An Introduction to Chaos in Topological Dynamical Systems

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Topological Dynamics

Definition (Topological Dynamical System).

- Let X be a non-empty compact metric space. A *topological dynamical system*, denoted (X, f), is given by a continuous map $f : X \to X$.
- The system starts at an initial point $x \in X$ and evolves through successive iterations of the map f.
- After $k \in \mathbb{N}$ iterations of f, the system can be described by $f^n := f \circ f \circ \cdots \circ f$, where x is mapped to the point $f^n(x)$.

Topological Dynamics

Definition (Orbit).

Let (X, f) be a topological dynamical system. The *orbit* of $x \in X$ under f is the set $\mathcal{O}_f(x) = \{f^n(x) : n \geq 0\} = \{x, f(x), f^2(x), \dots\}$ of iterates of x under the map f.

Definition (Periodic Point, Cycle).

- Let (X, f) be a topological dynamical system. A point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$.
- The *period* of a point x is the least positive integer k such that $f^k(x) = x$. If x has a period of k we say that x is a *period-k* point.
- The orbit $\mathcal{O}_f(x) = \{x, f(x), \dots, f^{k-1}(x)\}$ of a periodic point is a finite set of unique points, called a *periodic orbit* of period k or simply a *k-cycle*.

Topological Dynamics

Example (Logistic Map).

Define $F_\mu : [0,1] \to [0,1]$ to be the *logistic map*, where $F_\mu(x) = \mu x(1-x)$ and $\mu > 0$.

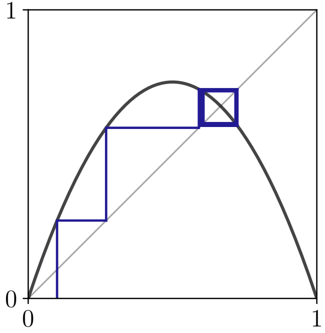

Figure: Logistic map F_μ with $\mu = 3$.

Topological Dynamics

Example (Doubling Map).

Define $\mathcal{D} : S^1 \to S^1$ to be the *doubling map on* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, where $\mathcal{D}(z) = z^2$, or equivalently $\mathcal{D}(e^{i\theta}) = e^{2i\theta}$ for some $\theta \in \mathbb{R}$.

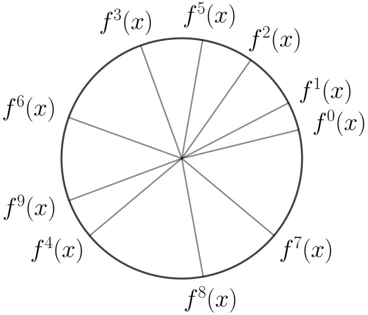

Figure: First ten iterations of the doubling map $\mathcal{D}$.

Topological Dynamics

Definition (Sequence Space).

- Let $\Sigma_2 = \{(s_1, s_2, \dots) : s_i \in \{0, 1\}\}$ be the set sequences of ones and zeros.
- Define (Σ_2, d) to be the *sequence space* where $d(s,t) = \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i}$ is a metric for $(s)_{i=1}^{\infty},\ (t)_{i=1}^{\infty} \in \Sigma_2$.
- The sequence space (Σ_2, d) is compact.

Proposition.

The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ given by $\sigma\left((s)_{i=1}^{\infty}\right) = (s)_{i=2}^{\infty}$ is continuous.

Proof.

Let $\varepsilon > 0$ and choose $\underline{s} = (s_i)_{i=1}^{\infty},\ \underline{t} = (t_i)_{i=1}^{\infty} \in \Sigma_2$ such that $d(\underline{s}, \underline{t}) = \Sigma_{i=1}^{\infty} |s_i - t_i| 2^{-i} < \delta$. Choose n such that $2^{-n} \leq \varepsilon$ and let $\delta = 2^{-(n+1)}$. Hence $\underline{s}$ and $\underline{t}$ agree on the first $n+1$ symbols and $\sigma(\underline{s})$ and $\sigma(\underline{t})$ agree on the first n symbols. Then $d(\sigma(\underline{s}), \sigma(\underline{t})) = d\left((s)_{i=n+1}^{\infty}, (t)_{i=n+1}^{\infty}\right) = \Sigma_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} \leq 2^{-n} \leq \varepsilon$. $\square$

Hence (Σ_2, σ) defines a topological dynamical system.

Proposition.
The shift map (Σ_2, σ) has a dense orbit. [2]

Proof.
Let $\underline{t} = (t)_{i=0}^{\infty} \in \Sigma_2$ be arbitrary and $\varepsilon > 0$. Consider the sequence $\underline{s} = (0, 1, \{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}, \{0, 0, 0\}, \{0, 0, 1\}, \dots)$ of 0s and 1s sorted in len-lex order. By construction we can perform some $k \in \mathbb{N}$ iterations of σ such that the first n symbols of $\sigma^k(\underline{s})$ and $\underline{t}$ agree. Choose $N \geq \log_2 \frac{1}{\varepsilon}$. Therefore for $n \geq N - 1$ we have that, $d(\sigma^k(\underline{s}), \underline{t}) = \sum_{i=n}^{\infty} |\sigma^k(\underline{s})_i - t_i| 2^{-i} \leq 2^{-(n+1)} < 2^{-N} \leq \varepsilon$. Hence the orbit of $\underline{s}$ is dense in Σ_2. $\square$

Proposition.

The periodic points of the shift map (Σ_2, σ) are dense in Σ_2. [2]

Proof.

Let $\underline{s} = (s)_{i=1}^{\infty}$ be an arbitrary point in Σ_2 and let $\varepsilon > 0$. Pick an n such that $2^{-n} \leq \varepsilon$ and define $t_n = (s_0, \ldots, s_n, s_0, \ldots, s_n, \ldots)$ to be an infinite repeating sequence where $t_i = s_i$ for $1 \leq i \leq n$. Then $d(s, t) = \sum_{i=0}^{n} |s_i - t_i| 2^{-i} + \sum_{i=n+1}^{\infty} |s_i - t_i| 2^{-i} = \sum_{i=n+1}^{\infty} 2^{-i} \leq 2^{-n} \leq \varepsilon$. Hence as $n \to \infty$ we have $t_n \to \underline{s}$. Since $\underline{s}$ was arbitrary, the periodic points of σ are dense. $\square$

Introduction to Chaos

- Many different definitions of chaos exist for topological dynamical systems.
- Devaney chaos, Li-Yorke chaos, Topological chaos, etc.
- These definitions rely on many different topological characteristics to define chaos in a natural way.
- Topological transitivity / existence of a dense orbit, sensitive dependence on initial conditions, dense periodic points, existence of an uncountable scrambled-set, and positive topological entropy.

Topological Characteristics of Chaos

Definition (Topological Transitivity).

Let (X, f) be a topological dynamical system. The map f is *topologically transitive* if for every pair of non-empty open sets $U, V \subseteq X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

In a topologically transitive mapping, points in an arbitrarily small set can be mapped into any other arbitrary small set under a repeated number of iterations of the map.

Proposition.

Let (X, f) be a topological dynamical system and suppose X has no isolated points. The map f is topologically transitive if and only if there exists some $x \in X$ such that $\mathcal{O}(x)$ is dense in X. [4]

Topological Characteristics of Chaos

Example.

Let $(S^1, \mathcal{D})$ be the doubling map over S^1, where $\mathcal{D}(z) = z^2$. Then $\mathcal{D}$ is topologically transitive.

Proof.

Let $z_1, z_2 \in S^1$. Let $(z_1, z_2) = U$ define an arc between z_1 and z_2. Suppose now $d\left(z_1, z_2\right) > \frac{2\pi}{2^k}$ for some $k \in \mathbb{N}$. Then $d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d\left(z_1, z_2\right) > 2^k \cdot \frac{2\pi}{2^k} = 2\pi$. Hence $\mathcal{D}^k((z_1, z_2))$ covers S^1 so for any $V \subseteq S^1$ we obtain $\mathcal{D}^k(U) \cap V \neq \emptyset$. Hence $\mathcal{D}$ is topologically transitive. $\square$

Topological Characteristics of Chaos

Definition (Sensitive Dependence on Initial Conditions).

Let (X, f) be a topological dynamical system and $\varepsilon > 0$. A point $x \in X$ is *ε-unstable* if, for every neighbourhood U of x, there exists a point $y \in U$ and $k \geq 0$ such that $d\left(f^k(x), f^k(y)\right) \geq \varepsilon$. The map f has *sensitive dependence on initial conditions* if for all points $x \in X$, x is ε-unstable.

In other words, there exist points arbitrary close to x that eventually get mapped at least ε far apart under multiple applications of the map.

Example.

Let $(S^1, \mathcal{D})$ be the doubling map over the S^1, where $\mathcal{D}(z) = z^2$. Then $\mathcal{D}$ has sensitive dependence on initial conditions.

Proof.

Let $z_1, z_2 \in S^1$ and $\varepsilon < 2^k\delta$ and suppose $d(z_1, z_2) = \delta$, then
$d\left(\mathcal{D}^k(z_1), \mathcal{D}^k(z_2)\right) = d\left(2^k z_1, 2^k z_2\right) = 2^k d(z_1, z_2) = 2^k\delta > \varepsilon$.
Hence we can always choose a k large enough so this holds, and so $\mathcal{D}$ has sensitive dependence on initial conditions. $\square$

Devaney Chaos

Definition (Devaney Chaos).

A topological dynamical system (X, f) is *chaotic in the sense of Devaney* if it is topologically transitive, has sensitive dependence on initial conditions, and if the periodic points of f are dense in X.

Devaney's definition considers:

- Unpredictability (sensitive dependence on initial conditions).
- Repetitiveness (dense periodic points).
- Indecomposability (topological transitivity).

Devaney Chaos

It turns out sensitive dependence on initial conditions is a redundant characteristic via the following proposition.

Proposition.

Let (X, f) be a topological dynamical system. If the map f is topologically transitive and has dense periodic points then f has sensitive dependence on initial conditions. [1] [3]

Devaney Chaos

Proposition.

The shift map (Σ_2, σ) is chaotic in the sense of Devaney.

Proof.

We previously showed that (Σ_2, σ) has a dense orbit and so is topologically transitive. We also showed that the periodic points of the shift map are dense in Σ_2. Hence, by the proposition above, we conclude that (Σ_2, σ) is Devaney chaotic. $\square$

Concluding Remarks

- We have defined chaos to be a mixture of unpredictability, repetitiveness and indecomposability.
- This was achieved using the properties of topological transitivity, sensitive dependence on initial conditions and dense periodic points.
- We have shown that the shift map (Σ_2, σ) is a Devaney chaotic topological dynamical system.
- Using symbolic dynamics and topological conjugacy we can also prove that $([0,1], F_\mu)$, $(S^1, \mathcal{D})$ and many more systems exhibit Devaney chaos.
- Furthermore, some of these systems also exhibit other types of chaos, such as Li-Yorke chaos and Topological chaos.

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