Chaos in Topological Dynamical Systems

by

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Abstract

A topological dynamical system is comprised of a continuous mapping acting on a compact metric space. Topological dynamical systems often give rise to complex and chaotic behaviour. This project examines various definitions of chaos, specifically covering Devaney chaos, Li-Yorke chaos and topological chaos. We will explore how these definitions apply to topological dynamical systems on the interval, the unit circle, in sequence space and on compact countable sets. We shall cover important topological properties of chaos, such as topological transitivity, sensitive dependence, dense periodic points, scrambled sets, Li-Yorke pairs and topological entropy. Finally, the text shall conclude by characterising various chaotic systems and comparing the various definitions of chaos.

Acknowledgements

Contents

Chapter 1

An Introduction to Topological Dynamics

The aim of this text is to introduce the reader to topological dynamical systems and explore the various interpretations of chaos through the lens of topology and topological dynamical systems. A discrete dynamical system is defined by a metric space with a corresponding continuous function, mapping the metric space to itself. This function is termed a map or mapping whereby points in the underlying metric space are mapped to other points in the set by the application of this function. Topological dynamical systems themselves are a subset of discrete dynamical systems, with the extra requirement that the underlying metric space is compact (i.e. complete and totally bounded). This extra condition for compactness is useful for investigating the limiting behaviour of the set of iterates of the map as it is repeatedly iterated to infinity; a relevant feature in the study of chaos. The term chaos itself, specifically deterministic chaos, has various definitions in mathematics and was first coined by Li and Yorke in their ubiquitous paper 'Period Three Implies Chaos' [17]. Since then, a number of authors have proposed their own definitions of chaos, hinging on the existence of various properties of the topological dynamical system. The properties we shall be studying in this text include topological transitivity, the existence of a dense orbit, the density of periodic points, the existence of an uncountable scrambled set, sensitive dependence on initial conditions and topological entropy. Due to the differences between definitions of chaos, topological dynamical systems can be chaotic according to one interpretation but not another. We shall aim to compare these definitions and understand their consequences, providing examples of topological dynamical systems that exhibit each type of chaos. Specifically, we shall restrict our attention to four compact metric spaces: closed intervals, the unit circle, sequence space and compact countable sets; as these spaces have been heavily studied in literature. This text assumes the reader to be a capable student of pure mathematics with a basic understanding of topology and analysis. The focus of this text is mainly topological; for the sake of brevity content from related areas of ergodic theory, group theory, measure theory,

etc. are excluded.

This chapter will briefly review some relevant results from topology before introducing ideas central to topological dynamics and the study of chaos in topological dynamical systems. We shall also introduce some popular topological dynamical systems that exhibit chaos, which we will be examining throughout this text. Subsequently, in Chapter 2 we will introduce notions of comparing and equivalating topological dynamical systems using the framework of topological conjugacy and symbolic dynamics. Here we will use these powerful results to prove topological results for the well-known tent map and logistic map using a topological conjugacy. Furthermore, we shall begin our study of chaos via Sharkovsky's theorem. Finally, in Chapter 3, we will study the three various interpretations of chaos from called Devaney chaos, Li and Yorke chaos, and topological chaos, analysing examples of topological dynamical systems which satisfy each definition. We shall look at the various properties topological dynamical systems require to be considered chaotic and compare the various definitions.

1.1 Topological Dynamical Systems and Discrete Dynamics

We shall start this chapter by giving the definition of a topological dynamical system, including some important definitions from topology and introducing some preliminary definitions from discrete dynamics. We shall refer back to these definitions constantly for the remainder of this text. Furthermore, prominent examples of topological dynamical systems will be presented. The reader should remember these examples as they will be integral to understanding several propositions in later chapters. Note that the definitions and results in this section apply generally to continuous maps, however, have been formulated in terms of topological dynamical systems for clarity and precision. Beginning, we shall introduce some preliminary definitions from topology.

Definition 1.1 (Dense Set). Let (Z, d) be a metric space and $X, Y \subseteq Z$ where $X \subseteq Y$. The set X is *dense* in Y if $\overline{X} = Y$, i.e. if for every $x \in Y$ there exists an open neighbourhood U of x with $y \in U$ such that $y \in X$.

Definition 1.2 (Finite Cover, Open Cover). Let (X, d) be a metric space. A *finite cover* is a collection of sets $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ such that $X \subseteq \bigcup_{i=1}^{n} C_i$. If the sets $C_1, C_2, \ldots C_n$ are all open then $\mathcal{C}$ is a *open cover*.

Definition 1.3 (Compact Space). A metric space (X, d) is *compact* if every open cover $\mathcal{U}$ of X has a finite subcover $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}\} \subseteq \mathcal{U}$, i.e. if $\{U_i\}_{i \in I}$ is a collection of open subsets of X, where $X \subseteq \bigcup_{i \in I} U_i$ then there exists a finite subcollection $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}\} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup_{j=1}^{n} U_{i(j)}$. Furthermore, a metric space is compact if and only if it is complete and totally bounded.

Note that on $\mathbb{R}$ with the standard metric, all closed intervals are compact. Since we now have a notion of what it means for a metric space to be compact, we can now define the main object we shall be studying in this paper, the topological dynamical system.

Definition 1.4 (Topological Dynamical System)**.** Let X be a non-empty compact metric space. A *topological dynamical system* denoted (X, f) is given by a continuous map $f : X \to X$. The system starts at an initial point $x \in X$ and evolves through successive iterations of the map f. After $k \in \mathbb{N}$ iterations of f, the system can be described by $f^n := f \circ f \circ \cdots \circ f$, where x is mapped to the point $f^n(x)$. By convention we take f^0 to be the identity map.

Having defined a topological dynamical system, we can now characterise the discrete dynamics of the underlying map through the following definitions.

Definition 1.5 (Orbit)**.** Let (X, f) be a topological dynamical system. The *orbit* or *forward orbit* of a point $x \in X$ under f is the set $\mathcal{O}_f(x) = \mathcal{O}_f^+(x) = \{f^n(x) : n \geq 0\} = \{x, f(x), f^2(x), \dots\}$ of iterates of x under the map f. If f is a homeomorphism (i.e f^{-1} exists and is continuous) then the *backward orbit* of x under f is similarly defined as $\mathcal{O}_f^-(x) = \{f^n(x) : n \leq 0\} = \{x, f^{-1}(x), f^{-2}(x), \dots\}$.

Note that in this text, unless otherwise stated, the term, orbit, will simply refer to the forward orbit, as we will mostly be dealing with forward dynamics. An interesting characteristic which can occur in topological dynamical systems is the existence of a dense orbit. We shall explore various consequences that systems with this property hold in Chapter 3. For now, note that from Definition 1.1 an orbit $\mathcal{O}_f(x)$ of f is said to be dense in X if for every $x \in X$ and $\varepsilon > 0$ there exists a $y \in X$ and $k \in \mathbb{N}$ such that $d(f^k(y), x) \leq \varepsilon$.

Definition 1.6 (Periodic Point, Cycle)**.** Let (X, f) be a topological dynamical system. A point $x \in X$ is *fixed* if $f(x) = x$ and *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$. The *period* of a point x is the least positive integer k such that $f^k(x) = x$. If x has a period of k we say that x is a *period-k* point. The set of all period-k points of f is denoted by $\mathrm{Per}_k(f)$ and the set of all periodic points of f is denoted $\mathrm{Per}(f)$. Moreover, $f^n(x) = x \iff n = lk$, for some $l \in \mathbb{N}$. The orbit $\mathcal{O}_f(x) = \{x, f(x), \dots, f^{k-1}(x)\}$ of a periodic point is a finite set of unique points, called a *periodic orbit* of period k or simply a *k-cycle*.

In most topological dynamical systems only a small subset of points are periodic. Most often a larger set of points either enter a periodic orbit after a certain number of iterations of f or converge asymptotically to a periodic orbit, leading us directly into the following definitions.

Definition 1.7 (Eventually Periodic, Asymptotically Periodic)**.** Let (X, f) be a topological dynamical system. A point $x \in X$ is *eventually periodic* of period k if the point x is not periodic and there exists a $n > 0$ such that $f^{k+i}(x) = f^i(x)$, for $i \geq n$. The point $x \in X$ is *asymptotically periodic* to a periodic point $p \in X$ if $\lim_{n \to \infty} d(f^n(x), f^n(p)) = 0$.

Proposition 1.8. *Let (X, f) be a topological dynamical system. If f is an invertible map, then every eventually periodic point is periodic.*

Proof. Suppose $x \in X$ is eventually periodic of period k in f. Then $f^{k+i}(x) = f^i(x)$ for some $n > 0$. By applying i iterations of f^{-1} we obtain, $f^{-i} \circ f^{k+i}(x) = f^{-i} \circ f^i(x) \implies f^k(x) = x$. Hence x is periodic with period k. $\qquad\square$

When studying chaos in topological dynamical systems it can be useful to understand how the system behaves for an increasing number of iterations. The ω-limit set, defined below as the set of limit points of a particular orbit, allows us to understand the behaviour of the system asymptotically.

Definition 1.9 (Omega-limit Set)**.** Let (X, f) be a topological dynamical system. The ω-*limit* set of $x \in X$, denoted $\omega(x, f)$ is the set of all limit points of the orbit $\mathcal{O}_f(x)$ given by

$$\omega(x, f) := \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k \geq n\}}$$

and the ω-*limit* set of the entire map f is defined as

$$\omega(f) := \bigcup_{x \in X} \omega(x, f)$$

By the definition, we can immediately see that the ω-limit set of a period-k point or eventually periodic point of period-k is simply the k-cycle. Furthermore, if a point is asymptotically periodic then the ω-limit set is clearly a cycle and hence finite. Now we are done with defining properties of discrete dynamics let's analyse the discrete dynamics of some popular topological dynamical systems.

Example 1.10 (Logistic Map)**.** Define $F_\mu : [0, 1] \to [0, 1]$ to be the *logistic map*, where $F_\mu(x) = \mu x(1 - x)$ and $\mu > 0$. Since $[0, 1]$ is a closed interval it is compact. Hence $([0, 1], F_\mu)$ describes a topological dynamical system.
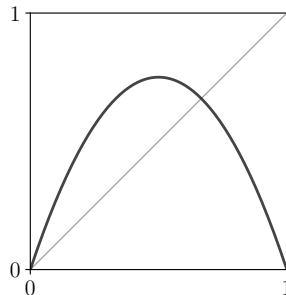

Figure 1.1: Logistic map F_μ with $\mu = 3$.

Example 1.11 (Tent Map). Define $T_s : [0, 1] \rightarrow [0, 1]$ to be the *tent map*, where $s \in (1, 2]$, $T_s(x) = sx$ for $x \in \left[0, \frac{1}{2}\right]$ and $T_s(x) = s(1 - x)$ for $x \in \left[\frac{1}{2}, 1\right]$. Since $[0, 1]$ is a closed interval it is compact. Hence $([0, 1], T_s)$ describes a topological dynamical system. The dynamics of this system becomes difficult to understand as the number of iterations of T_s increases due to the function's piecewise definition. In Chapter 2 we shall develop the technique of using symbolic dynamics to better analyse the topological dynamics of this system.
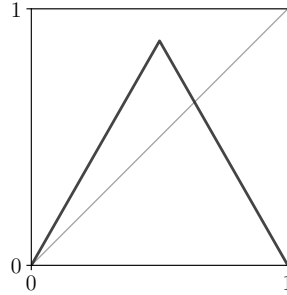

Figure 1.2: Tent map T_s with $s = \frac{7}{4}$.

Example 1.12 (Doubling Map on $[0, 1]$). Define $D : [0, 1] \rightarrow [0, 1]$ to be the *doubling map on* $[0, 1]$, where $D(x) = 2x$ for $x \in \left[0, \frac{1}{2}\right]$ and $D(x) = 2x - 1$ for $x \in \left[\frac{1}{2}, 1\right]$. Since $[0, 1]$ is a closed interval it is compact. However, the map D is not continuous as a discontinuity exists at $x = \frac{1}{2}$ and so this is not a dynamical system, however, we shall use this maps properties to establish results about other topological dynamical systems. Just like the tent map, the dynamics of this system become difficult to understand as the number of iterations of D increases.
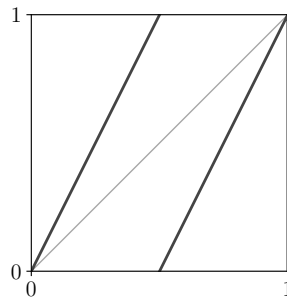

Figure 1.3: Doubling map D.

Note that throughout this text we shall define $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and is the unit circle in the complex plane. The metric space (S^1, d) is defined using the arc length metric, where $d(x, y)$ is the shortest arc connecting x to y.

Example 1.13 (Doubling Map on S^1). Define $\mathcal{D} : S^1 \to S^1$ to be the *doubling map on S^1*, where $\mathcal{D}(z) = z^2$, or equivalently $\mathcal{D}(e^{i\theta}) = e^{2i\theta}$ for some $\theta \in \mathbb{R}$.
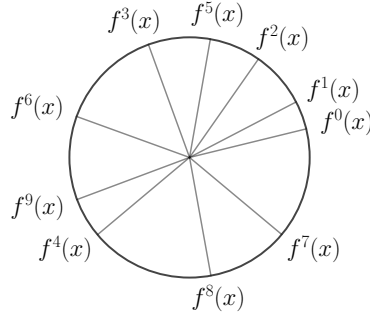

Figure 1.4: First ten iterations of the doubling map $\mathcal{D}$.

Example 1.14 (Rigid Rotations). Define $R_\alpha : S^1 \to S^1$ to be the *rigid rotations of the unit circle*, where $\alpha \in [0, 2\pi)$ and the function $R_\alpha(z) = ze^{i\alpha}$, or equivalently $R_\alpha(e^{i\theta}) = e^{i(\theta+\alpha)}$. Note that S^1 is compact as it is a closed subset of $\mathbb{R}$. Hence (S^1, R_α) describes a topological dynamical system.
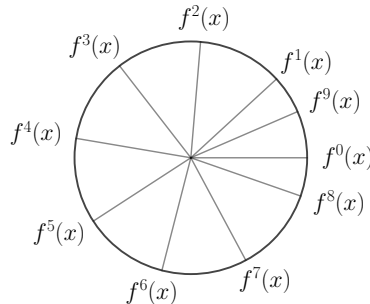

Figure 1.5: First ten iterations of the rigid rotations R_α.

An interesting property of the rigid rotations is how the behaviour of the dynamical system changes depending on the rationality or irrationality of α. This brings us to the following proposition.

Proposition 1.15. *If α is an irrational number, then for all $z \in S^1$, the orbit $\mathcal{O}_{R_\alpha}(z)$ is infinite and dense on S^1.*

Proof. Let $z \in S^1$ be arbitrary. If $R_\alpha^m(z) = R_\alpha^n(z)$ for some $m, n \in \mathbb{Z}$ then $ze^{(m-n)i\alpha} = z \implies (m-n)i\alpha = 0$. Since $\alpha \notin \mathbb{Q}$ and $(m-n)\alpha \neq 2\pi n$ for $n \in \mathbb{N}$ we have $m = n$. Hence all the points in the orbit are distinct and so $\mathcal{O}_{R_\alpha}(z)$ is infinite. Now let $w \in S^1$ be arbitrary and let $\varepsilon > 0$. Choose N such that $\frac{2\pi}{N} < \varepsilon$. Now there exists $0 \leq l, k \leq N$ such that $d\left(R_\alpha^k, R_\alpha^l\right) \leq \frac{2\pi}{N}$.

As R_α is an isometry i.e. $d(R_\alpha(x), R_\alpha(y)) = d(x, y)$ we obtain $d(R_\alpha^{(k-l)}(z), z) \leq \varepsilon$. Now since R_α is an isometry, the set $X = \{R_\alpha^{n(k-l)}(z) : n \in \mathbb{N}\}$ partitions S^1 into arcs of of length less than ε. Hence there must exist a $R_\alpha^{i(k-l)}(z) \in X$ such that $d(R_\alpha^{i(k-l)}(z), w) \leq \varepsilon$ and so $\mathcal{O}_{R_\alpha}(z)$ is dense on S^1. $\qquad\square$

Depending on the parameters s, μ and α the dynamical behaviour of these systems can range from predictable periodicity to chaotic. In Chapter 2 we shall subsequently prove that T_2 and F_4 are similar topologically speaking and share various topological properties. Moreover, both of these topological dynamical systems are described by simple mathematical equations, however, we shall prove later that they are chaotic and exhibit highly complex and irregular dynamics.

Chapter 2

Topological and Symbolic Relationships

In this chapter, we shall explore ways in which we can infer the topological properties of topological dynamical systems by using topological and symbolic relationships. In Section 2.1 we shall introduce topological conjugacy, a term used to describe when two maps exhibit the same topological behaviour. This section will include proving that various topological properties hold through topological conjugacy and introduce examples of topologically conjugate systems. In Section 2.2 we shall develop symbolic methods of characterising topological dynamical systems using infinite sequences of symbols. This proves useful for describing the topological dynamics of various topological dynamical systems where the dynamics are easier to explain using sequences of symbols. As a result, this section will allow us to prove compelling results about the tent and doubling maps. Section 2.3 will then introduce the reader to Sharkovsky's theorem, a very important result which explains when that if a topological dynamical system defined over intervals on $\mathbb{R}$ has a period three point, then it has points of all other periods. Furthermore, we shall then show how we can construct topological dynamical systems over closed intervals with a specific set of periods as defined by Sharkovsky's order.

2.1 Topological Conjugacy

As mentioned above, topological conjugacy defines when two maps exhibit the same topological behaviour and can be considered equivalent. If two maps are topologically conjugate, properties we have proved for one map can be applied to the other. Hence topological conjugacy is a powerful relationship when studying the dynamics of related topological dynamical systems. Note that just as in Section 1.1, the definitions below apply generally to continuous maps, however, have been formulated in terms of topological dynamical systems for clarity and precision. First, let's introduce our main definition.

Definition 2.1 (Topological Conjugacy)**.** Let (X, f) and (Y, g) be topological dynamical systems. The topological dynamical system (Y, g) is *topologically semi-conjugate* to (X, f) if there exists a

continuous, surjective map $\varphi : X \to Y$ termed a *topological semi-conjugacy* where $\varphi \circ f = g \circ \varphi$. Furthermore if φ is a homeomorphism i.e. φ is a bijection with φ and φ^{-1} both continuous then (Y, g) is *topologically conjugate* to (X, f). In this case φ is called a *topological conjugacy*.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X \\
\varphi \downarrow & & \downarrow \varphi \\
Y & \xrightarrow{\ g\ } & Y
\end{array}
$$

The diagram above is a termed commutative diagram and shows the relationship between the topological dynamical systems and the topological conjugacy. If φ is a topological semi-conjugacy then this diagram commutes. If φ is a just semi-conjugacy the topological dynamics of (X, f) are carried over to (Y, g), however, if φ is not a homeomorphism the reverse does not hold generally. Only when φ is homeomorphic and so a topological conjugacy do the dynamics of (Y, g) carry over into (X, f). Now we shall prove that the topological dynamical systems T_2 and F_4 are topologically conjugate and that T_2 and D are topologically semi-conjugate. We shall see later in Section 2.2 that they also share the same topological dynamics.

Proposition 2.2. *The tent map* $([0, 1], T_2)$ *and the logistic map* $([0, 1], F_4)$ *are topologically conjugate.*

Proof. Let $\varphi : [0, 1] \to [0, 1]$ be defined by $\varphi(x) = \sin^2(\frac{\pi x}{2})$ which is homeomorphic on $[0, 1]$ as is continuous, bijective and $\varphi^{-1} = \frac{2}{\pi} \sin^{-1}(\sqrt{x})$ exists and is continuous on $[0, 1]$. Clearly we have,

$$F_4 \circ \varphi(x) = 4 \sin^2 \left(\frac{\pi x}{2} \right) \cdot \left(1 - \sin^2 \left(\frac{\pi x}{2} \right) \right) = 4 \sin^2 \left(\frac{\pi x}{2} \right) \cos^2 \left(\frac{\pi x}{2} \right) = \sin^2 \pi x, \text{ for } x \in [0, 1]$$

$$\varphi \circ T_2(x) = \varphi(2x) = \sin^2 \pi x, \text{ for } x \in [0, 1/2]$$

$$\varphi \circ T_2(x) = \varphi(2(1 - x)) = \sin^2(\pi(1 - x)) = \sin^2 \pi x, \text{ for } x \in [1/2, 1]$$

Hence we have shown that $\varphi \circ T_2 = F_4 \circ \varphi$ so F_4 and T_2 are topologically conjugate. $\qquad \square$

Proposition 2.3. *The tent map* $([0, 1], T_2)$ *and the doubling map* $([0, 1], D)$ *are topologically semi-conjugate.*

Proof. Let $\varphi : [0, 1] \to [0, 1]$ be the tent map T_2 which is continuous and surjective. From the definition $\varphi(x) = 2x$ if $x \in \left[0, \frac{1}{2} \right]$ and $\varphi(x) = 2(1 - x)$ if $x \in \left[\frac{1}{2}, 1 \right]$. Then separating by cases we have,

$$\varphi \circ D(x) = 2(2x) = 4x \text{ and } T_2 \circ \varphi(x) = 2(2x) = 4x, \text{ if } x \in [0, 1/4]$$

$$\varphi \circ D(x) = 2(1 - (2x)) = 2 - 4x \text{ and } T_2 \circ \varphi(x) = 2(1 - 2x) = 2 - 4x, \text{ if } x \in [1/4, 1/2]$$

$$\varphi \circ D(x) = 2(2x - 1) = 4x - 2 \text{ and } T_2 \circ \varphi(x) = 2(1 - 2(1 - x)) = 4x - 2, \text{ if } x \in [1/2, 3/4]$$

$\varphi \circ D(x) = 2(1 - (2x - 1)) = -4x + 4$ and $T_2 \circ \varphi(x) = 2(2(1 - x)) = -4x + 4$, if $x \in [3/4, 1]$

Hence $\varphi \circ D = T_2 \circ \varphi$, so D and T_2 are topologically semi-conjugate. $\qquad\square$

Topological conjugacy preserves topological dynamical properties. For instance if topological dynamical systems (X, f) and (Y, g) are topologically conjugate through some map φ then if x is a fixed point of f, $\varphi(x)$ is a fixed point of g. This can be proved as $\varphi(x) = \varphi \circ f(x) = g \circ \varphi(x)$. This next lemma will prove the more general result for period-k points.

Proposition 2.4. *Let (X, f) and (Y, g) be topological dynamical systems and let $\varphi : X \to Y$ be a topological semi-conjugacy. If x is a period-k point of f then $\varphi(x)$ is a period-k point of g. Furthermore, if φ is a topological conjugacy then if $\varphi(x)$ is a period-k point of g then x is a period-k point of f.*

Proof. Suppose $f^k(x) = x$, then by induction $\varphi(x) = \varphi \circ f^k(x) = g^k \circ \varphi(x)$. Since φ is invertible $f^k(x) = \varphi \circ f^k \circ \varphi^{-1}(x)$. Now suppose $g^k(\varphi(x)) = \varphi(x)$, then $x = \varphi \circ \varphi^{-1}(x) = g^k \circ \varphi \circ \varphi(x)^{-1} = \varphi \circ f^k \circ \varphi^{-1}(x) = f^k(x)$. $\qquad\square$

Therefore, to understand the periodic points of one dynamical system we can analyse the periodic points of another system which is topologically conjugate to it. This becomes useful when dealing with dynamical systems which have particularly complex behaviour. This next proposition states that topological conjugacy holds for iterations of the maps f^n and g^n.

Proposition 2.5. *Let (X, f) and (Y, g) be topological dynamical systems and $n \in \mathbb{N}$. If $\varphi : X \to Y$ is a topological semi-conjugacy from (X, f) to (Y, g) then φ is a topological semi-conjugacy from (X, f^n) to (X, g^n)*

Proof. Let φ be a topological semi-conjugacy, so $\varphi \circ f = g \circ \varphi$. Hence, $\varphi \circ f^n = \varphi \circ f \circ f^{n-1} = g \circ \varphi \circ f \circ f^{n-2} = \cdots = g^{n-1} \circ \varphi \circ f = g^n \circ \varphi$. $\qquad\square$

Further topological properties that conjugacy preserves are the following.

Proposition 2.6. *Let (X, f) and (Y, g) be topological dynamical systems and let $\varphi : X \to Y$ be a topological semi-conjugacy with $x \in X$. If $\mathcal{O}_f(x)$ is an orbit of f then $\mathcal{O}_g(\varphi(x))$ is an orbit of g.*

Proof. Let $x \in X$ with $y = \varphi(x)$. By definition, $\mathcal{O}_f(x) = \{f^n(x) : n \geq 0\}$. Hence applying φ to the orbit we obtain $\varphi(\mathcal{O}_f(x)) = \{\varphi \circ f^n(x) : n \geq 0\} = \{g^n \circ \varphi(x) : n \geq 0\} = \mathcal{O}_g(\varphi(x))$ using Proposition 2.5. $\qquad\square$

Stated alternatively, topological conjugacy ensures orbits in (X, f) are sent to orbits in (Y, g). This next proposition follows directly from the proposition above.

Proposition 2.7. *Let (X, f) and (Y, g) be topological dynamical systems and let $\varphi : X \to Y$ be a topological semi-conjugacy. If f has a dense orbit in X then g has a dense orbit in Y.*

Proof. Before we begin, note that, from topology, if $f : X \to Y$ is a continuous, surjective function and $\overline{E} = X$ then $\overline{f(E)} = Y$. Now suppose $x \in X$ has a dense orbit in f, so $\overline{\mathcal{O}_f(x)} = X$. By definition, φ is continuous and surjective, hence $\overline{\varphi(\mathcal{O}_f(x))} = \overline{\mathcal{O}_g(\varphi(x))} = Y$ by Proposition 2.6. Hence g has a dense orbit in Y. $\qquad\square$

Proposition 2.8. *Let (X, f) and (Y, g) be topological dynamical systems and let $\varphi : X \to Y$ be a topological semi-conjugacy. If $Per(f)$ are dense in X then $Per(g)$ are dense in Y.*

Proof. Suppose $\overline{Per(f)} = X$. By continuity, $g(Y) \circ \varphi = \varphi \circ f(X) = \varphi \circ f(\overline{Per(f)}) \subseteq \varphi \circ \overline{f(Per(f))} = \overline{g(Per(g))} \circ \varphi$. Hence $g(Y) \subseteq \overline{g(Per(g))}$. Hence $Per(g)$ is dense in Y. $\qquad\square$

Both Proposition 2.7 and Proposition 2.8 turn out to be important in proving examples of topological dynamical systems are chaotic. Concluding this section, note the wide range of topological dynamics that is preserved between topologically conjugate systems. We shall apply these results in the next section on symbolic dynamics as we further characterise complex topological dynamics.

2.2 Symbolic Dynamics

Symbolic dynamics studies how the shift map affects infinite sequences of symbols, called itineraries, that describe the complex dynamics of specific topological dynamical systems. More specifically it is the assignment of a sequence of discrete symbols to the orbits of topological dynamical systems. This proves useful as in specific instances, like the tent map and doubling map over the unit interval, the behaviour of topological dynamical systems becomes easier to characterise using topologically conjugate, topological dynamical systems described symbolically using itineraries. In the process of applying symbolic dynamics to a topological dynamical system, we separate the underlying metric space into finitely many labelled partitions. We then generate a so-called itinerary by noting the infinite sequence of symbols obtained, describing the inclusion of a point in a partition for successive iterations of the map. This sequence is an element in the so-called sequence space, and it is in this space where we can study the discrete dynamics of the topological system. Let's now define this notion concretely.

Definition 2.9 (Itinerary). Let (X, f) be a topological dynamical system where X is a compact set. Let $x \in X$ and suppose $X = X_0 \cup X_1 \cup \cdots \cup X_n$ where X_i are compact sets. The *itinerary* of x is the sequence $\underline{s} = (s_1 s_2 \dots)$ where s_i denotes the set X_i such that $f^i(x) \in X_i$.

Figure 2.1 gives an example of useful itineraries for the tent map and doubling map. Properties of the itinerary reflect properties of the topological dynamical system it represents. If a point
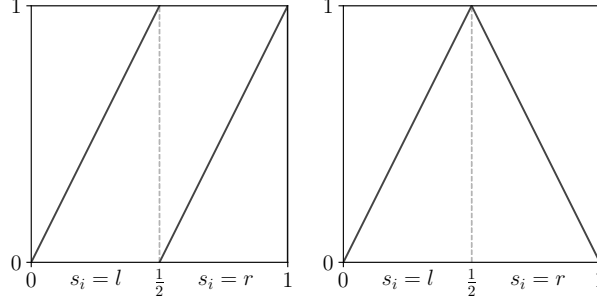
Figure 2.1: Itineraries for the tent map $([0,1], T_2)$ and the doubling map $([0,1], D)$ respectively.

the topological dynamical system is periodic, then that same point will give a periodic itinerary. Now we shall construct a metric space in which where the underlying set contains all possible itineraries of zeros and ones.

Definition 2.10 (Sequence Space). Let $\Sigma_2 = \{(s_1 s_2 \dots) : s_i \in \{0,1\}\}$ be the set of itineraries of zeros and ones. Define (Σ_2, d) to be the *sequence space* where $d(s,t) = \Sigma_{i=1}^{\infty}|s_i - t_i|2^{-i}$ is a metric for $(s)_{i=1}^{\infty} = (s_1 s_2 \dots) \in \Sigma_2$ and $(t)_{i=1}^{\infty} = (t_1 t_2 \dots) \in \Sigma_2$. It can be easily shown this is a metric space.

A useful topological dynamical system to use symbol dynamics on would be one that maps each point in a compact set to its itinerary in sequence space. Hence we have the following definition.

Definition 2.11 (Itinerary Map). Let X be a compact set and Σ_2 be the sequence space. Suppose $x \in X$. Define $\pi : X \to \Sigma_2$ to be the *itinerary map* where $\pi(x) = (s_1 s_2 \dots)$ i.e. π maps each $x \in X$ to its itinerary.

A property that we want the sequence space to have is for it to be compact. Hence we can then use it a the underlying metric space for some topological dynamical systems. Therefore, we shall now prove in the following proposition that it is compact.

Proposition 2.12. *The sequence space (Σ_2, d) where $d(s,t) = \Sigma_{i=1}^{\infty}|s_i - t_i|2^{-i}$ is compact.*

Proof. Let $(x_n)_n$ be a Cauchy sequence in Σ_2. Let $k \in \mathbb{N}$, where $x_i(k)$ denotes the k-th element of x_i. The sequence $x_1(k), x_2(k) \dots$ is eventually constant as if not we can find $n, m \in \mathbb{N}$ such that $x_n(N) \neq x_m(N)$ for some $N \in \mathbb{N}$ and hence $d(x_n, x_m) \geq \frac{1}{2^N}$, a contradiction. Let s_k be the eventually constant term in $(x_n)_n$. Hence we can construct $s = (s_k : k \in \mathbb{N}) \in \Sigma_2$ such that the sequence $(x_n)_n$ converges to s. We have proved Σ_2 is complete. Let $\varepsilon > 0$ and choose N such that $\frac{1}{2^{N-1}} \leq \varepsilon$. Hence $\{B_d(s,\varepsilon) : s \in \Sigma_2 \text{ where } s_k = 0, \forall k \geq N\}$ cover Σ_2. Hence Σ_2 is totally bounded and so is compact. $\qquad \square$

Now that we have proved (Σ_2, d) is a compact metric space, if we have an associated continuous mapping $f : \Sigma_2 \to \Sigma_2$ then we have a topological dynamical system. Hence we can now introduce an important topological dynamical system to the study of symbolic dynamics.

Definition 2.13 (Shift Map). Let $(s)_{i=1}^{\infty} \in \Sigma_2$. The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma\left((s)_{i=1}^{\infty}\right) = (s)_{i=2}^{\infty}$ and describes a topological system (Σ_2, σ).

Thereby, the shift map removes the first element from a sequence. We shall now prove that the shift map is continuous and so (Σ_2, σ) is a topological dynamical system.

Proposition 2.14. *The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is continuous.*

Proof. Let $\varepsilon > 0$ and choose $\underline{s} = (s_i)_{i=1}^{\infty}$, $\underline{t} = (t_i)_{i=1}^{\infty} \in \Sigma_2$ such that $d(\underline{s}, \underline{t}) = \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i} < \delta$. Choose n such that $2^{-n} \leq \varepsilon$ and let $\delta = 2^{-(n+1)}$. Hence $\underline{s}$ and $\underline{t}$ agree on the first $n+1$ symbols and $\sigma(\underline{s})$ and $\sigma(\underline{t})$ agree on the first n symbols. Then $d(\sigma(\underline{s}), \sigma(\underline{t})) = d\left((s)_{i=n+1}^{\infty}, (t)_{i=n+1}^{\infty}\right) = \Sigma_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} \leq 2^{-n} \leq \varepsilon$. $\qquad\square$

Since (Σ_2, σ) defines a topological dynamical system we shall use symbolic dynamics to prove a few important properties about it, which will come in useful when characterising chaos in Chapter 3. For now remember these results, from [11, §1.6], as we will be referring back to them.

Proposition 2.15. *The periodic points of the shift map (Σ_2, σ) are dense in Σ_2.*

Proof. Let $\underline{s} = (s)_{i=1}^{\infty}$ be an arbitrary point in Σ_2 and let $\varepsilon > 0$. Pick an n such that $2^{-n} \leq \varepsilon$ and define $t_n = (s_0, \ldots, s_n, s_0, \ldots, s_n, \ldots)$ to be an infinite repeating sequence where $t_i = s_i$ for $1 \leq i \leq n$. Then $d(s, t) = \sum_{i=0}^{n} |s_i - t_i| 2^{-i} + \sum_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} = \sum_{i=n+1}^{\infty} 2^{-i} \leq 2^{-n} \leq \varepsilon$. Hence as $n \to \infty$ we have $t_n \to \underline{s}$. Since $\underline{s}$ was arbitrary, the periodic points of σ are dense. $\quad\square$

Proposition 2.16. *The shift map (Σ_2, σ) has a dense orbit.*

Proof. Consider the sequence $\underline{s} = (0, 1, \{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}, \{0,0,0\}, \{0,0,1\}, \ldots)$ of 0s and 1s sorted in len-lex order. Let $\underline{t} = (t)_{i=0}^{\infty} \in \Sigma_2$ be arbitrary and $\varepsilon > 0$. By construction we can perform some $k \in \mathbb{N}$ iterations of σ such that the first n symbols of $\sigma^k(\underline{s})$ and $\underline{t}$ agree. Choose $N \geq \log_2 \frac{1}{\varepsilon}$. Therefore for $n \geq N - 1$ we have that, $d(\sigma^k(\underline{s}), \underline{t}) = \sum_{i=n}^{\infty} |\sigma^k(\underline{s})_i - t_i| 2^{-i} \leq 2^{-(n+1)} < 2^{-N} \leq \varepsilon$. Hence the orbit of $\underline{s}$ is dense in Σ_2. $\qquad\square$

Symbolic dynamics has an elegant application to various maps such as the tent map and doubling map. We shall first investigate the doubling map, then we can later apply these results to the tent map. The nature of the doubling map $D : [0,1] \to [0,1]$ makes binary expansions a suitable choice for expressing points in this map. For each $x \in [0,1]$ we can write $x = \sum_{i=1}^{\infty} b_i 2^{-i}$ where $b_i \in \{0,1\}$. This next proposition describes an important relationship between the doubling map and the shift map which we shall explore through the use of binary expansions.

Proposition 2.17. *The doubling map* $([0,1], D)$ *and the shift map* (Σ_2, σ) *are semi-conjugate via the semi-conjugacy* $\varphi : \Sigma_2 \to [0,1]$ *given by* $\varphi\left((b_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} b_i 2^{-i}$.

Proof. Let $\underline{b} = (b_i)_{i=1}^{\infty} \in \Sigma_2$ be a sequence of binary digits and let $\underline{c} = \sigma\left((b_i)_{i=1}^{\infty}\right) = (c_i)_{i=1}^{\infty} \in \Sigma_2$ where $c_i = b_{i+1}$ be the sequence $\underline{b}$ shifted once. Define the map $\varphi : \Sigma_2 \to [0,1]$ where $\varphi\left((b_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} b_i 2^{-i} \in [0,1]$. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map. The function φ is surjective, as every point $x \in [0,1]$ has at least one binary expansion denoted $x = \sum_{i=1}^{\infty} b_i 2^{-i}$ where $b_i \in \{0,1\}$.

$$
\begin{array}{ccc}
\Sigma_2 & \xrightarrow{\ \sigma\ } & \Sigma_2 \\
\varphi \downarrow & & \downarrow \varphi \\
[0,1] & \xrightarrow{\ D\ } & [0,1]
\end{array}
$$

Using an arbitrary $\underline{b} \in \Sigma_2$, $\varphi \circ \sigma\left((b_i)_{i=1}^{\infty}\right) = \varphi\left((c_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} c_i 2^{-i} = \sum_{i=1}^{\infty} b_{i+1} 2^{-i} \in [0,1]$. Similarly $D \circ \varphi\left((b_i)_{i=1}^{\infty}\right) = D\left(\sum_{i=1}^{\infty} b_i 2^{-i}\right) = \sum_{i=1}^{\infty} 2b_i 2^{-i} \pmod 1 = b_1 + \sum_{i=2}^{\infty} b_i 2^{-i+1} \pmod 1 = \sum_{i=2}^{\infty} b_i 2^{-i+1} = \sum_{j=1}^{\infty} b_{j+1} 2^{-j} \in [0,1]$. Hence we have shown $\varphi \circ \sigma = D \circ \varphi$. and so the doubling map D and the shift map σ are semi-conjugate via φ. We can also see φ is not injective as $\frac{1}{2} + \sum_{i=2}^{\infty} 0 \cdot 2^{-i} = \sum_{i=2}^{\infty} 2^{-i} = \frac{1}{2}$ and so φ is not a homeomorphism and hence is simply a semi-conjugacy. $\qquad \square$

Intuitively this makes sense as the doubling map acts as a shift on the binary expansion of a number. Moreover, the semi-conjugacy φ can be thought of as an inverse itinerary map as defined in Definition 2.11. We can use this semi-conjugacy gained via symbolic dynamics to now prove that the periodic points in the doubling map D and by extension the logistic map F_4 and the tent map T_2 are dense.

Proposition 2.18. *The periodic points of the logistic map* $([0,1], F_4)$, *the tent map* $([0,1], T_2)$ *and the doubling map* $([0,1], D)$ *are dense in* $[0,1]$.

Proof. By Proposition 2.2 the logistic map $([0,1], F_4)$ is conjugate to the tent map $([0,1], T_2)$. Similarly, by Proposition 2.3 the tent map $([0,1], T_2)$ is semi-conjugate to the doubling map $([0,1], D)$. By Proposition 2.17 the doubling map $([0,1], D)$ is semi-conjugate to the shift map (Σ_2, σ). In Proposition 2.15 the shift map σ has periodic orbits. Finally, by Proposition 2.7 dense orbits are preserved under semi-conjugacy so D, T_2 and F_4 all have periodic points that are dense in $[0,1]$. $\qquad \square$

$$
\begin{array}{ccc}
\Sigma_2 & \xrightarrow{\ \sigma\ } & \Sigma_2 \\
\varphi_1 \downarrow & & \downarrow \varphi_1 \\
[0,1] & \xrightarrow{\ D\ } & [0,1] \\
\varphi_2 \downarrow & & \downarrow \varphi_2 \\
[0,1] & \xrightarrow{\ T_2\ } & [0,1] \\
\varphi_3 \downarrow & & \downarrow \varphi_3 \\
[0,1] & \xrightarrow{\ F_4\ } & [0,1]
\end{array}
$$

The power of topological conjugacy and symbolic dynamics is clear from Proposition 2.18. It shows how we can use symbolic dynamics to build properties about a simple topological dynamical system like the shift map, then use topological conjugacy to show that these properties hold for more complex dynamical systems. Proposition 2.18 will be extremely useful when generating examples of chaotic topological dynamical systems in Chapter 3.

2.3 Sharkovsky's Theorem and Type

Sharkovsky's forcing theorem, often abbreviated to Sharkovsky's theorem [20], is a consequential theorem in the study of topological dynamics. The theorem only applies to topological dynamical systems defined over closed intervals, however, is still a powerful result. It states that if such a topological dynamical system has a period three point then periodic points of all other periods occur. Moreover the presence of a periodic point of given period implies the existence of other periods given by a total ordering called Sharkovsky's Order. Note that for the sake of brevity we shall note be proving Sharkovsky's theorem in this text. However, we shall prove a special case of Sharkovsky's theorem proved by Li and Yorke in their paper *'Period Three Implies Chaos'* [17] in which the existence of a period three point implies the existence of periodic points of all other periods. Note that as mentioned in the introduction this was the first paper that coined the term chaos in am mathematical context. First, we will introduce a theorem that proves the existence of fixed points on closed intervals. This theorem is a simplified version of Brouwer's fixed-point theorem [8]. The full theorem is used to prove the existence of fixed points in a topological dynamical system where the underlying set is a subset of $\mathbb{R}^n$ and has the condition of being convex. Since Sharkovsky's theorem only requires the simplified theorem, we shall once again only prove this specialised case.

Theorem 2.19 (Brouwer's Fixed Point Theorem)**.** *If (I, f) is a topological dynamical system where I is a closed interval and $I \subseteq f(I)$, then f has a fixed point in I.*

Proof. Let $I = [a, b]$ where $a, b \in \mathbb{R}$. Since $I \subseteq f(I)$ there exists $c, d \in I$ such that $f(c) = a$ and $f(d) = b$. Suppose $g(x) = f(x) - x$, then g is continuous. Also $g(c) = a - c \geq 0$ and $g(d) = b - d \leq 0$. By the Intermediate Value Theorem there exists some $x \in I$ with $a \leq x \leq b$ such that $g(x) = 0$. Hence $f(x) = x$. $\square$

Before we prove the theorem by Li and Yorke we shall explore intervals, coverings and loops through the following definition.

Definition 2.20 (Covering, N-loop)**.** Let $J_0, J_1, \ldots J_{n-1} \subseteq X$ be intervals and let $f : X \to X$ be a continuous map. The interval J_0 *covers* J_1 if $J_1 \subseteq F(J_0)$ and is denoted $J_0 \xrightarrow{f} J_1$. A series of coverings $J_0 \xrightarrow{f} \cdots \xrightarrow{f} J_{n-1} \xrightarrow{f} J_0$ which starts and ends at the same interval is called a *loop*, or specifically, *n-loop* of intervals. A point $x \in X$ *follows* the loop if $f^k(x) \in J_k$ for $0 \leq k \leq n - 1$ and $f^n(x) = x$.

Note that in this text, to denote a covering we will drop the f and just write $J_0 \to J_1$ as we shall only deal with one map at a time and hence don't need to be explicit. The proof of *'Period Three Implies Chaos'* [17] relies upon us first deriving a set of intervals with specific coverings between them to form a loop. Once we have a loop we can then use Theorem 2.19 to prove the existence of a periodic point with a period the size of this loop. This theorem is a specific case of Sharkovsky's theorem and hence is concise enough for us to prove in this text. The proof for this theorem follows work by Devaney [11, §1.10].

Theorem 2.21 (Period Three Implies Chaos)**.** *Let (I, f) be a topological dynamical system where I is a closed interval. If f has a point of period three, then f has period points of all other periods.*

Proof. Before we begin the proof note the following observation. If $A_0, A_1, \ldots$ are closed intervals with $A_{i+1} \subseteq f(A_i)$ for $0 \leq i \leq n - 1$, i.e. $A_0 \to A_1 \cdots \to A_{n-1}$, then there exists a subinterval $J_0 \subseteq A_0$ such that $f(J_0) \subseteq A_1$. Similarly, there exists a subinterval $J_1 \subseteq A_1$ such that $f(J_1) \subseteq A_2$. Hence there exists a subinterval $J_1 \subseteq J_0$ such that $f(J_1) \subseteq A_1$ and $f^2(J_1) = A_2$. Continuing we get a nested sequence of intervals which are mapped in order into each A_i. Hence there exists some $x \in A_0$ such that $f^i(x) \in A_i$ for all i.
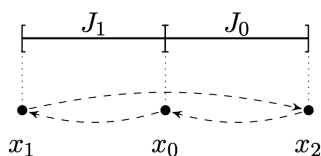
Now to begin the proof let $x_1, x_0, x_2 \in \mathbb{R}$ with $f(x_0) = x_1$, $f(x_1) = x_2$ and $f(x_2) = x_0$ and assume wlog that $x_1 < x_0 < x_2$. Let $J_1 = [x_1, x_0]$ and $J_0 = [x_0, x_2]$. Hence $J_1 \subseteq f(J_0)$ and $J_1 \cup J_0 \subseteq f(J_1)$. By Theorem 2.19, f has a fixed point between x_1 and x_0. Similarly, f^2 has at least one fixed point between x_0 and x_2. Hence f has a point of period 2. Now let $k \geq 2$.

Define the nested sequence of intervals $A_0, A_1, \ldots, A_{k-2} \subseteq J_1$ inductively. Let $A_0 = J_0$. Since $J_1 \subseteq f(J_1)$, there exists a subinterval $A_1 \subseteq A_0$ and $f(A_1) = A_0 = J_1$. Furthermore there exists a subinterval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$ and therefore $f^2(A_2) = A_0 = J_1$. Continuing we get a subinterval $A_{k-2} \subseteq A_{k-3}$ such that $f(A_{k-2}) = A_{k-3}$. By the observation above if $x \in A_{k-2}$ then $f(x), f^2(x), \ldots, f^{k-2}(x) \subseteq A_0$ so $f^{k-2}(A_{k-2}) = A_0 = J_1$. As $J_0 \subseteq f(J_1)$ there exists a subinterval $A_{k-1} \subseteq A_{k-2}$ such that $f^{k-1}(A_{k-1}) = J_0$. Since $J_1 \subseteq f(J_0)$, $J_1 \subseteq f^k(A_{k-1})$ so that $f^k(A_{k-1})$ covers A_{k-1}. Hence by our first observation f^k has a fixed point x in A_{k-1} and so f has a period-k point in A_{k-1}. The first $k-2$ iterates of x are in J_1, the $k-1$th iterate is in J_0 and the kth is x. If $f^{k-1}(x)$ is in the interior of J_0 then x has period k. However, if $f^{k-1}(x)$ is in the boundary of J_0 then $k = 2$ or 3.

$$J_0 \underset{f^{k-1}(x)}{\overset{f^k(x) = x}{\rightleftarrows}} J_1 \quad\circlearrowright\; f^i(x) \text{ for } 0 \le i \le k-2$$

$\square$

The diagram above demonstrates the behaviour. By Definition 2.20 take $I_i \to I_j$ to mean that $f(J_i)$ covers J_j i.e. $J_j \subseteq f(J_i)$. If $x \in J_1$ then we can produce a period-k cycle for any $k \in \mathbb{N}$. This is because we can find a sequence of open intervals $J_1 \to A_1 \to \cdots \to A_k \to J_1$ where $\{A_i \in \{J_1, J_0\} : 1 \le i \le k\}$, and hence by Theorem 2.19 find a fixed point for f^k in J_1 and hence a period-k point for f in J_1. Hence as $J_1 \subset I$, f has points of all other periods.

This consequences of this theorem are remarkable. If you can find a point of period three for a map f then there exists periodic points of all other periods in the natural numbers; and hence periodic points exist with infinite periods. This theorem however, does not show the bigger picture of whats going on here, for this we need to introduce Sharkovsky's theorem which is a full description of which periods imply the existence of other periods. However, before we get to the theorem, we require a total ordering termed Sharkovsky's order.

Definition 2.22 (Sharkovsky's Order)**.** The total ordering on the natural numbers, defined below, is named *Sharkovsky's order*.

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

Hence the Sharkovsky ordering first enumerates all the odd integers multiplied by 2^k for $k \in \mathbb{N}$. This exhausts all the natural numbers apart from powers of two, which are then ordered last in descending order. Finally, this brings us to Sharkovsky's theorem.

Theorem 2.23 (Sharkovsky's Forcing Theorem)**.** *Let (I, f) be a topological dynamical system where I is a closed interval. If f has a period point of period k then for all integers $l \lhd k$, f has periodic points of period l.*

The proof of this theorem will be omitted since it is outwith the scope of this text, however, can be found here [20]. Note however, that we proved the case where $k = 3$ and hence points with all other periods in Sharkovsky's order exist in Theorem 2.21. Hence, it can be seen that this theorem more general that the one proved by by Li and Yorke [17], as period three is the greatest period in Sharkovsky's ordering hence its presence implies the existence of all other periods. Noticing the Sharkovsky ordering it can be seen that if there exists a periodic point in f whose period is not a power of two, then f has infinitely many many periodic points. The converse statement also holds. If f has finitely many periodic points then they all must have periods that are powers of two. It can be useful to catagorise topological dynamical systems defined on a closed interval based on their periods. This leads us to the next definition from Ruette, [19, §3.3].

Definition 2.24 (Type). Let (I, f) be a topological dynamical system where I is a closed interval and let $n \in \mathbb{N}$. Define the map f to be of *type n* if the periods of periodic points of f are the set $\{m \in \mathbb{N} : m \trianglelefteq n\}$.

Clearly every topological dynamical system over the closed intervals has a type. Sharkovsky also proved the converse, that there exists topological dynamical system over the closed intervals of every type in another important theorem termed Sharkovsky's Realisation Theorem. We shall state this theorem after the following example where we prove we can construct a type five topological dynamical system defined over the closed intervals.

Example 2.25. Let $([0, 4], f)$ be a piecewise linear map with $f(0) = 2$, $f(2) = 3$, $f(3) = 1$, $f(1) = 4$ and $f(4) = 0$. The map is displayed in Figure 2.2. Clearly f is continuous and f has been constructed so that 0 is a period-5 point. Moreover $f^3[0, 1] = [1, 4]$, $f^3[1, 2] = [2, 4]$ and $f^3[3, 4] = [0, 3]$, so f^3 has no fixed points in those intervals. however, $f^3[2, 3] = [0, 4]$ hence by Theorem 2.19, f^3 has at least one fixed point in $[2, 3]$. Moreover $f : [2, 3] \to [1, 3]$, $f : [1, 3] \to [2, 5]$ and $f : [1, 4] \to [0, 4]$ are all monotonically decreasing on $[2, 3]$. Therefore f^3 is monotonically decreasing on $[2, 3]$ and so the fixed point is unique. Therefore this can only be the fixed point for f, and as such no period 3 point exists. Hence by Sharkovsky's forcing theorem, the map f contains periodic points with the period of every natural number apart from 3. Therefore f is of type 5.

Sharkovsky subsequently proved that topological dynamical systems over closed intervals can constructed to be of any type. In [20] he proves that you can construct maps with integer type, later in [21] he proves you can construct maps with type in 2^∞. This theorem is termed Sharkovsky's Realisation Theorem and is stated below along with an elegant proof proved by Alseda, Llibre, and Misiurewicz [2, §2.2] and outlined by Burns and Hasselblatt [9, §7]. This insightful proof uses properties of the truncated tent map to reveal one number in the Sharkovsky ordering at a time.
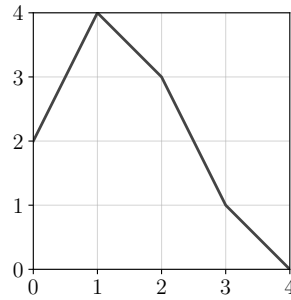
Figure 2.2: Piecewise linear map f with a period five point.

Theorem 2.26 (Sharkovsky's Realisation Theorem). *Every tail of the Sharkovsky order is the set of periods for some map f in a topological dynamical system (I, f) where I is a closed interval.*

Proof. Let $([0, 1], T_h)$ be a topological dynamical system, where $T_h(x) = \min(h, 1 - 2|x - 1/2|)$ is the family of truncated tent maps with $h \in [0, 1]$, shown in Figure 2.26. It is clear that T_0
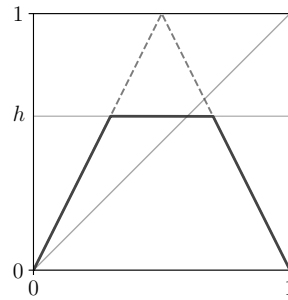
Figure 2.3: The truncated tent map T_h

has only one periodic point, a fixed point at $x = 0$. however, $T_1^3 \left(\frac{2}{7} \right) = T_1^2 \left(1 - 2 \left| \frac{2}{7} - \frac{1}{2} \right| \right) = T_1^2 \left(\frac{4}{7} \right) = T_1 \left(1 - 2 \left| \frac{4}{7} - \frac{1}{2} \right| \right) = T_1 \left(\frac{6}{7} \right) = \left(1 - 2 \left| \frac{6}{7} - \frac{1}{2} \right| \right) = \frac{2}{7}$. Hence T_1 has a period-3 point and so by Sharkovsky's Forcing Theorem has a periodic point for every period in the positive integers by Sharkovsky's Theorem. As T_1 and T_h are identical on the interval $[0, h)$ any k-cycle $\mathcal{O}_{T_h}$ is also a k-cycle for $\mathcal{O}_{T_1}$ and any k-cycle $\mathcal{O}_{T_1}$ is also a k-cycle for $\mathcal{O}_{T_h}$. Now define the map $h : \mathbb{N} \to [0, 1]$ where $h(m) = \min \{ \max \mathcal{O}_{T_1} : \mathcal{O}_{T_1}$ is m-cycle$\}$, noting that T_1^m has 2^m fixed points. Using the properties of the tent map we will find that the function h we have constructed orders the natural numbers according to Sharkovsky's ordering, with the set of periods of $T_h(m)$ being the tail of this ordering, starting from m. As the k-cycles for $\mathcal{O}_{T_h}$ and $\mathcal{O}_{T_1}$ are the same, T_h has the l-cycle $\mathcal{O}_{T_h}$ if and only if $h(l) < h$. Moreover, $\mathcal{O}_{T_{h(m)}}$ is a m-cycle for $T_{h(m)}$, and all other cycles $\mathcal{O}_{T_{h(m)}}$ are contained within $[0, h(m))$. Using Sharkovsky's forcing theorem it is clear that if $m \rhd l$ then $T_{h(m)}$ has a period-l orbit that lies in $[0, h(m))$ and hence $h(l) < h(m)$. Now by symmetry, $h(l) < h(m)$ if and only if $m \rhd l$. Hence we can see that for any positive

integer m the periodic points of $T_{h(m)}$ is the tail of the Sharkovsky order from $m \trianglerighteq l$. Now for powers of two. By above $h(2^\infty) = \sup_k(h(2^k)) > h(2^k)$ for all $k \in \mathbb{N}$, so $T_{h(2^\infty)}$ has 2^k-cycles for all $k \in \mathbb{N}$. Suppose $T_{h(2^\infty)}$ has an m-cycle with $m \neq 2^k$ for all $k \in \mathbb{N}$. Then by Sharkovsky's forcing theorem $T_{h(2^\infty)}$ has a $2m$-cycle. Because the m-cycle and $2m$-cycle are disjoint, at least one of them are contained in $[0, h(2^\infty))$ and hence in the interval $[0, h(2^k))$ for some $k \in \mathbb{N}$. $\square$

Thus there exists a map f and associated topological dynamical system (I, f) such that f has period-k points and is of type k, where $k \in \mathbb{N}$ is in the Sharkovsky ordering. This concludes our study of Sharkovsky's theorem and topological and symbolic relationships. In the next chapter on defining chaos we shall use the results we have developed here to prove that characteristics of chaos hold for certain topological dynamical systems.

Chapter 3

Defining Chaos

The term *chaos* in Mathematics is vague and has no universally accepted definition, hence various interpretations of chaos been invented throughout the years. As mentioned previously, the paper *'Period Three Implies Chaos'* by Li and Yorke [17] first introduced the term in a mathematical context. However, this paper never formally defines a notion of chaos. In subsequent years, various mathematicians have attempted to define their own interpretations of chaos. All these interpretations rely on the topological dynamical system exhibiting various defined topological characteristics. The properties relied upon by the definitions of chaos we will look at include: topological transitivity or the existence of a dense orbit, sensitive dependence on initial conditions, that periodic points are dense in the underlying metric space, and the existence of an uncountable scrambled set. In Section 3.2 we shall explore a widely accepted definition of chaos, termed *Devaney chaos*, developed by Devaney [11]. This definition relies upon the topological dynamical system having three properties, namely topological transitivity, sensitive dependence on initial conditions and that periodic points are dense in the underlying metric space. However, we shall show that if we have a topological dynamical system with no isolated points then the latter two properties are redundant, and are implied by topological transitivity. In Section 3.3 we shall explore *Li-Yorke chaos* [17]. This interpretation of chaos explores the types of topological dynamical systems Li and Yorke termed chaotic through the powerful definition of scrambled sets. Notably this type of chaos relies on the existence of an uncountable scrambled set. Finally, in Section 3.4 we shall study topological chaos and topological entropy. Specifically we will find that positive topological entropy is the only requirement for a topological dynamical system to be topologically chaotic. Topological dynamical systems that exhibit all these types of chaos will be explored. We shall see that even the most trivial topological dynamical systems such as the tent map and logistic map exhibit chaos according to multiple interpretations. Before we introduce these various types of chaos however, we first need to set up some preliminary definitions.

3.1 Topological Characteristics of Chaos

There are many different types of chaos in topological dynamical systems. All of these different types of chaos require the topologically dynamical system to display different topologically chaotic properties. This first definition is a prime characteristic of chaotic systems. In fact we shall show later that in topological dynamical systems defined over a closed interval it is the only characteristic needed to prove that a system is chaotic in the sense of Devaney.

Definition 3.1 (Topological Transitivity)**.** Let (X, f) be a topological dynamical system. The map f is *topologically transitive* if for every pair of non-empty open sets $U, V \subseteq X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Alternatively stated, in a topologically transitive mapping, points in an arbitrarily small neighbourhood can be mapped to any other arbitrary neighbourhood under a repeated number of iterations of the map. Hence the topological dynamical system cannot be partitioned into two disjoint non-empty open sets which are invariant under the map – i.e. if $U \in X$ then $f(U) \in U$.

Example 3.2. Let $(S^1, \mathcal{D})$ be the doubling map over S^1, where $\mathcal{D}(z) = z^2$. Let $z_1, z_2 \in S^1$. Let $(z_1, z_2) = U$ define an arc between z_1 and z_2. Suppose now $d(z_1, z_2) > \frac{2\pi}{2^k}$ for some $k \in \mathbb{N}$. Then $d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d(z_1, z_2) > 2^k \cdot \frac{2\pi}{2^k} = 2\pi$. Hence $\mathcal{D}^k((z_1, z_2))$ covers S^1 so for any $V \subseteq S^1$ we obtain $\mathcal{D}^k(U) \cap V \neq \emptyset$. Hence $\mathcal{D}$ is topologically transitive.

This next example, from [15], provides a creative way of producing topologically transitive topological dynamical systems by using the orbit of any other topological dynamical system.

Example 3.3. Let (X, f) be a topological dynamical system and let $x \in X$ be a periodic point of f. Clearly $\mathcal{O}_f(x) = Y$ is finite and so $(Y, f|_y)$ is a topological dynamical system. $(Y, f|_y)$ is transitive as if $U \subseteq Y$ is open, then $f^k(U) = Y$ for some $k > 0$ and so $f^k(U) \cap V \neq 0$ for all $V \subseteq Y$ open.

This next proposition, proved by Silverman [22], states that if (X, f) is a topological dynamical system and X has no isolated points, the existence of a dense orbit of f is equivalent to f being topologically transitive. In his paper, Silverman explicitly states that transitivity implies the existence of a dense orbit if X is separable and second-category. However, since our definition of a topological dynamical system takes X to be a compact metric space these properties automatically hold. This proposition is hugely important some authors give slight variations between definitions of chaos, some definitions ask for the existence of a dense orbit and others ask for topological transitivity. Hence using this proposition we can show that these definitions are equivalent in topological dynamical systems. The proof of this proposition follows Ruette [19, §2.1].

Proposition 3.4. *Let (X, f) be a topological dynamical system and suppose X has no isolated points. The map f is topologically transitive if and only if there exists some $x \in X$ such that $\mathcal{O}(x)$ is dense in X.*

Proof. Assume that f is transitive and let U be a non-empty open set. By transitivity, for every non-empty open set V there exists a $k > 0$ such that $f^k(U) \cap V \neq 0$. Hence, $\bigcup_{k \geq 0} f^{-n}(U)$ is dense in X. As X is compact, there exists a countable basis of non-empty open sets $(U_n)_{n \geq 0}$. For all $l \geq 0$, $\bigcup_{k \geq 0} f^{-k}(U_n)$ is dense by transitivity. Now define $G = \bigcap_{n \geq 0} \bigcup_{k \geq 0} f^{-k}(U_n)$. This is a dense G_δ set. If $x \in G_\delta$ then $f^k(x)$ enters any set U_n for some k. Hence $\mathcal{O}_f(x)$ is dense in X. For the reverse direction let $U, V \subseteq X$ be open with $U, V \neq \emptyset$. Let $x \in X$ such that $\overline{\mathcal{O}_f(x)} = X$. Then there exists $k \in \mathbb{N}$ such that $f^k(x) \in U$. Since X has no isolated points $V \setminus \{f^i(x) : 0 \leq i \leq k\}$ is open and non-empty. Hence there exists an $l \in \mathbb{N}$ such that $f^l(x) \in V \setminus \{f^i(x) : 0 \leq i \leq k\}$. Since $l > k$ and $f^l(x) = f^{l-k} \circ f^k(x) \in f^{l-k}(U) \cap V$ we get $f^{l-k}(U) \cap V \neq 0$. Hence f is transitive. $\square$

Since closed intervals have no isolated points, for topological dynamical systems (I, f) where I is a closed interval, topological transitivity and the existence of a dense orbit are interchangeable. Note that the condition for the topological dynamical system to have no isolated points is necessary. The following is a counterexample to show the definitions are not equivalent for topological dynamical systems in general.

Example 3.5. Let (X, f) be the topological dynamical system defined over the space $X = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ with the standard metric, and the map f where $f(0) = 0$ and $f(2^{-n}) = 2^{-n-1}$. It can be seen that (X, d) is a compact space, where d is the standard metric as X is a closed subset of $\mathbb{R}$, containing its limit point 0. The set X contains infinitely many isolated points as we can choose $B_d(x, \varepsilon)$ around each $x = 2^{-n} \in X$ with $\varepsilon < \frac{1}{4} \min \{2^{-n} - 2^{-n-1}, 2^{-n+1} - 2^{-n}\}$ such that the open balls are disjoint and $B_d(x, \varepsilon) = \{x\}$. Now let $U = \{\frac{1}{2}\}$ and $V = \{1\}$. Then $f^k(U) = f^k(\{\frac{1}{2}\}) = \{2^{-k-1}\}$. Hence $f^k(U) \cap V = \{2^{-k-1}\} \cap \{1\} = \emptyset, \forall k \in \mathbb{N}$. Hence f is not topologically transitive, however, $\mathcal{O}(1) = \{1, 2^{-1}, 2^{-2}, \dots\}$ is dense as $\overline{\{2^{-n} : n \in \mathbb{N}\}} = X$.

Now let's introduce and example of a topological dynamical system with no isolated points. By Proposition 3.4 we shall see that this max has both topological transitivity and the existence of a dense orbit.

Example 3.6. In Example 1.14 we introduced the rigid rotations, a topological dynamical system (S^1, R_α) where $R_\alpha(z) = ze^{i\alpha}$. Furthermore in Proposition 1.15 we proved that the irrational rotations gave rise to dense orbits and that these orbits where infinite. Hence S^1 does not contain an isolated point. Using Proposition 3.4 we see that (S^1, R_α) is topologically transitive.

Example 3.7. In Proposition 2.18 we proved that the periodic points of the logistic map $([0,1], F_4)$, the tent map $([0,1], T_2)$ and the doubling map $([0,1], D)$ are dense in $[0,1]$. Clearly all these systems have no isolated points. Hence, using Proposition 3.4 we can clearly see that all these maps are topologically transitive.

An important fact to note is that topological dynamical systems that are topologically transitive with an isolated point are in fact trivial, having only one periodic orbit. Hence in the rest of this text we shall restrict our study to the topological dynamical systems without an isolated point. Continuing, let's introduce our second topological characteristic of chaos, coming from Devaney [11].

Definition 3.8 (Sensitive Dependence On Initial Conditions)**.** Let (X, f) be a topological dynamical system and $\varepsilon > 0$. A point $x \in X$ is *ε-unstable* if, for every neighbourhood U of x, there exists a point $y \in U$ and $k \geq 0$ such that $d\left(f^k(x), f^k(y)\right) \geq \varepsilon$. The map f has *sensitive dependence on initial conditions* if for all points $x \in X$, x is ε-unstable.

In other words, there exist points arbitrary close to x that eventually get mapped at least ε far apart under multiple applications of the map. Hence this definition states that small perturbations between iterates may eventually increase through repeated iterations of the map to become wildly different over time; behaviour which hopefully feels notionally chaotic to the reader. Note that the definition states that at least one point contained within each neighbourhood of x gets mapped arbitrarily far apart, not all points. Here is an example of a topological dynamical system with sensitive dependence on initial conditions.

Example 3.9. Let $(S^1, \mathcal{D})$ be the doubling map over the S^1, where $\mathcal{D}(z) = z^2$. Let $z_1, z_2 \in S^1$, $\varepsilon < 2^k \delta$ and suppose $d(z_1, z_2) = \delta$, then $d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d(z_1, z_2) = 2^k \delta > \varepsilon$. Hence we can always choose a k large enough so this holds, and so f has sensitive dependence on initial conditions.

The same is true for the doubling map $([0,1], D)$ over the unit interval. Note the example above displays a strong type of sensitive dependence on initial conditions termed expansiveness. In a topological dynamical system which exhibits expansiveness, all points arbitrarily close together eventually get mapped arbitrarily far apart; not just a proper subset of points as for sensitive dependence on initial conditions. We can see that the rigid rotations do not satisfy the definition of having sensitive dependence on initial conditions.

Example 3.10. Let (S^1, R_α) be the rigid rotations. Let $z_1, z_2 \in S^1$, $\varepsilon > 0$ and suppose $d(z_1, z_2) < \varepsilon$, then since R_α is an isometry $d(R_\alpha^k(z_1), R_\alpha^k(z_2)) = d(z_1, z_2) < \varepsilon$. Hence (S^1, R_α) does not have sensitive dependence on initial conditions.

This concludes our study of topological characteristics of chaos. We shall now use the definitions of topological transitivity or the existence of dense orbit and sensitive dependence

on initial conditions to define notions of chaos in topological dynamical systems and investigate examples of chaotic systems.

3.2 Devaney Chaos

Our first notion of chaos was developed by Devaney [11] and is one of the most widely used definitions of chaos. Devaney's interpretation of chaos includes unpredictability via sensitive dependence on initial conditions, repetitive behaviour through periodic points being dense, and should be indecomposable through topological transitivity. Two characteristics which we developed in Section 3.1.

Definition 3.11 (Devaney Chaos). A topological dynamical system (X, f) is *chaotic in the sense of Devaney* if it is topologically transitive, has sensitive dependence on initial conditions, and if the periodic points of f are dense in X.

The main feature of Devaney chaos is topological transitivity. After Devaney released this definition Banks et al. [3] and Glasner et al. [14] showed that sensitive dependence on initial conditions is redundant. Note that this result holds even for a general mapping $f : X \to X$.

Proposition 3.12. *Let (X, f) be a topological dynamical system. If the map f is topologically transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

Proof. Let (X, d) be a metric space. Observe that we can find a $\delta_0 > 0$ such that for all $x \in X$ there exists a periodic point $q \in X$ such that $dist(\mathcal{O}_f(q), x) \geq \delta_0/2$. Proving this, take q_1, q_2 to be arbitrary periodic points where $\mathcal{O}_f(q_1) \cup \mathcal{O}_f(q_2) = \emptyset$. Let $\delta_0 = dist(\mathcal{O}_f(q_1), \mathcal{O}_f(q_2))$ and suppose $q_1' \in \mathcal{O}_f(q_1)$ and $q_2' \in \mathcal{O}_f(q_2)$ are points such that $d(q_1', q_2') = \delta_0$. For all $x \in X$ either $d(q_1', x) \leq d(q_2', x)$ or by symmetry $d(q_2', x) \leq (q_1', x)$. Using the triangle inequality we find $d(q_1', q_2') \leq d(q_1', x) + d(x, q_2')$ for all $x \in X$. Hence we either have $\delta_0 = d(q_1', q_2') \leq 2d(q_1', x)$ or $\delta_0 = d(q_1', q_2') \leq 2d(q_2', x)$. Therefore we either have $dist(\mathcal{O}_f(q_1), x) \geq \delta_0/2$ or $dist(\mathcal{O}_f(q_2), x) \geq \delta_0/2$. Using this observation we can now prove f has sensitive dependence on initial conditions. First let $\delta = \delta_0/8$ and let $x \in X$ be arbitrary, $x \in N$ where N is an open neighbourhood. Since the periodic points of f are dense in X there exists a period-n point $p \in U = N \cap B(x, \delta)$ open. By the observation above, there exists a periodic point $q \in X$ with $dist(\mathcal{O}_f(p), x) \geq 4\delta$. Now define $V = \bigcap_{i=0}^{n} f^{-i}(B(f^i(q), \delta))$. Since V is a finite intersection of open sets, it itself is open. Moreover $q \in V$ so V is non-empty. Since f is topologically transitive, there exists a $y \in U$ with natural number $k > 0$ such that $f^k(y) \in V$. Now suppose $j = \lfloor \frac{k}{n} \rfloor + 1$ such that $1 \leq nj - k \leq n$. Hence, $f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(V) \subseteq B(f^{nj-k}(q), \delta)$. We also have $f^{nj}(p) = p$, so by the triangle inequality $d(f^{nj}(p), f^{nj}(y)) = d(p, f^{nj}(y)) \geq d(x, f^{nj-k}(q)) - d(f^{nj-k}(q), f^{nj}(y)) - d(p, x)$. Now as $p \in B(x, \delta)$ and $f^{nj}(y) \in B(f^{nj-k}(q), \delta)$ we have $d(f^{nj}(p), f^{nj}(y)) > 4\delta - \delta - \delta = 2\delta$. Hence,

by the triangle inequality either $d(f^{nj}(x), f^{nj}(y)) > \delta$ or $d(f^{nj}(x), f^{nj}(p)) > \delta$. By definition, f has sensitive dependence on initial conditions. $\square$

For a general topological dynamical system this is the only superfluous property. Silverman [22] and Vellekoop and Berglund [24] later proved that for a topological dynamical system (I, f) where I is a closed interval, topological transitivity or equivalently the existence of a dense orbit, implies f has dense periodic points in X. We shall introduce and later prove this result, but first we need the following lemma, which can be found here [6, §4.1].

Lemma 3.13. *Let $f : I \to I$ be a continuous map and I an interval. Suppose $J \subseteq I$ is an interval which contains no periodic points of f. If $z, f^m(z), f^n(z) \in J$ where $m, n \in \mathbb{N}$, $m < n$ then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

Proof. Suppose there exists a $z \in J$ such that $z < f^m(z)$ and $f^m(z) > f^n(z)$. Define $g(x) = f^m(x)$, so $z < g(z)$. If $g^{k+1}(x) < g(z)$ for some $k \in \mathbb{N}$, $n \geq 1$ then $g^k(z) - z$ has a positive value in z and a negative value in $g(z)$ and by the Intermediate Value Theorem contain a point $c \in (z, g(z)) \subseteq J$ with $g^k(c) - c = 0$ and hence a km-periodic point. Therefore $z < g^k(z)$ for all positive integers k. Now let $k = n - m > 0$. Then $z < f^{(n-m)m}(z)$. Assuming $f^{(n-m)}(f^n(z)) < f^m(z)$ then taking $g = f^{n-m}(x)$ similarly yields $f^{(n-m)m}(f^m(z)) < f^m(z)$. However, this results in the function $f^{(n-m)m}(x) - x$ having a positive and negative value in $f^m(z)$. Hence, by the Intermediate Value Theorem a $(n-m)m$-periodic point exists in J, a contradiction. The other case for $z > f^m(z) > f^n(z)$ can be proved similarly. $\square$

Proposition 3.14. *Let (I, f) be a topological dynamical system. If the map f is topologically transitive then f has a dense set of periodic points.*

Proof. We shall aim for a contradiction. Suppose that the periodic points are not dense in I, so there exists an interval $J \subseteq I$ where J contains no periodic points. Let $x \in J$ where x is not an endpoint and let $N \subsetneq J$ be a neighbourhood of x. Also let $E = J \setminus N$. Since f is topologically transitive on I there exists a positive integer m with $f^m(N) \cap E \neq \emptyset$. Hence there exists a $y \in J$ such that $f^m(y) \in E \subsetneq J$ and since J contains no periodic points $y \neq f^m(y)$. Moreover, since f is continuous there exists an open neighbourhood U of y such that $f^m(U) \cap U \neq \emptyset$. Using topological transitivity again we can find a $n > m$ and a $z \in U$ with $f^n(z) \in U$. However, then $0 < m < n$ with $z \in f^n(U)$ and $z \notin f^m(U) \implies z \leq f^n(z) \leq f^m(z)$. This is a contradiction by Lemma 3.13. Hence the periodic points of f are dense. $\square$

Note that this result cannot hold generally because of the ordering over $\mathbb{R}$ used in Lemma 3.13. Using Propositions 3.12 and 3.14 we can clearly see that if (I, f) is a topologically transitive topological dynamical system defined over an closed interval then it is chaotic in the sense of Devaney, giving us the following important result.

Proposition 3.15. *Let (I, f) be a topological dynamical system where I is a closed interval. If f is topologically transitive then (I, f) is chaotic in the sense of Devaney.*

Proof. Suppose f is topologically transitive. Propositions 3.12 and 3.14 tell us that f has sensitive dependence on initial conditions and the periodic points of f are dense in I. Hence (I, f) is chaotic in the sense of Devaney. $\qquad\square$

As a result of this proposition, topological dynamical systems which have a topologically transitive map over a closed interval are automatically chaotic in the sense of Devaney. Here is a counterexample, proving that this result does not hold generally for topological dynamical systems.

Example 3.16. Let (S^1, R_α) where $R_\alpha(z) = ze^{i\alpha}$ be the topological dynamical systems described by rigid rotations, except we now we shall take α to be solely irrational. In Example 3.6 we proved these topological dynamical systems to be topologically transitive, as irrational rotations gave rise to infinite, dense orbits. However, we also proved in Example 3.10 that the system does not have sensitive dependence on initial conditions. Furthermore, irrational rotations give rise to infinite orbits and so no periodic points exist. Hence we have an example of a topological dynamical system which is topologically transitive but neither has dense periodic points nor sensitive dependence on initial conditions.

The example above gives great insight into the troubles involved in trying to develop an all encompassing definition of chaos. The irrational rotations of S^1 have dense orbits and so are topologically transitive, but the map is not particularly interesting as nearby points are constantly mapped near together, never giving way to erratic or uncontrollable behaviour. Looking back at the definition of Devaney chaos it is clear that all of the conditions for chaos are topological and so are preserved under topological conjugate maps. This makes looking for Devaney chaotic systems much easier by the following proposition.

Proposition 3.17. *Let (X, f) and (Y, g) be topologically conjugate, topological dynamical systems. If f is chaotic in the sense of Devaney then g is chaotic in the sense of Devaney.*

Proof. By Proposition 3.12 sensitive dependence on initial conditions was found to be redundant for topological dynamical systems. In Proposition 2.8 we proved that if $Per(f)$ are dense in X then $Per(g)$ are dense in Y. Hence we just need to prove that topological conjugacy preserves topological transitivity. Let $\varphi : X \to Y$ be a topological conjugacy between (X, f) and (Y, g) and suppose f is topologically transitive. Let $U, V \subseteq Y$ be non-empty open sets. Since φ is surjective $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are non-empty. As f is topologically transitive, there exists a positive integer $k > 0$ such that $f^k(\varphi^{-1}(U)) \cap \varphi^{-1}(V) \neq \emptyset$. Let $x \in \varphi^{-1}(U)$ such that $f^k(x) \in \varphi^{-1}(V)$. Now set $y = \varphi(x) \in U$ and note that $\varphi \circ f^k(x) = g^k \circ \varphi(x) = g^k(y)$. Therefore $g^k(y) = \varphi \circ f^k(x) \in V$ and so $g^k(U) \cap V \neq \emptyset$. $\qquad\square$

Now let's introduce some examples of topological dynamical systems that exhibit Devaney chaos, using topological conjugacy and building on the properties we have already observed in various topological dynamical systems.

Example 3.18. In Example 3.7 we showed that the periodic points of the logistic map $([0, 1], F_4)$, the tent map $([0, 1], T_2)$ and the doubling map $([0, 1], D)$ are all topologically transitive. By Proposition 3.15 these systems are all chaotic in the sense of Devaney.

3.3 Li-Yorke Chaos

Now onto our second definition of chaos. As mentioned in Section 2.3, the paper *'Period Three Implies Chaos'* by Li and Yorke [17] first introduced the term chaos in a mathematical context. In this paper they stated two properties of interval maps that lead to chaotic behaviour, namely sensitive dependence on initial conditions and the existence of an uncountable set with no periodic points. This was formally introduced in the following theorem.

Theorem 3.19. *If $f : I \to I$ be a continuous interval map with a period three point, then there exists an uncountable set $S \subseteq I$ (containing no periodic points) such that, for all $x, y \in S$ where $x \neq y$*

$$\limsup_{n \to +\infty} |f^n(x) - f^n(y)| > 0, \quad \liminf_{n \to +\infty} |f^n(x) - f^n(y)| = 0$$

and for all periodic points $z \in S$

$$\limsup_{n \to +\infty} |f^n(x) - f^n(z)| > 0. \tag{3.1}$$

Clearly by the requirement of (3.1) in this theorem, the set S contains no asymptotically stable points. Li and Yorke noted that interval maps which satisfied this equation displayed erratic and irregular behaviour. Hence within this theorem they have defined a sense of chaos in interval maps. Note that this theorem does not hold for general topological spaces. For instance, take the rigid rotations $(S^1, R_{2\pi/3})$ as an example. Every point $z \in S^1$ is a period three point as $R^3(z) = ze^{3i \cdot 2\pi/3} = z$, however, there does not exist an uncountable set $S \subseteq S^1$ containing no periodic points, so Theorem 3.19 does not hold. Since this theorem was published, various authors have generalised the definition of Li-Yorke chaos to the realm of topological dynamical systems. We shall be taking the definition from Blanchard et al. [5] which uses the notion of a Li-Yorke pair, defined as follows.

Definition 3.20 (Li-Yorke Pair)**.** Let (X, f) be a topological dynamical system with $x, y \in X$ and $\delta > 0$. The pair (x, y) is a *Li-Yorke pair* if

$$\limsup_{n \to +\infty} d\left(f^n(x), f^n(y)\right) \geq \delta \quad \text{and} \quad \liminf_{n \to +\infty} d\left(f^n(x), f^n(y)\right) = 0.$$

Hence if (x, y) is a Li-Yorke pair then x and y can be mapped at least δ far apart under multiple iterations of the map. This is the behaviour we defined, in Definition 3.8, as sensitive dependence on initial conditions. Furthermore if (x, y) is a Li-Yorke pair then x and y can be mapped to the same point under multiple iterations of the map. In this regard we can think of iterations of these two points being scrambled amongst the whole set X. Let's now define this behaviour generally over a whole set, to introduce a definition of Li-Yorke chaos

Definition 3.21 (Scrambled Set, Li-Yorke Chaos). A set $S \subseteq X$ is *scrambled* if for all distinct $x, y \in S$, (x, y) is a Li-Yorke pair. A topological dynamical system (X, f) is *chaotic in the sense of Li-Yorke* if there exists an uncountable scrambled set $S \subseteq X$.

Note that in this general version of Li-Yorke chaos the requirement for the set X to have a period three point has been excluded. Furthermore the last requirement, in (3.1), that no points converge asymptotically to periodic points has been removed. In fact, this extra requirement makes no difference for chaos in the sense of Li-Yorke as if S is a scrambled set then every point except, at most, one point of S satisfy this requirement. Recently it was shown by Lu et al. [18] that topological conjugacy does not preserve Li-Yorke chaos. Using Theorem 3.19 we can prove that the following topological dynamical systems are Li-Yorke chaotic.

Example 3.22. Take $([-1, 1], f)$ to be the topological dynamical system where $f(x) = 2|x| - 1$. Clearly $x = \frac{1}{9}$ is a period three point as $f^3 \left(\frac{1}{9} \right) = f^2 \left(\frac{-7}{9} \right) = f \left(\frac{5}{9} \right)$. Hence by Theorem 3.19 $([-1, 1], f)$ is chaotic in the sense of Li-Yorke.

Example 3.23. Let $([0, 1], F_\mu)$ where $1 + 2\sqrt{2} \leq \mu \leq 4$. Note that when $\mu = 1 + 2\sqrt{2}$ a period three point emerges, and hence by Theorem 3.19 is Li-Yorke chaotic.

3.4 Topological Chaos and Entropy

Next we shall explore our final definition of chaos: topological chaos. This definition relies heavily on topological entropy, a conjugacy invariant property exhibited by some topological dynamical systems. Topological entropy is a non-negative real number describing the complexity of a topological dynamical system by the asymptotic mean growth in the number of distinguishable collections of orbits at an arbitrarily fine yet finite resolution. The quantity was first outlined by Adler et al. [1] and uses the language of open covers. Later Bowen [7] and Dinaburg [12] reformulated this definition in terms of a metric and the separation of orbits. When the underling metric space is compact, i.e. in a topological dynamical system, these two definitions become equivalent. First let's introduce the former definition, which in an essence is more natural as it does not depend on the underlying metric space and so is more general in a topological sense.

Definition 3.24 (Topological Entropy - Adler et al.). Let (X, f) be a topological dynamical system, $\mathcal{C} = \{C_1, C_2, \ldots C_p\}$, $\mathcal{D} = \{D_1, D_2, \ldots D_q\}$ be finite covers and define the cover $\mathcal{C} \vee \mathcal{D} =$

$\{C_i \cap D_j : i \in [1,p], \ j \in [1,q]\}$. The cover $\mathcal{C}$ is *finer* than $\mathcal{D}$ if every element of $\mathcal{D}$ is also included in $\mathcal{C}$, and is expressed as $\mathcal{C} \prec \mathcal{D}$. Let $N(\mathcal{C})$ be the minimum cardinality of a subcover of $\mathcal{C}$, so $N(\mathcal{C}) = \min\left\{n : \exists i(1), \dots, i(n) \in [1,p], \ X = C_{i(1)} \cup \cdots \cup C_{i(n)}\right\}$. Then, for all integers $n \geq 1$ we can define $N_n(\mathcal{C}, f) = N\left(\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n-1)}(\mathcal{C})\right)$. The *topological entropy* of the finite cover $\mathcal{C}$ is given by

$$h(\mathcal{C}, f) = \lim_{n \to +\infty} \frac{\log N_n(\mathcal{C}, f)}{n} = \inf_{n \geq 1} \frac{\log N_n(\mathcal{C}, f)}{n}.$$

The *topological entropy* according to Adler et al., denoted $h_A(f)$, of the topological dynamical system (X, f) is given by

$$h_A(f) = \sup\left\{h(\mathcal{U}, f) : \mathcal{U} \text{ finite open cover of } X\right\}.$$

Now we shall introduce the Bowen-Dinaburg definition of topological entropy using the language of metric spaces. First we need to define the notion of (n, ε)-separated and (n, ε)-spanning sets, given by Bowen [7].

Definition 3.25 ((n, ε)-Separated, (n, ε)-Spanning)**.** Let (X, f) be a topological dynamical system defined on the metric space (X, d). A set $E \subseteq X$ is (n, ε)-*separated* if for all distinct $x, y \in E$ there exists a k with $0 \leq k < n$ such that $d(f^k(x), f^k(y)) \geq \varepsilon$. For $n \in \mathbb{N}$ where $n \geq 1$, define $d_n(x, y) = \max\left\{d(f^k(x), f^k(y)) : 0 \leq k < n\right\}$ and $B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}$. A set $E \subseteq X$ is (n, ε)-*spanning* if $X \subseteq \bigcup_{x \in E} B_n(x, \varepsilon)$. Let $r(n, \varepsilon)$ denote the minimum cardinality of an (n, ε)-spanning set and $s(n, \varepsilon)$ denote the maximum cardinality of an (n, ε)-separated set.

Note that since compactness guarantees we can find a finite subcover for X there always exists a (n, ε)-spanning set and a (n, ε)-spanning set with a finite cardinalities. In the above definition, ε can be considered the resolution, i.e. the minimum distance at which the two points become distinguishable, with $r(n, \varepsilon)$ describing the minimum number of collections of indistinguishable orbits and $s(n, \varepsilon)$ describing the maximum number of collections of distinguishable orbits. This is because we have defined an (n, ε)-separated set to be a set such that all points in the set get mapped at least ε away from each other in at least one of the first iterations of the map. Note that the following is an alternative form of the definition of a spanning set. If F is an (n, ε)-spanning set then for every $x \in X$ there is a $y \in F$ for which $d(f^k(x), f^k(y)) \leq \varepsilon$ for all $0 \leq k < n$. The following lemma, also from Bowen [7], is particularly useful to setup Bowen and Dinaburg's definition of topological entropy. The proof of this following lemma is adapted from analysis by Ruette [19, §4.1].

Lemma 3.26. *If (X, f) is a topological dynamical system with $\varepsilon > 0$ and $n \in \mathbb{N}$, then $r(n, \varepsilon) \leq s(n, \varepsilon) \leq r(n, \varepsilon/2) < \infty$.*

Proof. Suppose $E \subseteq X$ is an (n, ε)-separated set of maximum cardinality $s(n, \varepsilon)$. By the maximality of E, for every $y \in X \setminus E$, $E \cup \{y\}$ is not (n, ε)-separated, or alternatively $y \in$

$\bigcup_{x\in E} B_n(x,\varepsilon)$. Clearly $E \subseteq \bigcup_{x\in E} B_n(x,\varepsilon)$ so E is an (n,ε)-spanning set, and so $r(n,\varepsilon) \leq s(n,\varepsilon)$. Let F be an $(n,\varepsilon/2)$-spanning set of cardinality $r(n,\varepsilon/2)$. For every $x \in X$, there exists $y(x) \in F$ such that $x \in B_n(y(x),\varepsilon/2)$. If $x_1, x_2 \in E$ are distinct then we must have $y(x_1) \neq y(x_2)$ as then we would have $d(f^k(x_1), f^k(x_2)) < \varepsilon$ for $0 \leq k < n$. Hence $s(n,\varepsilon) \leq r(n,\varepsilon/2)$. $\qquad\square$

Finally we can now introduce Bowen and Dinaburg's definition of topological entropy.

Definition 3.27 (Topological Entropy - Bowen and Dinaburg)**.** Let (X,f) be a topological dynamical system defined on the metric space (X,d). Define

$$\overline{h}(\varepsilon,f) = \limsup_{n\to\infty} \frac{\log s(n,\varepsilon)}{n} = \limsup_{n\to\infty} \frac{\log r(n,\varepsilon)}{n}.$$

Note that these limits exists and is finite, as proved by Lemma 3.26. The *topological entropy* according to Bowen and Dinaburg, denoted $h_B(f)$, of the topological dynamical (X,f) is given by

$$h_B(f) = \sup_{\varepsilon > 0^+} \overline{h}(\varepsilon,f) = \lim_{\varepsilon \to 0^+} \overline{h}(\varepsilon,f).$$

By taking the logarithm of $s(n,\varepsilon)$ or $r(n,\varepsilon)$ and dividing by n we obtain the mean growth of distinguishable collections of orbits, which are at least ε far apart in the first n iterations of the map. Therefore by then taking the limit superior we obtain the mean asymptotic growth in the number of these distinguishable collections of orbits. Letting ε tend to zero we get the asymptotic mean growth in the number of collections of orbits at an arbitrarily fine resolution; this is the definition of topological entropy. As mentioned above, both definitions of topological entropy are equivalent in topological dynamical systems.

Proposition 3.28. *If (X,f) is a topological dynamical system, then $h_A(f) = h_B(f)$.*

Proof. Since X is a compact metric space X we can find an open cover $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of X with $\mathrm{diam}(U_i) \leq \varepsilon$ for all $i \in [1,n]$ and Lebesgue number 2δ. Hence we obtain $s(n,\varepsilon) \leq N(\mathcal{U}^n) \leq s(n,\delta)$ implying that $h_A(f) = h_B(f)$. $\qquad\square$

Throughout the rest of this text we shall write $h_{top}(f)$ to denote either $h_A(f)$ or $h_B(f)$, depending on circumstances. The entropy of a topological dynamical system may be zero. If the underlying map is an isometry we get the following result.

Proposition 3.29. *Let (X,f) be a topological dynamical system. If $f : X \to X$ is an isometry, then $h_{top}(f) = 0$.*

Proof. Let (X,d) be the underlying metric space of (X,f). Take $x, y \in X$ and let $\mathcal{U}$ be an open cover of X. As f is an isometry $d(x,y) = d(f^n(x), f^n(y))$ where $n \in \mathbb{N}$. Hence $N_n(\mathcal{U}, f)$ is remains constant, no matter the choice of n. Therefore $h_{top}(\mathcal{U}, f) = \inf_{n\geq 1} \frac{1}{n} \log N_n(\mathcal{C}, f) = 0$. $\qquad\square$

This result is clearly true as isometric mappings preserve distance, and so there should be a constant number of distinguishable orbits under repeated applications of the map. We shall now prove that topological entropy is a purely topological property, with its value being independent of the choice of metric.

Lemma 3.30. *Let (X, f) be a topological dynamical system defined over two equivalent metrics d and d'. The topological entropy of (X, f) with respect to d and d' is the same.*

Proof. Consider the following map $I : (X, d) \to (X, d')$ between metric spaces. As d and d' are equivalent, I is a homeomorphism. Since X is compact, I is uniformly continuous. Hence if $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ which implies that $d'(x, y) < \varepsilon$. Specifically if $d_n(x, y) \leq \delta$ then $d_n(x, y) \leq \varepsilon$ where $n \in \mathcal{N}$. Hence any (n, δ)-spanning set is also a (n, ε)-spanning set. So $s_d(n, \delta) \geq s_{d'}(n, \varepsilon)$ and therefore we obtain

$$h_{top_{d'}} = \lim_{\varepsilon \to 0^+} \overline{h}_d(\varepsilon, f) \leq \lim_{\delta \to 0^+} \overline{h}_{d'}(\delta, f) = h_{top_d}.$$

By repetition of the same argument, except with the map $I' : (X, d) \to (X, d')$ we obtain $h_{top_{d'}}(f) \geq h_{top_d}(f)$. Therefore $h_{top_{d'}}(f) = h_{top_d}(f)$. $\qquad\qquad\square$

This lemma proves particularly useful in proving that topological entropy is a conjugacy invariant property of topological dynamical systems.

Proposition 3.31. *If (X, f) and (Y, g) topological dynamical systems which are topologically conjugate via the conjugacy $\varphi : X \to Y$, then $h_{top}(f) = h_{top}(g)$.*

Proof. Let d be a metric on X and let d' be the metric on Y defined by $d'(y_1, y_2) = d(\varphi(y_1), \varphi(y_2))$ where $y_1, y_2 \in Y$. By Lemma 3.30 we know that $h_{top}(g)$ is independent on the definition of d'. Using the definition of d_n we get that $d'_n(y_1, y_2) = \max \left\{ d'(f^k(x), f^k(y)) : 0 \leq k \leq n - 1 \right\} = \max \left\{ d(\varphi(f^k(x)), \varphi(f^k(y))) : 0 \leq k \leq n - 1 \right\} = \max \left\{ d(f^k(\varphi(x)), f^k(\varphi(y))) : 0 \leq k \leq n - 1 \right\} = d_n(\varphi(y_1), \varphi(y_2))$. Hence as φ is a bijection (n, ε)-separated sets and (n, ε)-spanning sets have the same cardinality for X and Y. Therefore it follows that $h(g) = h(f)$. $\qquad\square$

The converse statement is not true generally. To see this, let (R_α, S^1) be the irrational rigid rotations and (R_β, S^1) be the rational rigid rotations. As the rigid rotations are an isometry, by Proposition 3.29 the topological entropy of both systems is zero, however, (R_α, S^1) and (R_β, S^1) are not topologically conjugate. In the following example we shall now calculate the topological entropy of the shift map (Σ_2, σ). Note that this example uses the following proposition, which we shall state without proof as this requires some insight into measure theory of which the reader is not assumed to be familiar with, for more detail see Walters [25].

Proposition 3.32. *Let (X, f) be a topological dynamical system and let C be a topological generator. Then $h_{top}(f) = h_{top}(C, f)$.*

Note that proof of the above just requires us to prove $h_{top}(C, f) \geq h_{top}(f)$ as the reverse is true by definition as a topological generator is simply a finite open cover. This next example follows work done by Adler et al. [1].

Example 3.33. Let (Σ_2, σ) denote the shift map and let $C = \{[0], [1]\}$ partition Σ_2. Then for $n \in \mathbb{N}$, $\bigvee_{j=0}^{n-1} \sigma^{-j}(C)$ is a partition of Σ_2 into 2^n sets. Since C is a topological generator, by Proposition 3.32, $h_{top}(\sigma) = h_{top}(C, \sigma) = \lim_{n \to +\infty} \frac{1}{n} \log N_n(C, \sigma) = \lim_{n \to +\infty} \frac{1}{n} \log 2^n = \log 2$.

Before we introduce the following example for finding the topological entropy of the doubling map $(S^1, \mathcal{D})$ we need to establish the following lemma and propositions, which follow working by Butt [10].

Lemma 3.34. *If $(S^1, \mathcal{D})$ is the doubling map with underlying metric space (S^1, d) where d is the arc length metric, then for $x, y \in S^1$ we have $d(x, y) \leq \frac{1}{4} \implies d(f(x), f(y)) = 2d(x, y)$.*

Proof. Clearly $d(x, y) = |x - y|$ when $|x - y| \leq \frac{1}{2}$. Let x, y be such that $d(x, y) \leq \frac{1}{4}$. Hence we have, $d(S^1(x), S^1(y)) = d(2x \pmod 1, 2y \pmod 1) = \min(|2x - 2y \pmod 1|, 1 - |2x - 2y \pmod 1|)$. As $|2x - 2y| \leq \frac{1}{2}$ whe have that $2x - 2y \mod 1 = 2x - 2y$. Therefore, $d(S^1(x), S^1(y)) = 2|x - y| = 2d(x, y)$. $\square$

Proposition 3.35. *The set $S_{n+k} = \left\{ \frac{i}{2^{n+k}} : 0 \leq i < 2^{n+k} - 1 \right\}$ is an (n, ε)-spanning set for the doubling map $(S^1, \mathcal{D})$.*

Proof. Let $\varepsilon > 0$ and choose $k \geq 2$ such that $\frac{1}{2^{k+1}} \leq \varepsilon < \frac{1}{2^k}$. Note that for any $x \in S^1$ we have that $x \in \left[\frac{i}{2^{n+k}}, \frac{i+1}{2^{n+k}} \right)$ where $0 \leq i < 2^{n+k} - 1$. Then choose $y \in S_{n+k}$ to be either endpoints of this dyadic interval so that $d(x, y) \leq \frac{1}{2^{n+k}} < \frac{1}{4}$. By Lemma 3.34 we obtain, $d(\mathcal{D}(x), \mathcal{D}(y)) = 2d(x, y) \leq \frac{2}{2^{n+k}} < \frac{1}{4}$. Applying Lemma 3.34 a total of j times, where $0 \leq j < n$ we obtain $d(\mathcal{D}^j(x), \mathcal{D}^j(y)) = 2^j d(x, y) \leq \frac{2^j}{2^{n-k}} \leq \frac{2^{n-1}}{2^{n-k}} < \frac{1}{2^{k+1}} \leq \varepsilon$. Hence for any $x \in S^1$ we have $d_n(x, y) = \max \left\{ d(\mathcal{D}^j(x), \mathcal{D}^j(y)) : 0 \leq j < n \right\} < \varepsilon$ for some $y \in S_{n+k}$, and so S_{n+k} is a (n, ε)-spanning set for $(S^1, \mathcal{D})$. $\square$

Proposition 3.36. *The set $S_{n-1+k} = \left\{ \frac{i}{2^{n-1+k}} : 0 \leq i < 2^{n-1+k} - 1 \right\}$ is an (n, ε)-separared set for the doubling map $(S^1, \mathcal{D})$.*

Proof. Let $\varepsilon > 0$ and choose $k \geq 2$ such that $\frac{1}{2^{k+1}} \leq \varepsilon < \frac{1}{2^k}$. Let $x, y \in S_{n-1+k}$ be distinct. Note that we want to prove that $d_n(x, y) \geq \varepsilon$, that is, prove that there exists a j where $0 \leq j < n$ such that $d(\mathcal{D}^j(x), \mathcal{D}^j(y)) \geq \varepsilon$ (this is the definition of being a (n, ε)-separated set). Now suppose there exists a j such that $d(\mathcal{D}^j(x), \mathcal{D}^j(y)) \geq \frac{1}{4}$, then we are done as $\varepsilon < \frac{1}{4}$ by assumption. Hence suppose $d(\mathcal{D}^j(x), \mathcal{D}^j(y)) \leq \frac{1}{4}$ for all $0 \leq j < n$. Therefore we can apply Lemma 3.34 a total of $n-1$ times to show $d(\mathcal{D}^{n-1}(x), \mathcal{D}^{n-1}(y)) = 2^{n-1} d(x, y)$. Now note that for distinct $x, y \in S_{n-1+k}$ we get $d(x, y) \geq \frac{1}{2^{n-1+k}}$, so $2^{n-1} d(x, y) \geq \frac{2^{n-1}}{2^{n-1+k}} = \frac{1}{2^k} \geq \varepsilon$. Hence S_{n-1+k} is a (n, ε)-separated set for $(S^1, \mathcal{D})$. $\square$

Example 3.37. Let $(S^1, \mathcal{D})$ be the doubling map over S^1. By Proposition 3.35 S_{n+k} is (n, ε)-spanning set. Clearly this set has cardinality 2^{n+k}, and so $r(n, \varepsilon) \leq 2^{n+k}$. Therefore we get that $\overline{h}(\varepsilon, \mathcal{D}) = \limsup_{n \to \infty} \frac{\log r(n, \varepsilon)}{n} \leq \limsup_{n \to \infty} \frac{(n+k) \log 2}{n} = \log 2$. Using Proposition 3.36, S_{n-1+k} is a (n, ε)-separared set with cardinality 2^{n-1+k}, so $s(n, \varepsilon) \geq 2^{n-1+k}$. Therefore we also get that $\overline{h}(\varepsilon, \mathcal{D}) = \limsup_{n \to \infty} \frac{\log s(n, \varepsilon)}{n} \geq \limsup_{n \to \infty} \frac{(n-1+k) \log 2}{n} = \log 2$. Hence using the squeeze theorem for limits we get that $h_{top}(\mathcal{D}) = \lim_{\varepsilon \to 0^+} \overline{h}(\varepsilon, \mathcal{D}) = \log 2$.

Generally it can be proved that for maps of the form $f(x) = \alpha x \pmod 1$, where $\alpha \in \mathbb{N}$, that $h(f) = \log \alpha$. Finally, with a detailed background in topological entropy we can now define topological chaos. Before the term chaos was coined by Li and Yorke in 'Period Three Implies Chaos', Furstenburg in [13], stated that all topological dynamical systems with zero topological entropy are 'deterministic'. In a later papers, Glasner and Weiss [14] and separtely Blanchard [4], defined that topological dynamical systems that have positive topological entropy exhibit topological chaos.

Definition 3.38. A topological dynamical system (X, f) exhibits topological chaos if it has positive topological entropy.

Using Proposition 3.31 we can clearly deduce that topological chaos is preserved under topological conjugacy. Now let's introduce some examples of topological dynamical systems that exhibit topological chaos.

Example 3.39. In Example 3.33 and Example 3.37 we showed that the shift map (Σ_2, σ) and the doubling map $(S^1, \mathcal{D})$ have topological entropy $h_{top}(\sigma) = h_{top}(\mathcal{D}) = \log 2 > 0$. Therefore (Σ_2, σ) and $(S^1, \mathcal{D})$ exhibit topological chaos.

Blanchard et al. [5] proved that in a topological dynamical system, topological chaos implies the existence of Li-Yorke chaos. We shall not cover the proof of this proposition here as it contains ideas from ergodic theory and is outwith the bounds of this text.

Proposition 3.40. *If a topological system (X, f) is topologically chaotic then it is also Li-Yorke chaotic.*

In [23] Smital proved the inverse of this statement to be false implying there exists zero entropy topological dynamical systems which are Li-Yorke chaotic. Furthermore Li [16] proved that for topological dynamical systems (I, f) where I is a closed interval positive topological entropy implies the system exhibits Devaney chaos, and vice versa.

Proposition 3.41. *Let (I, f) be a topological dynamical system and I be a closed interval. The (I, f) has positive topological entropy if and only if (I, f) is chaotic in the sense of Devaney.*

Hence for topological dynamical systems (I, f) where I is a closed interval we get the following relation between the different types of chaos mentioned in this text.

$$\text{Devaney chaos} \iff \text{Topological chaos} \implies \text{Li-Yorke chaos.}$$

Chapter 4

Conclusion

In Chapter 1, we defined the notion of a topological dynamical system (X, f) as being a non-empty compact metric space X with a corresponding continuous map $f : X \to X$. We then introduced some examples of topological dynamical systems, namely the logistic map $([0, 1], F_\mu)$, the tent map $([0, 1], T_s)$, the rigid rotations (S^1, R_α), the doubling maps defined over the unit interval $([0, 1], D)$ and unit circle the complex plane $(S^1, \mathcal{D})$. We also introduced important concepts from discrete dynamics, such as orbits, periodic points, cycles, eventually and asymptotically periodic points and ω-limit sets.

Subsequently in Chapter 2, we introduced symbolic dynamics and topological conjugacy. We first defined topological conjugation, a term used to describe when two maps exhibit the same topological behaviour. We prove that various topological properties such as the density of orbits and periodic points, are conjugate invariant properties. Next we introduced symbolic dynamics, a field which studies how the shift map effects infinite sequences of symbols, called itineraries, that describe the complex dynamics of some topological dynamical systems. Specifically it was the assignment of orbits of a topological dynamical system to a sequence of discrete symbols in sequence space, denoted Σ_2. We defined the shift map (Σ_2, σ) on sequence space and proved that it was in fact a topological dynamical system. Then we went on to prove the periodic points of the shift map are dense and that the shift map has a dense orbit. Furthermore, using topological conjugacy we proved that through various topological conjugacy's and semi-conjugacy's, shown in the diagram below (where an uni-directional arrow represents a semi-conjugacy and a bi-directional a full conjugacy), that the periodic points of logistic map $([0, 1], F_\mu)$, the tent map $([0, 1], T_2)$ and the doubling map $([0, 1], D)$ are dense in $[0, 1]$.

$$\begin{array}{ccc}
\Sigma_2 & \xrightarrow{\ \sigma\ } & \Sigma_2 \\
\varphi_1 \downarrow & & \downarrow \varphi_1 \\
[0,1] & \xrightarrow{\ D\ } & [0,1] \\
\varphi_2 \downarrow & & \downarrow \varphi_2 \\
[0,1] & \xrightarrow{\ T_2\ } & [0,1] \\
\varphi_3 \uparrow & & \uparrow \varphi_3 \\
[0,1] & \xrightarrow{\ F_4\ } & [0,1]
\end{array}$$

We next proved a simplified version of Sharkovsky's forcing theorem by Li and Yorke [17]. This theorem stated that if a topological dynamical system (I, f) where I is a closed interval, has a period three point then it has periodic points of every other period. The full version of this theorem states that if a topological dynamical system has a period-k point then there must exist a period-l point for every $l \lhd k$ in the Sharkovsky ordering. Finally, in this chapter we proved Sharkovsky's realisation theorem, which states that every tail of Sharkovsky order is the set of periods for some topological dynamical system (I, f).

In Chapter 3, we introduced three important definitions of chaos and the topological properties they required. The first topological property we came across was that of topological transitivity, a property that ensures that points in arbitrary small neighbourhoods are mapped outside their initial neighbourhood under a repeated number of iterations of the map. Furthermore we also proved that if the topological dynamical system has no isolated points, such as in a closed interval, then transitivity and the existence of a dense orbit are equivalent. The next topological property we came across was that of sensitive dependence on initial conditions. This property ensures that points that are arbitrarily close together are eventually mapped arbitrary far apart under repeated applications of the map. We also touched on another property called expansiveness, where all points arbitrarily close together are mapped arbitrarily far apart. Our first notion of chaos came from Devaney [11] and is widely accepted as a leading definition of chaos. Devaney's interpretation of chaos included unpredictability via sensitive dependence on initial conditions, repetitive behaviour through dense periodic points and indecomposability through topological transitivity. However, we proved that sensitive dependence on initial conditions is redundant. Furthermore if the topological dynamical system is defined over a closed interval the density of periodic points is also redundant. Hence on a topological dynamical system (I, f) where I is a closed interval, topological transitivity implies that the system is chaotic in the sense of Devaney. Furthermore, due to topological transitivity and the density of periodic points being purely topological properties, we proved that Devaney chaos is a conjugacy invariant property. Finally, we proved that $([0,1], F_4)$, $([0,1], T_2)$ and $([0,1], D)$ all exhibit Devaney chaos as they

are all topologically transitive. The second definition of chaos we uncovered was Li-Yorke chaos [17]. This definition originally only applied to topological dynamical systems defined over closed intervals and required that the topological dynamical system contains an uncountable set with no asymptotically stable points. This notion was later generalised by Blanchard et al.[4] using Li-Yorke pairs and scrambled sets. We defined a Li-Yorke pair to be two points which can be mapped arbitrarily far apart and mapped together under multiple iterations of the map, ensuring the iterates of these points are essentially scrambled throughout the entire set X. The final definition we covered was topological chaos, the definition of which being a topological dynamical system with positive topological entropy. We gave two equivalent definitions for topological entropy itself. The first was by Adler et al.[1] and defined topological entropy using open covers. The second was by Bowen and Dinaburg [7] [12] and uses the language of metric spaces. From these definitions we concluded that topological entropy is a non-negative real number describing the complexity of a topological dynamics system by the asymptotic mean growth in the number of distinguishable collections of orbits at an arbitrarily fine resolution. We then proved that if (X, f) is a topological dynamical system where f is an isometry we have that the topological entropy of the system is zero, and hence is not topologically chaotic. Furthermore we proved that topological entropy and hence topological chaos it a conjugacy invariant property of topological dynamical systems. We then proved that the the the doubling map $(S^1, \mathcal{D})$ and shift map (Σ_2, σ) are topologically chaotic with a positive topological entropy equal to $\log 2$. Finally, to conclude the chapter we noted that topological chaos implies Li-Yorke chaos, and if (I, f) is a topological dynamical system defined over a closed interval I then Devaney chaos is equivalent to topological chaos.

This concludes our study of different types of chaos in topological dynamics systems. Note that the types of chaos we have studied is not exhaustive. Other definitions of chaos in literature include distributional chaos, ω chaos, P chaos, Block chaos, Wiggins chaos, etc. As we have seen throughout this text, there is no one, universally accepted definition of chaos. However, all definitions we have covered in this text believe that chaotic topological dynamical systems are irregular and complex on even the most arbitrarily small scales. From our analysis, topological transitivity appears to be the most important property of topological dynamical systems which are Devaney chaotic. Hence it is important that chaotic topological dynamical systems can map points in any arbitrarily small neighbourhood to any other arbitrary neighbourhood under repeated iterations. Furthermore, the existence of an uncountable scrambled set is the important property in Li-Yorke chaos. This means that chaotic topological dynamical systems must map an uncountable set of pairs of points arbitrarily far apart and arbitrarily close together under repeated iterations. Furthermore, topological chaos has the important property of positive topological entropy. Hence, the asymptotic mean growth in the number of collections of orbits at arbitrarily fine resolutions must be positive in a chaotic topological dynamical system. All of these properties we have outlined ensure that chaos has the property of arbitrarily fine complexity.

The difference between Devaney chaos, Li-Yorke chaos and topological chaos is essentially how they characterise this arbitrarily fine complexity in their definition.

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