

# ARONDIGHT'S STANDARD CODE LIBRARY<sup>\*</sup>

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<sup>\*</sup><https://github.com/footoredo/Arondight>



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# Chapter 1

## 计算几何

```
1 int sign(DB x) {
2     return (x > eps) - (x < -eps);
3 }
4 DB msqrt(DB x) {
5     return sign(x) > 0 ? sqrt(x) : 0;
6 }
7
8 struct Point {
9     DB x, y;
10    Point rotate(DB ang) const { // 逆时针旋转 ang 弧度
11        return Point(cos(ang) * x - sin(ang) * y,
12                     cos(ang) * y + sin(ang) * x);
13    }
14    Point turn90() const { // 逆时针旋转 90 度
15        return Point(-y, x);
16    }
17    Point unit() const {
18        return *this / len();
19    }
20 };
21 DB dot(const Point& a, const Point& b) {
22     return a.x * b.x + a.y * b.y;
23 }
24 DB det(const Point& a, const Point& b) {
25     return a.x * b.y - a.y * b.x;
26 }
27 #define cross(p1,p2,p3) ((p2.x-p1.x)*(p3.y-p1.y)-(p3.x-p1.x)*(p2.y-p1.y))
28 #define crossOp(p1,p2,p3) sign(cross(p1,p2,p3))
29 bool isLL(const Line& l1, const Line& l2, Point& p) { // 直线与直线交点
30     DB s1 = det(l2.b - l2.a, l1.a - l2.a),
31         s2 = -det(l2.b - l2.a, l1.b - l2.a);
32     if (!sign(s1 + s2)) return false;
33     p = (l1.a * s2 + l1.b * s1) / (s1 + s2);
34     return true;
35 }
```

```

36 bool onSeg(const Line& l, const Point& p) { // 点在线段上
37     return sign(det(p - l.a, l.b - l.a)) == 0 && sign(dot(p - l.a, p - l.b)) <= 0;
38 }
39 Point projection(const Line & l, const Point& p) {
40     return l.a + (l.b - l.a) * (dot(p - l.a, l.b - l.a) / (l.b - l.a).len2());
41 }
42 DB distoLine(const Line& l, const Point& p) { // 点到 * 直线 * 距离
43     return fabs(det(p - l.a, l.b - l.a) / (l.b - l.a).len());
44 }
45 DB distoSeg(const Line& l, const Point& p) { // 点到线段距离
46     return sign(dot(p - l.a, l.b - l.a)) * sign(dot(p - l.b, l.a - l.b)) == 1 ? distoLine(l, p)
47     ↪ : std::min((p - l.a).len(), (p - l.b).len());
48 }
49 // 圆与直线交点
50 bool isCL(Circle a, Line l, Point& p1, Point& p2) {
51     DB x = dot(l.a - a.o, l.b - l.a),
52     y = (l.b - l.a).len2(),
53     d = x * x - y * ((l.a - a.o).len2() - a.r * a.r);
54     if (sign(d) < 0) return false;
55     Point p = l.a - ((l.b - l.a) * (x / y)), delta = (l.b - l.a) * (msqrt(d) / y);
56     p1 = p + delta; p2 = p - delta;
57     return true;
58 }
59 // 圆与圆的交面积
60 DB areaCC(const Circle& c1, const Circle& c2) {
61     DB d = (c1.o - c2.o).len();
62     if (sign(d - (c1.r + c2.r)) >= 0) return 0;
63     if (sign(d - std::abs(c1.r - c2.r)) <= 0) {
64         DB r = std::min(c1.r, c2.r);
65         return r * r * PI;
66     }
67     DB x = (d * d + c1.r * c1.r - c2.r * c2.r) / (2 * d),
68     t1 = acos(x / c1.r), t2 = acos((d - x) / c2.r);
69     return c1.r * c1.r * t1 + c2.r * c2.r * t2 - d * c1.r * sin(t1);
70 }
71 // 圆与圆交点
72 bool isCC(Circle a, Circle b, P& p1, P& p2) {
73     DB s1 = (a.o - b.o).len();
74     if (sign(s1 - a.r - b.r) > 0 || sign(s1 - std::abs(a.r - b.r)) < 0) return false;
75     DB s2 = (a.r * a.r - b.r * b.r) / s1;
76     DB aa = (s1 + s2) * 0.5, bb = (s1 - s2) * 0.5;
77     P o = (b.o - a.o) * (aa / (aa + bb)) + a.o;
78     P delta = (b.o - a.o).unit().turn90() * msqrt(a.r * a.r - aa * aa);
79     p1 = o + delta, p2 = o - delta;
80     return true;
81 }
82 // 求点到圆的切点, 按关于点的顺时针方向返回两个点
83 bool tanCP(const Circle &c, const Point &p0, Point &p1, Point &p2) {
84     double x = (p0 - c.o).len2(), d = x - c.r * c.r;

```

```

84     if (d < eps) return false; // 点在圆上认为没有切点
85     Point p = (p0 - c.o) * (c.r * c.r / x);
86     Point delta = ((p0 - c.o) * (-c.r * sqrt(d) / x)).turn90();
87     p1 = c.o + p + delta;
88     p2 = c.o + p - delta;
89     return true;
90 }
91 // 求圆到圆的外共切线, 按关于 c1.o 的顺时针方向返回两条线
92 vector<Line> extanCC(const Circle &c1, const Circle &c2) {
93     vector<Line> ret;
94     if (sign(c1.r - c2.r) == 0) {
95         Point dir = c2.o - c1.o;
96         dir = (dir * (c1.r / dir.len())).turn90();
97         ret.push_back(Line(c1.o + dir, c2.o + dir));
98         ret.push_back(Line(c1.o - dir, c2.o - dir));
99     } else {
100         Point p = (c1.o * -c2.r + c2.o * c1.r) / (c1.r - c2.r);
101         Point p1, p2, q1, q2;
102         if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) {
103             if (c1.r < c2.r) swap(p1, p2), swap(q1, q2);
104             ret.push_back(Line(p1, q1));
105             ret.push_back(Line(p2, q2));
106         }
107     }
108     return ret;
109 }
110 // 求圆到圆的内共切线, 按关于 c1.o 的顺时针方向返回两条线
111 std::vector<Line> intanCC(const Circle &c1, const Circle &c2) {
112     std::vector<Line> ret;
113     Point p = (c1.o * c2.r + c2.o * c1.r) / (c1.r + c2.r);
114     Point p1, p2, q1, q2;
115     if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) { // 两圆相切认为没有切线
116         ret.push_back(Line(p1, q1));
117         ret.push_back(Line(p2, q2));
118     }
119     return ret;
120 }
121 bool contain(vector<Point> polygon, Point p) { // 判断点 p 是否被多边形包含, 包括落在边界上
122     int ret = 0, n = polygon.size();
123     for(int i = 0; i < n; ++ i) {
124         Point u = polygon[i], v = polygon[(i + 1) % n];
125         if (onSeg(Line(u, v), p)) return true; // Here I guess.
126         if (sign(u.y - v.y) <= 0) swap(u, v);
127         if (sign(p.y - u.y) > 0 || sign(p.y - v.y) <= 0) continue;
128         ret += sign(det(p, v, u)) > 0;
129     }
130     return ret & 1;
131 }
132 // 用半平面 (q1,q2) 的逆时针方向去切凸多边形

```

```

133 std::vector<Point> convexCut(const std::vector<Point>&ps, Point q1, Point q2) {
134     std::vector<Point> qs; int n = ps.size();
135     for (int i = 0; i < n; ++i) {
136         Point p1 = ps[i], p2 = ps[(i + 1) % n];
137         int d1 = crossOp(q1, q2, p1), d2 = crossOp(q1, q2, p2);
138         if (d1 >= 0) qs.push_back(p1);
139         if (d1 * d2 < 0) qs.push_back(isSS(p1, p2, q1, q2));
140     }
141     return qs;
142 }
143 // 求凸包
144 std::vector<Point> convexHull(std::vector<Point> ps) {
145     int n = ps.size(); if (n <= 1) return ps;
146     std::sort(ps.begin(), ps.end());
147     std::vector<Point> qs;
148     for (int i = 0; i < n; qs.push_back(ps[i ++]))
149         while (qs.size() > 1 && sign(det(qs[qs.size() - 2], qs.back(), ps[i])) <= 0)
150             qs.pop_back();
151     for (int i = n - 2, t = qs.size(); i >= 0; qs.push_back(ps[i --]))
152         while ((int)qs.size() > t && sign(det(qs[qs.size() - 2], qs.back(), ps[i])) <= 0)
153             qs.pop_back();
154     return qs;

```

## 1.1 凸包

```

1 // 凸包中的点按逆时针方向
2 struct Convex {
3     int n;
4     std::vector<Point> a, upper, lower;
5     void make_shell(const std::vector<Point>& p,
6                     std::vector<Point>& shell) { // p needs to be sorted.
7         clear(shell); int n = p.size();
8         for (int i = 0, j = 0; i < n; i++, j++) {
9             for (; j >= 2 && sign(det(shell[j-1] - shell[j-2],
10                                     p[i] - shell[j-2])) <= 0; --j) shell.pop_back();
11             shell.push_back(p[i]);
12         }
13     }
14     void make_convex() {
15         std::sort(a.begin(), a.end());
16         make_shell(a, lower);
17         std::reverse(a.begin(), a.end());
18         make_shell(a, upper);
19         a = lower; a.pop_back();
20         a.insert(a.end(), upper.begin(), upper.end());
21         if ((int)a.size() >= 2) a.pop_back();
22         n = a.size();
23     }

```



```

24 void init(const std::vector<Point>& _a) {
25     clear(a); a = _a; n = a.size();
26     make_convex();
27 }
28 void read(int _n) { // Won't make convex.
29     clear(a); n = _n; a.resize(n);
30     for (int i = 0; i < n; i++)
31         a[i].read();
32 }
33 std::pair<DB, int> get_tangent(
34     const std::vector<Point>& convex, const Point& vec) {
35     int l = 0, r = (int)convex.size() - 2;
36     assert(r >= 0);
37     for (; l + 1 < r; ) {
38         int mid = (l + r) / 2;
39         if (sign(det(convex[mid + 1] - convex[mid], vec)) > 0)
40             r = mid;
41         else l = mid;
42     }
43     return std::max(std::make_pair(det(vec, convex[r]), r),
44         std::make_pair(det(vec, convex[0]), 0));
45 }
46 int binary_search(Point u, Point v, int l, int r) {
47     int s1 = sign(det(v - u, a[l % n] - u));
48     for (; l + 1 < r; ) {
49         int mid = (l + r) / 2;
50         int smid = sign(det(v - u, a[mid % n] - u));
51         if (smid == s1) l = mid;
52         else r = mid;
53     }
54     return l % n;
55 }
56 // 求凸包上和向量 vec 叉积最大的点, 返回编号, 共线的多个切点返回任意一个
57 int get_tangent(Point vec) {
58     std::pair<DB, int> ret = get_tangent(upper, vec);
59     ret.second = (ret.second + (int)lower.size() - 1) % n;
60     ret = std::max(ret, get_tangent(lower, vec));
61     return ret.second;
62 }
63 // 求凸包和直线 u, v 的交点, 如果不相交返回 false, 如果有则是和 (i, next(i)) 的交点, 交在点上不
64 // 确定返回前后两条边其中之一
65 bool get_intersection(Point u, Point v, int &i0, int &i1) {
66     int p0 = get_tangent(u - v), p1 = get_tangent(v - u);
67     if (sign(det(v - u, a[p0] - u)) * sign(det(v - u, a[p1] - u)) <= 0) {
68         if (p0 > p1) std::swap(p0, p1);
69         i0 = binary_search(u, v, p0, p1);
70         i1 = binary_search(u, v, p1, p0 + n);
71         return true;
72     }
73 }

```

```

72         else return false;
73     }
74 };

```

## 1.2 三角形的心

```

1 Point inCenter(const Point &A, const Point &B, const Point &C) { // 内心
2     double a = (B - C).len(), b = (C - A).len(), c = (A - B).len(),
3         s = fabs(det(B - A, C - A)),
4         r = s / p;
5     return (A * a + B * b + C * c) / (a + b + c);
6 }
7 Point circumCenter(const Point &a, const Point &b, const Point &c) { // 外心
8     Point bb = b - a, cc = c - a;
9     double db = bb.len2(), dc = cc.len2(), d = 2 * det(bb, cc);
10    return a - Point(bb.y * dc - cc.y * db, cc.x * db - bb.x * dc) / d;
11 }
12 Point orthoCenter(const Point &a, const Point &b, const Point &c) { // 垂心
13     Point ba = b - a, ca = c - a, bc = b - c;
14     double Y = ba.y * ca.y * bc.y,
15         A = ca.x * ba.y - ba.x * ca.y,
16         x0 = (Y + ca.x * ba.y * b.x - ba.x * ca.y * c.x) / A,
17         y0 = -ba.x * (x0 - c.x) / ba.y + ca.y;
18     return Point(x0, y0);
19 }

```

## 1.3 半平面交

```

1 struct Point {
2     int quad() const { return sign(y) == 1 || (sign(y) == 0 && sign(x) >= 0); }
3 };
4 struct Line {
5     bool include(const Point &p) const { return sign(det(b - a, p - a)) > 0; }
6     Line push() const { // 将半平面向外推 eps
7         const double eps = 1e-6;
8         Point delta = (b - a).turn90().norm() * eps;
9         return Line(a - delta, b - delta);
10    }
11 };
12 bool sameDir(const Line &l0, const Line &l1) { return parallel(l0, l1) && sign(dot(l0.b - l0.a,
13     ↪ l1.b - l1.a)) == 1; }
14 bool operator < (const Point &a, const Point &b) {
15     if (a.quad() != b.quad()) {
16         return a.quad() < b.quad();
17     } else {
18         return sign(det(a, b)) > 0;
19     }
20 }

```

```

19 }
20 bool operator < (const Line &l0, const Line &l1) {
21     if (sameDir(l0, l1)) {
22         return l1.include(l0.a);
23     } else {
24         return (l0.b - l0.a) < (l1.b - l1.a);
25     }
26 }
27 bool check(const Line &u, const Line &v, const Line &w) { return w.include(intersect(u, v)); }
28 vector<Point> intersection(vector<Line> &l) {
29     sort(l.begin(), l.end());
30     deque<Line> q;
31     for (int i = 0; i < (int)l.size(); ++i) {
32         if (i && sameDir(l[i], l[i - 1])) {
33             continue;
34         }
35         while (q.size() > 1 && !check(q[q.size() - 2], q[q.size() - 1], l[i])) q.pop_back();
36         while (q.size() > 1 && !check(q[1], q[0], l[i])) q.pop_front();
37         q.push_back(l[i]);
38     }
39     while (q.size() > 2 && !check(q[q.size() - 2], q[q.size() - 1], q[0])) q.pop_back();
40     while (q.size() > 2 && !check(q[1], q[0], q[q.size() - 1])) q.pop_front();
41     vector<Point> ret;
42     for (int i = 0; i < (int)q.size(); ++i) ret.push_back(intersect(q[i], q[(i + 1) %
    ↪ q.size()]));
43     return ret;
44 }

```

## 1.4 圆交面积及重心

```

1 struct Event {
2     Point p;
3     double ang;
4     int delta;
5     Event (Point p = Point(0, 0), double ang = 0, double delta = 0) : p(p), ang(ang),
    ↪ delta(delta) {}
6 };
7 bool operator < (const Event &a, const Event &b) {
8     return a.ang < b.ang;
9 }
10 void addEvent(const Circle &a, const Circle &b, vector<Event> &evt, int &cnt) {
11     double d2 = (a.o - b.o).len2(),
12         dRatio = ((a.r - b.r) * (a.r + b.r) / d2 + 1) / 2,
13         pRatio = sqrt(-(d2 - sqr(a.r - b.r)) * (d2 - sqr(a.r + b.r)) / (d2 * d2 * 4));
14     Point d = b.o - a.o, p = d.rotate(PI / 2),
15         q0 = a.o + d * dRatio + p * pRatio,
16         q1 = a.o + d * dRatio - p * pRatio;
17     double ang0 = (q0 - a.o).ang(),

```

```

18         ang1 = (q1 - a.o).ang();
19         evt.push_back(Event(q1, ang1, 1));
20         evt.push_back(Event(q0, ang0, -1));
21         cnt += ang1 > ang0;
22     }
23     bool issame(const Circle &a, const Circle &b) { return sign((a.o - b.o).len()) == 0 && sign(a.r
    ↪ - b.r) == 0; }
24     bool overlap(const Circle &a, const Circle &b) { return sign(a.r - b.r - (a.o - b.o).len()) >=
    ↪ 0; }
25     bool intersect(const Circle &a, const Circle &b) { return sign((a.o - b.o).len() - a.r - b.r) <
    ↪ 0; }
26     Circle c[N];
27     double area[N]; // area[k] -> area of intersections >= k.
28     Point centroid[N];
29     bool keep[N];
30     void add(int cnt, DB a, Point c) {
31         area[cnt] += a;
32         centroid[cnt] = centroid[cnt] + c * a;
33     }
34     void solve(int C) {
35         for (int i = 1; i <= C; ++i) {
36             area[i] = 0;
37             centroid[i] = Point(0, 0);
38         }
39         for (int i = 0; i < C; ++i) {
40             int cnt = 1;
41             vector<Event> evt;
42             for (int j = 0; j < i; ++j) if (issame(c[i], c[j])) ++cnt;
43             for (int j = 0; j < C; ++j) {
44                 if (j != i && !issame(c[i], c[j]) && overlap(c[j], c[i])) {
45                     ++cnt;
46                 }
47             }
48             for (int j = 0; j < C; ++j) {
49                 if (j != i && !overlap(c[j], c[i]) && !overlap(c[i], c[j]) && intersect(c[i], c[j]))
    ↪ {
50                     addEvent(c[i], c[j], evt, cnt);
51                 }
52             }
53             if (evt.size() == 0u) {
54                 add(cnt, PI * c[i].r * c[i].r, c[i].o);
55             } else {
56                 sort(evt.begin(), evt.end());
57                 evt.push_back(evt.front());
58                 for (int j = 0; j + 1 < (int)evt.size(); ++j) {
59                     cnt += evt[j].delta;
60                     add(cnt, det(evt[j].p, evt[j + 1].p) / 2, (evt[j].p + evt[j + 1].p) / 3);
61                     double ang = evt[j + 1].ang - evt[j].ang;
62                     if (ang < 0) {

```

```

63         ang += PI * 2;
64     }
65     if (sign(ang) == 0) continue;
66     add(cnt, ang * c[i].r * c[i].r / 2, c[i].o +
67         Point(sin(ang1) - sin(ang0), -cos(ang1) + cos(ang0)) * (2 / (3 * ang) *
↪ c[i].r));
68     add(cnt, -sin(ang) * c[i].r * c[i].r / 2, (c[i].o + evt[j].p + evt[j + 1].p) /
↪ 3);
69     }
70 }
71 }
72 for (int i = 1; i <= C; ++ i)
73     if (sign(area[i])) {
74         centroid[i] = centroid[i] / area[i];
75     }
76 }

```

## 1.5 极角排序

```

1 bool cmp(const Point &a, const Point &b)
2 {
3     int tmp1=(a.y>0 || (a.y==0 && a.x>0))?1:2;
4     int tmp2=(b.y>0 || (b.y==0 && b.x>0))?1:2;
5     if (tmp1!=tmp2) return tmp1<tmp2;
6     return det(a,b)>0;
7 }

```

## 1.6 极角求交

```

1 bool in(Point x, Point y, Point z)
2 {
3     if (sign(det(x,y))>0) swap(x,y);
4     return sign(det(z,y))<=0 && sign(det(x,z))<=0;
5 }
6 bool jiao(Point &x, Point &y, Point l, Point r)
7 {
8     if (!in(x,y,l) && !in(x,y,r)) swap(x,l), swap(y,r);
9     if (!in(x,y,l) && !in(x,y,r)) return false;
10    if (!in(x,y,l)) swap(l,r);
11    if (in(x,y,l) && in(x,y,r))
12    {
13        x=l;
14        y=r;
15        return true;
16    }
17    bool xx=in(l,r,x), yy=in(l,r,y);
18    if (xx && yy)

```

```

19 {
20     return true;
21 }
22 if (!xx && !yy)
23 {
24     x=1;
25     y=r;
26     return true;
27 }
28 if (xx && !yy)
29 {
30     y=1;
31     return true;
32 }
33 x=1;
34 return true;
35 }

```

## 1.7 最小圆覆盖

```

1 Point get_circle_center(const Point &a,const Point &b,const Point &c)
2 {
3     Point center;
4     double a1=b.x-a.x;
5     double b1=b.y-a.y;
6     double c1=(a1*a1+b1*b1)/2.0;
7     double a2=c.x-a.x;
8     double b2=c.y-a.y;
9     double c2=(a2*a2+b2*b2)/2.0;
10    double d=a1*b2-a2*b1;
11    center.x=a.x+(c1*b2-c2*b1)/d;
12    center.y=a.y+(a1*c2-a2*c1)/d;
13    return center;
14 }
15 inline bool inCircle(const Point &p,const Circle &c){
16     return sign((p-c.o).len()-c.r)<=0;
17 }
18 Circle Min_cover(vector<Point> p)
19 {//注意考虑没有点的情况
20     random_shuffle(p.begin(),p.end());
21     int n=p.size();
22     Circle ans;
23     ans.o=p[0];
24     ans.r=0;
25     for (int i=1;i<n;i++)
26     if (!inCircle(p[i],ans))
27     {
28         ans.o=p[i];

```

```

29     ans.r=0;
30     for (int j=0;j<i;j++)
31         if (!inCircle(p[j],ans))
32         {
33             ans.o=(p[j]+p[i])/2.0;
34             ans.r=((p[j]-p[i]).len())/2.0;
35             for (int k=0;k<j;k++)
36                 if (!inCircle(p[k],ans))
37                 {
38                     ans.o=get_circle_center(p[i],p[j],p[k]);
39                     ans.r=(p[i]-ans.o).len();
40                 }
41         }
42     }
43     return ans;
44 }

```

## 1.8 三维向量绕轴旋转

```

1 // 三维绕轴旋转, 大拇指指向 axis 向量方向, 四指弯曲方向转 w 弧度
2 Point rotate(const Point& s, const Point& axis, DB w) {
3     DB x = axis.x, y = axis.y, z = axis.z;
4     DB s1 = x * x + y * y + z * z, ss1 = msqrt(s1),
5         cosw = cos(w), sinw = sin(w);
6     DB a[4][4];
7     memset(a, 0, sizeof a);
8     a[3][3] = 1;
9     a[0][0] = ((y * y + z * z) * cosw + x * x) / s1;
10    a[0][1] = x * y * (1 - cosw) / s1 + z * sinw / ss1;
11    a[0][2] = x * z * (1 - cosw) / s1 - y * sinw / ss1;
12    a[1][0] = x * y * (1 - cosw) / s1 - z * sinw / ss1;
13    a[1][1] = ((x * x + z * z) * cosw + y * y) / s1;
14    a[1][2] = y * z * (1 - cosw) / s1 + x * sinw / ss1;
15    a[2][0] = x * z * (1 - cosw) / s1 + y * sinw / ss1;
16    a[2][1] = y * z * (1 - cosw) / s1 - x * sinw / ss1;
17    a[2][2] = ((x * x + y * y) * cosw + z * z) / s1;
18    DB ans[4] = {0, 0, 0, 0}, c[4] = {s.x, s.y, s.z, 1};
19    for (int i = 0; i < 4; ++ i)
20        for (int j = 0; j < 4; ++ j)
21            ans[i] += a[j][i] * c[j];
22    return Point(ans[0], ans[1], ans[2]);
23 }

```

## 1.9 三维凸包

```

1 __inline P cross(const P& a, const P& b) {
2     return P(

```

```

3         a.y * b.z - a.z * b.y,
4         a.z * b.x - a.x * b.z,
5         a.x * b.y - a.y * b.x
6     );
7 }
8
9 __inline DB mix(const P& a, const P& b, const P& c) {
10     return dot(cross(a, b), c);
11 }
12
13 __inline DB volume(const P& a, const P& b, const P& c, const P& d) {
14     return mix(b - a, c - a, d - a);
15 }
16
17 struct Face {
18     int a, b, c;
19     __inline Face() {}
20     __inline Face(int _a, int _b, int _c):
21         a(_a), b(_b), c(_c) {}
22     __inline DB area() const {
23         return 0.5 * cross(p[b] - p[a], p[c] - p[a]).len();
24     }
25     __inline P normal() const {
26         return cross(p[b] - p[a], p[c] - p[a]).unit();
27     }
28     __inline DB dis(const P& p0) const {
29         return dot(normal(), p0 - p[a]);
30     }
31 };
32
33 std::vector<Face> face, tmp; // Should be O(n).
34 int mark[N][N], Time, n;
35
36 __inline void add(int v) {
37     ++ Time;
38     clear(tmp);
39     for (int i = 0; i < (int)face.size(); ++ i) {
40         int a = face[i].a, b = face[i].b, c = face[i].c;
41         if (sign(volume(p[v], p[a], p[b], p[c])) > 0) {
42             mark[a][b] = mark[b][a] = mark[a][c] =
43                 mark[c][a] = mark[b][c] = mark[c][b] = Time;
44         }
45         else {
46             tmp.push_back(face[i]);
47         }
48     }
49     clear(face); face = tmp;
50     for (int i = 0; i < (int)tmp.size(); ++ i) {
51         int a = face[i].a, b = face[i].b, c = face[i].c;

```



```
52         if (mark[a][b] == Time) face.emplace_back(v, b, a);
53         if (mark[b][c] == Time) face.emplace_back(v, c, b);
54         if (mark[c][a] == Time) face.emplace_back(v, a, c);
55         assert(face.size() < 500u);
56     }
57 }
58
59 void reorder() {
60     for (int i = 2; i < n; ++ i) {
61         P tmp = cross(p[i] - p[0], p[i] - p[1]);
62         if (sign(tmp.len())) {
63             std::swap(p[i], p[2]);
64             for (int j = 3; j < n; ++ j)
65                 if (sign(volume(p[0], p[1], p[2], p[j]))) {
66                     std::swap(p[j], p[3]);
67                     return;
68                 }
69         }
70     }
71 }
72
73 void build_convex() {
74     reorder();
75     clear(face);
76     face.emplace_back(0, 1, 2);
77     face.emplace_back(0, 2, 1);
78     for (int i = 3; i < n; ++ i)
79         add(i);
80 }
```



## Chapter 2

# 数论

### 2.1 $O(m^2 \log n)$ 求线性递推数列第 $n$ 项

Given  $a_0, a_1, \dots, a_{m-1}$

$$a_n = c_0 \times a_{n-m} + \dots + c_{m-1} \times a_{n-1}$$

Solve for  $a_n = v_0 \times a_0 + v_1 \times a_1 + \dots + v_{m-1} \times a_{m-1}$

```
1 void linear_recurrence(long long n, int m, int a[], int c[], int p) {
2     long long v[M] = {1 % p}, u[M << 1], msk = !!n;
3     for(long long i(n); i > 1; i >= 1) {
4         msk <= 1;
5     }
6     for(long long x(0); msk; msk >>= 1, x <= 1) {
7         fill_n(u, m << 1, 0);
8         int b(!!(n & msk));
9         x |= b;
10        if(x < m) {
11            u[x] = 1 % p;
12        } else {
13            for(int i(0); i < m; i++) {
14                for(int j(0), t(i + b); j < m; j++, t++) {
15                    u[t] = (u[t] + v[i] * v[j]) % p;
16                }
17            }
18            for(int i((m << 1) - 1); i >= m; i--) {
19                for(int j(0), t(i - m); j < m; j++, t++) {
20                    u[t] = (u[t] + c[j] * u[i]) % p;
21                }
22            }
23        }
24        copy(u, u + m, v);
25    }
26    //a[n] = v[0] * a[0] + v[1] * a[1] + ... + v[m - 1] * a[m - 1].
27    for(int i(m); i < 2 * m; i++) {
28        a[i] = 0;
29        for(int j(0); j < m; j++) {
```

```

30     a[i] = (a[i] + (long long)c[j] * a[i + j - m]) % p;
31 }
32 }
33 for(int j(0); j < m; j++) {
34     b[j] = 0;
35     for(int i(0); i < m; i++) {
36         b[j] = (b[j] + v[i] * a[i + j]) % p;
37     }
38 }
39 for(int j(0); j < m; j++) {
40     a[j] = b[j];
41 }
42 }

```

## 2.2 求逆元

```

1 void ex_gcd(long long a, long long b, long long &x, long long &y) {
2     if (b == 0) {
3         x = 1;
4         y = 0;
5         return;
6     }
7     long long xx, yy;
8     ex_gcd(b, a % b, xx, yy);
9     y = xx - a / b * yy;
10    x = yy;
11 }
12
13 long long inv(long long x, long long MODN) {
14     long long inv_x, y;
15     ex_gcd(x, MODN, inv_x, y);
16     return (inv_x % MODN + MODN) % MODN;
17 }

```

## 2.3 中国剩余定理

```

1 // 返回 (ans, M), 其中 ans 是模 M 意义下的解
2 std::pair<long long, long long> CRT(const std::vector<long long>& m, const std::vector<long
  ↳ long>& a) {
3     long long M = 1, ans = 0;
4     int n = m.size();
5     for (int i = 0; i < n; i++) M *= m[i];
6     for (int i = 0; i < n; i++) {
7         ans = (ans + (M / m[i]) * a[i] % M * inv(M / m[i], m[i])) % M; // 可能需要大整数相乘取模
8     }
9     return std::make_pair(ans, M);
10 }

```

```

11 // 模数不互质的情况
12 bool solve(int n, std::pair<long long, long long> input[],
13           std::pair<long long, long long> &output) {
14     output = std::make_pair(1, 1);
15     for (int i = 0; i < n; ++i) {
16         long long number, useless;
17         // euclid(a, b, x, y)
18         euclid(output.second, input[i].second, number, useless);
19         long long divisor = std::__gcd(output.second, input[i].second);
20         if ((input[i].first - output.first) % divisor) return false;
21         number *= (input[i].first - output.first) / divisor;
22         fix(number, input[i].second); // fix 成正的
23         output.first += output.second * number;
24         output.second *= input[i].second / divisor;
25         fix(output.first, output.second);
26     }
27     return true;
28 }

```

## 2.4 魔法 CRT

```

1 // MOD is the given module
2 // Do not depend on LL * LL % LL
3 inline int CRT(int *a) {
4     static int x[N];
5     for (int i = 0; i < N; i++) {
6         x[i] = a[i];
7         for (int j = 0; j < i; j++) {
8             int t = (x[i] - x[j] + mod[i]) % mod[i];
9             if (t < 0) t += mod[i];
10            x[i] = 1LL * t * Inv[j][i] % mod[i];
11        }
12    }
13    int sum = 1, ret = x[0] % MOD;
14    for (int i = 1; i < N; i++) {
15        sum = 1LL * sum * mod[i - 1] % MOD;
16        ret += 1LL * x[i] * sum % MOD;
17        if (ret >= MOD) ret -= MOD;
18    }
19    return ret;
20 }
21 for (int i = 0; i < N; i++)
22     for (int j = i + 1; j < N; j++) {
23         Inv[i][j] = fpw(mod[i], mod[j] - 2, mod[j]);
24     }

```

## 2.5 素性测试

```

1 int strong_pseudo_primetest(long long n,int base) {
2     long long n2=n-1,res;
3     int s=0;
4     while(n2%2==0) n2>>=1,s++;
5     res=powmod(base,n2,n);
6     if((res==1)|| (res==n-1)) return 1;
7     s--;
8     while(s>=0) {
9         res=mulmod(res,res,n);
10        if(res==n-1) return 1;
11        s--;
12    }
13    return 0; // n is not a strong pseudo prime
14 }
15 int isprime(long long n) {
16     static LL testNum[]={2,3,5,7,11,13,17,19,23,29,31,37};
17     static LL lim[]={4,0,1373653LL,25326001LL,25000000000LL,2152302898747LL,
18     ↪ 3474749660383LL,341550071728321LL,0,0,0,0};
19     if(n<2||n==3215031751LL) return 0;
20     for(int i=0;i<12;++i){
21         if(n<lim[i]) return 1;
22         if(strong_pseudo_primetest(n,testNum[i])==0) return 0;
23     }
24     return 1;
25 }

```

## 2.6 EX-BSGS

```

1 /*
2  * a^x = b (mod p)
3  * p may not be a prime
4  */
5 unordered_map<int,int> mp;
6
7 ll exbsgs(ll a,ll b,ll p)
8 {
9     if (b == 1) return 0;
10    ll t, d = 1, k = 0;
11    while ((t = __gcd(a, p)) != 1) {
12        if (b % t) return -1;
13        ++k, b /= t, p /= t, d = d * (a / t) % p;
14        if (b == d) return k;
15    }
16    mp.clear();
17    ll m = std::ceil(std::sqrt(p));
18    ll a_m = powmod(a, m, p);

```

```

19     ll mul = b;
20     for (ll j = 1; j <= m; ++j) {
21         mul = mul * a % p;
22         mp[mul] = j;
23     }
24     for (ll i = 1; i <= m; ++i) {
25         d = d * a_m % p;
26         if (mp.count(d)) return i * m - mp[d] + k;
27     }
28     return -1;
29 }

```

## 2.7 质因数分解

```

1 int ansn; LL ans[1000];
2 LL func(LL x, LL n) { return (mod_mul(x, x, n) + 1) % n; }
3 LL Pollard(LL n) {
4     LL i, x, y, p;
5     if (Rabin_Miller(n)) return n;
6     if (!(n & 1)) return 2;
7     for (i = 1; i < 20; i++) {
8         x = i; y = func(x, n); p = gcd(y - x, n);
9         while (p == 1) { x = func(x, n); y = func(func(y, n), n); p = gcd((y - x + n) % n, n); }
10        if (p == 0 || p == n) continue;
11        return p;
12    }
13 }
14 void factor(LL n) {
15     LL x;
16     x = Pollard(n);
17     if (x == n) { ans[ansn++] = x; return; }
18     factor(x), factor(n / x);
19 }

```

## 2.8 线下整点

```

1 //  $\sum_{i=0}^{n-1} \lfloor \frac{a+bi}{m} \rfloor, n, m, a, b > 0$ 
2 LL solve(LL n, LL a, LL b, LL m) {
3     if (b == 0) return n * (a / m);
4     if (a >= m) return n * (a / m) + solve(n, a % m, b, m);
5     if (b >= m) return (n - 1) * n / 2 * (b / m) + solve(n, a, b % m, m);
6     return solve((a + b * n) / m, (a + b * n) % m, m, b);
7 }

```

## 2.9 线性同余不等式

```

1 // Find the minimal non-negative solutions for  $l \leq d \cdot x \bmod m \leq r$ 
2 //  $0 \leq d, l, r < m; l \leq r, O(\log n)$ 
3 ll cal(ll m, ll d, ll l, ll r)
4 {
5     if (l == 0) return 0;
6     if (d == 0) return MXL; // 无解
7     if (d * 2 > m) return cal(m, m - d, m - r, m - l);
8     if ((l - 1) / d < r / d) return (l - 1) / d + 1;
9     ll k = cal(d, (-m % d + d) % d, l % d, r % d);
10    return k == MXL ? MXL : (k * m + l - 1) / d + 1; // 无解 2
11 }

```

## 2.10 原根相关

1. 模  $m$  有原根的充要条件:  $m = 2, 4, p^a, 2p^a$ , 其中  $p$  是奇素数;
2. 求任意数  $p$  原根的方法: 对  $\phi(p)$  因式分解, 即  $\phi(p) = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ , 若恒成立:

$$g^{\frac{p-1}{p_i}} \not\equiv 1 \pmod{p}$$

那么  $g$  就是  $p$  的原根。

3. 若模  $m$  有原根, 那么它一共有  $\Phi(\Phi(m))$  个原根。



## Chapter 3

# 代数

### 3.1 快速傅里叶变换

```
1 int prepare(int n) {
2     int len = 1;
3     for (; len <= 2 * n; len <= 1);
4     for (int i = 0; i < len; i++) {
5         e[0][i] = Complex(cos(2 * pi * i / len), sin(2 * pi * i / len));
6         e[1][i] = Complex(cos(2 * pi * i / len), -sin(2 * pi * i / len));
7     }
8     return len;
9 }
10 void DFT(Complex *a, int n, int f) {
11     for (int i = 0, j = 0; i < n; i++) {
12         if (i > j) std::swap(a[i], a[j]);
13         for (int t = n >> 1; (j ^= t) < t; t >>= 1);
14     }
15     for (int i = 2; i <= n; i <= 1)
16         for (int j = 0; j < n; j += i)
17             for (int k = 0; k < (i >> 1); k++) {
18                 Complex A = a[j + k];
19                 Complex B = e[f][n / i * k] * a[j + k + (i >> 1)];
20                 a[j + k] = A + B;
21                 a[j + k + (i >> 1)] = A - B;
22             }
23     if (f == 1) {
24         for (int i = 0; i < n; i++)
25             a[i].a /= n;
26     }
27 }
```

## 3.2 任意数快速傅里叶变换

```

1  #include<bits/stdc++.h>
2  using namespace std;
3
4  typedef long double db;
5  typedef long long ll;
6  const int mask=(1<<15)-1;
7  const int maxn=100010;
8  const int maxl=262144;
9  const db pi=acos(-1.0);
10 const int mo=998244353;
11
12 struct Complex
13 {
14     db r,i;
15     Complex(db x=0.0,db y=0.0)
16     {
17         r=x,i=y;
18     }
19     Complex operator+(const Complex &o)    const
20     {
21         return Complex(r+o.r,i+o.i);
22     }
23     Complex operator-(const Complex &o)    const
24     {
25         return Complex(r-o.r,i-o.i);
26     }
27     Complex operator*(const Complex &o)    const
28     {
29         return Complex(r*o.r-i*o.i,r*o.i+i*o.r);
30     }
31     Complex conj()    const
32     {
33         return Complex(r,-i);
34     }
35 }    A[maxl],B[maxl],C[maxl],D[maxl];
36
37 int a[maxn<<1],b[maxn<<1],c[maxn<<1];
38
39 int q;
40 void FFT(Complex *a,int n,int isdft)
41 {
42     for (register int i=0,j=0;i<n;i++)
43     {
44         if (i>j)    swap(a[i],a[j]);
45         for (int t=n>>1;(j^=t)<t;t>>=1);
46     }
47     for (int i=2;i<=n;i<=<=1)

```

```

48 {
49     Complex wn(cos(isdft*2*pi/i),sin(isdft*2*pi/i));
50     for (int j=0;j<n;j+=i)
51     {
52         Complex w(1,0),u,v;
53         for (int k=j;k<j+i/2;k++)
54         {
55             u=a[k],v=a[k+i/2]*w;
56             a[k]=u+v;
57             a[k+i/2]=u-v;
58             w=w*wn;
59         }
60     }
61 }
62 if (isdft== -1)
63     for (int i=0;i<n;i++)    a[i].r/=n;
64 }
65 inline void cheng(int *x,int *y,int *z,int len)
66 {
67     // len++;
68     int l=1;
69     q=0;
70     while (l<len*2)    l*=2,q++;
71     for (int i=0;i<len;i++)
72         A[i]=Complex(x[i]>>15,x[i]&mask);
73     for (int i=len;i<l;i++)
74         A[i]=Complex(0,0);
75     for (int i=0;i<len;i++)
76         B[i]=Complex(y[i]>>15,y[i]&mask);
77     for (int i=len;i<l;i++)
78         B[i]=Complex(0,0);
79     FFT(A,l,1);
80     FFT(B,l,1);
81     for (int i=0;i<l;i++)
82     {
83         int j=(l-i)%l;
84         Complex _a=(A[i]-A[j].conj())*Complex(0,-0.5);
85         Complex _b=(A[i]+A[j].conj())*Complex(0.5,0);
86         Complex _c=(B[i]-B[j].conj())*Complex(0,-0.5);
87         Complex _d=(B[i]+B[j].conj())*Complex(0.5,0);
88         C[j]=_a*_d+_a*_c*Complex(0,1);
89         D[j]=_b*_d+_b*_c*Complex(0,1);
90     }
91     FFT(C,l,1);
92     FFT(D,l,1);
93     for (int i=0;i<l;i++)
94     {
95         ll _a=((ll)(C[i].i/l+0.5))%mo;
96         ll _b=((ll)(C[i].r/l+0.5))%mo;

```

```

97     ll _c=((ll)(D[i].i/l+0.5))%mo;
98     ll _d=((ll)(D[i].r/l+0.5))%mo;
99     z[i]=((_d<<30)+((_b+_c)<<15)+_a)%mo;
100 }
101 }
102 int main()
103 {
104     int n,m;
105     scanf("%d%d",&n,&m);
106     n++,m++;
107     for (int i=0;i<n;i++)
108         scanf("%d",&a[i]);
109     for (int i=0;i<m;i++)
110         scanf("%d",&b[i]);
111     cheng(a,b,c,max(n,m));
112     for (int i=0;i<n+m-1;i++)
113         printf("%d\n",c[i]);
114     return 0;
115 }

```

### 3.3 fwt

```

1  /*
2  xor(A) = (xor(A0+A1),xor(A0-A1))
3  _xor(A) = (_xor((A0+A1)/2),_xor((A0-A1)/2))
4
5  and(A) = (and(A0+A1),and(A1))
6  _and(A) = (_and(A0-A1),_and(A1))
7
8  or(A) = (or(A0),or(A0+A1))
9  _or(A) = (_or(A0),_or(A1-A0))
10 */
11
12 void FWT(int *a,int n)
13 {
14     for (int h=2;h<=n;h<<=1)
15         for (int j=0;j<n;j+=h)
16             for (int k=j;k<j+h/2;k++)
17             {
18                 int u=a[k],v=a[k+h/2];
19                 //xor : a[k]=(u+v)%mo; a[k+h/2]=(u-v+mo)%mo;
20                 //and : a[k]=(u+v)%mo; a[k+h/2]=v;
21                 //or : a[k]=u; a[k+h/2]=(u+v)%mo;
22             }
23 }
24 void IFWT(int *a,int n)
25 {
26     for (int h=2;h<=n;h<<=1)

```

```

27     for (int j=0;j<n;j+=h)
28         for (int k=j;k<j+h/2;k++)
29             {
30                 int u=a[k],v=a[k+h/2];
31                 //xor : a[k]=((u+v)*1LL*inv_2)%mo; a[k+h/2]=((u-v)*1LL*inv_2 %mo + mo)%mo;
32                 //and : a[k]=(u-v+mo)%mo; a[k+h/2]=v;
33                 //or : a[k]=u; a[k+h/2]=(u-v+mo)%mo;
34             }
35 }
36 void cheng(int *a,int *b,int *c,int len)
37 {
38     int l=1;
39     while (l<len)    l<<=1;
40     len=l;
41     FWT(a,len);
42     FWT(b,len);
43     for (int i=0;i<len;i++)
44         c[i]=(a[i]*1LL*b[i])%mo;
45     IFWT(c,len);
46 }

```

### 3.4 快速数论变换

```

1 // meminit(A, l, r) 是将数组 A 的 [l, r) 清 0。
2 // memcpy(target, source, l, r) 是将 source 的 [l, r) 复制到 target 的 [l, r)
3 #define meminit(A, l, r) memset(A + (l), 0, sizeof(*A) * ((r) - (l)))
4 #define memcpy(B, A, l, r) memcpy(B, A + (l), sizeof(*A) * ((r) - (l)))
5 void DFT(int *a, int n, int f) { // 封闭形式, 常数小 (107 跑 2.23 秒)
6     for (register int i = 0, j = 0; i < n; i++) {
7         if (i > j) std::swap(a[i], a[j]);
8         for (register int t = n >> 1; (j ^= t) < t; t >>= 1);
9     }
10    for (register int i = 2; i <= n; i <<= 1) {
11        static int exp[MAXN];
12        exp[0] = 1; exp[1] = fpm(PRT, (MOD - 1) / i);
13        if (f == 1) exp[1] = fpm(exp[1], MOD - 2);
14        for (register int k = 2; k < (i >> 1); k++) {
15            exp[k] = 1ll * exp[k - 1] * exp[1] % MOD;
16        }
17        for (register int j = 0; j < n; j += i) {
18            for (register int k = 0; k < (i >> 1); k++) {
19                register int &pA = a[j + k], &pB = a[j + k + (i >> 1)];
20                register int A = pA, B = 1ll * pB * exp[k] % MOD;
21                pA = (A + B) % MOD;
22                pB = (A - B + MOD) % MOD;
23            }
24        }
25    }
}

```

```

26     if (f == 1) {
27         register int rev = fpm(n, MOD - 2, MOD);
28         for (register int i = 0; i < n; i++) {
29             a[i] = 1ll * a[i] * rev % MOD;
30         }
31     }
32 }
33 // 在不写高精度的情况下合并 FFT 所得结果对 MOD 取模后的答案
34 // 值得注意的是, 这个东西不能最后再合并, 而是应该每做一次多项式乘法就 CRT 一次
35 int CRT(int *a) {
36     static int x[3];
37     for (int i = 0; i < 3; i++) {
38         x[i] = a[i];
39         for (int j = 0; j < i; j++) {
40             int t = (x[i] - x[j] + FFT[i] -> MOD) % FFT[i] -> MOD;
41             if (t < 0) t += FFT[i] -> MOD;
42             x[i] = 1LL * t * inv[j][i] % FFT[i] -> MOD;
43         }
44     }
45     int sum = 1, ret = x[0] % MOD;
46     for (int i = 1; i < 3; i++) {
47         sum = 1LL * sum * FFT[i] -> MOD % MOD;
48         ret += 1LL * x[i] * sum % MOD;
49         if (ret >= MOD) ret -= MOD;
50     }
51     return ret;
52 }
53 for (int i = 0; i < 3; i++) // inv 数组的预处理过程, inverse(x, p) 表示求 x 在 p 下逆元
54     for (int j = 0; j < 3; j++)
55         inv[i][j] = inverse(FFT[i] -> MOD, FFT[j] -> MOD);

```

### 3.5 自适应辛普森积分

```

1 namespace adaptive_simpson {
2     template<typename function>
3     inline double area(function f, const double &left, const double &right) {
4         double mid = (left + right) / 2;
5         return (right - left) * (f(left) + 4 * f(mid) + f(right)) / 6;
6     }
7
8     template<typename function>
9     inline double simpson(function f, const double &left, const double &right, const double
10     ↪ &eps, const double &area_sum) {
11         double mid = (left + right) / 2;
12         double area_left = area(f, left, mid);
13         double area_right = area(f, mid, right);
14         double area_total = area_left + area_right;
15         if (fabs(area_total - area_sum) <= 15 * eps) {

```

```

15         return area_total + (area_total - area_sum) / 15;
16     }
17     return simpson(f, left, right, eps / 2, area_left) + simpson(f, mid, right, eps / 2,
↪ area_right);
18 }
19
20 template<typename function>
21 inline double simpson(function f, const double &left, const double &right, const double
↪ &eps) {
22     return simpson(f, left, right, eps, area(f, left, right));
23 }
24 }

```

## 3.6 单纯形

```

1  const double eps = 1e-8;
2  // max{c * x | Ax <= b, x >= 0} 的解, 无解返回空的 vector, 否则就是解.
3  vector<double> simplex(vector<vector<double>> &A, vector<double> b, vector<double> c) {
4      int n = A.size(), m = A[0].size() + 1, r = n, s = m - 1;
5      vector<vector<double>> D(n + 2, vector<double>(m + 1));
6      vector<int> ix(n + m);
7      for(int i = 0; i < n + m; i++) {
8          ix[i] = i;
9      }
10     for(int i = 0; i < n; i++) {
11         for(int j = 0; j < m - 1; j++) {
12             D[i][j] = -A[i][j];
13         }
14         D[i][m - 1] = 1;
15         D[i][m] = b[i];
16         if (D[r][m] > D[i][m]) {
17             r = i;
18         }
19     }
20
21     for(int j = 0; j < m - 1; j++) {
22         D[n][j] = c[j];
23     }
24     D[n + 1][m - 1] = -1;
25     for(double d; ;) {
26         if (r < n) {
27             swap(ix[s], ix[r + m]);
28             D[r][s] = 1. / D[r][s];
29             for(int j = 0; j <= m; j++) {
30                 if (j != s) {
31                     D[r][j] *= -D[r][s];
32                 }
33             }

```

```

34     for(int i = 0; i <= n + 1; i++) {
35         if (i != r) {
36             for(int j = 0; j <= m; j++) {
37                 if (j != s) {
38                     D[i][j] += D[r][j] * D[i][s];
39                 }
40             }
41             D[i][s] *= D[r][s];
42         }
43     }
44 }
45 r = -1, s = -1;
46 for(int j = 0; j < m; j++) {
47     if (s < 0 || ix[s] > ix[j]) {
48         if (D[n + 1][j] > eps || D[n + 1][j] > -eps && D[n][j] > eps) {
49             s = j;
50         }
51     }
52 }
53 if (s < 0) {
54     break;
55 }
56 for(int i = 0; i < n; i++) {
57     if (D[i][s] < -eps) {
58         if (r < 0 || (d = D[r][m] / D[r][s] - D[i][m] / D[i][s]) < -eps
59             || d < eps && ix[r + m] > ix[i + m]) {
60
61             r = i;
62         }
63     }
64 }
65
66 if (r < 0) {
67     return vector<double> ();
68 }
69 }
70 if (D[n + 1][m] < -eps) {
71     return vector<double> ();
72 }
73
74 vector<double> x(m - 1);
75 for(int i = m; i < n + m; i++) {
76     if (ix[i] < m - 1) {
77         x[ix[i]] = D[i - m][m];
78     }
79 }
80 return x;
81 }

```



## Chapter 4

# 字符串

### 4.1 后缀数组

```
1 const int MAXN = MAXL * 2 + 1;
2 int a[MAXN], x[MAXN], y[MAXN], c[MAXN], sa[MAXN], rank[MAXN], height[MAXN];
3 void calc_sa(int n) {
4     int m = alphabet, k = 1;
5     memset(c, 0, sizeof(*c) * (m + 1));
6     for (int i = 1; i <= n; ++i) c[x[i] = a[i]]++;
7     for (int i = 1; i <= m; ++i) c[i] += c[i - 1];
8     for (int i = n; i; --i) sa[c[x[i]]--] = i;
9     for (; k <= n; k <= 1) {
10         int tot = k;
11         for (int i = n - k + 1; i <= n; ++i) y[i - n + k] = i;
12         for (int i = 1; i <= n; ++i)
13             if (sa[i] > k) y[++tot] = sa[i] - k;
14         memset(c, 0, sizeof(*c) * (m + 1));
15         for (int i = 1; i <= n; ++i) c[x[i]]++;
16         for (int i = 1; i <= m; ++i) c[i] += c[i - 1];
17         for (int i = n; i; --i) sa[c[x[y[i]]]--] = y[i];
18         for (int i = 1; i <= n; ++i) y[i] = x[i];
19         tot = 1; x[sa[1]] = 1;
20         for (int i = 2; i <= n; ++i) {
21             if (max(sa[i], sa[i - 1]) + k > n || y[sa[i]] != y[sa[i - 1]] || y[sa[i] + k] !=
↪ y[sa[i - 1] + k]) ++tot;
22             x[sa[i]] = tot;
23         }
24         if (tot == n) break; else m = tot;
25     }
26 }
27 void calc_height(int n) {
28     for (int i = 1; i <= n; ++i) rank[sa[i]] = i;
29     for (int i = 1; i <= n; ++i) {
30         height[rank[i]] = max(0, height[rank[i - 1]] - 1);
31         if (rank[i] == 1) continue;
32         int j = sa[rank[i] - 1];
```

```

33     while (max(i, j) + height[rank[i]] <= n && a[i + height[rank[i]]] == a[j +
↪ height[rank[i]]]) ++height[rank[i]];
34 }
35 }

```

## 4.2 后缀自动机

```

1 static const int MAXL = MAXN * 2; // MAXN is original length
2 static const int alphabet = 26; // sometimes need changing
3 int l, last, cnt, trans[MAXL][alphabet], par[MAXL], sum[MAXL], seq[MAXL], mxl[MAXL], size[MAXL];
↪ // mxl is maxlength, size is the size of right
4 char str[MAXL];
5 inline void init() {
6     l = strlen(str + 1); cnt = last = 1;
7     for (int i = 0; i <= l * 2; ++i) memset(trans[i], 0, sizeof(trans[i]));
8     memset(par, 0, sizeof(*par) * (l * 2 + 1));
9     memset(mxl, 0, sizeof(*mxl) * (l * 2 + 1));
10    memset(size, 0, sizeof(*size) * (l * 2 + 1));
11 }
12 inline void extend(int pos, int c) {
13     int p = last, np = last = ++cnt;
14     mxl[np] = mxl[p] + 1; size[np] = 1;
15     for (; p && !trans[p][c]; p = par[p]) trans[p][c] = np;
16     if (!p) par[np] = 1;
17     else {
18         int q = trans[p][c];
19         if (mxl[p] + 1 == mxl[q]) par[np] = q;
20         else {
21             int nq = ++cnt;
22             mxl[nq] = mxl[p] + 1;
23             memcpy(trans[nq], trans[q], sizeof(trans[nq]));
24             par[nq] = par[q];
25             par[np] = par[q] = nq;
26             for (; trans[p][c] == q; p = par[p]) trans[p][c] = nq;
27         }
28     }
29 }
30 inline void buildsam() {
31     for (int i = 1; i <= l; ++i) extend(i, str[i] - 'a');
32     memset(sum, 0, sizeof(*sum) * (l * 2 + 1));
33     for (int i = 1; i <= cnt; ++i) sum[mxl[i]]++;
34     for (int i = 1; i <= l; ++i) sum[i] += sum[i - 1];
35     for (int i = cnt; i; --i) seq[sum[mxl[i]]--] = i;
36     for (int i = cnt; i; --i) size[par[seq[i]]] += size[seq[i]];
37 }

```

## 4.3 EX 后缀自动机

```

1 inline void add_node(int x, int &last) {
2     int lastnode = last;
3     if (c[lastnode][x]) {
4         int nownode = c[lastnode][x];
5         if (l[nownode] == l[lastnode] + 1) last = nownode;
6         else {
7             int auxnode = ++cnt; l[auxnode] = l[lastnode] + 1;
8             for (int i = 0; i < alphabet; ++i) c[auxnode][i] = c[nownode][i];
9             par[auxnode] = par[nownode]; par[nownode] = auxnode;
10            for (; lastnode && c[lastnode][x] == nownode; lastnode = par[lastnode]) {
11                c[lastnode][x] = auxnode;
12            }
13            last = auxnode;
14        }
15    } else {
16        int newnode = ++cnt; l[newnode] = l[lastnode] + 1;
17        for (; lastnode && !c[lastnode][x]; lastnode = par[lastnode]) c[lastnode][x] = newnode;
18        if (!lastnode) par[newnode] = 1;
19        else {
20            int nownode = c[lastnode][x];
21            if (l[lastnode] + 1 == l[nownode]) par[newnode] = nownode;
22            else {
23                int auxnode = ++cnt; l[auxnode] = l[lastnode] + 1;
24                for (int i = 0; i < alphabet; ++i) c[auxnode][i] = c[nownode][i];
25                par[auxnode] = par[nownode]; par[nownode] = par[newnode] = auxnode;
26                for (; lastnode && c[lastnode][x] == nownode; lastnode = par[lastnode]) {
27                    c[lastnode][x] = auxnode;
28                }
29            }
30        }
31        last = newnode;
32    }
33 }

```

## 4.4 后缀树

1. 边上的字符区间是左闭右开区间；
2. 如果要建立关于多个串的后缀树，请用不同的分隔符，并且对于每个叶子结点，去掉和它父亲的连边上出现的第一个分隔符之后的所有字符；

## 4.5 回文自动机

```

1 int nT, nStr, last, c[MAXT][26], fail[MAXT], r[MAXN], l[MAXN], s[MAXN];
2 int allocate(int len) {
3     l[nT] = len;

```

```

4     r[nT] = 0;
5     fail[nT] = 0;
6     memset(c[nT], 0, sizeof(c[nT]));
7     return nT++;
8 }
9 void init() {
10     nT = nStr = 0;
11     int newE = allocate(0);
12     int newO = allocate(-1);
13     last = newE;
14     fail[newE] = newO;
15     fail[newO] = newE;
16     s[0] = -1;
17 }
18 void add(int x) {
19     s[++nStr] = x;
20     int now = last;
21     while (s[nStr - l[now] - 1] != s[nStr]) now = fail[now];
22     if (!c[now][x]) {
23         int newnode = allocate(l[now] + 2), &newfail = fail[newnode];
24         newfail = fail[now];
25         while (s[nStr - l[newfail] - 1] != s[nStr]) newfail = fail[newfail];
26         newfail = c[newfail][x];
27         c[now][x] = newnode;
28     }
29     last = c[now][x];
30     r[last]++;
31 }
32 void count() {
33     for (int i = nT - 1; i >= 0; i--) {
34         r[fail[i]] += r[i];
35     }
36 }

```

## 4.6 回文自动机-mxh

```

1 struct PAM
2 {
3     int trans[maxn][maxc];
4     int fail[maxn];
5     int cnt[maxn]; //出现次数
6     int num[maxn];
7     int len[maxn];
8     int s[maxn];
9     int last, n, tot;
10    int newnode(int l)
11    {
12        for (int i=0; i<maxc; i++) trans[tot][i]=0;

```

```
13     cnt[tot]=num[tot]=len[tot]=0;
14     len[tot]=1;
15     return tot++;
16 }
17 void clear()
18 {
19     last=n=tot=0;
20     newnode(0);newnode(-1);
21     s[0]=-1;fail[0]=1;
22 }
23 int get_fail(int x)
24 {
25     while (s[n-len[x]-1]!=s[n])    x=fail[x];
26     return x;
27 }
28 void add(int c)
29 {
30     s[++n]=c;
31     int cur=get_fail(last);
32     if (!trans[cur][c])
33     {
34         int now=newnode(len[cur]+2);
35         fail[now]=trans[get_fail(fail[cur])][c];
36         trans[cur][c]=now;
37         num[now]=num[fail[now]]+1;
38     }
39     last=trans[cur][c];
40     cnt[last]++;
41 }
42 void count()
43 {
44     for (int i=tot-1;i>=0;i--)    cnt[fail[i]]+=cnt[i];
45 }
46 } pam;
```



## Chapter 5

# 数据结构

### 5.1 KD-Tree

```
1 long long norm(const long long &x) {
2     // For manhattan distance
3     return std::abs(x);
4     // For euclid distance
5     return x * x;
6 }
7
8 struct Point {
9     int x, y, id;
10
11     const int& operator [] (int index) const {
12         if (index == 0) {
13             return x;
14         } else {
15             return y;
16         }
17     }
18
19     friend long long dist(const Point &a, const Point &b) {
20         long long result = 0;
21         for (int i = 0; i < 2; ++i) {
22             result += norm(a[i] - b[i]);
23         }
24         return result;
25     }
26 } point[N];
27
28 struct Rectangle {
29     int min[2], max[2];
30
31     Rectangle() {
32         min[0] = min[1] = INT_MAX; // sometimes int is not enough
33         max[0] = max[1] = INT_MIN;
```

```

34     }
35
36     void add(const Point &p) {
37         for (int i = 0; i < 2; ++i) {
38             min[i] = std::min(min[i], p[i]);
39             max[i] = std::max(max[i], p[i]);
40         }
41     }
42
43     long long dist(const Point &p) {
44         long long result = 0;
45         for (int i = 0; i < 2; ++i) {
46             // For minimum distance
47             result += norm(std::min(std::max(p[i], min[i]), max[i]) - p[i]);
48             // For maximum distance
49             result += std::max(norm(max[i] - p[i]), norm(min[i] - p[i]));
50         }
51         return result;
52     }
53 };
54
55 struct Node {
56     Point separator;
57     Rectangle rectangle;
58     int child[2];
59
60     void reset(const Point &p) {
61         separator = p;
62         rectangle = Rectangle();
63         rectangle.add(p);
64         child[0] = child[1] = 0;
65     }
66 } tree[N << 1];
67
68 int size, pivot;
69
70 bool compare(const Point &a, const Point &b) {
71     if (a[pivot] != b[pivot]) {
72         return a[pivot] < b[pivot];
73     }
74     return a.id < b.id;
75 }
76
77 // 左閉右開: build(1, n + 1)
78 int build(int l, int r, int type = 1) {
79     pivot = type;
80     if (l >= r) {
81         return 0;
82     }

```



```

83     int x = ++size;
84     int mid = l + r >> 1;
85     std::nth_element(point + l, point + mid, point + r, compare);
86     tree[x].reset(point[mid]);
87     for (int i = l; i < r; ++i) {
88         tree[x].rectangle.add(point[i]);
89     }
90     tree[x].child[0] = build(l, mid, type ^ 1);
91     tree[x].child[1] = build(mid + 1, r, type ^ 1);
92     return x;
93 }
94
95 int insert(int x, const Point &p, int type = 1) {
96     pivot = type;
97     if (x == 0) {
98         tree[++size].reset(p);
99         return size;
100     }
101     tree[x].rectangle.add(p);
102     if (compare(p, tree[x].separator)) {
103         tree[x].child[0] = insert(tree[x].child[0], p, type ^ 1);
104     } else {
105         tree[x].child[1] = insert(tree[x].child[1], p, type ^ 1);
106     }
107     return x;
108 }
109
110 // For minimum distance
111 // For maximum: 下面递归 query 时 0, 1 换顺序;< and >;min and max
112 void query(int x, const Point &p, std::pair<long long, int> &answer, int type = 1) {
113     pivot = type;
114     if (x == 0 || tree[x].rectangle.dist(p) > answer.first) {
115         return;
116     }
117     answer = std::min(answer,
118         std::make_pair(dist(tree[x].separator, p), tree[x].separator.id));
119     if (compare(p, tree[x].separator)) {
120         query(tree[x].child[0], p, answer, type ^ 1);
121         query(tree[x].child[1], p, answer, type ^ 1);
122     } else {
123         query(tree[x].child[1], p, answer, type ^ 1);
124         query(tree[x].child[0], p, answer, type ^ 1);
125     }
126 }
127
128 std::priority_queue<std::pair<long long, int> > answer;
129
130 void query(int x, const Point &p, int k, int type = 1) {
131     pivot = type;

```

```

132     if (x == 0 || (int)answer.size() == k && tree[x].rectangle.dist(p) > answer.top().first) {
133         return;
134     }
135     answer.push(std::make_pair(dist(tree[x].separator, p), tree[x].separator.id));
136     if ((int)answer.size() > k) {
137         answer.pop();
138     }
139     if (compare(p, tree[x].separator)) {
140         query(tree[x].child[0], p, k, type ^ 1);
141         query(tree[x].child[1], p, k, type ^ 1);
142     } else {
143         query(tree[x].child[1], p, k, type ^ 1);
144         query(tree[x].child[0], p, k, type ^ 1);
145     }
146 }

```

## 5.2 Treap

```

1 struct Node{
2     int mn, key, size, tag;
3     bool rev;
4     Node* ch[2];
5     Node(int mn, int key, int size): mn(mn), key(key), size(size), rev(0), tag(0){}
6     void downtag();
7     Node* update(){
8         mn = min(ch[0] -> mn, min(key, ch[1] -> mn));
9         size = ch[0] -> size + 1 + ch[1] -> size;
10        return this;
11    }
12 };
13 typedef pair<Node*, Node*> Pair;
14 Node *null, *root;
15 void Node::downtag(){
16     if(rev){
17         for(int i = 0; i < 2; i++)
18             if(ch[i] != null){
19                 ch[i] -> rev ^= 1;
20                 swap(ch[i] -> ch[0], ch[i] -> ch[1]);
21             }
22         rev = 0;
23     }
24     if(tag){
25         for(int i = 0; i < 2; i++)
26             if(ch[i] != null){
27                 ch[i] -> key += tag;
28                 ch[i] -> mn += tag;
29                 ch[i] -> tag += tag;
30             }

```

```

31     tag = 0;
32 }
33 }
34 int r(){
35     static int s = 3023192386;
36     return (s += (s << 3) + 1) & (~0u >> 1);
37 }
38 bool random(int x, int y){
39     return r() % (x + y) < x;
40 }
41 Node* merge(Node *p, Node *q){
42     if(p == null) return q;
43     if(q == null) return p;
44     p -> downntag();
45     q -> downntag();
46     if(random(p -> size, q -> size)){
47         p -> ch[1] = merge(p -> ch[1], q);
48         return p -> update();
49     }else{
50         q -> ch[0] = merge(p, q -> ch[0]);
51         return q -> update();
52     }
53 }
54 Pair split(Node *x, int n){
55     if(x == null) return make_pair(null, null);
56     x -> downntag();
57     if(n <= x -> ch[0] -> size){
58         Pair ret = split(x -> ch[0], n);
59         x -> ch[0] = ret.second;
60         return make_pair(ret.first, x -> update());
61     }
62     Pair ret = split(x -> ch[1], n - x -> ch[0] -> size - 1);
63     x -> ch[1] = ret.first;
64     return make_pair(x -> update(), ret.second);
65 }
66 pair<Node*, Pair> get_segment(int l, int r){
67     Pair ret = split(root, l - 1);
68     return make_pair(ret.first, split(ret.second, r - l + 1));
69 }
70 int main(){
71     null = new Node(INF, INF, 0);
72     null -> ch[0] = null -> ch[1] = null;
73     root = null;
74 }

```

### 5.3 Link/cut Tree

```

1 inline void reverse(int x) {
2     tr[x].rev ^= 1; swap(tr[x].c[0], tr[x].c[1]);
3 }
4
5 inline void rotate(int x, int k) {
6     int y = tr[x].fa, z = tr[y].fa;
7     tr[x].fa = z; tr[z].c[tr[z].c[1] == y] = x;
8     tr[tr[x].c[k ^ 1]].fa = y; tr[y].c[k] = tr[x].c[k ^ 1];
9     tr[x].c[k ^ 1] = y; tr[y].fa = x;
10 }
11
12 inline void splay(int x, int w) {
13     int z = x; pushdown(x);
14     while (tr[x].fa != w) {
15         int y = tr[x].fa; z = tr[y].fa;
16         if (z == w) {
17             pushdown(z = y); pushdown(x);
18             rotate(x, tr[y].c[1] == x);
19             update(y); update(x);
20         } else {
21             pushdown(z); pushdown(y); pushdown(x);
22             int t1 = tr[y].c[1] == x, t2 = tr[z].c[1] == y;
23             if (t1 == t2) rotate(y, t2), rotate(x, t1);
24             else rotate(x, t1), rotate(x, t2);
25             update(z); update(y); update(x);
26         }
27     }
28     update(x);
29     if (x != z) par[x] = par[z], par[z] = 0;
30 }
31
32 inline void access(int x) {
33     for (int y = 0; x; y = x, x = par[x]) {
34         splay(x, 0);
35         if (tr[x].c[1]) par[tr[x].c[1]] = x, tr[tr[x].c[1]].fa = 0;
36         tr[x].c[1] = y; par[y] = 0; tr[y].fa = x; update(x);
37     }
38 }
39
40 inline void makeroot(int x) {
41     access(x); splay(x, 0); reverse(x);
42 }
43
44 inline void link(int x, int y) {
45     makeroot(x); par[x] = y;
46 }
47

```

```
48 inline void cut(int x, int y) {
49     access(x); splay(y, 0);
50     if (par[y] != x) swap(x, y), access(x), splay(y, 0);
51     par[y] = 0;
52 }
53
54 inline void split(int x, int y) { // x will be the root of the tree
55     makeroot(y); access(x); splay(x, 0);
56 }
```

## 5.4 树状数组查询第 $k$ 小元素

```
1 int find(int k){
2     int cnt=0,ans=0;
3     for(int i=22;i>=0;i--){
4         ans+=(1<<i);
5         if(ans>n || cnt+d[ans]>=k)ans--(1<<i);
6         else cnt+=d[ans];
7     }
8     return ans+1;
9 }
```



## Chapter 6

# 图论

### 6.1 基础

```
1 struct Graph { // Remember to call .init()!
2     int e, nxt[M], v[M], adj[N], n;
3     bool base;
4     __inline void init(bool _base, int _n = 0) {
5         assert(n < N);
6         n = _n; base = _base;
7         e = 0; memset(adj + base, -1, sizeof(*adj) * n);
8     }
9     __inline int new_node() {
10         adj[n + base] = -1;
11         assert(n + base + 1 < N);
12         return n++ + base;
13     }
14     __inline void ins(int u0, int v0) { // directional
15         assert(u0 < n + base && v0 < n + base);
16         v[e] = v0; nxt[e] = adj[u0]; adj[u0] = e++;
17         assert(e < M);
18     }
19     __inline void bi_ins(int u0, int v0) { // bi-directional
20         ins(u0, v0); ins(v0, u0);
21     }
22 };
```

### 6.2 KM

```
1 struct KM {
2     // Truly  $O(n^3)$ 
3     // 邻接矩阵, 不能连的边设为 -INF, 求最小权匹配时边权取负, 但不能连的还是 -INF, 使用时先对 1 -> n
    ↪ 调用 hungary(), 再 get_ans() 求值
4     int w[N][N];
5     int lx[N], ly[N], match[N], way[N], slack[N];
```

```

6  bool used[N];
7  void init() {
8      for (int i = 1; i <= n; i++) {
9          match[i] = 0;
10         lx[i] = 0;
11         ly[i] = 0;
12         way[i] = 0;
13     }
14 }
15 void hungary(int x) {
16     match[0] = x;
17     int j0 = 0;
18     for (int j = 0; j <= n; j++) {
19         slack[j] = INF;
20         used[j] = false;
21     }
22
23     do {
24         used[j0] = true;
25         int i0 = match[j0], delta = INF, j1 = 0;
26         for (int j = 1; j <= n; j++) {
27             if (used[j] == false) {
28                 int cur = -w[i0][j] - lx[i0] - ly[j];
29                 if (cur < slack[j]) {
30                     slack[j] = cur;
31                     way[j] = j0;
32                 }
33                 if (slack[j] < delta) {
34                     delta = slack[j];
35                     j1 = j;
36                 }
37             }
38         }
39         for (int j = 0; j <= n; j++) {
40             if (used[j]) {
41                 lx[match[j]] += delta;
42                 ly[j] -= delta;
43             }
44             else slack[j] -= delta;
45         }
46         j0 = j1;
47     } while (match[j0] != 0);
48
49     do {
50         int j1 = way[j0];
51         match[j0] = match[j1];
52         j0 = j1;
53     } while (j0);
54 }

```



```

55
56     int get_ans() {
57         int sum = 0;
58         for(int i = 1; i <= n; i++) {
59             if (w[match[i]][i] == -INF) ; // 无解
60             if (match[i] > 0) sum += w[match[i]][i];
61         }
62         return sum;
63     }
64 } km;

```

## 6.3 HK

```

1  int matchx[N], matchy[N], level[N];
2  vector<int> edge[N];
3  bool dfs(int x) {
4      for (int i = 0; i < (int)edge[x].size(); ++i) {
5          int y = edge[x][i];
6          int w = matchy[y];
7          if (w == -1 || level[x] + 1 == level[w] && dfs(w)) {
8              matchx[x] = y;
9              matchy[y] = x;
10             return true;
11         }
12     }
13     level[x] = -1;
14     return false;
15 }
16 int solve() {
17     memset(matchx, -1, sizeof(*matchx) * n);
18     memset(matchy, -1, sizeof(*matchy) * m);
19     for (int ans = 0; ; ) {
20         std::vector<int> q;
21         for (int i = 0; i < n; ++i) {
22             if (matchx[i] == -1) {
23                 level[i] = 0;
24                 q.push_back(i);
25             } else {
26                 level[i] = -1;
27             }
28         }
29         for (int head = 0; head < (int)q.size(); ++head) {
30             int x = q[head];
31             for (int i = 0; i < (int)edge[x].size(); ++i) {
32                 int y = edge[x][i];
33                 int w = matchy[y];
34                 if (w != -1 && level[w] < 0) {
35                     level[w] = level[x] + 1;

```

```

36         q.push_back(w);
37     }
38 }
39 }
40 int delta = 0;
41 for (int i = 0; i < n; ++i) {
42     if (matchx[i] == -1 && dfs(i)) {
43         delta++;
44     }
45 }
46 if (delta == 0) {
47     return ans;
48 } else {
49     ans += delta;
50 }
51 }
52 }

```

## 6.4 点双连通分量

bcc.forest is a set of connected tree whose vertices are chequered with cut-vertex and BCC.

```

1  const bool BCC_VERTEX = 0, BCC_EDGE = 1;
2  struct BCC { // N = NO + MO. Remember to call init(&raw_graph).
3      Graph *g, forest; // g is raw graph ptr.
4      int dfn[N], DFN, low[N];
5      int stack[N], top;
6      int expand_to[N]; // Where edge i is expanded to in expanded graph.
7      // Vertex i expanded to i.
8      int compress_to[N]; // Where vertex i is compressed to.
9      bool vertex_type[N], cut[N], compress_cut[N], branch[M];
10     //std::vector<int> BCC_component[N]; // Cut vertex belongs to none.
11     __inline void init(Graph *raw_graph) {
12         g = raw_graph;
13     }
14     void DFS(int u, int pe) {
15         dfn[u] = low[u] = ++DFN; cut[u] = false;
16         if (!g->adj[u]) {
17             cut[u] = 1;
18             compress_to[u] = forest.new_node();
19             compress_cut[compress_to[u]] = 1;
20         }
21         for (int e = g->adj[u]; ~e; e = g->nxt[e]) {
22             int v = g->v[e];
23             if ((e ^ pe) > 1 && dfn[v] > 0 && dfn[v] < dfn[u]) {
24                 stack[top++] = e;
25                 low[u] = std::min(low[u], dfn[v]);
26             }
27             else if (!dfn[v]) {

```

```

28         stack[top++] = e; branch[e] = 1;
29         DFS(v, e);
30         low[u] = std::min(low[v], low[u]);
31         if (low[v] >= dfn[u]) {
32             if (!cut[u]) {
33                 cut[u] = 1;
34                 compress_to[u] = forest.new_node();
35                 compress_cut[compress_to[u]] = 1;
36             }
37             int cc = forest.new_node();
38             forest.bi_ins(compress_to[u], cc);
39             compress_cut[cc] = 0;
40             //BCC_component[cc].clear();
41             do {
42                 int cur_e = stack[--top];
43                 compress_to[expand_to[cur_e]] = cc;
44                 compress_to[expand_to[cur_e^1]] = cc;
45                 if (branch[cur_e]) {
46                     int v = g->v[cur_e];
47                     if (cut[v])
48                         forest.bi_ins(cc, compress_to[v]);
49                     else {
50                         //BCC_component[cc].push_back(v);
51                         compress_to[v] = cc;
52                     }
53                 }
54             } while (stack[top] != e);
55         }
56     }
57 }
58 }
59 void solve() {
60     forest.init(g->base);
61     int n = g->n;
62     for (int i = 0; i < g->e; i++) {
63         expand_to[i] = g->new_node();
64     }
65     memset(branch, 0, sizeof(*branch) * g->e);
66     memset(dfn + g->base, 0, sizeof(*dfn) * n); DFN = 0;
67     for (int i = 0; i < n; i++)
68         if (!dfn[i + g->base]) {
69             top = 0;
70             DFS(i + g->base, -1);
71         }
72 }
73 } bcc;
74
75 bcc.init(&raw_graph);
76 bcc.solve();

```

```
77 // Do something with bcc.forest ...
```

## 6.5 边双连通分量

```
1 struct BCC {
2     Graph *g, forest;
3     int dfn[N], low[N], stack[N], tot[N], belong[N], vis[N], top, dfs_clock;
4     // tot[] is the size of each BCC, belong[] is the BCC that each node belongs to
5     pair<int, int> ori[M]; // bridge in raw_graph(raw node)
6     bool is_bridge[M];
7     __inline void init(Graph *raw_graph) {
8         g = raw_graph;
9         memset(is_bridge, false, sizeof(*is_bridge) * g -> e);
10        memset(vis + g -> base, 0, sizeof(*vis) * g -> n);
11    }
12    void tarjan(int u, int from) {
13        dfn[u] = low[u] = ++dfs_clock; vis[u] = 1; stack[++top] = u;
14        for (int p = g -> adj[u]; ~p; p = g -> nxt[p]) {
15            if ((p ^ 1) == from) continue;
16            int v = g -> v[p];
17            if (vis[v]) {
18                if (vis[v] == 1) low[u] = min(low[u], dfn[v]);
19            } else {
20                tarjan(v, p);
21                low[u] = min(low[u], low[v]);
22                if (low[v] > dfn[u]) is_bridge[p / 2] = true;
23            }
24        }
25        if (dfn[u] != low[u]) return;
26        tot[forest.new_node()] = 0;
27        do {
28            belong[stack[top]] = forest.n;
29            vis[stack[top]] = 2;
30            tot[forest.n]++;
31            --top;
32        } while (stack[top + 1] != u);
33    }
34    void solve() {
35        forest.init(g -> base);
36        int n = g -> n;
37        for (int i = 0; i < n; ++i)
38            if (!vis[i + g -> base]) {
39                top = dfs_clock = 0;
40                tarjan(i + g -> base, -1);
41            }
42        for (int i = 0; i < g -> e / 2; ++i)
43            if (is_bridge[i]) {
44                int e = forest.e;
```

```

45         forest.bi_ins(belong[g -> v[i * 2]], belong[g -> v[i * 2 + 1]], g -> w[i * 2]);
46         ori[e] = make_pair(g -> v[i * 2 + 1], g -> v[i * 2]);
47         ori[e + 1] = make_pair(g -> v[i * 2], g -> v[i * 2 + 1]);
48     }
49 }
50 } bcc;

```

## 6.6 最小树形图

```

1  const int MAXN, INF; // INF >= sum( W_ij )
2  int from[MAXN + 10][MAXN * 2 + 10], n, m, edge[MAXN + 10][MAXN * 2 + 10];
3  int sel[MAXN * 2 + 10], fa[MAXN * 2 + 10], vis[MAXN * 2 + 10];
4  int getfa(int x){if(x == fa[x]) return x; return fa[x] = getfa(fa[x]);}
5  void liuzhu(){ // 1-base: root is 1, answer = (sel[i], i) for i in [2..n]
6      fa[1] = 1;
7      for(int i = 2; i <= n; ++i){
8          sel[i] = 1; fa[i] = i;
9          for(int j = 1; j <= n; ++j) if(fa[j] != i)
10             if(from[j][i] == i, edge[sel[i]][i] > edge[j][i]) sel[i] = j;
11     }
12     int limit = n;
13     while(1){
14         int prelimit = limit; memset(vis, 0, sizeof(vis)); vis[1] = 1;
15         for(int i = 2; i <= prelimit; ++i) if(fa[i] == i && !vis[i]){
16             int j = i; while(!vis[j]) vis[j] = i, j = getfa(sel[j]);
17             if(j == 1 || vis[j] != i) continue; vector<int> C; int k = j;
18             do C.push_back(k), k = getfa(sel[k]); while(k != j);
19             ++limit;
20             for(int i = 1; i <= n; ++i){
21                 edge[i][limit] = INF, from[i][limit] = limit;
22             }
23             fa[limit] = vis[limit] = limit;
24             for(int i = 0; i < int(C.size()); ++i){
25                 int x = C[i], fa[x] = limit;
26                 for(int j = 1; j <= n; ++j)
27                     if(edge[j][x] != INF && edge[j][limit] > edge[j][x] - edge[sel[x]][x]){
28                         edge[j][limit] = edge[j][x] - edge[sel[x]][x];
29                         from[j][limit] = x;
30                     }
31             }
32             for(int j = 1; j <= n; ++j) if(getfa(j) == limit) edge[j][limit] = INF;
33             sel[limit] = 1;
34             for(int j = 1; j <= n; ++j)
35                 if(edge[sel[limit]][limit] > edge[j][limit]) sel[limit] = j;
36         }
37         if(prelimit == limit) break;
38     }
39     for(int i = limit; i > 1; --i) sel[from[sel[i]][i]] = sel[i];

```

40 }

## 6.7 带花树

```

1 vector<int> link[maxn];
2 int n,match[maxn],Queue[maxn],head,tail;
3 int pred[maxn],base[maxn],start,finish,newbase;
4 bool InQueue[maxn],InBlossom[maxn];
5 void push(int u){ Queue[tail++]=u;InQueue[u]=true; }
6 int pop(){ return Queue[head++]; }
7 int FindCommonAncestor(int u,int v){
8     bool InPath[maxn];
9     for(int i=0;i<n;i++) InPath[i]=0;
10    while(true){ u=base[u];InPath[u]=true;if(u==start) break;u=pred[match[u]]; }
11    while(true){ v=base[v];if(InPath[v]) break;v=pred[match[v]]; }
12    return v;
13 }
14 void ResetTrace(int u){
15     int v;
16     while(base[u]!=newbase){
17         v=match[u];
18         InBlossom[base[u]]=InBlossom[base[v]]=true;
19         u=pred[v];
20         if(base[u]!=newbase) pred[u]=v;
21     }
22 }
23 void BlossomContract(int u,int v){
24     newbase=FindCommonAncestor(u,v);
25     for (int i=0;i<n;i++)
26         InBlossom[i]=0;
27     ResetTrace(u);ResetTrace(v);
28     if(base[u]!=newbase) pred[u]=v;
29     if(base[v]!=newbase) pred[v]=u;
30     for(int i=0;i<n;++i)
31         if(InBlossom[base[i]]){
32             base[i]=newbase;
33             if(!InQueue[i]) push(i);
34         }
35 }
36 bool FindAugmentingPath(int u){
37     bool found=false;
38     for(int i=0;i<n;++i) pred[i]=-1,base[i]=i;
39     for (int i=0;i<n;i++) InQueue[i]=0;
40     start=u;finish=-1; head=tail=0; push(start);
41     while(head<tail){
42         int u=pop();
43         for(int i=link[u].size()-1;i>=0;i--){
44             int v=link[u][i];

```

```

45         if(base[u]!=base[v]&&match[u]!=v)
46             if(v==start||(match[v]>=0&&pred[match[v]]>=0))
47                 BlossomContract(u,v);
48             else if(pred[v]==-1){
49                 pred[v]=u;
50                 if(match[v]>=0) push(match[v]);
51                 else{ finish=v; return true; }
52             }
53     }
54 }
55 return found;
56 }
57 void AugmentPath(){
58     int u=finish,v,w;
59     while(u>=0){ v=pred[u];w=match[v];match[v]=u;match[u]=v;u=w; }
60 }
61 void FindMaxMatching(){
62     for(int i=0;i<n;++i) match[i]=-1;
63     for(int i=0;i<n;++i) if(match[i]==-1) if(FindAugmentingPath(i)) AugmentPath();
64 }

```

## 6.8 带权带花树

```

1 //maximum weight blossom, change g[u][v].w to INF - g[u][v].w when minimum weight blossom is
  ↪ needed
2 //type of ans is long long
3 //replace all int to long long if weight of edge is long long
4
5 struct WeightGraph {
6     static const int INF = INT_MAX;
7     static const int MAXN = 400;
8     struct edge{
9         int u, v, w;
10        edge() {}
11        edge(int u, int v, int w): u(u), v(v), w(w) {}
12    };
13    int n, n_x;
14    edge g[MAXN * 2 + 1][MAXN * 2 + 1];
15    int lab[MAXN * 2 + 1];
16    int match[MAXN * 2 + 1], slack[MAXN * 2 + 1], st[MAXN * 2 + 1], pa[MAXN * 2 + 1];
17    int flower_from[MAXN * 2 + 1][MAXN+1], S[MAXN * 2 + 1], vis[MAXN * 2 + 1];
18    vector<int> flower[MAXN * 2 + 1];
19    queue<int> q;
20    inline int e_delta(const edge &e){ // does not work inside blossoms
21        return lab[e.u] + lab[e.v] - g[e.u][e.v].w * 2;
22    }
23    inline void update_slack(int u, int x){
24        if(!slack[x] || e_delta(g[u][x]) < e_delta(g[slack[x]][x]))

```

```

25         slack[x] = u;
26     }
27     inline void set_slack(int x){
28         slack[x] = 0;
29         for(int u = 1; u <= n; ++u)
30             if(g[u][x].w > 0 && st[u] != x && S[st[u]] == 0)
31                 update_slack(u, x);
32     }
33     void q_push(int x){
34         if(x <= n)q.push(x);
35         else for(size_t i = 0; i < flower[x].size(); i++)
36             q_push(flower[x][i]);
37     }
38     inline void set_st(int x, int b){
39         st[x]=b;
40         if(x > n) for(size_t i = 0; i < flower[x].size(); ++i)
41             set_st(flower[x][i], b);
42     }
43     inline int get_pr(int b, int xr){
44         int pr = find(flower[b].begin(), flower[b].end(), xr) - flower[b].begin();
45         if(pr % 2 == 1){
46             reverse(flower[b].begin() + 1, flower[b].end());
47             return (int)flower[b].size() - pr;
48         } else return pr;
49     }
50     inline void set_match(int u, int v){
51         match[u]=g[u][v].v;
52         if(u > n){
53             edge e=g[u][v];
54             int xr = flower_from[u][e.u], pr=get_pr(u, xr);
55             for(int i = 0; i < pr; ++i)
56                 set_match(flower[u][i], flower[u][i ^ 1]);
57             set_match(xr, v);
58             rotate(flower[u].begin(), flower[u].begin()+pr, flower[u].end());
59         }
60     }
61     inline void augment(int u, int v){
62         for(;;){
63             int xnv=st[match[u]];
64             set_match(u, v);
65             if(!xnv)return;
66             set_match(xnv, st[pa[xnv]]);
67             u=st[pa[xnv]], v=xnv;
68         }
69     }
70     inline int get_lca(int u, int v){
71         static int t=0;
72         for(++t; u || v; swap(u, v)){
73             if(u == 0)continue;

```



```

74         if(vis[u] == t)return u;
75         vis[u] = t;
76         u = st[match[u]];
77         if(u) u = st[pa[u]];
78     }
79     return 0;
80 }
81 inline void add_blossom(int u, int lca, int v){
82     int b = n + 1;
83     while(b <= n_x && st[b]) ++b;
84     if(b > n_x) ++n_x;
85     lab[b] = 0, S[b] = 0;
86     match[b] = match[lca];
87     flower[b].clear();
88     flower[b].push_back(lca);
89     for(int x = u, y; x != lca; x = st[pa[y]]) {
90         flower[b].push_back(x),
91         flower[b].push_back(y = st[match[x]]),
92         q_push(y);
93     }
94     reverse(flower[b].begin() + 1, flower[b].end());
95     for(int x = v, y; x != lca; x = st[pa[y]]) {
96         flower[b].push_back(x),
97         flower[b].push_back(y = st[match[x]]),
98         q_push(y);
99     }
100     set_st(b, b);
101     for(int x = 1; x <= n_x; ++x) g[b][x].w = g[x][b].w = 0;
102     for(int x = 1; x <= n; ++x) flower_from[b][x] = 0;
103     for(size_t i = 0; i < flower[b].size(); ++i){
104         int xs = flower[b][i];
105         for(int x = 1; x <= n_x; ++x)
106             if(g[b][x].w == 0 || e_delta(g[xs][x]) < e_delta(g[b][x]))
107                 g[b][x] = g[xs][x], g[x][b] = g[x][xs];
108         for(int x = 1; x <= n; ++x)
109             if(flower_from[xs][x]) flower_from[b][x] = xs;
110     }
111     set_slack(b);
112 }
113 inline void expand_blossom(int b){ // S[b] == 1
114     for(size_t i = 0; i < flower[b].size(); ++i)
115         set_st(flower[b][i], flower[b][i]);
116     int xr = flower_from[b][g[b][pa[b]].u], pr = get_pr(b, xr);
117     for(int i = 0; i < pr; i += 2){
118         int xs = flower[b][i], xns = flower[b][i + 1];
119         pa[xs] = g[xns][xs].u;
120         S[xs] = 1, S[xns] = 0;
121         slack[xs] = 0, set_slack(xns);
122         q_push(xns);

```

```

123     }
124     S[xr] = 1, pa[xr] = pa[b];
125     for(size_t i = pr + 1; i < flower[b].size(); ++i){
126         int xs = flower[b][i];
127         S[xs] = -1, set_slack(xs);
128     }
129     st[b] = 0;
130 }
131 inline bool on_found_edge(const edge &e){
132     int u = st[e.u], v = st[e.v];
133     if(S[v] == -1){
134         pa[v] = e.u, S[v] = 1;
135         int nu = st[match[v]];
136         slack[v] = slack[nu] = 0;
137         S[nu] = 0, q_push(nu);
138     }else if(S[v] == 0){
139         int lca = get_lca(u, v);
140         if(!lca) return augment(u, v), augment(v, u), true;
141         else add_blossom(u, lca, v);
142     }
143     return false;
144 }
145 inline bool matching(){
146     memset(S + 1, -1, sizeof(int) * n_x);
147     memset(slack + 1, 0, sizeof(int) * n_x);
148     q = queue<int>();
149     for(int x = 1; x <= n_x; ++x)
150         if(st[x] == x && !match[x]) pa[x]=0, S[x]=0, q_push(x);
151     if(q.empty())return false;
152     for(;;){
153         while(q.size()){
154             int u = q.front();q.pop();
155             if(S[st[u]] == 1)continue;
156             for(int v = 1; v <= n; ++v)
157                 if(g[u][v].w > 0 && st[u] != st[v]){
158                     if(e_delta(g[u][v]) == 0){
159                         if(on_found_edge(g[u][v]))return true;
160                     }else update_slack(u, st[v]);
161                 }
162         }
163         int d = INF;
164         for(int b = n + 1; b <= n_x; ++b)
165             if(st[b] == b && S[b] == 1)d = min(d, lab[b]/2);
166         for(int x = 1; x <= n_x; ++x)
167             if(st[x] == x && slack[x]){
168                 if(S[x] == -1)d = min(d, e_delta(g[slack[x]][x]));
169                 else if(S[x] == 0)d = min(d, e_delta(g[slack[x]][x])/2);
170             }
171         for(int u = 1; u <= n; ++u){

```

```

172         if(S[st[u]] == 0){
173             if(lab[u] <= d)return 0;
174             lab[u] -= d;
175         }else if(S[st[u]] == 1)lab[u] += d;
176     }
177     for(int b = n+1; b <= n_x; ++b)
178         if(st[b] == b){
179             if(S[st[b]] == 0) lab[b] += d * 2;
180             else if(S[st[b]] == 1) lab[b] -= d * 2;
181         }
182     q=queue<int>();
183     for(int x = 1; x <= n_x; ++x)
184         if(st[x] == x && slack[x] && st[slack[x]] != x && e_delta(g[slack[x]][x]) == 0)
185             if(on_found_edge(g[slack[x]][x]))return true;
186     for(int b = n + 1; b <= n_x; ++b)
187         if(st[b] == b && S[b] == 1 && lab[b] == 0)expand_blossom(b);
188 }
189 return false;
190 }
191 inline pair<long long, int> solve(){
192     memset(match + 1, 0, sizeof(int) * n);
193     n_x = n;
194     int n_matches = 0;
195     long long tot_weight = 0;
196     for(int u = 0; u <= n; ++u) st[u] = u, flower[u].clear();
197     int w_max = 0;
198     for(int u = 1; u <= n; ++u)
199         for(int v = 1; v <= n; ++v){
200             flower_from[u][v] = (u == v ? u : 0);
201             w_max = max(w_max, g[u][v].w);
202         }
203     for(int u = 1; u <= n; ++u) lab[u] = w_max;
204     while(matching()) ++n_matches;
205     for(int u = 1; u <= n; ++u)
206         if(match[u] && match[u] < u)
207             tot_weight += g[u][match[u]].w;
208     return make_pair(tot_weight, n_matches);
209 }
210 inline void init(){
211     for(int u = 1; u <= n; ++u)
212         for(int v = 1; v <= n; ++v)
213             g[u][v]=edge(u, v, 0);
214 }
215 };

```

## 6.9 Dominator Tree

```

1 vector<int> prec[N], succ[N];
2 vector<int> ord;
3 int stamp, vis[N];
4 int num[N];
5 int fa[N];
6 void dfs(int u) {
7     vis[u] = stamp;
8     num[u] = ord.size();
9     ord.push_back(u);
10    for (int i = 0; i < (int)succ[u].size(); ++i) {
11        int v = succ[u][i];
12        if (vis[v] != stamp) {
13            fa[v] = u;
14            dfs(v);
15        }
16    }
17 }
18 int fs[N], mins[N], dom[N], sem[N];
19 int find(int u) {
20     if (u != fs[u]) {
21         int v = fs[u];
22         fs[u] = find(fs[u]);
23         if (mins[v] != -1 && num[sem[mins[v]]] < num[sem[mins[u]]]) {
24             mins[u] = mins[v];
25         }
26     }
27     return fs[u];
28 }
29 void merge(int u, int v) { fs[u] = v; }
30 vector<int> buf[N];
31 int buf2[N];
32 void mark(int source) {
33     ord.clear();
34     ++stamp;
35     dfs(source);
36     for (int i = 0; i < (int)ord.size(); ++i) {
37         int u = ord[i];
38         fs[u] = u, mins[u] = -1, buf2[u] = -1;
39     }
40     for (int i = (int)ord.size() - 1; i > 0; --i) {
41         int u = ord[i], p = fa[u];
42         sem[u] = p;
43         for (int j = 0; j < (int)prec[u].size(); ++j) {
44             int v = prec[u][j];
45             if (use[v] != stamp) continue;
46             if (num[v] > num[u]) {
47                 find(v); v = sem[mins[v]];

```

```

48         }
49         if (num[v] < num[sem[u]]) {
50             sem[u] = v;
51         }
52     }
53     buf[sem[u]].push_back(u);
54     mins[u] = u;
55     merge(u, p);
56     while (buf[p].size()) {
57         int v = buf[p].back();
58         buf[p].pop_back();
59         find(v);
60         if (sem[v] == sem[mins[v]]) {
61             dom[v] = sem[v];
62         } else {
63             buf2[v] = mins[v];
64         }
65     }
66 }
67 dom[ord[0]] = ord[0];
68 for (int i = 0; i < (int)ord.size(); ++i) {
69     int u = ord[i];
70     if (~buf2[u]) {
71         dom[u] = dom[buf2[u]];
72     }
73 }
74 }

```

## 6.10 树 hash

```

1  const unsigned long long MAGIC = 4423;
2
3  unsigned long long magic[N];
4  std::pair<unsigned long long, int> hash[N];
5
6  void solve(int root) {
7      magic[0] = 1;
8      for (int i = 1; i <= n; ++i) {
9          magic[i] = magic[i - 1] * MAGIC;
10     }
11     std::vector<int> queue;
12     queue.push_back(root);
13     for (int head = 0; head < (int)queue.size(); ++head) {
14         int x = queue[head];
15         for (int i = 0; i < (int)son[x].size(); ++i) {
16             int y = son[x][i];
17             queue.push_back(y);
18         }

```

```

19 }
20 for (int index = n - 1; index >= 0; --index) {
21     int x = queue[index];
22     hash[x] = std::make_pair(0, 0);
23
24     std::vector<std::pair<unsigned long long, int> > value;
25     for (int i = 0; i < (int)son[x].size(); ++i) {
26         int y = son[x][i];
27         value.push_back(hash[y]);
28     }
29     std::sort(value.begin(), value.end());
30
31     hash[x].first = hash[x].first * magic[1] + 37;
32     hash[x].second++;
33     for (int i = 0; i < (int)value.size(); ++i) {
34         hash[x].first = hash[x].first * magic[value[i].second] + value[i].first;
35         hash[x].second += value[i].second;
36     }
37     hash[x].first = hash[x].first * magic[1] + 41;
38     hash[x].second++;
39 }
40 }

```

## 6.11 无向图最小割

```

1 int cost[maxn][maxn], seq[maxn], len[maxn], n, m, pop, ans;
2 bool used[maxn];
3 void Init(){
4     int i, j, a, b, c;
5     for(i=0; i<n; i++) for(j=0; j<n; j++) cost[i][j]=0;
6     for(i=0; i<m; i++){
7         scanf("%d %d %d", &a, &b, &c); cost[a][b]+=c; cost[b][a]+=c;
8     }
9     pop=n; for(i=0; i<n; i++) seq[i]=i;
10 }
11 void Work(){
12     ans=inf; int i, j, k, l, mm, sum, pk;
13     while(pop > 1){
14         for(i=1; i<pop; i++) used[seq[i]]=0; used[seq[0]]=1;
15         for(i=1; i<pop; i++) len[seq[i]]=cost[seq[0]][seq[i]];
16         pk=0; mm=-inf; k=-1;
17         for(i=1; i<pop; i++) if(len[seq[i]] > mm){ mm=len[seq[i]]; k=i; }
18         for(i=1; i<pop; i++){
19             used[seq[l=k]]=1;
20             if(i==pop-2) pk=k;
21             if(i==pop-1) break;
22             mm=-inf;
23             for(j=1; j<pop; j++) if(!used[seq[j]])

```

```

24         if((len[seq[j]]+=cost[seq[l]][seq[j]]) > mm)
25             mm=len[seq[j]], k=j;
26     }
27     sum=0;
28     for(i=0;i<pop;i++) if(i != k) sum+=cost[seq[k]][seq[i]];
29     ans=min(ans,sum);
30     for(i=0;i<pop;i++)
31         cost[seq[k]][seq[i]]=cost[seq[i]][seq[k]]+=cost[seq[pk]][seq[i]];
32     seq[pk]=seq[--pop];
33 }
34 printf("%d\n",ans);
35 }

```

## 6.12 ISAP 最大流

```

1  const unsigned long long MAGIC = 4423;
2
3  unsigned long long magic[N];
4  std::pair<unsigned long long, int> hash[N];
5
6  void solve(int root) {
7      magic[0] = 1;
8      for (int i = 1; i <= n; ++i) {
9          magic[i] = magic[i - 1] * MAGIC;
10     }
11     std::vector<int> queue;
12     queue.push_back(root);
13     for (int head = 0; head < (int)queue.size(); ++head) {
14         int x = queue[head];
15         for (int i = 0; i < (int)son[x].size(); ++i) {
16             int y = son[x][i];
17             queue.push_back(y);
18         }
19     }
20     for (int index = n - 1; index >= 0; --index) {
21         int x = queue[index];
22         hash[x] = std::make_pair(0, 0);
23
24         std::vector<std::pair<unsigned long long, int> > value;
25         for (int i = 0; i < (int)son[x].size(); ++i) {
26             int y = son[x][i];
27             value.push_back(hash[y]);
28         }
29         std::sort(value.begin(), value.end());
30
31         hash[x].first = hash[x].first * magic[1] + 37;
32         hash[x].second++;
33         for (int i = 0; i < (int)value.size(); ++i) {

```

```

34         hash[x].first = hash[x].first * magic[value[i].second] + value[i].first;
35         hash[x].second += value[i].second;
36     }
37     hash[x].first = hash[x].first * magic[1] + 41;
38     hash[x].second++;
39 }
40 }

```

### 6.13 重口味费用流

```

1  int S, T, totFlow, totCost;
2
3  int dis[N], slack[N], visit[N];
4
5  int modlable () {
6      int delta = INF;
7      for (int i = 1; i <= T; i++) {
8          if (!visit[i] && slack[i] < delta) delta = slack[i];
9          slack[i] = INF;
10     }
11     if (delta == INF) return 1;
12     for (int i = 1; i <= T; i++)
13         if (visit[i]) dis[i] += delta;
14     return 0;
15 }
16
17 int dfs (int x, int flow) {
18     if (x == T) {
19         totFlow += flow;
20         totCost += flow * (dis[S] - dis[T]);
21         return flow;
22     }
23     visit[x] = 1;
24     int left = flow;
25     for (int i = e.last[x]; ~i; i = e.succ[i])
26         if (e.cap[i] > 0 && !visit[e.other[i]]) {
27             int y = e.other[i];
28             if (dis[y] + e.cost[i] == dis[x]) {
29                 int delta = dfs (y, min (left, e.cap[i]));
30                 e.cap[i] -= delta;
31                 e.cap[i ^ 1] += delta;
32                 left -= delta;
33                 if (!left) { visit[x] = 0; return flow; }
34             } else {
35                 slack[y] = min (slack[y], dis[y] + e.cost[i] - dis[x]);
36             }
37         }
38     return flow - left;

```



```

39 }
40
41 pair <int, int> minCost () {
42     totFlow = 0; totCost = 0;
43     fill (dis + 1, dis + T + 1, 0);
44     do {
45         do {
46             fill (visit + 1, visit + T + 1, 0);
47             } while (dfs (S, INF));
48     } while (!modlable ());
49     return make_pair (totFlow, totCost);
50 }

```

## 6.14 2-SAT

```

1  int stamp, comps, top;
2  int dfn[N], low[N], comp[N], stack[N];
3
4  void add(int x, int a, int y, int b) {
5      edge[x << 1 | a].push_back(y << 1 | b);
6  }
7
8  void tarjan(int x) {
9      dfn[x] = low[x] = ++stamp;
10     stack[top++] = x;
11     for (int i = 0; i < (int)edge[x].size(); ++i) {
12         int y = edge[x][i];
13         if (!dfn[y]) {
14             tarjan(y);
15             low[x] = std::min(low[x], low[y]);
16         } else if (!comp[y]) {
17             low[x] = std::min(low[x], dfn[y]);
18         }
19     }
20     if (low[x] == dfn[x]) {
21         comps++;
22         do {
23             int y = stack[--top];
24             comp[y] = comps;
25         } while (stack[top] != x);
26     }
27 }
28
29 bool solve() {
30     int counter = n + n + 1;
31     stamp = top = comps = 0;
32     std::fill(dfn, dfn + counter, 0);
33     std::fill(comp, comp + counter, 0);

```

```

34     for (int i = 0; i < counter; ++i) {
35         if (!dfn[i]) {
36             tarjan(i);
37         }
38     }
39     for (int i = 0; i < n; ++i) {
40         if (comp[i << 1] == comp[i << 1 | 1]) {
41             return false;
42         }
43         answer[i] = (comp[i << 1 | 1] < comp[i << 1]);
44     }
45     return true;
46 }

```

## 6.15 欧拉遍历

```

1 //从一个奇度点 dfs, sqn 即为回路/路径
2 //first 存点, second 存边的编号, 正反边编号一致
3 //清空 cur、used 数组
4 void getCycle(int u)
5 {
6     for(int &i=cur[u]; i < (int)adj[u].size(); ++ i){
7         int id = adj[u][i].second;
8         if (used[id]) continue;
9         used[id] = true;
10        getCycle(adj[u][i].first);
11    }
12    sqn.push_back(u);
13 }

```

## 6.16 最大团搜索

```

1 /*
2 Int g[][] 为图的邻接矩阵。
3 MC(V) 表示点集 V 的最大团
4 令 Si={vi, vi+1, ..., vn}, mc[i] 表示 MC(Si)
5 倒着算 mc[i], 那么显然 MC(V)=mc[1]
6 此外有 mc[i]=mc[i+1] or mc[i]=mc[i+1]+1
7 */
8 void init(){
9     int i, j;
10    for (i=1; i<=n; ++i) for (j=1; j<=n; ++j) scanf("%d", &g[i][j]);
11 }
12 void dfs(int size){
13     int i, j, k;
14     if (len[size]==0) {
15         if (size>ans) {

```

```

16         ans=size; found=true;
17     }
18     return;
19 }
20 for (k=0; k<len[size] && !found; ++k) {
21     if (size+len[size]-k<=ans) break;
22     i=list[size][k];
23     if (size+mc[i]<=ans) break;
24     for (j=k+1, len[size+1]=0; j<len[size]; ++j)
25         if (g[i][list[size][j]]) list[size+1][len[size+1]++]=list[size][j];
26     dfs(size+1);
27 }
28 }
29 void work(){
30     int i, j;
31     mc[n]=ans=1;
32     for (i=n-1; i; --i) {
33         found=false;
34         len[1]=0;
35         for (j=i+1; j<=n; ++j) if (g[i][j]) list[1][len[1]++]=j;
36         dfs(1);
37         mc[i]=ans;
38     }
39 }
40 void print(){
41     printf("%d\n", ans);
42 }

```

## 6.17 线性规划

```

1 void sieve(){
2     f[1]=mu[1]=phi[1]=1;
3     for(int i=2; i<maxn; i++){
4         if(!minp[i]){
5             minp[i]=i;
6             minpw[i]=i;
7             mu[i]=-1;
8             phi[i]=i-1;
9             f[i]=i-1;
10            p[++p[0]]=i; //Case 1 prime
11        }
12        for(int j=1; j<=p[0] && (LL)i*p[j]<maxn; j++){
13            minp[i*p[j]]=p[j];
14            if(i%p[j]==0){
15                //Case 2 not coprime
16                minpw[i*p[j]]=minpw[i]*p[j];
17                phi[i*p[j]]=phi[i]*p[j];
18                mu[i*p[j]]=0;

```

```
19     if(i==minpw[i]){
20         f[i*p[j]]=i*p[j]-i; //Special Case for  $f(p^k)$ 
21     }else{
22         f[i*p[j]]=f[i/minpw[i]]*f[minpw[i]*p[j]];
23     }
24     break;
25 }else{
26     //Case 3 coprime
27     minpw[i*p[j]]=p[j];
28     f[i*p[j]]=f[i]*f[p[j]];
29     phi[i*p[j]]=phi[i]*(p[j]-1);
30     mu[i*p[j]]=-mu[i];
31 }
32 }
33 }
34 }
```

# Chapter 7

## 其他

### 7.1 Dancing Links

```
1 struct Node {
2     Node *l, *r, *u, *d, *col;
3     int size, line_no;
4     Node() {
5         size = 0; line_no = -1;
6         l = r = u = d = col = NULL;
7     }
8 } *root;
9
10 void cover(Node *c) {
11     c->l->r = c->r; c->r->l = c->l;
12     for (Node *u = c->d; u != c; u = u->d)
13         for (Node *v = u->r; v != u; v = v->r) {
14             v->d->u = v->u;
15             v->u->d = v->d;
16             -- v->col->size;
17         }
18 }
19
20 void uncover(Node *c) {
21     for (Node *u = c->u; u != c; u = u->u) {
22         for (Node *v = u->l; v != u; v = v->l) {
23             ++ v->col->size;
24             v->u->d = v;
25             v->d->u = v;
26         }
27     }
28     c->l->r = c; c->r->l = c;
29 }
30
31 std::vector<int> answer;
32 bool search(int k) {
33     if (root->r == root) return true;
```

```

34 Node *r = NULL;
35 for (Node *u = root->r; u != root; u = u->r)
36     if (r == NULL || u->size < r->size)
37         r = u;
38 if (r == NULL || r->size == 0) return false;
39 else {
40     cover(r);
41     bool succ = false;
42     for (Node *u = r->d; u != r && !succ; u = u->d) {
43         answer.push_back(u->line_no);
44         for (Node *v = u->r; v != u; v = v->r) // Cover row
45             cover(v->col);
46         succ |= search(k + 1);
47         for (Node *v = u->l; v != u; v = v->l)
48             uncover(v->col);
49         if (!succ) answer.pop_back();
50     }
51     uncover(r);
52     return succ;
53 }
54 }
55
56 bool entry[CR][CC];
57 Node *who[CR][CC];
58 int cr, cc;
59
60 void construct() {
61     root = new Node();
62     Node *last = root;
63     for (int i = 0; i < cc; ++i) {
64         Node *u = new Node();
65         last->r = u; u->l = last;
66         Node *v = u; u->line_no = i;
67         last = u;
68         for (int j = 0; j < cr; ++j)
69             if (entry[j][i]) {
70                 ++u->size;
71                 Node *cur = new Node();
72                 who[j][i] = cur;
73                 cur->line_no = j;
74                 cur->col = u;
75                 cur->u = v; v->d = cur;
76                 v = cur;
77             }
78         v->d = u; u->u = v;
79     }
80     last->r = root; root->l = last;
81     for (int j = 0; j < cr; ++j) {
82         Node *last = NULL;

```

```

83     for (int i = cc - 1; i >= 0; -- i)
84         if (entry[j][i]) {
85             last = who[j][i];
86             break;
87         }
88     for (int i = 0; i < cc; ++ i)
89         if (entry[j][i]) {
90             last->r = who[j][i];
91             who[j][i]->l = last;
92             last = who[j][i];
93         }
94     }
95 }
96
97 void destruct() {
98     for (Node *u = root->r; u != root; ) {
99         for (Node *v = u->d; v != u; ) {
100             Node *nxt = v->d;
101             delete(v);
102             v = nxt;
103         }
104         Node *nxt = u->r;
105         delete(u); u = nxt;
106     }
107     delete root;
108 }

```

## 7.2 Dancing Links 可重覆盖

```

1  #include<bits/stdc++.h>
2  using namespace std;
3  const int maxnode = 51111, MaxN = 55555, MaxM = 55555;
4  int n, m;
5  struct DLX{
6      int n,m,SIZE;
7      int U[maxnode],D[maxnode],R[maxnode],L[maxnode],Row[maxnode],Col[maxnode];
8      int H[MaxN],S[MaxM];
9      int ansd, ans[MaxN];
10     void init(int _n,int _m) {
11         n = _n;
12         m = _m;
13         for (int i = 0; i <= m; ++i) {
14             S[i] = 0;
15             U[i] = D[i] = i;
16             L[i] = i - 1;
17             R[i] = i + 1;
18         }
19         R[m] = 0; L[0] = m;

```

```

20     SIZE = m;
21     for (int i = 1; i <= n; ++i) H[i] = -1;
22 }
23 void Link(int r, int c) {
24     ++S[Col[++SIZE]=c];
25     Row[SIZE] = r;
26     D[SIZE] = D[c];
27     U[D[c]] = SIZE;
28     U[SIZE] = c;
29     D[c] = SIZE;
30     if (H[r] < 0) H[r] = L[SIZE] = R[SIZE] = SIZE;
31     else {
32         R[SIZE] = R[H[r]];
33         L[R[H[r]]] = SIZE;
34         L[SIZE] = H[r];
35         R[H[r]] = SIZE;
36     }
37 }
38 void repeat_remove(int c) {
39     for (int i = D[c]; i != c; i = D[i])
40         L[R[i]] = L[i], R[L[i]] = R[i];
41 }
42 void repeat_resume(int c) {
43     for (int i = U[c]; i != c; i = U[i])
44         L[R[i]] = R[L[i]] = i;
45 }
46 int f() {
47     bool vv[MaxM];
48     int ret = 0, c, i, j;
49     for (c = R[0]; c != 0; c = R[c]) vv[c] = 1;
50     for (c = R[0]; c != 0; c = R[c])
51         if (vv[c]) {
52             ++ret, vv[c] = 0;
53             for (i = D[c]; i != c; i = D[i])
54                 for (j = R[i]; j != i; j = R[j]) vv[Col[j]] = 0;
55         }
56
57     return ret;
58 }
59 void repeat_dance(int d) {
60     if (d + f() >= ansd) return;
61     if (R[0] == 0) {
62         if (d < ansd) ansd = d;
63         return;
64     }
65     int c = R[0], i, j;
66     for (i = R[0]; i; i = R[i]) if (S[i] < S[c]) c = i;
67     for (i = D[c]; i != c; i = D[i]) {
68         repeat_remove(i);

```



```

69         for (j = R[i]; j != i; j = R[j]) repeat_remove(j);
70         repeat_dance(d + 1);
71         for (j = L[i]; j != i; j = L[j]) repeat_resume(j);
72         repeat_resume(i);
73     }
74 }
75 };
76 DLX g;
77 void work() {
78     g.init(n, m);
79     for (int i = 1; i <= m; ++i) {
80         int x, y;
81         scanf("%d%d", &x, &y);
82         g.Link(x, i);
83         g.Link(y, i);
84     }
85     g.ansd = n;
86     g.repeat_dance(0);
87     cout << g.ansd << endl;
88 }
89 int main() {
90     while (~scanf("%d%d", &n, &m)) work();
91     return 0;
92 }

```

## 7.3 蔡勒公式

0 for Sunday. Day and month is 1-based.

```

1 int zeller(int y,int m,int d) {
2     if (m<=2) y--,m+=12; int c=y/100; y%=100;
3     int w=((c>>2)-(c<<1)+y+(y>>2)+(13*(m+1)/5)+d-1)%7;
4     if (w<0) w+=7; return(w);
5 }

```

## 7.4 蔡勒公式 new

0 for Sunday. Day and month is 1-based.

```

1 int zeller(int y,int m,int d)
2 {
3     if (m<=2)    y--,m+=12;
4     int c=y/100;
5     y%=100;
6     int w;
7     if (y<1582 || (y==1582 && (m<10 || (m==10 && d<=4))))
8         w=((c/4)-2*c+y+(y/4)+(13*(m+1)/5)+d-1)%7;
9     else

```

```
10     w=y+(y/4)+(c/4)-2*c+(13*(m+1)/5)+d+2;  
11     retur w;  
12 }
```

## Chapter 8

# 技巧

### 8.1 真正的释放 STL 容器内存空间

```
1 template <typename T>
2 __inline void clear(T& container) {
3     container.clear(); // 或者删除了一堆元素
4     T(container).swap(container);
5 }
```

### 8.2 无敌的大整数相乘取模

Time complexity  $O(1)$ .

```
1 // 需要保证 x 和 y 非负
2 long long mult(long long x, long long y, long long MODN) {
3     long long t = (x * y - (long long)((long double)x / MODN * y + 1e-3) * MODN) % MODN;
4     return t < 0 ? t + MODN : t;
5 }
```

### 8.3 无敌的读入优化

```
1 // getchar() 读入优化 << 关同步 cin << 此优化
2 // 用 isdigit() 会小幅变慢
3 // 返回 false 表示读到文件尾
4 namespace Reader {
5     const int L = (1 << 15) + 5;
6     char buffer[L], *S, *T;
7     __inline bool getchar(char &ch) {
8         if (S == T) {
9             T = (S = buffer) + fread(buffer, 1, L, stdin);
10            if (S == T) {
11                ch = EOF;
12                return false;
13            }
14        }
15    }
```

```
14     }
15     ch = *S++;
16     return true;
17 }
18 __inline bool getint(int &x) {
19     char ch; bool neg = 0;
20     for (; getchar(ch) && (ch < '0' || ch > '9'); ) neg ^= ch == '-';
21     if (ch == EOF) return false;
22     x = ch - '0';
23     for (; getchar(ch), ch >= '0' && ch <= '9'; )
24         x = x * 10 + ch - '0';
25     if (neg) x = -x;
26     return true;
27 }
28 }
```

## 8.4 梅森旋转算法

High quality pseudorandom number generator, twice as efficient as `rand()` with `-O2`. C++11 required.

```
1 #include <random>
2
3 int main() {
4     std::mt19937 g(seed); // std::mt19937_64
5     std::cout << g() << std::endl;
6 }
```

## Chapter 9

# 提示

### 9.1 反演相关

$$\sigma_k(n) = \sum_{d|n} d^k = \prod_{i=1}^{\omega(n)} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1}$$

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right)$$

$J_k(n)$  is the number of  $k$ -tuples of positive integers all less than or equal to  $n$  that form a coprime  $(k+1)$ -tuple together with  $n$ .

$$\sum_{\delta|n} J_k(\delta) = n^k$$

$$\sum_{\delta|n} \delta^s J_r(\delta) J_s\left(\frac{n}{\delta}\right) = J_{r+s}(n)$$

$$\sum_{\delta|n} \varphi(\delta) d\left(\frac{n}{\delta}\right) = \sigma(n), \quad \sum_{\delta|n} |\mu(\delta)| = 2^{\omega(n)}$$

$$\sum_{\delta|n} 2^{\omega(\delta)} = d(n^2), \quad \sum_{\delta|n} d(\delta^2) = d^2(n)$$

$$\sum_{\delta|n} d\left(\frac{n}{\delta}\right) 2^{\omega(\delta)} = d^2(n), \quad \sum_{\delta|n} \frac{\mu(\delta)}{\delta} = \frac{\varphi(n)}{n}$$

$$\sum_{\delta|n} \frac{\mu(\delta)}{\varphi(\delta)} = d(n), \quad \sum_{\delta|n} \frac{\mu^2(\delta)}{\varphi(\delta)} = \frac{n}{\varphi(n)}$$

$$n | \varphi(a^n - 1)$$

$$\sum_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} f(\gcd(k-1, n)) = \varphi(n) \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)}$$

$$\varphi(\text{lcm}(m, n)) \varphi(\gcd(m, n)) = \varphi(m) \varphi(n)$$

$$\begin{aligned}
\sum_{\delta|n} d^3(\delta) &= \left( \sum_{\delta|n} d(\delta) \right)^2 \\
d(uv) &= \sum_{\delta|\gcd(u,v)} \mu(\delta) d\left(\frac{u}{\delta}\right) d\left(\frac{v}{\delta}\right) \\
\sigma_k(u)\sigma_k(v) &= \sum_{\delta|\gcd(u,v)} \delta^k \sigma_k\left(\frac{uv}{\delta^2}\right) \\
\mu(n) &= \sum_{k=1}^n [\gcd(k, n) = 1] \cos 2\pi \frac{k}{n} \\
\varphi(n) &= \sum_{k=1}^n [\gcd(k, n) = 1] = \sum_{k=1}^n \gcd(k, n) \cos 2\pi \frac{k}{n} \\
\begin{cases} S(n) = \sum_{k=1}^n (f * g)(k) \\ \sum_{k=1}^n S(\lfloor \frac{n}{k} \rfloor) = \sum_{i=1}^n f(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} (g * 1)(j) \end{cases} \\
\begin{cases} S(n) = \sum_{k=1}^n (f \cdot g)(k), g \text{ completely multiplicative} \\ \sum_{k=1}^n S(\lfloor \frac{n}{k} \rfloor) g(k) = \sum_{k=1}^n (f * 1)(k) g(k) \end{cases}
\end{aligned}$$

## 9.2 第二类斯特林数

$$\begin{aligned}
\begin{Bmatrix} n+1 \\ k \end{Bmatrix} &= k \begin{Bmatrix} n \\ k \end{Bmatrix} + \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \\
x^n &= \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}} = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\bar{k}} \\
m! \begin{Bmatrix} n \\ m \end{Bmatrix} &= \sum_k \binom{m}{k} k^n (-1)^{m-k}
\end{aligned}$$

## 9.3 控制 cout 输出实数精度

```
1 std::cout << std::fixed << std::setprecision(5);
```

## 9.4 vimrc

```
1 set nu
2 set sw=4
3 set sts=4
4 set ts=4
```

```
5 | syntax on
6 | set cindent
```

9.5 让 make 支持 c++ 11

In .bashrc or whatever:  
export CXXFLAGS='-std=c++11 -Wall'

9.6 tuple 相关

```
1 | mytuple = std::make_tuple (10, 2.6, 'a');           // packing values into tuple
2 | std::tie (myint, std::ignore, mychar) = mytuple;    // unpacking tuple into variables
3 | std::get<I>(mytuple) = 20;
4 | std::cout << std::get<I>(mytuple) << std::endl;    // get the Ith(const) element
```

9.7 线性规划转对偶

maximize  $\mathbf{c}^T \mathbf{x}$   
subject to  $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0$   $\iff$  minimize  $\mathbf{y}^T \mathbf{b}$   
subject to  $\mathbf{y}^T \mathbf{A} \geq \mathbf{c}^T, \mathbf{y} \geq 0$

9.8 32-bit/64-bit 随机素数

32-bit	64-bit
73550053	1249292846855685773
148898719	1701750434419805569
189560747	3605499878424114901
459874703	5648316673387803781
1202316001	6125342570814357977
1431183547	6215155308775851301
1438011109	6294606778040623451
1538762023	6347330550446020547
1557944263	7429632924303725207
1981315913	8524720079480389849

9.9 NTT 素数及其原根

Prime	Primitive root
1053818881	7
1051721729	6
1045430273	3
1012924417	5
1007681537	3

9.10 Java Hints

```
1 | import java.io.*;
2 | import java.lang.*;
```

```

3 import java.math.*;
4 import java.util.*;
5
6 /* Regular usage:
7     Slower IO :
8         Scanner in = new Scanner (System.in);
9         Scanner in = new Scanner (new BufferedInputStream (System.in));
10        Input :
11            in.nextInt () / in.nextBigInteger () / in.nextBigDecimal () / in.nextDouble ()
12            in.nextLine () / in.hasNext ()
13        Output :
14            System.out.print (...);
15            System.out.println (...);
16            System.out.printf (...);
17    Faster IO :
18        Shown below.
19    BigInteger :
20        BigInteger.valueOf (int) : convert to BigInteger.
21        abs / negate () / max / min / add / subtract / multiply /
22        divide / remainder (BigInteger) : BigInteger algebraic.
23        gcd (BigInteger) / modInverse (BigInteger mod) /
24        modPow (BigInteger ex, BigInteger mod) / pow (int ex) : Number Theory.
25        not () / and / or / xor (BigInteger) / shiftLeft / shiftRight (int) : Bit operation.
26        compareTo (BigInteger) : comparison.
27        intValue () / longValue () / toString (int radix) : converts to other types.
28        isProbablePrime (int certainty) / nextProbablePrime () : checks primitive.
29    BigDecimal :
30        consists of a BigInteger value and a scale.
31        The scale is the number of digits to the right of the decimal point.
32        divide (BigDecimal) : exact divide.
33        divide (BigDecimal, int scale, RoundingMode roundingMode) :
34            divide with roundingMode, which may be:
35                CEILING / DOWN / FLOOR / HALF_DOWN / HALF_EVEN / HALF_UP / UNNECESSARY / UP.
36        BigDecimal setScale (int newScale, RoundingMode roundingMode) :
37            returns a BigDecimal with newScale.
38        doubleValue () / toPlainString () : converts to other types.
39    Arrays :
40        Arrays.sort (T [] a);
41        Arrays.sort (T [] a, int fromIndex, int toIndex);
42        Arrays.sort (T [] a, int fromIndex, int toIndex, Comparator <? super T> comparator);
43    LinkedList <E> :
44        addFirst / addLast (E) / getFirst / getLast / removeFirst / removeLast () :
45            deque implementation.
46        clear () / add (int, E) / remove (int) : clear, add & remove.
47        size () / contains / removeFirstOccurrence / removeLastOccurrence (E) :
48            deque methods.
49        ListIterator <E> listIterator (int index) : returns an iterator :
50            E next / previous () : accesses and iterates.
51            hasNext / hasPrevious () : checks availability.

```



```

52         nextIndex / previousIndex () : returns the index of a subsequent call.
53         add / set (E) / remove () : changes element.
54     PriorityQueue <E> (int initcap, Comparator <? super E> comparator) :
55         add (E) / clear () / iterator () / peek () / poll () / size () :
56         priority queue implementations.
57     TreeMap <K, V> (Comparator <? super K> comparator) :
58         Map.Entry <K, V> ceilingEntry / floorEntry / higherEntry / lowerEntry (K):
59         getKey / getValue () / setValue (V) : entries.
60         clear () / put (K, V) / get (K) / remove (K) : basic operation.
61         size () : size.
62     StringBuilder :
63         Mutable string.
64         StringBuilder (string) : generates a builder.
65         append (int, string, ...) / insert (int offset, ...) : adds objects.
66         charAt (int) / setCharAt (int, char) : accesses a char.
67         delete (int, int) : removes a substring.
68         reverse () : reverses itself.
69         length () : returns the length.
70         toString () : converts to string.
71     String :
72         Immutable string.
73         String.format (String, ...) : formats a string. i.e. sprintf.
74         toLowerCase / toUpperCase () : changes the case of letters.
75 */
76
77 /* Examples on Comparator :
78 public class Main {
79     public static class Point {
80         public int x;
81         public int y;
82         public Point () {
83             x = 0;
84             y = 0;
85         }
86         public Point (int xx, int yy) {
87             x = xx;
88             y = yy;
89         }
90     };
91     public static class Cmp implements Comparator <Point> {
92         public int compare (Point a, Point b) {
93             if (a.x < b.x) return -1;
94             if (a.x == b.x) {
95                 if (a.y < b.y) return -1;
96                 if (a.y == b.y) return 0;
97             }
98             return 1;
99         }
100     };

```

```

101     public static void main (String [] args) {
102         Cmp c = new Cmp ();
103         TreeMap <Point, Point> t = new TreeMap <Point, Point> (c);
104         return;
105     }
106 };
107 */
108
109 /*  Another way to implement is to use Comparable.
110    However, equalTo and hashCode must be rewritten.
111    Otherwise, containers may fail.
112    Example :
113    public static class Point implements Comparable <Point> {
114        public int x;
115        public int y;
116        public Point () {
117            x = 0;
118            y = 0;
119        }
120        public Point (int xx, int yy) {
121            x = xx;
122            y = yy;
123        }
124        public int compareTo (Point p) {
125            if (x < p.x) return -1;
126            if (x == p.x) {
127                if (y < p.y) return -1;
128                if (y == p.y) return 0;
129            }
130            return 1;
131        }
132        public boolean equalTo (Point p) {
133            return (x == p.x && y == p.y);
134        }
135        public int hashCode () {
136            return x + y;
137        }
138    };
139 */
140
141 //Faster IO :
142
143 public class Main {
144
145     static class InputReader {
146         public BufferedReader reader;
147         public StringTokenizer tokenizer;
148         public InputReader (InputStream stream) {
149             reader = new BufferedReader (new InputStreamReader (stream), 32768);

```

```

150         tokenizer = null;
151     }
152     public String next() {
153         while (tokenizer == null || !tokenizer.hasMoreTokens()) {
154             try {
155                 String line = reader.readLine();
156                 tokenizer = new StringTokenizer (line);
157             } catch (IOException e) {
158                 throw new RuntimeException (e);
159             }
160         }
161         return tokenizer.nextToken();
162     }
163     public BigInteger nextBigInteger() {
164         return new BigInteger (next (), 10);    //    customize the radix here.
165     }
166     public int nextInt() {
167         return Integer.parseInt (next());
168     }
169     public double nextDouble() {
170         return Double.parseDouble (next());
171     }
172 }
173
174 public static void main (String[] args) {
175     InputReader in = new InputReader (System.in);
176
177     //    Put your code here.
178
179 }
180 }
181
182 // Arrays
183 int a[];
184 .fill(a[, int fromIndex, int toIndex],val) | .sort(a[, int fromIndex, int toIndex])
185 // String
186 String s;
187 .charAt(int i) | compareTo(String) | compareToIgnoreCase () | contains(String) |
188 length () | substring(int l, int len)
189 // BigInteger
190 .abs() | .add() | bitLength () | subtract () | divide () | remainder () | divideAndRemainder ()
191     ↪ | modPow(b, c) |
192 pow(int) | multiply () | compareTo () |
193 gcd() | intValue () | longValue () | isProbablePrime(int c) (1 - 1/2^c) |
194 nextProbablePrime () | shiftLeft(int) | valueOf ()
195 // BigDecimal
196 .ROUND_CEILING | ROUND_DOWN_FLOOR | ROUND_HALF_DOWN | ROUND_HALF_EVEN | ROUND_HALF_UP | ROUND_UP
197 .divide(BigDecimal b, int scale , int round_mode) | doubleValue () | movePointLeft(int) |
198     ↪ pow(int) |

```

```
197 setScale(int scale(精度) , int round_mode) | stripTrailingZeros ()
198 // StringBuilder
199 StringBuilder sb = new StringBuilder ();
200 sb.append(elem) | out.println(sb)
201 // Collection
202 .add(Object o) => boolean | .clear() | .isEmpty() | .contains() | .size() | .toArray() |
    ↪ .remove()
203 for (Object o: s) {}
204 // Set
205 Set s = new HashSet () | TreeSet ();
206 // List
207 List s = new LinkedList () | ArrayList ();
208 .add([int index,] E e) | .get (int index) | .indexOf (Object o) | .set (int index, E e) |
    ↪ .subList (int l, int r) | .remove (int index)
```

# Theoretical Computer Science Cheat Sheet

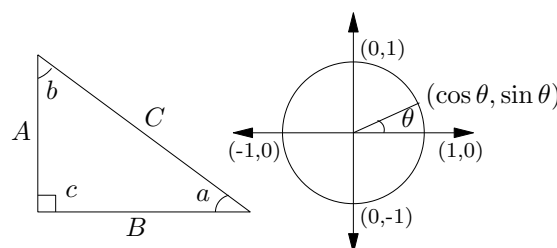
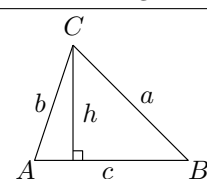
Definitions		Series
$f(n) = O(g(n))$	iff $\exists$ positive $c, n_0$ such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$ .	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$
$f(n) = \Omega(g(n))$	iff $\exists$ positive $c, n_0$ such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$ .	In general:
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ .	$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[ (n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$
$f(n) = o(g(n))$	iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$ .	$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a  < \epsilon, \forall n \geq n_0$ .	Geometric series:
$\sup S$	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$ .	$\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad  c  < 1,$
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$ .	$\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad  c  < 1.$
$\liminf_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}.$	Harmonic series:
$\limsup_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}.$	$H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$
$\binom{n}{k}$	Combinations: Size $k$ sub-sets of a size $n$ set.	$\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left( H_{n+1} - \frac{1}{m+1} \right).$
$[n_k]$	Stirling numbers (1st kind): Arrangements of an $n$ element set into $k$ cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$
$\{n_k\}$	Stirling numbers (2nd kind): Partitions of an $n$ element set into $k$ non-empty sets.	4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$
$\langle n_k \rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with $k$ ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$
$\langle\langle n_k \rangle\rangle$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$
$C_n$	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1,$
14. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!,$	15. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$	12. $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1, \quad 13. \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\},$
16. $\begin{bmatrix} n \\ n \end{bmatrix} = 1,$	17. $\begin{bmatrix} n \\ k \end{bmatrix} \geq \left\{ \begin{matrix} n \\ k \end{matrix} \right\},$	
18. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$	19. $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2},$	20. $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$
22. $\langle n_0 \rangle = \langle n_{n-1} \rangle = 1,$	23. $\langle n_k \rangle = \langle n_{n-1-k} \rangle,$	24. $\langle n_k \rangle = (k+1) \langle n-1_k \rangle + (n-k) \langle n-1_{k-1} \rangle,$
25. $\langle n_k \rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\langle n_1 \rangle = 2^n - n - 1,$	27. $\langle n_2 \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$
28. $x^n = \sum_{k=0}^n \langle n_k \rangle \binom{x+k}{n},$	29. $\langle n_m \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	30. $m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^n \langle n_k \rangle \binom{k}{n-m},$
31. $\langle n_m \rangle = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\langle\langle n_0 \rangle\rangle = 1,$	33. $\langle\langle n_n \rangle\rangle = 0 \quad \text{for } n \neq 0,$
34. $\langle\langle n_k \rangle\rangle = (k+1) \langle\langle n-1_k \rangle\rangle + (2n-1-k) \langle\langle n-1_{k-1} \rangle\rangle,$	35. $\sum_{k=0}^n \langle\langle n_k \rangle\rangle = \frac{(2n)^n}{2^n},$	
36. $\left\{ \begin{matrix} x \\ x-n \end{matrix} \right\} = \sum_{k=0}^n \langle\langle n_k \rangle\rangle \binom{x+n-1-k}{2n},$	37. $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \sum_{k=0}^n \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (m+1)^{n-k},$	

# Theoretical Computer Science Cheat Sheet

Identities Cont.		Trees
<p>38. <math>\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},</math></p> <p>40. <math>\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k},</math></p> <p>42. <math>\left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} = \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\},</math></p> <p>44. <math>\binom{n}{m} = \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},</math></p> <p>46. <math>\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},</math></p> <p>48. <math>\left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k},</math></p>	<p>39. <math>\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{2n},</math></p> <p>41. <math>\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k},</math></p> <p>43. <math>\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},</math></p> <p>45. <math>(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k},</math> for <math>n \geq m,</math></p> <p>47. <math>\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\},</math></p> <p>49. <math>\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.</math></p>	<p>Every tree with <math>n</math> vertices has <math>n-1</math> edges.</p> <p>Kraft inequality: If the depths of the leaves of a binary tree are <math>d_1, \dots, d_n</math>:</p> $\sum_{i=1}^n 2^{-d_i} \leq 1,$ <p>and equality holds only if every internal node has 2 sons.</p>
Recurrences		
<p>Master method:</p> $T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$ <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = O(n^{\log_b a - \epsilon})</math> then</p> $T(n) = \Theta(n^{\log_b a}).$ <p>If <math>f(n) = \Theta(n^{\log_b a})</math> then</p> $T(n) = \Theta(n^{\log_b a} \log_2 n).$ <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = \Omega(n^{\log_b a + \epsilon})</math>, and <math>\exists c &lt; 1</math> such that <math>af(n/b) \leq cf(n)</math> for large <math>n</math>, then</p> $T(n) = \Theta(f(n)).$ <p>Substitution (example): Consider the following recurrence</p> $T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$ <p>Note that <math>T_i</math> is always a power of two. Let <math>t_i = \log_2 T_i</math>. Then we have</p> $t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$ <p>Let <math>u_i = t_i/2^i</math>. Dividing both sides of the previous equation by <math>2^{i+1}</math> we get</p> $\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$ <p>Substituting we find</p> $u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$ <p>which is simply <math>u_i = i/2</math>. So we find that <math>T_i</math> has the closed form <math>T_i = 2^{2^{i-1}}</math>.</p> <p>Summing factors (example): Consider the following recurrence</p> $T(n) = 3T(n/2) + n, \quad T(1) = 1.$ <p>Rewrite so that all terms involving <math>T</math> are on the left side</p> $T(n) - 3T(n/2) = n.$ <p>Now expand the recurrence, and choose a factor which makes the left side “telescope”</p>	$\begin{aligned} 1(T(n) - 3T(n/2) &= n) \\ 3(T(n/2) - 3T(n/4) &= n/2) \\ \vdots \quad \quad \quad \vdots & \\ 3^{\log_2 n - 1}(T(2) - 3T(1) &= 2) \end{aligned}$ <p>Let <math>m = \log_2 n</math>. Summing the left side we get <math>T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k</math> where <math>k = \log_2 3 \approx 1.58496</math>. Summing the right side we get</p> $\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$ <p>Let <math>c = \frac{3}{2}</math>. Then we have</p> $\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n \left( \frac{c^m - 1}{c - 1} \right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1) \log_2 n} - 1) \\ &= 2n^k - 2n, \end{aligned}$ <p>and so <math>T(n) = 3n^k - 2n</math>. Full history recurrences can often be changed to limited history ones (example): Consider</p> $T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$ <p>Note that</p> $T_{i+1} = 1 + \sum_{j=0}^i T_j.$ <p>Subtracting we find</p> $\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$ <p>And so <math>T_{i+1} = 2T_i = 2^{i+1}</math>.</p>	<p>Generating functions:</p> <ol style="list-style-type: none"> <li>Multiply both sides of the equation by <math>x^i</math>.</li> <li>Sum both sides over all <math>i</math> for which the equation is valid.</li> <li>Choose a generating function <math>G(x)</math>. Usually <math>G(x) = \sum_{i=0}^{\infty} x^i g_i</math>.</li> <li>Rewrite the equation in terms of the generating function <math>G(x)</math>.</li> <li>Solve for <math>G(x)</math>.</li> <li>The coefficient of <math>x^i</math> in <math>G(x)</math> is <math>g_i</math>.</li> </ol> <p>Example:</p> $g_{i+1} = 2g_i + 1, \quad g_0 = 0.$ <p>Multiply and sum:</p> $\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$ <p>We choose <math>G(x) = \sum_{i \geq 0} x^i g_i</math>. Rewrite in terms of <math>G(x)</math>:</p> $\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$ <p>Simplify:</p> $\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$ <p>Solve for <math>G(x)</math>:</p> $G(x) = \frac{x}{(1-x)(1-2x)}.$ <p>Expand this using partial fractions:</p> $\begin{aligned} G(x) &= x \left( \frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left( 2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\ &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$ <p>So <math>g_i = 2^i - 1</math>.</p>

Theoretical Computer Science Cheat Sheet					
$\pi \approx 3.14159,$		$e \approx 2.71828,$	$\gamma \approx 0.57721,$	$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$	$\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$
$i$	$2^i$	$p_i$	General	Probability	
1	2	2	Bernoulli Numbers ( $B_i = 0$ , odd $i \neq 1$ ):	Continuous distributions: If	
2	4	3	$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$	$\Pr[a < X < b] = \int_a^b p(x) dx,$	
3	8	5	$B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	then $p$ is the probability density function of $X$ . If	
4	16	7	Change of base, quadratic formula:	$\Pr[X < a] = P(a),$	
5	32	11	$\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	then $P$ is the distribution function of $X$ . If $P$ and $p$ both exist then	
6	64	13	Euler's number $e$ :	$P(a) = \int_{-\infty}^a p(x) dx.$	
7	128	17	$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$	Expectation: If $X$ is discrete	
8	256	19	$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	$E[g(X)] = \sum_x g(x) \Pr[X = x].$	
9	512	23	$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$	If $X$ continuous then	
10	1,024	29	$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$	$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$	
11	2,048	31	Harmonic numbers:	Variance, standard deviation:	
12	4,096	37	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	$\text{VAR}[X] = E[X^2] - E[X]^2,$	
13	8,192	41	$\ln n < H_n < \ln n + 1,$	$\sigma = \sqrt{\text{VAR}[X]}.$	
14	16,384	43	$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$	For events $A$ and $B$ :	
15	32,768	47	Factorial, Stirling's approximation:	$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$	
16	65,536	53	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$	
17	131,072	59	$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$	iff $A$ and $B$ are independent.	
18	262,144	61	Ackermann's function and inverse:	$\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$	
19	524,288	67	$a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$	For random variables $X$ and $Y$ :	
20	1,048,576	71	$\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	$E[X \cdot Y] = E[X] \cdot E[Y],$	
21	2,097,152	73	Binomial distribution:	if $X$ and $Y$ are independent.	
22	4,194,304	79	$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$	$E[X + Y] = E[X] + E[Y],$	
23	8,388,608	83	$E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$	$E[cX] = cE[X].$	
24	16,777,216	89	Poisson distribution:	Bayes' theorem:	
25	33,554,432	97	$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	$\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$	
26	67,108,864	101	Normal (Gaussian) distribution:	Inclusion-exclusion:	
27	134,217,728	103	$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$	
28	268,435,456	107	The "coupon collector": We are given a random coupon each day, and there are $n$ different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all $n$ types is	$\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$	
29	536,870,912	109	$nH_n.$	Moment inequalities:	
30	1,073,741,824	113		$\Pr[ X  \geq \lambda E[X]] \leq \frac{1}{\lambda},$	
31	2,147,483,648	127		$\Pr[ X - E[X]  \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$	
32	4,294,967,296	131		Geometric distribution:	
Pascal's Triangle				$\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$	
1				$E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$	
1 1					
1 2 1					
1 3 3 1					
1 4 6 4 1					
1 5 10 10 5 1					
1 6 15 20 15 6 1					
1 7 21 35 35 21 7 1					
1 8 28 56 70 56 28 8 1					
1 9 36 84 126 126 84 36 9 1					
1 10 45 120 210 252 210 120 45 10 1					

# Theoretical Computer Science Cheat Sheet

Trigonometry	Matrices	More Trig.																								
<div></div> <p>Pythagorean theorem: <math>C^2 = A^2 + B^2</math>.</p> <p>Definitions:</p> $\sin a = A/C, \quad \cos a = B/C,$ $\csc a = C/A, \quad \sec a = C/B,$ $\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$ <p>Area, radius of inscribed circle:</p> $\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$ <p>Identities:</p> $\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$ $\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$ $1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$ $\sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \sin x = \sin(\pi - x),$ $\cos x = -\cos(\pi - x), \quad \tan x = \cot\left(\frac{\pi}{2} - x\right),$ $\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{x}{2} - \cot x,$ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$ $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$ $\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$ $\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$ $\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$ $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$ $\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y,$ $\cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$ <p>Euler's equation:</p> $e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$	<p>Multiplication:</p> $C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$ <p>Determinants: <math>\det A \neq 0</math> iff <math>A</math> is non-singular.</p> $\det A \cdot B = \det A \cdot \det B,$ $\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$ <p><math>2 \times 2</math> and <math>3 \times 3</math> determinant:</p> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ $= aei + bfg + cdh - ceg - fha - ibd.$ <p>Permanents:</p> $\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$	<div></div> <p>Law of cosines: <math>c^2 = a^2 + b^2 - 2ab \cos C</math>.</p> <p>Area:</p> $A = \frac{1}{2}hc,$ $= \frac{1}{2}ab \sin C,$ $= \frac{c^2 \sin A \sin B}{2 \sin C}.$ <p>Heron's formula:</p> $A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$ $s = \frac{1}{2}(a + b + c),$ $s_a = s - a,$ $s_b = s - b,$ $s_c = s - c.$ <p>More identities:</p> $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$ $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$ $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$ $= \frac{1 - \cos x}{\sin x},$ $= \frac{\sin x}{1 + \cos x},$ $\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$ $= \frac{1 + \cos x}{\sin x},$ $= \frac{\sin x}{1 - \cos x},$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i},$ $\cos x = \frac{e^{ix} + e^{-ix}}{2},$ $\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$ $= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$ $\sin x = \frac{\sinh ix}{i},$ $\cos x = \cosh ix,$ $\tan x = \frac{\tanh ix}{i}.$																								
	<p>Hyperbolic Functions</p> <p>Definitions:</p> $\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$ $\text{sech } x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$ <p>Identities:</p> $\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$ $\coth^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$ $\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$ $\sinh 2x = 2 \sinh x \cosh x,$ $\cosh 2x = \cosh^2 x + \sinh^2 x,$ $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$ $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$ $2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$																									
	<table><tr><th><math>\theta</math></th><th><math>\sin \theta</math></th><th><math>\cos \theta</math></th><th><math>\tan \theta</math></th></tr><tr><td>0</td><td>0</td><td>1</td><td>0</td></tr><tr><td><math>\frac{\pi}{6}</math></td><td><math>\frac{1}{2}</math></td><td><math>\frac{\sqrt{3}}{2}</math></td><td><math>\frac{\sqrt{3}}{3}</math></td></tr><tr><td><math>\frac{\pi}{4}</math></td><td><math>\frac{\sqrt{2}}{2}</math></td><td><math>\frac{\sqrt{2}}{2}</math></td><td>1</td></tr><tr><td><math>\frac{\pi}{3}</math></td><td><math>\frac{\sqrt{3}}{2}</math></td><td><math>\frac{1}{2}</math></td><td><math>\sqrt{3}</math></td></tr><tr><td><math>\frac{\pi}{2}</math></td><td>1</td><td>0</td><td><math>\infty</math></td></tr></table>	$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	0	0	1	0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\pi}{2}$	1	0	$\infty$	<p>... in mathematics you don't understand things, you just get used to them.</p> <p>– J. von Neumann</p>
$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$																							
0	0	1	0																							
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$																							
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$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$																							
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<p>v2.02 ©1994 by Steve Seiden</p> <p>sseiden@acm.org</p> <p><a href="http://www.csc.lsu.edu/~seiden">http://www.csc.lsu.edu/~seiden</a></p>																										



# Theoretical Computer Science Cheat Sheet

## Number Theory

The Chinese remainder theorem: There exists a number  $C$  such that:

$$C \equiv r_1 \pmod{m_1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$C \equiv r_n \pmod{m_n}$$

if  $m_i$  and  $m_j$  are relatively prime for  $i \neq j$ .

Euler's function:  $\phi(x)$  is the number of positive integers less than  $x$  relatively prime to  $x$ . If  $\prod_{i=1}^n p_i^{e_i}$  is the prime factorization of  $x$  then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$$

Euler's theorem: If  $a$  and  $b$  are relatively prime then

$$1 \equiv a^{\phi(b)} \pmod{b}.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \pmod{p}.$$

The Euclidean algorithm: if  $a > b$  are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If  $\prod_{i=1}^n p_i^{e_i}$  is the prime factorization of  $x$  then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers:  $x$  is an even perfect number iff  $x = 2^{n-1}(2^n - 1)$  and  $2^n - 1$  is prime.

Wilson's theorem:  $n$  is a prime iff

$$(n-1)! \equiv -1 \pmod{n}.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$

$$+ O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3}$$

$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

## Graph Theory

### Definitions:

*Loop* An edge connecting a vertex to itself.

*Directed* Each edge has a direction.

*Simple* Graph with no loops or multi-edges.

*Walk* A sequence  $v_0 e_1 v_1 \dots e_\ell v_\ell$ .

*Trail* A walk with distinct edges.

*Path* A trail with distinct vertices.

*Connected* A graph where there exists a path between any two vertices.

*Component* A maximal connected subgraph.

*Tree* A connected acyclic graph.

*Free tree* A tree with no root.

*DAG* Directed acyclic graph.

*Eulerian* Graph with a trail visiting each edge exactly once.

*Hamiltonian* Graph with a cycle visiting each vertex exactly once.

*Cut* A set of edges whose removal increases the number of components.

*Cut-set* A minimal cut.

*Cut edge* A size 1 cut.

*k-Connected* A graph connected with the removal of any  $k-1$  vertices.

*k-Tough*  $\forall S \subseteq V, S \neq \emptyset$  we have  $k \cdot c(G-S) \leq |S|$ .

*k-Regular* A graph where all vertices have degree  $k$ .

*k-Factor* A  $k$ -regular spanning subgraph.

*Matching* A set of edges, no two of which are adjacent.

*Clique* A set of vertices, all of which are adjacent.

*Ind. set* A set of vertices, none of which are adjacent.

*Vertex cover* A set of vertices which cover all edges.

*Planar graph* A graph which can be embedded in the plane.

*Plane graph* An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If  $G$  is planar then  $n - m + f = 2$ , so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree  $\leq 5$ .

### Notation:

$E(G)$  Edge set

$V(G)$  Vertex set

$c(G)$  Number of components

$G[S]$  Induced subgraph

$\deg(v)$  Degree of  $v$

$\Delta(G)$  Maximum degree

$\delta(G)$  Minimum degree

$\chi(G)$  Chromatic number

$\chi_E(G)$  Edge chromatic number

$G^c$  Complement graph

$K_n$  Complete graph

$K_{n_1, n_2}$  Complete bipartite graph

$r(k, \ell)$  Ramsey number

### Geometry

Projective coordinates: triples  $(x, y, z)$ , not all  $x, y$  and  $z$  zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian Projective

$$(x, y) \quad (x, y, 1)$$

$$y = mx + b \quad (m, -1, b)$$

$$x = c \quad (1, 0, -c)$$

Distance formula,  $L_p$  and  $L_\infty$  metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

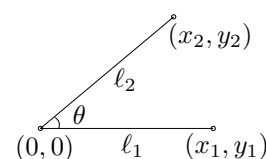
$$[|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p},$$

$$\lim_{p \rightarrow \infty} [|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p}.$$

Area of triangle  $(x_0, y_0), (x_1, y_1)$  and  $(x_2, y_2)$ :

$$\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points  $(x_0, y_0)$  and  $(x_1, y_1)$ :

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Issac Newton

# Theoretical Computer Science Cheat Sheet

$\pi$

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left( 1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

## Partial Fractions

Let  $N(x)$  and  $D(x)$  be polynomial functions of  $x$ . We can break down  $N(x)/D(x)$  using partial fraction expansion. First, if the degree of  $N$  is greater than or equal to the degree of  $D$ , divide  $N$  by  $D$ , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of  $N'$  is less than that of  $D$ . Second, factor  $D(x)$ . Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[ \frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[ \frac{d^k}{dx^k} \left( \frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.  
– George Bernard Shaw

## Calculus

Derivatives:

1.  $\frac{d(cu)}{dx} = c \frac{du}{dx},$
2.  $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx},$
3.  $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$
4.  $\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx},$
5.  $\frac{d(u/v)}{dx} = \frac{v \left( \frac{du}{dx} \right) - u \left( \frac{dv}{dx} \right)}{v^2},$
6.  $\frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$
7.  $\frac{d(c^u)}{dx} = (\ln c) c^u \frac{du}{dx},$
8.  $\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$
9.  $\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx},$
10.  $\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$
11.  $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx},$
12.  $\frac{d(\cot u)}{dx} = \csc^2 u \frac{du}{dx},$
13.  $\frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx},$
14.  $\frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$
15.  $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx},$
16.  $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$
17.  $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx},$
18.  $\frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$
19.  $\frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx},$
20.  $\frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$
21.  $\frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx},$
22.  $\frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$
23.  $\frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx},$
24.  $\frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$
25.  $\frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx},$
26.  $\frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},$
27.  $\frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx},$
28.  $\frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$
29.  $\frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx},$
30.  $\frac{d(\operatorname{arccoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$
31.  $\frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$
32.  $\frac{d(\operatorname{arcsch} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}.$

Integrals:

1.  $\int cu \, dx = c \int u \, dx,$
2.  $\int (u+v) \, dx = \int u \, dx + \int v \, dx,$
3.  $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1,$
4.  $\int \frac{1}{x} \, dx = \ln x,$
5.  $\int e^x \, dx = e^x,$
6.  $\int \frac{dx}{1+x^2} = \arctan x,$
7.  $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$
8.  $\int \sin x \, dx = -\cos x,$
9.  $\int \cos x \, dx = \sin x,$
10.  $\int \tan x \, dx = -\ln |\cos x|,$
11.  $\int \cot x \, dx = \ln |\cos x|,$
12.  $\int \sec x \, dx = \ln |\sec x + \tan x|,$
13.  $\int \csc x \, dx = \ln |\csc x + \cot x|,$
14.  $\int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$

# Theoretical Computer Science Cheat Sheet

## Calculus Cont.

15.  $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16.  $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17.  $\int \sin^2(ax) dx = \frac{1}{2a} (ax - \sin(ax) \cos(ax)),$
18.  $\int \cos^2(ax) dx = \frac{1}{2a} (ax + \sin(ax) \cos(ax)),$
19.  $\int \sec^2 x dx = \tan x,$
20.  $\int \csc^2 x dx = -\cot x,$
21.  $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22.  $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23.  $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24.  $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25.  $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26.  $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27.  $\int \sinh x dx = \cosh x,$
28.  $\int \cosh x dx = \sinh x,$
29.  $\int \tanh x dx = \ln |\cosh x|,$
30.  $\int \coth x dx = \ln |\sinh x|,$
31.  $\int \operatorname{sech} x dx = \arctan \sinh x,$
32.  $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33.  $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34.  $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35.  $\int \operatorname{sech}^2 x dx = \tanh x,$
36.  $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37.  $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38.  $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39.  $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left( x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40.  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41.  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42.  $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44.  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45.  $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46.  $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47.  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48.  $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49.  $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50.  $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51.  $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52.  $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53.  $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54.  $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56.  $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58.  $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59.  $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60.  $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61.  $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$