Probability Models

Advanced Econometrics 2

Petra Tomanová

Winter Semester 2020/2021

KEKO, University of Economics Prague

Motivation

”Simple linear models everyone can estimate! (click buttons)
Econometricians (are expected to) do more!”

”Econometricians are suppose to be specialists in cutting-edge,
state-of-the-art models (not available in software packages).”

— Francisco Blasques —

Motivation

First, we need to learn some basic concepts of set theory and measure theory in order to find an appropriate definition of probability model and data generating process.

Unfortunately (or fortunately? ☺), we will not be able to dig in depth. Actually, we will barely scratch its surface. Our objective is to become familiar with certain basic concepts and terminology used in research articles. In particular, we need to speak a common language, understand why researchers refer to models as sets, or why two different models may both be correct.

Probability Models

Probability space

Definition 1: (Probability space)

A probability space is a triplet $(\mathcal{E}, \mathcal{F}, P)$ where $\mathcal{E}$ is the 'event space', $\mathcal{F}$ is a σ-algebra defined on the event space $\mathcal{E}$ and P is a probability measure defined on the σ-algebra $\mathcal{F}$.

Event space $\mathcal{E}$ is the collection of all possible outcomes for the random variable.

Probability measure P defines probability associated to each event and each collection of events in $\mathcal{E}$.

σ-field $\mathcal{F}$ contains all the relevant collections of events.

Note that $P : \mathcal{F} \to [0, 1]$ maps elements of $\mathcal{F}$ to the interval $[0, 1]$.

Probability space

Definition 1: (Probability space)

A probability space is a triplet $(\mathcal{E}, \mathcal{F}, P)$ where $\mathcal{E}$ is the 'event space', $\mathcal{F}$ is a σ-algebra defined on the event space $\mathcal{E}$ and P is a probability measure defined on the σ-algebra $\mathcal{F}$.

Example of event spaces $\mathcal{E}$:

- Coin tosses: $\mathcal{E} = \{heads, tails\}$
- Dice tosses: $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$
- Gaussian random variable: $\mathcal{E} = \mathbb{R}$

Probability space

Definition 1: (Probability space)

A probability space is a triplet $(\mathcal{E}, \mathcal{F}, P)$ where $\mathcal{E}$ is the 'event space', $\mathcal{F}$ is a σ-algebra defined on the event space $\mathcal{E}$ and P is a probability measure defined on the σ-algebra $\mathcal{F}$.

Example of event spaces $\mathcal{E}$:

- Coin tosses: $\mathcal{E} = \{heads, tails\}$
- Dice tosses: $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$
- Gaussian random variable: $\mathcal{E} = \mathbb{R}$

WHY IS P DEFINED ON COLLECTIONS OF SETS IN $\mathcal{F}$?

Probability space

Definition 1: (Probability space)

A probability space is a triplet $(\mathcal{E}, \mathcal{F}, P)$ where $\mathcal{E}$ is the 'event space', $\mathcal{F}$ is a σ-algebra defined on the event space $\mathcal{E}$ and P is a probability measure defined on the σ-algebra $\mathcal{F}$.

Example of event spaces $\mathcal{E}$:

- Coin tosses: $\mathcal{E} = \{heads, tails\}$
- Dice tosses: $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$
- Gaussian random variable: $\mathcal{E} = \mathbb{R}$

WHY IS P DEFINED ON COLLECTIONS OF SETS IN $\mathcal{F}$?

Describes what are joint and disjoint sets.

Probability space

Coin toss example:

A σ-field $\mathcal{F}$ of the event space $\mathcal{E} = \{heads, tails\}$ is

$$\mathcal{F} \coloneqq \Big\{\{\emptyset\}, \{heads\}, \{tails\}, \{heads, tails\}\Big\}.$$

Note that $\mathcal{F}$ contains the empty set $\{\emptyset\}$, each element of $\mathcal{E}$, and the event space $\mathcal{E} = \{heads, tails\}$.

Probability measure P must define a probability of

- Nothing happening $P(\emptyset)$
- Drawing heads $P(heads)$
- Drawing tails $P(tails)$
- Drawing either heads or tails $P(\{heads, tails\})$

σ-algebra (σ-field)

There are certain rules that must be followed for constructing a σ-algebra $\mathcal{F}$.

Banach-Tarsky Paradox (1924):
It is possible to take a ball, cut it into pieces, and re-arrange those pieces in such a manner as to obtain two balls of the exact same size, that have no parts missing! The σ-algebra solves this problem!

Definition 2: (σ-field)

A σ-algebra $\mathcal{F}$ of a set $\mathcal{E}$ is a collection of subsets of $\mathcal{E}$ satisfying:

(i) $\mathcal{E} \in \mathcal{F}$.

(ii) If $F \in \mathcal{F}$, then $F^c \in \mathcal{F}$, where F^c denotes the complement of F in $\mathcal{F}$.

(iii) If $\{F_n\}_{n\in\mathbb{N}}$ is a collection of disjoints sets in $\mathcal{F}$, then $\cup_{n=1}^{\infty} F_n \in \mathcal{F}$.

Measurable spaces and probability measures

Definition 3: (Measurable space)

A measurable space is just a pair $(\mathcal{E}, \mathcal{F})$ composed of an event space $\mathcal{E}$ and respective σ-algebra $\mathcal{F}$.

Definition 4: (Probability measure)

A probability measure P defined on a measurable space $(\mathcal{E}, \mathcal{F})$ is a function $P : \mathcal{F} \to [0; 1]$ satisfying:

(i) $P(F) \geq 0 \; \forall F \in \mathcal{F}$.

(ii) $P(\mathcal{E}) = 1$.

(iii) If $\{F_n\}_{n \in \mathbb{N}}$ is a collection of disjoints sets in $\mathcal{F}$, then $P\left(\cup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$.

Random variable

Definition 5: (Measurable function)

Given two measurable spaces $(\mathbb{A}, \mathcal{F}_{\mathbb{A}})$ and $(\mathbb{B}, \mathcal{F}_{\mathbb{B}})$, a function $f : \mathbb{A} \to \mathbb{B}$ is said to be measurable if f is such that every element $b \in \mathcal{F}_{\mathbb{B}}$ satisfies $f^{-1}(b) \in \mathcal{F}_{\mathbb{A}}$; i.e. the inverse image of each element of $\mathcal{F}_{\mathbb{B}}$ is in $\mathcal{F}_{\mathbb{A}}$.

Note that inverse map f^{-1} exists always, it may just not be a function! Do you remember the properties of a function?

Definition 6: (Random variable)

Given a probability space $(\mathcal{E}, \mathcal{F}, P)$ and the measurable space $(\mathbb{R}, \mathcal{F}_{\mathbb{R}})$, a random-variable x_t is a measurable map $x_t : \mathcal{E} \to \mathbb{R}$ that maps elements of $\mathcal{E}$ to the real numbers $\mathbb{R}$; i.e. the inverse map $x_t^{-1} : \mathcal{F}_{\mathbb{R}} \to \mathcal{F}$ satisfies $x_t^{-1}(r) \in \mathcal{F}$ for every element $r \in \mathcal{F}_{\mathbb{R}}$.

Random variable

The definition of random variable is very intuitive.

Measurability of $x_t : \mathcal{E} \to \mathbb{R}$ implies that we can assign probabilities for each interval $R \subseteq \mathbb{R}$ of the real line

- $P_{\mathbb{R}}(R) = P(x_t^{-1}(R)) = P(e \in \mathcal{E} : x_t(e) \in R)$,
- we obtain a new probability space $(\mathbb{R}, \mathcal{F}_{\mathbb{R}}, P_{\mathbb{R}})$.

Now, we can define the cumulative distribution function F that you know so well as

$$F(a) = P_{\mathbb{R}}(x \leq a) \ \forall \ a \in \mathbb{R}.$$

x_t is a random variable. $x_t(e) \in \mathbb{R}$ is the realization of the random variable produced by event $e \in \mathcal{E}$.

Random vectors and random elements

The concept of random variable is easy to generalize.

Definition 7: (Random vector)

Given a probability space $(\mathcal{E}, \mathcal{F}, P)$ and a measurable space $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n})$ with $n \in \mathbb{N}$, an n-variate random-vector x_t is a measurable map $x_t : \mathcal{E} \to \mathbb{R}^n$ that maps elements of $\mathcal{E}$ to $\mathbb{R}^n$.

Definition 8: (Random element)

Given a probability space $(\mathcal{E}, \mathcal{F}, P)$ and a measurable space $(\mathbb{A}, \mathcal{F}_{\mathbb{A}})$, a random-element at taking values in $\mathbb{A}$ is a measurable map $a_t : \mathcal{E} \to \mathbb{A}$ that maps elements of $\mathcal{E}$ to $\mathbb{A}$.

Random variable and Borel σ-algebra

THE DEFINITION OF RANDOM VARIABLE DEPENDS ON THE σ-ALGEBRA THAT ONE IS USING.

Consider the case where x_t is a normal random variable $x_t \sim N(0, \sigma^2)$. Is x_t^2 also a random variable? How about $exp(x_t)$?

Random variable and Borel σ-algebra

The definition of random variable depends on the σ-algebra that one is using.

Consider the case where x_t is a normal random variable $x_t \sim N(0, \sigma^2)$. Is x_t^2 also a random variable? How about $exp(x_t)$?

Yes, if we use the Borel σ-algebra.

Definition 9: (Borel σ-algebra)

Given set $\mathbb{A}$ the Borel σ-algebra $\mathcal{B}_{\mathbb{A}}$ is the smallest σ-algebra containing all open sets of $\mathbb{A}$.

Random variable and continuous functions

All continuous functions are measurable under the Borel σ-algebra. Any continuous transformation $f(x_t)$ of a random variable x_t is also a random variable.

It is obvious that all continuous functions are measurable. Just look at the definition of continuous function.

Definition 10: (Continuous function)

Let $(\mathbb{A}, \mathcal{T}_{\mathbb{A}})$ and $(\mathbb{B}, \mathcal{T}_{\mathbb{B}})$ be topological spaces. A function $f : \mathbb{A} \to \mathbb{B}$ is said to be continuous if its inverse f^{-1} maps open sets to open sets; i.e. if for every $b \in \mathcal{T}_{\mathbb{B}}$ we have $f^{-1}(b) \in \mathcal{T}_{\mathbb{A}}$.

What is a probability model?

What exactly is a model?

Coin tosses:

Given T tosses of a coin, it is reasonable to suppose that $x_1, \ldots, x_T$ are realizations of T Bernoulli random variables, $x_t \sim \text{Bern}(\theta)$ with unknown probability parameter $\theta \in [0, 1]$.

Each θ defines a probability distribution for the random vector $(x_1, \ldots, x_T)$ taking values in $\mathbb{R}_T$. Our model is a collection of probability distributions on $\mathbb{R}_T$.

This definition of model is the one you have been always using. Even if you did not realize it!

What exactly is a model?

Gaussian linear AR(1) model:

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2_\varepsilon), \qquad \forall\, t \in \mathbb{Z}.$$

Each $\theta = (\alpha, \beta, \sigma^2_\varepsilon)$ defines a distribution for the time-series $\{y_t\}_{t \in \mathbb{Z}}$. Our model is a collection of probability distributions on $\mathbb{R}^\infty$.

This definition of model is the one you have been always using. Even if you did not realize it!

Probability model

Definition 11: (Probability model)

Given the measurable space $(\mathcal{E}, \mathcal{F})$ and a parameter space Θ, a probability model is a collection $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ of probability measures defined on $\mathcal{F}$.

Definition 12: (Time-series probability model)

Given the measurable space $(\mathbb{R}^\infty, \mathcal{F}_{\mathbb{R}^\infty})$ and a parameter space Θ, a probability model is a collection $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ of probability measures defined on $\mathcal{F}_{\mathbb{R}^\infty}$.

Probability model

Definition 12: (Parametric and nonparametric models)

A probability model $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ is said to be:

- parametric if the parameter space Θ is finite dimensional;
- nonparametric if Θ is infinite dimensional;
- semi-parametric if $\Theta = \Theta_1 \times \Theta_2$ where Θ_1 is finite dimensional and Θ_2 is infinite dimensional;
- semi-nonparametric if Θ_T is indexed by the sample size T with 'sieves' $\{\Theta_T\}_{T \in \mathbb{N}}$ with increasing dimension.

Definition 13: (Nested model)

Given a measurable space $\mathcal{E}, \mathcal{F}$ and two parametric models $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ and $\mathbb{P}^*{}_{\Theta^*} := \{P^*_{\theta^*}, \theta^* \in \Theta^*\}$, we say that model $\mathbb{P}_\Theta$ nests model $\mathbb{P}^*{}_{\Theta^*}$ if and only if $\mathbb{P}^*{}_{\Theta^*} \subseteq \mathbb{P}_\Theta$.

Probability model

Why should we work with probability models?

A very brief history of econometrics:

1. In 1936, Jan Tinbergen published macroeconometric model of The Netherlands with hundreds of regressions.
2. In 1939, John Maynard Keynes published critical review of Tinbergen's work: the model could not be proven wrong! Error term accounts for anything!
3. In 1944, Haavelmo published The Probability Approach in Econometrics which solved the problem: unlikely errors consitute evidence against the model!
4. John Maynard Keynes shaped the discipline of econometrics.

What is a DGP?

Definition 14: (Data generating process)

Given a measurable space $(\mathcal{E}, \mathcal{F})$, a data generating process is a probability measure P_0 defined on $\mathcal{F}_{\mathbb{R}^\infty}$.

Definition 15: (Time-series data generating process)

Given a measurable space $(\mathbb{R}^\infty, \mathcal{F}_{\mathbb{R}^\infty})$, a data generating process is a probability measure P_0 defined on $\mathcal{F}$.

Intuition: The DGP is the 'unknown mechanism' that 'generates the data'.

Note that in economics, the DGP is most likely very complex involving millions of agents, factors, variables, decisions, etc.

Correct specification

Definition 16: (Correctly specified model)

A model $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ is said to be correctly specified if the data generating process P_0 is an element of the model $\mathbb{P}_\Theta$; i.e. if there exists $\theta_0 \in \Theta$ such that $\mathbb{P}_{\Theta_0} = P_0$. When this parameter θ_0 exists, then it is called the 'true parameter'.

SHORTER NOTATION: a model is correctly specified if $\exists\, \theta_0 \in \Theta : P_{\theta_0} = P_0$ which means that $P_0 \in \mathbb{P}_\Theta$.

Definition 17: (Mis-specified model)

A model $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ is said to be mis-specified (or incorrectly specified) if the data generating process P_0 is not an element of the model $\mathbb{P}_\Theta$; i.e. if $P_\theta \neq P_0 \;\forall\, \theta \in \Theta$.

Correct specification

Example: (Linear Gaussian AR(1) model)

For every $\theta := (\alpha, \beta, \sigma_\varepsilon^2) \in \Theta \subseteq \mathbb{R}^4$ let

$$x_t = \alpha + \beta x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2), \qquad \forall\, t \in \mathbb{Z}.$$

Note: This is a model because θ can take many values in Θ.

Note: The collection of probability measures on σ-algebra of $\mathbb{R}^\infty$ is specified implicitly.

Specification: Given a time series $\{y_t\}_{t\in\mathbb{Z}}$ with probability measure P_0, the AR(1) model is well specified if $\exists\, \theta_0 \in \Theta$ such that the linear AR(1) generates a random sequence with measure $P_{\theta_0} = P_0$.

Generality of linear dynamic models

Definition 18: (Weak stationarity)

A time series $\{x_t\}$ is weakly stationary if $\mu_t = \mathbb{E}(x_t)$ and $\gamma_t(h) = \mathbb{C}ov(x_t, x_{t-h})$ satisfy $\mu_t = \mu$ and $\gamma_t(h) = \gamma(h)\ \forall\ (t, h)$.

Definition 19: (White Noise)

A random sequence $\{x_t\}_{t\in\mathbb{Z}}$ is said to be a white noise process, if $\mathbb{C}ov(x_t, x_{t-h}) = 0$, $\mathbb{E}(x_t) = 0$ and $\mathbb{V}ar(x_t) = \sigma^2\ \forall\ (t, h)$.

Definition 20: (Linear process)

A time-series $\{x_t\}$ is said to be a linear if it can be represented as,

$$x_t = \sum_{j=-\infty}^{\infty} \psi_j z_{t-j} \qquad \text{where } \{z_t\} \sim WN(0, \sigma^2),$$

and $\{\psi_j\}_{-\infty}^{\infty}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

Linear model

Definition 21: (Linear time-series model)

A time-series model $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ is said to be linear if every measure P_θ defines a stochastic process that is linear.

Definition 22: (Wold's representation)

Let $\{x_t\}$ be a weakly stationary process. Then it admits the following representation,

$$x_t = \sum_{j=0}^{\infty} \psi_j z_{t-j} + v_t, \qquad \text{where}$$

(i) $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$,

(ii) $\{z_t\}_{t \in \mathbb{Z}} \sim WN(0, \sigma^2)$,

(iii) $\{v_t\}_{t \in \mathbb{Z}}$ is deterministic (non-random).

Linear model

Wold's theorem can be used as a justification for the adoption of linear dynamic models.

Example: (ARMA(p, q) model)

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_p \varepsilon_{t-q},$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim N(0, \sigma_\varepsilon^2)$.

ARMA(p, q) is linear if the autoregressive polynomial $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ is invertible, because then it can be re-written in the infinite MA representation

$$x_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \text{with } \psi_0 = 1 \text{ and } \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

Problems with this justification...

Wold's representation + MA(∞) Representation of ARMA

Seems that ARMA models can describe any WS process!

This is not true! We must look at the details:

1. Stochastic component of Wold's theorem $\sum_{j=0}^{\infty} \psi_j z_{t-j}$ is not necessarily a linear process.
2. Stochastic component of Wold's theorem $\sum_{j=0}^{\infty} \psi_j z_{t-j}$ involves infinitely many parameters $\{\psi_j\}$ that cannot be estimated from a finite sample of data.
3. Wold's theorem features a deterministic component $\{v_t\}_{t\in\mathbb{Z}}$ that is unknown and potentially very complex.
4. Distribution of the white noise sequence $\{z_t\}_{t\in\mathbb{Z}}$ is unknown and possibly very complex!

Nonlinear dynamic model

Definition 23: (Nonlinear time-series model)

A time-series model $\mathbb{P}_\Theta := \{P_\theta, \theta \in \Theta\}$ is said to be nonlinear if at least some measure $P_\theta \in \mathbb{P}_\Theta$ defines a stochastic process that is not linear.

Example: (Quadratic AR model)

$$x_t = \alpha + \beta x_{t-1} + \gamma x_{t-1}^2 + \varepsilon_t, \qquad \varepsilon_t \sim NID(0, \sigma_\varepsilon^2)$$

is nonlinear model if Θ allows for $\gamma \neq 0$.

Example: (SESTAR model)

$$x_t = \frac{\gamma}{1 + \exp(\alpha + \beta x_{t-1})} x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim NID(0, \sigma_\varepsilon^2)$$

is nonlinear model if $\gamma \neq 0$ and $\beta \neq 0$.

NOTE that both models above nest the linear AR(1) model.

Nonlinear dynamic model

"In the end of the day, **THE QUESTION OF WHICH MODEL TO USE SHOULD ALWAYS BE LEFT TO THE DATA TO ANSWER.** *As we shall see, in many cases, ARMA models actually do well in describing economic data! Model specification tests may show that the ARMA model provides a good description of the data, and it may happen that nonlinear models do not provide better results. In other cases, nonlinear models are clearly needed (e.g. for modeling time-varying volatility in financial applications). In a considerable number of cases there is at least room for improvement by adopting nonlinear models."*

— Francisco Blasques —

Examples of Nonlinear Dynamic Models

Nonlinear autoregressions: GNLAR

Let $\theta \in \Theta$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be (iid) innovations with $\varepsilon_t \sim p_\varepsilon(\theta)$.

Example: (GNLAR(1) model)

The general nonlinear autoregressive (GNLAR) model is given by,

$$x_t = f(x_{t-1}, \varepsilon_t; \theta) \quad \forall\, t \in \mathbb{Z}.$$

Example: (GNLAR(p) model)

The GNLAR(p) takes the form

$$x_t = f(x_{t-1}, \dots, x_{t-p}, \varepsilon_t; \theta) \quad \forall\, t \in \mathbb{Z}.$$

Example: (GNLARMA(p, q) model)

General nonlinear autoregressive moving-average GNLARMA(p, q) model takes the form

$$x_t = f(x_{t-1}, \dots, x_{t-p}, \varepsilon_t, \dots, \varepsilon_{t-p}; \theta) \quad \forall\, t \in \mathbb{Z}.$$

Nonlinear autoregressions: NLAR model

Example: (NLAR(1) model)

The nonlinear autoregressive (NLAR) model with additive innovations

$$x_t = f(x_{t-1}; \theta) + \varepsilon_t \quad \forall\, t \in \mathbb{Z}.$$

Note that NLAR(1) models are often written in the equivalent form

$$x_t = g(x_{t-1}; \theta) x_{t-1} + \varepsilon_t \quad \forall\, t \in \mathbb{Z}$$

by defining $g(x_{t-1}; \theta) := f(x_{t-1}; \theta)/x_{t-1}$, or

$$x_t = \beta_t x_{t-1} + \varepsilon_t \quad \forall\, t \in \mathbb{Z}$$

by defining $\beta_t := g(x_{t-1}; \theta)$.

Nonlinear autoregressions: STAR model

Example: (STAR model)

A famous NLAR(1) model is the smooth transition autoregressive (STAR) model,

$$x_t = g(z_{t-1};\theta)x_{t-1} + \varepsilon_t \quad \forall\, t \in \mathbb{Z},$$

where $\{\varepsilon_t\}$ are innovations with some specified distribution and

$$g(z_{t-1};\theta) := \delta + \frac{\gamma}{1 + exp(\alpha + \beta z_{t-1})} \quad \forall\, t \in \mathbb{Z}.$$

Alternative STAR models:

(i) exogenous STAR model (exogenous driver $\{z_t\}$),

(ii) logistic self-excited STAR (endogenous driver $z_t = x_t$),

(iii) exponential self-excited STAR (endogenous driver $z_t = x_t^2$).

Logistic SESTAR model

Example: (Logistic SESTAR model)

$$g(z_{t-1};\theta) := \delta + \frac{\gamma}{1 + exp(\alpha + \beta x_{t-1})} \quad \forall\, t \in \mathbb{Z}.$$

Note that the logistic SESTAR model allows us to model changes in the dependence of the time-series $\{x_{t-1}\}_{t\in\mathbb{Z}}$.

Practice: crucial for modeling higher dependence of macro variables in recessions ($x_{t-1} < 0$) than expansions ($x_{t-1} > 0$).

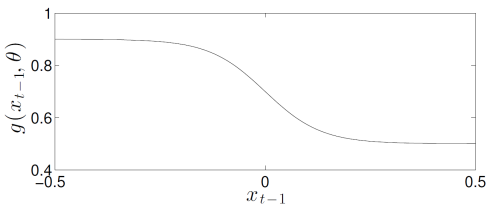

Figure 1: Plot of $g(x_{t-1};\theta)$ for $(\delta,\gamma,\alpha,\beta) = (0.5, 0.4, 0, 15)$.

Example: (Exponential SESTAR model)

$$g(z_{t-1};\theta) := \delta + \frac{\gamma}{1 + exp(\alpha + \beta(x_{t-1} - \mu)^2)} \quad \forall\, t \in \mathbb{Z}.$$

Practice: crucial for modeling temporal dependence of real foreign exchange (FX) rate: high-dependence (no mean reversion) at rates close to 1 and low-dependence (mean reverting behavior) for rates far from 1.

Note that behavior of real FX is justified by the law of one price.

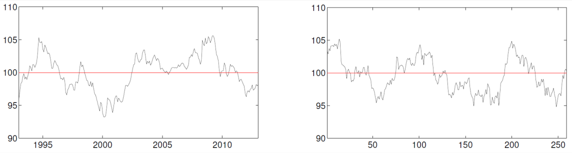

Figure 2: Real exchange rate of EU 15 vs Danish Kroner (left) and data simulated from an exponential SESTAR model (right).

Distributed SESTAR model

Example: (Distributed SESTAR model)

$$x_t = a_t x_{t-1} + b_t w_t + \varepsilon_t \quad \forall\, t \in \mathbb{Z},$$

$$a_t := \delta + \frac{\gamma}{1 + exp(\alpha + \beta x_{t-1})}, \qquad b_t := \delta^* + \frac{\gamma}{1 + exp(\alpha^* + \beta^* x_{t-1})}.$$

NOTE that this model generates time-varying multiplier effect.

PRACTICE: crucial extension to ADL model for explaining changes in government expenditure multiplier.

ECONOMIC THEORY: predicts that multiplier is larger when economy is substantially below potential.

Random coefficient autoregression

Example: (RCAR(1) model)

The random coefficient autoregressive (RCAR) model is given by

$$x_t = \phi_{t-1} x_{t-1} + \varepsilon_t \quad \forall\, t \in \mathbb{Z},$$

where both $\{\phi_t\}_{t\in\mathbb{Z}}$ and $\{\varepsilon\}_{t\in\mathbb{Z}}$ are exogenous iid sequences with a certain distribution.

Note that first proposed by in Quinn (1980) has several important applications in finance and biology as it allows for a time-varying conditional mean and variance.

$$\phi_{t-1} \sim N(\phi, \sigma_\phi) \qquad \text{and} \qquad \varepsilon_t \sim N(0, \sigma^2) \qquad \forall\, t \in \mathbb{Z}.$$

$$x_t | x_{t-1} \sim N\left(\phi x_{t-1}, \sigma_\phi^2 x_{t-1}^2 + \varepsilon_t^2\right).$$

The End