In Class Work - April 07

(1) Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$. Show that $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$, then express $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(2) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$. Show that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, then express $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

(3) Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

(4) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

(5) Let $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

(6) Let $\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Let W be the subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

(7) Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(8) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal but $\mathbf{u}_3$ is not orthogonal to $\mathbf{u}_1$ or $\mathbf{u}_2$. It can be shown that $\mathbf{u}_3$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

(9) Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

(a) Solve $A\mathbf{x} = \mathbf{b}$. Is the system consistent, i.e. is $\mathbf{b} \in \mathrm{Col}A$?
(b) Use Gram-Schmidt to produce an orthonormal basis of Col$A$.
(c) Find $\hat{\mathbf{b}} = \mathrm{proj}_{\mathrm{Col}A}\mathbf{b}$.
(d) Solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$.
(d) Use the normal equations to find the least-squares solution for $A\mathbf{x} = \mathbf{b}$. Compare your solution to your solution in part (d).

(10) Plot the data points $\{(0,1),\ (1,1),\ (2,2),\ (3,2)\}$. Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best fits the given data points.

(11) A certain experiment produces the data $\{(1,1.8),\ (2,2.7),\ (3,3,4),\ (4,3.8),\ (5,3.9)\}$. Describe the model that produces a least-squares fit of these points by a function of the form

$$y = \beta_1 x + \beta_2 x^2.$$

Such a function might arise, for example, as the revenue from the sale of x units of a product, when the amount offered for sale affects the price to be set for the product.
a. Give the design matrix, the observation vector, and the unknown parameter vector.
b. Find the associated least-squares curve for the data.

(12) Suppose radioactive substances A and B have decay constants of 0.02 and 0.07, respectively. If a mixture of these two substances at time $t = 0$ contains M_{A} grams of A and M_{B} grams of B, then a model for the total amount y of the mixture present at time t is

$$y = M_{\mathrm{A}}\mathrm{e}^{-0.02t} + M_{\mathrm{B}}\mathrm{e}^{-0.07t}.$$

Suppose the initial amounts M_{A}, M_{B} are unknown, but a scientist is able to measure the total amount present at several times and records the following points (t_i, y_i) :. $(10, 21.34)$, $(11, 20.68)$, $(12, 20.05)$, $(14, 18.87)$, and $(15, 18.30)$.
a. Describe a linear model that can be used to estimate M_{A} and M_{B}.
b. Find the least-squares curve based on the model.