An Introduction to Chaos in Topological Dynamical Systems

Fraser Robert Love

School of Mathematics and Statistics
University of St Andrews

March 27, 2023

Table of Contents

Topological Dynamics

Definition (Topological Dynamical System).

- Let X be a non-empty compact metric space. A *topological dynamical system*, denoted (X, f), is given by a continuous map $f : X \to X$.
- The system starts at an initial point $x \in X$ and evolves through successive iterations of the map f.
- After $k \in \mathbb{N}$ iterations of f, the system can be described by $f^n := f \circ f \circ \cdots \circ f$, where x is mapped to the point $f^n(x)$.

Topological Dynamics

Definition (Orbit).

Let (X, f) be a topological dynamical system. The *orbit* of $x \in X$ under f is the set $\mathcal{O}_f(x) = \{f^n(x) : n \geq 0\} = \{x, f(x), f^2(x), \dots\}$ of iterates of x under the map f.

Definition (Periodic Point, Cycle).

- Let (X, f) be a topological dynamical system. A point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$.
- The *period* of a point x is the least positive integer k such that $f^k(x) = x$. If x has a period of k we say that x is a *period-k* point.
- The orbit $\mathcal{O}_f(x) = \{x, f(x), \dots, f^{k-1}(x)\}$ of a periodic point is a finite set of unique points, called a *periodic orbit* of period k or simply a *k-cycle*.

Topological Dynamics

Example (Logistic Map).

Define $F_\mu : [0, 1] \to [0, 1]$ to be the *logistic map*, where $F_\mu(x) = \mu x(1 - x)$ and $\mu > 0$.

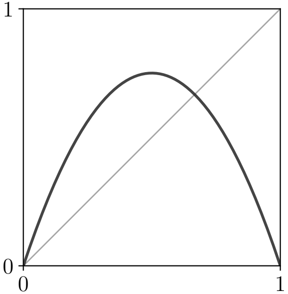

Figure: Logistic map F_μ with $\mu = 3$.

Topological Dynamics

Example (Doubling Map).

Define $\mathcal{D} : S^1 \to S^1$ to be the *doubling map on* S^1, where $\mathcal{D}(z) = z^2$, or equivalently $\mathcal{D}(e^{i\theta}) = e^{2i\theta}$ for some $\theta \in \mathbb{R}$.

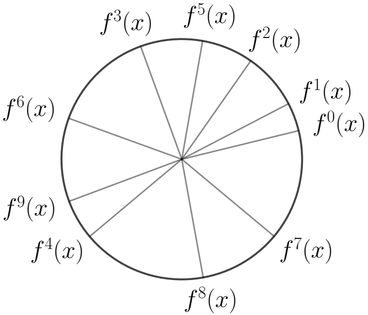

Figure: First ten iterations of the doubling map $\mathcal{D}$.

Topological Dynamics

Definition (Sequence Space).

- Let $\Sigma_2 = \{(s_1, s_2, \dots) : s_i \in \{0, 1\}\}$ be the set sequences of zeros and ones.
- Define (Σ_2, d) to be the *sequence space* where $d(s, t) = \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i}$ is a metric for $(s)_{i=1}^{\infty}, \ (t)_{i=1}^{\infty} \in \Sigma_2$.
- The sequence space (Σ_2, d) is compact.

Example (Shift Map).

Let $(s)_{i=1}^{\infty} \in \Sigma_2$. The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma\left((s)_{i=1}^{\infty}\right) = (s)_{i=2}^{\infty}$.

Proposition.

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is continuous.

Proof.

Let $\varepsilon > 0$ and suppose $\underline{s} = (s_i)_{i=1}^{\infty},\ \underline{t} = (t_i)_{i=1}^{\infty} \in \Sigma_2$. Choose $\delta = \varepsilon$ and suppose $d(\underline{s}, \underline{t}) = \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i} < \delta$. Then $d\left(\sigma\left(\underline{s}\right) - \sigma\left(\underline{t}\right)\right) = d\left(\left(s\right)_{i=2}^{\infty} - \left(t\right)_{i=2}^{\infty}\right) = \Sigma_{i=2}^{\infty} |s_i - t_i| 2^{-i} \leq \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i} < \delta = \varepsilon$. $\square$

Hence (Σ_2, σ) defines a topological dynamical system.

Proposition.

The shift map (Σ_2, σ) has a dense orbit. [2]

Proof.

Consider the sequence $\underline{s} = (0, 1, 00, 01, 10, 11, 000, 001, \dots)$ of 0s and 1s sorted in len-lex order. Let $\underline{t} = (t)_{i=0}^{\infty} \in \Sigma_2$ be arbitrary. Let $\varepsilon > 0$. By construction we can perform some k number of iterations of σ such that if $n > N + k = \frac{1}{\varepsilon} + k$ iterations of σ such that $d(\underline{s}, \underline{t}) = \sum_{i=k}^{n} |s_i - t_i| 2^{-i} + \sum_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} \leq \sum_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} = 2^{-n} < 2^{-N} < \frac{1}{N} = \varepsilon$. Hence the orbit $\underline{s}$ is dense in Σ_2. □

Proposition.

The periodic points of the shift map (Σ_2, σ) are dense in Σ_2. [2]

Proof.

Let $\underline{s} = (s)_{i=1}^{\infty}$ be an arbitrary point in Σ_2. Define $t_n = (s_0, \ldots, s_n, s_0, \ldots, s_n, \ldots)$ to be an infinite repeating sequence where $t_{n_i} = s_i$ for $1 \leq i \leq n$. Then $d(s, t) = \sum_{i=0}^{n} |s_i - s_i| 2^{-i} + \sum_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} \leq \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n}$. Hence as $n \to \infty$ we have $t_n \to \underline{s}$. Since $\underline{s}$ was arbitrary, the periodic points of σ are dense. $\square$

Introduction to Chaos

- Many different definitions of chaos exist for topological dynamical systems.
- Devaney chaos, Li-Yorke chaos, Topological chaos, etc.
- These definitions rely on many different topological characteristics to define chaos in a natural way.
- Topological transitivity / existence of a dense orbit, sensitive dependence on initial conditions, dense periodic points, existence of an uncountable scrambled-set, and positive topological entropy.

Topological Characteristics of Chaos

Definition (Topological Transitivity).

Let (X, f) be a topological dynamical system. The map f is *topologically transitive* if for every pair of non-empty open sets $U, V \subseteq X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

In a topologically transitive mapping, points in an arbitrarily small set can be mapped into any other arbitrary small set under a repeated number of iterations of the map.

Proposition.

Let (X, f) be a topological dynamical system and suppose X has no isolated points. The map f is topologically transitive if and only if there exists some $x \in X$ such that $\mathcal{O}(x)$ is dense in X. [4]

Topological Characteristics of Chaos

Example.

Let $(S^1, \mathcal{D})$ be the doubling map over S^1, where $\mathcal{D}(z) = z^2$. Then $\mathcal{D}$ is topologically transitive.

Proof.

Let $z_1, z_2 \in S^1$. Let $(z_1, z_2) = U$ define an arc between z_1 and z_2. Suppose now $d\left(z_1, z_2\right) > \frac{2\pi}{2^k}$ for some $k \in \mathbb{N}$. Then $d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d\left(z_1, z_2\right) > 2^k \cdot \frac{2\pi}{2^k} = 2\pi$. Hence $\mathcal{D}^k((z_1, z_2))$ covers S^1 so for any $V \subseteq S^1$ we obtain $\mathcal{D}^k(U) \cap V \neq \emptyset$. Hence $\mathcal{D}$ is topologically transitive. $\square$

Topological Characteristics of Chaos

Definition (Sensitive Dependence on Initial Conditions).

Let (X, f) be a topological dynamical system and $\varepsilon > 0$. A point $x \in X$ is *ε-unstable* if, for every neighbourhood U of x, there exists a point $y \in U$ and $k \geq 0$ such that $d\left(f^k(x), f^k(y)\right) \geq \varepsilon$. The map f has *sensitive dependence on initial conditions* if for all points $x \in X$, x is ε-unstable.

In other words, there exist points arbitrary close to x that eventually get mapped at least ε far apart under multiple applications of the map.

Topological Characteristics of Chaos

Example.

Let $(S^1, \mathcal{D})$ be the doubling map over the S^1, where $\mathcal{D}(z) = z^2$. Then $\mathcal{D}$ has sensitive dependence on initial conditions.

Proof.

Let $z_1, z_2 \in S^1$ and $\varepsilon < 2^k\delta$ and suppose $d(z_1, z_2) = \delta$, then $d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d(z_1, z_2) = 2^k\delta > \varepsilon$. Hence we can always choose a k large enough so this holds, and so $\mathcal{D}$ has sensitive dependence on initial conditions. $\square$

Devaney Chaos

Definition (Devaney Chaos).

A topological dynamical system (X, f) is *chaotic in the sense of Devaney* if it is topologically transitive, has sensitive dependence on initial conditions, and if the periodic points of f are dense in X.

Devaney's definition considers:

- Unpredictability (sensitive dependence on initial conditions).
- Repetitiveness (dense periodic points).
- Indecomposability (topological transitivity).

Devaney Chaos

It turns out sensitive dependence on initial conditions is a redundant characteristic via the following proposition.

Proposition.

Let (X, f) be a topological dynamical system. If the map f is topologically transitive and has dense periodic points then f has sensitive dependence on initial conditions. [1] [3]

Devaney Chaos

Proposition.

The shift map (Σ_2, σ) is chaotic in the sense of Devaney.

Proof.

We previously showed that (Σ_2, σ) has a dense orbit and so is topologically transitive. We also showed that the periodic points of the shift map are dense in Σ_2. Hence, by the proposition above, we conclude that (Σ_2, σ) is Devaney chaotic. $\square$

Concluding Remarks

- We have defined chaos to be a mixture of unpredictability, repetitiveness and indecomposability.
- This was achieved using the properties of topological transitivity, sensitive dependence on initial conditions and dense periodic points.
- We have shown that the shift map (Σ_2, σ) is a Devaney chaotic topological dynamical system.
- Using symbolic dynamics and topological conjugacy we can also prove that $(F_\mu, [0, 1])$, $(\mathcal{D}, S^1)$ and many more systems exhibit Devaney chaos.
- Furthermore, some of these systems also exhibit other types of chaos, such as Li-Yorke chaos and Topological chaos.

Bibliography

J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey.
On devaney's definition of chaos.
The American Mathematical Monthly, 99(4):332–334, 1992.

R. L. Devaney.
An introduction to Chaotic Dynamical Systems.
Addison-Wesley Publishing Company Advanced Book Program, Redwood City, California, second edition, 1989.

E. Glasner and B. Weiss.
Sensitive dependence on initial conditions.
Nonlinearity, 6(6):1067–1075, 1993.

S. Silverman.
On maps with dense orbits and the definition of chaos.
Rocky Mountain Journal of Mathematics, 22(1):353–375, 1992.