Math 425 Computation Linear Algebra

HW1, Part A

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***See Appendix for hand calcluations**

In [1]:
```
# environment setup
import numpy as np # Let's use numpy for this
from sympy.matrices import Matrix # Include this in case we want some pretty ma
trices
from sympy.solvers.solveset import linsolve
from sympy import * # import the entire namespace of sympy at root, NOT the bes
t of practices
from math import e, pi
init_printing()  # initialize pretty printing
```

1. Solve the system

$u + v + w = -2$

$3u + 3v - w = 6$

$u - v + w = -1$

In [2]:
```
# use linalg.solve to solve system
A = np.array([[1,1,1],[3,3,-1],[1,-1,1]])
B = np.array([-2,6,-1])
print('Solution: [u,v,w] =', np.linalg.solve(A,B))
```

```
Solution: [u,v,w] = [ 1.5 -0.5 -3. ]
```

In [3]:
```
# let's make this prettier with sympy

u,v,w = symbols('u v w')

# quick and clean
r=linsolve([u+v+w+2, 3*u+3*v-w-6, u-v+w+1], (u,v,w))

# more verbose
A = Matrix([[1,1,1],[3,3,-1],[1,-1,1]])
B = Matrix([-2,6,-1])

rr= linsolve((A,B), (u,v,w))
rr, r #show both results are the same, not really any prettier though
```

Out[3]: $$\left(\left\{\left(\frac{3}{2},\ -\frac{1}{2},\ -3\right)\right\},\ \left\{\left(\frac{3}{2},\ -\frac{1}{2},\ -3\right)\right\}\right)$$

2. Choose h and k such that the system below has

(a) no solution,

(b) a unique solution, and

(c) many solutions.

Give separate answers for each part.

$x_1 + 3x_2 = 2$

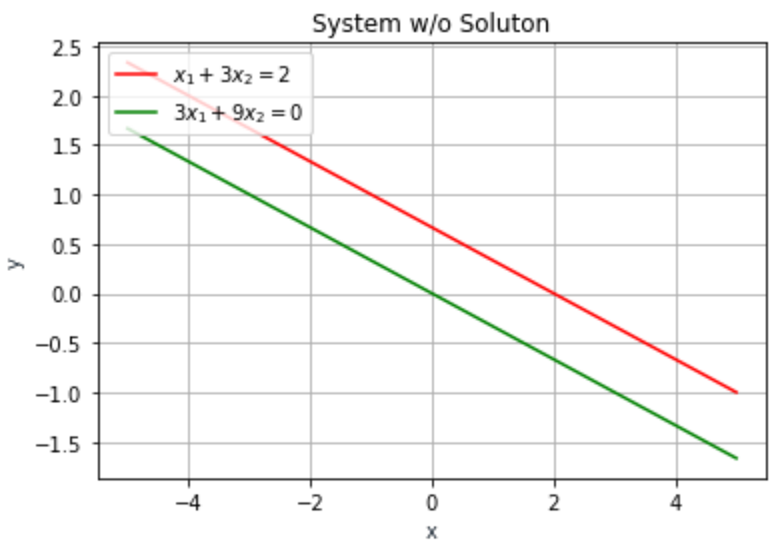
$3x_1 + hx_2 = k$

In [4]:
```
# (a) no solution (parallel)
h=3*3
k=0
print('h = ', h, ', k = ', k)

A = np.array([[1,3], [3,h]])
B = np.array([2,k])

try:
    np.linalg.solve(A,B) # a Singular matrix is expected if no solution
except:
    print('Singular matrix, no solution in this case, but not a suffienct tes
t.')
```

```
h =  9 , k =  0
Singular matrix, no solution in this case, but not a suffienct test.
```

In [5]:

```python
# (a) no solution (parallel) continuned...
# Let's try plotting the equations now
import matplotlib.pyplot as plt
import numpy as np

x1 = np.linspace(-5,5,100)
x2 = (2 - x1)/3
xx2 = (0 - 3*x1)/9
plt.plot(x1, x2, '-r', label='$x_1+3x_2=2$')
plt.plot(x1, xx2, '-g', label='$3x_1+9x_2=0$')

plt.title('System w/o Soluton')
plt.xlabel('x', color='#1C2833')
plt.ylabel('y', color='#1C2833')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

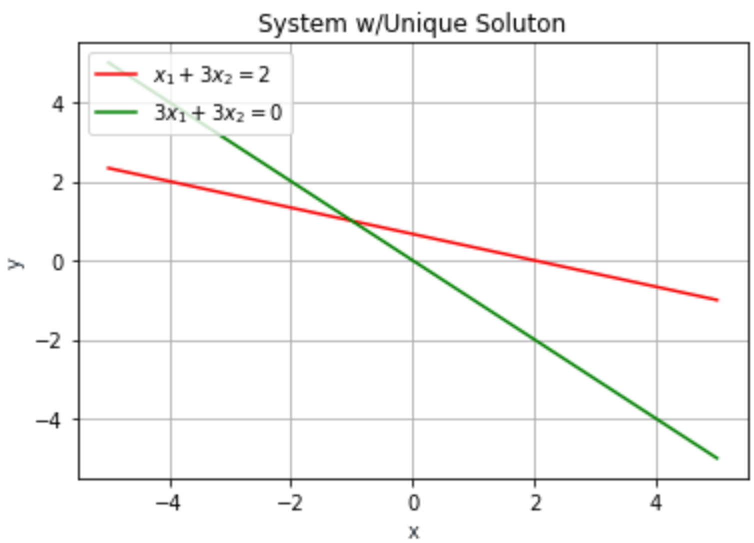


In [6]:

```python
# (b) a unique solution (beams are crossed)
h=3
k=0
print('h = ', h, ', k = ', k)

A = np.array([[1,3], [3,h]])
B = np.array([2,k])

try:
    print('Solution', np.linalg.solve(A,B))
except:
    print('Singular matrix, no solution or many solutions.')
```

```
h =  3 , k =  0
Solution [-1.  1.]
```

In [7]:
```python
# (b) a unique solution (beams are crossed) continuned...
# Let's try plotting the equations now
import matplotlib.pyplot as plt
import numpy as np
h=3
k=0

x1 = np.linspace(-5,5,100)
x2 = (2 - x1)/3
xx2 = (k - 3*x1)/h
plt.plot(x1, x2, '-r', label='$x_1+3x_2=2$')
plt.plot(x1, xx2, '-g', label='$3x_1+3x_2=0$')

plt.title('System w/Unique Soluton')
plt.xlabel('x', color='#1C2833')
plt.ylabel('y', color='#1C2833')
plt.legend(loc='upper left')
plt.grid()
plt.show()

print('BTW: How do you catch an unique animal?')
print('     Unique up on it, of course.')
```

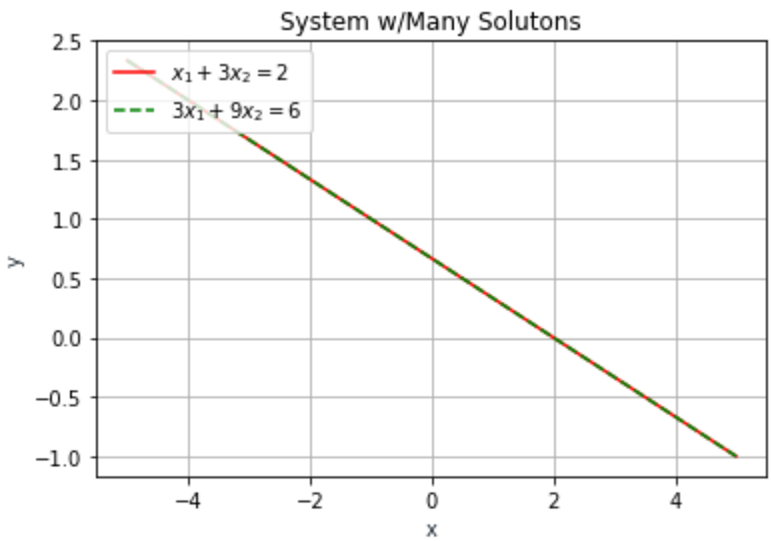



```
BTW: How do you catch an unique animal?
     Unique up on it, of course.
```

In [8]:
```python
# (c) many solutions (coeffs are of equal ratios, meaning lines are on top of e
ach other)
h=3*3
k=2*3
print('h = ', h, ', k = ', k)

A = np.array([[1,3], [3,h]])
B = np.array([2,k])
try:
    print('Solution', np.linalg.solve(A,B))
except:
    print('Singular matrix, many solutions in this case.\nAgain not sufficient
test but demonstrated graphically below.')
```

```
h =  9 , k =  6
Singular matrix, many solutions in this case.
Again not sufficient test but demonstrated graphically below.
```

In [9]:
```
# (c) many solutions continuned...
# Let's try plotting the equations now
import matplotlib.pyplot as plt
import numpy as np
h=9
k=6

x1 = np.linspace(-5,5,100)
x2 = (2 - x1)/3
xx2 = (k - 3*x1)/h
plt.plot(x1, x2, '-r', label='$x_1+3x_2=2$')
plt.plot(x1, xx2, '--g', label='$3x_1+9x_2=6$')

plt.title('System w/Many Solutons')
plt.xlabel('x', color='#1C2833')
plt.ylabel('y', color='#1C2833')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```



3. Consider the system of equations below.

$4x_1 + x_2 + 3x_3 = 9$

$x_1 - 7x_2 - 2x_3 = 12$

$8x_1 + 6x_2 - 5x_3 = 15$

(a) Column-space view:

Find the vectors v_1, v_2, v_3 and write the system as a vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$

In [10]:
```
# think about how best to show this using python
# of course we can just do this by hand, which we will

# use sympy to make some pretty stuff
v1=Matrix([[4],[1],[8]])
v2=Matrix([[1],[-7],[6]])
v3=Matrix([[3],[-2],[-5]])

B = Matrix([[9],[12],[15]])
x1,x2,x3= symbols('x_1 x_2 x_3')
A = x1*v1 + x2*v2 + x3*v3

# show our varibles to demonstrate that A and B bits are equalivent to the syst
em
[v1, v2, v3, A, B]
```

Out[10]:

$$\left[\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 4x_1 + x_2 + 3x_3 \\ x_1 - 7x_2 - 2x_3 \\ 8x_1 + 6x_2 - 5x_3 \end{bmatrix}, \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} \right]$$

(b) Row-space view:

Find the vectors w_1, w_2, w_3 and x such that the system is equivalent to

$w_1 \cdot x = 9$

$w_2 \cdot x = 12$

$w_3 \cdot x = 15$

In [11]:
```
# same idea as above but now with row vectors

# use sympy to make some pretty stuff
w1 = Matrix([[4,1,3]])
w2 = Matrix([[1,-7,-2]])
w3 = Matrix([[8,6,-5]])
b1 = 9
b2 = 12
b3 = 15

x1,x2,x3 = symbols('x_1 x_2 x_3')
x = Matrix([[x1], [x2], [x3]])

a1 = w1*x
a2 = w2*x
a3 = w3*x

# show our varibles to demonstrate that our system is equalivent
[w1,a1,b1], [w2,a2,b2], [w3,a3,b3], x
```

Out[11]:

$$\Bigg([[4 \quad 1 \quad 3], \ [4x_1 + x_2 + 3x_3], \ 9], \ [[1 \quad -7 \quad -2], \ [x_1 - 7x_2 - 2x_3], \ 12], \ [[8 \quad 6 \quad -5],$$

4. Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 6 \\ 1 & -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

In [12]:
```
# by hand we would row reduce the augmented matrix and check if it has a soluti
on
# here we simply show that a solution exits
A = np.array([[1,-2,-6],[0,3,6],[1,-2,5]])
B = np.array([11,-5,9])
np.linalg.solve(A,B)  # it exists therefore it is a linear combination, isn't L
inear Algebra fun with Python?!!
```

Out[12]:
```
array([ 7.3030303 , -1.3030303 , -0.18181818])
```

In [13]:
```
# 4. continued (use symp rref as example)
A = Matrix([[1,-2,-6],[0,3,6],[1,-2,5]])
B = Matrix([11,-5,9])
M=A.col_insert(4,B)

x1,x2,x3 = symbols('x_1 x_2 x_3')
Mrref = M.rref(pivots=False)
R = Mrref.col(-1)
print('Result as float:\n', np.array(Matrix.tolist(R)).astype(np.float64)) # sh
ow float numbers are same as above
Mrref.col(-1), linsolve((A,B), (x1,x2,x3)) # show rref and linsolve provide sam
e result
```

```
Result as float:
 [[ 7.3030303 ]
 [-1.3030303 ]
 [-0.18181818]]
```

Out[13]:

$$\left(\begin{bmatrix} \frac{241}{33} \\ -\frac{43}{33} \\ -\frac{2}{11} \end{bmatrix}, \ \left\{ \left(\frac{241}{33}, \ -\frac{43}{33}, \ -\frac{2}{11} \right) \right\} \right)$$

5. Let $f(z) = az + b$ where $z \in \mathbb{C}$.

Find a and b if $f(z)$ translates z one unit up and one unit to the right,

rotates the result by $\frac{\pi}{2}$ clockwise and

scales the resulting complex number by 2.

In [14]:
```
# Don't we wish we did this HW problem before the quiz?  Anyway, this is for th
e joy of it all.
# The operations again are translate, rotate and scale.
def f(z): return ( (z + (1+1j) * e**(-1j*pi/2)) * 2)
z=0
t= f(z)

a = -2j
b = 2-2j
tt= a * z + b
(t.real, t.imag), (tt.real, tt.imag) # show both hand calc and script are equal
```

Out[14]: $((2.0, \ -2.0), \ (2.0, \ -2.0))$

Appendix 1. Hand Calculations

Time to break out the Cognac and compute like it's 1855.

1

P1) Solve System

$$u + v + w = -2$$
$$3u + 3v - w = 6$$
$$u - v + w = -1$$

Use elimination method on augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 3 & 3 & -1 & 6 \\ 1 & -1 & 1 & -1 \end{array}\right] \Rightarrow \begin{array}{l} r_1 = r_1 \\ r_2 = r_2 - 3r_1 \\ r_3 = r_3 - r_1 \end{array} \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 0 & -4 & 12 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow r_1 = r_1 + \frac{r_2}{2} \begin{bmatrix} 1 & 0 & 1 & -3/2 \\ 0 & 0 & -4 & 12 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow r_1 = r_1 + r_3 \quad [\,1 \;\; 0 \;\; 0 \;\; 1\tfrac{1}{2}$$

$$\begin{bmatrix} 0 & 0 & -4 & 12 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} r_1 = r_1 \\ r_2 = \frac{r_2}{-4} \\ r_3 = \frac{r_3}{-2} \end{array} \begin{bmatrix} 1 & 0 & 0 & 1\frac{1}{2} \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\boxed{\begin{array}{l} u = 1\frac{1}{2} \\ v = -\frac{1}{2} \\ w = -3 \end{array}}$$

P2) Choose h and k such that has given properties

$$x_1 + 3x_2 = 2$$
$$3x_1 + hx_2 = k$$

a) no solution

- this will occurs when eq's are //
- we are going to set the coefficients to be equal ratios of each other but not the result

like this

$$x_1 + 3x_2 = 2$$
$$3x_1 + 3\cdot 3x_2 = 0$$

again w/ the elimination method

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & 9 & 0 \end{array}\right] \Rightarrow \begin{array}{l} r_1 = r_1 \\ r_3 = r_3 - 3r_1 \end{array} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & -6 \end{array}\right]$$

$0 \neq -6$ thus no solution

$\boxed{\text{thus } h=9 \text{ and } k=0 \text{ works!}}$

P2) Continued ...

b) a unique solution

- this occurs when eq's cross
- let's pick a couple easy numbers to calculate.

$$x_1 + 3x_2 = 2$$
$$3x_1 + 3x_2 = 0$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & 3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} r_1 = r_1 \\ r_3 = r_3 - 3r_1 \end{array} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -6 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} r_1 = r_1 \\ r_3 = \frac{r_3}{-6} \end{array} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow r_1 = r_1 - 3r_3 \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$(x_1, x_2) = (-1, 1)$$
is indeed a unique solution

4

P2 continued ...

c) many solution

- this will happen when eq's are on top of each other
- we can cook up some values that work by making the coeff and result have equal ratios, like so...

$x_1 + 3x_2 = 2$

$3x_1 + 3\cdot 3x_2 = 2\cdot 3$

again w/ augmented matrix reduction

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & 9 & 6 \end{array}\right] \Rightarrow \begin{array}{l} r_1 = r_1 \\ r_2 = r_2 - 3r_1 \end{array} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 = 2$ is are the solutions

5

3. Consider system of eq

$4x_1 + x_2 + 3x_3 = 9$

$x_1 - 7x_2 - 2x_3 = 12$

$8x_1 + 6x_2 - 5x_3 = 15$

a) Col space

Find vector v_1, v_2, v_3 and write system as vector eq

$v_1 = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$

$$\boxed{x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}}$$ ✓

b) Row space

Find vector w_1, w_2, w_3 and x

$w_1 = [4\ 1\ 3], \quad w_2 = [1\ -7\ -2] \quad w_3 = [8\ 6\ -5]$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 9 \Rightarrow 4x_1 + x_2 + 3x_3 = 9$$

$$\begin{bmatrix} 1 & 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 12 \Rightarrow x_1 - 7x_2 - 2x_2 = 12$$

$$\begin{bmatrix} 8 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 15 \Rightarrow 8x_1 + 6x_2 - 5x_2 = 15$$

end p3

p4) Determine if b is Lin Combo of A

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 6 \\ 1 & -2 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 6 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \Rightarrow \begin{array}{c} \\ r_2/3 \\ r_3 - r_1 \end{array} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 1 & 2 & -5/3 \\ 0 & 0 & 11 & -2 \end{bmatrix}$$

$-43/33$

$$\Rightarrow \begin{array}{c} r_1 \\ r_2 - \frac{2r_3}{11} \\ r_3/11 \end{array} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 1 & 0 & -5/3 + 4/11 \\ 0 & 0 & 1 & -2/11 \end{bmatrix}$$

$277/33$

$$\begin{array}{c} r_1 + 2r_2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -6 & (11 + 2(-43/33)) \\ 0 & 1 & 0 & -43/33 \\ 0 & 0 & 1 & -2/11 \end{bmatrix}$$

$$\begin{array}{c} r_1 + 6r_3 \\ \\ \end{array} \begin{bmatrix} 1 & & & 277/33 + 6(-2/11) \\ & 1 & & -43/33 \end{bmatrix}$$

$$1 \quad -2/11$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & & & 241/33 \\ & 1 & & -43/33 \\ & & 1 & -2/11 \end{array}\right]$$

✓ There is a solution, so yes this is a Linear Combination

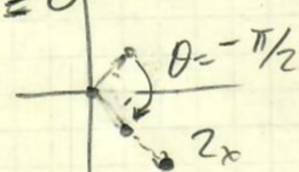
7

P5) $f(z) = az + b$ when $z \in \mathbb{C}$

Find a & b if $f(z)$:

* translation z one unit up & one unit right
* rotation result by $\pi/2$ clockwise
* and scale result by 2

Consider $z = 0$



$$az + b = (z + 1 + i) \cdot e^{-i\pi/2} \cdot 2$$

$$= (2z + 2 + 2i) \cdot \left(\cancel{\cos(-\tfrac{\pi}{2})}^{\,0} + i\left(\cancel{\sin(-\tfrac{\pi}{2})}^{\,-1}\right)\right)$$

$$= (2z + 2 + 2i)(-i)$$

$$= (-2z - 2 - 2i)i$$

$$= (-2i)z - 2i + 2$$

$$\boxed{\begin{aligned} az+b &= (-2i)z + (2 - 2i) \\ a &= -2i \\ b &= (2 - 2i) \end{aligned}}$$

✓