This contains a list of expressions for the mean and variance of the (Gaussian) slice distribution for a range of different SDE models.

SDE: 1. Cubic polynomial in time, linear in X, constant diffusion

In[557]:= $dX_t = (\alpha_2 + \alpha_3\, t + \alpha_4\, t^2 + \alpha_5\, t^3 + \alpha_1\, X_t)\, dt + \sigma\, dW_t$

Out[557]= $\sigma\, dW_t + dt\,(X_t\, \alpha_1 + \alpha_2 + t\, \alpha_3 + t^2\, \alpha_4 + t^3\, \alpha_5)$

In[558]:=
```
mySDE = ItoProcess[{a2 + a3 t + a4 t^2 + a5 t^3 + a1 x  , sigma }, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
  a4 → 0.6, a5 → 0.7, sigma → 0.9, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
  a4 → 0.6, a5 → 0.7, sigma → 0.9, x → 0, t → 1}
```

Out[558]= ItoProcess[{{a2 + a3 t + a4 t² + a5 t³ + a1 x[t]}, {{sigma}}, x[t]}, {{x}, {x0}}, {t, t0}]

Out[559]=
$$\frac{1}{a1^4}\Big(-6\,a5 - a1\,(2\,a4 + 6\,a5\,t + a1\,(a3 + t\,(2\,a4 + 3\,a5\,t))) + a1^2\,(a2 + t\,(a3 + t\,(a4 + a5\,t))) + e^{a1\,(t-t0)}\,(6\,a5 + a1\,(2\,a4 + 6\,a5\,t0 + a1\,(a3 + t0\,(2\,a4 + 3\,a5\,t0)) + a1^2\,(a2 + t0\,(a3 + t0\,(a4 + a5\,t0))) + a1^3\,x0))\Big)$$

Out[560]=
$$\frac{\left(-1 + e^{2\,a1\,(t-t0)}\right)\,sigma^2}{2\,a1}$$

Out[561]= 0.00868439

Out[562]= 0.00868439

SDE: 2. Quartic polynomial in time, linear in X, constant diffusion

In[563]:= $dX_t = (\alpha_2 + \alpha_3\, t + \alpha_4\, t^2 + \alpha_5\, t^3 + \alpha_6\, t^4 + \alpha_1\, X_t)\, dt + \sigma\, dW_t$

Out[563]= $\sigma\, dW_t + dt\,(X_t\, \alpha_1 + \alpha_2 + t\, \alpha_3 + t^2\, \alpha_4 + t^3\, \alpha_5 + t^4\, \alpha_6)$

In[564]:=
```
mySDE =
 ItoProcess[{a2 + a3 t + a4 t^2 + a5 t^3 + a6 t^4 + a1 x  , sigma }, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
  a4 → 0.6, a5 → 0.7, a6 → 0.1, sigma → 0.9, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
  a4 → 0.6, a5 → 0.7, a6 → 0.1, sigma → 0.9, x → 0, t → 1}
```

Out[564]= $\text{ItoProcess}\left[\left\{\left\{a2 + a3\,t + a4\,t^2 + a5\,t^3 + a6\,t^4 + a1\,x[t]\right\}, \{\{sigma\}\}, x[t]\right\}, \{\{x\}, \{x0\}\}, \{t, t0\}\right]$

Out[565]=
$$-\frac{1}{a1^5}\left(24\,a6 + 6\,a1\,(a5 + 4\,a6\,t) + 2\,a1^2\,(a4 + 3\,t\,(a5 + 2\,a6\,t)) + a1^3\,(a3 + t\,(2\,a4 + 3\,a5\,t + 4\,a6\,t^2)) + a1^4\,(a2 + t\,(a3 + t\,(a4 + t\,(a5 + a6\,t))))\right) + \frac{1}{a1^5}e^{a1\,(t-t0)}\left(24\,a6 + 6\,a1\,(a5 + 4\,a6\,t0) + 2\,a1^2\,(a4 + 3\,t0\,(a5 + 2\,a6\,t0)) + a1^3\,(a3 + t0\,(2\,a4 + 3\,a5\,t0 + 4\,a6\,t0^2)) + a1^4\,(a2 + t0\,(a3 + t0\,(a4 + t0\,(a5 + a6\,t0)))) + a1^5\,x0\right)$$

Out[566]=
$$\frac{\left(-1 + e^{2\,a1\,(t-t0)}\right)\,sigma^2}{2\,a1}$$

Out[567]= 0.00818208

Out[568]= 0.00818208

SDE: 3. Cubic polynomial in time, linear in X, diffusion linear in time

In[569]:= $dX_t = (\alpha_2 + \alpha_3\,t + \alpha_4\,t^2 + \alpha_5\,t^3 + \alpha_1\,X_t)\,dt + \sigma\,t\,dW_t$

Out[569]= $t\,\sigma\,dW_t + dt\,(X_t\,\alpha_1 + \alpha_2 + t\,\alpha_3 + t^2\,\alpha_4 + t^3\,\alpha_5)$

In[570]:=
```
mySDE = ItoProcess[{a2 + a3 t + a4 t^2 + a5 t^3 + a1 x  , sigma t}, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
   a4 → 0.6, a5 → 0.7, a6 → 0.1, sigma → 10, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5,
   a4 → 0.6, a5 → 0.7, a6 → 0.1, sigma → 10, x → 0, t → 1}
```

Out[570]= ItoProcess[{{a2 + a3 t + a4 t² + a5 t³ + a1 x[t]}, {{sigma t}}, x[t]}, {{x}, {x0}}, {t, t0}]

Out[571]= $$\frac{1}{a1^4}\left(-6\,a5 - a1\,(2\,a4 + 6\,a5\,t + a1\,(a3 + t\,(2\,a4 + 3\,a5\,t))) + a1^2\,(a2 + t\,(a3 + t\,(a4 + a5\,t)))\right) + e^{a1\,(t-t0)}\left(6\,a5 + a1\,(2\,a4 + 6\,a5\,t0 + a1\,(a3 + t0\,(2\,a4 + 3\,a5\,t0)) + a1^2\,(a2 + t0\,(a3 + t0\,(a4 + a5\,t0))) + a1^3\,x0))\right)$$

Out[572]= $$\frac{1}{4\,a1^3}e^{-2\,a1\,t0}\,sigma^2\left(-e^{2\,a1\,t0}\,(1 + 2\,a1\,t\,(1 + a1\,t)) + e^{2\,a1\,t}\,(1 + 2\,a1\,t0\,(1 + a1\,t0))\right)$$

Out[573]= 0.0581119

Out[574]= 0.0581119

SDE: 4. Cubic polynomial in time, linear in X, diffusion poly in time

In[575]:= $dX_t = \left(\alpha_2 + \alpha_3\,t + \alpha_4\,t^2 + \alpha_5\,t^3 + \alpha_1\,X_t\right)dt + \sigma\left(\alpha_6 + \alpha_7\,t + \alpha_8\,t^2\right)dW_t$

Out[575]= $dt\left(X_t\,\alpha_1 + \alpha_2 + t\,\alpha_3 + t^2\,\alpha_4 + t^3\,\alpha_5\right) + \sigma\,dW_t\left(\alpha_6 + t\,\alpha_7 + t^2\,\alpha_8\right)$

In[576]:=
```
mySDE = ItoProcess[
   {a2 + a3 t + a4 t^2 + a5 t^3 + a1 x  , sigma (a6 + a7 t + a8 t^2)}, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, 1 → 0.3, a2 → 0.4, a3 → 0.5, a4 → 0.6,
   a5 → 0.7, a6 → 0.1, a7 → 0.8, a8 → 0.3, sigma → 10, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, a3 → 0.5, a4 → 0.6,
   a5 → 0.7, a6 → 0.1, a7 → 0.8, a8 → 0.3, sigma → 10, x → 0, t → 1}
```

Out[576]=
$$\mathrm{ItoProcess}\left[\left\{\left\{a2 + a3\,t + a4\,t^2 + a5\,t^3 + a1\,x[t]\right\}, \left\{\left\{sigma\left(a6 + a7\,t + a8\,t^2\right)\right\}\right\}, x[t]\right\}, \{\{x\}, \{x0\}\}, \{t, t0\}\right]$$

Out[577]=
$$\frac{1}{a1^4}\Big(-6\,a5 - a1\left(2\,a4 + 6\,a5\,t + a1\left(a3 + t\left(2\,a4 + 3\,a5\,t\right)\right) + a1^2\left(a2 + t\left(a3 + t\left(a4 + a5\,t\right)\right)\right)\right) + e^{a1\,(t-t0)}\Big(6\,a5 + a1\Big(2\,a4 + 6\,a5\,t0 + a1\left(a3 + t0\left(2\,a4 + 3\,a5\,t0\right)\right) + a1^2\left(a2 + t0\left(a3 + t0\left(a4 + a5\,t0\right)\right)\right) + a1^3\,x0\Big)\Big)\Big)$$

Out[578]=
$$\frac{1}{4\,a1^5}e^{-2\,a1\,t0}\,sigma^2\Big(3\,a8^2\left(e^{2\,a1\,t} - e^{2\,a1\,t0}\right) + 3\,a1\,a8\left(-e^{2\,a1\,t0}\left(a7 + 2\,a8\,t\right) + e^{2\,a1\,t}\left(a7 + 2\,a8\,t0\right)\right) + 2\,a1^3\left(-e^{2\,a1\,t0}\left(a7 + 2\,a8\,t\right)\left(a6 + t\left(a7 + a8\,t\right)\right) + e^{2\,a1\,t}\left(a7 + 2\,a8\,t0\right)\left(a6 + t0\left(a7 + a8\,t0\right)\right)\right) + 2\,a1^4\left(-e^{2\,a1\,t0}\left(a6 + t\left(a7 + a8\,t\right)\right)^2 + e^{2\,a1\,t}\left(a6 + t0\left(a7 + a8\,t0\right)\right)^2\right) + a1^2\left(-e^{2\,a1\,t0}\left(a7^2 + 6\,a7\,a8\,t + 2\,a8\left(a6 + 3\,a8\,t^2\right)\right) + e^{2\,a1\,t}\left(a7^2 + 6\,a7\,a8\,t0 + 2\,a8\left(a6 + 3\,a8\,t0^2\right)\right)\right)\Big)$$

Out[579]=
$$e^{\frac{a1^5\,e^{2.\,a1}\left(1.25 + \frac{4.2 + a1\left(1.62 + 0.641\,a1 + 0.4567\,a1^2\right) - \left(4.2 + a1\left(5.4 + 3.8\,a1 + 2.2\,a1^2\right)\right)e^{-0.9\,a1}}{a1^4}\right)^2}{50\left(0.27\,e^{0.2\,a1} + 1.26\,a1\,e^{0.2\,a1} + 2.68\,a1^2\,e^{0.2\,a1} + 3.36\,a1^3\,e^{0.2\,a1} + 2.88\,a1^4\,e^{0.2\,a1} - 0.27\,e^{2\,a1} - 0.774\,a1\,e^{2\,a1} - 0.8494\,a1^2\,e^{2\,a1} - 0.31476\,a1^3\,e^{2\,a1} - 0.066978\,a1^4\,e^{2\,a1}\right)}} \Big/ \left(5\sqrt{\left(-\frac{1}{a1^5}e^{-0.2\,a1}\left(0.27\,e^{0.2\,a1} + 1.26\,a1\,e^{0.2\,a1} + 2.68\,a1^2\,e^{0.2\,a1} + 3.36\,a1^3\,e^{0.2\,a1} + 2.88\,a1^4\,e^{0.2\,a1} - 0.27\,e^{2\,a1} - 0.774\,a1\,e^{2\,a1} - 0.8494\,a1^2\,e^{2\,a1} - 0.31476\,a1^3\,e^{2\,a1} - 0.066978\,a1^4\,e^{2\,a1}\right)\right)}\,\sqrt{2\,\pi}\right)$$

Out[580]= 0.05077

SDE: 5. product of time and state, constant diffusion

In[581]:= $dX_t = X_t\ (\alpha_1 + \alpha_2\, t)\ dt\ +\ \sigma\ dW_t$

Out[581]= $\sigma\, dW_t + dt\, X_t\, (\alpha_1 + t\, \alpha_2)$

In[582]:=
```
mySDE = ItoProcess[{x (a1 + a2 t)  , sigma }, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, sigma → 0.9, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, sigma → 0.9, x → 0, t → 1}
```

Out[582]= ItoProcess[{{(a1 + a2 t) x[t]}, {{sigma}}, x[t]}, {{x}, {x0}}, {t, t0}]

Out[583]= $e^{\frac{1}{2}(t-t0)(2\,a1+a2\,(t+t0))}\,x0$

Out[584]= $$\frac{e^{\frac{(a1+a2\,t)^2}{a2}}\sqrt{\pi}\,sigma^2\left(Erf\left[\frac{a1+a2\,t}{\sqrt{a2}}\right]-Erf\left[\frac{a1+a2\,t0}{\sqrt{a2}}\right]\right)}{2\sqrt{a2}}$$

Out[585]= 0.073969

Out[586]= 0.073969

SDE: 6. product of time and state, constant diffusion

In[587]:= $dX_t = X_t\,\alpha_1\,dt + (\sigma / (\alpha_2 + t))\,dW_t$

Out[587]= $$dt\,X_t\,\alpha_1 + \frac{\sigma\,dW_t}{t+\alpha_2}$$

```
In[588]:= mySDE = ItoProcess[{a1 * x[t] , sigma / (a2 + t) }, {x, x0}, {t, t0}]
f1 = PDF[mySDE[t], x];
(* not printed - PDF of slice distribution at time t - Gaussian *)
E1 = FullSimplify[Mean[mySDE[t]]]
(* printed below - expression for mean of slice distribution at time t *)
FullSimplify[Variance[mySDE[t]]]
(* printed below - expression for variance of slice distribution at time t *)
f2 = PDF[NormalDistribution[Mean[mySDE[t]], StandardDeviation[mySDE[t]]], x];
(* not printed *)
(* a numerical check that slice distribution is Gaussian -
 these should give identical estimates *)
f1 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, sigma → 0.9, x → 0, t → 1}
f2 /. {x0 → 1.25, t0 → 0.1, a1 → 0.3, a2 → 0.4, sigma → 0.9, x → 0, t → 1}
```

Out[588]= $\text{ItoProcess}\left[\left\{\{a1\, x[t]\}, \left\{\left\{\frac{\text{sigma}}{a2+t}\right\}\right\}, x[t]\right\}, \{\{x\}, \{x0\}\}, \{t, t0\}\right]$

Out[589]= $e^{a1\,(t-t0)}\, x0$

Out[590]= $-2\, a1\, e^{2\, a1\,(a2+t)}\, \text{sigma}^2 \left(\text{Gamma}[-1, 2\, a1\,(a2+t)] - \text{Gamma}[-1, 2\, a1\,(a2+t0)] + \text{Log}[a2+t] - \text{Log}[a1\,(a2+t)] - \text{Log}[a2+t0] + \text{Log}[a1\,(a2+t0)]\right)$

Out[591]= 0.133516

Out[592]= 0.133516