Block 4
Instrumental variable regression (IVR)
Two stage least squares (2SLS)
Simultaneous Equation Models

Advanced econometrics 1 4EK608
Pokročilá ekonometrie 1 4EK416

Vysoká škola ekonomická v Praze

Outline

Introduction: endogenous regressors

- CS model: $y_i = \boldsymbol{x}_i \boldsymbol{\beta} + u_i \quad$ and $\quad E[\boldsymbol{x}_i, u_i] \neq 0$.
  - If important regressors cannot be measured (thus make part of u_i) and are correlated with observed regressors of LRM.
  - Endogeneity can be caused by measurement errors.
  - Always present in simultaneous equations models (SEMs).
- With endogenous regressors, OLS is biased & inconsistent.

Endogeneity in regressors can sometimes be solved

- By means of proxy variables (if uncorrelated to u_i).
- More detailed (multi-equation) specification, if possible.
- Using panel data methods (data availability permitting).
- Using instrumental variable regression (IVR)
  (we need "good" instruments, assumptions apply).

Introduction: instrumental variables

Example: $\log(wage_i) = \beta_0 + \beta_1 educ_i + [abil_i + u_i]$

Instrumental variables

1. Not in the main (structural) equation: no effect on the dependent variable after controlling for observed regressors.
2. Correlated (positively or negatively) with the endogenous regressor (this can be tested).
3. Not correlated with the error term (in some cases, this can be tested, see Sargan test discussed next).

- Possible IVs: father's education, mother's education, number of siblings, etc.

  Usually, IQ is not a good IV - it's often correlated with $abil$, i.e. with the error term $[abil_i + u_i]$.

Instrumental variables

- $y_i = \beta_0 + \beta_1 x_i + u_i$ SLRM with exogenous regressor x:

$$\begin{array}{ccc} y & \leftarrow & x \\ & \nwarrow & \\ & & u \end{array} \quad \text{and} \quad \frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \beta_1 = \frac{\mathrm{cov}(y,x)}{\mathrm{var}(x)}$$

- $y_i = \boldsymbol{x}_i \boldsymbol{\beta} + u_i$ MLRM with exogenous regressor(s):

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} \qquad \mid \text{subs. for } \boldsymbol{y}$$

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}) \qquad \mid \text{rearr. \& take expects.}$$

$$E[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta} + E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}] = \boldsymbol{\beta}$$

- With exogenous regressors, OLS is unbiased.

Instrumental variables

- $y_i = \beta_0 + \beta_1 x_i + u_i$ SLRM with endogenous regressor x:

$$\begin{array}{ccc} y & \leftarrow & x \\ & \nwarrow & | \\ & & u \end{array} \qquad \text{and} \qquad \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \beta_1 + \frac{\mathrm{d}\,u}{\mathrm{d}\,x}$$

- $y_i = \boldsymbol{x}_i \boldsymbol{\beta} + u_i$ MLRM with endogenous regressor(s):

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} && \mid \text{subs. for } \boldsymbol{y} \\ \hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}) && \mid \text{rearr. \& take expects.} \\ E[\hat{\boldsymbol{\beta}}] &= \boldsymbol{\beta} + E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}] \neq \boldsymbol{\beta} \end{aligned}$$

- With endogenous regressors, $E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}] \neq \boldsymbol{0}$.
  Thus, OLS is biased (and asymptotically biased).

Instrumental variables

- $y_i = \beta_0 + \beta_1 x_i + u_i$ IVR principle (SLRM):

$$\begin{array}{ccccc} y & \leftarrow & x & \leftarrow & z \\ & \nwarrow & | & & \\ & & u & & \end{array} \qquad \text{and} \qquad \beta_1 = \frac{\mathrm{cov}(z, y)}{\mathrm{cov}(z, x)}$$

- $y_i = \boldsymbol{x}_i \boldsymbol{\beta} + u_i$ IVR in MLRMs:

$$\hat{\boldsymbol{\beta}}_{\text{OLS}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$
$$\hat{\boldsymbol{\beta}}_{\text{IV}} = (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{y}$$

where $\boldsymbol{Z}$ is a matrix of instruments, same dimensions as $\boldsymbol{X}$.

- Exact identification: # endogenous regressors = # IVs,
- $\boldsymbol{Z}$ follows from $\boldsymbol{X}$, each endogenous regressor (column) is replaced by unique instrument (full column ranks of $\boldsymbol{X}$,$\boldsymbol{Z}$),
- in IVR, R^2 has no interpretation (SST $\neq$ SSE + SSR),
- for IVR, we use specialized robust standard errors,
- **IVR estimator is biased and consistent.**

Instrumental variables: IVR as MM estimator

Exogenous regressors:

- MM: replace $E[\boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})] = \boldsymbol{0}$ by $\frac{1}{n}[\boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})] = \boldsymbol{0}$ and solve moment equations
- OLS provides identical estimate: $\hat{\boldsymbol{\beta}}_{\mathrm{OLS}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$

With endogenous regressors (exact identification), moment conditions change:

- MM: replace $E[\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})] = \boldsymbol{0}$ by $\frac{1}{n}[\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})] = \boldsymbol{0}$ and solve moment equations
- IVR provides identical estimate: $\hat{\boldsymbol{\beta}}_{\mathrm{IV}} = (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{y}$

Instrumental variables: IVR as MM estimator

$$y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + u_i \quad | \; z_1 \text{ is IV for } y_2$$

$$n^{-1} \sum_{i=1}^{n} (y_{i1} - \hat{\beta}_0 - \hat{\beta}_1 y_{i2} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_k x_{ik}) = 0$$

$$n^{-1} \sum_{i=1}^{n} z_{i1} \cdot (y_{i1} - \hat{\beta}_0 - \hat{\beta}_1 y_{i2} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_k x_{ik}) = 0$$

$$n^{-1} \sum_{i=1}^{n} x_{i2} \cdot (y_{i1} - \hat{\beta}_0 - \hat{\beta}_1 y_{i2} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_k x_{ik}) = 0$$

$$\cdots$$

$$n^{-1} \sum_{i=1}^{n} x_{ik} \cdot (y_{i1} - \hat{\beta}_0 - \hat{\beta}_1 y_{i2} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_k x_{ik}) = 0$$

- In moment equations, y_{i2} is replaced by z_{i1}
- Exogenous regressors serve as their own instruments.

IVR estimator is consistent

$$
\begin{aligned}
\hat{\boldsymbol{\beta}}_{\text{IV}} &= (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{y} && \mid \text{subs. for } \boldsymbol{y} \\
\hat{\boldsymbol{\beta}}_{\text{IV}} &= (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'(\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{u}) && \mid \text{rearrange} \\
\hat{\boldsymbol{\beta}}_{\text{IV}} &= \boldsymbol{\beta} + (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{u}
\end{aligned}
$$

- If consistency condition holds: $\operatorname{plim}\left[\frac{1}{n}\boldsymbol{Z}'\boldsymbol{u}\right] = \mathbf{0}$, $\hat{\boldsymbol{\beta}}_{\text{IV}}$ is consistent.
- This can be seen from expansion of $[(\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{u}]$:

$$
\hat{\boldsymbol{\beta}}_{\text{IV}} = \boldsymbol{\beta} + (n^{-1}\boldsymbol{Z}'\boldsymbol{X})^{-1}\, n^{-1}\boldsymbol{Z}'\boldsymbol{u}
$$

Instrumental variables: over-identification

$$y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + u_i \quad | \; z_1, z_2, z_3 \text{ are IVs for } y_2$$

- By choosing any of the z_1, z_2, z_3 IVs (or any linear combination of), we perform IVR
- $\hat{\boldsymbol{\beta}}_{\text{IV}}$ values change, as IV in moment equations changes.
- We cannot "simply" use all three instruments. If # columns in $\boldsymbol{Z}$ (l) $>$ # columns in $\boldsymbol{X}$ (k), $\boldsymbol{Z}'\boldsymbol{X}$ is $(l \times k)$ with rank k and no inverse: $\hat{\boldsymbol{\beta}}_{\text{IV}} = (\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{Z}'\boldsymbol{y}$ cannot be calculated
- Solution: Project $\boldsymbol{X}$ to the space column of $\boldsymbol{Z}$ (GMM). ($\boldsymbol{X}$ has an endogenous column, $\boldsymbol{Z}$ is purely exogenous).

Instrumental variables: over-identification

Projection matrices (exogenous $\boldsymbol{X}$) – repetition

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{P}\boldsymbol{y}$$
$$\boldsymbol{y} = \hat{\boldsymbol{y}} + \hat{\boldsymbol{u}} = \boldsymbol{P}\boldsymbol{y} + \boldsymbol{M}\boldsymbol{y}, \text{ where}$$
$$\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' = \boldsymbol{I} - \boldsymbol{P}$$

- Projection of columns of $\boldsymbol{X}$ in the column space of $\boldsymbol{Z}$:

$$\hat{\boldsymbol{X}} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{X} = \boldsymbol{P_Z}\boldsymbol{X},$$

- Columns of $\hat{\boldsymbol{X}}$ are linear combinations of columns in $\boldsymbol{Z}$.
- Exogenous columns in $\boldsymbol{X}$ are repeated in $\boldsymbol{Z}$, hence projected on themselves & therefore do not change between $\boldsymbol{X}$ and $\boldsymbol{Z}$.
- General form of the IV estimator (over-identification):

$$\hat{\boldsymbol{\beta}}_{\mathrm{IV}} = (\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y}$$

Instrumental variables: over-identification

- Projection of columns of $\boldsymbol{X}$ in the column space of $\boldsymbol{Z}$:

$$\hat{\boldsymbol{X}} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{X},$$

- It may be shown that IVR is equivalent to OLS regression $\boldsymbol{y} \leftarrow \hat{\boldsymbol{X}}$:

$$\begin{aligned}\hat{\beta}_{\mathrm{IV}} &= (\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y} \\ &= (\boldsymbol{X}'(\boldsymbol{I} - \boldsymbol{M}_Z)\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{I} - \boldsymbol{M}_Z)\boldsymbol{y} \\ &= (\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y}\end{aligned}$$

- $\boldsymbol{y} \leftarrow \hat{\boldsymbol{X}}$ is part of a two-stage LS (2SLS) method, (discussed next).

Instrumental variables: identification conditions

- In $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}$, multiple $\boldsymbol{x}_j$ regressors may be endogenous.
- Identification (estimability) conditions:
  - **Order condition:** We need at least as many IVs (excluded exogenous variables) as there are included endogenous regressors in the main (structural) equation.

    This is a necessary condition for identification.
  - **Rank condition:** $\hat{\boldsymbol{X}} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{X}$ has full column rank (k) so that $(\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}$ or $(\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}})^{-1}$ can be calculated in the IV estimator $\hat{\boldsymbol{\beta}}_{\mathrm{IV}} = (\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y}$ (will be discussed in detail with respect to 2SLS method and for SEM models).

    This is a necessary and sufficient condition for identification.

Instrumental variables: statistical properties

SLRM: $y_{i1} = \beta_0 + \beta_1 x_{i1} + u_i \quad | \; x_{i1}$ endog., z_{i1} exists

- Asymptotic variance of the IV estimator decreases with increasing correlation between z and x.
- IV-related routines & tests are implemented in R, ...
- Both endogenous explanatory variables and IVs can be binary variables.
- R^2 can be negative and has no interpretation nor relevance if IVR is used.

Instrumental variables: statistical properties

SLRM: $y_{i1} = \beta_0 + \beta_1 x_{i1} + u_i \quad | \; x_{i1}$ endog., z_{i1} exists

- In large samples, IV estimator has approximately normal distribution (MM/GMM properties).
- For calculation of standard errors, we usually need assumption of homoscedasticity conditional on IV(s). Alternatively, we calculate robust errors.
- Asymptotic variance of the IV estimator is always higher than of the OLS estimator.

$$\text{var}(\hat{\beta}_{1,IV}) = \frac{\hat{\sigma}^2}{SST_x \cdot R^2_{x,z}} \quad > \quad \text{var}(\hat{\beta}_{1,OLS}) = \frac{\hat{\sigma}^2}{SST_x}$$

Instrumental variables: statistical properties

SLRM: $y_{i1} = \beta_0 + \beta_1 x_{i1} + u_i \quad | \; x_{i1}$ endog., z_{i1} exists

- If (small) correlation between u and instrument z is possible, inconsistency in the IV estimator can be much higher than in the OLS estimator:

$$\text{plim}\hat{\beta}_{1,OLS} = \beta_1 + \text{corr}(x,u) \cdot \frac{\sigma_u}{\sigma_x}$$

$$\text{plim}\hat{\beta}_{1,IV} = \beta_1 + \frac{\text{corr}(z,u)}{\text{corr}(z,x)} \cdot \frac{\sigma_u}{\sigma_x}$$

- Weak instrument: if correlation between z and x is small.

Instrumental variables: statistical properties

MLRM: $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u} \quad |$ valid $\boldsymbol{Z}$ exists

- IVR method is a "trick" for consistent estimation of the ceteris paribus effects, i.e. $\hat{\beta}_{j,\mathrm{IV}}$.
- Fitted values are generated as $\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\mathrm{IV}}$ (NOT from $\hat{\boldsymbol{y}} = \hat{\boldsymbol{X}}\hat{\boldsymbol{\beta}}_{\mathrm{IV}}$).
- Similarly: $\mathrm{var}(\hat{u}_i) = \hat{\sigma}^2 = \frac{1}{n-k}\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i\hat{\boldsymbol{\beta}}_{\mathrm{IV}})^2$ d.f. correction is superfluous (asymptotic use only).
- $\mathrm{Asy.Var}(\hat{\boldsymbol{\beta}}_{\mathrm{IV}}) = \hat{\sigma}^2(\boldsymbol{Z}'\boldsymbol{X})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{X}'\boldsymbol{Z})^{-1}$ for the exactly identified & homoscedastic case.
- With heteroscedasticity and/or over-identification, the $\mathrm{Asy.Var}(\hat{\boldsymbol{\beta}}_{\mathrm{IV}})$ formula is complex and built into all SW packages.

2SLS as a special case of IVR

$$\hat{\boldsymbol{\beta}}_{\mathrm{IV}} = (\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y} = (\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y}$$

2SLS:

- Structural equation (as in SEMs)

  $y_1 = \beta_0 + \beta_1 y_2 + \beta_2 x_2 + \cdots + \beta_k x_k + u \quad | \; z_1$ exists

- Reduced form for y_2 – endogenous variable as function of all exogenous variables (including IVs)

  $y_2 = \pi_0 + \pi_1 z_1 + \pi_2 x_2 + \cdots + \pi_k x_k + \varepsilon$

- 1st stage of 2SLS: Estimate reduced form by OLS
  - Order condition for identification of the structural equation: at least one instrument for each endogenous regressor).
  - If z_1 is an IV for y_2, its coefficient must not be zero (rank condition for identification) in the reduced form equation - see stage 2 of 2SLS.

2SLS as a special case of IVR

$$\hat{\boldsymbol{\beta}}_{\text{IV}} = (\hat{\boldsymbol{X}}'\boldsymbol{X})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y} = (\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y}$$

2SLS:

- Structural equation

  $y_1 = \beta_0 + \beta_1 y_2 + \beta_2 x_2 + \cdots + \beta_k x_k + u \quad | \ z_1$ exists

- 1^{st} stage of 2SLS: estimate reduced form for y_2:

  $\hat{y}_2 = \hat{\pi}_0 + \hat{\pi}_1 z_1 + \hat{\pi}_2 x_2 + \cdots + \hat{\pi}_k x_k$

- 2^{nd} stage of 2SLS: Use $\hat{y}_2$ to estimate structural equation:

  $y_1 = \beta_0 + \beta_1 \hat{y}_2 + \beta_2 x_2 + \cdots + \beta_k x_k + u$

- Note that RHS in the 2^{nd} stage contains all exogenous regressors repeated from $\boldsymbol{X}$, while $\hat{y}_2$ is y_2 "projected" onto $\boldsymbol{Z}$ and thus uncorrelated with u.

- Order condition fulfilled. Rank condition explained: if $\pi_1 = 0$, $\hat{y}_2$ is a perfect linear combination of the remaining RHS regressors in 2^{nd} stage.

Instrumental variables: summary

- Excluded from the main / structural equation
- Must be correlated with endogenous regressor(s)
- Must not be correlated with u

All IVs used in IVR / 2SLS estimation must fulfill the conditions above.

In 2SLS, 1$^{\text{st}}$ stage is used to generate the "best" IV.
With multiple endogenous regressors, reduced forms for each endogenous regressor must be constructed and estimated, rank and order conditions apply.

2SLS properties

- The standard errors from the OLS second stage regression are biased and inconsistent estimators with respect to the original structural equation (SW handles this problem automatically).
- If there is one endogenous variable and one instrument then 2SLS = IVR
- With multiple endogenous variables and/or multiple instruments, 2SLS is a special case of IVR.

  Example:
  Consider MLRM with one endogenous regressor and 3 relevant IVs. Choosing any IV (or any ad-hoc linear combination of IVs) results in IVR (MM-type & consistent estimator). 2SLS (GMM-type approach) provides the "best" IVR estimator – lowest variance in the 2^{nd} stage comes from best fit between IVs and endogenous regressor in 1^{st} stage.

Statistical properties of the 2SLS/IV estimator

- Under assumptions completely analogous to OLS, but conditioning on z_i rather than on x_i , 2SLS/IV is consistent and asymptotically normal.
- 2SLS/IV estimator is typically much less efficient than the OLS estimator because there is more multicollinearity and less explanatory variation in the second stage regression
- Problem of multicollinearity is much more serious with 2SLS than with OLS

Statistical properties of the 2SLS/IV estimator

- Corrections for heteroscedasticity/serial correlation analogous to OLS
- 2SLS/IVR estimamtion easily extends to time series and panel data situations

IVR diagnostic tests: introduction

LRM: $y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i$; $\boldsymbol{z}$ instruments exist

IV regression advantages for endogenous y_2:

$\rightarrow \hat{\beta}_{1,\text{OLS}}$ is a biased and inconsistent estimator (asymptotic errors)

$\rightarrow \hat{\beta}_{1,\text{IV}}$ is a biased and consistent estimator (increased sample size (n) lowers estimator bias and s.e.)

IVR disadvantages (price for the IVR):

- s.e.$(\hat{\beta}_{1,\text{IV}}) >$ s.e.$(\hat{\beta}_{1,\text{OLS}})$
- $\hat{\beta}_{1,\text{IV}}$ is biased, even if y_2 is actually exogenous

  $\hat{\beta}_{1,\text{OLS}}$ is unbiased for exogenous regressors (potentially, pending other G-M conditions).

IVR diagnostic tests: introduction

LRM: $y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i$; $\quad \boldsymbol{z}$ instruments exist

- Is the regressor y_2 endogenous / $\text{corr}(y_2, u) \neq 0$ / ?
  Is it meaningful to use IVR (considering IVRs "price")?
  Durbin-Wu-Hausman endogeneity test
- Are the instruments actually helpful
  (weakly or strongly correlated with endogenous regressors)?
  Weak instruments test
- Are the instruments really exogenous / $\text{corr}(z_j, u) = 0$ / ?
  Sargan test (only applicable in case of over-identification)

Different types & specifications for IV-tests exist, often focusing on the distribution of the difference between IVR and OLS estimators $(\hat{\boldsymbol{\beta}}_{\text{IV}} - \hat{\boldsymbol{\beta}}_{\text{OLS}})$ under the corresponding H_0.

Durbin-Wu-Hausman endogeneity test

$$y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i \qquad | \; z_{i1},$$

DWH test motivation:

If z_1 is a proper instrument (uncorrelated with u), then y_2 is endogenous (correlated with u) if and only if ε (error from reduced form equation) is correlated with u.

- If y_2 is endogenous $\quad \Leftrightarrow \quad \text{corr}(y_2, u) \neq 0$
- Reduced form: $y_2 = l.f.(x_1, z_1) + \varepsilon \;\; \Rightarrow \;\; y_2 = \hat{y}_2 + \hat{\varepsilon}$
- $\text{corr}(y_2, u) \neq 0 \; \wedge \; \text{corr}(\hat{y}_2, u) = 0 \;\; \Rightarrow \;\; \text{corr}(\hat{\varepsilon}, u) \neq 0$
- y_1 is always correlated with u.
- Hence, $\hat{\varepsilon}$ is significant in an auxiliary regression
  $y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + \delta \hat{\varepsilon}_i + u_i,$
  if y_2 is an endogenous regressor.
- IV/IVs being uncorrelated with u is an essential condition for DWH test to "work".

Note: other variants of the DWH test exist...

Durbin-Wu-Hausman endogeneity test

Structural equation:

$$y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i; \quad \text{IVs: } z_1 \text{ and } z_2 \tag{1}$$

Reduced form for y_2:

$$y_{i2} = \pi_0 + \pi_1 z_{i1} + \pi_2 z_{i2} + \pi_3 x_{i1} + \varepsilon_i \tag{2}$$

H_0: y_2 is exogenous $\leftrightarrow$ $\hat{\varepsilon}$ is not significant when added to equation (1)

H_1: y_2 is endogenous $\rightarrow$ OLS is not consistent for (1) estimation, use IVR (2SLS).

Testing algorithm:

1. Estimate equation (2) and save residuals $\hat{\varepsilon}$.
2. Add residuals $\hat{\varepsilon}$ into equation (1) and estimate using OLS (use HC inference).
3. H_0 is rejected if $\hat{\varepsilon}$ in the modified equation (1) is statistically significant (t-test).

Weak instruments

Motivation for Weak instruments and Sargan tests:

LRM: $y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i$; z instrument exists

- IVR is consistent if $\text{cov}(z, y_2) \neq 0$ and $\text{cov}(z, u) = 0$
- If we allow for (weak) correlation between z and u, the asymptotic error of IV estimator is:

$$\text{plim}(\hat{\beta}_{1,IV}) = \beta_1 + \frac{\text{corr}(z,u)}{\text{corr}(z,y_2)} \cdot \frac{\sigma_u}{\sigma_{y_2}}$$

- If $\text{corr}(z, y_2)$ is too weak (too close to zero in absolute value), OLS may be better than IV. The asymptotic bias for OLS (LRM with endogenous y_2):

$$\text{plim}(\hat{\beta}_{1,OLS}) = \beta_1 + \text{corr}(y_2, u) \cdot \frac{\sigma_u}{\sigma_{y_2}}$$

Rule of thumb: IF $|\text{corr}(z, y_2)| < |\text{corr}(y_2, u)|$, do not use IVR.

Weak instruments

Structural equation:

$$y_1 = \beta_0 + \beta_1 y_2 + \beta_2 x_1 + \cdots + \beta_{k+1} x_k + u; \quad \text{IVs: } z_1, z_2, \ldots, z_m$$

The reduced form for y_2:

$$y_2 = \pi_0 + \pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_k x_k + \theta_1 z_1 + \cdots + \theta_m z_m + \varepsilon$$

H_0: $\theta_1 = \theta_2 = \cdots = \theta_m = 0$
interpretation: “instruments are weak”.

H_1: $\neg \, H_0$

Testing for weak instruments:

Use F-test (heteroscedasticity-robust).
Note: multiple testing approaches & exist.

Sargan test (over-identification only)

Structural equation:

$$y_{i1} = \beta_0 + \beta_1 y_{i2} + \beta_2 x_{i1} + u_i; \quad \text{IVs: } z_1, z_2, \ldots \tag{3}$$

H_0: all IVs are uncorrelated with u

H_1: at least one instrument is endogenous

Testing algorithm:

1. Estimate equation (3) using IVR and save the $\hat{u}$ residuals.
2. Use OLS to estimate auxiliary regression: $\hat{u} \leftarrow f(\boldsymbol{x}, \boldsymbol{z})$ and save the R_a^2
3. Under H_0: $nR_a^2 \sim \chi_q^2$ where
   $q =$ (number of IVs) - (number of endogenous regressors)
   i.e. q is the number of over-identifying variables.
4. If the observed test statistic exceeds its critical value (at a given significance level), we reject H_0.

IVR diagnostic tests: example

Wooldridge, bwght dataset
R code, {AER} package

```
Call:
ivreg(formula = lbwght ~ packs + male | faminc + motheduc + male,
    data = bwght)

Residuals:
      Min        1Q    Median        3Q       Max
 -1.66291  -0.09793   0.01717   0.11616   0.82793

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   4.77419    0.01099 434.478  < 2e-16 ***
packs        -0.25584    0.07613  -3.361 0.000798 ***
male          0.02422    0.01048   2.311 0.021003 *

Diagnostic tests:
                   df1  df2 statistic  p-value
Weak instruments     2 1383    38.732  <2e-16 ***
Wu-Hausman           1 1383     5.385  0.0205 *
Sargan               1   NA     4.476  0.0344 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1

Residual std. error: 0.195 on 1384 d.f.
Multiple R-Squared: -0.04371,   Adj R-sqr: -0.04522
Wald test: 8.342 on 2 and 1384 DF,  p-value: 0.0002504
```

IVs

Regressors explicitly included in equation

✓Reject H_0: IVs are weak

✓Reject H_0: pack are exogenous (IVR adequate)

!! Reject H_0: all IVs are uncorrelated with u (!DWH assumptions!)

Simlultaneous equation model (SEM)

- SEM: outline
- SEM: identification
- Identification conditions
- SEMs with more than two equations

Simultaneity is another important form of endogeneity

Simultaneity occurs if at least two variables are jointly determined. A typical case is when observed outcomes are the result of separate behavioral mechanisms that are coordinated in an equilibrium.

Prototypical case: a system of demand and supply equations:

- $D(p)$ how high *would* demand be if the price was set to p?
- $S(p)$ how high *would* supply be if the price was set to p?

- Both mechanisms have a ceteris paribus interpretation.
- Observed quantity and price will be determined in equilibrium, where $D(p) = S(p)$.

Simultaneous equations systems can be estimated by 2SLS/IVR … Identification conditions apply.

SEM examples

Example 1: Labor supply and demand in agriculture

$$h_s = \alpha_1 w + \beta_1 z_1 + u_1$$
$$h_d = \alpha_2 w + \beta_2 z_2 + u_2$$

- Endogenous variables, exogenous variables, observed and unobserved supply shifter, observed and unobserved demand shifter

- We have n regions, market sets equilibrium price and quantity in each. We observe the equilibrium values only

$$h_{is} = h_{id} \Rightarrow (h_i, w_i)$$

SEM examples

Example 1: Labor supply and demand in agriculture contnd.

$$h_i = \alpha_1 w_i + \beta_1 z_{i1} + u_{i1}$$
$$h_i = \alpha_2 w_i + \beta_2 z_{i2} + u_{i2}$$

- If we have the same exogenous variables in each equation, we cannot identify (distinguish) equations.
- We assume independence between errors in structural equations & exogenous regressors.

SEM examples

Example 1: Labor supply and demand in agriculture contnd.

If we estimate the structural equation with OLS method, estimators will be biased – so called "simultaneity bias".

$$y_1 = \alpha_1 y_2 + \beta_1 z_1 + u_1$$

$$y_2 = \alpha_2 y_1 + \beta_2 z_2 + u_2$$

y_2 is dependent on u_1
(substitute RHS of the 1^{st} equation for y_1 in the 2^{nd} eq.)

$$\Rightarrow y_2 = \left[\frac{\alpha_2\beta_1}{1-\alpha_2\alpha_1}\right] z_1 + \left[\frac{\beta_2}{1-\alpha_2\alpha_1}\right] z_2 + \left[\frac{\alpha_2 u_1 + u_2}{1-\alpha_2\alpha_1}\right]$$

Structural and reduced form equations, 2SLS method

Structural equations (example)

$$y_1 = \beta_{10} + \beta_{11} y_2 + \beta_{12} z_1 + u_1$$

$$y_2 = \beta_{20} + \beta_{21} y_1 + \beta_{22} z_2 + u_2$$

Reduced form equations

$$y_1 = \pi_{10} + \pi_{11} z_1 + \pi_{12} z_2 + \varepsilon_1 \quad \Rightarrow \quad \hat{y}_1 \text{ by OLS}$$

$$y_2 = \pi_{20} + \pi_{21} z_1 + \pi_{22} z_2 + \varepsilon_2 \quad \Rightarrow \quad \hat{y}_2 \text{ by OLS}$$

2SLS (a special case of IVR)

- 1^{st} stage: Estimate reduced forms, get $\hat{y}_1$ and $\hat{y}_2$.
- 2^{nd} stage: Replace endogenous regressors in structural equations by fitted values from 1^{st} stage, estimate by OLS.

Estimation assumptions and “problems” involved:

… Identification of structural equations,

… Statistical inference in structural equations (2^{nd} stage).

SEM examples

Example 2: (Structural equations)
Estimation of murder rates

$$
\begin{aligned}
murdpc &= \beta_{10} + \alpha_1 polpc + \beta_{11} incpc + u_1 \\
polpc &= \beta_{20} + \alpha_2 murdpc + \boldsymbol{\beta}(other\ factors) + u_2
\end{aligned}
$$

- 1^{st} equation describes the behaviour of murderers, 2^{nd} one the behaviour of municipalities. Each one has its ceteris paribus interpretation.

- For the municipality policy, the 1^{st} equation is interesting: what is the impact of exogenous increase of police force on the murder rate?
- However, the number of police officers is not exogenous (simultaneity problem).

SEM examples

SEM equation properties (for each equation):

- Variables with proper ceteris paribus interpretation
- Structural equations describe process from different perspectives
  - Labor market: employees vs. employers
  - Criminality: authorities vs. "criminals"

Counter example: households' saving and housing expendituress:

$$\begin{aligned} housing &= \beta_{10} + \beta_{11} saving + \beta_{12} income + \cdots + u_1 \\ saving &= \beta_{20} + \beta_{21} housing + \beta_{22} income + \cdots + u_2 \end{aligned}$$

- Both equations model household behavior
- Both endogenous variables chosen by the same agent
- Cannot reasonably change *income* and hold *saving* fixed (first equation)

Example 3: (Identification)
Identification problem in a SEM

- Example: Supply and demand for milk

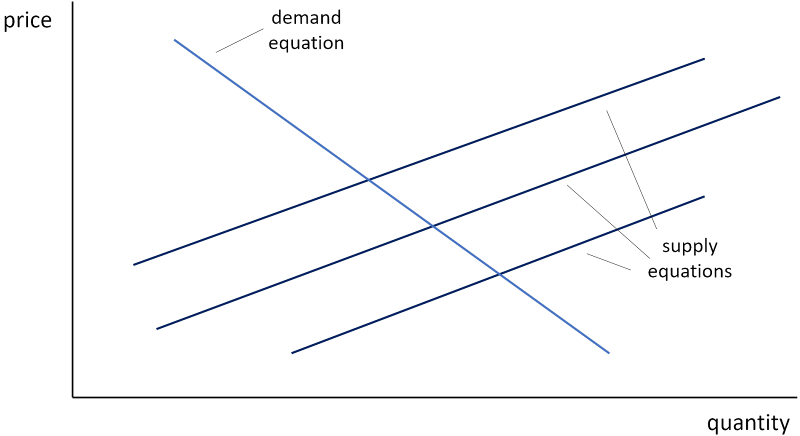
  Supply of milk: $q = \alpha_1 p + \beta_1 z_1 + u_1$
  Demand for milk: $q = \alpha_2 p + u_2$

- Supply of milk cannot be consistently estimated because we do not have (at least) one exogenous variable "available" to be used as instrument for p in the supply equation.

- Demand for milk can be consistently estimated because we can use exogenous variable z_1 as instrument for p in the demand equation.

SEM identification

- Ilustration



Identification conditions

Identification conditions for a sample 2-equation SEM (individual i subscripts omitted)

$$
\begin{aligned}
y_1 &= \beta_{10} + \alpha_1 y_2 + \beta_{11} z_{11} + \beta_{12} z_{12} + \cdots + \beta_{1k} z_{1k} + u_1 \\
y_2 &= \beta_{20} + \alpha_2 y_1 + \beta_{21} z_{21} + \beta_{22} z_{22} + \cdots + \beta_{2k} z_{2k} + u_2
\end{aligned}
$$

- Order condition (necessary): 1^{st} equation is identified if at least one exogenous variable z is excluded from 1^{st} equation (yet in the SEM).
- Rank condition (necessary and sufficient): 1^{st} equation is identified if and only if the second equation includes at least one exogenous variable excluded from the first equation with a nonzero coefficient, so that it actually appears in the reduced form.
- For the second equation, the conditions are analogous.
- Some estimation approaches allow for identification through IVs not explicitly included in the SEM.

Example 4: (Identification)
Labor supply of married working women

Supply (workers):

$$
\begin{aligned}
hours = \alpha_1 \log(wage) + \beta_{10} + \beta_{11}\, educ + \beta_{12}\, age + \beta_{13}\, kidslt6 \\
+ \beta_{14}\, nwifeinc + u_1
\end{aligned}
$$

Demand (enterprises):

$$
\log(wage) = \alpha_2 hours + \beta_{20} + \beta_{21}\, educ + \beta_{22}\, exper + \beta_{23}\, exper^2 + u_2
$$

Order condition is fulfilled in both equations.

Example 4: (Identification)

Labor supply of married working women contnd.

- Identification of the first equation (Supply). For the rank condition, either β_{22} or β_{23} non-zero population coefficient (in the second equation) is required – so that $exper$, $exper^2$ (or both) can be used in the reduced form.
- To evaluate the rank condition for supply equation, we estimate the reduced form for $\log(wage)$ and test if we can reject the null hypothesis that coefficients for both $exper$ and $exper^2$ are zero.
  If H_0 is rejected, the rank condition is fulfilled.
- We would do the evaluation of the rank condition for the demand equation analogically.

Estimation

- We can consistently estimate identified equations with the 2SLS method.
- In the 1^{st} stage, we regress each endogenous variable on all exogenous variables ("reduced forms").
- In the 2^{nd} stage we put into the structural equations instead of endogenous variables their predictions from the 1^{st} stage and estimate with the OLS method.
- The reduced form can be always estimated (by OLS).
- In the 2^{nd} stage, we cannot estimate unidentified structural equations.
- With some additional assumptions, we can use a more efficient estimation method than 2SLS: 3SLS.

Systems with more than two equations

Example 5: Keynesian macroeconomic model

$$
\begin{aligned}
C_t &= \beta_0 + \beta_1(Y_t - T_t) + \beta_2 r_t + u_{t1} \\
I_t &= \gamma_0 + \gamma_1 r_t + u_{t2} \\
Y_t &\equiv C_t + I_t + G_t
\end{aligned}
$$

Endogenous: C_t, I_t, Y_t Exogenous: T_t, G_t, r_t

- Order condition for identification is the same as for two-equation systems, rank condition is more complicated.
- Complex models based on macroeconomic time series are sometimes used. Problems with these models: series are usually not weakly dependent, it is difficult to find enough exogenous variables as instruments. Question is, if any macroeconomic variables are exogenous at all.

Identification in SEMs with more than two equations

$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{u}_i$ is the i-th equation of a SEM.

K - number of exogenous/predetermined variables in the SEM,
K_i - number of K in the i-th equation,
G_i - number of endogenous variables in the i-th equation.

Order condition for the i-th equation:
necessary, not sufficient condition for identification

$K - K_i \geq G_i - 1$

Condition evaluates as:

- $=$ Equation i is just-identified,
- $>$ Equation i is over-identified,
- $<$ Equation i is not identified,
  structural equation i cannot be estimated by 2SLS/IVR.

Identification in SEMs with more than two equations

Rank condition: based on matrix algebra & IV estimator

Consider IVR for an identified i-th equation of SEM

$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{u}_i$

$\boldsymbol{X}_i$ is a $(n \times k)$ matrix, includes the intercept column and all endogenous regressors of the i-th equation,

$\hat{\boldsymbol{X}}_i$ is a $(n \times k)$ matrix, includes the intercept column. Exogenous regressors are repeated from $\boldsymbol{X}_i$, endogenous are projected to the column space of $\boldsymbol{Z}$: a $(n \times l)$ matrix of all exogenous variables in the SEM.

Single equation (limited information) estimator for each i-th equation:

- $\hat{\boldsymbol{\beta}}_{IVR} = \hat{\boldsymbol{\beta}}_{2SLS,i} = \left(\hat{\boldsymbol{X}}_i'\boldsymbol{X}_i\right)^{-1}\hat{\boldsymbol{X}}_i'\boldsymbol{y}$
- GMM – moment equations can be used

Identification in SEMs with more than two equations

Rank condition: based on matrix algebra & IV estimator (cont.)

$\hat{\beta}_{IVR} = \left(\hat{\boldsymbol{X}}_i' \boldsymbol{X}_i \right)^{-1} \hat{\boldsymbol{X}}_i' \boldsymbol{y}$

- **Order condition**: The necessary condition for the i-th equation to be identified is that the number of columns (exogenous variables of SEM) in $\boldsymbol{Z}$ should be no less than the number of columns (explanatory variables) in $\boldsymbol{X}_i$.

- **Rank condition**: The necessary and sufficient condition for identification of the i-th equation is that $\hat{\boldsymbol{X}}_i'$ has full column rank of $\boldsymbol{X}_i$.
  ...ensures the existence of $\left(\hat{\boldsymbol{X}}_i' \boldsymbol{X}_i \right)^{-1}$.

Identification in SEMs with more than two equations

Identification: recap & final remarks

- Reduced form equations can always be estimated.
- Structural equations can be estimated (IV/2SLS) only if identified: i.e. if rank condition is met.

- With SW, checking rank condition for $\left(\hat{\boldsymbol{X}}_i'\boldsymbol{X}_i\right)^{-1}$ is easy for finite datasets.
- Asymptotic identification may be "tricky": because some columns in $\boldsymbol{X}_i$ are endogenous, plim $n^{-1}\hat{\boldsymbol{X}}_i'\boldsymbol{X}_i$

  depends on the parameters of the DGP.

  ...see Davidson-MacKinnon (2009) Econometric theory and methods