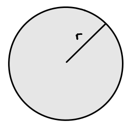
# Measure Theory

Concerned with:

- measuring the volumes of sets,
- integration of functions.

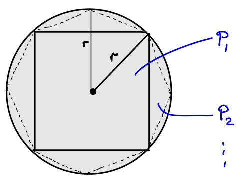
## 1. Intro

Measuring volume can be tricky:




volume = ?

Idea: approximate circle by simpler objects whose volume we can compute:




$m(P_k) :=$ volume of $P_k$, e.g. $m(P_1) = \left(\frac{r}{\sqrt{2}}\right)^2 \cdot 4$.

Clearly: • $m(P_k)$ increasing with $k$ (since $P_k \subset P_{k+1}$)

• $m(P_k)$ bounded above

$\Rightarrow \lim\limits_{k\to\infty} m(P_k)$ exists!

$$m(\text{disc}) := \lim_{k\to\infty} m(P_k)$$

$\rightsquigarrow$ what about even weirder sets, e.g. $\mathbb{Q} \cap [0,1]$ ?

---

Integration of functions can be tricky:

- $f(x) := \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{else} \end{cases}$

  is not Riemann integrable $\rightsquigarrow$ Drawback

- Riemann integral of sequence $(f_n)$:

  $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$ only if $f_n \to f$ uniformly.

very strong requirement $\rightsquigarrow$ Drawback

Abstract measure theory solves these issues!

## 2. σ-Algebras and Measures

Conventions:

$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$

$\infty \cdot \infty = \infty$

$a + \infty = \infty \quad \forall a \in \mathbb{R}$

$a \cdot \infty = \begin{cases} \infty, & a > 0 \\ -\infty, & a < 0 \end{cases}$

$0 \cdot \infty = 0$

$\infty - \infty$ not defined!

Notation:

$X$ set, $\mathcal{P}(X)$ = power set of $X$, for $A \subset X$: $A^c = X \setminus A$.

Disjoint union: $C = A \dot{\cup} B \iff C = A \cup B$ and $A \cap B = \emptyset$.

Def 2.1:

Let $X$ be a set. A <u>σ-algebra</u> on $X$ is a family $\Sigma \subset \mathcal{P}(X)$ s.t.

(i) $X \in \Sigma$

(ii) $S \in \Sigma \Rightarrow X \setminus S \in \Sigma$

(iii) $S_n \in \Sigma \;\forall n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \Sigma$

Def 2.2:

A <u>measurable space</u> is a pair $(X, \Sigma)$, where $X$ is a set and $\Sigma$ is a σ-algebra on $X$.

Def 2.3:

$\Sigma \subset \mathcal{P}(X)$ is called an <u>algebra</u> iff

(i) $X \in \Sigma$

(ii) $S \in \Sigma \Rightarrow X \setminus S \in \Sigma$

(iii) $S, S' \in \Sigma \Rightarrow S \cup S' \in \Sigma$.

**Remark:**

If $\Sigma$ is a $\sigma$-algebra on $X$, then

(a) $\emptyset \in \Sigma$

(b) $S_n \in \Sigma \Rightarrow \bigcap_{n=1}^{\infty} S_n \in \Sigma$

(c) $\Sigma$ is an algebra

(d) $A, B \in \Sigma \Rightarrow A \setminus B \in \Sigma$.

**Example:**

$X$ any set, then

$\Sigma = \{X, \emptyset\}$

- $\Sigma = \mathcal{P}(X)$

are $\sigma$-algebras.

**Lemma 2.6:**

$X$ a set, $I$ set of indices, $\{\Sigma_\alpha\}_{\alpha \in I}$ any family of $\sigma$-algebras on $X$. Then $\bigcap_{\alpha \in I} \Sigma_\alpha$ is a $\sigma$-algebra.

**Proof:** clear. $\square$

**Caution:** Unions of $\sigma$-algebras are not $\sigma$-algebras in general!

**Def 2.7:**

$X$ a set, $\mathcal{S} \subset \mathcal{P}(X)$. Then

$$\sigma(\mathcal{S}) := \bigcap_{\substack{\Sigma \ \sigma\text{-algebra} \\ \mathcal{S} \subset \Sigma}} \Sigma$$

is called the <u>$\sigma$-algebra generated by $\mathcal{S}$</u>.

Def 2.8

Let $X$ metric space. Then the $\sigma$-algebra generated by open sets of $X$ is called the Borel $\sigma$-algebra of $X$, denoted $\mathcal{B}(X)$.
$S \in \mathcal{B}(X)$ is called Borel set.

Lemma 2.9:

$X$ metric space, then every closed $S \subset X$ is a Borel set.

Def 2.10:

(i) Countable intersections of open sets are $\mathcal{G}_\delta$ - sets

(ii) Countable unions of closed sets are $\mathcal{F}_\sigma$ - sets.

Example:

$$[0,1) = \bigcup_{n=2}^{\infty} [0, 1-\tfrac{1}{n}] = \bigcap_{n=1}^{\infty} (-\tfrac{1}{n}, 1)$$

$\Longrightarrow$ $[0,1]$ is both $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$.

**Lemma 2.11**

Let $I := \{(\alpha, \infty) \mid \alpha \in \mathbb{R}\}$. Then $\sigma(I) = \mathcal{B}(\mathbb{R})$.

**Proof:** Exercise.

Sketch: Step 1: all open intervals are in $\sigma(I)$:

- $(-\infty, \beta] \in \sigma(I) \Rightarrow (-\infty, \beta) \in \sigma(I)$
  $\Rightarrow (\alpha, \beta) \in \sigma(I) \quad \forall \alpha, \beta \in \mathbb{R}$.

Step 2: any open set is union of open intervals with rational endpoints.

$\square$

**Def 2.12:**

Let $(X, \Sigma)$ be a measurable space. A map $\mu : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ is called a <u>measure</u> on $(X, \Sigma)$ if

(i) $\mu \geq 0$

(ii) $\mu(\emptyset) = 0$

(iii) if $\{S_n\}_{n=1}^{\infty} \subset \Sigma$ are <u>disjoint,</u> then $\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n)$.

The triple $(X, \Sigma, \mu)$ is called a <u>measure space</u>.

**Def 2.13:**

Let $(X, \Sigma, \mu)$ be a measure space.

(i) Elements of $\Sigma$ will often be called <u>measurable sets</u>

(ii) If $\mu(X) < \infty$ then $\mu$ is called a <u>finite</u> measure

(iii) If $X$ can be written as the union of countably many sets of finite measure, then $\mu$ is said to be <u>$\sigma$-finite</u>

Examples:

- Dirac measure:

  Let $X$ be a set, $x_0 \in X$ and

$$\delta_{x_0} : \mathcal{P}(X) \longrightarrow [0,\infty),$$
$$\delta_{x_0}(S) = \begin{cases} 1, & x_0 \in S \\ 0, & x_0 \notin S . \end{cases}$$

  Then $(X, \mathcal{P}(X), \delta_{x_0})$ is a measure space.

- Counting measure:

  Let $X$ be a set. The counting measure on $\mathcal{P}(X)$ is defined by setting

$$\mu(S) = \begin{cases} \text{number of elements of } S, & \text{if } S \text{ finite} \\ \infty & , \text{ else} \end{cases}$$

**Theorem 2.15:**

Let $(X, \Sigma, \mu)$ be a measure space.

(i) If $U, V \in \Sigma$ with $U \subset V$ and $\mu(U) < \infty$, then $\mu(V \setminus U) = \mu(V) - \mu(U)$.

(ii) if $U, V \in \Sigma$ with $U \subset V$, then $\mu(U) \leq \mu(V)$. (monotonicity)

(iii) if $(V_n)_{n=1}^{\infty}$ is a sequence in $\Sigma$, then $\mu\left(\bigcup_{n=1}^{\infty} V_n\right) \leq \sum_{n=1}^{\infty} \mu(V_n)$ (countable subadditivity)

(iv) if $U, V \in \Sigma$ and $\mu(U \cap V) < \infty$, then $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(U \cap V)$

(v) if $V_1, \dots, V_n \in \Sigma$ and $\mu(V_i) < \infty \ \forall i$, then

$$\mu\left(\bigcup_{i=1}^{n} V_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \mu\left(\bigcap_{j=1}^{k} V_{i_j}\right)$$

**Proof:**

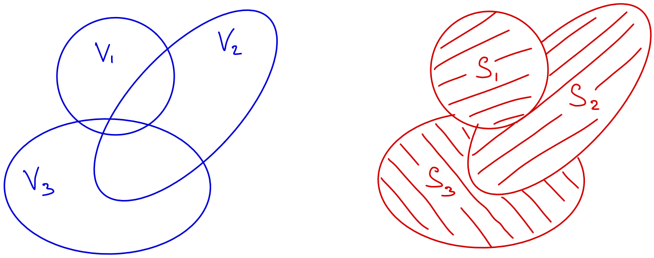
(i): $V = \underbrace{U \cup (V \setminus U)}_{\text{disjoint}} \implies \mu(V) = \underbrace{\mu(U)}_{< \infty} + \mu(V \setminus U).$

(ii): $\mu(V) = \mu(U) + \underbrace{\mu(V \setminus U)}_{\geq 0}$

$\geq \mu(U)$

(iii): Write $S_1 = V_1$, $S_2 = V_2 \setminus V_1$, ..., $S_n = V_n \setminus (V_1 \cup \dots \cup V_{n-1})$

$\rightsquigarrow$ all $S_n$ are disjoint members of $\Sigma$ and $\bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} V_n$




$$\Rightarrow \quad \mu\left(\bigcup_{n=1}^{\infty} V_n\right) = \mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n) \underset{\text{monotonicity}}{\leq} \sum_{n=1}^{\infty} \mu(V_n)$$

(iv): $U \cup V = U \cup (V \setminus (U \cap V))$

$$\Rightarrow \mu(U \cup V) = \mu(U) + \mu(V \setminus (U \cap V))$$
$$\overset{(i)}{=} \mu(U) + \mu(V) - \mu(U \cap V)$$

(v): Follows from (iv) by induction (exercise).

$\square$

Theorem 2.16 (Monotone convergence theorem for sets):
Let $(X, \Sigma, \mu)$ be a measure space.
(i) Let $S_1 \subset S_2 \subset \cdots$ be an increasing sequence in $\Sigma$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \lim_{n \to \infty} \mu(S_n) \qquad \text{(continuity from below)}$$

(ii) Let $S_1 \supset S_2 \supset \cdots$ be a decreasing sequence in $\Sigma$ with $\mu(S_k) < \infty$ for some $k$.

$$\mu\left(\bigcap_{n=1}^{\infty} S_n\right) = \lim_{n \to \infty} \mu(S_n)$$

**Proof:**

(i): Write $A_1 = S_1$ and $A_n = S_n \setminus S_{n-1}$ for $n \geq 2$ ("onion slices").

$\Rightarrow$ $A_n$ disjoint, measurable and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} S_n$.

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n)$$

$$= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right)$$

$$= \lim_{n\to\infty} \mu(S_n)$$

(ii): W.l.o.g. $\mu(S_1) < \infty$. Then $(S_1 \setminus S_n)_{n=1}^{\infty}$ is increasing sequence in $\Sigma$ and

$S_1 \setminus \bigcup_{n=1}^{\infty} (S_1 \setminus S_n) = \bigcap_{n=1}^{\infty} S_n$.

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} S_n\right) = \mu\left(S_1 \setminus \bigcup_{n=1}^{\infty} (S_1 \setminus S_n)\right)$$

$$= \mu(S_1) - \mu\left(\bigcup_{n=1}^{\infty} (S_1 \setminus S_n)\right) \qquad (\text{since } \mu(S_1) < \infty)$$

$$= \mu(S_1) - \lim_{n\to\infty} \mu(S_1 \setminus S_n) \qquad (\text{by (i)})$$

$$= \mu(S_1) - \lim_{n\to\infty} \left(\mu(S_1) - \mu(S_n)\right)$$

$$= \lim_{n\to\infty} \mu(S_n)$$

$\square$

- If anyone has done problem sheet 1, pleas hand in.

**Def 2.17:**

Let $(X,\Sigma,\mu)$ be a measure space.

(i) A set $T\in\Sigma$ with $\mu(T)=0$ is called <u>$(\mu\text{-})$ null set</u>

(ii) A statement about points in $X$ is said to hold <u>almost everywhere</u> (a.e.), if the set of points for which the statement is false is a $\mu$-null set.

**Example:**

$\mu$ measure on $\mathbb{R}$. $f:\mathbb{R}\to\mathbb{R}$ is continuous a.e. iff $\mu(\{x \mid f \text{ not continuous at } x\})=0$.

**Def 2.18**

A measure space $(X,\Sigma,\mu)$ is called <u>complete</u> if

$$\left(S\subset T,\ T\in\Sigma \text{ and } \mu(T)=0\right) \Rightarrow S\in\Sigma,$$

i.e. every subset of a null set is measurable. (necessarily $\mu(S)=0$ then).

**Lemma 2.19:**

Let $(X,\Sigma,\mu)$ be a measure space. Define the <u>completion</u> $(X,\Sigma',\mu')$ of $(X,\Sigma,\mu)$ by

$$\Sigma' := \{A\cup S \mid A\in\Sigma,\ S\subset T \text{ for some null set } T\in\Sigma\}$$

$$\mu'(A')=\mu(A) \quad \text{when } A'=A\cup S \text{ for } A\in\Sigma,\ S\subset T \text{ for some } T\in\Sigma.$$

Then $(X,\Sigma',\mu')$ is a complete measure space.

**Proof:**

<u>Step 1: $\Sigma'$ is a $\sigma$-algebra:</u>

(i) $X\in\Sigma'$ ✓

(ii) $\Sigma'$ is closed under complements:

Let $A,T\in\Sigma$, $\mu(T)=0$, $S\subset T$. Then

$$
\begin{aligned}
X \setminus (A \cup S) &= (X\setminus A) \cap (X \setminus S) \\
&= (X\setminus A) \cap \left[ ((X\setminus S)\setminus T) \cup ((X\setminus S)\cap T) \right] \\
&\overset{S\subset T}{=} (X\setminus A) \cap \left[ (X\setminus T) \cup ((X\setminus S)\cap T) \right] \\
&= \underbrace{\left[ (X\setminus A) \cap (X\setminus T) \right]}_{\in \Sigma} \cup \underbrace{\left[ (X\setminus A) \cap (X\setminus S) \cap T \right]}_{\subset T} \\
&\in \Sigma'
\end{aligned}
$$

(iii) $\Sigma'$ is closed under countable unions:

Let $A_n' \in \Sigma'$ $\forall n \in \mathbb{N}$. Then $\exists A_n, S_n$ : $A_n' = A_n \cup S_n$, $A_n \in \Sigma$, $S_n \subset T_n$ null set.

$$
\bigcup_{n=1}^{\infty} A_n' = \bigcup_{n=1}^{\infty} (A_n \cup S_n) = \underbrace{\left( \bigcup_{n=1}^{\infty} A_n \right)}_{\in \Sigma} \cup \underbrace{\left( \bigcup_{n=1}^{\infty} S_n \right)}_{\subset \bigcup_{n=1}^{\infty} T_n \text{ null set}}
$$

<u>Step 2: $(X, \Sigma', \mu')$ is complete:</u>

Suppose $\exists\, Y \subset X$ with $Y \subset R$, $R \in \Sigma'$, $\mu'(R) = 0$.

By definition $\exists\, A, T \in \Sigma$, $S \subset X$ with $\mu(T) = 0$, $S \subset T$ and $R = A \cup S$. Now

$$
0 = \mu'(R) = \mu(A).
$$

$\Rightarrow$ $\hat{T} := A \cup T$ is $\mu$-null set with $Y \subset R \subset \hat{T}$.

$$
\Rightarrow \quad Y = \underbrace{\emptyset}_{\in \Sigma} \cup \underbrace{Y}_{\subset \hat{T}}
$$

$$
\Rightarrow \quad Y \in \Sigma'
$$

$\square$

# 3. Lebesgue Measure on $\mathbb{R}^N$

## 3.1 Lebesgue outer measure

Notation:

$I = (a_1, b_1) \times \cdots \times (a_N, b_N)$ open rectangle in $\mathbb{R}^N$. ($I = \emptyset$ admitted)

$|I| := (b_1 - a_1) \cdots (b_N - a_N)$ volume of $I$.

**Def 3.1:**

Let $S \subset \mathbb{R}^N$. The Lebesgue outer measure of $S$ is defined by

$$m_0(S) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \;\middle|\; \{I_n\}_{n=1}^{\infty} \text{ is a countable cover of } S \text{ by open rectangles} \right\}$$

("cover" means $S \subset \bigcup_{n=1}^{\infty} I_n$).

**Remark 3.2:**

$m_0$ is **not** a measure on $\mathcal{P}(\mathbb{R}^N)$.

**Thm 3.3:**

(i) $m_0(S) \geq 0 \quad \forall S \subset \mathbb{R}^N$

(ii) $m_0(\emptyset) = 0$

(iii) if $A \subset B$, then $m_0(A) \leq m_0(B)$

(iv) if $(S_n)_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}^N$ then

$$m_0\left(\bigcup_{n=1}^{\infty} S_n\right) \leq \sum_{n=1}^{\infty} m_0(S_n)$$

**Proof:**

(i): clear.

(ii): For any $\varepsilon > 0$ one has $\emptyset \subset (0, \varepsilon)^N$.

$\Rightarrow \quad 0 \leq m_0(\emptyset) \leq \varepsilon^N$ for all $\varepsilon > 0 \quad \Rightarrow \quad m_0(\emptyset) = 0$.

(iii): Any cover $\{I_n\}$ of $B$ is also a cover of $A \Rightarrow m_0(A) \leq \sum_{n=1}^{\infty} |I_n|$.

Taking infimum $\Rightarrow \quad m_0(A) \leq m_0(B)$.

(iv): Let $\varepsilon > 0$. By definition of $m_0$, for each $n$ there ex. cover $\{I_{mn}\}_{m=1}^{\infty}$ s.t.

$$\sum_{m=1}^{\infty} |I_{mn}| \le m_0(S_n) + \varepsilon 2^{-n}.$$

Then $\{I_{mn}\}_{m,n=1}^{\infty}$ is a countable cover of $\bigcup_{n=1}^{\infty} S_n$

$$\Rightarrow m_0\left(\bigcup_{n=1}^{\infty} S_n\right) \le \sum_{m,n=1}^{\infty} |I_{mn}| \le \sum_{n=1}^{\infty} \left(m_0(S_n) + \varepsilon 2^{-n}\right)$$
$$= \left(\sum_{n=1}^{\infty} m_0(S_n)\right) + \varepsilon$$

$$\Rightarrow m_0\left(\bigcup_{n=1}^{\infty} S_n\right) \le \sum_{n=1}^{\infty} m_0(S_n).$$

$\square$

## Example:

1) Countable subsets have Lebesgue outer measure 0:

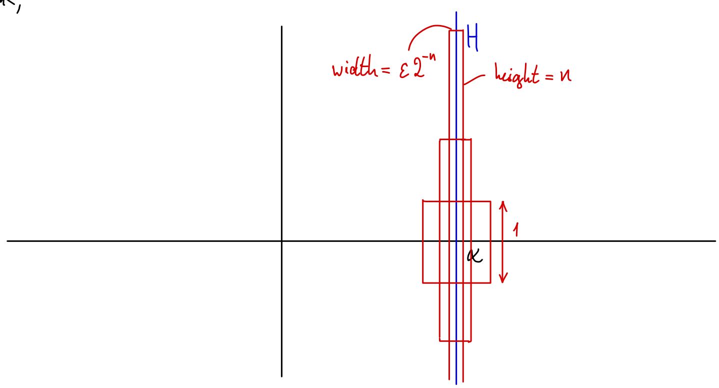
$m_0(\{x\}) = 0$ for single points $x$, since $\{x\} \subset (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon)$.

Let $S \subset \mathbb{R}^N$ be countable, $S = \{y_n \mid n \in \mathbb{N}\}$. Then

$$m_0(S) \le \sum_{n=1}^{\infty} m_0(\{y_n\}) = 0.$$

2) Coordinate hyperplanes $H$ have $m_0(H) = 0$.

$\alpha \in \mathbb{R}$, $\quad H = \{x \in \mathbb{R}^N \mid x_k = \alpha\}$



**Prop 3.4:**

If $I \subset \mathbb{R}^N$ is an open rectangle, then $m_0(I) = |I|$.

**Proof:**

Not proved here; technical! $\square$

**Def 3.5:**

A set $S \subset \mathbb{R}^N$ is called <u>Lebesgue measurable</u> if

$$m_0(U \cap S) + m_0(U \setminus S) = m_0(U)$$

for all $U \in \mathcal{P}(\mathbb{R}^N)$. The family of Lebesgue measurable subsets of $\mathbb{R}^N$ is denoted $\Sigma_L$.

**Plan:** Show that $\Sigma_L$ is a $\sigma$-algebra and $m_0$ restricted to $\Sigma_L$ is a measure.

**Lemma 3.6**

If $S \subset \mathbb{R}^N$, $m_0(S) = 0$, then $S \in \Sigma_L$.

**Proof:**

Let $S \subset \mathbb{R}^N$ and $m_0(S) = 0$. Let $U \subset \mathbb{R}^N$. Then

$$m_0(U \cap S) + m_0(U \setminus S) \underset{\text{monotonicity}}{\leq} m_0(S) + m_0(U) = m_0(U)$$

and

$$m_0(U \cap S) + m_0(U \setminus S) \underset{\text{countable subadditivity}}{\geq} m_0(U)$$

$$\Longrightarrow \quad m_0(U \cap S) + m_0(U \setminus S) = m_0(U) \quad \forall\, U \subset \mathbb{R}^N$$

$$\Longleftrightarrow \quad S \in \Sigma_L. \qquad \square$$

**Lemma 3.7**

$\Sigma_L$ is an algebra.

**Proof:**

(i) $\mathbb{R}^N \in \Sigma_L$: clear from definition

(ii) $S \in \Sigma_L \Rightarrow \mathbb{R}^N \setminus S \in \Sigma_L$:

Let $S \in \Sigma_L$ and write $W := \mathbb{R}^N \setminus S$. Then for all $U \subset \mathbb{R}^N$

$$m_0(U\cap W) + m_0(U\setminus W) = m_0(U\setminus S) + m_0(U\cap S)$$
$$\overset{S\in\Sigma_L}{=} m_0(U)$$
$$\Longrightarrow W \in \Sigma_L.$$

(iii) Let $S_1, S_2 \in \Sigma_L$ and $S = S_1 \cup S_2$. Then for all $U \subset \mathbb{R}^N$

(1) $m_0(U\cap S\cap S_1) + m_0((U\cap S)\setminus S_1) = m_0(U\cap S)$, since $S_1 \in \Sigma_L$

(2) $m_0((U\setminus S_1)\cap S_2) + m_0((U\setminus S_1)\setminus S_2) = m_0(U\setminus S_1)$

Moreover,

(3) $$(U\cap S)\setminus S_1 = (U\setminus S_1)\cap S_2$$

Now check measurability condition:

$$m_0(U\cap S) + m_0(U\setminus S) \overset{(1)}{=} m_0(U\cap S\cap S_1) + m_0((U\cap S)\setminus S_1) + m_0(U\setminus S)$$
$$\overset{(3)}{=} m_0(U\cap S_1) + \underline{m_0((U\setminus S_1)\cap S_2) + m_0((U\setminus S_1)\setminus S_2)}$$
$$\overset{(2)}{=} m_0(U\cap S_1) + m_0(U\setminus S_1)$$
$$= m_0(U)$$

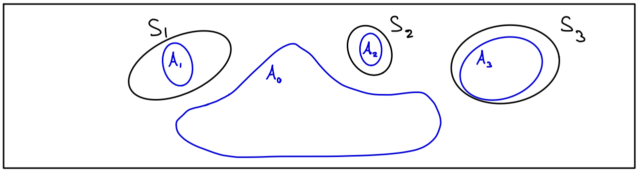
$$\Longrightarrow S \in \Sigma_L$$

$\square$

## Lemma 3.8

(i) Let $S_1, \dots, S_n \in \Sigma_L$ disjoint and $A_r \subset S_r$ $\forall r = 1, \dots, n$ and let $A_0 \subset \mathbb{R}^N \setminus \bigcup_{r=1}^n S_r$. Then $m_0\left(\bigcup_{r=0}^n A_r\right) = \sum_{r=0}^n m_0(A_r)$.

(ii) Let $(S_r)_{r\in\mathbb{N}}$ sequence, $S_r \in \Sigma_L$ and $A_r \subset S_r$ $\forall r \in \mathbb{N}$ and $A_0 \subset \mathbb{R}^N \setminus \bigcup_{r=1}^\infty S_r$. Then $m_0\left(\bigcup_{r=0}^\infty A_r\right) = \sum_{r=0}^\infty m_0(A_r)$.




**Proof:**

(i):
$$m_0\left(\bigcup_{r=0}^n A_r\right) = m_0\left(\bigcup_{r=0}^n (A_r \setminus S_n)\right) + m_0\left(\bigcup_{r=0}^n A_r \cap S_n\right) \quad (S_n \in \Sigma_L)$$
$$= m_0\left(\bigcup_{r=0}^{n-1} A_r\right) + m_0(A_n)$$
$$= m_0\left(\bigcup_{r=0}^{n-2} A_r\right) + m_0(A_{n-1}) + m_0(A_n)$$
$$= \cdots$$
$$= m_0(A_0) + m_0(A_1) + \cdots + m_0(A_n).$$

(ii): Only need to show $m_0\left(\bigcup_{r=0}^\infty A_r\right) \geq \sum_{r=0}^\infty m_0(A_r)$.

Monotonicity and (i) $\Longrightarrow$
$$m_0\left(\bigcup_{r=0}^\infty A_r\right) \geq m_0\left(\bigcup_{r=0}^n A_r\right) = \sum_{r=0}^n m_0(A_r)$$
$$\Longrightarrow \lim_{n\to\infty} \sum_{r=0}^n m_0(A_r) \leq m_0\left(\bigcup_{r=0}^\infty A_r\right)$$

$\square$

## 3.2 The Lebesgue Measure

**Thm 3.9:**

$\Sigma_L$ is a $\sigma$-algebra and $m_0|_{\Sigma_L}$ is a measure.

**Proof:**

**Step 1:** $\Sigma_L$ closed under countable disjoint unions

Let $S_n \in \Sigma_L \ \forall n \in \mathbb{N}$, $S := \bigcup_{n=1}^{\infty} S_n$.

For $A \subset \mathbb{R}^N$, let $A_n := A \cap S_n \ \forall n$ and $A_0 := A \setminus S$.

Moreover, let $A_n' := A_n$ for $n \in \mathbb{N}$, $A_0' := \emptyset$.

Then $\{A_n\}_{n=0}^{\infty}$ and $\{A_n'\}_{n=0}^{\infty}$ satisfy hypotheses of Lemma 3.8.

$$\Rightarrow \quad m_0\left(\bigcup_{n=1}^{\infty} A_n\right) = m_0\left(\bigcup_{n=0}^{\infty} A_n'\right) \underset{\text{Lemma 3.8 (ii)}}{=} \sum_{n=0}^{\infty} m_0(A_n') = \sum_{n=1}^{\infty} m_0(A_n) \qquad (*)$$

$$\Rightarrow \quad m_0(A \cap S) + m_0(A \setminus S) = m_0\left(\bigcup_{n=1}^{\infty} A_n\right) + m_0(A_0)$$

$$\overset{(*)}{=} \sum_{n=1}^{\infty} m_0(A_n) + m_0(A_0)$$

$$\overset{\text{Lemma 3.8 (ii)}}{=} m_0\left(A_0 \cup \bigcup_{n=1}^{\infty} A_n\right)$$

$$= m_0(A)$$

**Step 2:** general countable unions.

$S_n \in \Sigma_L \ \forall n \in \mathbb{N}$; Write $S_1' := S_1$ and $S_n' := S_n \setminus (S_1 \cup \cdots \cup S_{n-1})$.

Then $S_n' \in \Sigma_L$, since $\Sigma_L$ is algebra and

$$\bigcup_{n=1}^{\infty} S_n = \dot{\bigcup_{n=1}^{\infty}} S_n'$$

Step 1 $\Rightarrow$ $\bigcup_{n=1}^{\infty} S_n = \dot{\bigcup_{n=1}^{\infty}} S_n' \in \Sigma_L$.

**Step 3:** $m_0|_{\Sigma_L}$ is measure:

$\{S_n\}_{n=1}^{\infty}$ disjoint sets in $\Sigma_L$, then $m_0\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} m_0(S_n)$, by using

Lemma 3.8 (ii) with $A_n = S_n$, $n \in \mathbb{N}$ and $A_0 = \emptyset$. $\square$

**Def 3.10:**

The restriction $m := m_0|_{\Sigma_L}$ is called <u>Lebesgue measure</u>.

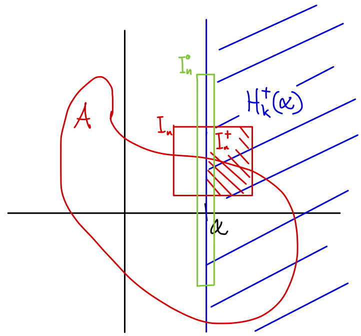
**Thm 3.11:**

$\mathcal{B}(\mathbb{R}^N) \subset \Sigma_L$.

**Proof:**

<u>Step 1</u>: Open half spaces $\subset \Sigma_L$:

Let $k \in \{1, \dots, N\}$ and $\alpha \in \mathbb{R}$ and define




$$H_k^{\pm}(\alpha) = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm(x_k - \alpha) > 0\}$$

$$H_k^0(\alpha) = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_k - \alpha = 0\}$$

Let $A \subset \mathbb{R}^N$, $\varepsilon > 0$. Consider countable covers

$$A \subset \bigcup_{n=1}^{\infty} I_n$$

by open rectangles. Set $I_n^{\pm} := I_n \cap H_k^{\pm}(\alpha)$ $\big($then $|I_n| = |I_n^+| + |I_n^-|\big)$.

Moreover, choose cover $H_k^0(\alpha) \subset \bigcup_{n=1}^{\infty} I_n^0$ with $\sum_{n=1}^{\infty} |I_n^0| < \varepsilon$.

Now

- $\{I_n^-, I_n^0\}_{n=1}^{\infty}$ covers $A \setminus H_k^+(\alpha)$
- $\{I_n^+\}_{n=1}^{\infty}$ covers $A \cap H_k^+(\alpha)$.

Hence

$$m_0(A \cap H_k^+(\alpha)) + m_0(A \setminus H_k^+(\alpha)) \le \sum_{n=1}^{\infty} |I_n^+| + \sum_{n=1}^{\infty} |I_n^-| + \sum_{n=1}^{\infty} |I_n^0|$$
$$\le \sum_{n=1}^{\infty} |I_n| + \varepsilon$$

$$\Longrightarrow \quad m_0(A \cap H_k^+(\alpha)) + m_0(A \setminus H_k^+(\alpha)) \le m_0(A) + \varepsilon$$

$\varepsilon$ arbitrary $\Longrightarrow$

$$m_0(A \cap H_k^+(\alpha)) + m_0(A \setminus H_k^+(\alpha)) \le m_0(A). \qquad (**)$$

"$\geq$" follows from subadditivity $\Longrightarrow$ "$=$" holds in $(**)$

$\Longrightarrow H_k^+(\alpha) \in \Sigma_L$

Step 2:

Moreover:

- $H_k^o(\alpha) \in \Sigma_L$ (Lemma 3.6)
- $H_k^-(\alpha) = \mathbb{R}^N \setminus \left( H_k^+(\alpha) \cup H_k^o(\alpha) \right)$

$\Longrightarrow$ Open rectangles are in $\Sigma_L$

Step 3:

Any open subset of $\mathbb{R}^N$ is union of countably many open rectangles

$\Longrightarrow \{\text{open sets}\} \subset \Sigma_L$

$\Longrightarrow \mathcal{B}(\mathbb{R}^N) \subset \Sigma_L$. $\square$

---

Problem sheet 2, Part B

2. $\mu$ finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $F: \mathbb{R} \to \mathbb{R}$; $t \mapsto \mu(-\infty, t)$

(i) Show that $F$ nondecreasing.

(ii) $F(t+) := \lim_{s \downarrow t} F(s)$. Show: $F(t+) = F(t)$ $\forall t \in \mathbb{R}$.

(iii) Show: $\lim_{t \to -\infty} F(t) = 0$

(iv) Show: $F$ continuous in $t \iff \mu(\{t\}) = 0$

(v) Find $\mu$ s.t. $F$ not continuous at $t = 1$

Solution:

(i) $\mu$ monotone $\Longrightarrow F(s) = \mu((-\infty, s]) \leq \mu((-\infty, t]) = F(t)$ if $s \leq t$.

(ii) $\lim_{s \downarrow t} F(s)$ exists, because $F$ nondecreasing.

$$\Longrightarrow \lim_{s \downarrow t} F(s) = \lim_{n \to \infty} F(t + \tfrac{1}{n}) = \lim_{n \to \infty} \mu\left(\left(-\infty, t + \tfrac{1}{n}\right]\right) \overset{\text{Th. 2.16, } \mu \text{ finite}}{=} \mu\left( \bigcap_{n=1}^{\infty} \left(-\infty, t + \tfrac{1}{n}\right] \right) = \mu((-\infty, t]) = F(t)$$

(iii) Same as (ii):

$$\lim_{t\to-\infty} F(t) = \lim_{n\to\infty} F(-n) = \lim_{n\to\infty} \mu((-\infty,-n]) \overset{\text{Th. 2.16}}{=} \mu\left(\bigcap_{n=1}^{\infty}(-\infty,-n]\right) = \mu(\emptyset) = 0$$

(iv) Let $F$ be continuous.

$\Rightarrow F(t-) := \lim_{s\nearrow t} F(s)$ exists for all $t\in\mathbb{R}$ and $F(t-) = F(t+)$.

$$F(t-) = \lim_{n\to\infty} \mu\left(\left(-\infty, s-\tfrac{1}{n}\right]\right) \overset{\text{Th. 2.16}}{=} \mu((-\infty,t))$$

$$F(t+) = \mu((-\infty,t]) = \mu((-\infty,t)) + \mu(\{t\})$$

$\rightsquigarrow$ $F(t+) = F(t-) \iff \mu((-\infty,t)) = \mu((-\infty,t)) + \mu(\{t\})$

$$\overset{\mu \text{ finite}}{\iff} \mu(\{t\}) = 0.$$

Let $\mu(\{t\}) = 0 \ \forall t\in\mathbb{R}$.

Same reasoning as above backwards $\Rightarrow$ $F$ continuous.

(v) $\mu = \delta_1$, Dirac measure at 1. By (iv) $\Rightarrow$ $F$ not continuous.

---

<u>Remark 3.12:</u>

(i) $(\mathbb{R}^N, m, \Sigma_L)$ is $\sigma$-finite: $\mathbb{R}^N = \bigcup_{n=1}^{\infty} [-n,n]^N$

(ii) $\Sigma_L$ contains all $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ - sets!

<u>Thm 3.13:</u>

Let $S\in\Sigma_L$. Then for all $\varepsilon>0$ there ex. $G\subset\mathbb{R}^N$ open and $F\subset\mathbb{R}^N$ closed s.t.

$$F\subset S\subset G, \quad m(G\setminus S)<\varepsilon, \quad m(S\setminus F)<\varepsilon. \qquad (*)$$

(i.e. $m$ is "regular").

Proof:

Step 1: $m(S) < \infty$.

Choose open cover $S \subset \bigcup_{n=1}^{\infty} I_n$ s.t. $\sum_{n=1}^{\infty} |I_n| < m(S) + \varepsilon$.

Set $G := \bigcup_{n=1}^{\infty} I_n$. Then $m(G) \leq \sum_{n=1}^{\infty} |I_n| < m(S) + \varepsilon$.

$$\Longrightarrow \quad m(G \setminus S) \overset{m(S)\text{ finite}}{=} m(G) - m(S) < m(S) + \varepsilon - m(S) = \varepsilon.$$

Step 2: $m(S) = \infty$.

Write $S = \bigcup_{n=1}^{\infty} S_n$, $\quad m(S_n) < \infty \ \forall n$.

By Step 1: Can choose open $G_n \supset S_n$ s.t. $m(G_n \setminus S_n) < \varepsilon 2^{-n}$.

$G := \bigcup_{n=1}^{\infty} G_n$. Then

$$\begin{aligned}
m(G \setminus S) &= m\left(\left(\bigcup_{n=1}^{\infty} G_n\right) \setminus S\right) \\
&= m\left(\bigcup_{n=1}^{\infty} (G_n \setminus S)\right) \\
&\leq m\left(\bigcup_{n=1}^{\infty} (G_n \setminus S_n)\right) \\
&\leq \sum_{n=1}^{\infty} m(G_n \setminus S_n) \\
&< \varepsilon \sum_{n=1}^{\infty} 2^{-n} \\
&= \varepsilon.
\end{aligned}$$

Step 3: Ex. of $F \subset S$ closed:

Set $W := \mathbb{R}^n \setminus S$, choose open $G \supset W$ with $m(G \setminus W) < \varepsilon$.

Then $F := \mathbb{R}^n \setminus G$ and

$$\begin{aligned}
m(S \setminus F) = m((\mathbb{R}^n \setminus W) \setminus F) &= m((\mathbb{R}^n \setminus W) \cap G) \\
&= m(G \setminus W) \\
&< \varepsilon
\end{aligned}$$

$\square$

Remark 3.14:

A measure which satisfies (*) and is finite on compact sets is called <u>Radon measure</u>. $m$ and $\delta_0$ are Radon measures; counting measure is not.

Thm 3.15

Let $S \in \Sigma_L$. Then

(i) $\exists$ $\mathcal{F}_\sigma$-set $U$ and $\mathcal{G}_\delta$-set $V$ s.t.

$$U \subset S \subset V, \quad m(V \setminus S) = 0, \quad m(S \setminus U) = 0$$

(ii) $\exists$ $U \in \mathcal{B}(\mathbb{R}^N)$ and $N \in \Sigma_L$ with $m(N) = 0$ s.t.

- $N$ contained in Borel set of zero Lebesgue measure
- $S = U \cup N$.

Proof:

(i) Choose open sets $G_n$ s.t. $S \subset G_n$ and $m(G_n \setminus S) < \frac{1}{n}$.

Then set $V := \bigcap_{n=1}^{\infty} G_n$. Then

- $V$ is $\mathcal{G}_\delta$
- $S \subset V$
- $0 \leq m(V \setminus S) \leq m(G_n \setminus S) < \frac{1}{n}$ for all $n \in \mathbb{N}$ $\Rightarrow$ $m(V \setminus S) = 0$.

Existence of $\mathcal{F}_\sigma$-set $U$: Apply the above to $\mathbb{R}^N \setminus S$.

(ii) $\mathcal{F}_\sigma$- and $\mathcal{G}_\delta$-sets are Borel.

$\rightsquigarrow$ Choose $U, V$ as in (i) and $N := S \setminus U$. Then $N \subset \underbrace{V \setminus U}_{\text{Borel}}$ and

$$0 \leq m(V \setminus U) = m\big((V \setminus S) \cup (S \setminus U)\big) \leq m(V \setminus S) + m(S \setminus U) = 0.$$

$\square$

Remark 3.16:

(i) $m$ is <u>translation invariant</u> (homework)

(ii) $m$ is invariant under rotations $SO(N)$ (difficult to prove ...)

**Prop 3.17**

$f:\mathbb{R}^{N-1}\to\mathbb{R}$ continuous and $G_f := \{(x,f(x)) \mid x\in\mathbb{R}^{N-1}\}$.

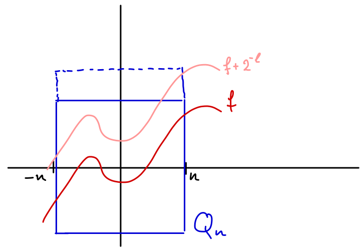
Then $G_f \in \mathcal{B}(\mathbb{R}^N)$ and $m(G_f)=0$.

**Proof:**

$G_f \subset \mathbb{R}^N$ closed $\Longrightarrow$ $G_f$ Borel.

Define $Q_n := (-n,n)^N$ and $C_{n,\ell} := (0,\dots,0,2^{-\ell}) + (G_f\cap Q_n)$.

Then if we set $B_n := (-n,n)^{N-1}\times(-n,n+2)$, then




$$\bigcup_{\ell=1}^{\infty} C_{n,\ell} \subset B_n$$

- $m(C_{n,\ell}) = m(G_f\cap Q_n)$, $\forall \ell$ by translation invariance;
- $C_{n,\ell}$ disjoint for different $\ell$

$$\Longrightarrow \quad \infty > m(B_n) \geq m\left(\bigcup_{\ell=1}^{K} C_{n,\ell}\right) = K\cdot m(G_f\cap Q_n) \quad \forall K\in\mathbb{N}$$

$$\Longrightarrow \quad m(G_f\cap Q_n)=0$$

$n$ arbitrary $\Longrightarrow$ $m(G_f)=0$.

$\square$

## Corollary:

1) $A \subset$ hyperplane $\Rightarrow m(A) = 0$ (hyperplane = graph of continuous fcn.)

2) Sphere: $S = \{x \in \mathbb{R}^n \mid |x| = 1\}$. Then $m(S) = 0$, since $S$ is union of graphs of continuous fcns.

## Prop 3.18:

Define

$$C := \left\{ \sum_{n=1}^{\infty} \alpha_n 3^{-n} \,\middle|\, \alpha_n \in \{0,2\} \ \forall n \in \mathbb{N} \right\}.$$

Then $C \subset [0,1]$ is uncountable, $C \in \mathcal{B}(\mathbb{R})$ and $m(C) = 0$.

## Proof:

Uncountability: $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n 3^{-n}$ is bijection from $\{0,2\}^{\mathbb{N}} \to C$. On the other hand,

$(\alpha_n) \mapsto \sum_{n=1}^{\infty} \frac{\alpha_n}{2} 2^{-n}$ is bijection from $\{0,2\}^{\mathbb{N}} \to [0,1]$.

$\rightsquigarrow$ bijection $C \to [0,1]$

$m(C) = 0$:

$$\sum_{n=1}^{\infty} \alpha_n 3^{-n} = \sum_{n=1}^{k} \alpha_n 3^{-n} + \sum_{n=k+1}^{\infty} \alpha_n 3^{-n}$$
$$\in \sum_{n=1}^{k} \alpha_n 3^{-n} + [0, 3^{-k}]$$

$2^k$ possibilities for $(\alpha_1, \dots, \alpha_k) \in \{0,2\}^k \Rightarrow C \subset$ union of $2^k$ intervals of length $3^{-k}$.

$\rightsquigarrow C = \bigcap_{k=1}^{\infty} C_k$, where $C_1 = [0,1]$, $C_2 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$,


$\Rightarrow C$ closed $\Rightarrow C \in \mathcal{B}(\mathbb{R})$

Clearly, $m(C_k) = \frac{2^k}{3^k}$

$$\Rightarrow m(C) = m\left(\bigcap_{k=1}^{\infty} C_k\right) \overset{\text{Thm 2.16}}{=} \lim_{k \to \infty} m(C_k) = \lim_{k \to \infty} \left(\frac{2}{3}\right)^k = 0$$

$\square$

Existence of non-Lebesgue measurable sets:

Prop 3.19:

$\Sigma_L(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$, i.e. there exist subsets of $\mathbb{R}$ which are not Lebesgue measurable.

Proof:

See Lecture notes. $\square$

Proof uses axiom of choice! Without AOC, one cannot disprove $\Sigma_L = \mathcal{P}(\mathbb{R})$!

Remark 3.21:

(i) $(\mathbb{R}^N, \Sigma_L, m)$ is complete, since $m_o$ monotonic & sets of zero Lebesgue outer measure are in $\Sigma_L$.

(ii) $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), m|_{\mathcal{B}(\mathbb{R}^N)})$ is **not** complete.

(iii) $(\mathbb{R}^N, \Sigma_L, m)$ is the completion of $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), m|_{\mathcal{B}(\mathbb{R}^N)})$, (cf. Thm. 3.15)

## 3.4 Hausdorff Measure

Def 3.25

For $A \subset \mathbb{R}^N$, the diameter of $A$ is $\operatorname{diam}(A) := \sup\{|x-y| : x, y \in A\}$.

Let $\alpha \in [0,\infty)$, $\delta > 0$, $S \subset \mathbb{R}^N$. Define the map

$$H^\alpha_\delta(S) = \inf\left\{ \sum_{i=1}^\infty \operatorname{diam}(A_i)^\alpha \;\middle|\; S \subset \bigcup_{i=1}^\infty A_i,\ \operatorname{diam}(A_i) < \delta\ \forall i \in \mathbb{N} \right\}.$$

Clearly, $H^\alpha_{\delta_1}(S) \geqslant H^\alpha_{\delta_2}(S)$ for $\delta_1 < \delta_2$.

$H^\alpha_\delta$ is **not** a measure!

Def 3.26:

$S \in \mathcal{B}(\mathbb{R}^N)$, $\alpha > 0$. Define

$$H^\alpha(S) := \lim_{\delta \to 0} H^\alpha_\delta(S).$$

Sheet 3, Part B,

1. $a, b \in \mathbb{R}$, $a < b$. Show that $m_0([a,b]) = b-a$.

Solution :

Let $\varepsilon > 0$. $(a-\varepsilon, b+\varepsilon)$ is open cover of $[a,b]$.

$$\Rightarrow \quad m_0([a,b]) \leq |(a-\varepsilon, b+\varepsilon)| = b-a+2\varepsilon$$

$$\overset{\varepsilon \to 0}{\Longrightarrow} \quad m_0([a,b]) \leq b-a$$

Conversely, let $\varepsilon > 0$ and choose cover $\{I_k\}_{k\in\mathbb{N}}$ of $[a,b]$ by open intervals s.t.

$$\sum_{k=1}^{\infty} |I_k| \leq m_0([a,b]) + \varepsilon .$$

$[a,b]$ compact $\Rightarrow$ $[a,b] \subset \bigcup_{j=1}^{n} I_{k_j}$ for some $\{k_1, \dots, k_n\}$

$$\Rightarrow \sum_{j=1}^{n} |I_k| \leq m_0([a,b]) + \varepsilon$$

$I_{k_j}$ open $\Rightarrow$ have to overlap in order to cover $[a,b]$

$$\Rightarrow \sum_{j=1}^{n} |I_k| \geq b-a$$

$$\Rightarrow \quad b-a \leq m_0([a,b]) + \varepsilon$$

$$\overset{\varepsilon \to 0}{\Longrightarrow} \quad b-a \leq m_0([a,b]).$$

**Remark:**

$H^d$ is a measure on $\mathcal{B}(\mathbb{R}^N)$, called <u>Hausdorff measure.</u>

**Prop 3.27:**

For $\lambda > 0$, $S \in \mathcal{B}(\mathbb{R}^N)$ one has
$$H^d(\lambda S) = \lambda^d H^d(S).$$

**Prop 3.28:**

On $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ one has $H^N(S) = c_N m(S)$ for some $c_N > 0$.

**Def 3.29:**

$S \in \mathcal{B}(\mathbb{R}^N)$.
$$\dim_H(S) := \inf\{d \geq 0 \mid H^d(S) = 0\}$$
is called <u>Hausdorff dimension</u> of $S$.

$\dim_H$ generalises the notion of dimension to sets which are not manifolds!

# 4. Integration

**Notation:**

Let $X$ be a set. For $S \subset X$, let
$$\chi_S(x) := \mathbb{1}_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$
(characteristic function of $S$).

Recall: $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$

**Def 4.1:**

Let $(X, \Sigma)$, $(Y, \hat{\Sigma})$ be measurable spaces. A function $f : X \to Y$ is called $\Sigma$-$\hat{\Sigma}$-measurable if $f^{-1}(A) \in \Sigma$ for all $A \in \hat{\Sigma}$.

**Def 4.2:**

Let $(X, \Sigma, \mu)$ be a measure space, $(Y, \hat{\Sigma})$ a measurable space, and $f: X \to Y$ be $\Sigma$-$\hat{\Sigma}$ measurable. Then

$$\nu : \hat{\Sigma} \to [0, \infty]$$
$$\nu(A) = \mu(f^{-1}(A))$$

is a measure on $\hat{\Sigma}$, called <u>image measure</u>.

**Def 4.3:**

Let $(X, \Sigma)$ be a measurable space and let $f: X \to \mathbb{R} \cup \{\pm\infty\}$.

(i) $f$ is called <u>$\Sigma$-measurable</u> if

$$\{x \in X \mid f(x) > \alpha\} \in \Sigma \quad \text{for all } \alpha \in \mathbb{R}$$

(ii) $f: \mathbb{R}^N \to \mathbb{R}$ is called <u>(Lebesgue) measurable</u> if $f$ is $\Sigma_L$-measurable

(iii) A function $f: X \to \mathbb{R}$, where $X$ is a metric space, is called <u>Borel measurable</u> if $f$ is $\mathcal{B}(X)$-measurable.

**Lemma 4.4:**

(i) Every real continuous function on a metric space is Borel measurable.

(ii) Any monotone function $f: \mathbb{R} \to \mathbb{R}$ is Borel measurable

(iii) Every Borel measurable function on $\mathbb{R}^N$ is Lebesgue measurable

(iv) There exist non-Lebesgue measurable functions

**Proof:**

(i) Problem sheet 1.

(ii) $f$ monotone $\Rightarrow$ $f^{-1}((\alpha, \infty])$ is interval.

(iii) $\mathcal{B}(\mathbb{R}^N) \subset \Sigma_L$

(iv) Let $S$ non Lebesgue measurable set and take $f = \chi_S$. $\square$

**Lemma 4.5:**

Let $(X,\Sigma)$ be a measurable space and $f: X \to \overline{\mathbb{R}}$ measurable.

Then $f^{-1}(\infty) \in \Sigma$, $f^{-1}(-\infty) \in \Sigma$ and $f^{-1}(S) \in \Sigma$ for every Borel set $S \subset \mathbb{R}$.

**Proof:**

$$f^{-1}(\infty) = \bigcap_{n=1}^{\infty} f^{-1}((n,\infty]) \qquad \in \Sigma$$

$$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} \left(X \setminus f^{-1}((-n,\infty])\right) \in \Sigma$$

Let $\mathcal{A} := \{ S \subset \mathbb{R} \mid f^{-1}(S) \in \Sigma \}$. Then $\mathcal{A}$ is $\sigma$-algebra and $(\alpha,\infty) \in \mathcal{A}$ $\forall \alpha \in \mathbb{R}$,

since $f^{-1}((\alpha,\infty)) = f^{-1}((\alpha,\infty]) \setminus f^{-1}(\infty) \in \Sigma$.

$\Longrightarrow$ $\mathcal{A}$ contains generator of $\mathcal{B}(\mathbb{R})$

$\Longrightarrow$ $\mathcal{A}$ contains $\mathcal{B}(\mathbb{R})$. $\square$

**Remark 4.6:**

The family $\overline{\mathcal{B}}(\mathbb{R}) := \{ A \subset \overline{\mathbb{R}} \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \}$ is a $\sigma$-algebra and a function $f: X \to \overline{\mathbb{R}}$ is measurable in the sense of Def 4.3 iff it is $\Sigma$-$\overline{\mathcal{B}}(\mathbb{R})$ measurable in the sense of Def. 4.1.

**Lemma 4.7:**

Let $(X,\Sigma)$ measurable space, $f_1,\dots,f_N : X \to \overline{\mathbb{R}}$ measurable.

(i) If $G: \mathbb{R}^N \to \mathbb{R}$ is continuous, then $g: X \to \mathbb{R}$; $g(x) = G(f_1(x),\dots,f_N(x))$ is measurable.

(ii) $f_1 + \dots + f_N$, $f_1 \cdot \dots \cdot f_N$, $c \cdot f_i$ $(i \in \{1,\dots,N\}, c \in \mathbb{R})$ are all measurable.

**Proof:**

(ii) follows from (i).

Proof of (i): Let $\alpha \in \mathbb{R}$. $G^{-1}((\alpha,\infty)) \subset \mathbb{R}^N$ open $\Rightarrow$ $G^{-1}((\alpha,\infty)) = \bigcup_{n=1}^{\infty} I_n$,

$I_n = (a_1^n, b_1^n) \times \dots \times (a_N^n, b_N^n)$, $a_i^n, b_i^n \in \mathbb{Q}$. Now

$$\{x \in X \mid (f_1(x),\dots,f_N(x)) \in I_n\} = f_1^{-1}((a_1^n,b_1^n)) \cap \dots \cap f_N^{-1}((a_N^n,b_N^n)) \in \Sigma$$

$$\Rightarrow g^{-1}((\alpha,\infty)) = \bigcup_{n=1}^{\infty} \{x \in X \mid (f_1(x),\dots,f_N(x)) \in I_n\} \in \Sigma.$$

$\square$

## Lemma 4.8

Let $(X, \Sigma)$ measurable space, $f : X \to \overline{\mathbb{R}}$ be $\Sigma$-measurable and $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ Borel measurable. Then $\varphi \circ f : X \to \overline{\mathbb{R}}$ is $\Sigma$-measurable.

**Proof:**

$$\varphi^{-1}((\alpha,\infty]) \in \mathcal{B}(\overline{\mathbb{R}}) \ \forall \alpha \in \overline{\mathbb{R}} \implies f^{-1}(\varphi^{-1}((\alpha,\infty])) \in \Sigma \text{ by Lemma 4.5.}$$

$$\iff \varphi \circ f \ \ \Sigma\text{-measurable.}$$

□

## Notation:

Limits of sequences of functions are always pointwise, i.e. $\left(\lim\limits_{n\to\infty} f_n\right)(x) := \lim\limits_{n\to\infty}\left(f_n(x)\right)$.

## Lemma 4.10:

Let $(X, \Sigma)$ measurable space, $f_1, f_2, \ldots : X \to \overline{\mathbb{R}}$ measurable functions. Then

(i) $x \mapsto \sup\limits_{n\in\mathbb{N}} f_n(x)$ and $x \mapsto \inf\limits_{n\in\mathbb{N}} f_n(x)$ are measurable functions,

(ii) $x \mapsto \limsup\limits_{n\to\infty} f_n(x)$ and $x \mapsto \liminf\limits_{n\to\infty} f_n(x)$ are measurable functions,

(iii) $x \mapsto \lim\limits_{n\to\infty} f_n(x)$ is measurable, if it exists,

(iv) the set $\{x \in X \mid \lim\limits_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

Proof:

(i) Let $\alpha \in \mathbb{R}$. Then

$$\{x \in X \mid \sup_{n\in\mathbb{N}} f_n(x) > \alpha\} = \bigcup_{n\in\mathbb{N}} \{x \in X \mid f_n(x) > \alpha\} \qquad \in \Sigma$$

$$\{x \in X \mid \inf_{n\in\mathbb{N}} f_n(x) > \alpha\} = \bigcup_{k\in\mathbb{N}} \{x \in X \mid f_n(x) > \alpha + \tfrac{1}{k} \ \forall n \in \mathbb{N}\}$$

$$= \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \{x \in X \mid f_n(x) > \alpha + \tfrac{1}{k}\}$$

$$= \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} f_n^{-1}\left(\left(\alpha + \tfrac{1}{k}, \infty\right]\right) \ \in \Sigma$$

(ii) $\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k$, $\rightsquigarrow$ apply (i). Similarly for liminf.

(iii) follows from (ii)

(iv) Let $S := \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Then

$$X \setminus S = \{x \in X \mid \limsup_{n\to\infty} f_n(x) > \liminf_{n\to\infty} f_n(x)\}$$

$$= \bigcup_{r\in\mathbb{Q}} \underbrace{\{x \in X \mid \limsup_{n\to\infty} f_n(x) > r\}}_{\in \Sigma} \cap \underbrace{\{x \in X \mid \liminf_{n\to\infty} f_n(x) < r\}}_{\in \Sigma}$$

$\Rightarrow \ S \in \Sigma$. $\square$

Def 4.11:

Let $(X, \Sigma)$ measurable space. A (non-negative) simple function on $X$ is a function

$$\varphi = \sum_{n=1}^{K} c_n \chi_{A_n},$$

where $c_1, \dots, c_K \in (0, \infty)$, $A_1, \dots, A_K \in \Sigma$.

Remark 4.12

(i) $A_n$ are not necessarily disjoint, but one can always find $A_n'$ s.t. $A_n'$ disjoint and $\varphi = \sum_{n=1}^{K} c_n \chi_{A_n'}$.

(ii) For any two simple functions $\varphi, \psi$, one can always find $\{A_n\}$ s.t.

$$\varphi = \sum_n c_n \chi_{A_n}, \quad \psi = \sum_n d_n \chi_{A_n}$$

(see e.g. Bauer, p. 56)

## Thm 4.13:

Let $(X, \Sigma)$ measurable space, $f: X \longrightarrow [0,\infty]$ measurable. Then there exists an increasing sequence of simple functions $\varphi_n : X \longrightarrow [0,\infty]$ with $\varphi_n \longrightarrow f$.

**Proof:**

Let $n \in \mathbb{N}$ and

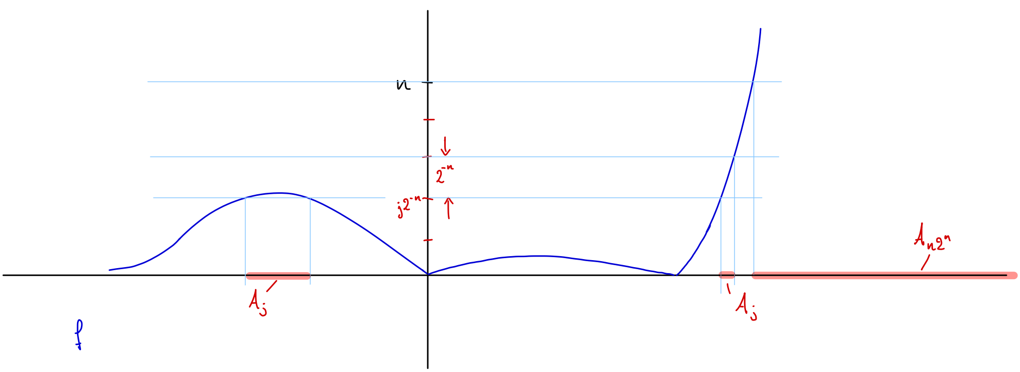
$$A_j := \{x \in X \mid j 2^{-n} \leq f(x) < (j+1) 2^{-n}\}, \qquad 0 \leq j < n 2^n$$

$$A_{n2^n} = \{x \in X \mid f(x) \geq n\}.$$

Then all $A_j$ measurable, hence

$$\varphi_n := \sum_{j=0}^{n2^n} j 2^{-n} \chi_{A_j}$$

is a simple function with $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$ $\forall n \in \mathbb{N}$ and $\varphi_n \to f$ as $n \to \infty$, since by construction $|\varphi_n(x) - f(x)| \leq 2^{-n}$ on each $A_j$.




$\square$

## Def 4.14:

$(X, \Sigma, \mu)$ measure space, $\varphi = \sum_{n=1}^{K} c_n \chi_{A_n}$ simple function on $X$. Define

$$\int_X \varphi \, d\mu := \sum_{n=1}^{K} c_n \mu(A_n)$$

(note that $\int_X \varphi \, d\mu$ is independent of representation of $\varphi$ (see Bauer, Lemma 10.2))

$A \in \mathcal{B}(\mathbb{R}^N)$, $d > 0$. Show:

(a) $\exists d_0 > 0 : H^{d_0}(A) < \infty \implies H^d(A) = 0 \quad \forall d > d_0$

(b) $\dim_H(A) \leq N$

(c) $A$ has nonempty interior $\implies \dim_H(A) = N$

(d) $A$ countable $\implies \dim_H(A) = 0$

<u>Solution:</u>

(a) Let $H^{d_0}(A) < \infty$.

$\implies \exists\, C > 0$ s.t. $H^{d_0}_\delta(A) \leq C \quad \forall \delta > 0$.

$\implies \exists$ coverings $\{A_{i,\delta}\}_{i \in \mathbb{N}}$ of $A$ s.t. $\operatorname{diam}(A_{i,\delta}) < \delta$ and

$$\sum_{i=1}^{\infty} \operatorname{diam}(A_{i,\delta})^{d_0} < C + 1.$$

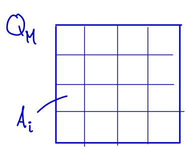
$$\begin{aligned}
\implies H^d_\delta(A) \leq \sum_{i=1}^{\infty} \operatorname{diam}(A_{i,\delta})^d &= \sum_{i=1}^{\infty} \operatorname{diam}(A_{i,\delta})^{d_0} \operatorname{diam}(A_{i,\delta})^{d-d_0} \\
&\leq \sum_{i=1}^{\infty} \operatorname{diam}(A_{i,\delta})^{d_0} \delta^{d-d_0} \\
&\leq \delta^{d-d_0}(C+1) \\
&\xrightarrow[\delta \to 0]{} 0
\end{aligned}$$

(b) Sufficient to show $H^d(A) = 0 \quad \forall d > N$.

Let $M \in \mathbb{N}$ and consider cube $Q_M$ of edge length $M$, centered at $0$.

Let $k \in \mathbb{N}$ and choose covering $\{A_i\}$ of $Q_M$ by cubes of edge length $\frac{1}{k}$.

Then $\{A_i\}$ is a cover for $Q_M \cap A$ and




$$\begin{aligned}
\text{Then } \sum_{i=1}^{(Mk)^N} \operatorname{diam}(A_i)^d &= \sum_{i=1}^{(Mk)^N} \sqrt{N}\, k^{-d} \\
&= M^N k^N \sqrt{N}\, k^{-d} \\
&= M^N \sqrt{N}\, k^{N-d} \\
&\xrightarrow{k \to \infty} 0.
\end{aligned}$$

$\implies H^d(Q_M \cap A) = 0$

$$\implies H^d(A) = H^d\Big(\bigcup_{M=1}^{\infty} (Q_M \cap A)\Big) \leq \sum_{M=1}^{\infty} H^d(Q_M \cap A) = 0.$$

(c) Sufficient to show $H^N(A) \neq 0$. But

$H^N(A) = c_N\, m(A)$ by Prop. 3.28, and

$m(A) \neq 0$, since $A$ contains cube of size $\varepsilon$ for some $\varepsilon > 0$.

(d) $H^d(\{x\}) = 0$, since $\{x\} \subset [x-\delta, x+\delta]^N \Rightarrow H^d_\delta(\{x\}) \leq (2\sqrt{N}\delta)^d \longrightarrow 0 \quad (\delta \to 0)$

for all $d > 0$

$$\Rightarrow H^d(\{x_1, x_2, \dots\}) = H^d\left(\bigcup_{i=1}^{\infty} \{x_i\}\right)$$

$$\leq \sum_{i=1}^{\infty} H^d(\{x_i\})$$

$$= 0$$

for all $d > 0 \Rightarrow \dim_H(\{x_1, x_2, \dots\}) = 0$.

**Def 4.15:**

Let $(X, \Sigma, \mu)$ measure space, $f: X \to [0, \infty]$ measurable. Define

$$\int_X f \, d\mu := \sup\left\{ \int_X \varphi \, d\mu \;\middle|\; \varphi \text{ simple}, \; 0 \le \varphi \le f \right\} \in [0, \infty]. \qquad (*)$$

**Remark 4.16:**

If $f = \varphi$ is simple, then Def 4.14 and Def 4.15 give the same result, as the following lemma shows.

**Lemma 4.17**

$(X, \Sigma, \mu)$ measure space.

(i) If $\varphi, \psi : X \to [0, \infty)$ are simple and $\varphi \le \psi$, then $\int_X \varphi \, d\mu \le \int_X \psi \, d\mu$;

(ii) $f, g : X \to [0, \infty]$ measurable, $f \le g$. Then $\int_X f \, d\mu \le \int_X g \, d\mu$.

**Proof:**

(i): Choose common sets $A_n \in \Sigma$ s.t. $\varphi = \sum_{n=1}^{K} c_n \chi_{A_n}$, $\psi = \sum_{n=1}^{K} d_n \chi_{A_n}$. Then $c_n \le d_n$ $\forall n$ and

$$\sum_n c_n \mu(A_n) \le \sum_n d_n \mu(A_n)$$

(ii): Let $\varphi$ simple with $\varphi \le f$. Then $\varphi \le g$ and by $(*)$ $\int_X \varphi \, d\mu \le \int_X g \, d\mu$.

Taking supremum $\Rightarrow$ $\int_X f \, d\mu \le \int_X g \, d\mu$ $\square$

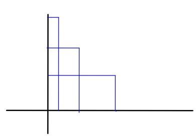
**Theorem 4.18 (Beppo Levi)**

$(X, \Sigma, \mu)$ measure space, $(f_n)$ <u>increasing</u> sequence of <u>non-negative</u> functions and $f_n \nearrow f$ as $n \to \infty$. Then

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

**Remark:**

(i) "Increasing" is essential. Counterexample:



(ii) The assumption "$f_n \ge 0$" can be replaced by $\sup_n \int_X f_n \, d\mu < \infty$.

Proof:

Let $f(x) := \lim_{n\to\infty} f_n(x) \in [0,\infty]$. Then $f$ measurable by Lemma 4.10. Now,

Lemma 4.17 $\Longrightarrow$ $\int_X f_n \, d\mu \le \int_X f_{n+1} \, d\mu \le \int_X f \, d\mu \quad \forall n \in \mathbb{N}$

$$\Longrightarrow \lim_{n\to\infty} \int_X f_n \, d\mu \le \int_X f \, d\mu.$$

Opposite inequality:

Let $\varphi = \sum_{j=1}^{K} a_j \chi_{S_j}$ be simple and $0 \le c < 1$. Define

$$E_n := \{x \in X \mid c\varphi(x) \le f_n(x)\} \in \Sigma$$

Then $E_n \subset E_{n+1}$ $\forall n \in \mathbb{N}$ and $X = \bigcup_{n=1}^{\infty} E_n$ (since $c < 1$). Now

$$c\chi_{E_n}\varphi = \sum_{j=1}^{K} c a_j \chi_{E_n \cap S_j}$$

is simple and $c\chi_{E_n}\varphi \le f_n$ $\forall n$

$$\Longrightarrow \int_X c\chi_{E_n}\varphi \, d\mu \le \int_X f_n \, d\mu. \qquad (*)$$

Note that

$$\int_X c\chi_{E_n}\varphi \, d\mu = \sum_{j=1}^{K} c a_j \mu(E_n \cap S_j) \xrightarrow[n\to\infty]{\text{Th. 2.16}} \sum_{j=1}^{K} c a_j \mu(S_j) = c\int_X \varphi \, d\mu \qquad (**)$$

$(*)$ & $(**)$ $\overset{n\to\infty}{\Longrightarrow}$ $c\int_X \varphi \, d\mu \le \lim_{n\to\infty} \int_X f_n \, d\mu.$

since $\varphi$ arbitrary $\Longrightarrow$ $\sup\left\{\int_X \varphi \, d\mu \mid \varphi \text{ simple}, \varphi \le f\right\} \le c^{-1} \lim_{n\to\infty} \int_X f_n \, d\mu$

$$\Longleftrightarrow \int_X f \, d\mu \le c^{-1} \lim_{n\to\infty} \int_X f_n \, d\mu$$

$$\overset{c\to 1}{\Longrightarrow} \int_X f \, d\mu \le \lim_{n\to\infty} \int_X f_n \, d\mu.$$

$\square$

Thm 4.20 (Fatou's Lemma):

Let $(X,\Sigma,\mu)$ measure space, $(f_n)$ a sequence of non-negative measurable functions on $X$. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu .$$

Proof:

Define $g_n := \inf_{k\geq n} f_k$ for $n\in\mathbb{N}$. Then $(g_n)$ increasing, non-negative and $g_n \to \liminf_{n\to\infty} f_n$

$$\overset{\text{Thm 4.18}}{\Longrightarrow} \quad \lim_{n\to\infty} \int_X g_n \, d\mu = \int_X f \, d\mu .$$

But $g_n \leq f_n \implies \liminf_{n\to\infty} \int_X f_n \, d\mu \geq \liminf_{n\to\infty} \int_X g_n \, d\mu = \int_X f \, d\mu$ $\square$

Thm 4.21:

$(X,\Sigma,\mu)$ measure space, $f,g: X \to [0,\infty]$, $\alpha,\beta\geq 0$. Then $\alpha f+\beta g$ measurable and

$$\int_X (\alpha f+\beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g \, d\mu$$

Proof:

Step 1: $f,g$ simple: $f=\sum_{i=1}^{K} a_i \chi_{A_i}$, $g=\sum_{i=1}^{K} b_i \chi_{A_i}$ (same sets $A_i$ for $f$ and $g$).

$$\begin{aligned}\int_X (\alpha f+\beta g)\, d\mu &= \sum_{i=1}^{K} (\alpha a_i + \beta g_i)\mu(A_i) \\ &= \alpha \sum_{i=1}^{K} a_i \mu(A_i) + \beta \sum_{i=1}^{K} b_i \mu(A_i) \\ &= \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu\end{aligned}$$

Step 2: Let $\varphi_n \nearrow f$, $\psi_n \nearrow g$ be simple. Then $\alpha\varphi_n + \beta\psi_n \nearrow \alpha f+\beta g$ and

$$\int_X (\alpha\varphi_n+\beta\psi_n)\, d\mu = \alpha\int_X \varphi_n \, d\mu + \beta \int_X \psi_n \, d\mu$$

Let $n\to\infty$ with Thm 4.18:

$$\implies \int_X (\alpha f+\beta g)\, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

$\square$

Corollary 4.22:

$(X,\Sigma,\mu)$ measure space, $(f_n)$ seq. of non-neg. measurable fcns. Then

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Def 4.23:

$f \geq 0$ on measure space $(X,\Sigma,\mu)$ and $E \in \Sigma$.

$$\int_E f \, d\mu := \int_X \chi_E f \, d\mu$$

Lemma 4.24:

$(X,\Sigma,\mu)$ measure space, $f: X \to [0,\infty]$ measurable, $\{E_n\}_{n=1}^{\infty} \subset \Sigma$ disjoint, $E := \bigcup_{n=1}^{\infty} E_n$.

Then

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$$

Proof:

Apply Cor. 4.22 to $f_n = \chi_{E_n} f$. $\square$

Remark 4.25:

Any non-neg. meas. fcn $f$ on $(X,\Sigma,\mu)$ defines a new measure $\mu_f$ on $\Sigma$ by

$$\mu_f(E) = \int_E f \, d\mu.$$

Clearly, $\mu(E) = 0 \Rightarrow \mu_f(E) = 0$. The converse is true:

Theorem 4.26 (Radon-Nikodym):

Let $(X,\Sigma)$ measurable space. If $\mu, \nu$ are $\sigma$-finite measures on $(X,\Sigma)$ and

$$\mu(E) = 0 \implies \nu(E) = 0 \qquad \forall E \in \Sigma.$$

Then $\exists$ non-neg. meas. fcn $f$ s.t.

$$\nu(E) = \int_E f \, d\mu \qquad \forall E \in \Sigma.$$

$f$ is called $\mu$-density of $\nu$ or Radon-Nikodym derivative of $\nu$ and sometimes denoted $\frac{d\nu}{d\mu}$.

Proof: see Bauer, Theorem 17.10. ☐

**Prop 4.28:**

Let $(X, \Sigma, \mu)$ be a measure space and $f : X \longrightarrow [0, \infty]$ measurable. Then

$$\int_X f \, d\mu = 0 \iff f = 0 \text{ almost everywhere on } X.$$

Proof: Homework. ☐

**Prop 4.29:**

Let $(X, \Sigma, \mu)$ be a measure space, $f, g : X \longrightarrow [0, \infty]$ measurable and $f = g$ a.e. Then

$$\int_X f \, d\mu = \int_X g \, d\mu.$$

**Proof:**

Choose $Z \in \Sigma$ s.t. $\mu(Z) = 0$ and $f = g$ on $X \setminus Z$. Then

$$\int_X f \, d\mu \underset{\text{Lemma 4.24}}{=} \int_{X\setminus Z} f \, d\mu + \int_Z f \, d\mu \underset{\text{Prop 4.28}}{=} \int_{X \setminus Z} f \, d\mu = \int_{X \setminus Z} g \, d\mu = \int_X g \, d\mu$$

☐

**Example:**

$$\int_0^1 \chi_{\mathbb{Q}} \, dm = 0, \qquad \text{since } m(\mathbb{Q}) = 0.$$

## 4.2 The Lebesgue Integral

**Def 4.31:**

$f : X \longrightarrow \overline{\mathbb{R}}$. Positive part: $f_+(x) := \max\{f(x), 0\}$

Negative part: $f_-(x) := \max\{-f(x), 0\}$ for $x \in X$.

**Remark 4.32:** $f = f_+ - f_-$, $|f| = f_+ + f_-$.

Def 4.33:

$(X,\Sigma,\mu)$ measure space, $f: X \to \overline{\mathbb{R}}$ measurable. Assume that either $\int_X f_+ \, d\mu < \infty$ or $\int_X f_- \, d\mu < \infty$. Then define

$$\int_X f \, d\mu := \int_X f_+ \, d\mu - \int_X f_- \, d\mu.$$

$f$ is called <u>integrable</u> if $\int_X f \, d\mu$ is finite.

The set of integrable functions is denoted $\mathcal{L}^1(X,\Sigma,\mu)$

Remark 4.34:

(i) $f$ integrable $\iff$ $f$ measurable and $\int_X |f| \, d\mu < \infty$.

(ii) Common notations for the Lebesgue integral are

$$\int_X f \, d\mu, \quad \int_X f(x) \, d\mu(x), \quad \int_X f.$$

If $\mu = m$, then $\int f(x) \, dx$ instead of $\int_X f(x) \, dm(x)$ is common

Remark 4.35:

(i) $f \in \mathcal{L}^1(X,\Sigma,\mu) \implies |f| < \infty$ a.e.

Proof: If $f = \infty$ on $A \in \Sigma$ and $\mu(A) > 0$, then $\int_X |f| \, d\mu \geq \int_X n \cdot \chi_A \, d\mu = n \cdot \mu(A) \quad \forall n \in \mathbb{N}$.

(ii) $|f| < \infty$ a.e. $\not\Rightarrow$ $f \in \mathcal{L}^1(X,\Sigma,\mu)$.

Prop 4.36:

$(X,\Sigma,\mu)$ measure space, $f \in \mathcal{L}^1(X,\Sigma,\mu)$, then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proof:

$$\left| \int_X f \, d\mu \right| = \left| \int_X f_+ \, d\mu - \int_X f_- \, d\mu \right| \leq \int_X f_+ \, d\mu + \int_X f_- \, d\mu = \int_X |f| \, d\mu.$$

$\square$

Prop 4.37:

Let $(X,\Sigma,\mu)$ measure space, $f,g \in \mathcal{L}^1(X,\Sigma,\mu)$, $\alpha,\beta \in \mathbb{R}$. Then $\alpha f+\beta g \in \mathcal{L}^1(X,\Sigma,\mu)$ and

$$\int_X (\alpha f+\beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

Proof:

Step 1: $f \in \mathcal{L}^1(X,\Sigma,\mu)$, $\alpha \geq 0$. Then

$$\begin{aligned}
\int_X \alpha f\, d\mu &= \int_X (\alpha f)_+\, d\mu - \int_X (\alpha f)_-\, d\mu \\
&= \int_X \alpha f_+\, d\mu - \int_X \alpha f_-\, d\mu \\
&= \alpha \int_X f_+\, d\mu - \alpha \int_X f_-\, d\mu \\
&= \alpha \int_X f\, d\mu
\end{aligned}$$

Step 2: $\alpha = -1$:

$$\begin{aligned}
\int_X -f\, d\mu &= \int_X (-f)_+\, d\mu - \int_X (-f)_-\, d\mu \\
&= \int_X f_-\, d\mu - \int_X f_+\, d\mu \\
&= -\int_X f\, d\mu
\end{aligned}$$

Step 1 + Step 2 $\Rightarrow$ $\int_X \alpha f\, d\mu = \alpha \int_X f\, d\mu \quad \forall \alpha \in \mathbb{R}$.

Step 3: Additivity. Let $f,g \in \mathcal{L}^1(X,\Sigma,\mu)$. Then $h := f+g$ defined almost everywhere (since $|f|, |g| < \infty$ a.e. by Remark 4.35) and measurable.

Moreover $h_+ \leq f_+ + g_+$, $h_- \leq f_- + g_-$ (easy to see from definition of $f_\pm$).

$\Rightarrow h \in \mathcal{L}^1(X,\Sigma,\mu)$ and

$$h_+ - h_- = f_+ - f_- + g_+ - g_- \quad \text{a.e.}$$

$$\Rightarrow h_+ + f_- + g_- = h_- + f_+ + g_+ \quad \text{a.e.}$$

$$\Longrightarrow \int_X h_+ \, d\mu + \int_X f_- \, d\mu + \int_X g_- \, d\mu = \int_X h_- \, d\mu + \int_X f_+ \, d\mu + \int_X g_+ \, d\mu$$

$$\Longrightarrow \int_X h_+ \, d\mu - \int_X h_- \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu + \int_X g_+ \, d\mu - \int_X g_- \, d\mu$$

$$\Longleftrightarrow \int_X h \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu .$$

Step 1 + Step 2 + Step 3 $\Longrightarrow$ Linearity. $\square$

**Prop 4.38**

Let $(X, \Sigma, \mu)$ measure space.

(i) If $h \in \mathcal{L}^1(X, \Sigma, \mu)$, $h \geq 0$ a.e., then

$$\int_X h \, d\mu \geq 0 .$$

(ii) If $f, g \in \mathcal{L}^1(X, \Sigma, \mu)$ and $f \leq g$ a.e then

$$\int_X f \, d\mu \leq \int_X g \, d\mu .$$

**Proof:**

(i) $h \geq 0$ a.e. $\Rightarrow \exists\, E \in \Sigma : h \geq 0$ on $X \setminus E$, $\mu(E) = 0$.

$$\Longrightarrow \int_X h \, d\mu = \int_X h_+ \, d\mu - \int_X h_- \, d\mu$$

$$= \int_X h_+ \, d\mu - \underbrace{\int_E h_- \, d\mu}_{=0}$$

$$\geq 0$$

(ii) $g - f \geq 0$ a.e. $\Longrightarrow 0 \leq \int_X (g - f) \, d\mu = \int_X g \, d\mu - \int_X f \, d\mu .$ $\square$

## Theorem 4.39 (Lebesgue's Dominated Convergence Theorem):

Let $(X, \Sigma, \mu)$ measure space, $0 \leq g \in \mathcal{L}^1(X, \Sigma, \mu)$ and $(f_n)_{n \in \mathbb{N}}$ a sequence of measurable fcns on $X$ satisfying

- $|f_n| \leq g$ a.e. $\forall n \in \mathbb{N}$ and
- $\exists f : X \to \overline{\mathbb{R}}$ measurable s.t. $f_n \to f$ a.e. as $n \to \infty$.

Then $f \in \mathcal{L}^1(X, \Sigma, \mu)$ and

$$\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu \qquad \text{as } n \to \infty.$$

Proof:

W.l.o.g. $f_n \to f$ and $|f_n| \le g$ everywhere, because:

If $E := \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\} \cap \bigcap_{n=1}^{\infty} \{x \in X \mid |f_n(x)| \le g(x)\} \cap \{x \in X \mid g(x) < \infty\}$,

then $E \in \Sigma$ and $\mu(X \setminus E) = 0$. Then $\tilde{f}_n := \chi_E f_n$ satisfies

- $\tilde{f}_n \to \chi_E \cdot f$ everywhere
- $\int_X \tilde{f}_n \, d\mu = \int_X f_n \, d\mu$
- $\int_X \chi_E f \, d\mu = \int_X f \, d\mu$.

In particular $\int_X f_n \, d\mu \to \int_X f \, d\mu$ iff $\int_X \tilde{f}_n \, d\mu \to \int_X \chi_E f \, d\mu$.

$\rightsquigarrow$ Assume w.l.o.g. that $f_n \to f$, $|f_n| \le g < \infty$ everywhere.

Step 1: Prove assertion for $f_n \ge 0$.

Fatou's Lemma $\Longrightarrow$ $\int_X f \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$, $\quad (*)$

$$\int_X g \, d\mu - \int_X f \, d\mu = \int_X (g-f) \, d\mu$$

$$\le \liminf_{n\to\infty} \int_X (g - f_n) \, d\mu$$

$$= \int_X g \, d\mu - \limsup_{n\to\infty} \int_X f_n \, d\mu$$

$\int_X g \, d\mu < \infty$ $\Longrightarrow$ $\int_X f \, d\mu \ge \limsup_{n\to\infty} \int_X f_n \, d\mu$ $\quad (**)$

$(*), (**) \Longrightarrow \liminf_{n\to\infty} \int_X f_n \, d\mu \ge \int_X f \, d\mu \ge \limsup_{n\to\infty} \int_X f_n \, d\mu$

$$\Longrightarrow \int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$$

Step 2: $f_n$ possibly negative:

Write $f_n = (f_n)_+ - (f_n)_-$. By step 1: $\int_X (f_n)_+ \, d\mu \to \int_X f_+ \, d\mu$,

$$\int_X (f_n)_- \, d\mu \to \int_X f_- \, d\mu$$

$$\Rightarrow \int_X f_n \, d\mu = \int_X (f_n)_+ \, d\mu - \int_X (f_n)_- \, d\mu$$

$$\to \int_X f_+ \, d\mu - \int_X f_- \, d\mu$$

$$= \int_X f \, d\mu$$

□

Cor 4.40:

Let $(X, \Sigma, \mu)$ measure space, $f \in \mathcal{L}^1(X, \Sigma, \mu)$, $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ disjoint. Then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu.$$

Proof:

$f_n := \chi_{\bigcup_{k=1}^{n} E_k} f = \sum_{k=1}^{n} \chi_{E_k} f$. Then $f_n \to \chi_E f$ and $|f_n| \leq |f|$ a.e.

Thm 4.39 $\Rightarrow \int_X \chi_E f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f \, d\mu.$

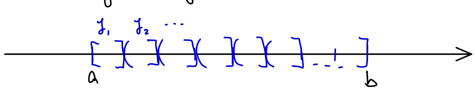
□

Theorem 4.43:

Let $f : [a,b] \to \mathbb{R}$ be bounded and Riemann integrable. Then $f$ is Lebesgue integrable on $[a,b]$ and the Riemann and Lebesgue integrals of $f$ are equal.

Proof: W.l.o.g. $f \geq 0$, because can consider $f + c$

Let $\Delta_n$ be equal partition of $[a,b]$ into $2^n$ intervals $I_1, \dots, I_{2^n}$, with $I_1$ closed and $I_2, \dots, I_{2^n}$ closed at the right edge:



Define:

$$m_k = \inf f(\overline{J}_k)$$
$$M_k = \sup f(\overline{J}_k)$$
$$L(f, \Delta_n) = \sum_{k=1}^{2^n} m_k 2^{-n}$$
$$U(f, \Delta_n) = \sum_{k=1}^{2^n} M_k 2^{-n}$$

Notation in this proof:

$(R) \int_a^b f \, dx$ - Riemann integral

$(L) \int_a^b f \, dx$ - Lebesgue integral

Since $f$ Riemann integrable:

$$U(f, \Delta_n) - L(f, \Delta_n) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Define

$$\varphi_n := \sum_{k=1}^{2^n} m_k \chi_{J_k}$$
$$\psi_n := \sum_{k=1}^{2^n} M_k \chi_{J_k}.$$

Then $\varphi_n, \psi_n$ are simple, $\varphi_n \le f \le \psi_n$ and $\varphi_n \nearrow$, $\psi_n \searrow$.

---

$\Rightarrow$ $\exists\, g, h$ measurable: $\varphi_n \nearrow g$, $\psi_n \searrow h$, $g \le f \le h$. Now

$$L(f, \Delta_n) = (L)\int_a^b \varphi_n \, dx \le (L)\int_a^b g \, dx$$
$$U(f, \Delta_n) = (L)\int_a^b \psi_n \, dx \ge (L)\int_a^b h \, dx$$

$$\overset{\text{Fatou}}{\Longrightarrow} (L)\int_a^b (h-g)\, dx \le \lim_{n\to\infty} (L)\int_a^b (\psi_n - \varphi_n)\, dx$$
$$= \lim_{n\to\infty} \big(U(f, \Delta_n) - L(f, \Delta_n)\big)$$
$$= 0$$

$$\Longrightarrow (L)\int_a^b (h-g)\,dx = 0 \quad \Longrightarrow \quad h = g \ \text{a.e.}$$

$$\Longrightarrow f = g = h \ \text{a.e.}$$

$$\Longrightarrow f \text{ Lebesgue measurable. } (\Sigma_L \text{ complete})$$

By Riemann integrability: $L(f,\Delta_n) \longrightarrow (R)\int_a^b f\,dx$

$$U(f,\Delta_n) \longrightarrow (R)\int_a^b f\,dx$$

$$\Longrightarrow \quad L(f,\Delta_n) = (L)\int_a^b \varphi_n\,dx \le (L)\int_a^b f\,dx \le (L)\int_a^b \psi_n\,dx = U(f,\Delta_n)$$

$$\overset{n\to\infty}{\Longrightarrow} \quad (R)\int_a^b f\,dx \le (L)\int_a^b f\,dx \le (R)\int_a^b f\,dx$$

$\square$

---

Part B

3. (proof of Prop. 4.28): measure space $(X,\Sigma,\mu)$.

(a) $S \in \Sigma$, $\mu(S) = 0$, $f(x) = \begin{cases} \infty, & x \in S \\ 0, & x \notin S \end{cases}$.

Prove that $\int_X f\,d\mu = 0$.

(b) If $g : X \longrightarrow [0,\infty]$ measurable, then

$$\int_X g\,d\mu = 0 \iff \mu(\{x \in X \mid g(x) > 0\}) > 0$$

<u>Solution:</u>

(a) $\varphi_n(x) = n\chi_S$ simple, $\varphi_n \nearrow f$ pointwise.

Monotone convergence theorem $\Longrightarrow$ $0 = n\mu(S) = n\int_X \chi_S\,d\mu = \int_X \varphi_n\,d\mu \longrightarrow \int_X f\,d\mu$

$$\Longrightarrow 0 = \int_X f\,d\mu$$

(b) "$\Longrightarrow$":

$E_n := \{x \in X \mid g(x) > \frac{1}{n}\}$, $\varphi_n(x) := \frac{1}{n}\chi_{E_n}$.

Then $\varphi_n \leq g$ and

$$\frac{1}{n}\mu(E_n) = \int_X \varphi_n \, d\mu \leq \int_X g \, d\mu = 0$$

$$\Longrightarrow \quad \mu(E_n) = 0 \quad \forall n.$$

But $E_n \subset E_{n+1}$ $\forall n$ $\rightsquigarrow$ Mon. Conv. Thm. for sets:

$$\mu\left(\{x \in X \mid g(x) > 0\}\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n) = 0.$$

"$\Longleftarrow$":

Choose $S = \{x \in X \mid g(x) > 0\}$ and let $f$ be as in (a). Then

$$\int_X g \, d\mu \leq \int_X f \, d\mu = 0.$$

---

**Theorem 4.44:**

Let $f : [a,b] \longrightarrow \mathbb{R}$ bounded. Then

$f$ Riemann integrable $\Longleftrightarrow$ $f$ continuous $m$-almost everywhere.

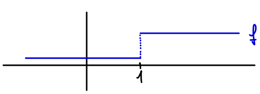
**Proof:**

Lecture notes. $\square$

**Remark 4.45:**

(i) $f$ continuous a.e. $\cancel{\Longleftrightarrow}$ $\exists\, g$ continuous s.t. $f = g$ a.e.

Counterexample:




$\{x \in \mathbb{R} \mid f \text{ not cont. at } x\} = \{1\}$, $m(\{1\}) = 0$, but $\nexists\, g$ continuous s.t. $m(\{x \in \mathbb{R} \mid f \neq g\}) = 0$.

(ii) There exist functions which are <u>improperly</u> Riemann integrable, but <u>not</u> Lebesgue integrable

# 5. Spaces of integrable functions

## 5.1 Integral inequalities

**Def 5.1:**

Let $(X, \Sigma, \mu)$ measure space, $f : X \to \overline{\mathbb{R}}$ $\Sigma$-measurable.

$$\operatorname*{ess\,sup}_X f := \inf \{ \xi \in \overline{\mathbb{R}} \mid f \leq \xi \text{ a.e.} \} \quad \text{"essential supremum"}$$

$$\operatorname*{ess\,inf}_X f := \sup \{ \xi \in \overline{\mathbb{R}} \mid f \geq \xi \text{ a.e.} \} \quad \text{"essential infimum"}$$

$f$ essentially bounded $:\Leftrightarrow$ $\operatorname*{ess\,sup}_X f$ and $\operatorname*{ess\,inf}_X f$ are finite.

**Remark 5.2:**

(i) $f \leq g$ a.e. $\Rightarrow$ $\operatorname*{ess\,sup}_X f \leq \operatorname*{ess\,sup}_X g$, $\operatorname*{ess\,inf}_X f \leq \operatorname*{ess\,inf}_X g$

(ii) $\operatorname*{ess\,inf}_X f \leq f \leq \operatorname*{ess\,sup}_X f$ a.e.

**Proof:** Homework.

**Example:**

$f = \chi_{\mathbb{Q}}$ on $(\mathbb{R}, \Sigma_L, m)$. Then $f \leq 0$ a.e., since $m(\mathbb{Q}) = 0$ $\Rightarrow$ $\operatorname*{ess\,sup}_{\mathbb{R}} f \leq 0$

and $f \geq 0$ (everywhere) $\Rightarrow$ $\operatorname*{ess\,inf}_{\mathbb{R}} f \geq 0$ $\Rightarrow$ $\operatorname*{ess\,sup}_{\mathbb{R}} f = \operatorname*{ess\,inf}_{\mathbb{R}} = 0$.

**Def 5.3:**

Let $(X, \Sigma, \mu)$ measure space, $1 \leq p < \infty$.

$$\mathcal{L}^p(X, \Sigma, \mu) := \left\{ f : X \to \overline{\mathbb{R}} \;\middle|\; f \ \Sigma\text{-measurable and } \int_X |f|^p \, d\mu < \infty \right\},$$

$$\|f\|_p := \left( \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \quad \text{for } f \in \mathcal{L}^p(X, \Sigma, \mu)$$

$$\mathcal{L}^\infty(X, \Sigma, \mu) := \{ f : X \to \overline{\mathbb{R}} \mid f \ \Sigma\text{-measurable and essentially bounded} \}$$

$$\|f\|_\infty := \operatorname*{ess\,sup}_X |f| \quad \text{for } f \in \mathcal{L}^\infty(X, \Sigma, \mu)$$

If $(X, \Sigma, \mu) = (\mathbb{R}^N, \Sigma_L, m)$, one writes $\mathcal{L}^p(X)$, $1 \leq p \leq \infty$.

**Example:**

$(X,\Sigma,\mu) = (\mathbb{R},\Sigma_L,m)$, $f(x) = x^{-\frac{1}{2}}\cdot \chi_{(0,1)}(x)$. Then $f$ measurable, since

$\{x\in\mathbb{R} \mid f(x)>\alpha\}$ is interval for all $\alpha>0$.

**Claim:** $f\in\mathcal{L}^1(\mathbb{R})$, but $f\notin\mathcal{L}^2(\mathbb{R})$.

**Proof:** $f\in\mathcal{L}^1$: consider $f_n := \chi_{[\frac{1}{n},1]}\cdot f$. Then $f_n\nearrow f$ and $f_n$ is Riemann integrable on $[0,1]$. Hence

$$\int_{\mathbb{R}} f_n\,dx = \int_{\frac{1}{n}}^{1} f(x)\,dx \overset{\text{Riemann}}{=} 2\sqrt{x}\Big|_{\frac{1}{n}}^{1} = 2-\frac{2}{\sqrt{n}}$$

Monotone convergence theorem $\Longrightarrow \int_{\mathbb{R}}|f|\,dx = \lim_{n\to\infty}\left(2-\frac{2}{\sqrt{n}}\right) = 2 < \infty$

$$\Longrightarrow f\in\mathcal{L}^1(\mathbb{R})$$

$f\notin\mathcal{L}^2(\mathbb{R})$:

Need to show: $\int_{\mathbb{R}}|f|^2\,dx = \infty$. Again, define $f_n := \chi_{[\frac{1}{n},1]}\cdot|f|^2$. Then $f\geq f_n\ \forall n$

$$\Longrightarrow \int_{\mathbb{R}}|f|^2 dx \geq \int_{\mathbb{R}} f_n\,dx \quad \forall n$$
$$= \int_{\frac{1}{n}}^{1}\frac{1}{x}\,dx$$
$$\overset{\text{Riemann}}{=} -\log\frac{1}{n} \quad \forall n \qquad \Longrightarrow \int_{\mathbb{R}}|f|^2\,dx = \infty.$$

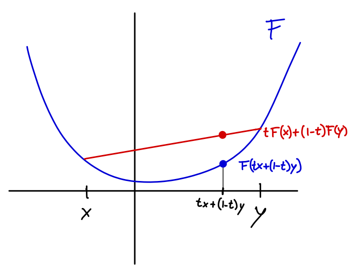
**Remark 5.4:**

$\mathcal{L}^p$ are not vector spaces, because $f+g$ makes no sense if we allow infinities;
$\|f\|_p$ is not a norm, because $\|f\|_p = 0 \not\Longrightarrow f=0$.

**Def 5.5:**

$I\subset\mathbb{R}$ interval. $F: I\to\mathbb{R}$ is called <u>convex</u>, if

$$F(tx+(1-t)y) \leq tF(x)+(1-t)F(y)$$



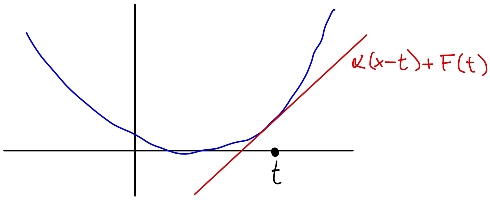

**Example:** $x\longmapsto |x|^p$ is convex for $p\geq 1$

**Remark 5.6:**

(i) If $F$ twice differentiable, then $F'' \geq 0 \implies F$ convex.

(ii) $F$ convex $\implies F$ continuous

(iii) $F$ convex $\implies \forall t \in I \; \exists \alpha > 0: \; F(x) \geq \alpha(x-t) + F(t) \quad \forall x \in I$




**Lemma 5.7 (Jensen's inequality):**

Let $(X, \Sigma, \mu)$ measure space, $\mu(X) = 1$. Let $f: X \to \mathbb{R}$ integrable, $F: \mathbb{R} \to \mathbb{R}$ convex.

Then $F \circ f$ measurable and if

- either $F \circ f$ bounded below, or
- $F \circ f$ integrable,

then

$$F\left(\int_X f \, d\mu\right) \leq \int_X F \circ f \, d\mu .$$

**Proof:**

By assumption $M := \int_X f \, d\mu$ finite. Since $F$ convex $\exists \, \alpha = \alpha(M) > 0$:

$$F(f(x)) \geq \alpha\big(f(x) - M\big) + F(M) \qquad \forall x \in X$$

$$\implies \int_X F \circ f \, d\mu \geq \int_X \big[\alpha(f(x) - M) + F(M)\big] \, d\mu(x)$$

$$= \alpha \int_X f \, d\mu - \alpha M \int_X d\mu + F(M) \int_X d\mu$$

$$= \alpha M - \alpha M + F(M).$$

$\square$

$f \notin \mathcal{L}^2(\mathbb{R})$:

Need to show: $\int_{\mathbb{R}} |f|^2 \, dx = \infty$. Again, define $f_n := \chi_{[\frac{1}{n},1]} \cdot |f|^2$. Then $f \geq f_n \ \forall n$

$$\Longrightarrow \quad \int_{\mathbb{R}} |f|^2 dx \geq \int_{\mathbb{R}} f_n \, dx \quad \forall n$$

$$= \int_{\frac{1}{n}}^{1} \frac{1}{x} \, dx$$

$$\overset{\text{Riemann}}{=} -\log \frac{1}{n} \quad \forall n \quad \Longrightarrow \int_{\mathbb{R}} |f|^2 dx = \infty.$$

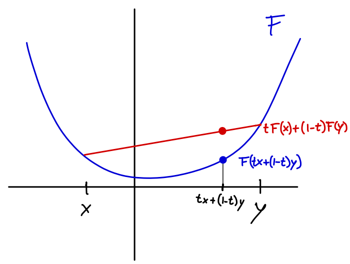
## Remark 5.4:

$\mathcal{L}^p$ are not vector spaces, because $f+g$ makes no sense if we allow infinities; $\|f\|_p$ is not a norm, because $\|f\|_p = 0 \not\Longrightarrow f = 0$.

## Def 5.5:

$I \subset \mathbb{R}$ interval. $F: I \to \mathbb{R}$ is called <u>convex</u>, if

$$F(tx + (1-t)y) \leq t F(x) + (1-t) F(y)$$



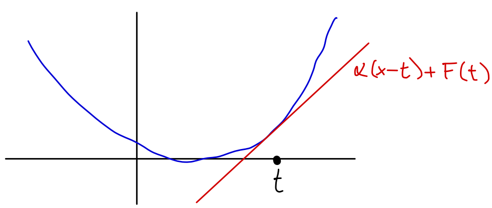

Example: $x \mapsto |x|^p$ is convex for $p \geq 1$

## Remark 5.6:

(i) If $F$ twice differentiable, then $F'' \geq 0 \Longrightarrow F$ convex.

(ii) $F$ convex $\Longrightarrow F$ continuous

(iii) $F$ convex $\Longrightarrow \forall t \in I \ \exists \alpha > 0: F(x) \geq \alpha(x-t) + F(t) \quad \forall x \in I$



**Lemma 5.7 (Jensen's inequality):**

Let $(X, \Sigma, \mu)$ measure space, $\mu(X) = 1$. Let $f : X \to \mathbb{R}$ integrable, $F : \mathbb{R} \to \mathbb{R}$ convex.
Then $F \circ f$ measurable and if

- either $F$ bounded below, or
- $F \circ f$ integrable,

then

$$F\left(\int_X f \, d\mu\right) \leq \int_X F \circ f \, d\mu .$$

**Proof:**

By assumption $M := \int_X f \, d\mu$ finite. Since $F$ convex $\exists \, \alpha = \alpha(M) > 0$ :

$$F(f(x)) \geq \alpha\big(f(x) - M\big) + F(M) \qquad \forall x \in X$$

$$\Rightarrow \int_X F \circ f \, d\mu \geq \int_X \big[\alpha(f(x) - M) + F(M)\big] \, d\mu(x)$$

$$= \alpha \int_X f \, d\mu - \alpha M \int_X d\mu + F(M) \int_X d\mu$$

$$= \alpha M - \alpha M + F(M).$$

$\square$

**Lemma 5.8 (Young's inequality)**

Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \geq 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} .$$

**Proof:**

$ab = 0$ : clear.

$a, b > 0$ : Let $F(x) := -\log(x)$. Then $F''(x) = x^{-2} > 0$ for $x > 0 \Rightarrow F$ convex.

$$\Rightarrow \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)$$

$$= \frac{1}{p} p \log(a) + \frac{1}{q} q \log(b)$$

$$= \log(a) + \log(b)$$

$$= \log(ab) \qquad \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\square$

**Terminology:** If $\frac{1}{p}+\frac{1}{q}=1$, then $p,q$ are called <u>conjugate exponents</u>

**Theorem 5.9 (Hölder's inequality):**

Let $(X,\Sigma,\mu)$ measure space, $1\leq p,q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and $f\in\mathcal{L}^p(X,\Sigma,\mu)$, $g\in\mathcal{L}^q(X,\Sigma,\mu)$. Then $fg\in\mathcal{L}^1(X,\Sigma,\mu)$ and

$$\int_X |fg|\,d\mu \leq \|f\|_p\|g\|_q$$

**Proof:**

<u>Case 1</u>: $1<p<\infty$.

W.l.o.g. $\|f\|_p, \|g\|_q>0$ (otherwise $f=g=0$ a.e.).

Young's inequality $\Rightarrow$

$$\left|\frac{f}{\|f\|_p}\right|\left|\frac{g}{\|g\|_q}\right| \leq \frac{1}{p}\left|\frac{f}{\|f\|_p}\right|^p+\frac{1}{q}\left|\frac{g}{\|g\|_q}\right|^q$$

$$\Rightarrow \frac{1}{\|f\|_p}\frac{1}{\|g\|_q}\int_X|fg|\,d\mu \leq \frac{1}{p}\frac{1}{\|f\|_p^p}\int_X|f|^p\,d\mu+\frac{1}{q}\frac{1}{\|g\|_q^q}\int_X|g|^q\,d\mu$$

$$=\frac{1}{p}+\frac{1}{q}$$

$$=1$$

<u>Case 2</u>: $p=\infty$ or $q=\infty$.

W.l.o.g. $p=1$, $q=\infty$.

$$\int_X|fg|\,d\mu \leq \int_X|f|\,\|g\|_\infty\,d\mu=\|g\|_\infty\int_X|f|\,d\mu=\|g\|_\infty\|f\|_1.$$

$\square$

Sheet 7, Part B:

3. (a) $f \in \mathcal{L}^1(\mathbb{R}, \Sigma_L, m)$, $f(x) = 0$ for $x \le 0$.

Show: $\int_{(0,\infty)} f \, dm = \lim_{n \to \infty} \int_{(0,n)} f \, dm$

(b) Example of $f$ s.t. $\lim_{n\to\infty} \int_{(0,n)} f \, dm$ exists, but $f \notin \mathcal{L}^1$.

Solution:

(a) $f_n := \chi_{(0,n)} f$. Then $|f_n| \le |f|$ and $f_n \to f$ pointwise.

Dominated convergence theorem $\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{(0,n)} f \, dm = \int_{\mathbb{R}} \lim_{n\to\infty} \chi_{(0,n)} f \, dm = \int_{(0,\infty)} f \, dm$.

(b) $f := \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{(n-1,n]}$. Then

$$\lim_{k\to\infty} \int_{(0,k)} f \, dm = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{(-1)^n}{n} = \log(2) - 1,$$

but $\int_{(0,\infty)} |f| \, dm = \infty$, because

$$\int_{(0,\infty)} |f| \, dm \geq \int_{(0,k)} |f| \, dm = \sum_{n=1}^{k} \frac{1}{n} \longrightarrow \infty \quad (k \to \infty).$$

4.

(b) $(\mathbb{R}^N, \mathcal{P}(\mathbb{R}^N), \delta_0)$, $f: \mathbb{R} \to \mathbb{R}$. Show: $f$ integrable and $\int_{\mathbb{R}^N} f \, d\delta_0 = f(0)$.

Solution:

$f$ measurable: clear. Integrability:

$$\int_{\mathbb{R}^N} f_+ \, d\delta_0 = \int_{\mathbb{R}^N} \underbrace{\chi_{\mathbb{R}^N \setminus \{0\}} f_+}_{=0 \text{ a.e.}} \, d\delta_0 + \int_{\mathbb{R}^N} \underbrace{\chi_{\{0\}} f_+}_{\text{simple}} \, d\delta_0 = 0 + f_+(0) \cdot \delta_0(\{0\})$$

$$= f_+(0)$$

$$< \infty.$$

Same for $f_-$.

$$\Rightarrow \int_{\mathbb{R}^N} f \, d\delta_0 = f_+(0) - f_-(0) = f(0).$$

**Theorem 5.10 (Minkowski's inequality):**
Let $(X,\Sigma,\mu)$ measure space, $1 \le p \le \infty$. If $f,g \in \mathcal{L}^p(X,\Sigma,\mu)$, then $f+g \in \mathcal{L}^p(X,\Sigma,\mu)$ and

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

**Proof:** W.l.o.g. $f,g < \infty$ everywhere $\rightsquigarrow$ $f+g$ defined and measurable.

If $p=1$: follows from triangle inequality.

$\infty > p > 1$: Choose $q = \left(1-\frac{1}{p}\right)^{-1} = \frac{p}{p-1}$. Then $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g|^p \, d\mu \\
&= \int_X |f+g|\,|f+g|^{p-1} \, d\mu \\
&\le \int_X |f|\,|f+g|^{p-1} \, d\mu + \int_X |g|\,|f+g|^{p-1} \, d\mu \\
&\overset{\text{Hölder}}{\le} \|f\|_p \, \||f+g|^{p-1}\|_q + \|g\|_p \, \||f+g|^{p-1}\|_q \\
&= \left(\|f\|_p + \|g\|_q\right)\left(\int_X |f+g|^{(p-1)\cdot q} \, d\mu\right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_q\right)\left(\int_X |f+g|^{p} \, d\mu\right)^{\frac{p-1}{p}} \\
&= \left(\|f\|_p + \|g\|_q\right)\|f+g\|_p^{p-1}
\end{aligned}$$

$p=\infty$: $|f+g| \le |f|+|g| \implies |f+g| \le \|f\|_\infty + \|g\|_\infty$ a.e.

$$\implies \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty .$$

$\square$

## 5.2 Theory of $L^p$ spaces

**Def 5.11:**

Let $(X,\Sigma,\mu)$ measure space, $1 \le p \le \infty$. Define the equivalence relation

$$f \sim g \iff f = g \text{ a.e.}$$

and let $L^p(X,\Sigma,\mu) := \mathcal{L}^p(X,\Sigma,\mu)/_\sim$ be the set of equivalence classes.

Define

$$[f] + [g] := [f+g],$$

$$\lambda[f] := [\lambda f],$$

where $f,g$ are finite valued representatives of $[f],[g]$ and $\lambda \in \mathbb{R}$.

For $[f] \in L^p(X,\Sigma,\mu)$, define

$$\|[f]\|_{L^p} := \|f\|_p .$$

**Lemma 5.12:**

$L^p(X,\Sigma,\mu)$ are normed vector spaces over $\mathbb{R}$.

**Proof:**

(i) Well defined:

- finite rep's always exist, since $f \in \mathcal{L}^p$ finite a.e.
- $[f]+[g]$ well-def: Let $[f]=[f']$, $[g]=[g']$. Then

$$f = f' \text{ a.e.}, \; g = g' \text{ a.e.} \implies f+g = f'+g' \text{ a.e.} \implies [f+g] = [f'+g']$$

$\lambda[f]$: similar.

(ii) $L^p$ is vector space: Clear (Minkowski's inequality).

(iii) $\|\cdot\|_{L^p}$ is norm on $L^p$:

- Triangle inequality: Minkowski.
- Homogeneity: $\|\lambda[f]\|_{L^p} = \|[\lambda f]\|_{L^p} = \|\lambda f\|_p = |\lambda|\,\|f\|_p = |\lambda|\,\|[f]\|_{L^p}$
- $\|[f]\|_{L^p} = 0 \iff \|f\|_p = 0 \implies f = 0 \text{ a.e.} \implies [f] = [0].$

$\square$

## Remark 5.13:

(i) Equivalence classes have no point values, i.e. $[f(x)]$ makes no sense.

(ii) In practice people don't distinguish between functions and their equivalence classes, i.e. instead of $[f] \in L^p$, one writes $f \in L^p$.

## Lemma 5.15:

Let $(X, \Sigma, \mu)$ measure space, $1 \leq p \leq \infty$, $([f_n])_{n\in\mathbb{N}}$ Cauchy sequence in $L^p(X, \Sigma, \mu)$.
Then $\exists\, f \in L^p(X, \Sigma, \mu)$ s.t. $\|[f_n] - [f]\|_{L^p} \to 0$ as $n \to \infty$.

## Proof:

$p = \infty$:

$[f_n]$ Cauchy $\Rightarrow \forall k \in \mathbb{N}\ \exists N_k > 0 : \|[f_n] - [f_m]\|_\infty \leq \frac{1}{k} \quad \forall m, n > N_k$

$$\Leftrightarrow \qquad |f_m(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in X \setminus E_k, \quad \mu(E_k) = 0. \qquad (*)$$

Then $E := \bigcup_{k=1}^{\infty} E_k$ is null set and

$\Rightarrow$ $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ for all $x \in X \setminus E$.

$\Rightarrow$ $\lim_{n\to\infty} f_n(x)$ exists a.e.

$\rightsquigarrow$ $f(x) := \lim_{n\to\infty} \chi_{X\setminus E} f_n(x)$, then let $m \to \infty$ in $(*)$:

$$|f(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in X \setminus E \quad \forall n > N_k$$

$$\Rightarrow \quad \|f - f_n\|_\infty \leq \frac{1}{k} \quad \forall n > N_k$$

$$\Rightarrow \quad \|[f] - [f_n]\|_{L^\infty} \to 0.$$

$\underline{p < \infty:}$

Choose subsequence $([f_{n_k}])_{k\in\mathbb{N}}$ s.t.

$$\|[f_{n_{k+1}}] - [f_{n_k}]\|_p \leq 2^{-k} \quad \forall k \geq 1.$$

Define $\quad g_n(x) := \sum_{k=1}^{n} |f_{n_{k+1}}(x) - f_{n_k}(x)| \quad$ (for finite rep's $f_{n_k}$).

$\Longrightarrow \quad \|g_n\|_p \leq \sum_{k=1}^{\infty} 2^{-k} = 1 \quad$ and $\quad g_n \geq 0$ increasing.

Let $g := \lim_{n\to\infty} g_n$. Then Mon. Conv. Th. $\Longrightarrow \|g\|_p = \lim_{n\to\infty} \|g_n\|_p \leq 1$

$\Longrightarrow g \in \mathcal{L}^p$

$\Longrightarrow |g| < \infty$ a.e.

Claim: $(f_{n_i}(x))$ is Cauchy in $\mathbb{R}$ for a.e. $x$.

Proof: Let $m, k \in \mathbb{N}$. Then

$$|f_{n_m}(x) - f_{n_k}(x)| \leq |f_{n_m}(x) - f_{n_{m-1}}(x)| + \cdots + |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

$$\leq g(x) - g_{n_{k-1}}(x) \qquad \text{a.e. for all } m \in \mathbb{N}. \qquad (*)$$

$$\longrightarrow 0 \quad \text{as} \quad k \to \infty$$

$\Longrightarrow \quad (f_{n_k}(x))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ $\forall x \in X\setminus E$, $\mu(E) = 0$.

$\Longrightarrow \quad f(x) := \lim_{k\to\infty} \chi_{X\setminus E} f_{n_k}(x)$ measurable and

- $|\chi_{X\setminus E} f_{n_k} - f| \overset{(*)}{\leq} g - g_{n_{k-1}} \leq g \quad \Longrightarrow \quad \chi_{X\setminus E} f_{n_k} - f \in \mathcal{L}^p \Longrightarrow f \in \mathcal{L}^p.$
- $|\chi_{X\setminus E} f_{n_k}|^p \longrightarrow |f|^p$

Dom. Conv. Th. $\Longrightarrow \|\chi_{X\setminus E} f_{n_k} - f\|_p \longrightarrow 0$

$$\Longleftrightarrow \|[f_{n_k}] - [f]\|_{L^p} \longrightarrow 0.$$

Entire sequence: Let $\varepsilon>0$, $n, n_k \in \mathbb{N}$ s.t. $\|[f_n]-[f_{n_k}]\|_p < \varepsilon$.

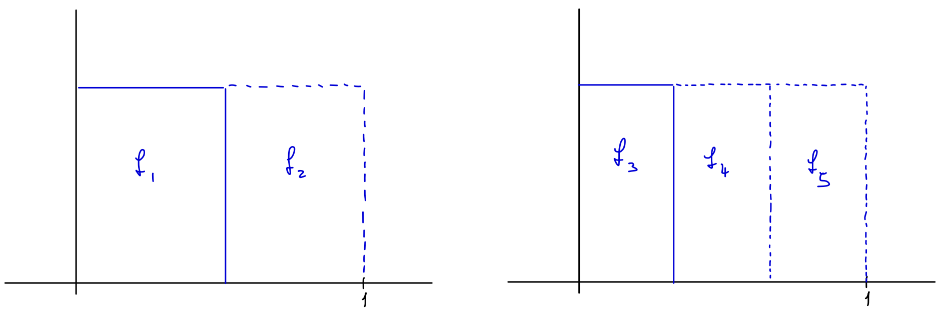
$$\|[f_n]-[f]\|_p \le \|[f_n]-[f_{n_k}]\|_p + \|[f_{n_k}]-[f]\|_p$$
$$< \varepsilon + \|[f_{n_k}]-[f]\|_p$$
$$\xrightarrow{k\to\infty} \varepsilon$$
$$\overset{\varepsilon\to 0}{\Longrightarrow} \|[f_n]-[f]\|_p \longrightarrow 0 \qquad (n\to\infty)$$

□

**Cor. 5.16:**
Let $1\le p\le\infty$ and assume $[f_n]\to[f]$ in $L^p(X,\Sigma,\mu)$. Let $(f_n)_{n\in\mathbb{N}}$ be any sequence of representatives of $([f_n])_{n\in\mathbb{N}}$. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k}\xrightarrow{k\to\infty} f$ pointwise a.e.

**Proof:**
Take $f_{n_k}$ from the previous proof.

**Remark:**
The "subsequence" in Cor. 5.16 cannot be omitted if $p<\infty$! Example:




$f_n$ converges to $0$ in $\mathcal{L}^p$, $1\le p<\infty$, but not pointwise.

**Theorem (Riesz-Fischer-Young):**
Let $(X,\Sigma,\mu)$ be a measure space and $1\le p\le\infty$. Then $L^p(X,\Sigma,\mu)$ is a Banach space.

**Proof:**
Consequence of Lemma 5.12 and Lemma 5.15.

Example:

- $L^2(X,\Sigma,\mu)$ is a Hilbert space: If we define

$$\langle [f],[g]\rangle := \int_X fg \, d\mu,$$

then $\|[f]\|_{L^2} = \sqrt{\langle [f],[f]\rangle}$.

- Let $(X,\Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $c$ the counting measure.

Then $\emptyset$ is the only null set $\Rightarrow$ $f=g$ a.e. tantamount to $f=g$ everywhere

$$\Rightarrow [f] = \{f\} \text{ in } L^p$$

$$\Rightarrow \mathcal{L}^p(\mathbb{N},\mathcal{P}(\mathbb{N}),\mu) \cong L^p(\mathbb{N},\mathcal{P}(\mathbb{N}),\mu).$$

If $1 \leq p < \infty$, then $L^p = \{f : \mathbb{N} \to \mathbb{R} \mid \sum_{n=1}^{\infty} |f(n)|^p < \infty\}$ and $\|f\|_{L^p} = \left(\sum_{n=1}^{\infty} |f(n)|^p\right)^{\frac{1}{p}}$.

If $p=\infty$, then $L^p = \{f : \mathbb{N} \to \mathbb{R} \mid \sup_{n\in\mathbb{N}} |f(n)| < \infty\}$ and $\|f\|_{L^\infty} = \sup_n |f(n)|$.

$L^p(\mathbb{N},\mathcal{P}(\mathbb{N}),c)$ is usually denoted $\ell^p$.

- If $X = (x_1,\dots,x_N)$ is a finite set, $\Sigma = \mathcal{P}(X)$, $c$ counting measure, then any $f : X \to \mathbb{R}$ can be written

$$f = (f(x_1),\dots,f(x_N)) =: (y_1,\dots,y_N)$$

and $\|f\|_{L^2} = \sqrt{y_1^2 + \dots + y_N^2}$.

$\Rightarrow$ $L^2(X,\mathcal{P}(X),c) \cong \mathbb{R}^N$ with Euclidean norm.

## Sheet 8, Part A:

1. $(X, \Sigma, \mu)$ measure space. Are the following true or false?

(a) $\operatorname{essinf}_X(f)$ depends on the chosen measure $\mu$

(b) $\operatorname{essinf}_X(f+g) = \operatorname{essinf}_X(f) + \operatorname{essinf}_X(g)$

(c) If $f \geq 1$ a.e., then $\operatorname{esssup}_X(f) \geq 1$

(d) $\operatorname{esssup}_X(f+g) \leq \operatorname{esssup}_X(f) + \operatorname{esssup}_X(g)$

**Reminder:**

$$\operatorname{esssup}_X(f) := \inf\{\xi \in \mathbb{R} \mid f \leq \xi \text{ a.e.}\}$$

$$\operatorname{essinf}_X(f) := \sup\{\xi \in \mathbb{R} \mid f \geq \xi \text{ a.e.}\}$$

**Solution:**

(a) TRUE

(b) FALSE, take $f = -g$

(c) TRUE, $f \geq 1$ a.e. $\Rightarrow$ "$f \leq 1-\varepsilon$ a.e." is false for all $\varepsilon > 0$

$$\Rightarrow 1-\varepsilon \notin \{\xi \mid f \leq \xi \text{ a.e.}\} \quad \forall \varepsilon > 0$$

$$\Rightarrow \inf\{\xi \mid f \leq \xi \text{ a.e.}\} \geq 1-\varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \inf\{\xi \mid f \leq \xi \text{ a.e.}\} \geq 1$$

(d) TRUE:

$f \leq \operatorname{esssup}_X(f)$ a.e., $\quad g \leq \operatorname{esssup}_X(g)$ a.e.

$$\Rightarrow f+g \leq \operatorname{esssup}_X(f) + \operatorname{esssup}_X(g) \text{ a.e.}$$

$$\Rightarrow \operatorname{esssup}_X(f) + \operatorname{esssup}_X(g) \in \{\xi \mid f+g \leq \xi \text{ a.e.}\}$$

$$\Rightarrow \operatorname{esssup}_X(f+g) \leq \operatorname{esssup}_X(f) + \operatorname{esssup}_X(g)$$

# 6. Uniqueness Lemma and Monotone Class Theorem

**Question:**

If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mu_1 = \mu_2$ on $\mathcal{A}$, then does one have $\mu_1 = \mu_2$ on $\sigma(\mathcal{A})$?

**Def. 6.1:**

Let $X$ be a set, $\mathcal{S} \subset \mathcal{P}(X)$.

(i) $\mathcal{S}$ is called a <u>$\pi$-system</u> on $X$ if $\mathcal{S}$ is closed under finite intersections

(ii) $\mathcal{S}$ is called a <u>$d$-system</u> if

(a) $X \in \mathcal{S}$

(b) $A, B \in \mathcal{S}$ and $A \subset B \Rightarrow B \setminus A \in \mathcal{S}$

(c) $S_1 \subset S_2 \subset \dots \in \mathcal{S} \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \mathcal{S}$

**Def 6.2:**

If $\mathcal{A} \subset \mathcal{P}(X)$, the intersection of all $d$-systems containing $\mathcal{A}$ is denoted $d(\mathcal{A})$ (and is a $d$-system). $d(\mathcal{A})$ is called the $d$-system <u>generated by $\mathcal{A}$</u>.

**Lemma 6.3:**

$\mathcal{S} \subset \mathcal{P}(X)$ is a $\sigma$-algebra if and only if $\mathcal{S}$ is both a $\pi$-system and a $d$-system.

**Proof:**

$\mathcal{S}$ $\sigma$-algebra $\Longrightarrow$ $\mathcal{S}$ $\pi$- and $d$-system: Trivial.

Let $\mathcal{S}$ be both $\pi$- and $d$-system. Only need to show that $\mathcal{S}$ closed under countable unions. Given $\{S_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$, write

$$S_n' := \bigcup_{k=1}^{n} S_k = X \setminus \underbrace{\bigcap_{k=1}^{n} (X \setminus S_k)}_{\in \mathcal{S}}.$$

$\Longrightarrow$ $S_n'$ is increasing sequence in $\mathcal{S}$

$\overset{(c)}{\Longrightarrow}$ $\bigcup_{n=1}^{\infty} S_n' = \bigcup_{n=1}^{\infty} S_n \in \mathcal{S}$.

$\square$

## Lemma 6.4 (Dynkin's Lemma)

Let $\mathcal{A}$ $\pi$-system on a set $X$. Then $d(\mathcal{A})$ is also a $\pi$-system; hence $d(\mathcal{A})$ is a $\sigma$-algebra and $d(\mathcal{A}) = \sigma(\mathcal{A})$.

**Proof:**

Claim: $A \in \mathcal{A}, S \in d(\mathcal{A}) \Rightarrow A \cap S \in d(\mathcal{A})$.

Proof: Let $\mathcal{D} := \{S \in d(\mathcal{A}) \mid \forall A \in \mathcal{A} : S \cap A \in d(\mathcal{A})\}$. Then $\mathcal{D}$ is a $d$-system, because:

$X \in \mathcal{D}$: trivial. If $S_1, S_2 \in \mathcal{D}$, $S_1 \subset S_2$, then for $A \in \mathcal{A}$

$$(S_2 \setminus S_1) \cap A = \underbrace{(S_2 \cap A)}_{\in d(\mathcal{A})} \setminus \underbrace{(S_1 \cap A)}_{\in d(\mathcal{A})} \in d(\mathcal{A}) \Rightarrow S_2 \setminus S_1 \in \mathcal{D}.$$

Finally, if $\{S_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ increasing, then

$$\left(\bigcup_{n=1}^{\infty} S_n\right) \cap A = \bigcup_{n=1}^{\infty} \underbrace{(S_n \cap A)}_{\in d(\mathcal{A})} \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \mathcal{D}$$

$\Rightarrow$ $\mathcal{D}$ is $d$-system and $\mathcal{A} \subset \mathcal{D}$, since $\mathcal{A}$ is $\pi$-system.

$\Rightarrow$ $\mathcal{D} = d(\mathcal{A})$ $\Rightarrow$ Claim.

To prove Lemma, it is sufficient to show $\mathcal{E} = d(\mathcal{A})$, where

$$\mathcal{E} = \{S \in d(\mathcal{A}) \mid \forall A \in d(\mathcal{A}) : S \cap A \in d(\mathcal{A})\} \subset d(\mathcal{A})$$

$\rightsquigarrow$ Show that $\mathcal{E}$ is $d$-system:

- $X \in \mathcal{E}$ clear.
- $S_1, S_2 \in \mathcal{E}$, $S_1 \subset S_2$. Then for $A \in d(\mathcal{A})$:

$$(S_2 \setminus S_1) \cap A = \underbrace{(S_2 \cap A)}_{\in d(\mathcal{A})} \setminus \underbrace{(S_1 \cap A)}_{\in d(\mathcal{A})} \Rightarrow S_2 \setminus S_1 \in \mathcal{E}$$

- $\{S_n\}_{n \in \mathbb{N}} \subset \mathcal{E}$ increasing. Then for $A \in d(\mathcal{A})$:

$$\left(\bigcup_{n=1}^{\infty} S_n\right) \cap A = \bigcup_{n=1}^{\infty} \underbrace{(S_n \cap A)}_{\in d(\mathcal{A})} \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \mathcal{E}$$

$\Rightarrow$ $\mathcal{E}$ is a $d$-system and $\mathcal{A} \subset \mathcal{E}$, by Claim.

$\Rightarrow$ $\mathcal{E} = d(\mathcal{A})$.

$\square$

**Lemma 6.5 (Uniqueness Lemma):**

Let $X$ be a set, $\mathcal{A}$ a $\pi$-system on $X$, and $\Sigma = \sigma(\mathcal{A})$. Let $\mu_1, \mu_2$ be measures on $\Sigma$ s.t.

(i) $\mu_1(X) = \mu_2(X) < \infty$ and

(ii) $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{A}$.

Then $\mu_1(S) = \mu_2(S)$ for all $S \in \Sigma$.

**Proof:**

Let $\mathcal{D} := \{ S \in \Sigma \mid \nu_1(S) = \nu_2(S) \} \subset \Sigma$.

Claim: $\mathcal{D}$ is d-system.

Proof:
- $\nu_1(X) = \nu_2(X) \Rightarrow X \in \mathcal{D}$
- $A, B \in \mathcal{D},\ A \subset B \Rightarrow \nu_1(A \setminus B) \overset{\nu_1 \text{ finite}}{=} \nu_1(A) - \nu_1(B)$

$$= \nu_2(A) - \nu_2(B)$$
$$= \nu_2(A \setminus B) \qquad \Rightarrow B \setminus A \in \mathcal{D}$$

- $\{S_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ increasing $\Rightarrow$

$$\nu_1\left(\bigcup_{n=1}^{\infty} S_n\right) = \lim_{k \to \infty} \nu_1(S_k) = \lim_{k \to \infty} \nu_2(S_k) = \nu_2\left(\bigcup_{n=1}^{\infty} S_n\right).$$

$$\Rightarrow \bigcup_{n=1}^{\infty} S_n \in \mathcal{D}$$

$\Rightarrow$ Claim.

$\Rightarrow \mathcal{D}$ is d-system and $\mathcal{A} \subset \mathcal{D}$ by hypothesis

$\Rightarrow d(\mathcal{A}) \subset \mathcal{D}$

Dynkin's Lemma $\Rightarrow \Sigma = \sigma(\mathcal{A}) = d(\mathcal{A}) \subset \mathcal{D}$

$\Rightarrow \Sigma = \mathcal{D}$.

$\square$

Example:

$\mathcal{A} = \{$ open subintervals of $(0,1)\} \cup \emptyset$ is $\pi$-system that generates $\mathcal{B}((0,1))$.

$\Longrightarrow$ Lebesgue measure is the only measure on $\mathcal{B}((0,1))$ s.t $m((a,b)) = b-a$.

Remark 6.6:

The uniqueness lemma holds for measures which are $\sigma$-finite but not finite, if we replace the condition $\mu_1(X) = \mu_2(X) < \infty$ by the condition that $\exists$ sequence $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ s.t. $E_n \nearrow X$ and $\mu_1(E_n) = \mu_2(E_n) < \infty$ $\forall n \in \mathbb{N}$.

**Remark 6.6:**

Uniqueness Lemma holds if assumption $\mu_1(X) = \mu_2(X) < \infty$ is replaced by

$$\exists\; E_n \in \Sigma \text{ s.t. } E_n \nearrow X,\; \mu_1(E_n) = \mu_2(E_n) < \infty \quad \forall n \in \mathbb{N}.$$

**Thm 6.7 (Monotone class theorem)**

Let $X$ set, $\mathcal{A}$ $\pi$-system on $X$, $\Sigma = \sigma(\mathcal{A})$, $\mathcal{H}$ vector subspace of bounded real valued functions on $X$ s.t.

(i) $\chi_X \in \mathcal{H}$

(ii) $\chi_A \in \mathcal{H}$ for all $A \in \mathcal{A}$

(iii) $f : X \to \mathbb{R}$ bdd, $\exists f_n \in \mathcal{H} : 0 \leq f_n \nearrow f \implies f \in \mathcal{H}$.

Then $\{\text{bdd } \Sigma\text{-meas. fcns on } X\} \subset \mathcal{H}$.

**Proof:**

$\mathcal{D} := \{ S \in \Sigma \mid \chi_S \in \mathcal{H} \} \subset \Sigma$. Then $\mathcal{A} \subset \mathcal{D}$ by (ii).

$\rightsquigarrow$ Show that $\mathcal{D}$ is d-system:

- $X \in \mathcal{D}$, clear by (i).
- $A, B \in \mathcal{D}$, $A \subset B$. $\implies \chi_{B \setminus A} = \chi_B - \chi_A \in \mathcal{H}$ (vector space) $\implies B \setminus A \in \mathcal{D}$.
- $\mathcal{D}$ closed under countable unions by (iii) with $f_n = \chi_{S_n}$.

$\implies \mathcal{D} \supset d(\mathcal{A}) = \Sigma$ (Dynkin's Lemma).

$\implies \mathcal{D} = \Sigma$

$\iff \chi_S \in \mathcal{H} \quad \forall S \in \Sigma$.

$\implies \mathcal{H} \supset \{\text{simple functions}\}$

$\overset{\text{(iii)}}{\implies} \mathcal{H} \supset \{\text{nonneg. bdd. meas. fcns}\}$

$\implies \mathcal{H} \supset \{\text{bdd. meas. fcns}\}$ (vector space). $\square$

# 7. Product Measures and Iterated Integrals

## 7.1 Product σ-algebras and product measures

**Def 7.1**

Let $(X,\Sigma_X)$, $(Y,\Sigma_Y)$ meas. spaces. The <u>product σ-algebra $\Sigma_X \times \Sigma_Y$</u> on $X \times Y$ is defined as

$$\Sigma_X \times \Sigma_Y := \sigma\big(\{\underbrace{A \times B}_{\text{"measurable rectangle"}} \mid A \in \Sigma_X,\ B \in \Sigma_Y\}\big).$$

Caution: $\Sigma_X \times \Sigma_Y$ is <u>not</u> the cartesian product of $\Sigma_X$ and $\Sigma_Y$ !

**Def 7.2:**

Let $X$ set, $\{(Y_i,\Sigma_i)\}_{i\in I}$ be family of meas. spaces and let $f_i : X \to Y_i$ family of fcns. Then the σ-alg. generated by $\{f_i \mid i \in I\}$ is the σ-alg. generated by the sets $\{f_i^{-1}(S) \mid i \in I,\ S \in \Sigma_i\}$.

**Remark 7.3:**

Let $\pi_X : X \times Y \to X$, $\pi_X(x,y) = x$, $\pi_Y : X \times Y \to Y$, $\pi_Y(x,y) = y$ projections. Then $\Sigma_X \times \Sigma_Y$ is the σ-alg. generated by $\{\pi_X, \pi_Y\}$.

**Lemma 7.5:**

Let $(X,\Sigma_X)$, $(Y,\Sigma_Y)$ meas. spaces, and $f : X \times Y \to \overline{\mathbb{R}}$ be $\Sigma_X \times \Sigma_Y$-measurable. Then $f(\cdot,y) : X \to \overline{\mathbb{R}}$ is $\Sigma_X$-measurable $\forall\, y \in Y$ and $f(x,\cdot) : Y \to \overline{\mathbb{R}}$ is $\Sigma_Y$-measurable $\forall\, x \in X$.

**Proof:**

Let $y \in Y$ and $\iota_y : X \to X \times Y$; $\iota_y(x) = (x,y)$.

Claim: $\iota_y$ is $\Sigma_X - \Sigma_X \times \Sigma_Y$-measurable.

Proof: Need to show $\iota_y^{-1}(S) \in \Sigma_X$ $\forall\, S \in \Sigma_X \times \Sigma_Y$.

$\leadsto$ $C := \{S \in \Sigma_X \times \Sigma_Y \mid \iota_y^{-1}(S) \in \Sigma_X\} \subset \Sigma_X \times \Sigma_Y$

$C$ is σ-alg. (PS 1) and {meas. rectangles} $\subset C$, since

$$\iota_y^{-1}(A \times B) = \begin{cases} A, & y \in B \\ \emptyset, & y \notin B. \end{cases}$$

$$\Longrightarrow \quad \Sigma_X \times \Sigma_Y = \sigma(\{\text{meas. rectangles}\}) \subset C$$

$$\Longrightarrow \quad \Sigma_X \times \Sigma_Y = C$$

$$\Longrightarrow \quad \text{Claim.}$$

$\Longrightarrow$ For $\alpha \in \mathbb{R}$: $\{x \in X \mid f(x,y) > \alpha\} = \iota_y^{-1}\Big(\underbrace{f^{-1}((\alpha,\infty])}_{\in \Sigma_X \times \Sigma_Y}\Big) \overset{\text{Claim}}{\in} \Sigma_X$

$\Longrightarrow$ $f(\cdot, y)$ measurable. $\square$

Sheet 9, Part A

1. $(X, \Sigma, \mu)$ measure space, $1 \leq p < \infty$. Which of the following are true?

(a) $(f_n)$ bdd seq. in $L^1((0,1))$ and convergent a.e. Then $\exists f : \|f_n - f\|_{L^1} \to 0$. F

(b) $(f_n)$ seq in $\mathcal{L}^p$ and $\forall \varepsilon > 0 \; \exists N$ s.t. $\int_X |f_n - f_m|^p d\mu < \varepsilon$ for $m, n > N$. T

Then $\exists f \in \mathcal{L}^p$ s.t. $\int_X |f - f_n|^p d\mu \to 0$.

(c) For $f_n$ as in (b) $\exists$ meas. fcn $f$ s.t. $\lim_{n\to\infty} f_n = f$ a.e. F

(d) For $f_n$ as in (b) $\exists$ meas. fcn $f$ s.t. $\lim_{k\to\infty} f_{n_k} = f$ a.e. for some subsequence $(f_{n_k})_{k \in \mathbb{N}}$. T

Solution:

(a) $n \chi_{(0, \frac{1}{n})}$

(b) $[f_n]$ Cauchy in $L^p \Rightarrow [f_n]$ conv. in $L^p \Rightarrow \exists [f] \in L^p$ s.t. $\|[f_n] - [f]\|_{L^p} \to 0$

$\Longleftrightarrow \|f_n - f\|_p \to 0 \Longleftrightarrow \int_X |f_n - f|^p d\mu \to 0$

(c) NO, see counterexample from lecture

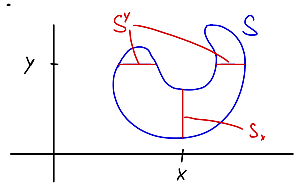
(d) Lemma 5.15

Lemma 7.6:

Let $(X, \Sigma_X, \mu)$, $(Y, \Sigma_Y, \nu)$ $\sigma$-finite measure spaces and $S \in \Sigma_X \times \Sigma_Y$. Define

$$S_x := \{ y \in Y \mid (x,y) \in S \} \quad \text{for } x \in X$$
$$S^y := \{ x \in X \mid (x,y) \in S \} \quad \text{for } y \in Y.$$

"slices"




Then

(i) $x \mapsto \nu(S_x)$ is $\Sigma_X$-measurable,

(ii) $y \mapsto \mu(S^y)$ is $\Sigma_Y$-measurable,

(iii) $\int_X \nu(S_x)\, d\mu(x) = \int_Y \mu(S^y)\, d\nu(y)$.

Proof:

$S_x$, $S^y$ measurable by Lemma 7.5 (with $f = \chi_S$).

Step 1: Assume $\mu(X), \nu(Y) < \infty$.

Define $\mathcal{D} := \{ S \in \Sigma_X \times \Sigma_Y \mid \text{(i)} - \text{(iii) hold} \}$. Aim: Show $\mathcal{D} = \Sigma_X \times \Sigma_Y$.

Claim 1: $\{\text{meas. rectangles}\} \subset \mathcal{D}$.

Proof: Let $A \in \Sigma_X$, $B \in \Sigma_Y$. Then

$$(A \times B)_x = \begin{cases} B, & x \in A \\ \emptyset, & x \notin A \end{cases}$$

$$(A \times B)^y = \begin{cases} A, & y \in B \\ \emptyset, & y \notin B \end{cases}$$

and $\nu((A\times B)_x) = \nu(B)\chi_A(x)$, $\mu((A\times B)^y) = \mu(A)\chi_B(y)$ simple functions and

$$\int_X \nu((A\times B)_x)\, d\mu(x) = \nu(B)\mu(A) = \int_Y \mu((A\times B)^y)\, d\nu(y)$$

$\Rightarrow$ (i)-(iii) hold

$\Rightarrow$ Claim 1.

Claim 2: $\mathcal{D}$ is d-system.

Proof: • $X \times Y \in \mathcal{D}$: Claim 1.

• $S, T \in \mathcal{D}$, $S \subset T$.

$$\nu((T\setminus S)_x) = \nu(T_x \setminus S_x) \overset{\nu \text{ finite}}{=} \nu(T_x) - \nu(S_x)$$

$$\mu((T\setminus S)^y) = \mu(T^y \setminus S^y) \underset{\mu \text{ finite}}{=} \underbrace{\mu(T^y)}_{\text{meas.}} - \underbrace{\mu(S^y)}_{\text{meas.}} \quad \text{measurable.}$$

$$\int_X \nu((T\setminus S)_x)\, d\mu(x) = \int_X \nu(T_x)\, d\mu(x) - \int_X \nu(S_x)\, d\mu(x)$$
$$= \int_Y \mu(T^y)\, d\nu(y) - \int_Y \mu(S^y)\, d\nu(x)$$
$$= \int_Y \mu((T\setminus S)^y)\, d\nu(y)$$

$\Rightarrow$ (i)-(iii) hold

$\Rightarrow$ $T\setminus S \in \mathcal{D}$.

• $\{S_n\}_{n=1}^\infty \subset \mathcal{D}$ increasing, $S := \bigcup_{n=1}^\infty S_n$. Then for $x \in X$, $y \in Y$

$$\begin{matrix} (S_n)_x \nearrow S_x \\ (S_n)^y \nearrow S^y \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \nu((S_n)_x) \nearrow \nu(S_x) \\ \mu((S_n)^y) \nearrow \mu(S^y) \end{matrix}$$

by Mon. Conv. for sets, $\Rightarrow$ $x \mapsto \nu(S_x)$, $y \mapsto \mu(S^y)$ measurable.

$$\int_X \nu((S_n)_x)\, d\mu(x) = \int_Y \mu((S_n)^y)\, d\nu(y)$$

$$\underset{n\to\infty}{\Longrightarrow} \quad \int_X \nu(S_x)\, d\mu(x) = \int_Y \mu(S^y)\, d\nu(y),$$

by Mon. Conv. Th.

$\Rightarrow$ $S \in \mathcal{D}$.

$\Rightarrow$ Claim.

Apply Dynkin's Lemma to $\mathcal{A} = \{\text{meas. rectangles}\}$

$\Rightarrow$ $\Sigma_X \times \Sigma_Y = \sigma(\mathcal{A}) = d(\mathcal{A}) \subset \mathcal{D}$

$\Rightarrow$ $\mathcal{D} = \Sigma_X \times \Sigma_Y$.

<u>Step 2:</u> $\mu(X), \nu(Y) = \infty$ :

Write $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$. For $S \subset X \times Y$, apply Step 1 to $S \cap (X_n \times Y_n)$ and then use monotone convergence theorem.

$\square$

<u>Def 7.7:</u>

Let $(X, \Sigma_X, \mu)$, $(Y, \Sigma_Y, \nu)$ $\sigma$-finite measure spaces. The <u>product measure</u> $\mu \times \nu$ on $(X \times Y, \Sigma_X \times \Sigma_Y)$ is defined by

$$(\mu \times \nu)(S) = \int_X \nu(S_x)\, d\mu(x) = \int_Y \mu(S^y)\, d\nu(y) \qquad \text{for } S \in \Sigma_X \times \Sigma_Y.$$

<u>Prop 7.8:</u>

$(X, \Sigma_X, \mu)$, $(Y, \Sigma_Y, \nu)$ $\sigma$-finite measure spaces. Then $\mu \times \nu$ is a measure.

<u>Proof:</u>

- $\mu \times \nu \geq 0$ : clear
- $\mu \times \nu(\emptyset) = 0$ : clear
- Let $\{S_n\}_{n \in \mathbb{N}} \subset \Sigma_X \times \Sigma_Y$ disjoint, $S := \bigcup_{n=1}^{\infty} S_n$. Then $\{(S_n)_x\}_{n \in \mathbb{N}}$ disjoint and $S_x = \bigcup_{n=1}^{\infty} (S_n)_x$. Thus

$$\begin{aligned}(\mu \times \nu)(S) &= \int_X \nu(S_x)\, d\mu(x) \\ &= \int_X \sum_{n=1}^{\infty} \nu((S_n)_x)\, d\mu(x) \\ &\overset{\text{Mon. conv.}}{=} \sum_{n=1}^{\infty} \int_X \nu((S_n)_x)\, d\mu(x) \\ &= \sum_{n=1}^{\infty} (\mu \times \nu)(S_n)\end{aligned}$$

$\square$

Remark:

- $\sigma$-finiteness assumption in Lemma 7.6 is crucial (counterexample in lecture notes).

Example:

Measure which is not a product measure:

$\mu$ on $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ defined by

$$\mu(S) := \int_{\mathbb{R}} \chi_S(x,x) \, dm(x) \qquad \text{for } S \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$$

cannot be expressed as a product measure (proof in lecture notes).

## 7.2. Iterated Integrals

Q: When do we have $\int_X \int_Y f(x,y) \, d\mu(x) \, d\nu(y) = \int_Y \int_X f(x,y) \, d\nu(y) \, d\mu(x)$ ?

How do these double integrals relate to $\int_{X \times Y} f \, d(\mu \times \nu)$ ?

Example:

$$\int_0^1 \int_0^1 \frac{y^2 - x^2}{(x^2+y^2)^2} \, dx \, dy = \frac{\pi}{4}$$

($\rightsquigarrow$ wtf ??)

$$\int_0^1 \int_0^1 \frac{y^2 - x^2}{(x^2+y^2)^2} \, dy \, dx = -\frac{\pi}{4}$$

Theorem 7.13 (Tonelli):

Let $(X, \Sigma_X, \mu)$, $(Y, \Sigma_Y, \nu)$ be $\sigma$-finite measure spaces. If $f : X \times Y \to [0, \infty]$ is $\Sigma_X \times \Sigma_Y$-measurable, then

$$y \mapsto \int_X f(x,y) \, d\mu(x) \quad \text{is } \Sigma_Y\text{-measurable on } Y \tag{1}$$

$$x \mapsto \int_Y f(x,y) \, d\nu(y) \quad \text{is } \Sigma_X\text{-measurable on } X \tag{2}$$

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f(x,y) \, d\nu(y) \right) d\mu(x) = \int_Y \left( \int_X f(x,y) \, d\mu(x) \right) d\nu(y). \tag{3}$$

Proof:

For $f = \chi_S$, $S \in \Sigma_X \times \Sigma_Y$: follows from Lemma 7.6.

$\Rightarrow$ Thm holds for simple functions.

Let $f : X \times Y \to [0,\infty]$ product-measurable. $\exists$ sequence $f_n \nearrow f$ of simple fcns. (Th. 4.13)

Mon. Conv. Th. $\Rightarrow$

$$\int_Y f_n(x,y)\, d\nu(y) \nearrow \int_Y f(x,y)\, d\nu(y) \qquad \forall x \in X$$

$$\int_X f_n(x,y)\, d\mu(x) \nearrow \int_X f(x,y)\, d\mu(x) \qquad \forall y \in Y$$

$\Rightarrow$ (1), (2).

(3) follows by applying Mon. Conv. Th. again to

$$\int_Y f_n(x,y)\, d\nu(y), \quad \int_X f_n(x,y)\, d\mu(x), \quad \int_{X\times Y} f_n(x,y)\, d(\mu\times\nu).$$

$\square$

Recall: Tonelli applies to nonnegative functions only.

**Theorem 7.14 (Fubini):**

Let $(X, \Sigma_X, \mu)$, $(Y, \Sigma_Y, \nu)$ be $\sigma$-finite measure spaces. If $f \in \mathcal{L}^1(X \times Y, \Sigma_X \times \Sigma_Y, \mu \times \nu)$, then $y \mapsto f(x,y)$ is $\nu$-integrable for $\mu$-almost all $x \in X$, and $x \mapsto f(x,y)$ is $\mu$-integrable for $\nu$-almost all $y \in Y$ and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f(x,y) \, d\nu(y) \right) d\mu(x) = \int_Y \left( \int_X f(x,y) \, d\mu(x) \right) d\nu(y),$$

and all integrals are well defined.

*Sketch of proof:*

$$f \in \mathcal{L}^1(\mu \times \nu) \implies \int_{X \times Y} |f| \, d(\mu \times \nu) < \infty.$$

$$\text{Tonelli} \implies \int_X \int_Y |f(x,y)| \, d\nu(y) \, d\mu(x) < \infty$$

$$\implies \int_Y |f(x,y)| \, d\nu(y) < \infty \quad \text{for } \mu\text{-a.e. } x \in X$$

$$\iff f(x, \cdot) \ \nu\text{-integrable for } \mu\text{-a.e. } x \in X.$$

$$\text{Tonelli} \implies \int_{X \times Y} f_\pm \, d(\mu \times \nu) = \int_X \int_Y f_\pm \, d\nu \, d\mu \,, \quad \text{since } f_\pm \geq 0$$

$$\implies \int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f \, d\nu \, d\mu.$$

$\square$

**Remark 7.15:**

Recall that $f$ integrable $\iff$ $|f|$ integrable. Since $|f| \geq 0$, Tonelli applies and we get the criterion

$$f \in \mathcal{L}^1(X \times Y, \Sigma_X \times \Sigma_Y, \mu \times \nu) \iff \begin{cases} \int_{X \times Y} |f| \, d(\mu \times \nu) < \infty & \text{or} \\ \int_X \left( \int_Y |f(x,y)| \, d\nu(y) \right) d\mu(x) < \infty & \text{or} \\ \int_Y \left( \int_X |f(x,y)| \, d\mu(x) \right) d\nu(y) < \infty, \end{cases}$$

i.e. in order to apply Fubini, it suffices to check iterated integral over absolute value.

## Example:

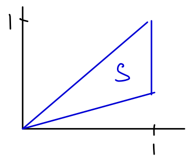
(1) Measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}), m \times m)$. The function

$$(x,y) \longmapsto \frac{y^2 - x^2}{(x^2+y^2)^2}$$

is not integrable w.r.t. $m \times m$ over $[0,1] \times [0,1]$ ($\rightsquigarrow$ Fubini does not apply).

**Proof:**

Let $S := \{(x,y) \in (0,1) \times (0,1) \mid \frac{y}{4} \le x \le \frac{y}{2}\}$.




Then

$$\left.\begin{array}{l} y^2 - x^2 \geq y^2 - \frac{y^2}{4} = \frac{3}{4} y^2 \\ x^2 + y^2 \le \frac{5}{4} y^2 \end{array}\right\} \text{ on } S.$$

Hence by Tonelli:

$$\int_{[0,1]^2} \frac{|y^2 - x^2|}{(x^2+y^2)^2} \, d(m \times m) \geq \int_0^1 \left( \int_0^1 \chi_S(x,y) \frac{3}{4} y^2 \cdot \left(\frac{5}{4} y^2\right)^{-2} dm(x) \right) dm(y)$$

$$= \frac{3}{25} \int_0^1 \frac{1}{y^2} \int_{\frac{y}{4}}^{\frac{y}{2}} 1 \, dm(x) \, dm(y)$$

$$= \frac{3}{25} \int_0^1 \frac{1}{y^2} \left[\frac{y}{2} - \frac{y}{4}\right] dm(y)$$

$$= \frac{3}{100} \int_0^1 \frac{1}{y} \, dm(y)$$

$$= \infty$$

## Example for product $\sigma$-algebras:

- $\mathcal{B}(\mathbb{R}^{M+N}) = \mathcal{B}(\mathbb{R}^M) \times \mathcal{B}(\mathbb{R}^N)$ and

$$m\big|_{\mathcal{B}(\mathbb{R}^{M+N})} = \left(m\big|_{\mathcal{B}(\mathbb{R}^M)}\right) \times \left(m\big|_{\mathcal{B}(\mathbb{R}^N)}\right)$$

In fact, by uniqueness lemma, $m$ is the unique measure on $\mathcal{B}(\mathbb{R}^{M+N})$ s.t.

$$m(A \times B) = m(A) \cdot m(B)$$

for all measurable rectangles $A \times B$.

- $\Sigma_L(\mathbb{R}^M) \times \Sigma_L(\mathbb{R}^N) \subsetneq \Sigma_L(\mathbb{R}^{M+N})$ (proper subset!)

Proof that they're not equal:

Let $S \subset \mathbb{R}^M$ not Lebesgue measurable and consider

$$A := S \times \{0\}.$$

Then $A \subset \mathbb{R}^M \times \{0\}$ and $m(\mathbb{R}^M \times \{0\}) = 0$. Since $\Sigma_L(\mathbb{R}^{M+N})$ complete, $A \in \Sigma_L(\mathbb{R}^{M+N})$.
But $A \notin \Sigma_L(\mathbb{R}^M) \times \Sigma_L(\mathbb{R}^N)$ by Lemma 7.6 (the slice $A^0$ is not measurable). $\square$

The Lebesgue measure $m^{M+N}$ on $\Sigma_L(\mathbb{R}^{M+N})$ is the completion of $m^M \times m^N$.

# Sheet 10, Part B

1. $(X,\Sigma_X,\mu)=(\mathbb{R},\mathcal{P}(\mathbb{R}),\delta_0)$, $(Y,\Sigma_Y,\nu)=(\mathbb{R},\Sigma_L,m)$. Compute

(a) $(m\times\delta_0)(\{(x,y)\in\mathbb{R}^2\mid x^2+y^2=1\})$

(b) $(m\times\delta_0)(\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\leq 1\})$

## Solution:

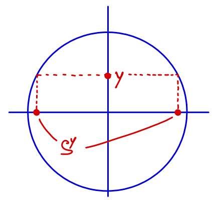
(a) By Def 7.7:

$$(\mu\times\nu)(S)=\int_X \nu(S_x)\,d\mu(x)=\int_Y \mu(S^y)\,d\nu(y)$$

Determine slice:

$S:=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2=1\}$. Then for $y\in\mathbb{R}$

$$\begin{aligned} S^y &= \{x\in\mathbb{R}\mid (x,y)\in S\} \\ &= \{x\in\mathbb{R}\mid x^2+y^2=1\} \\ &= \{\pm\sqrt{1-y^2}\} \end{aligned}$$




$$\Longrightarrow \quad \delta_0(S^y)=\begin{cases}1 & \text{if } 0\in S^y \\ 0 & \text{else}\end{cases}$$

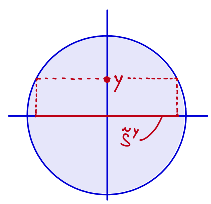
We have $0\in S^y$ iff $y=\pm 1$.

$$\Longrightarrow \quad \delta_0(S^y)=\chi_{\{1,-1\}}$$

$$\begin{aligned}\Longrightarrow \quad (m\times\delta_0)(S) &= \int_{\mathbb{R}} \delta_0(S^y)\,dm(y) \\ &= \int_{\mathbb{R}} \chi_{\{1,-1\}}(y)\,dm(y) \\ &= 0\end{aligned}$$

(b) Denote $\tilde{S}:=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\leq 1\}$. Then for $y\in\mathbb{R}$

$$\tilde{S}^y=\{x\in\mathbb{R}^2\mid x^2+y^2\leq 1\}$$




$$\delta_0(S^y)=\begin{cases}1 & \text{if } 0\in\tilde{S}^y \\ 0 & \text{else}\end{cases}$$

$$= \chi_{[-1,1]}$$

$$\Longrightarrow \quad (m \times \delta_0)(\tilde{S}) = \int_{\mathbb{R}} \delta_0(\tilde{S}^y)\, dm(y)$$

$$= \int_{\mathbb{R}} \chi_{[-1,1]}(y)\, dm(y)$$

$$= \int_{-1}^{1} 1\, dy$$

$$= 2.$$

## Remark 4.46:

If $(X, \Sigma, \mu)$ is a measure space and $E \in \Sigma$, then $E$ can be regarded as a measure space in its own right if we define

$$\Sigma|_E := \{A \subset E \mid A \in \Sigma\}$$
$$\mu|_E(A) := \mu(A) \qquad \text{for } A \in \Sigma|_E.$$

<u>Example:</u> $\Sigma = \mathcal{B}(\mathbb{R}^N)$ or $\Sigma = \Sigma_L(\mathbb{R}^N)$ and $\mu = m$, $E \subset \mathbb{R}^N$ measurable (e.g. ball or rectangle). Usually, one just writes $\Sigma_L$ for $\Sigma_L|_E$ and $m$ for $m|_E$

$\leadsto$ Now things like $([0,1], \Sigma_L, m)$ or $\mathcal{L}^1([0,1] \times [0,\infty), \Sigma_L, m)$ make sense.

## Example: Application of Fubini's theorem

<u>Problem:</u> Show that $\lim_{R \to \infty} \int_0^R \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}$.

<u>Solution:</u> Define $f: \mathbb{R}^2 \to \mathbb{R}$; $f(t,x) = e^{-xt} \sin(x)$. Then $\int_0^\infty e^{-xt} \, dt = \frac{1}{x}$

and hence for $R > 0$

$$\int_0^R \frac{\sin(x)}{x} \, dx = \int_0^R \left( \int_0^\infty e^{-xt} \, dt \right) \cdot \sin(x) \, dx. \qquad (*)$$

Want to apply Fubini to swap integrals.

$\leadsto$ Have to show that $f \in \mathcal{L}^1([0,\infty) \times [0,R])$.

$\leadsto$ Apply Tonelli to $|f|$:

We have $|\sin(x)| \leq x$ for $x \geq 0$ and thus $|f(x)| \leq x e^{-tx}$.

$$\Longrightarrow \int_0^R \int_0^\infty x e^{-tx} \, dt \, dx = \int_0^R x \cdot \frac{1}{x} \, dx = R < \infty$$

$\Longrightarrow f \in \mathcal{L}^1([0,\infty) \times [0,R])$ by Tonelli

$\Longrightarrow$ We can apply Fubini to $(*)$ and swap integrals.

$$\Longrightarrow \int_0^R \frac{\sin(x)}{x}\,dx = \int_0^\infty \int_0^R \sin(x)\, e^{-tx}\,dx\,dt$$

Integration by parts gives

$$\int_0^R \sin(x)\, e^{-tx}\,dx = \Big[-\cos(x)\, e^{-tx}\Big]_0^R - \int_0^R + \cos(x)\, t e^{-tx}\,dx$$

$$= \Big[-\cos(x)\, e^{-tx}\Big]_0^R - \left( \Big[\sin(x)\, t\, e^{-tx}\Big]_0^R - \int_0^R \sin(x)\, t^2 e^{-tx}\,dx \right)$$

$$(1+t^2) \int_0^R \sin(x)\, e^{-tx}\,dx = -\cos(R)\, e^{-tR} + 1 - \sin(R)\, t\, e^{-tR} + 0$$

$\rightsquigarrow$ Integrating in $t$ we get

$$\int_0^\infty \int_0^R \frac{\sin(x)}{x}\,dx\,dt = \int_0^\infty \frac{1}{1+t^2}\,dt \underbrace{- \int_0^\infty \underbrace{\frac{\cos(R)}{1+t^2}}_{|\cdots| \le 1} e^{-tR}\,dt - \int_0^\infty \underbrace{\frac{\sin(R)\, t}{1+t^2}}_{|\cdots| \le 1} e^{-tR}\,dt}_{=:\, I(R)}$$

$$= \int_0^\infty \frac{1}{1+t^2}\,dt - I(R)$$

$$|I(R)| \le 2 \int_0^\infty e^{-tR}\,dt = \frac{2}{R} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty$$

$$\Longrightarrow \lim_{R \to \infty} \int_0^\infty \int_0^R \frac{\sin(x)}{x}\,dx\,dt = \int_0^\infty \frac{1}{1+t^2}\,dt$$

$$= \arctan(\infty) - \arctan(0)$$

$$= \frac{\pi}{2}$$

## Sheet 9, Part B

4. Lebesgue measure is the only translation invariant measure $\mu$ on $\mathcal{B}(\mathbb{R})$ s.t. $\mu((0,1))=1$.

Solution:

Let $\mu$ be translation invariant measure on $\mathcal{B}(\mathbb{R})$ with $\mu((0,1))=1$.

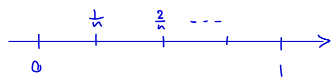
Consider the set of intervals

$$\mathcal{A} := \{(a,b) \mid a \leq b,\ a,b \in \mathbb{Q}\}.$$

Then $\mathcal{A}$ is a $\pi$-system and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$.

Aim: Show that $\mu((a,b)) = m((a,b))$ for all $(a,b) \in \mathcal{A}$, then the $\sigma$-finite uniqueness lemma (Remark 6.6) implies $\mu = m$ on $\mathcal{B}(\mathbb{R})$.

$$\mu\left(\left(0,\tfrac{1}{n}\right)\right) = \frac{1}{n}\cdot\sum_{k=0}^{n-1}\mu\left(\left(\tfrac{k}{n},\tfrac{k+1}{n}\right)\right) \quad \text{by translation invariance}$$




additivity $\quad = \frac{1}{n}\,\mu\left((0,1)\setminus\bigcup_{k=1}^{n-1}\left\{\tfrac{k}{n}\right\}\right) \qquad (*)$

monotonicity $\quad \leq \frac{1}{n}\,\mu((0,1))$

$$= \frac{1}{n}$$

$$\Longrightarrow \mu\left(-\tfrac{1}{2n},\tfrac{1}{2n}\right) \leq \frac{1}{n} \quad \text{by translation invariance}$$

$$\Longrightarrow \mu(\{0\}) = \mu\left(\bigcap_{n=1}^{\infty}\left(-\tfrac{1}{2n},\tfrac{1}{2n}\right)\right) = \lim_{n\to\infty}\mu\left(\left(-\tfrac{1}{2n},\tfrac{1}{2n}\right)\right) = \lim_{n\to\infty}\frac{1}{n} = 0$$

$$\Longrightarrow \mu(\{x\}) = 0 \quad \forall x\in\mathbb{R} \quad \text{by translation invariance}$$

$$\Longrightarrow \mu\left((0,1)\setminus\bigcup_{k=1}^{n-1}\left\{\tfrac{k}{n}\right\}\right) = \mu((0,1)) - \underbrace{\mu\left(\bigcup_{k=1}^{n-1}\left\{\tfrac{k}{n}\right\}\right)}_{=0} = \mu((0,1))$$

From $(*)$: $\mu\left(\left(0,\tfrac{1}{n}\right)\right) = \frac{1}{n}\mu((0,1)) = \frac{1}{n}$

$$\Longrightarrow \mu\left(\left(\tfrac{k}{n},\tfrac{k+1}{n}\right)\right) = \frac{1}{n} \quad \text{by translation invariance}$$

$$\Longrightarrow \mu\left(\left(\tfrac{p}{q},\tfrac{r}{s}\right)\right) = \frac{r}{s}-\frac{p}{q} \quad \text{for } \tfrac{r}{s},\tfrac{p}{q}\in\mathbb{Q}, \text{ by calculation.}$$

$$= m\left(\left(\tfrac{p}{q},\tfrac{r}{s}\right)\right).$$