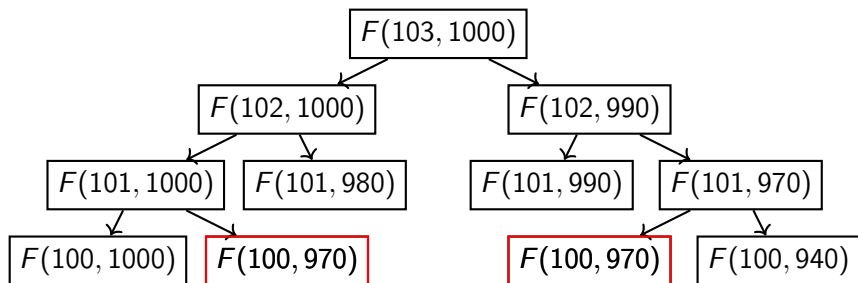
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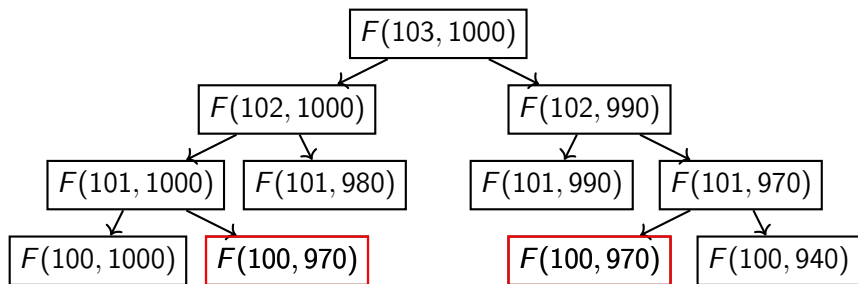
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The same subproblems might be solved multiple times!

Dynamic programming: recursive implementation (top-down)

Same algorithm as before, but now we store the results of the subproblems to avoid solving multiple times the same subproblems:

```
procedure F(w, T, j, c)
  if T[j][c] == NULL then
    if j == 0 then
      if c == 0 then
        T[j][c] ← True
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        T[j][c] ← False
    else if c < w[j] then
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      T[j][c] ← F(j - 1, c) or F(j - 1, c - w[j])
  return T[j][c]

procedure subsetsum(w, C)
  T ← array of size (n + 1) × (C + 1) initialized at NULL
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► Time complexity?
 $O(nC)$

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Dynamic programming

Dynamic programming

Solving a problem recursively and storing the results of the subproblems to avoid recomputing them multiple times.

Dynamic programming: iterative implementation (bottom-up)

```
procedure subsetsum( $w, C$ )  
   $T \leftarrow$  array of size  $(n + 1) \times (C + 1)$  initialized at NULL  
   $T[0][0] \leftarrow \text{True}$   
  for  $c = 1, \dots, C$  do  
     $T[0][c] \leftarrow \text{False}$   
  for  $j = 1, \dots, n$  do  
    for  $c = 0, \dots, w[j] - 1$  do  
       $T[j][c] \leftarrow T[j - 1][c]$   
    for  $c = w[j], \dots, C$  do  
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- Time complexity?

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- Time complexity? $O(nC)$

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- ▶ Time complexity? $O(nC)$
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procedure subsetsum( $w, C$ )  
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         $T[0][c] \leftarrow \text{False}$   
    for  $j = 1, \dots, n$  do  
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```

- ▶ Time complexity? $O(nC)$
- ▶ Space complexity? $O(nC)$
- ▶ In practice, 10 times faster than the recursive implementation

Dynamic programming example

Instance:

- ▶ $n = 5, w = \{4, 11, 6, 8, 7\}$
- ▶ $C = 17$

Dynamic programming example

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Reminder:

$$F(j, c) = \begin{cases} \text{True} & \text{if } j = 0 \text{ and } c = 0 \\ \text{False} & \text{if } j = 0 \text{ and } c \neq 0 \\ F(j-1, c) & \text{if } j \neq 0 \text{ and } c < w_j \\ F(j-1, c) & \text{otherwise} \\ \text{or } F(j-1, c - w_j) \end{cases}$$

j / c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
-------	---	---	---	---	---	---	---	---	---	---	----	----	----	----	----	----	----	----

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j / c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	T	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F

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Dynamic programming example

Instance:

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$$S = \{5, 3, 1\}$$

Retrieving the solution with backtracking

```
function SSPbacktracking( $w, C, T$ )  
   $S \leftarrow \{\}$   
   $c \leftarrow C$   
   $i \leftarrow n$   
  while  $i > 0$  do  
    if not  $T[i - 1][c]$  then  
       $S \leftarrow S \cup \{i\}$   
       $c \leftarrow c - w[i]$   
     $i \leftarrow i - 1$   
  return  $S$ 
```

Going further

- ▶ Write an algorithm computing $F(n, C)$ which only keeps two lines of the array in memory (spatial complexity $O(C)$)
- ▶ Write an algorithm computing $F(n, C)$ which only keeps a single line of the array in memory.
- ▶ How to return a solution when keeping only a single line in memory?
 - ▶ if the array is stored as an array of integers
 - ▶ if the array is stored as an array of bits

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Knapsack problem

- ▶ Instance:
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- ▶ Problem: find a subset of items such that the total weight of the subset is less than or equal to C .
- ▶ Objective: maximize the total profit of the selected items.

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Link between the subset sum problem and the knapsack problem?
The subset sum problem is a knapsack problem with $p_j = w_j$ for all $j = 1, \dots, n$.

Recursive function

For all $j = 0, \dots, n$, $c = 0, \dots, C$, let us define $F(j, c)$ the maximum profit of a subset of items $1, \dots, j$ with total weight less than or equal to c .

Recursive function

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What is the value of

► $F(0, 0)$?

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- ▶ $F(0, 0)$? 0
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What is the relation between the knapsack problem and F ?

The knapsack problem is equivalent to determining the value of $F(n, C)$.

Computing $F(n, C)$

We compute $F(j, c)$ with the following recursive formula:

$$F(j, c) = \begin{cases} 0 & \text{if } j = 0 \\ F(j-1, c) & \text{if } j \neq 0 \text{ and } c < w_j \\ \max \begin{cases} F(j-1, c) \\ F(j-1, c - w_j) + p_j \end{cases} & \text{otherwise} \end{cases}$$

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$$S = \{3, 1\}$$

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Introduction

The partition problem and the subset sum problem

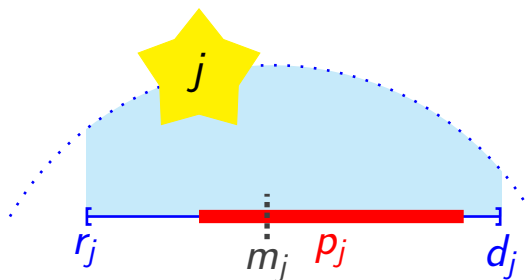
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A star

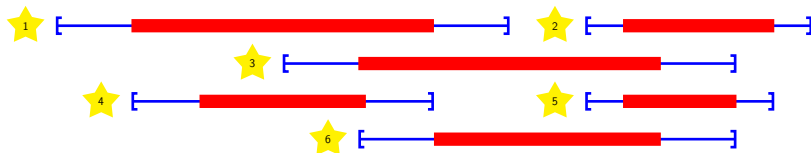


- ▶ $[r_j, d_j]$: visibility interval
- ▶ p_j : observation time
- ▶ w_j : scientific interest of the observation

Every observation j has a meridian $m_j \in [r_j, d_j[$ which is a **mandatory instant** of the observation.

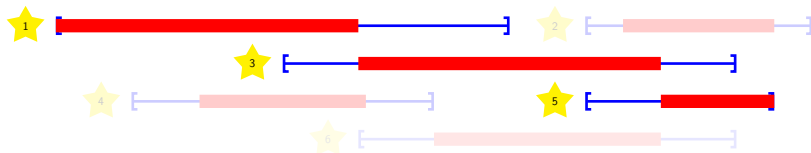
Problem definition

Instance: a set of stars \mathcal{N} ; each star $j \in \mathcal{N}$ has a scientific interest w_j , an observation time p_j and a time window $[r_j, d_j[$



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Problem: find a subset $\mathcal{N}' \subset \mathcal{N}$ as well as the start date s_j of each selected observation $j \in \mathcal{N}'$ such that:

- ▶ for all $j \in \mathcal{N}'$: $[s_j, s_j + p_j] \subset [r_j, d_j[$
- ▶ for all $(j_1, j_2) \in \mathcal{N}'^2$: $[s_{j_1}, s_{j_1} + p_{j_1}] \cap [s_{j_2}, s_{j_2} + p_{j_2}] = \emptyset$

Objective: maximize $\sum_{j \in \mathcal{N}'} w_j$ the profit of the selected observations.

Properties

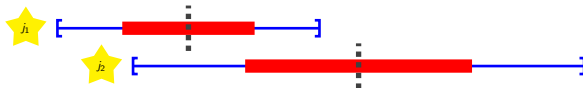
Property 1

There exists an optimal solution in which selected observations are scheduled in non-decreasing order of their mandatory instant.

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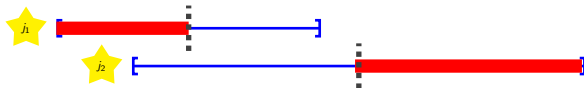
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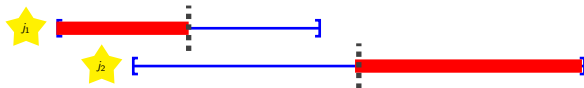
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Properties

Property 1

There exists an optimal solution in which selected observations are scheduled in non-decreasing order of their mandatory instant.



Property 2

Consider a subset $\mathcal{N}' \subset \mathcal{N}$ and an observation j_{\max} such that $d_{j_{\max}} = \max_{j \in \mathcal{N}'} d_j$. If there exists a feasible solution with selected observations \mathcal{N}' , then there exists a feasible solution with selected observations \mathcal{N}' such that $s_{j_{\max}} = d_{j_{\max}} - p_{j_{\max}}$.

Recursive function

We consider that the observations are numbered in non-decreasing order of their mandatory instant.

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For all $j = 0, \dots, n$, $t = 0, \dots, T$, let us define $F(j, t)$ the maximum scientific interest of a subset of observations of $1, \dots, j$ during the interval $[0, T]$.

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Complexity: $O(nT)$

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Sometimes, a bit of work is needed in order to exhibit the structure to design an efficient algorithm based on dynamic programming.

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Breadth first search: example (knapsack problem)

Example:

$$C = 5$$

Item	Weight	Profit
1	3	2
2	2	2
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(j, w, p)

$(0,0,0)$

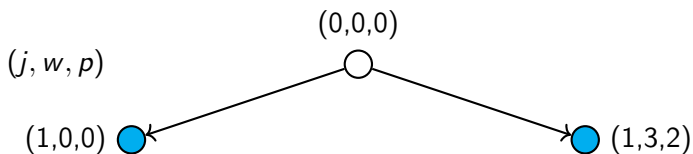


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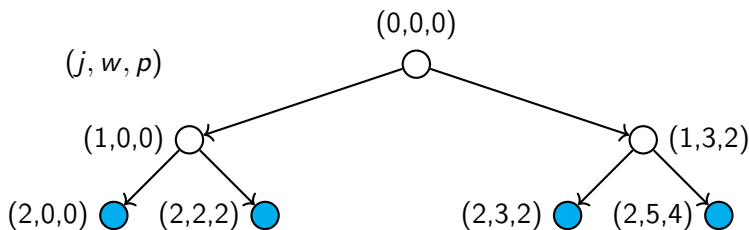


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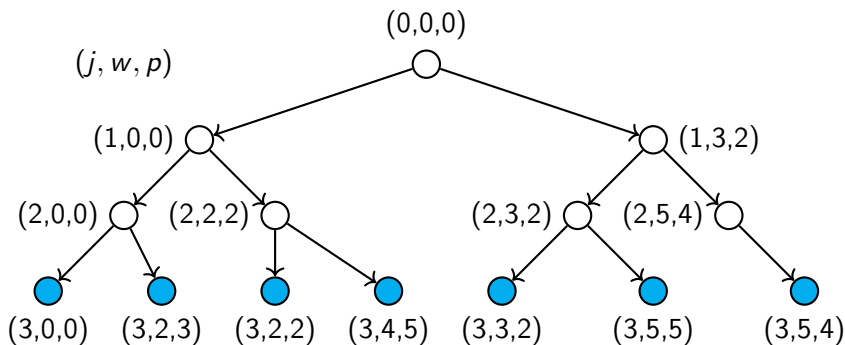


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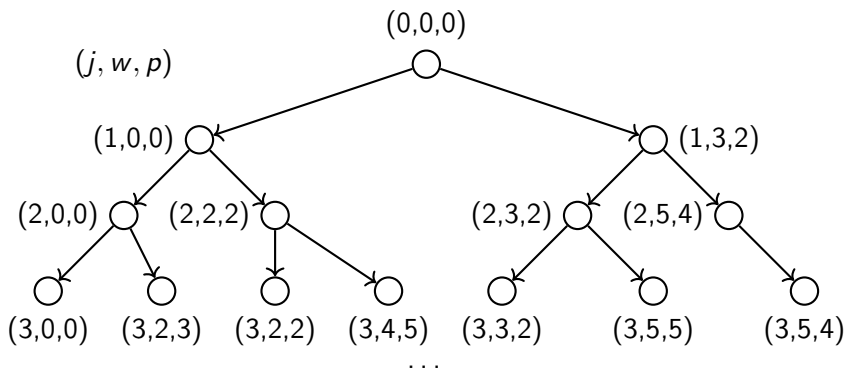


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Breadth first search (knapsack problem)

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procedure BreadthFirstSearch( $w, C$ )  
   $L_0 \leftarrow ((0, 0, 0))$   
  for  $j = 1, \dots, n$  do  
    for  $(j, w, p) \in L_{j-1}$  do  
       $L_j \leftarrow L_j \cup ((j, w, p))$   
      if  $w + w[j] \leq C$  then  
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- ▶ Space complexity? $O(2^n)$

Dominance rule (knapsack problem)

We express dynamic programming as a dominance rule: Consider two nodes $n_1 = (j_1, w_1, p_1)$ and $n_2 = (j_2, w_2, p_2)$. If

$$j_1 \leq j_2 \quad \text{and} \quad w_1 \leq w_2 \quad \text{and} \quad p_1 \geq p_2$$

then node n_1 dominates node n_2 and therefore node n_2 can be safely pruned.

Breadth first search + dominance rule: example (knapsack problem)

Example:

$$C = 5$$

Item	Weight	Profit
1	3	2
2	2	2
3	2	3
4	1	2

Breadth first search + dominance rule: example (knapsack problem)

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$$C = 5$$

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(j, w, p)

$(0,0,0)$

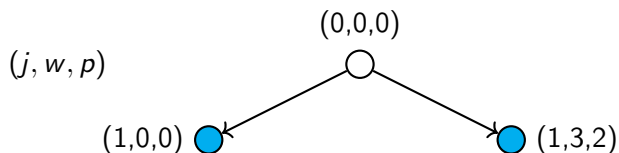


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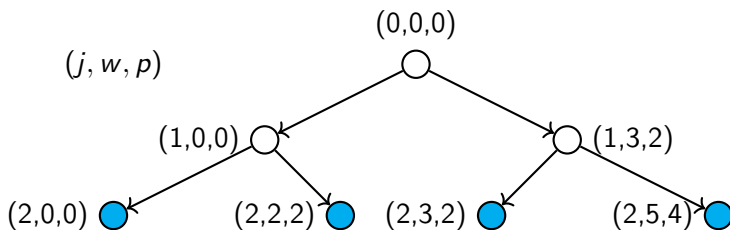


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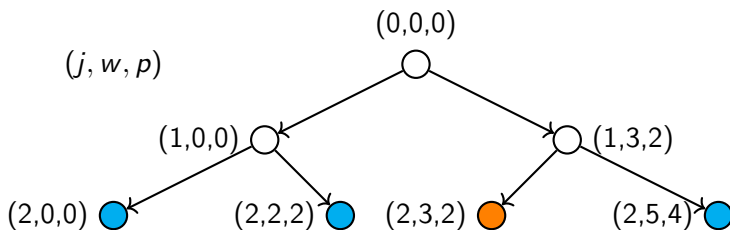


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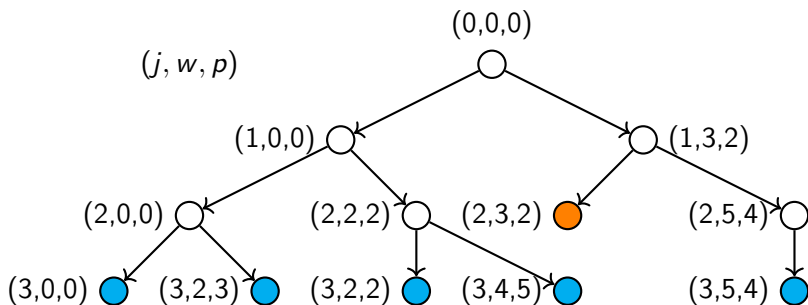


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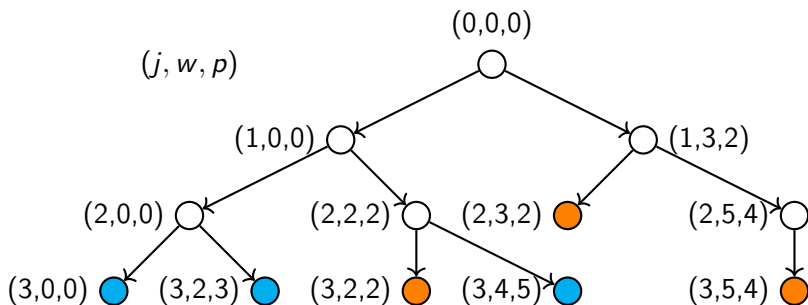


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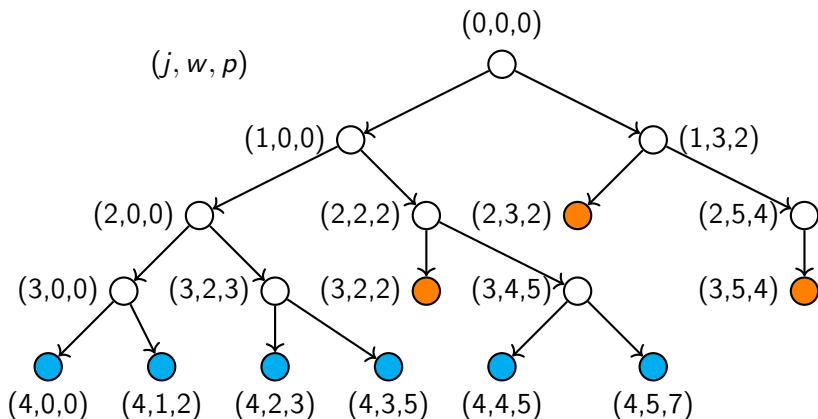


Breadth first search + dominance rule: example (knapsack problem)

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Breadth first search + dominance rule (knapsack problem)

```
procedure BreadthFirstSearch( $w, C$ )  
   $L_0 \leftarrow ((0, 0, 0))$   
  for  $j = 1, \dots, n$  do  
    for  $(j, w, p) \in L_{j-1}$  do  
       $L_j \leftarrow L_j \cup ((j, w, p))$   
      if  $w + w[j] \leq C$  then  
         $L_j \leftarrow L_j \cup ((j, w + w_j, p + p_j))$   
  Remove all dominated nodes from  $L_j$ 
```

Breadth first search + dominance rule (knapsack problem)

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- Time complexity?

Breadth first search + dominance rule (knapsack problem)

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- Time complexity? The complexity depends on the complexity of applying the dominance rule!
In this case, it is possible to implement it in $O(C)$ and keep the complexity of the whole algorithm to $O(nC)$ as for the iterative implementation. But in some cases, the complexity might increase.

Table of contents

Introduction

The partition problem and the subset sum problem

The knapsack problem

The single-night star observation scheduling problem

Dynamic programming as a tree search

Conclusion

Conclusion

- ▶ Dynamic programming: solving a problem recursively and storing the results of the subproblems to avoid recomputing them multiple times.
- ▶ It requires the problem to have a specific structure. It might not be applicable to all problems.
- ▶ Sometimes, a bit of work is needed in order to exhibit this structure
- ▶ Multiple possible implementations with their advantages and drawbacks (recursive, iterative, tree search)
- ▶ “Knapsack Problems” (Kellerer, Pferschy et Pisinger, 2004)
- ▶ Dynamic programming through path problems: <https://moodle.caseine.org/mod/page/view.php?id=30723>

Advanced Models and Methods in Operations
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