Math 425 Computation Linear Algebra

HW2, Part A

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Linear independence, Span, and Vector spaces.

In [1]:
```
# environment setup
import numpy as np # nice arrays and other stuff

#from sympy import * # import the entire namespace of sympy at root, NOT the be
st of practices
import sympy as sym # make is clear which lib I'm using for what
from sympy.matrices import Matrix # Include this in case we want some pretty ma
trices
from sympy.solvers.solveset import linsolve

from math import e, pi
from IPython.display import display, Math, Latex # used to display formatted re
sults
sym.init_printing()  # initialize pretty printing
```

1. List five vectors in $span\{v_1, v_2\}$. For each vector, show the weights on v_1, v_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

In [2]:
```
a = [-1, 0, 1, 2, 3]
b = [1, 0 , -1, -2, -3]

v_1 = Matrix([3, 0, 2])
v_2 = Matrix([-2, 0, 3])

display(Latex(f'Five vectors in the $span({sym.latex(v_1)},{sym.latex(v_2)})$ a
re:'))
[display(Math(f'({a[i]})v_1 + ({b[i]})v_2 = {sym.latex(a[i] * v_1 + b[i] * v_
2)}' )) for i in range(len(a))]
display(Latex('Sweet!!! We have $\LaTeX$ formating to working. This will come i
n handy later.'))
```

Five vectors in the $span(\begin{bmatrix}3\\0\\2\end{bmatrix},\begin{bmatrix}-2\\0\\3\end{bmatrix})$ are:

$$(-1)v_1 + (1)v_2 = \begin{bmatrix}-5\\0\\1\end{bmatrix}$$

$$(0)v_1 + (0)v_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$(1)v_1 + (-1)v_2 = \begin{bmatrix}5\\0\\-1\end{bmatrix}$$

$$(2)v_1 + (-2)v_2 = \begin{bmatrix}10\\0\\-2\end{bmatrix}$$

$$(3)v_1 + (-3)v_2 = \begin{bmatrix}15\\0\\-3\end{bmatrix}$$

Sweet!!! We have $\LaTeX$ formating to working. This will come in handy later.

2. Decide whether the following sets of vectors are linearly independent or linearly dependent. Give reasons for your choices.

2.17 **Definition** ***linearly independent***

- A list $v_1, \ldots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list ( ) is also declared to be linearly independent.

2.19 **Definition** ***linearly dependent***

- A list of vectors in V is called ***linearly dependent*** if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

- Def and Lemma from 'Linear Algebra Done Right', By Sheldon Axler

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -11 \\ -8 \end{bmatrix} \right\}$ ***Not obvious multiples so let's see what a row reduction looks like.**

In [3]:
```
display(Latex('Our method here is to compute the nullspace of a the matrix crea
ted from the vectors and compare it to the column space.'))
A = Matrix([[1,0,3],[-3,2,-7],[2,-11,-8]])
display(Math(f'$A={sym.latex(A)}'))
display(Math(f'{ sym.latex(A.rref(pivots=False))}'))
display(Latex('a) is indeed linearly independent, and spans $\mathbb{R^3}$. (Ou
r display formatting foo grows stronger by the hour.)'))
A, A.rank(), A.nullspace() # nullspace says there is nowhere we cannot go in R^
3
```

Our method here is to compute the nullspace of a the matrix created from the vectors and compare it to the column space.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & -7 \\ 2 & -11 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) is indeed linearly independent, and spans $\mathbb{R}^3$. (Our display formatting foo grows stronger by the hour.)

Out[3]:

$$\left(\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & -7 \\ 2 & -11 & -8 \end{bmatrix}, \ 3, \ [] \right)$$

b) $\left\{ \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -11 \\ -8 \end{bmatrix} \right\}$ ***Not multiples, so we know it's Independent, but lets do a rref and see what happens anyway.**

In [4]:
```python
display(Latex('Again, our method is to compute the nullspace of a the matrix cr
eated from the vectors and compare it to the column space.'))
B = Matrix([[-3,2,-7],[2,-11,-8]])
display(Latex(f'$B.rref={sym.latex(B.rref(pivots=False))}$'))
display(Latex('b) is indeed a linearly independent in $\mathbb{R}^3$, but does
not span $\mathbb{R^3}.$'))
B, B.rank(), B.nullspace()
```

Again, our method is to compute the nullspace of a the matrix created from the vectors and compare it to the column space.

$$B.rref = \begin{bmatrix} 1 & 0 & \frac{93}{29} \\ 0 & 1 & \frac{38}{29} \end{bmatrix}$$

b) is indeed a linearly independent in $\mathbb{R}^3$, but does not span $\mathbb{R}^3$.

Out[4]:

$$\left(\begin{bmatrix} -3 & 2 & -7 \\ 2 & -11 & -8 \end{bmatrix}, \ 2, \ \left[\begin{bmatrix} -\frac{93}{29} \\ -\frac{38}{29} \\ 1 \end{bmatrix} \right] \right)$$

c) $\left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{3} \end{bmatrix}, \begin{bmatrix} \mathbf{-3} \\ \mathbf{2} \\ \mathbf{-7} \end{bmatrix}, \begin{bmatrix} \mathbf{2} \\ \mathbf{-11} \\ \mathbf{-8} \end{bmatrix}, \begin{bmatrix} \mathbf{9} \\ \mathbf{12} \\ \mathbf{13} \end{bmatrix} \right\}$

In [5]:
```python
display(Latex('Use our hammer again... compute the nullspace of a the matrix cr
eated from the vectors and compare it to the column space.'))
C = Matrix([[1,0,3],[-3,2,-7],[2,-11,-8],[9,12,13]])
display(Math(f'{ sym.latex(C.rref(pivots=False))}'))
display(Latex('c) is linearly dependent in $\mathbb{R}^3$, which is clear as th
e list is longer than the a spanning list for $\mathbb{R}^3.$ '))
C.rank()
```

Use our hammer again... compute the nullspace of a the matrix created from the vectors and compare it to the column space.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

c) is linearly dependent in $\mathbb{R}^3$, which is clear as the list is longer than the a spanning list for $\mathbb{R}^3$.

Out[5]: 3

d) $\left\{\begin{bmatrix}1\\4\\9\\10\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}5\\7\\1\\0\end{bmatrix}\right\}$

In [6]:
```
display(Latex('Compute the nullspace of a the matrix created from the vectors a
nd compare it to the column space.'))
D = Matrix([[1,0,5],[4,0,7],[9,0,1],[10,0,0]])
display(Math(f'{ sym.latex(D.rref(pivots=False))}'))
display(Latex('d) is linearly independent in $\mathbb{F^4}$, as shown by the pi
vots.'))
```

Compute the nullspace of a the matrix created from the vectors and compare it to the column space.

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

d) is linearly independent in $\mathbb{F}^4$, as shown by the pivots.

3. Determine if the columns of the matrix form a linearly independent set. Justify each answer.

a) $\begin{bmatrix}-4 & -3 & 0\\0 & -1 & 4\\1 & 0 & 3\\5 & 4 & 6\end{bmatrix}$

In [7]:
```
A = Matrix([[-4,-3,0],[0,-1,4],[1,0,3],[5,4,6]])
display(Math('A=%s' %sym.latex(A) ))
display(Math('A.rref=%s' %sym.latex(A.rref(pivots=False)) ))
display(Latex('a) is linearly independant as there is are more rows than rank
(in other words, the column space is larger than rank). Note this is R^4 subspa
ce.'))
A.nullspace(), A.columnspace(), A.rank()
```

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

$$A.rref = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a) is linearly independant as there is are more rows than rank (in other words, the column space is larger than rank). Note this is R^4 subspace.

Out[7]:

$$\left([], \left[\begin{bmatrix} -4 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \\ 6 \end{bmatrix} \right], 3 \right)$$

b) $\begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{7} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$

In [8]:
```
B = Matrix([[1,-3,3,2],[-3,7,-1,2],[0,1,-4,3]])
display(Latex(f'$B.rref={sym.latex(B.rref())}$'))
display(Latex('BTW: rref.rref returns a tuple of two elements. The first is the
reduced row echelon form, and the second is a tuple of indices of the pivot col
umns.)'))
display(Latex('b) is linearily independent because there are fewer pivots than
row space.'))
```

$$B.rref = \left(\begin{bmatrix} 1 & 0 & -9 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (0,\ 1,\ 3) \right)$$

BTW: rref.rref returns a tuple of two elements. The first is the reduced row echelon form, and the second is a tuple of indices of the pivot columns.)

b) is linearily independent because there are fewer pivots than row space.

4. Let $v_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 9 \\ h \end{bmatrix}$

a) For what values of h is v_3 in $Span\{v_1, v_2\}$

In [9]:
```
v_1 = Matrix([1,-5,-3])
v_2 = Matrix([-2,10,6])
h = sym.symbols('h')
v_3 = Matrix([2,-9,h])

display(Latex('a) $v_1$ and $v_2$ are multiples of each other so there is no mu
ltiple of $v_3$ in that subspace.'))
lactation = sym.latex(v_3) + 'not^{*} \in Span^{**}\{' + sym.latex(v_1) + sym.l
atex(v_2) + '\} \Leftrightarrow Span\{' + sym.latex(v_1) + '\}'
display(Latex('No value of $h$ puts $v_3$ in $Span\{v_1,v_2\}$, or rather $%
s$.' %lactation))
display(Latex("* \$\\notin\$ isn't rendered in sympy.latex(), file bug report a
s there are lot of standard notation messing."))
display(Latex('** Ask about about this and learn the correct Maths terminology.
Maybe I already know it.  How much do any of us know anyway?'))
display(Latex('*** Ask your Mathematician if ${subspaces}$ are right for yo
u.'))
```

a) v_1 and v_2 are multiples of each other so there is no multiple of v_3 in that subspace.

No value of h puts v_3 in $Span\{v_1, v_2\}$, or rather $\begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix} not^{*} \in Span^{**}\{\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}\begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}\} \Leftrightarrow Span\{\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}\}$.

* $\notin$ isn't rendered in sympy.latex(), file bug report as there are lot of standard notation messing.

** Ask about about this and learn the correct Maths terminology. Maybe I already know it. How much do any of us know anyway?

*** Ask your Mathematician if $subspaces$ are right for you.

b) For what values of h is v_1, v_2, v_3 lineraly dependent?

In [10]:
```
print('FIXME!!!  Ask Henry more about this.')
h = sym.symbols('h')

A = Matrix([[1,-5,-3],[-2,10,6],[2,-9,h]])
B = Matrix([0,0,0])
x_1,x_2,x_3 = sym.symbols('x_1, x_2, x_3')

display(Latex("Let's use the computer to enhance our understanding..."))
display(Latex(f'Recall our Linear system is $A={sym.latex(A)}$'))
display(Latex(f'Row reduction results in $A.rref={sym.latex(A.rref())}$ and sho
ws two pivots and one free varible.'))
x = Matrix(list(sym.linsolve((A,B), (x_1,x_2,x_3))))
display(Latex(f'Solving this system for zero roots we get, $x = {sym.latex
(x)}$'))
display(Latex(f'BTW: The nullspace is ${sym.latex(A.nullspace())}$, you can see
how the zero roots are related to this.'))
h_nogo = sym.latex([-6, sym.Rational(-27,5)])  # if we're not right then at lea
st we can make it pretty
display(Latex(f'By inspection$^*$ we can see that $h$ ≠ ${h_nogo}$. *Ask Henery
if there is more Mathematical way to do this'))
```

```
FIXME!!!  Ask Henry more about this.
```

Let's use the computer to enhance our understanding...

Recall our Linear system is $A = \begin{bmatrix} 1 & -5 & -3 \\ -2 & 10 & 6 \\ 2 & -9 & h \end{bmatrix}$

Row reduction results in $A.rref = \left(\begin{bmatrix} 1 & 0 & 5h+27 \\ 0 & 1 & h+6 \\ 0 & 0 & 0 \end{bmatrix}, (0, 1) \right)$ and shows two pivots and one free varible.

Solving this system for zero roots we get, $x = \begin{bmatrix} x_3(-5h-27) & x_3(-h-6) & x_3 \end{bmatrix}$

BTW: The nullspace is $\left[\begin{bmatrix} -5h-27 \\ -h-6 \\ 1 \end{bmatrix} \right]$, you can see how the zero roots are related to this.

By inspection * we can see that $h \neq \left[-6, -\frac{27}{5} \right]$. *Ask Henery if there is more Mathematical way to do this

5. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $Ax = 0$.

In [11]:
```
A = Matrix([[4,1,6], [-7,5,3], [9,-3,3]])
display(Math(f'A.rref={sym.latex(A.rref(pivots=False))}'))
display(Latex('Okay, looks like we have $x_1+x_3 =0$ and $x_2+2x_3=0$.'))
x_1,x_2,x_3 = sym.symbols('x_1 x_2 x_3')
x = Matrix([-x_3,-2*x_3,x_3])
display(Latex(f"Thus, our non-trival solution is $x={sym.latex(x)}$"))
```

$$A.rref = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Okay, looks like we have $x_1 + x_3 = 0$ and $x_2 + 2x_3 = 0$.

Thus, our non-trival solution is $x = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$

In [12]:
```
display(Latex("Let's explore the Eigeness of this all.")) # show verbose way
lambda_ = sym.symbols('lambda_')
A = Matrix([[4,1,6], [-7,5,3], [9,-3,3]])
P = sym.det(A-lambda_*sym.eye(3))
P_ = sym.factor(sym.Eq(P,0))
display(Latex("FIXME!!! Discuss results below and get Mathmatical."))
display(Latex("We ought to discuss change of basis and the like, but there is s
o much that we've forgotten!"))
A, A.rref(), A.rank(), P, P_, sym.solve(P_, lambda_) # chararteristic polynomia
l and roots
```

Let's explore the Eigeness of this all.

FIXME!!! Discuss results below and get Mathmatical.

We ought to discuss change of basis and the like, but there is so much that we've forgotten!

Out[12]:

$$\left(\begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}, \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, (0,\ 1)\right), 2,\ -\lambda^3 + 12\lambda^2 - 9\lambda,\ -\lambda\left(\lambda^2 - 12\lambda + 9\right) = 0,\ \left[0,\ 6 - 3\sqrt{3},\ 3\sqrt{3} + 6\right]\right)$$

In [13]:
```
display(Latex('Now do it symbolically, as we are questing for a deeper understa
nding.'))
x_1,x_2,x_3 = sym.symbols('x_1 x_2 x_3')
x = Matrix([x_1,x_2,x_3])
M = A*x
B= Matrix([0,0,0])
sym.linsolve((A,B), (x_1,x_2,x_3)) # confirming our results above
```

Now do it symbolically, as we are questing for a deeper understanding.

Out[13]:

$$\left\{\left(-x_3,\ -2x_3,\ x_3\right)\right\}$$

6. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

In [14]:
```
display(Latex("No, we'll need at least 4 independent vector to span $\mathbb{R^
4}$.  Like I was saying, we need $n$ independent vectors, $n \geq m$ to span $\
mathbb{R^m}$.  Otherwise we'll just have a subspace of $\mathbb{R^4}$ and won't
be able to reach every point."))
```

No, we'll need at least 4 independent vector to span $\mathbb{R^4}$. Like I was saying, we need n independent vectors, $n \geq m$ to span $\mathbb{R^m}$. Otherwise we'll just have a subspace of $\mathbb{R^4}$ and won't be able to reach every point.

7. Let $u = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is u in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or not?

In [15]:
```
u = Matrix([2,-3,2])
A = Matrix([[5,8,7],[0,1,-1],[1,3,0]])
display(Math(f'A.ref={sym.latex(A.rref(pivots=False))}, A.columnspace={sym.late
x(A.columnspace())}, A.rank={sym.latex(A.rank())}'))
display(Latex("Now append $u$ to A and see how our span and rank changes."))
A=A.col_insert(3,u)
display(Math(f'A.ref={sym.latex(A.rref(pivots=False))}, A.columnspace={sym.late
x(A.columnspace())}, A.rank={sym.latex(A.rank())}'))
display(Latex("Now by inspection$^{*}$ we can clearly see that $u$ is not in th
e subspace formed by $A$."))
display(Latex("* engineers are rather lazy like this, physicists are even wors
t, and don't even ask about artists."))
display(Latex("Anyway, no $u$ is not in the $span(A)$ by the aforementioned obs
erations$^{**}$ of rank and span.\n"))
display(Latex("** Honestly, it really is at lot of fun to play with Linear Alge
bra in this way."))
```

$$A.ref = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, A.columnspace = \left[\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} \right], A.rank = 2$$

Now append u to A and see how our span and rank changes.

$$A.ref = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A.columnspace = \left[\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right], A.rank = 3$$

Now by inspection * we can clearly see that u is not in the subspace formed by A.

* engineers are rather lazy like this, physicists are even worst, and don't even ask about artists.

Anyway, no u is not in the $span(A)$ by the aforementioned obserations ** of rank and span.

** Honestly, it really is at lot of fun to play with Linear Algebra in this way.

8. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible b, and describe the set of b for which $Ax = b$ does have a solution.

In [16]:
```
b_1,b_2,b_3 = sym.symbols('b_1 b_2 b_3')
b = Matrix([b_1,b_2,b_3])
A = Matrix([[1,-3,-4],[-3,2,6],[5,-1,-8]])
display(Latex(f'$A.rref={sym.latex(A.rref()) }$')) # Two pivots
display(Latex("By inspection we can see the rank of $A$ is 2, thus $A$ a subspa
ce of $\mathbb{R^3}$. This means we can only 'reach' into the $span\{A\}$ where
$b_3=0$."))
```

$$A.rref = \left(\begin{bmatrix} 1 & 0 & -\frac{10}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & 0 \end{bmatrix}, (0,\ 1) \right)$$

By inspection we can see the rank of A is 2, thus A a subspace of $\mathbb{R}^3$. This means we can only 'reach' into the $span\{A\}$ where $b_3 = 0$.

9.Let $\boldsymbol{B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}.}$

Do the columns of B span $\mathbb{R}^4$?

Does the equation $Bx = y$ have a solution for each $y \in \mathbb{R}^4$?

In [17]:
```
display(Latex("Now that I've4 got this hammer everything looks a nails."))
B = Matrix([[1,3,-2,2],[0,1,1,-5],[1,2,-3,7],[-2,-8,2,-1]])
display(Math(f"B={sym.latex(B)}, B.rref={sym.latex(B.rref(pivots=False))}, B.ra
nk={sym.latex(B.rank())}"))
display(Latex("Nope, $B$ doesn't span{$\mathbb{R^4}$}, as evident by it's ran
k."))
display(Latex("No, $Bx=y$ only has solution where $y_4=0$.  FIXME!!! Ask Henry
about this."))
```

Now that I've4 got this hammer everything looks a nails.

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}, B.rref = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B.rank = 3$$

Nope, B doesn't span{$\mathbb{R}^4$}, as evident by it's rank.

No, $Bx = y$ only has solution where $y_4 = 0$. FIXME!!! Ask Henry about this.

Appendix 1. Playing with displaying with LaTex

In [18]:
```
from IPython.display import display, Math, Latex
display(Math(r'F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx'))
```

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx$$

In [19]:
```
x = sym.symbols('x')
expr = sym.sqrt(3) * x**3

display(Math('\\frac{1}{2} ') )
display(expr)
display(Math('\\frac{1}{2} %s' %sym.latex(expr)))
display(Math('\\frac{1}{2} %s' %sym.latex(expr)))
```

$$\frac{1}{2}$$

$$\sqrt{3}x^3$$

$$\frac{1}{2}\sqrt{3}x^3$$

$$\frac{1}{2}\sqrt{3}x^3$$

Appendix 2. Exploring the Nullspace and Span relationships

In [20]:

```
A=Matrix([[1,1,1,-1],[2,4,5,6],[3,9,5,4]])
display(Math(f'A={sym.latex(A)}'))
display(Math(f'A.rref={sym.latex(A.rref(pivots=False))}'))
lactation = sym.latex(A.nullspace())  # Milk it, like a cow!
x_1,x_2,x_3,x_4 = sym.symbols('x_1, x_2, x_3, x_4') # Sprinkle some sugar.
x = Matrix([x_1, x_2, x_3, x_4])  # X marks the spot.
display(Math(f'{sym.latex(x)}=x_4*{sym.latex(A.rref(pivots=False).col(3))} '))
display(Math(f'N(A)=Span{lactation}') ) # Moo!
n = A.shape[1]
rank = A.rank()
nullity = n - rank
print("Nullity: ", nullity)
print("Rank: ", rank)
```

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & 5 & 6 \\ 3 & 9 & 5 & 4 \end{bmatrix}$$

$$A.rref = \begin{bmatrix} 1 & 0 & 0 & -\frac{53}{14} \\ 0 & 1 & 0 & \frac{5}{14} \\ 0 & 0 & 1 & \frac{17}{7} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 * \begin{bmatrix} -\frac{53}{14} \\ \frac{5}{14} \\ \frac{17}{7} \end{bmatrix}$$

$$N(A) = Span \left[\begin{bmatrix} \frac{53}{14} \\ -\frac{5}{14} \\ -\frac{17}{7} \\ 1 \end{bmatrix} \right]$$

```
Nullity:  1
Rank:  3
```