

Advanced Models and Methods in Operations
Research
Column Generation Heuristics

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Cutting Stock Problem, Description

Input:

- ▶ a capacity C
- ▶ n item types; for each item type $j = 1, \dots, n$, a weight w_j and a demand q_j

Problem:

- ▶ Pack all items such that the total weight of the items in a bin does not exceed the capacity.

Objective:

- ▶ Minimize the number of bin used.

Cutting Stock Problem, Formulation

Let us define the K possible patterns such that $x_j^k = q$ iff pattern k , $k = 1 \dots K$ contains q copies of item type j

► Variables:

► $y^k \in \mathbb{N}$, $\forall k = 1 \dots K$.

$y^k = q$ iff q copies of pattern k are used

► Objective:

$$\min \sum_{k=1}^K y^k$$

► Constraints:

$$\sum_{k=1}^K x_j^k y^k = q_j \quad \forall j = 1 \dots n$$

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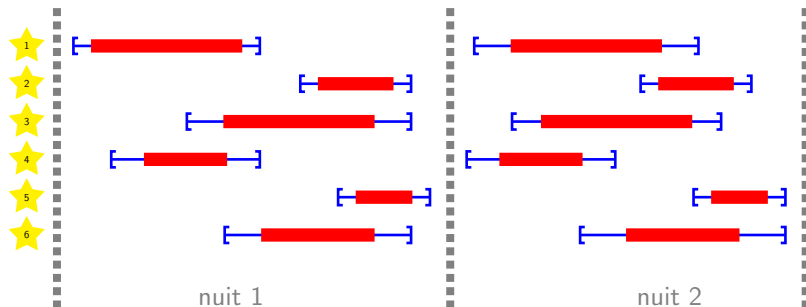
$$\sum_{k=1}^K x_j^k y^k = q_j \quad \forall j = 1 \dots n$$

Why is this formulation good compared to the classical one?

- ▶ No big-M constraint
- ▶ Better relaxation
- ▶ Easier to write

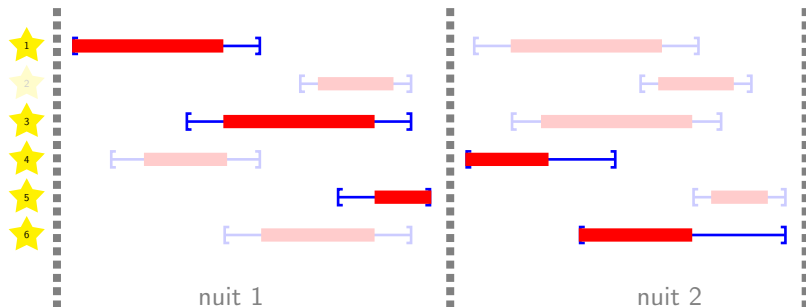
Star Observation Scheduling Problem, Description

Input: a set \mathcal{M} of nights and a set \mathcal{N} of stars; for each star $j \in \mathcal{N}$, a scientific interest w_j , an observation duration p_j^i and a visibility window $[r_j^i, d_j^i]$, depending on the night i of the observation.



Star Observation Scheduling Problem, Description

Input: a set \mathcal{M} of nights and a set \mathcal{N} of stars; for each star $j \in \mathcal{N}$, a scientific interest w_j , an observation duration p_j^i and a visibility window $[r_j^i, d_j^i]$, depending on the night i of the observation.



Star Observation Scheduling Problem, Formulation

For each night i , $i = 1 \dots m$, let us define the K_i possible schedules such that $x_{i,j}^k = 1$ iff schedule k , $k = 1 \dots K_i$ of night i contains star j

► Variables:

- $y_i^k \in \{0, 1\}$, $\forall i = 1 \dots m$, $\forall k = 1 \dots K_i$.
 $y_i^k = 1$ iff scheduled k of night i is selected

► Objective:

$$\max \sum_{i=1}^m \sum_{k=1}^{K_i} \sum_{j=1}^n w_j x_{i,j}^k y_i^k$$

► Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \quad \forall i = 1 \dots m$$

$$\sum_{i=1}^m \sum_{k=1}^{K_i} x_{i,j}^k y_i^k \leq 1 \quad \forall j = 1 \dots n$$

2D Guillotine Variable-sized Bin Packing, Description

Input:

- ▶ n item types; for each item type $j = 1, \dots, n$, a width w_j , a height h_j and a demand q_j
- ▶ m bin types; for each bin type $i = 1, \dots, m$, a width W_i , a height H_i , a lower bound l_i , an upper bound u_i and a cost c_i

Problem:

- ▶ Find a subset of guillotine patterns such that all item type demands and bin type use bounds are satisfied

Objective:

- ▶ Minimize the cost of the selected bins.

2D Guillotine Variable-sized Bin Packing, Formulation

For each bin type i , $i = 1 \dots m$, let us define the K_i possible patterns such that $x_{i,j}^k = q$ iff pattern k , $k = 1 \dots K_i$ of bin type i contains q copies of item type j

► Variables:

► $y_i^k \in \mathbb{N}$, $\forall i = 1 \dots m$, $\forall k = 1 \dots K_i$.

$y_i^k = q$ iff q copies of pattern k of bin type i are used

► Objective:

$$\min \sum_{i=1}^m \sum_{k=1}^{K_i} c_i y_i^k$$

► Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \quad \forall i = 1 \dots m$$

$$\sum_{i=1}^n \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \quad \forall j = 1 \dots n$$

Other examples

Usually, variables represent:

- ▶ A bin/knapsack (for packing problems)
- ▶ The schedule of a machine (for parallel scheduling problems)
- ▶ The route of a vehicle (for vehicle routing problems)
- ▶ ...

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Introduction

- ▶ With these formulations, generating all the variables is generally not possible since their number grows exponentially with the size of the problem.
- ▶ First we focus on solving the **linear relaxation**

The Column Generation procedure

- ▶ We use the **simplex algorithm**.
 - ▶ At each iteration, it adds a variable of negative reduced cost to the current basis

- ▶ Objective:

$$\min \sum_{j=1}^n c_j x_j$$

- ▶ Constraints:

$$\sum_{j=1}^n a_{i,j} x_j \leq b_i \quad \forall i = 1 \dots m$$

- ▶ Reduced cost of variable x_j :

$$c_j - \sum_{i=1}^m a_{i,j} v_i$$

with v_i the dual value of constraint i .

- ▶ It stops when there are no variable of negative reduced cost
- ▶ The difference with the traditional simplex algorithm, is that here, it is not possible to loop through all the variables to find a variable of negative reduced cost, since they have not been all generated.

The Column Generation procedure

- ▶ Instead, finding a variable of negative reduced cost becomes an optimization problem
- ▶ Example with the Cutting Stock Problem

- ▶ Objective:

$$\min \sum_{k=1}^K y^k$$

- ▶ Constraints:

$$\sum_{k=1}^K x_j^k y^k = q_j \quad \forall j = 1 \dots n$$

- ▶ Reduced cost of y^k :

$$1 - \sum_{j=1}^n x_j^k v_j$$

with v_j the dual value of constraint j .

The Column Generation procedure

- ▶ We look for a variable y^k such that

$$1 - \sum_{j=1}^n x_j^k v_j < 0$$

- ▶ Finding a variable of negative reduced cost is equivalent to finding a pattern with total profit ≥ 1 with the profit of item type j being equal to v_j .
- ▶ In practice, we solve the problem as an optimization problem: we find the best solution of the Knapsack Problem and check if the reduced cost of the corresponding variable is negative.

The Column Generation procedure

Summary:

function ColumnGeneration(P)

$Y \leftarrow$ initial set of columns

while True **do**

Solve the Linear Program P' with variables from Y

Look for a variable of negative reduced cost (**Pricing Problem**)

if there is one **then**

Add it to Y

else

return the solution of P'

Initial set of columns

- ▶ To get dual values, the LP needs to be feasible
- ▶ With 0 variable, the LP might be infeasible
 - ▶ Example: Cutting Stock Problem, demand constraints are not satisfied
- ▶ Therefore, we need to find a way to get an initial set of columns such that the LP is feasible
 - ▶ Find a feasible solution and add the corresponding columns
 - ▶ Example: Cutting Stock, Best Fit algorithm
 - ▶ Drawback: Problem specific, additional work for the implementation of the heuristic
 - ▶ Advantage: if the solution is good, it might speed up the column generation procedure
 - ▶ Find manually a set of columns that ensures the LP to be feasible
 - ▶ Example: create n columns with only one item
 - ▶ Generate a dummy column with very high cost for each problematic constraint
 - ▶ Advantage: not problem specific
 - ▶ Drawback: numerical issue is the cost of the dummy columns is not well calibrated

Star Observation Scheduling Problem, Pricing

- Objective:

$$\max \sum_{i=1}^m \sum_{k=1}^{K_i} \sum_{j=1}^n w_j x_{i,j}^k y_i^k$$

- Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \quad \forall i = 1 \dots m \quad \text{dual } u_i$$

$$\sum_{i=1}^n \sum_{k=1}^{K_i} x_{i,j}^k y_i^k \leq 1 \quad \forall j = 1 \dots n \quad \text{dual } v_j$$

Star Observation Scheduling Problem, Pricing

- Objective:

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- Reduced cost of y_i^k :

$$\sum_{j=1}^n w_j x_{i,j}^k - u_i - \sum_{j=1}^n x_{i,j}^k v_j = \sum_{j=1}^n (w_j - v_j) x_{i,j}^k - u_i$$

Star Observation Scheduling Problem, Pricing

- Objective:

$$\max \sum_{i=1}^m \sum_{k=1}^{K_i} \sum_{j=1}^n w_j x_{i,j}^k y_i^k$$

- Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \quad \forall i = 1 \dots m \quad \text{dual } u_i$$

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- Reduced cost of y_i^k :

$$\sum_{j=1}^n w_j x_{i,j}^k - u_i - \sum_{j=1}^n x_{i,j}^k v_j = \sum_{j=1}^n (w_j - v_j) x_{i,j}^k - u_i$$

- Finding a variable of maximum reduced cost reduces to solving m Single Night Star Observation Scheduling Problems with targets with profit $w_j - v_j$.

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^m \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \quad \forall i = 1 \dots m \quad \text{dual } u'_i$$

$$\sum_{i=1}^n \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \quad \forall j = 1 \dots n \quad \text{dual } v_j$$

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^m \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \quad \forall i = 1 \dots m \quad \text{dual } u'_i$$

$$\sum_{i=1}^n \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \quad \forall j = 1 \dots n \quad \text{dual } v_j$$

- Reduced cost of y_i^k :

$$c_i - u'_i - \sum_{j=1}^n x_j^k v_j$$

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^m \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \quad \forall i = 1 \dots m \quad \text{dual } u'_i$$

$$\sum_{i=1}^n \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \quad \forall j = 1 \dots n \quad \text{dual } v_j$$

- Reduced cost of y_i^k :

$$c_i - u'_i - \sum_{j=1}^n x_{i,j}^k v_j$$

- Finding a variable of minimum reduced cost reduces to solving m 2D Guillotine Knapsack Problems with items with profit v_j for each bin type.

Transition

- ▶ The Column Generation procedure solves the relaxation of the exponential formulation

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- ▶ The Column Generation procedure solves the relaxation of the exponential formulation
- ▶ Thus, it provides a valid bound

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- ▶ But it generally does not provide a feasible solution

Transition

- ▶ The Column Generation procedure solves the relaxation of the exponential formulation
- ▶ Thus, it provides a valid bound
- ▶ But it generally does not provide a feasible solution
- ▶ How to exploit the Column Generation to get feasible solutions?

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The Branch-and-Price algorithm (1)

- ▶ LP-based branch-and-bound, the relaxation is solved by the Column Generation procedure in each node
- ▶ How to branch?
 - ▶ Branching on columns of the exponential formulation? No, the pricing problem becomes too difficult
 - ▶ Branching on the variables of the compact formulation?
 - ▶ Bin Packing: branch on whether item j is packed in bin i or not. If yes, then the available bins have now different capacities and a Knapsack Problem for each capacity needs to be computed
 - ▶ Best solution for the Bin Packing: branch on whether two items are packed in the same bin or not. If yes, then they are merged into a single item. If no, then the subproblem becomes a Knapsack Problem with Conflicts which is strongly NP-hard instead of the Knapsack Problem
 - ▶ \implies Branching rules are usually problem dependent and might change the pricing problem, making it harder to solve

The Branch-and-Price algorithm (2)

- ▶ It can be combined with cuts (Branch-and-Price-and-Cut).
 - ▶ The added cuts might also change the pricing problem \implies even more complex to implement
- ▶ Only exact method based on Column Generation, state-of-the-art exact method for many Vehicle Routing and Parallel Machine Scheduling Problems

Solving the restricted master

- ▶ The Column Generation procedure is executed once
- ▶ Solve the exponential formulation with a MILP solver using only the columns generated during the Column Generation procedure
- ▶ No guarantee to find the optimal solution (or even a feasible solution)
- ▶ Solving the MILP is computationally expensive if many columns have been generated. It can take some time before finding a first solution
- ▶ It requires a good MILP solver

Heuristic Tree Search

Branching scheme:

- ▶ Root node: no column has been fixed
- ▶ Children: solve the relaxation by column generation, select the variable y with the most integral value $v \neq 0$, for each possible value v' of y create one child.
- ▶ The discrepancy of a child is computed as:

$$\text{disc}_{\text{child}} = \text{disc}_{\text{father}} + |v' - v|$$

Algorithms:

- ▶ Greedy
- ▶ Limited Discrepancy Search

Note that the depth of the tree is of the order of the number of columns in a solution.

Additional tricks

- ▶ Using a fast heuristic algorithm to solve the pricing problem. If the heuristic doesn't find a column of negative reduced cost:
 - ▶ Case 1: Try with a more expensive exact algorithm
 - ▶ Case 2: Stop the column generation procedure. The bound is not valid, therefore, it is not possible to use an exact Branch-and-price in this case. But the heuristics still work.
- ▶ Generating columns without the simplex algorithm
 - ▶ It might be faster than the column generation procedure
 - ▶ It might be difficult to generate columns that fit well together
 - ▶ No bound
 - ▶ Then solve the restricted master or use a Heuristic Tree Search algorithm
- ▶ Solving the restricted master with a heuristic algorithm
 - ▶ Often, the master problem is a set covering or set packing problem for which heuristic algorithms have already been developed
 - ▶ It might be faster than a MILP solver

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columngenerationsolverpy

- ▶ A package that simplifies the implementation of Column Generation based algorithms
- ▶ Written in Python3 (original version in C++)
- ▶ <https://github.com/fontanf/columngenerationsolverpy>
- ▶ Install with: `pip3 install columngenerationsolverpy`
- ▶ It includes:
 - ▶ The Column Generation algorithm
 - ▶ The Greedy algorithm
 - ▶ The Limited Discrepancy Search algorithm
- ▶ To solve a problem, one needs to provide the exponential formulation and the solver for the Pricing Problem (able to take as input the currently fixed columns)
- ▶ The implementation of the Greedy algorithm and the Limited Discrepancy Search algorithm relies on the `treesearchsolverpy` package

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Conclusion

- ▶ Column Generation: solving the relaxation of exponential formulations by generating the columns dynamically
- ▶ It can be embedded in a classical Branch-and-bound
 - ▶ State-of-the art exact method for many Vehicle Routing and Parallel Machine Scheduling Problems
 - ▶ Cumbersome to implement
- ▶ It can be embedded in a Heuristic Tree Search framework
 - ▶ Also state-of-the-art heuristics for several problems
 - ▶ Easier to implement
- ▶ Works better when the number of elements in columns is small (≤ 20)

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