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Information and coding theory - Project 1

Appendix - Details of the calculation by hand

Exercise 1

We know that

$$P(x, y) = P(x) \cdot P(y)$$

This yields

X_i \ Y_j	0	1	2	3
0	$\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$
1	$\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$
2	$\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$
3	$\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$	$\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$

We know that

$$P(S = S_i) = \sum_{X_j, Y_k \mid S_i = X_j + Y_k} P(x = X_j, y = Y_k)$$

Hence,

S_i	$P(S = S_i)$
0	$1/8$
1	$(1/16) + (1/8) = 3/16$
2	$(1/32) + (1/16) + (1/8) = 7/32$
3	$2 \cdot (1/32) + (1/16) + (1/8) = 8/32 = 1/4$
4	$2 \cdot (1/32) + (1/16) = 4/32 = 1/8$
5	$2 \cdot (1/32) = 1/16$
6	$1/32$

We know that

$$P(Z = Z_i) = \sum_{X_j, Y_k \mid Z_i = 1(X_j = Y_k)} P(X = X_j, Y = Y_k)$$

Hence,

Z_i	$P(Z = Z_i)$
0	$6(1/32) + 3(1/16) + 3(1/8) = 24/32 = 3/4$
1	$2(1/32) + (1/16) + (1/8) = 8/32 = 1/4$

Exercise 2

- $H(X) = -\sum_{X_i} P(X_i) \log_2 P(X_i)$

$$= -\left[\left(\frac{1}{4}\log_2 \frac{1}{4}\right) \cdot 4\right]$$

$$= 2$$

- $H(Y) = -\sum_{Y_i} P(Y_i) \log_2 P(Y_i)$

$$= -\left[\frac{1}{2}\log_2 \frac{1}{2} + \frac{1}{4}\log_2 \frac{1}{4} + 2\left(\frac{1}{8}\log_2 \frac{1}{8}\right)\right]$$

$$= 1,75$$

- $H(S) = -\sum_{S_i} P(S_i) \cdot \log_2 P(S_i)$

$$= -\left[\frac{1}{8} \cdot \log_2 \frac{1}{8} + \frac{3}{16}\log_2 \frac{3}{16} + \frac{7}{32}\log_2 \frac{7}{32} + \frac{1}{4}\log_2 \frac{1}{4} + \frac{1}{8}\log_2 \frac{1}{8} + \frac{1}{16}\log_2 \frac{1}{16} + \frac{1}{32}\log_2 \frac{1}{32}\right]$$

$$= 2,5887$$

- $H(Z) = -\sum_{Z_i} P(Z_i) \cdot \log_2 P(Z_i)$

$$= -\left[\frac{3}{4}\log_2 \frac{3}{4} + \frac{1}{4}\log_2 \frac{1}{4}\right]$$

$$= 0,8113$$

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Calculations

For the rest of the developments, we will need:

- $P(X, S)$
- $P(Y, Z)$
- $P(S, Z)$
- $P(Z, X)$

Let's calculate them.

We know that

- $P(X, S) = \sum_Y P(X, Y, S)$
- $P(Y, Z) = \sum_X P(X, Y, Z)$
- $P(Z, X) = \sum_Y P(X, Y, Z)$

so let's calculate $P(X, Y, S)$ and $P(X, Y, Z)$.

$P(X, Y, S)$

$P(X, Y, S) = P(X, Y) . 1 (S = X + Y)$

→ equals 0 when $S \neq X + Y$ so we only calculate value when $X + Y = S$

S_k	X_i	Y_j	$P(X_i, Y_j, S_k)$ when $S_k = X_i + Y_j$
0	0	0	1/8
1	0	1	1/16
	1	0	1/8
2	0	2	1/32
	2	0	1/8
	1	1	1/16
3	0	3	1/32
	3	0	1/8
	1	2	1/32
	2	1	1/16
4	1	3	1/32
	3	1	1/16
	2	2	1/32
5	2	3	1/32
	3	2	1/32
6	3	3	1/32

$P(X, Y, Z)$

When $Z_k = 1$, we only keep values of $P(X, Y)$ where $X_i = Y_j$ (the rest is equal to 0)

When $Z_k = 0$, we only keep values of $P(X, Y)$ where $X_i \neq Y_j$ (the rest is equal to 0)

So, we have

Z_k	X_i	Y_j	$P(X_i, Y_j, Z_k)$
1	0	0	1/8
	1	1	1/16
	2	2	1/32
	3	3	1/32
0	0	1	1/16
	0	2	1/32
	0	3	1/32
	1	0	1/8
	1	2	1/32
	1	3	1/32
	2	0	1/8
	2	1	1/16
	2	3	1/32
	3	0	1/8
	3	1	1/16
	3	2	1/32

$P(X, S)$

$$P(X, S) = \sum_Y P(X, Y, S)$$

It gives the same values as in table $P(X, Y, S)$ because for one couple (X_i, S_j) we only have one value Y_k.

$P(Y, Z)$

$$P(Y, Z) = \sum_X P(X, Y, Z)$$

		Y_i 0	1	2	3
Z_j	0	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{8}$	$\frac{1}{16}+\frac{1}{16}+\frac{1}{16} = \frac{3}{16}$	$\frac{1}{32}+\frac{1}{32}+\frac{1}{32} = \frac{3}{32}$	$\frac{1}{32}+\frac{1}{32}+\frac{1}{32} = \frac{3}{32}$
	1	1/8	1/16	1/32	1/32

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$P(Z, X)$

$$P(Z, X) = \sum_{Y} P(X, Y, Z)$$

Z_j \ X_i	0	1	2	3
0	$\frac{1}{16} + \frac{1}{32} + \frac{1}{32} = \frac{4}{32}$	$\frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{6}{32}$	$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{7}{32}$	$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{7}{32}$
1	1/8	1/16	1/32	1/32

$P(S, Z)$

$$P(S, Z) = \sum_{X, Y} P(X, Y, S, Z)$$

To calculate these values, one thinks according to S and Z.

When $S = S_i$, we take all the values in $P(X, Y)$ such that $X_j + Y_k = S_i$. Let's call this set of values W.

Then, when we have W, if $Z_i = 0$, we only keep values in W such that $X_j \neq Y_k$ and we sum them. If $Z_i = 1$, we do the same but with values such that $X_j = Y_k$.

If there is no value, the sum equals 0.

This gives us:

Z_j \ S_i	0	1	2	3	4	5	6
0	0	3/16	5/32	8/32	3/32	2/32	0
1	1/8	0	1/16	0	1/32	0	1/32

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Exercise 3

- $H(X,Y) = -\sum_{x_i, y_j} P(x_i, y_j) \log_2 P(x_i, y_j)$

$$= -\left[4\left(\frac{1}{8}\log_2\frac{1}{8}\right) + 4\left(\frac{1}{16}\log_2\frac{1}{16}\right) + 8\left(\frac{1}{32}\log_2\frac{1}{32}\right)\right]$$

$= 3{,}75$

- $H(X,S) = -\sum_{x_i, s_j} P(x_i, s_j) \log_2 P(x_i, s_j)$

$$= -\left[\left(\frac{1}{8}\cdot\log_2\frac{1}{8}\right)\cdot 4 + \left(\frac{1}{16}\cdot\log_2\frac{1}{16}\right)\cdot 4 + \left(\frac{1}{32}\cdot\log_2\frac{1}{32}\right)\cdot 8\right]$$

$= 3{,}75$

- $H(Y,Z) = -\sum_{y_i, z_j} P(y_i, z_j) \cdot \log_2 P(y_i, z_j)$

$$= -\left[\frac{3}{8}\log_2\frac{3}{8} + \frac{3}{16}\log_2\frac{3}{16} + 2\left(\frac{3}{32}\log_2\frac{3}{32}\right)\right.$$

$$\left. + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + 2\left(\frac{1}{32}\log_2\frac{1}{32}\right)\right]$$

$= 2{,}5613$

- $H(S,Z) = -\sum_{s_i, z_j} P(s_i, z_j) \log_2 P(s_i, z_j)$

$$= -\left[\frac{3}{16}\log_2\frac{3}{16} + \frac{5}{32}\log_2\frac{5}{32} + \frac{8}{32}\log_2\frac{8}{32} + \frac{3}{32}\log_2\frac{3}{32}\right.$$

$$\left. + \frac{2}{32}\log_2\frac{2}{32} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + 2\left(\frac{1}{32}\log_2\frac{1}{32}\right)\right]$$

$= 2{,}8789$

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Exercise 4

- $H(X|Y) = H(X,Y) - H(Y)$

$= 3{,}75 - 1{,}75$

$= 2$

- $H(Z|X) = \underbrace{H(Z,X)} - H(X)$

We need $H(Z,X) = -\sum_{z_i, x_j} \underbrace{P(z_i, x_j)} \log_2 \underbrace{P(z_i, x_j)}$

We need $P(z_i, x_j)$!
→ already calculated !

and so,

$$H(Z,X) = -\left[2\left(\frac{7}{32}\log_2\frac{7}{32}\right) + 2\left(\frac{1}{32}\log_2\frac{1}{32}\right) + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + \frac{4}{32}\log_2\frac{4}{32} + \frac{6}{32}\log_2\frac{6}{32}\right] = 2{,}7246$$

finally,

$H(Z|X) = 0{,}7246 = 2{,}7246 - 2$

- $H(S|X) = H(S,X) - H(X)$

$= H(X,S) - H(X)$

$= 3{,}75 - 2$

$= 1{,}75$

- $H(S|Z) = H(S,Z) - H(Z)$

$= 2{,}8789 - 0{,}8113$

$= 2{,}0676$

Exercise 5

- $H(X, Y \mid S) = H(X, Y, S) - H(S)$
- $H(S, Y \mid X) = \underbrace{H(S, Y, X)}_{} - H(X)$

We need $H(X, Y, S)$!

$$H(X, Y, S) = -\sum_{x_i, y_j, s_k} \underbrace{P(x_i, y_j, s_k)}_{} \log_2 \underbrace{P(x_i, y_j, s_k)}_{}$$

We need $P(x_i, y_j, s_k)$!

$$P(X, Y, S) = P(X, Y) \cdot 1(S = X + Y)$$

→ equals 0 when $S \neq X + Y$ so we only compute value when $X + Y = S$.

s_k	x_i	y_j	$P(x_i, y_j, s_k)$ when $s_k = x_i + y_j$
0	0	0	1/8
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	1	0	1/8
2	0	2	1/32
	2	0	1/8
	1	1	1/16
3	0	3	1/32
	3	0	1/8
	1	2	1/32
	2	1	1/16
4	1	3	1/32
	3	1	1/16
	2	2	1/32
5	2	3	1/32
	3	2	1/32
6	3	3	1/32

(already calculated previously in fact)

and so,

$$H(X, Y, S) = -\left[4\left(\frac{1}{8}\log_2\frac{1}{8}\right) + 4\left(\frac{1}{16}\log_2\frac{1}{16}\right) + 8\left(\frac{1}{32}\log_2\frac{1}{32}\right)\right]$$

$$= 3{,}75$$

finally,

$$H(X, Y \mid S) = 3{,}75 - 2{,}5887 = 1{,}1613$$

$$H(S, Y \mid X) = 3{,}75 - 2 = 1{,}75$$

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Exercise 6

- $I(X;Y) = H(X) - H(X|Y)$
$= 2 - 2$
$= 0$

- $I(X;S) = H(S) - H(S|X)$
$= 2,5887 - 1,75$
$= 0,8387$

- $I(Y;Z) = H(Y) + H(Z) - H(Y,Z)$
$= 1,75 + 0,8113 - 2,5613$
$= 0$

- $I(S;Z) = H(S) - H(S|Z)$
$= 2,5887 - 2,0676$
$= 0,5211$

Exercise 7

- $I(X;Y|S) = H(X|S) - H(X|Y,S)$
$= [H(X,S) - H(S)] - [H(X,Y,S) - H(Y,S)^{*}]$
$= (3,75 - 2,5887) - (3,75 - 3,75)$
$= 1,1613$

- $I(S;Y|X) = H(S|X) - H(S|Y,X)$
$= H(S|X) - [H(S,Y,X) - H(Y,X)]$
$= 1,75 - (3,75 - 3,75)$
$= 1,75$

* $H(Y,S)$ was calculated in the same way as $H(X,S)$ in y (the value of $P(X)$ has been replace by those of $P(Y)$).

A. S.