Math 425 Computation Linear Algebra

HW2, Part A

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Linear independence, Span, and Vector spaces.

In [1]:
```
# environment setup
import numpy as np # nice arrays and other stuff

#from sympy import * # import the entire namespace of sympy at root,
NOT the best of practices
import sympy as sym # make is clear which lib I'm using for what
from sympy.matrices import Matrix # Include this in case we want som
e pretty matrices
from sympy.solvers.solveset import linsolve

from math import e, pi
from IPython.display import display, Math, Latex # used to display f
ormatted results
sym.init_printing()  # initialize pretty printing
```

1. List five vectors in $span\{v_1, v_2\}$. For each vector, show the weights on v_1, v_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

In [2]:
```
# our varibles
a = [1,1,1,1,1]
b = [1,0,-1,1j,-1j]  # "You may say I'm a dreamer", Imagine by The B
eatles

#our vectors
v_1 = Matrix([3, 0, 2])
v_2 = Matrix([-2, 0, 3])

display(Latex(f'Five vectors in the $span({sym.latex(v_1)},{sym.late
x(v_2)})$ are:'))
[display(Math(f'({a[i]})v_1 + ({b[i]})v_2 = {sym.latex(a[i] * v_1 +
 b[i] * v_2)}' )) for i in range(len(a))]
display(Latex('Sweet!!! We have $\LaTeX$ formating to working. This
 will come in handy later.'))
```

Five vectors in the $span(\left[\begin{matrix}3\\0\\2\end{matrix}\right],\left[\begin{matrix}-2\\0\\3\end{matrix}\right])$ are:

$$(1)v_1 + (1)v_2 = \left[\begin{matrix}1\\0\\5\end{matrix}\right]$$

$$(1)v_1 + (0)v_2 = \left[\begin{matrix}3\\0\\2\end{matrix}\right]$$

$$(1)v_1 + (-1)v_2 = \left[\begin{matrix}5\\0\\-1\end{matrix}\right]$$

$$(1)v_1 + (1j)v_2 = \left[\begin{matrix}3 - 2.0i\\0\\2 + 3.0i\end{matrix}\right]$$

$$(1)v_1 + ((-0 - 1j))v_2 = \left[\begin{matrix}3 + 2.0i\\0\\2 - 3.0i\end{matrix}\right]$$

Sweet!!! We have $\LaTeX$ formating to working. This will come in handy later.

2. Decide whether the following sets of vectors are linearly independent or linearly dependent. Give reasons for your choices.

2.17 Definition ***linearly independent***

- A list $v_1, \ldots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list ( ) is also declared to be linearly independent.

2.19 Definition ***linearly dependent***

- A list of vectors in V is called ***linearly dependent*** if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

- Def and Lemma from 'Linear Algebra Done Right', By Sheldon Axler

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -11 \\ -8 \end{bmatrix} \right\}$

In [3]:
```
display(Latex('Drawing upon the previous question, our assumption is
that these vectors form the coefficient column Matrix '))
v_1 = Matrix([1,0,3])
v_2 = Matrix([-3,2,-7])
v_3 = Matrix([2,-11,-8])
A = Matrix([v_1.T, v_2.T, v_3.T]).T # form a coefficient matrix
display(Math(f'$A={sym.latex(A)}'))
display(Math(f'A.rref={ sym.latex(A.rref(pivots=False))}'))
display(Latex('a) is indeed linearly independent, and spans $\mathbb
{R^3}$.'))
display(Latex(f"We can further see from the empty nullspace ($A.null
space={sym.latex(A.nullspace())}$) and full rank ($A.rank={sym.latex
(A.rank())}$) that this set Spans %s" %'$\mathbb{R^3}$'))
```

Drawing upon the previous question, our assumption is that these vectors form the coefficient column Matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -11 \\ 3 & -7 & -8 \end{bmatrix}$$

$$A.rref = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) is indeed linearly independent, and spans $\mathbb{R}^3$.

We can further see from the empty nullspace ($A.nullspace = []$) and full rank ($A.rank = 3$) that this set Spans $\mathbb{R}^3$

b) $\left\{ \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -11 \\ -8 \end{bmatrix} \right\}$ ***Not multiples, so we know it's Independent.**

In [4]:
```
display(Latex('Again, our method is to compute the nullspace of a the matrix created from the vectors and compare it to the column space.'))
v_1 = Matrix([-3,2,-7])
v_2 = Matrix([2,-11,-8])
x_1,x_2 = sym.symbols('x_1 x_2')

B = Matrix([v_1.T,v_2.T]).T
display(Latex(f'$B={sym.latex(B.rref(pivots=False))}$'))
display(Latex(f'$B.rref={sym.latex(B.rref(pivots=False))}$'))
display(Latex('b) is indeed a linearly idependent in $\mathbb{R}^2$, and spans $\mathbb{R^2}.$'))
display(Latex(f"We can further see from the empty nullspace ($B.nullspace={sym.latex(B.nullspace())}$) and rank ($B.rank={sym.latex(B.rank())}$) that these vectors Span %s" %'$\mathbb{R^2}$'))
B, B.rank(), B.nullspace()
```

Again, our method is to compute the nullspace of a the matrix created from the vectors and compare it to the column space.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B.rref = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

b) is indeed a linearly idependent in $\mathbb{R}^2$, and spans $\mathbb{R}^2$.

We can further see from the empty nullspace ($B.nullspace = []$) and rank ($B.rank = 2$) that these vectors Span $\mathbb{R}^2$

Out[4]:

$$\left(\begin{bmatrix} -3 & 2 \\ 2 & -11 \\ -7 & -8 \end{bmatrix}, \ 2, \ [] \right)$$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -11 \\ -8 \end{bmatrix}, \begin{bmatrix} 9 \\ 12 \\ 13 \end{bmatrix} \right\}$

In [5]:
```
display(Latex('Drawing upon the previous question, our assumption is that these vectors form the coefficient column Matrix '))
v_1 = Matrix([1,0,3])
v_2 = Matrix([-3,2,-7])
v_3 = Matrix([2,-11,-8])
v_4 = Matrix([9,12,13])


C = Matrix([v_1.T, v_2.T, v_3.T, v_4.T]).T # form a coefficient matrix
display(Math(f'$C={sym.latex(C)}'))
display(Math(f'C.rref={ sym.latex(C.rref(pivots=False))}'))
display(Latex('c) is linearly dependent, and a subspace of $\mathbb{R^4}$.'))
display(Latex(f"We can further see from the empty nullspace ($C.nullspace={sym.latex(C.nullspace())}$) and rank ($C.rank={sym.latex(C.rank())}$) that this set forms a subspace of %s" %'$\mathbb{R^4}$'))
```

Drawing upon the previous question, our assumption is that these vectors form the coefficient column Matrix

$$C = \begin{bmatrix} 1 & -3 & 2 & 9 \\ 0 & 2 & -11 & 12 \\ 3 & -7 & -8 & 13 \end{bmatrix}$$

$$C.rref = \begin{bmatrix} 1 & 0 & 0 & \frac{458}{3} \\ 0 & 1 & 0 & \frac{161}{3} \\ 0 & 0 & 1 & \frac{26}{3} \end{bmatrix}$$

c) is linearly dependent, and a subspace of $\mathbb{R}^4$.

We can further see from the empty nullspace ($C.nullspace = \left[\begin{bmatrix} -\frac{458}{3} \\ -\frac{161}{3} \\ -\frac{26}{3} \\ 1 \end{bmatrix}\right]$) and rank ($C.rank = 3$) that this set forms a subspace of $\mathbb{R}^4$

d) $\left\{ \begin{bmatrix} 1 \\ 4 \\ 9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 1 \\ 0 \end{bmatrix} \right\}$

In [6]:
```
display(Latex('Drawing upon the previous question, our assumption is that these vectors form the coefficient column Matrix '))
v_1 = Matrix([1,4,9,10])
v_2 = Matrix([0,0,0,0])
v_3 = Matrix([5,7,1,0])

D = Matrix([v_1.T, v_2.T, v_3.T]).T # form a coefficient matrix
display(Math(f'$D={sym.latex(D)}'))
display(Math(f'D.rref={ sym.latex(D.rref(pivots=False))}'))
display(Latex('d) is linearly dependent, and a subspace of $\mathbb{R^3}$.'))
display(Latex(f"We can further see from the empty nullspace ($D.nullspace={sym.latex(D.nullspace())}$) and rank ($D.rank={sym.latex(D.rank())}$) that this set forms a subspace of %s" %'$\mathbb{R^3}$'))
display(Latex('Note zero vector, makes this dependent.'))
```

Drawing upon the previous question, our assumption is that these vectors form the coefficient column Matrix

$$D = \begin{bmatrix} 1 & 0 & 5 \\ 4 & 0 & 7 \\ 9 & 0 & 1 \\ 10 & 0 & 0 \end{bmatrix}$$

$$D.rref = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) is linearly dependent, and a subspace of $\mathbb{R}^3$.

We can further see from the empty nullspace ($D.nullspace = \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$) and rank ($D.rank = 2$) that this set forms a subspace of $\mathbb{R}^3$

Note zero vector, makes this dependent.

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

3. Determine if the columns of the matrix form a linearly independent set. Justify each answer.

a) $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$

In [7]:
```python
A = Matrix([[-4,-3,0],[0,-1,4],[1,0,3],[5,4,6]])
display(Math('A=%s' %sym.latex(A) ))
display(Math('A.rref=%s' %sym.latex(A.rref(pivots=False)) ))
display(Latex('a) is linearly independant as there is are pivots in
 each column). Note this is R^3 subspace.'))
display(Latex(f"We can further see from the empty nullspace ($A.null
space={sym.latex(A.nullspace())}$) and rank ($A.rank={sym.latex(A.ra
nk())}$) that this set forms a subspace of %s" %'$\mathbb{R^3}$'))
```

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

$$A.rref = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a) is linearly independant as there is are pivots in each column). Note this is R^3 subspace.

We can further see from the empty nullspace ($A.nullspace = []$) and rank ($A.rank = 3$) that this set forms a subspace of $\mathbb{R}^3$

b) $\begin{bmatrix} 1 & -3 & 3 & 2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

In [8]:
```
B = Matrix([[1,-3,3,2],[-3,7,-1,2],[0,1,-4,3]])
display(Math(f'$B={sym.latex(B)}'))
display(Math(f'B.rref={ sym.latex(B.rref(pivots=False))}'))
display(Latex('b) is linearly dependent, and an inconsistent subspac
e of $\mathbb{R^4}$.'))
display(Latex(f"We can further see from the empty nullspace ($B.null
space={sym.latex(B.nullspace())}$) and rank ($B.rank={sym.latex(B.ra
nk())}$) that this set forms a subspace of %s" %'$\mathbb{R^4}$'))
```

$$B=\begin{bmatrix}1 & -3 & 3 & 2\\ -3 & 7 & -1 & 2\\ 0 & 1 & -4 & 3\end{bmatrix}$$

$$B.rref=\begin{bmatrix}1 & 0 & -9 & 0\\ 0 & 1 & -4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

b) is linearly dependent, and an inconsistent subspace of $\mathbb{R^4}$.

We can further see from the empty nullspace ($B.nullspace=\left[\begin{bmatrix}9\\4\\1\\0\end{bmatrix}\right]$) and rank ($B.rank=3$) that this set forms a subspace of $\mathbb{R^4}$

4. Let $v_1=\begin{bmatrix}1\\-5\\-3\end{bmatrix}, v_2=\begin{bmatrix}-2\\10\\6\end{bmatrix}, v_3=\begin{bmatrix}2\\9\\h\end{bmatrix}$

a) For what values of h **is** v_3 **in** $Span\{v_1, v_2\}$

In [9]:
```
x_1,x_2,x_3,h = sym.symbols('x_1 x_2 x_3 h')
v_1 = Matrix([1,-5,-3])
v_2 = Matrix([-2,10,6])
v_3 = Matrix([2,-9,h])

display(Latex('a) $v_1$ and $v_2$ are multiples of each other so the
re is no multiple of $v_3$ in that subspace.'))
lactation = sym.latex(v_3) + 'not^{*} \in Span^{**}\{' + sym.latex(v
_1) + sym.latex(v_2) + '\} \Leftrightarrow Span\{' + sym.latex(v_1)
+ '\}'
display(Latex('No value of $h$ puts $v_3$ in $Span\{v_1,v_2\}$, or r
ather $%s$.' %lactation))
display(Latex("* \$\\notin\$ isn't rendered in sympy.latex(), file b
ug report as there are lot of standard notation messing."))
display(Latex('** Ask about about this and learn the correct Maths t
erminology.  Maybe I already know it.  How much do any of us know an
yway?'))
display(Latex('*** Ask your Mathematician if ${subspaces}$ are right
for you.'))
```

a) v_1 and v_2 are multiples of each other so there is no multiple of v_3 in that subspace.

No value of h puts v_3 in $Span\{v_1,v_2\}$, or rather $\left[\begin{matrix}2\\-9\\h\end{matrix}\right]not^{*} \in Span^{**}\{\left[\begin{matrix}1\\-5\\-3\end{matrix}\right]\left[\begin{matrix}-2\\10\\6\end{matrix}\right]\} \Leftrightarrow Span\{\left[\begin{matrix}1\\-5\\-3\end{matrix}\right]\}$.

* $\notin$ isn't rendered in sympy.latex(), file bug report as there are lot of standard notation messing.

** Ask about about this and learn the correct Maths terminology. Maybe I already know it. How much do any of us know anyway?

*** Ask your Mathematician if $subspaces$ are right for you.

b) For what values of h is v_1, v_2, v_3 lineraly dependent?

In [10]:
```
x_1,x_2,x_3, h = sym.symbols('x_1 x_2 x_3 h')
v_1 = Matrix([1,-5,-3])
v_2 = Matrix([-2,10,6])
v_3 = Matrix([2,-9,h])
A = Matrix([v_1.T,v_2.T,v_3.T]).T
B = Matrix([0,0,0])

display(Latex(f'Thus our Linear system is $A={sym.latex(A)}$'))
display(Latex(f'Row reduction results in $A.rref={sym.latex(A.rref
())}$ This system inconsistant no value of $h$ is going to help this
system.'))
```

Thus our Linear system is $A = \begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix}$

Row reduction results in $A.rref = \left(\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ (0, \ 2) \right)$ This system inconsistant no value of h is going to help this system.

5. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $Ax = 0$.

In [11]:
```
A = Matrix([[4,1,6], [-7,5,3], [9,-3,3]])
display(Math(f'A.rref={sym.latex(A.rref(pivots=False))}'))
display(Latex('Okay, looks like we have $x_1+x_3 =0$ and $x_2+2x_3=0
$.'))
x_1,x_2,x_3 = sym.symbols('x_1 x_2 x_3')
x = Matrix([-x_3,-2*x_3,x_3])
display(Latex(f"Thus, our non-trival solution is $x={sym.latex(x)}$"
))
display(Latex(f"Also, note how the nullspace is related to this solu
tion, $N={sym.latex(A.nullspace())}$"))
```

$$A.rref = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Okay, looks like we have $x_1+x_3 =0$ and $x_2+2x_3=0$.

Thus, our non-trival solution is $x=\begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$

Also, note how the nullspace is related to this solution, $N=\left[\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right]$

In [12]:
```
display(Latex('Now do it symbolically, as we are questing for a deep
er understanding.'))
x_1,x_2,x_3 = sym.symbols('x_1 x_2 x_3')
x = Matrix([x_1,x_2,x_3])
M = A*x
B= Matrix([0,0,0])
moo = list(sym.linsolve((A,B), (x_1,x_2,x_3))) #
display(Latex(f'$x={sym.latex(moo)}$, confirming our results from ab
ove.'))
```

Now do it symbolically, as we are questing for a deeper understanding.

$x=\left[\left(-x_3,\ -2x_3,\ x_3\right)\right]$, confirming our results from above.

6. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

In [13]:
```
display(Latex("No, we'll need at least 4 independent vector to span
 $\mathbb{R^4}$.  Like I was saying, we need $n$ independent vector
s, $n \geq m$ to span $\mathbb{R^m}$.  Otherwise we'll just have a s
ubspace of $\mathbb{R^n}$ and won't be able to reach every point."))
```

No, we'll need at least 4 independent vector to span $\mathbb{R^4}$. Like I was saying, we need n independent vectors, $n \geq m$ to span $\mathbb{R^m}$. Otherwise we'll just have a subspace of $\mathbb{R^n}$ and won't be able to reach every point.

7. Let $u = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is u in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or not?

In [14]:
```
u = Matrix([2,-3,2])
A = Matrix([[5,8,7],[0,1,-1],[1,3,0]])
display(Latex(f'$A={sym.latex(A)}$'))
display(Math(f'A.ref={sym.latex(A.rref(pivots=False))}, A.rank={sym.
latex(A.rank())}'))
display(Latex("Now append $u$ to A and see how our span and rank cha
nges."))
Au=A.col_insert(3,u)  # right?
display(Latex(f'$Au={sym.latex(Au)}$'))
display(Math(f'Au.ref={sym.latex(Au.rref(pivots=False))},  Au.rank=
{sym.latex(Au.rank())}'))
display(Latex("Now by inspection$^{*}$ we can clearly see that $u$ i
s not in the subspace formed by $A$."))
display(Latex("* engineers are rather lazy like this, physicists are
even worst, and don't even ask about artists."))
display(Latex("Anyway, no $u$ is not in the $span(A)$ by the aforeme
ntioned obserations$^{**}$ of rank and span.\n"))
display(Latex("** Honestly, it really is at lot of fun to play with
 Linear Algebra in this way."))
```

$$A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

$$A.ref = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, A.rank = 2$$

Now append u to A and see how our span and rank changes.

$$Au = \begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix}$$

$$Au.ref = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Au.rank = 3$$

Now by inspection* we can clearly see that u is not in the subspace formed by A.

* engineers are rather lazy like this, physicists are even worst, and don't even ask about artists.

Anyway, no u is not in the $span(A)$ by the aforementioned obserations** of rank and span.

** Honestly, it really is at lot of fun to play with Linear Algebra in this way.

8. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible b, and describe the set of b for which $Ax = b$ does have a solution.

In [15]:
```
print('FIXME!!! Discribe set of b where eq has solution')
b_1,b_2,b_3 = sym.symbols('b_1 b_2 b_3')
b = Matrix([b_1,b_2,b_3])
A = Matrix([[1,-3,-4],[-3,2,6],[5,-1,-8]])
display(Latex(f'$A.rref={sym.latex(A.rref()) }$')) # Two pivots
display(Latex("By inspection we can see the rank of $A$ is 2, thus
 $A$ a subspace of $\mathbb{R^3}$. This means we can only 'reach' in
to the $span\{A\}$ where $b_3=0$."))
sym.linsolve((A,B), (b_1,b_2,b_3)) # oh that's interesting

display(Latex("This system is only consistent if b3 = 0."))
display(Latex("The counter example of b=[1,1,1] is inconsistant."))
```

```
FIXME!!! Discribe set of b where eq has solution
```

$$A.rref = \left(\begin{bmatrix} 1 & 0 & -\frac{10}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & 0 \end{bmatrix}, \ (0,\ 1) \right)$$

By inspection we can see the rank of A is 2, thus A a subspace of $\mathbb{R}^3$. This means we can only 'reach' into the $span\{A\}$ where $b_3 = 0$.

This system is only consistent if b3 = 0.

The counter example of b=[1,1,1] is inconsistant.

9.Let $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$.

Do the columns of B span $\mathbb{R}^4$?

Does the equation $Bx = y$ have a solution for each $y \in \mathbb{R}^4$?

In [16]:
```
display(Latex("Now that I've got this hammer everything looks a nail
s."))
B = Matrix([[1,3,-2,2],[0,1,1,-5],[1,2,-3,7],[-2,-8,2,-1]])
display(Math(f"B={sym.latex(B)}, B.rref={sym.latex(B.rref(pivots=Fal
se))}, B.rank={sym.latex(B.rank())}"))
display(Latex("Nope, $B$ doesn't span{$\mathbb{R^4}$}, as evident by
it's rank."))
display(Latex("No, $Bx=y$ only has solution where $x_4=0$, so this s
ystem is inconsistent"))
display(Latex(f"Looking at $B.nullspace={sym.latex(B.nullspace())}$,
provides further evience of the conclusion."))
```

Now that I've got this hammer everything looks a nails.

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}, B.rref = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B.rank = 3$$

Nope, B doesn't span{$\mathbb{R}^4$}, as evident by it's rank.

No, $Bx = y$ only has solution where $x_4 = 0$, so this system is inconsistent

Looking at $B.nullspace = \left[\begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right]$, provides further evience of the conclusion.

Appendix 1. Playing with displaying with LaTex

In [17]:

```
from IPython.display import display, Math, Latex
display(Math(r'F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx'
))
```

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx$$

In [18]:

```
x = sym.symbols('x')
expr = sym.sqrt(3) * x**3

display(Math('\\frac{1}{2} ') )
display(expr)
display(Math('\\frac{1}{2} %s' %sym.latex(expr)))
display(Math('\\frac{1}{2} %s' %sym.latex(expr)))
```

$$\frac{1}{2}$$

$$\sqrt{3} x^{3}$$

$$\frac{1}{2} \sqrt{3} x^{3}$$

$$\frac{1}{2} \sqrt{3} x^{3}$$

Appendix 2. Exploring the Nullspace and Span relationships

In [19]:
```
A=Matrix([[1,1,1,-1],[2,4,5,6],[3,9,5,4]])
display(Math(f'A={sym.latex(A)}'))
display(Math(f'A.rref={sym.latex(A.rref(pivots=False))}'))
lactation = sym.latex(A.nullspace())  # Milk it, like a cow!
x_1,x_2,x_3,x_4 = sym.symbols('x_1, x_2, x_3, x_4') # Sprinkle some
 sugar.
x = Matrix([x_1, x_2, x_3, x_4])  # X marks the spot.
display(Math(f'{sym.latex(x)}=x_4*{sym.latex(A.rref(pivots=False).co
l(3))} '))
display(Math(f'N(A)=Span{lactation}') ) # Moo!
n = A.shape[1]
rank = A.rank()
nullity = n - rank
print("Nullity: ", nullity)
print("Rank: ", rank)
```

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & 5 & 6 \\ 3 & 9 & 5 & 4 \end{bmatrix}$$

$$A.rref = \begin{bmatrix} 1 & 0 & 0 & -\frac{53}{14} \\ 0 & 1 & 0 & \frac{5}{14} \\ 0 & 0 & 1 & \frac{17}{7} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 * \begin{bmatrix} -\frac{53}{14} \\ \frac{5}{14} \\ \frac{17}{7} \end{bmatrix}$$

$$N(A) = Span \left[\begin{bmatrix} \frac{53}{14} \\ -\frac{5}{14} \\ -\frac{17}{7} \\ 1 \end{bmatrix} \right]$$

```
Nullity:  1
Rank:  3
```

Appendix 3. Eigeness

In [20]:
```
display(Latex("Let's explore the Eigeness of this all.")) # show ver
bose way
lambda_ = sym.symbols('lambda_')
A = Matrix([[4,1,6], [-7,5,3], [9,-3,3]])
P = sym.det(A-lambda_*sym.eye(3))
P_ = sym.factor(sym.Eq(P,0))
display(Latex("Discuss results below and get Mathmatical."))
display(Latex("We ought to discuss change of basis$^*$ and the like,
but there is so much that we've forgotten!"))
display(Latex(" *This is out of scope for this problem."))
A, A.rref(), A.rank(), P, P_, sym.solve(P_, lambda_) # chararteristi
c polynomial and roots
```

Let's explore the Eigeness of this all.

Discuss results below and get Mathmatical.

We ought to discuss change of basis* and the like, but there is so much that we've forgotten!

*This is out of scope for this problem.

Out[20]:

$$\left(\begin{bmatrix}4 & 1 & 6\\-7 & 5 & 3\\9 & -3 & 3\end{bmatrix},\ \left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix},\ (0,\ 1)\right),\ 2,\ -\lambda^3 + 12\lambda^2 - 9\lambda,\ -\lambda$$