Fast Doubling Fibonacci Series Calculation

Sampath Singamsetty

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1 Fibonacci Series

The usual **Fibonacci Sequence** is defined as below which obviously is a difference equation.

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-2} + F_{n-1} \end{aligned} \tag{1}$$

The same when expressed in a matrix format can be written as below

$$\begin{aligned} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \end{aligned} \tag{2}$$

That is, each Fibonacci number F_n is the sum of the two previous `Fibonacci numbers`, except for the very first two numbers which are defined to be `0` and `1`.

With the above definition, it appears that computing F_n requires one to always compute F_{n-1} and F_{n-2}.

2 Difference equations

A `Difference equation` may be regarded as an equation consisting of a finite differences of an unknown equation. The *Differential equations* are suitable for `continuous systems`, but cannot be used for `discrete variables`. Difference equations are the discrete equivalent of differential equations and arise whenever an independent variable can have only discrete values.

They have a wide area of importance in engineering particularly due to their applications in discrete time-systems used in association with microprocessors.

They are also widely used in studying the mathematical modeling of biological systems. Mathematical models of biological systems are often expressed in terms of `differential equations`. The reason for this being that biological systems are dynamical, changing with respect to time, space, or stage of development.

Difference equations are relationships between quantities as they change over discrete time intervals ($t = 0, 1, 2, \ldots$). Differential equations describe changes in quantities over continuous time intervals ($0 \leq t \leq \infty$). The quantities modeled by the difference or differential equations are called `states` of the system.

2.1 Mathematical properties of the Difference operator

The **Forward Difference Operator** Δ is defined as $\Delta f(x) = f(x+h) - f(x)$, where `h` is step size and this this formula uses the values at x and $x+h$ to calculate the point at the next step.

For $x = x_i$, the above equation may be represented as $\Delta f(x_i) = f(x_i + h) - f(x_i)$.

2.2 First Order Differences

The values, $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ are called the `First Order Differences`.

$$\begin{aligned}
\Delta y_i &= y_{i+1} - y_i \quad \text{By taking h = 1 unit, and for i = 0, 1, 2, ..., n - 1} \\
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
&\vdots \\
\Delta y_n &= y_{n+1} - y_n
\end{aligned} \tag{3}$$

2.3 Second Order Differences

The differences of the `First Order Differences` are called `Second Order Differences` denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$ which are described next.

From the first order difference relation 3 described above, we can write and expand the second order difference relation $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ as below:

$$
\begin{aligned}
\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) \\
&= (y_2 - 2y_1 + y_0) \\
\Delta^2 y_1 &= \Delta y_2 - \Delta y_1 \\
&= (y_3 - y_2) - (y_2 - y_1) \\
&= (y_3 - 2y_2 + y_1) \\
&\vdots
\end{aligned}
\tag{4}
$$

2.4 Higher Order Differences

Similar to the `First` and `Second` order, the `Higher` order differences may be derived as below.

$$\begin{aligned}
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\
&= (y_3 - 3y_2 + 3y_1 - y_0) \\
\Delta^3 y_1 &= (y_4 - 3y_3 + 3y_2 - y_1) \\
&\vdots \\
\Delta^n y_0 &= y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^{n-1n} C_{n-1} y_1 + (-1)^n y_0
\end{aligned}$$

Generalizing on the above, we get

$$\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r \tag{5}$$

In general, from the relation 5 we can write $\Delta^{n+1} y_i = \Delta^n y_{i+1} - \Delta^n y_i$ for $n = 0, 1, 2, \ldots$

Also, it must be noted that Δ^0 is an `Identity Operator`. i.e., $\Delta^0 f(x) = f(x)$ and $\Delta^1 = \Delta$.

2.5 Difference table

All the *forward differences* may be represented in a tabular form called the `Forward Difference` or `Diagonal Difference` table. If $x_0, x_1, x_2, x_3 and x_4$ be four arguments, then all the forward differences of these arguments can be represented as shown in the table 1.

Table 1: Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_2$	
		Δy_3			
x_4	y_4				

2.6 Propagation of errors in Difference table

If any of the entries in the difference table throws an error, the same would spread across the table in a convex manner. The propagation of error in a difference table is illustrated in the table 2. As an example considering y_3 to be erroneous with an error amount of ϵ, the following observations may be noted.

Table 2: Error Propagation in Finite Difference Table

x	y	Δ y	Δ^2y	Δ^3y	Δ^4y	Δ^5y
x_0	y_0					
		Δ y_0				
x_1	y_1		$\Delta^2 y_0$			
		Δ y_1		$\Delta^3 y_0 + \epsilon$		
x_2	y_2		$\Delta^2 y_1 + \epsilon$		$\Delta^4 y_0 - 4\epsilon$	
		Δ $y_2 + \epsilon$		Δ^3$y_1 - 3\epsilon$		$\Delta^5 y_0 + 10\epsilon$
x_3	$y_3 - \epsilon$		$\Delta^2 y_2 - 2\epsilon$		$\Delta^4 y_1 + 6\epsilon$	
		Δ $y_3 - \epsilon$		Δ^3$y_2 + 3\epsilon$		$\Delta^5 y_0 - 10\epsilon$
x_4	y_4		$\Delta^2 y_3 + \epsilon$		$\Delta^4 y_2 - 4\epsilon$	
		Δ y_4		$\Delta^3 y_3 - \epsilon$		
x_5	y_5		$\Delta^2 y_4$			
		Δ y_5				
x_6	y_6					

From the table 2, it can be observed that:

- As the order of difference increases, the error increases
- The coefficients of errors in the k^{th} difference column are the binomial coefficients in the expansion of $(1-x)^n$. Particularly, the 2^{nd} difference column has errors $\epsilon, -2\epsilon, \epsilon$ and the 3^{rd} difference column has errors $\epsilon, -3\epsilon, 3\epsilon, \epsilon$ and so on.
- The algebraic sum of errors in any full column is zero.
- Wherever there was an error, the differences did not follow a smooth pattern.

2.7 *Some common properties of the Forward Difference Operator ("Δ")*

Below are some of the properties which the forward difference operator Δ satisfies, that might be useful:

$$\begin{aligned}
\Delta c &= 0, && \text{c being a constant} \\
\Delta[cf(x)] &= c\Delta f(x) \\
\Delta[af(x) \pm bg(x)] &= a\Delta f(x) \pm b\Delta g(x) \\
\Delta^m \Delta^n f(x) &= \Delta^{m+n} f(x) \\
&= \Delta^{n+m} f(x) \\
&= \Delta^k \Delta^{m+n-k} f(x) && \text{k = 0, 1, 2, ..., m or n} \\
\Delta[c^x] &= c^{x+h} - c^x \\
&= c^x(c^h - 1) && \text{for some constant c}
\end{aligned} \tag{6}$$

3 Fast Doubling method

The `Fast Doubling Method` is perhaps the fastest way of computing a Fibonacci sequence and the method uses a matrix multiplication approach for the fast calculation of Fibonacci number sequence.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \tag{7}$$