Advanced Models and Methods in Operations Research
Dynamic Programming

Florian Fontan

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Me

- Former student of ORCO (2015–2016)
- PhD in Operations Research
- Engineer at Artelys: `https://www.artelys.com/`
  - Artelys is an independent company specialized in optimization, decision support and modeling
  - Design and implementation of optimization algorithms for industrial clients
  - Development for the NLP/MINLP solver Artelys Knitro `https://www.artelys.com/solvers/knitro/`
- My GitHub: `https://github.com/fontanf/`

Organization

- 4 classes with me:
  - Dynamic Programming
  - Heuristic Tree Search
  - Column Generation Heuristics
  - Project Presentations
- 1.5 hours lecture / 1.5 hours practical training
- Goal of the classes: understanding the theory of the methods and being able to implement them to solve practical problems

Organization

- Evaluation:
  - Not in the final exam
  - Project by groups of 3 or 4
  - Implementation of the algorithms studied in the classes
  - First deadline with feedbacks
  - Second deadline with final grade
  - Please do not put your code on a public repository
  - You can keep it on a private repository. It can be valuable if you decide to apply for a company some day
- All materials (slides, projects...) are online `https://github.com/fontanf/teaching`
- E-mail: dev@florian-fontan.fr

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Problem definition

Partition Problem

- Instance: n items with weight w_j, $j = 1, \ldots, n$.
- Question: is it possible to partition the set of items into two subsets of equal weights?

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Subset Sum Problem (decision version)

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  - n items with weight w_j, $j = 1, \ldots, n$
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Link between the Partition Problem and the Subset Sum Problem?
The Partition Problem is a Subset Sum Problem with capacity $C = \frac{1}{2} \sum_{j=1}^{n} w_j$.

Recursive function

For all $j = 0, \ldots, n$, $c = 0, \ldots, C$, let us define:

$$F(j,c) = \begin{cases} \text{True} & \text{if among items } 1, \ldots, j, \text{ there exists} \\ & \text{a subset of items with total weight } c \\ \text{False} & \text{otherwise} \end{cases}$$

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What is the value of

- $F(0, 0)$?

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What is the relation between the Subset Sum Problem and F?

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What is the value of

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- $F(0, c)$? True for $c = 0$, False otherwise
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What is the relation between the Subset Sum Problem and F?
The Subset Sum Problem is equivalent to determining the value of $F(n, C)$.

Computing $F(n, C)$

We compute $F(j, c)$ with the following recursive formula:

$$F(j,c) = \begin{cases} \text{True} & \text{if } j = 0 \text{ and } c = 0 \\ \text{False} & \text{if } j = 0 \text{ and } c \neq 0 \\ F(j-1,c) & \textit{if}\, j \neq 0 \text{ and } c < w_j \\ F(j-1,c) & \text{otherwise} \\ \text{or } F(j-1,c-w_j) & \end{cases}$$

Divide-and-conquer

We can now solve the Subset Sum Problem with a divide-and-conquer algorithm:

function F(w, j, c)
    if $j == 0$ **then**
        if $c == 0$ **then**
            return *True*
        else
            return *False*
    else if $c < w_j$ **then**
        return $F(j-1, c)$
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function subsetsum(w, C)
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- Time complexity? $O(2^n)$
- Space complexity? $O(n)$

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```
function F(w, j, c)
    if j == 0 then
        if c == 0 then
            return True
        else
            return False
    else if c < w_j then
        return F(j − 1, c)
    else
        return F(j − 1, c) or F(j − 1, c − w[j])
function subsetsum(w, C)
    return F(w, n, C)
```

- Time complexity? $O(2^n)$
- Space complexity? $O(n)$

Is there a way to improve the time complexity?

Divide-and-conquer example

Consider an instance I of the Subset Sum Problem with

- $n = 103$
- $C = 1000$
- $w_{101} = 30$, $w_{102} = 20$, $w_{103} = 10$.

$$\boxed{F(103, 1000)}$$

Divide-and-conquer example

Consider an instance I of the Subset Sum Problem with

- $n = 103$
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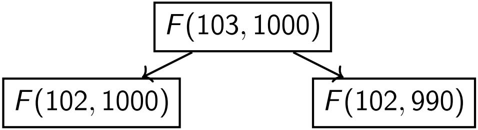

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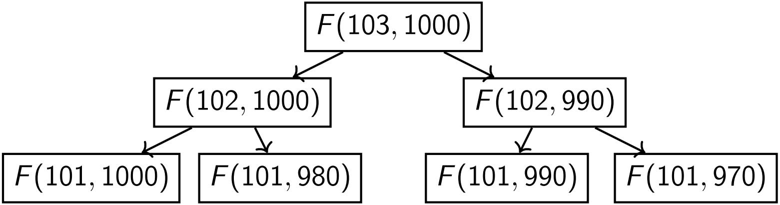

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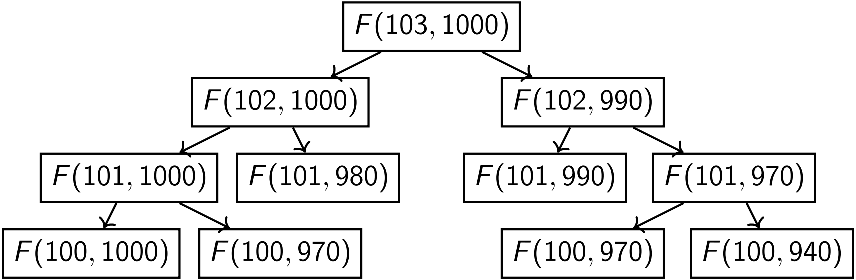

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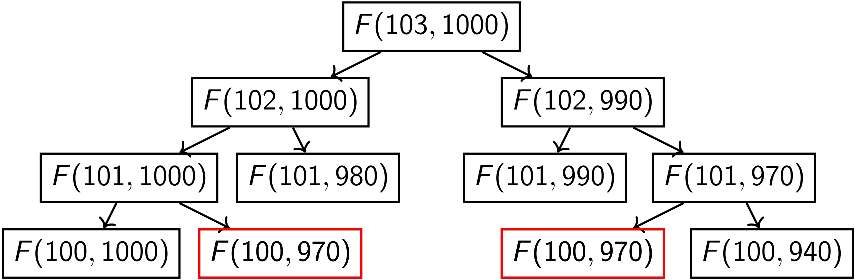

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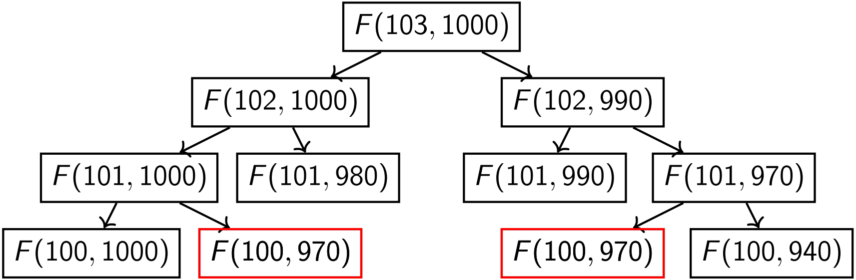

The same subproblems might be solved multiple times!

Dynamic Programming: recursive implementation (top-down)

Same algorithm as before, but now we store the results of the subproblems to avoid solving multiple times the same subproblems:

procedure F(w, T, j, c)
  if $T[j][c] == \mathrm{NULL}$ **then**
    if $j == 0$ **then**
      if $c == 0$ **then**
        $T[j][c] \leftarrow$ True
      else
        $T[j][c] \leftarrow$ False
    else if $c < w[j]$ **then**
      $T[j][c] \leftarrow F(j-1, c)$
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procedure subsetsum(w, C)
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procedure subsetsum(w, C)
  $T \leftarrow$ array of size $(n+1)\times(C+1)$ initialized at NULL
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- Time complexity? $O(nC)$

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procedure subsetsum(w, C)
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- Time complexity? $O(nC)$
- Space complexity? $O(nC)$

Dynamic Programming

Dynamic Programming

Solving a problem recursively and storing the results of the subproblems to avoid recomputing them multiple times.

Dynamic Programming: iterative implementation (bottom-up)

procedure subsetsum(w, C)
  $T \leftarrow$ array of size $(n+1) \times (C+1)$ initialized at NULL
  $T[0][0] \leftarrow$ True
  for $c = 1, \ldots, C$ **do**
    $T[0][c] \leftarrow$ False
  for $j = 1, \ldots, n$ **do**
    for $c = 0, \ldots, w[j]-1$ **do**
      $T[j][c] \leftarrow T[j-1][c]$
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- Time complexity?

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    for $c = 0, \ldots, w[j] - 1$ **do**
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- Time complexity? $O(nC)$

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- Space complexity?

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- Time complexity? $O(nC)$
- Space complexity? $O(nC)$
- In practice, 10 times faster than the recursive implementation

Dynamic Programming example

Instance:

- $n = 5$, $w = \{4, 11, 6, 8, 7\}$
- $C = 17$

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Reminder:

$$F(j,c) = \begin{cases} \text{True} & \text{if } j = 0 \text{ and } c = 0 \\ \text{False} & \text{if } j = 0 \text{ and } c \neq 0 \\ F(j-1,c) & if\, j \neq 0 \text{ and } c < w_j \\ F(j-1,c) & \text{otherwise} \\ \text{or } F(j-1, c-w_j) & \end{cases}$$

j / c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

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0	T	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F

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Dynamic Programming example

Instance:

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- $C = 17$

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$S = \{5, 3, 1\}$

Retrieving the solution with backtracking

function SSPbacktracking(w, C, T)
    $S \leftarrow \{\}$
    $c \leftarrow C$
    $i \leftarrow n$
    while $i > 0$ **do**
        if not $T[i-1][c]$ **then**
            $S \leftarrow S \cup \{i\}$
            $c \leftarrow c - w[j]$
        $i \leftarrow i - 1$
    return S

Going further

- Write an algorithm computing $F(n, C)$ which only keeps two lines of the array in memory (spatial complexity $O(C)$)
- Write an algorithm computing $F(n, C)$ which only keeps a single line of the array in memory.
- How to return a solution when keeping only a single line in memory?
  - if the array is stored as an array of integers
  - if the array is stored as an array of bits

Table of contents

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Knapsack Problem

- Instance:
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Link between the Subset Sum Problem and the Knapsack Problem? The Subset Sum Problem is a Knapsack Problem with $p_j = w_j$ for all $j = 1, \ldots, n$.

Recursive function

For all $j = 0, \ldots, n$, $c = 0, \ldots, C$, let us define $F(j, c)$ the maximum profit of a subset of items $1, \ldots, j$ with total weight less than or equal to c.

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What is the relation between the Knapsack Problem and F?
The Knapsack Problem is equivalent to determining the value of $F(n, C)$.

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We compute $F(j, c)$ with the following recursive formula:

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j / c	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0

Computing $F(n, C)$

We compute $F(j, c)$ with the following recursive formula:

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Example: $C = 7$

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2	5	6
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0	0	0	0	0	0	0	0	0
1	0	0	0	4	4	4	4	4

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$S = \{3, 1\}$

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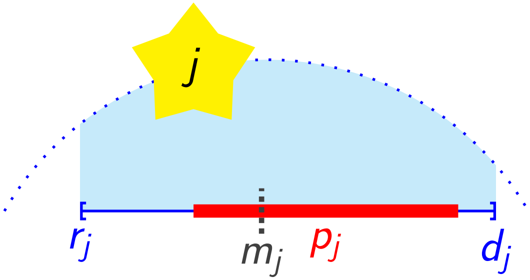
A star

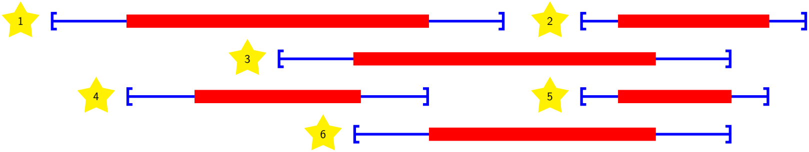


- $[r_j, d_j[$: visibility interval
- p_j: observation time
- w_j: scientific interest of the observation

Every observation j has a meridian $m_j \in [r_j, d_j[$ which is a mandatory instant of the observation.

Problem definition

Instance: a set of stars $\mathcal{N}$; each star $j \in \mathcal{N}$ has a scientific interest w_j, an observation time p_j and a time window $[r_j, d_j[$



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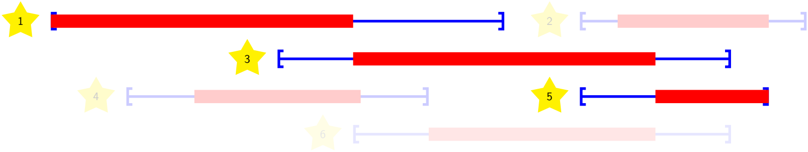

Problem: find a subset $\mathcal{N}' \subset \mathcal{N}$ as well as the start date s_j of each selected observation $j \in \mathcal{N}'$ such that:

- for all $j \in \mathcal{N}'$: $[s_j, s_j + p_j[\subset [r_j, d_j[$
- for all $(j_1, j_2) \in \mathcal{N}'^2$: $[s_{j_1}, s_{j_1} + p_{j_1}[\cap [s_{j_2}, s_{j_2} + p_{j_2}[= \emptyset$

Objective: maximize $\sum_{j \in \mathcal{N}'} w_j$ the profit of the selected observations.

Properties

Property 1

There exists an optimal solution in which selected observations are scheduled in non-decreasing order of their mandatory instant.

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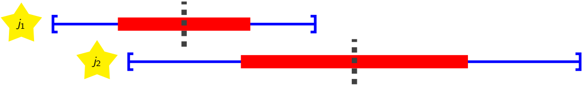

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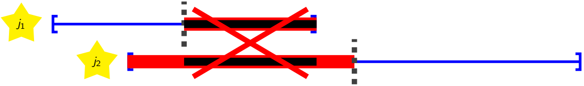

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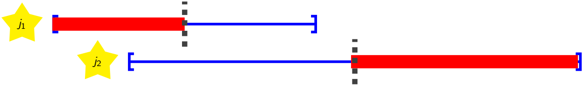

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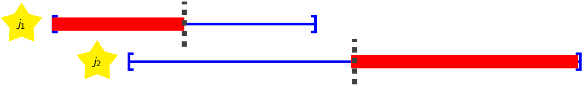

Property 2

Consider a subset $\mathcal{N}' \subset \mathcal{N}$ and an observation $j_{\max}$ such that $d_{j_{\max}} = \max_{j \in \mathcal{N}'} d_j$. If there exists a feasible solution with selected observations $\mathcal{N}'$, then there exists a feasible solution with selected observations $\mathcal{N}'$ such that $s_{j_{\max}} = d_{j_{\max}} - p_{j_{\max}}$.

Recursive function

We consider than the observations are numbered in non-decreasing order of their mandatory instant.

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For all $j = 0, \ldots, n$, $t = 0, \ldots, T$, let us define $F(j, t)$ the maximum scientific interest of a subset of observations of $1, \ldots, j$ during the interval $[0, T]$.

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Complexity: $O(nT)$

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Sometimes, a bit of work is needed in order to exhibit the structure to design an efficient algorithm based on Dynamic Programming.

Table of contents

Breadth First Search: Example (Knapsack Problem)

Example:
$C = 5$

Item	Weight	Profit
1	3	2
2	2	2
3	2	3
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(j, w, p)

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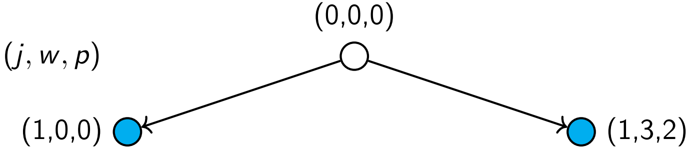

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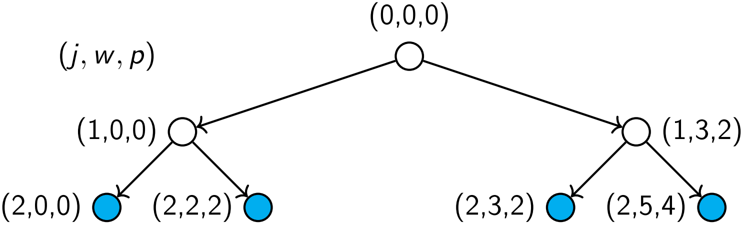

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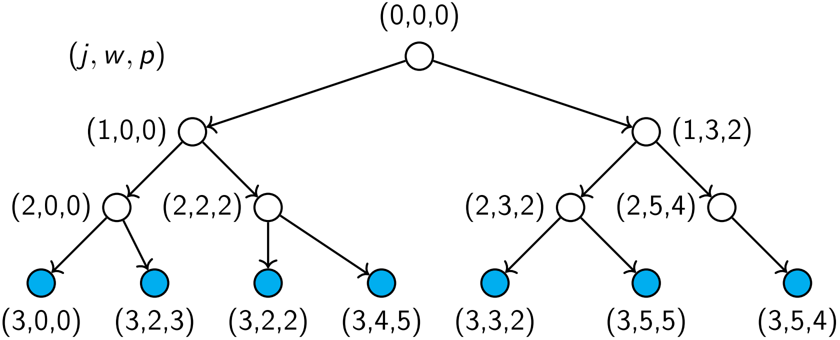

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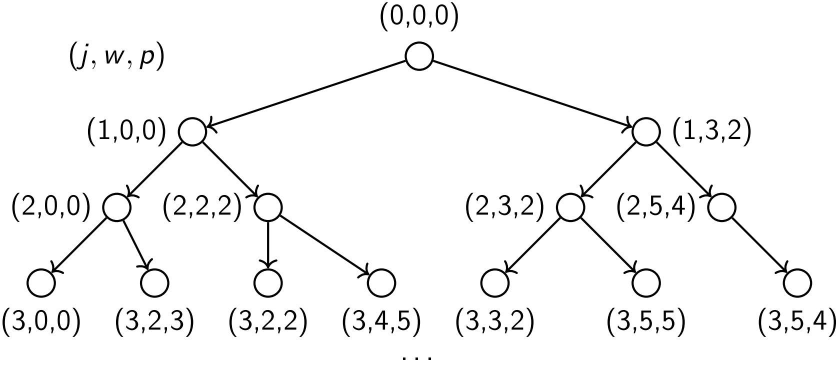

Breadth First Search (Knapsack Problem)

procedure BreadthFirstSearch(w, C)
$L_0 \leftarrow ((0,0,0))$
for $j = 1, \ldots, n$ **do**
for $(j, w, p) \in L_{j-1}$ **do**
$L_j \leftarrow L_j \cup ((j, w, p))$
if $w + w[j] \leq C$ **then**
$L_j \leftarrow L_j \cup ((j, w + w_j, p + p_j))$

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- Time complexity?

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- Space complexity?

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- Time complexity? $O(2^n)$
- Space complexity? $O(2^n)$

Dominance rule (Knapsack Problem)

We express Dynamic Programming as a dominance rule: Consider two nodes $n_1 = (j_1, w_1, p_1)$ and $n_2 = (j_2, w_2, p_2)$. If

$$j_1 \leq j_2 \quad \text{and} \quad w_1 \leq w_2 \quad \text{and} \quad p_1 \geq p_2$$

then node n_1 dominates node n_2 and therefore node n_2 can be safely pruned.

Breadth First Search + Dominance rule: Example (Knapsack Problem)

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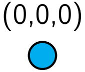

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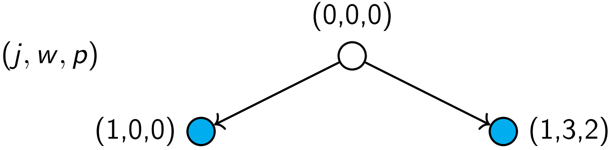
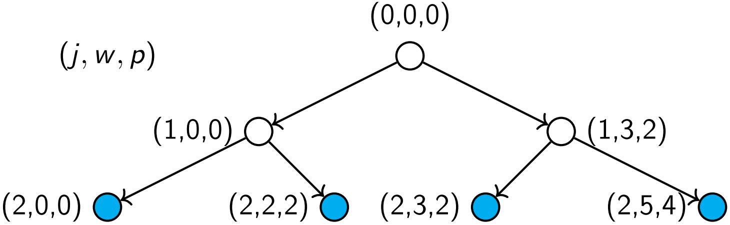
Example:
$C = 5$



Breadth First Search + Dominance rule: Example (Knapsack Problem)

Item	Weight	Profit
1	3	2
2	2	2
3	2	3
4	1	2

Example:
$C = 5$



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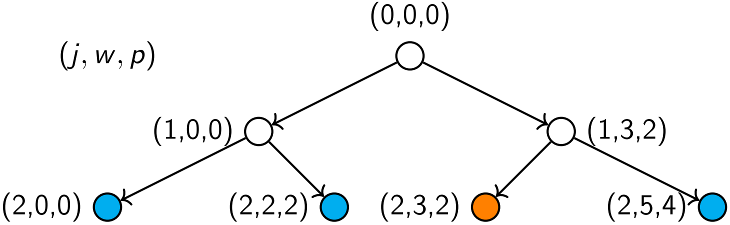

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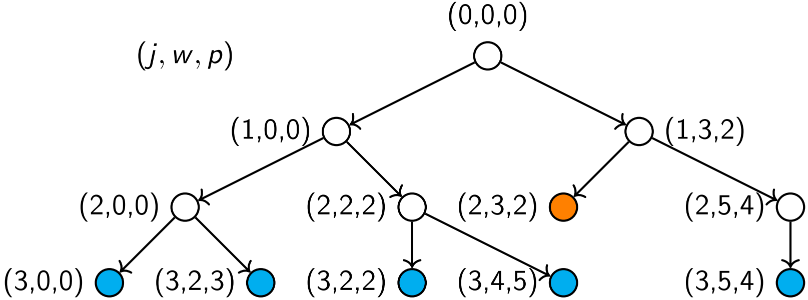

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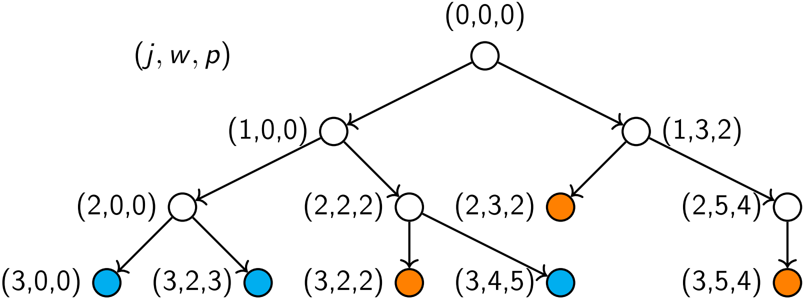

Breadth First Search + Dominance rule: Example (Knapsack Problem)

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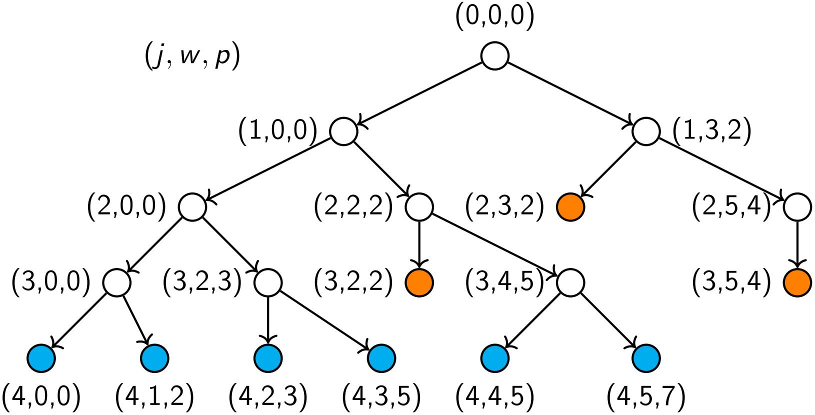

Breadth First Search + Dominance rule (Knapsack Problem)

procedure BreadthFirstSearch(w, C)
    $L_0 \leftarrow ((0, 0, 0))$
    for $j = 1, \ldots, n$ **do**
        for $(j, w, p) \in L_{j-1}$ **do**
            $L_j \leftarrow L_j \cup ((j, w, p))$
            if $w + w[j] \leq C$ **then**
                $L_j \leftarrow L_j \cup ((j, w + w_j, p + p_j))$
        Remove all dominated nodes from L_j

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        Remove all dominated nodes from L_j

- Time complexity?

Breadth First Search + Dominance rule (Knapsack Problem)

```
procedure BreadthFirstSearch(w, C)
    L_0 ← ((0, 0, 0))
    for j = 1, ..., n do
        for (j, w, p) ∈ L_{j-1} do
            L_j ← L_j ∪ ((j, w, p))
            if w + w[j] ≤ C then
                L_j ← L_j ∪ ((j, w + w_j, p + p_j))
        Remove all dominated nodes from L_j
```

- Time complexity? The complexity depends on the complexity of applying the dominance rule!
  In this case, it is possible to implement it in $O(C)$ and keep the complexity of the whole algorithm to $O(nC)$ as for the iterative implementation. But in some cases, the complexity might increase.

Table of contents

Conclusion

- Dynamic Programming: solving a problem recursively and storing the results of the subproblems to avoid recomputing them multiple times.
- It requires the problem to have a specific structure. It might not be applicable to all problems.
- Sometimes, a bit of work is needed in order to exhibit this structure
- Multiple possible implementations with their advantages and drawbacks (recursive, iterative, tree search)
- "Knapsack Problems" (Kellerer, Pferschy et Pisinger, 2004)
- Dynamic Programming through path problems: `https://moodle.caseine.org/mod/page/view.php?id=30723`

Advanced Models and Methods in Operations Research
Dynamic Programming

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