Advanced Models and Methods in Operations Research
Column Generation Heuristics

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Table of contents

Cutting Stock Problem, Description

Input:

- a capacity C
- n item types; for each item type $j = 1, \ldots, n$, a weight w_j and a demand q_j

Problem:

- Pack all items such that the total weight of the items in a bin does not exceed the capacity.

Objective:

- Minimize the number of bin used.

Cutting Stock Problem, Formulation

Let us define the K possible patterns such that $x_j^k = q$ iff pattern k, $k = 1 \dots K$ contains q copies of item type j

- Variables:
  - $y^k \in \mathrm{N}$, $\forall k = 1 \dots K$.
    $y^k = q$ iff q copies of pattern k are used
- Objective:

$$\min \sum_{k=1}^{K} y^k$$

- Constraints:

$$\sum_{k=1}^{K} x_j^k y^k = q_j \qquad\qquad \forall j = 1 \dots n$$

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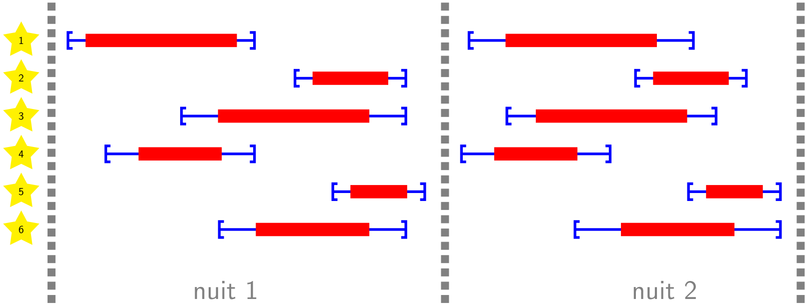
Why is this formulation good compared to the classical one?

- No big-M constraint
- Better relaxation
- Easier to write

Star Observation Scheduling Problem, Description

Input: a set $\mathcal{M}$ of nights and a set $\mathcal{N}$ of stars; for each star $j \in \mathcal{N}$, a scientific interest w_j, an observation duration p_j^i and a visibility window $[r_j^i, d_j^i[$, depending on the night i of the observation.



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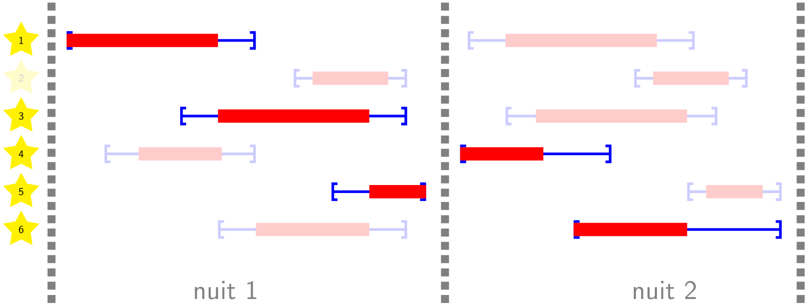

Star Observation Scheduling Problem, Formulation

For each night i, $i = 1 \dots m$, let us define the K_i possible schedules such that $x_{i,j}^k = 1$ iff schedule k, $k = 1 \dots K_i$ of night i contains star j

- Variables:
  - $y_i^k \in \{0, 1\}$, $\forall i = 1 \dots m$, $\forall k = 1 \dots K_i$.
    $y_i^k = 1$ iff scheduled k of night i is selected
- Objective:

$$\max \sum_{i=1}^{m} \sum_{k=1}^{K_i} \sum_{j=1}^{n} w_j x_{i,j}^k y_i^k$$

- Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \qquad \forall i = 1 \dots m$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x_{i,j}^k y_i^k \leq 1 \qquad \forall j = 1 \dots n$$

2D Guillotine Variable-sized Bin Packing, Description

Input:

- n item types; for each item type $j = 1, \ldots, n$, a width w_j, a height h_j and a demand q_j
- m bin types; for each bin type $i = 1, \ldots, m$, a width W_i, a height H_i, a lower bound l_i, an upper bound u_i and a cost c_i

Problem:

- Find a subset of guillotine patterns such that all item type demands and bin type use bounds are satisfied

Objective:

- Minimize the cost of the selected bins.

2D Guillotine Variable-sized Bin Packing, Formulation

For each bin type i, $i = 1 \dots m$, let us define the K_i possible patterns such that $x^k_{i,j} = q$ iff pattern k, $k = 1 \dots K_i$ of bin type i contains q copies of item type j

- Variables:
  - $y^k_i \in \mathrm{N}$, $\forall i = 1 \dots m$, $\forall k = 1 \dots K_i$.
    $y^k_i = q$ iff q copies of pattern k of bin type i are used
- Objective:

$$\min \sum_{i=1}^{m} \sum_{k=1}^{K_i} c_i y^k_i$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y^k_i \leq u_i \qquad \forall i = 1 \dots m$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x^k_{i,j} y^k_i = q_j \qquad \forall j = 1 \dots n$$

Other examples

Usually, variables represent:

- A bin/knapsack (for packing problems)
- The schedule of a machine (for parallel scheduling problems)
- The route of a vehicle (for vehicle routing problems)
- ...

Table of contents

Introduction

- With these formulations, generating all the variables is generally not possible since their number grows exponentially with the size of the problem.
- First we focus on solving the linear relaxation

The Column Generation procedure

- We use the simplex algorithm.
  - At each iteration, it adds a variable of negative reduced cost to the current basis
    - Objective:

$$\min \sum_{j=1}^{n} c_j x_j$$

    - Constraints:

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_j \qquad \forall j = 1 \ldots n$$

    - Reduced cost of variable x_j:

$$c_j - \sum_{i=1}^{m} a_{i,j} v_i$$

      with v_i the dual value of constraint i.
  - It stops when there are no variable of negative reduced cost
- The difference with the traditional simplex algorithm, is that here, it is not possible to loop through all the variables to find a variable of negative reduced cost, since they have not been all generated.

The Column Generation procedure

- Instead, finding a variable of negative reduced cost becomes an optimization problem
- Example with the Cutting Stock Problem
  - Objective:

$$\min \sum_{k=1}^{K} y^k$$

  - Constraints:

$$\sum_{k=1}^{K} x_j^k y^k = q_j \qquad \forall j = 1 \dots n$$

- Reduced cost of y^k:

$$1 - \sum_{j=1}^{n} x_j^k v_j$$

with v_j the dual value of constraint j.

The Column Generation procedure

- We look for a variable y^k such that

$$1 - \sum_{j=1}^{n} x_j^k v_j < 0$$

- Finding a variable of negative reduced cost is equivalent to finding a pattern with total profit ≥ 1 with the profit of item type j being equal to v_j.
- In practice, we solve the problem as an optimization problem: we find the best solution of the Knapsack Problem and check if the reduced cost of the corresponding variable is negative.

The Column Generation procedure

Summary:

```
function ColumnGeneration(P)
    Y ← initial set of columns
    while True do
        Solve the Linear Program P′ with variables from Y
        Look for a variable of negative reduced cost (Pricing Problem)
        if there is one then
            Add it to Y
        else
            return the solution of P′
```

Initial set of columns

- To get dual values, the LP needs to be feasible
- With 0 variable, the LP might be infeasible
  - Example: Cutting Stock Problem, demand constraints are not satisfied
- Therefore, we need to find a way to get an initial set of columns such that the LP is feasible
  - Find a feasible solution and add the corresponding columns
    - Example: Cutting Stock, Best Fit algorithm
    - Drawback: Problem specific, additional work for the implementation of the heuristic
    - Advantage: if the solution is good, it might speed up the column generation proceudre
  - Find manually a set of columns that ensures the LP to be feasible
    - Example: create n columns with only one item
  - Generate a dummy column with very high cost for each problematic constraint
    - Advantage: not problem specific
    - Drawback: numerical issue is the cost of the dummy columns is not well calibrated

Star Observation Scheduling Problem, Pricing

- Objective:

$$\max \sum_{i=1}^{m} \sum_{k=1}^{K_i} \sum_{j=1}^{n} w_j x_{i,j}^k y_i^k$$

- Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \qquad \forall i = 1 \dots m \qquad \text{dual } u_i$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x_{i,j}^k y_i^k \leq 1 \qquad \forall j = 1 \dots n \qquad \text{dual } v_j$$

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- Reduced cost of y_i^k:

$$\sum_{j=1}^{n} w_j x_{i,j}^k - u_i - \sum_{j=1}^{n} x_{i,j}^k v_j = \sum_{j=1}^{n} (w_j - v_j) x_{i,j}^k - u_i$$

Star Observation Scheduling Problem, Pricing

- Objective:

$$\max \sum_{i=1}^{m} \sum_{k=1}^{K_i} \sum_{j=1}^{n} w_j x_{i,j}^k y_i^k$$

- Constraints:

$$\sum_{k=1}^{K_i} y_i^k = 1 \qquad \forall i = 1 \dots m \qquad \text{dual } u_i$$

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- Reduced cost of y_i^k:

$$\sum_{j=1}^{n} w_j x_{i,j}^k - u_i - \sum_{j=1}^{n} x_{i,j}^k v_j = \sum_{j=1}^{n} (w_j - v_j) x_{i,j}^k - u_i$$

- Finding a variable of maximum reduced cost reduces to solving m Single Night Star Observation Scheduling Problems with targets with profit $w_j - v_j$.

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^{m} \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \qquad \forall i = 1 \ldots m \qquad \text{dual } u_i'$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \qquad \forall j = 1 \ldots n \qquad \text{dual } v_j$$

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^{m} \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \qquad \forall i = 1 \dots m \qquad \text{dual } u_i'$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \qquad \forall j = 1 \dots n \qquad \text{dual } v_j$$

- Reduced cost of y_i^k:

$$c_i - u_i' - \sum_{j=1}^{n} x_j^k v_j$$

2D Guillotine Variable-sized Bin Packing, Pricing

- Objective:

$$\min \sum_{i=1}^{m} \sum_{k=1}^{K_i} c_i y_i^k$$

- Constraints:

$$l_i \leq \sum_{k=1}^{K_i} y_i^k \leq u_i \qquad \forall i = 1 \ldots m \qquad \text{dual } u_i'$$

$$\sum_{i=1}^{n} \sum_{k=1}^{K_i} x_{i,j}^k y_i^k = q_j \qquad \forall j = 1 \ldots n \qquad \text{dual } v_j$$

- Reduced cost of y_i^k:

$$c_i - u_i' - \sum_{j=1}^{n} x_j^k v_j$$

- Finding a variable of minium reduced cost reduces to solving m 2D Guillotine Knapsack Problems with items with profit v_j for each bin type.

Transition

- The Column Generation procedure solves the relaxation of the exponential formulation

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Transition

- The Column Generation procedure solves the relaxation of the exponential formulation
- Thus, it provides a valid bound
- But it generally does not provide a feasible solution
- How to exploit the Column Generation to get feasible solutions?

Table of contents

The Branch-and-Price algorithm (1)

- LP-based branch-and-bound, the relaxation is solved by the Column Generation procedure in each node
- How to branch?
  - Branching on columns of the exponential formulation? No, the pricing problem becomes to difficult
  - Branching on the variables of the compact formulation?
    - Bin Packing: branch on whether item j is packed in bin i or not. If yes, then the available bins have now different capacities and a Knapsack Problem for each capacity needs to be computed
  - Best solution for the Bin Packing: branch on whether two items are packed in the same bin or not. If yes, then they are merged into a single item. If no, then the subproblem becomes a Knapsack Problem with Conflicts which is strongly NP-hard instead of the Knapsack Problem
  - $\Longrightarrow$ Branching rules are usually problem dependent and might change the pricing problem, making it harder to solve

The Branch-and-Price algorithm (2)

- It can be combined with cuts (Branch-and-Price-and-Cut).
  - The added cuts might also change the pricing problem $\implies$ even more complex to implement
- Only exact method based on Column Generation, state-of-the-art exact method for many Vehicle Routing and Parallel Machine Scheduling Problems

Solving the restricted master

- The Column Generation procedure is executed once
- Solve the exponential formulation with a MILP solver using only the columns generated during the Column Generation procedure
- No guarantee to find the optimal solution (or even a feasible solution)
- Solving the MILP is computationaly expensive if many columns have been generated. It can take some time before finding a first solution
- It requires a good MILP solver

Heuristic Tree Search

Branching scheme:

- Root node: no column has been fixed
- Children: solve the relaxation by column generation, select the variable y with the most integral value $v \neq 0$, for each possible value v' of y create one child.
- The discrepancy of a child is computed as:

$$\mathrm{disc}_{\mathsf{child}} = \mathrm{disc}_{\mathsf{father}} + |v' - v|$$

Algorithms:

- Greedy
- Limited Discrepancy Search

Note that the depth of the tree is of the order of the number of columns in a solution.

Additional tricks

- Using a fast heuristic algorithm to solve the pricing problem. If the heuristic doesn't find a column of negative reduced cost:
  - Case 1: Try with a more expensive exact algorithm
  - Case 2: Stop the column generation procedure. The bound is not valid, therefore, it is not possible to use an exact Branch-and-price in this case. But the heuristics still work.
- Generating columns without the simplex algorithm
  - It might be faster than the column generation procedure
  - It might be difficult to generate columns that fit well together
  - No bound
  - Then solve the restricted master or use a Heuristic Tree Search algorithm
- Solving the restricted master with a heuristic algorithm
  - Often, the master problem is a set covering or set packing problem for which heuristic algorithms have already been developed
  - It might be faster than a MILP solver

Table of contents

columngenerationsolverpy

- A package that simplifies the implementation of Column Generation based algorithms
- Written in `Python3` (original version in `C++`)
- `https://github.com/fontanf/columngenerationsolverpy`
- Install with: `pip3 install columngenerationsolverpy`
- It includes:
  - The Column Generation algorithm
  - The Greedy algorithm
  - The Limited Discrepancy Search algorithm
- To solve a problem, one needs to provide the exponential formulation and the solver for the Pricing Problem (able to take as input the currently fixed columns)
- The implementation of the Greedy algorithm and the Limited Discpreancy Search algorithm relies on the `treesearchsolverpy` package

Table of contents

Conclusion

- Column Generation: solving the relaxation of exponential formulations by generating the columns dynamically
- It can be embedded in a classical Branch-and-bound
  - State-of-the art exact method for many Vehicle Routing and Parallel Machine Scheduling Problems
  - Cumbersome to implement
- It can be embedded in a Heuristic Tree Search framework
  - Also state-of-the-art heuristics for several problems
  - Easier to implement
- Works better when the number of elements in columns is small (≤ 20)

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