Math 425 Computation Linear Algebra

HW4

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Vector Spaces, Basis and Othogonality.

In [1]:
```
# environment setup, try to make it clear which library I'm using for what
import numpy as np  # nice arrays and other stuff
import sympy as sym # symbollic maths
from sympy.matrices import Matrix # pretty matrices
from sympy import Eq # pretty equations
from sympy.physics.quantum.dagger import Dagger # we'll want this later...
from math import e, pi, sqrt # Mathy math math
from mpl_toolkits.mplot3d import Axes3D # we like 3d quivers for tutorials
import matplotlib.pyplot as plt # old standby for plotting like a villian
from IPython.display import display, Math, Latex # used to display formatted results in
sym.init_printing()  # initialize pretty printing
```

1. Prove that the set $\{\,(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\,\}$ is a subspace of $\mathbb{F}^3$.

By notation convention, $(x_1, x_2, 0)$ is a list of length 1 of vectors in $\mathbb{F}^3$.

Let U = $(x_1, x_2, 0)$ and V = $\mathbb{F}^3$.

The conditions for a subspace are that a subset of U of V is a subspace of V if and only if U satisfies these three conditions:

$\textbf{additive identity}$: $0 \in U$

$\textbf{closed under addition}$: $u, w \in U$ implies $u + w \in U$

$\textbf{closed under scalar multiplication}$: $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

Proof (Adapted from 'Linear Algebra Done Right', by Sheldon Axler)

If U is a subspace of V, then U satisfies the three conditions above by defination of vector space.

Conversely, suppose U satisfies the three conditions above. The first condition ensures that the additive identity of V is in U. The second ensures that addition holds true on U. The thrid ensures that scalar multiplication holds true on U.

2. Let $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{bmatrix}$.

(i) Find a basis for RowA and NulA.

In [42]:

```
A = Matrix([[1,-2,0,3],[2,-3,1,-4],[3,-5,-1,-1]])
display(Latex(f'$A={sym.latex(A)}$, $A^T={sym.latex(A.T)}$, \
$A^T.rref()={sym.latex(A.T.rref(pivots = False))}$'))
display(Latex(f"$RowA=ColA^T={sym.latex((A.T).columnspace())}$ *"))
display(Latex("*Above we use Gilbert Strang's convention for rowspace, which shows a bas
of column vectors."))
display(Latex(f"$RowA={sym.latex(A.rowspace())}$ **"))
display(Latex("**The above we show the results of Sympy's rowspace for comparison, which
however naive application of row reduction to this matrix would not be valid, hence the
Strang's basis vector convention."))
display(Latex(f"$NullA={sym.latex(A.nullspace())}$ ***Our nullspace (aka Kernel)."))

display(Latex("Below we show the product of $Au=0$, thus demostrating this is indeed the
u = Matrix(A.nullspace()) # 'I have become nullspace, annihilator of worlds' -misquoted
A,u, A*u
```

$A=\begin{bmatrix}1 & -2 & 0 & 3\\ 2 & -3 & 1 & -4\\ 3 & -5 & -1 & -1\end{bmatrix}$, $A^T=\begin{bmatrix}1 & 2 & 3\\ -2 & -3 & -5\\ 0 & 1 & -1\\ 3 & -4 & -1\end{bmatrix}$, $A^T.rref()=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$

$RowA=ColA^T=\left[\begin{bmatrix}1\\ -2\\ 0\\ 3\end{bmatrix}, \begin{bmatrix}2\\ -3\\ 1\\ -4\end{bmatrix}, \begin{bmatrix}3\\ -5\\ -1\\ -1\end{bmatrix}\right]$ *

*Above we use Gilbert Strang's convention for rowspace, which shows a basis vector as composed of column vectors.

$RowA=\left[\begin{bmatrix}1 & -2 & 0 & 3\end{bmatrix}, \begin{bmatrix}0 & 1 & 1 & -10\end{bmatrix}, \begin{bmatrix}0 & 0 & -2 & 0\end{bmatrix}\right]$ **

**The above we show the results of Sympy's rowspace for comparison, which is 'correct', however naive application of row reduction to this matrix would not be valid, hence the adoption of Strang's basis vector convention.

$NullA=\left[\begin{bmatrix}17\\ 10\\ 0\\ 1\end{bmatrix}\right]$ ***Our nullspace (aka Kernel).

Below we show the product of $Au=0$, thus demostrating this is indeed the nullspace of A.

Out[42]:

$$\left(\begin{bmatrix}1 & -2 & 0 & 3\\ 2 & -3 & 1 & -4\\ 3 & -5 & -1 & -1\end{bmatrix}, \begin{bmatrix}17\\ 10\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\right)$$

In [47]:
```python
# Be more verbose
A, u
# x1 - 17 x4 = 0
# x2 - 10 x4 = 0
# x3  = 0

# x1 = 17 x4
# x2 = 10 x4
# x3 = 0
# x4 = x4
nullA = Matrix([17, 10, 0, 1])
nullA
```

Out[47]:

$$\begin{bmatrix}17\\10\\0\\1\end{bmatrix}$$

(ii) Find the inner product of each vector in the basis of RowA with each vector in basis of NulA.

In [3]:
```python
print('Our prediction is that these spaces are orthogonal...')
rowA = Matrix([A.T.columnspace()])
nullA = Matrix(A.nullspace())
#rowA, nullA, rowA.T*nullA
display(Latex(f'${sym.latex(rowA.T)}*{sym.latex(nullA)}={sym.latex(rowA.T*nullA)}$ *Here

rowA = Matrix(A.rowspace()).T
display(Latex(f'${sym.latex(rowA.T)}*{sym.latex(nullA)}={sym.latex(rowA.T*nullA)}$ *Here
print('...indeed, our prediction was correct.\n')
```

```
Our prediction is that these spaces are orthogonal...
```

$\begin{bmatrix}1&-2&0&3\\2&-3&1&-4\\3&-5&-1&-1\end{bmatrix}*\begin{bmatrix}17\\10\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ *Here we show RowA as ColA^T

$\begin{bmatrix}1&-2&0&3\\0&1&1&-10\\0&0&-2&0\end{bmatrix}*\begin{bmatrix}17\\10\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ *Here we show same result using Sympy rowspace().

```
...indeed, our prediction was correct.
```

3. Let $u_1=\begin{bmatrix}3\\-3\\0\end{bmatrix}$, $u_2=\begin{bmatrix}2\\2\\-1\end{bmatrix}$, $u_3=\begin{bmatrix}1\\1\\4\end{bmatrix}$, and $x=\begin{bmatrix}5\\-3\\1\end{bmatrix}$.

Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$ then express x as a linear comibination of u_1, u_2 and u_3.

In [4]:
```
u1 = Matrix([3,-3,0])
u2 = Matrix([2,2,-1])
u3 = Matrix([1,1,4])
x = Matrix([5,-3,1])
display(Latex(f'$u_1 \circ u_2={sym.latex(u1.dot(u2))}$, \
$u_2 \circ u_3={sym.latex(u2.dot(u3))}$, \
$u_3 \circ u_1={sym.latex(u3.dot(u1))}$, *dot product shows all vectors are $\perp$. \
'))
M = Matrix([list(v) for v in (u1,u2,u3,x)]).T # We're just play about, same as row join.
M, M.rref(), M.rref(pivots=False).col(3) # The last column of row reduced augmented matr
C =  M.rref(pivots=False).col(3)
c1,c2,c3 = list(C)
c1*u1+c2*u2+c3*u3
display(Latex(f'$c_1u_1+c_2u_2+c_3u_3={sym.latex(c1*u1+c2*u2+c3*u3)}$, $x={sym.latex(x)}
```

$u_1 \circ u_2 = 0$, $u_2 \circ u_3 = 0$, $u_3 \circ u_1 = 0$, *dot product shows all vectors are $\perp$.

$c_1u_1 + c_2u_2 + c_3u_3 = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$, voilà

4. Suppose W is a subspace of $\mathbb{R}^{\mathrm{n}}$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^{\mathrm{n}}$.

The statement above asserts W is a subspace of $\mathbb{R}^{\mathrm{n}}$ and indeed by the definition of span we know the following:

The set of all linear combinations of a list of vectors $w_1, \ldots, w_n$ in W is a called the $\mathbf{span}$ of $w_1, \ldots, w_n$, and denoted $span(w_1, \ldots, w_n)$. Also, the span of the empty list ( ) is defined to be $\{0\}$.

Thus, $span(w_1, \ldots, w_n)$ = $\{a_1w_1 + \cdots + a_nw_n : a_1, \ldots, a_n \in \mathbb{R}\}$ demonstrating that indeed $W = \mathbb{R}^{\mathrm{n}}$ and thusly increasing the total Existential Angst of the Universe.

5. Let $W = Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ and $y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

Write y as a vector in W and a vector in $W^{\perp}$.

Recall: The orthogonal projection of y onto W is $\hat{y}$=$\frac{y \circ u_1}{u_1 \circ u_1} u_1 + \frac{y \circ u_2}{u_2 \circ u_2} u_2$ + $\frac{y \circ u_3}{u_3 \circ u_3} u_3$.

Also, $y - \hat{y}$ is in $W^{\perp}$. (by Orthogonal Decomposition Theorem)

Also, recall: If y is in $W = Span\{u_1, \ldots u_p\}$, then $proj_W y = y$.

```
In [5]: W = Matrix([[1,1,0],[1,0,-1],[0,1,1],[-1,1,-1]])
y = Matrix([3,4,5,6])
print(f'W is an orthogonal basis. Lie detector says, "{W.rref(pivots=False)==sym.eye(4,3

# Let x1...x3 = W.col(0...2)
u1 = W.col(0)
u2 = W.col(1)
u3 = W.col(2)

v1 = (y.dot(u1)/(u1.dot(u1)))*u1
v2 = (y.dot(u2)/(u2.dot(u2)))*u2
v3 = (y.dot(u3)/(u3.dot(u3)))*u3

y_hat = v1+v2+v3
z = y - y_hat
y, y_hat, z.dot(y_hat)
W.row_join(z).rref()
display(Latex('$y=proj_Wy+proj_{W^\perp}y=$'+f'${sym.latex(y_hat)}+{sym.latex(z)}={sym.l
display(Latex(f'${sym.latex(y_hat)}\circ{sym.latex(z)}={sym.latex(y_hat.dot(z))}$, indee

display(Latex(f'${sym.latex(z.T)}*{sym.latex(W)}={sym.latex(z.T*W)}$'+', indeed $proj_{W
display(Latex('*This ought to be satisfying to the Mathematician as it is quite obvious.
```

```
W is an orthogonal basis. Lie detector says, "True"
```

$y=proj_Wy+proj_{W^\perp}y=\begin{bmatrix}5\\2\\3\\6\end{bmatrix}+\begin{bmatrix}-2\\2\\2\\0\end{bmatrix}=\begin{bmatrix}3\\4\\5\\6\end{bmatrix}$

$\begin{bmatrix}5\\2\\3\\6\end{bmatrix}\circ\begin{bmatrix}-2\\2\\2\\0\end{bmatrix}=0$, indeed these vector are $\perp$.

$\begin{bmatrix}-2 & 2 & 2 & 0\end{bmatrix}*\begin{bmatrix}1 & 1 & 0\\1 & 0 & -1\\0 & 1 & 1\\-1 & 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$, indeed $proj_{W^\perp}y$ is $\perp$ to W. *

*This ought to be satisfying to the Mathematician as it is quite obvious. ;-)

6. Let $z=\begin{bmatrix}2\\4\\0\\-1\end{bmatrix}$, $v_1=\begin{bmatrix}2\\0\\-1\\-3\end{bmatrix}$ and $v_2=\begin{bmatrix}5\\-2\\4\\2\end{bmatrix}$.

Find the best approximation to z by vectors of the form $c_1v_1+c_2v_2$.

In [6]:
```
z = Matrix([2,4,0,-1])
v1 = Matrix([2,0,-1,-3])
v2 = Matrix([5,-2,4,2])

display(Latex(f'$v_1\circ v_2={sym.latex(v1.dot(v2))}$, *Show $v_1$ and $v_2$ are orthog
u1 = (z.dot(v1)/(v1.dot(v1)))*v1
u2 = (z.dot(v2)/(v2.dot(v2)))*v2
z_hat = u1+u2

A = Matrix([v1.T,v2.T]).T # use least-square to validate our results
c = (A.T*A).inv()*A.T*z

display(Latex('$z≈\hat{z}=c_1 v_1+c_2 v_2=$'+\
f'${sym.latex(c[0])}{sym.latex(v1)}+{sym.latex(c[1])}{sym.latex(v2)}=$\
${sym.latex(v1.row_join(v2)*c)}$, '+\
'$\hat{z}=$'+ f'${sym.latex(z_hat)}$'))
```

$v_1 \circ v_2 = 0$, *Show v_1 and v_2 are orthogonal

$$z \approx \hat{z} = c_1 v_1 + c_2 v_2 = \frac{1}{2}\begin{bmatrix}2\\0\\-1\\-3\end{bmatrix} + 0\begin{bmatrix}5\\-2\\4\\2\end{bmatrix} = \begin{bmatrix}1\\0\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}, \hat{z} = \begin{bmatrix}1\\0\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}$$

7. Let $u_1 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$, $u_2 = \begin{bmatrix}5\\-1\\2\end{bmatrix}$ and $u_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

In [7]:
```
u1 = Matrix([1,1,-2])
u2 = Matrix([5,-1,2])
u3 = Matrix([0,1,0])
u1,u2,u3
```

Out[7]:

$$\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}5\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

a) Show that $\{u_1, u_2\}$ is a basis for $W = Span\{u_1, u_2\}$. Do not use row reduction.

In [8]:
```
display(Latex(f'$u1\circ u2 = {sym.latex(u1.dot(u2))}$, * thus this is an orthognal basi
```

$u1 \circ u2 = 0$, * thus this is an orthognal basis for W

b) Show that $u_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ is not in W.

In [9]:
```
W = Matrix([u1.T,u2.T]).T # We continue to use the Strang basis convention which is inde
Wu = W.row_join(u3)
display(Latex(f'Augment $W$ and $u_2$ to form ${sym.latex(Wu)}$.'))
display(Latex(f'Row reducing this matrix yeilds, ${sym.latex(Wu.rref())}$, \
Which shows three pivots, thus $u_3$ is not in $W$.'))
```

Augment W and u_2 to form $\begin{bmatrix} 1 & 5 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}$.

Row reducing this matrix yeilds, $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ (0,\ 1,\ 2) \right)$, Which shows three pivots, thus u_3 is not in W.

c) Use the fact that u_3 is not in W to construct a nonzero vector v in $\mathbb{R}^3$ that is orthogonal to u_1 and u_2.

In [10]:
```
c = Matrix(sym.symbols('c1 c2 c3'))
display(Latex(f"Any scalar multiple of $u_3$ is orthogonal to $u_1$ and $u_2$, also\
${sym.latex(u3)}\circ{sym.latex(c)}= {sym.latex(u3.dot(c))}$."))
display(Latex('Thus by assigning α=$c_2$, α$u_3$ constructs all nonzero vectors $\perp$
```

Any scalar multiple of u_3 is orthogonal to u_1 and u_2, also $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_2$.

Thus by assigning α=c_2, αu_3 constructs all nonzero vectors $\perp$ to u_1 and u_2.

8. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

In [41]:
```
A= Matrix([[-1,6,6],[3,-8,3],[1,-2,6],[1,-4,-3]])
display(Latex('GramSchmidt:'))
x1 = A.col(0)
x2 = A.col(1)
x3 = A.col(2)

v1 = x1
v2 = x2 - (x2.dot(v1)/v1.dot(v1))*v1
proj_w_x3 = (x3.dot(v1)/v1.dot(v1))*v1 + (x3.dot(v2)/v2.dot(v2))*v2
v3 = x3 - proj_w_x3

v1.T*v2, v2.T*v3, v3.T*v1
v1, v2, v3
```

GramSchmidt:

Out[41]:

$$\left(\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right)$$

9. Find an orthonormal basis for the column space of the matrix

$$A = \begin{bmatrix} 3 & -3 & 0 \\ -4 & 14 & 10 \\ 5 & -7 & -2 \end{bmatrix}.$$

In [12]:
```
A = Matrix([[3,-3,0],[-4,14,10],[5,-7,-2]])
display(Latex("Firstly let's determine an orthonormal basis by selecting the pivot colum
display(Latex(f'$A.rref()={sym.latex(A.rref())}$, \
thus our colA basis is ${sym.latex(A.col(0))},{sym.latex(A.col(1))}$.'+\
' *note that this set does not span $\mathbb{R^3}$.'))

colA = Matrix([A.columnspace()])
display(Latex(f'$colA = {sym.latex(colA)}$ * as confirmed using sympy.'))

display(Latex("Finally we want to normalize this basis..."))
u1 = colA.col(0)/sym.sqrt((colA.col(0).T*colA.col(0))[0]) # a rather complicated method,
u2 = colA.col(1)/sym.sqrt((colA.col(1).T*colA.col(1))[0])
U = Matrix([u1.T,u2.T]).T # note that this set doesn't span R^3, no way this is invertab

display(Latex(f'After normalizing each column, our colA orthonormal basis is ${sym.latex
display(Latex('The normality of these basis vectors is evident by dotting each vector wi
u1.dot(u1), u2.dot(u2)
```

Firstly let's determine an orthonormal basis by selecting the pivot columns of matrix A.

$A.rref() = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ (0, \ 1) \right)$, thus our colA basis is $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$. *note that this set does not span $\mathbb{R}^3$.

$colA = \begin{bmatrix} 3 & -3 \\ -4 & 14 \\ 5 & -7 \end{bmatrix}$ * as confirmed using sympy.

Finally we want to normalize this basis...

After normalizing each column, our colA orthonormal basis is $\begin{bmatrix} \frac{3\sqrt{2}}{10} \\ -\frac{2\sqrt{2}}{5} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{3\sqrt{254}}{254} \\ \frac{7\sqrt{254}}{127} \\ -\frac{7\sqrt{254}}{254} \end{bmatrix}$.

The normality of these basis vectors is evident by dotting each vector with itself as seen below.

Out[12]: $(1, \ 1)$

10. Let $u_1, \ldots, u_p$ be an orthogonal basis for the subspace W of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by $T(x) = proj_W x$. Show that T is a linear transformation.

If $\{u_1, \ldots, u_p\}$ is and orthogonal basis for the subspace W of $\mathbb{R}^n$, then each x in W can be form as a linear combination of each basis vector.

$x = c_1u_1 + c_2u_2 + \ldots + c_pu_p$

From the above we see that the weights of the constants are given by,

$c_j = \frac{x \circ u_j}{u_j \circ u_j}$ $(j = 1, \ldots, p)$, which by construction is $proj_W x_j$.

defining another linear combination y as,

$y = b_1v_1 + b_2v_2 + \ldots + b_pv_p$

...we can clearly see $T(cx + y) = cT(x) + T(y)$, thus proving T is a linear tranformation.

Appendix 1. Practice Problems

In [13]:
```
# Ex 6.2.5
v1 = 1/sym.sqrt(11) * Matrix([3,1,1])
v2 = 1/sym.sqrt(6) * Matrix([-1,2,1])
v3 = 1/sym.sqrt(66) * Matrix([-1,-4,7])

U = Matrix([v1.T,v2.T,v3.T]).T

U.T*U, v1.dot(v1), v2.dot(v2), v3.dot(v3)
```

Out[13]:

$$\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, 1, 1, 1\right)$$

In [14]:
```
# Lay Ex 6.4.2
x1 = Matrix([1,1,1,1])
x2 = Matrix([0,1,1,1])
x3 = Matrix([0,0,1,1])

# W1 = Span{x1} = Span{v1}
v1 = x1
v2 = x2 - (x2.dot(v1)/v1.dot(v1))*v1 # don't bother scaling v2, we are computer
v2 = 4 *v2 # alright we'll scale it to match the book

# W2
v3 = x3 - ((x3.dot(v1)/v1.dot(v1))*v1 + (x3.dot(v2)/v2.dot(v2))*v2)

v1,v2,v3
```

Out[14]:

$$\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}\right)$$

In [15]:
```
# Lay PP 5.8
A = Matrix([[5,8,4],[8,3,-1],[4,-1,2]])
x = Matrix([1.0,-4.3,8.1])
Ax = A*x
A = A.applyfunc(lambda a: float(a)) # case A as float
x_inv = x.applyfunc(lambda x: 1/x)
Ax_x_ratios = sym.matrix_multiply_elementwise(Ax,x_inv)
Ax, x_inv, Ax_x_ratios, A.eigenvals() # note, strictly dominant eigenvalue
```

Out[15]:

$$\left(\begin{bmatrix} 3.0 \\ -13.0 \\ 24.5 \end{bmatrix}, \begin{bmatrix} 1.0 \\ -0.232558139534884 \\ 0.123456790123457 \end{bmatrix}, \begin{bmatrix} 3.0 \\ 3.02325581395349 \\ 3.02469135802469 \end{bmatrix}, \{-5.63682475389763 : 1,\ 3.0240\right.$$

Lay P 6.3.7

Recall: The orthogonal projection of y onto W is $\hat{y}$=$\frac{y \circ u_1}{u_1 \circ u_1} u_1 + \frac{y \circ u_2}{u_2 \circ u_2} u_2$.

Also, $y - \hat{y}$ is in $W^{\perp}$. (by Orthogonal Decomposition Theorem)

Also, recall: If y is in $W = Span\{\,u_1, \ldots u_p\,\}$, then $proj_W y = y$.

In [18]:
```
# Lay Ex 6.3.7
y = Matrix([1,3,5]); u1 = Matrix([1,3,-2]); u2 = Matrix([5,1,4])
#y = Matrix([1,2,3]); u1 = Matrix([2,5,-1]); u2 = Matrix([-2,1,1]) # validate code using
y = Matrix([-1,4,3]); u1 = Matrix([1,1,0]); u2 = Matrix([-1,1,0])
W = u1.row_join(u2)
W.rref() # yes, Span{u1,u2} is an orthogonal basis for W

y_hat = (y.dot(u1)/u1.dot(u1))*u1 + (y.dot(u2)/u2.dot(u2))*u2
y_hat, y-y_hat
display(Latex(f'$y={sym.latex(y)}={sym.latex(y_hat)}+{sym.latex(y-y_hat)}$'))
#W,y, W.norm()
#W.columnspace()
```

$$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Appendix 2. Galois Connection

(adapted from Rainbox Connection sang by Kermit the Frog)

see also: https://en.wikipedia.org/wiki/Galois_connection

Why aren't there many songs about Galois
And what's on the other side
Galois are visions
But only illusions
And Galois have nothing to hide
So we've been told
And some choose to believe it
I know they're wrong, wait and see

Someday we'll find it
The Galois connection
The lovers, the dreamers, and me

Who said that every wish
Would be heard and answered
When wished on the morning star
Somebody thought of that
And someone believed it
Look what it's done so far

What's so amazing that keeps us stargazing
And what do we think we might see
Someday we'll find it
The Galois connection
The lovers, the dreamers, and me

All of us under its spell
We know that it's probably magic

Have you been half asleep
And have you heard voices?
I've heard them calling my name
Is this the sweet sound
That called the young sailors?
The voice might be one and the same

I've heard it too many times to ignore it
It's something that I'm supposed to be
Some day we'll find it
The Galois connection
The lovers, the dreamers, and me

La da da di da da dooo, la da da da da di da doooooh