

Algebraic Recipes for Integer Programming

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ABSTRACT. Integer programming is concerned with solving linear systems of equations over the non-negative integers. The basic question is to find a solution which minimizes a given linear objective function for a fixed right hand side. Here we also consider parametric versions where the objective function and the right hand side are allowed to vary. The main emphasis is on Gröbner bases, rational generating functions, and how to use existing software packages. Concrete applications to problems in statistical modeling will be presented.

1. An Introductory Coin Problem

This lecture is about solving linear equations over the non-negative integers. Our point of departure is the integer programming problem in *standard form*:

$$(1.1) \quad \text{Minimize } c \cdot u \text{ subject to } A \cdot u = b \text{ and } u \in \mathbb{N}^n.$$

The given instance consists of an integer matrix $A \in \mathbb{Z}^{d \times n}$, a row vector $c \in \mathbb{Z}^n$ and a column vector $b \in \mathbb{Z}^d$. The unknown is the column vector $u = (u_1, \dots, u_n)$. What makes the problem hard is the requirement that the u_i be non-negative integers.

As an example consider the following simple *coin problem*. Suppose you are carrying a large collection of coins in your pocket. The allowed coins are pennies (1 cent), nickels (5 cents), dimes (10 cents) and quarters (25 cents). The problem is to replace your “portfolio” by an equal number of coins having the same monetary value, but such that the number of nickels plus the number of quarters is minimized.

This problem can be expressed in the standard form (1.1) by setting

$$(1.2) \quad d = 2, n = 4, \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 10 & 25 \end{pmatrix} \text{ and } c = (0, 1, 0, 1).$$

The right hand side vector $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is left unspecified. Its coordinates b_1 and b_2 are the number of coins and value (in pennies) of the portfolio respectively.

EXAMPLE 1.1. For instance, if $b = \begin{pmatrix} 10 \\ 114 \end{pmatrix}$, then we seek to express one dollar and fourteen cents with ten coins. The optimal solution to this instance of (1.1) is $u = (4, 2, 0, 4)$ and the *optimal value* is $c \cdot u = 6$. In words, you can take four

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pennies, two nickels and four quarters to make one dollar and fourteen cents with ten coins, but it is impossible to do it with less than six nickels or quarters. \square

A parametric solution to our problem is provided by the *Gröbner basis*

$$(1.3) \quad \mathcal{G} = \{ n^3q - d^4, n^6 - p^5q, n^3d^4 - p^5q^2, p^5q^3 - d^8 \}.$$

Our Gröbner basis in (1.3) is expressed as a set of *monomial differences*, which is how they usually appear in computer algebra systems. We note that there are two alternative but entirely equivalent ways of writing our Gröbner basis. In the optimization literature, it is more common to express \mathcal{G} as a *set of lattice vectors*;

$$(1.4) \quad \mathcal{G}' = \{ (0, 3, -4, 1), (-5, 6, 0, -1), (-5, 3, 4, -2), (5, 0, -8, 3) \}.$$

The Gröbner basis is a set of *exchange rules* which you can use to successively improve your portfolio. For instance, the first rule $n^3q - d^4$ says that you can replace three nickels and one quarter with four dimes. Each of the four moves in \mathcal{G} changes neither the number of coins nor their value but it decreases the objective function value. The crucial property of being a Gröbner basis says that *if none of the exchange rules can be applied then your portfolio is guaranteed to be optimal*.

There is a third way of encoding the Gröbner basis, which will be of importance in Section 4. Namely, we can also express \mathcal{G} as the following *generating function*:

$$(1.5) \quad \mathcal{G}'' = n^3D^4q + P^5n^6Q + P^5n^3d^4Q^2 + p^5D^8q^3.$$

In the last representation there are two variables for each column of the matrix A , and each monomial represents one exchange rule. The lower case variable represents the gain and the upper case variable represents the loss in the exchange of coins.

In Section 2 we explain how the Gröbner basis is constructed for an arbitrary matrix A and cost function c , and in Section 3 we discuss the relationship to other notions of test sets in integer programming, including *Hilbert bases* and *Graver bases*, and we introduce Hemmecke's easy-to-use software `4ti2` for computing these test sets. In Section 4 we address complexity issues. In particular, we show how the Gröbner basis can be computed in polynomial time when d and n are fixed.

The power of algebraic methods in integer programming stems from the fact that they can answer parametric questions like: *What are all the optimal portfolios in our coin problem?* Each portfolio is given as a vector $u = (u_1, u_2, u_3, u_4)$ or as a monomial $p^{u_1}n^{u_2}d^{u_3}q^{u_4}$, and we wish to encode all portfolios that are optimal solutions of (1.1) with $b = Au$. The following three conditions are equivalent:

- (a) The vector $u \in \mathbb{N}^4$ is an optimal portfolio.
- (b) None of the four monomials n^3q , n^6 , n^3d^4 or p^5q^3 divides $p^{u_1}n^{u_2}d^{u_3}q^{u_4}$.
- (c) $(u_2 \leq 2 \text{ or } u_4 = 0)$ and $(u_2 \leq 5)$ and $(u_2 \leq 2 \text{ or } u_3 \leq 3)$ and $(u_1 \leq 4 \text{ or } u_4 \leq 2)$.

The *Hilbert series of all optimal solutions* is the formal sum of these monomials:

$$\sum_{u \text{ optimal}} p^{u_1}n^{u_2}d^{u_3}q^{u_4}$$

This generating function is equal to the following rational function:

$$\frac{n^3q^3p^5 - n^6qd^4 + n^6d^4 + n^3qd^4 - q^3p^5 + n^6q - n^3d^4 - n^6 - n^3q + 1}{(1-p)(1-n)(1-d)(1-q)}.$$

In Section 4, we will see that such Hilbert series can be computed in polynomial time (for fixed d and n). In Sections 5 and 6 we will focus on applications of integer

programming to *statistics*, and we will argue that Gröbner bases and generating functions are useful tools for the optimization problems arising in this context.

The use of Gröbner bases as a tool for integer programming first appeared in the paper [1] by Conti and Traverso. Their approach was further developed in two doctoral dissertations in the Cornell Operations Research Department, written by Thomas (see [Tho]) and Hoşten (see [HS1]) in 1994 and 1997 respectively. Subsequently, Hoşten and Thomas [HT1] developed an algebraic theory of group relaxations, extending the foundational work in integer programming theory which was done by Gomory in the 1960's. These and many other important topics will not be discussed in this lecture, which aims to be introductory and self-contained. Readers wishing to learn more about commutative algebra methods in integer programming are referred to the book [Stu] and the survey articles [HT2] and [Tho2].

2. Gröbner Bases

We consider the integer programming problem in standard form (1.1) where A and c are fixed and b is arbitrary. In this section we further assume that c is *generic* in the sense that (1.1) has a unique optimal solution for every feasible right hand side b . In practise, **this can always be accomplished** by lexicographically perturbing the given cost vector c . Consider the infinite set of all optimal solutions,

$$\text{Opt}_{A,c} = \{ u \in \mathbb{N}^n : u \text{ is the optimal solution of (1.1) for } b = Au \}.$$

Suppose that u and u' are vectors in \mathbb{N}^n such that $u' \leq u$ (coordinatewise) and $u \in \text{Opt}_{A,c}$. Then **it can be seen that** $u' \in \text{Opt}_{A,c}$. We paraphrase this observation in the following lemma, using the language of partially ordered sets (= posets).

LEMMA 2.1. *The set $\text{Opt}_{A,c}$ is an **order ideal** in the partially ordered set \mathbb{N}^n .*

A basic result about order ideals in the poset \mathbb{N}^n , known as **Dickson's Lemma**, states that the set of minimal elements in the complementary set $\text{Opt}_{A,c} \setminus \mathbb{N}^n$ is finite. We write $\text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c})$ for this finite set. Its elements are called the *minimally non-optimal points* of the integer programming family (1.1). Recall that our introductory coin example had precisely four minimally non-optimal points:

$$\text{Min}(\mathbb{N}^4 \setminus \text{Opt}_{A,c}) = \{ (0, 3, 0, 1), (0, 6, 0, 0), (0, 3, 4, 0), (5, 0, 0, 3) \}.$$

For every $g^+ \in \text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c})$ there exists a unique vector $g^- \in \text{Opt}_{A,c}$ such that $Ag^+ = Ag^-$. Namely, g^- is the optimal solution to (1.1) with $b = Ag^+$.

DEFINITION 2.2. The *Gröbner basis* for the matrix A and cost vector c is

$$\mathcal{G}_{A,c} = \{ g^+ - g^- : g^+ \in \text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c}) \}.$$

This is a finite set of lattice vectors in the kernel of A . We can also regard them as monomial differences in n unknowns x_i or as monomials in $2n$ unknowns x_i, y_i via

$$(2, 13, 0, -8, 5, -7) \longleftrightarrow x_1^2 x_2^{13} x_5^5 - x_4^8 x_6^7 \longleftrightarrow x_1^2 x_2^{13} y_4^8 x_5^5 y_6^7.$$

The following theorem states that the Gröbner basis is a minimal test set for the family of integer programs specified by the matrix A and the cost vector c .

THEOREM 2.3. *Let u be a feasible solution of (1.1). Then u is non-optimal if and only if there exists $g \in \mathcal{G}_{A,c}$ with $g^+ \leq u$, and in this case $u - g$ is a better feasible solution than u . There is no smaller set than $\mathcal{G}_{A,c}$ which has this property.*

PROOF. By construction, every element $g = g^+ - g^-$ satisfies $c \cdot g = c \cdot g^+ - c \cdot g^- > 0$. The if-direction follows because $(u - g) \cdot c < u \cdot c$ and $g^+ \leq u$ is equivalent to $u - g$ being feasible. For the only-if direction suppose that no $g \in \mathcal{G}_{A,c}$ satisfies $g^+ \leq u$. This means that no element of $\text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c})$ lies below u in the poset \mathbb{N}^n . But this means that $u \in \text{Opt}_{A,c}$. The minimality of $\mathcal{G}_{A,c}$ holds because every element of $\text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c})$ has to be reducible by some vector in the test set. \square

Under a certain genericity hypothesis on the matrix A , the elements in the Gröbner basis are in bijection with the *neighbors of the origin*, which is a test set for integer programming introduced by Herbert Scarf [Sca]. The connection between neighbors and Gröbner bases was studied in a commutative algebra setting in [PS].

Let us assume that the Gröbner basis $\mathcal{G}_{A,c}$ is known to us in some explicit or implicit form. If we are given any feasible solution $u \in \mathbb{N}^n$ then the integer programming problem (1.1) can be solved by the following one-line algorithm:

(2.1) **While** there exists $g \in \mathcal{G}_{A,c}$ with $g^+ \leq u$ **do** replace u by $u - g$.

The problem of constructing a first feasible solution u from the right hand side b can be solved by the same reduction process but for a different Gröbner basis. The idea is completely analogous to *Phase One in the Simplex Algorithm*. To keep our discussion simple, we will assume that some feasible solution u is known beforehand.

One of the objectives of this lecture is to dispel the belief, held by many experts in complexity theory and combinatorial optimization, that the algebraic notion of Gröbner bases is utterly useless when it comes to designing efficient algorithms. Let me begin by pointing out that **computing Gröbner bases is easy and fun**.

My currently favorite tool for producing the Gröbner basis $\mathcal{G}_{A,c}$ from the matrix A and the cost vector c is the software **4ti2** developed by Raymond Hemmecke. It can be found at www.4ti2.de and is ridiculously easy to download and run. It took me (= a technologically challenged individual) precisely three minutes to install **4ti2** on my (ancient) computer, and another minute later I was already enjoying my first Gröbner basis on the screen. Actually, I don't recall having ever encountered a piece of mathematical software that was simpler to use than **4ti2**.

The first non-coin example I tried had $d = 3$ and $n = 7$. The matrix A was filled snakewise by prime numbers and the vector c was filled by square integers. The input to **4ti2** consists of a matrix in a file named **example** in the format

```
7 3
 2 3 5 7 11 13 17
43 41 37 31 29 23 19
47 53 59 61 67 71 73
```

and a cost vector in a file **example.cost** in the format

```
7 1
1 4 9 16 25 36 49
```

After typing **groebner example** and hitting the return key, about three seconds later, the Gröbner basis appeared in a new file called **example.gro**:

```
7 241
-6 14 -11 2 1 0 0
-6 13 -9 2 0 -1 1
0 -1 2 0 -1 -1 1
-4 11 -7 -1 -2 3 0
```

10 -3 -35 38 -1 -7 0
 0 22 -53 39 -2 -4 0
 4 11 -46 40 0 -7 0

This Gröbner basis consists of 241 vectors in \mathbb{Z}^7 , and it represents a solution to the parametric problem of minimizing $\sum_{i=1}^7 i^2 \cdot u_i$ over all vectors $u \in \mathbb{N}^7$ that satisfy

$$(2.2) \quad \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 \\ 43 & 41 & 37 & 31 & 29 & 23 & 19 \\ 47 & 53 & 59 & 61 & 67 & 71 & 73 \end{pmatrix} \cdot u = b.$$

Knowing the 241 vectors in the Gröbner basis, we can now apply the reduction algorithm (2.1) starting with any given feasible solution u . Take, for instance, $u = (100, 100, 100, 100, 100, 100)^T$. The corresponding right hand side is $b = A \cdot u = (5800, 22300, 43100)$. The algorithm in (2.1) reduces u to the optimal solution $u^* = (62, 8, 176, 17, 423, 0, 0)$. The optimal value is found to be $c^* = 12,525$.

Knowledge of the Gröbner basis allows us to answer more advanced structural questions about the system (2.2). One such question is that of finding the *integer programming gap*, a topic to be discussed in Section 4. Another example is the question of *sensitivity analysis* with respect to the cost function. Suppose that the cost vector is allowed to vary in a neighborhood of the given vector c . Then the Gröbner basis $\mathcal{G} = \mathcal{G}_{A,c}$ remains unchanged provided c ranges in the *Gröbner cone*, which is defined by the following linear inequalities in the unknowns c_1, \dots, c_n :

$$(2.3) \quad c \cdot g > 0 \quad \text{for all } g \in \mathcal{G}.$$

For instance, in our coin example, the Gröbner cone is the set of all solutions to

$$3c_2 + c_4 > 3c_3, \quad 6c_2 > 5c_1 + c_4, \quad 3c_2 + 4c_3 > 5c_1 + 2c_4, \quad 5c_1 + 3c_4 > 8c_3.$$

The collection of all Gröbner cones in \mathbb{R}^n forms the *Gröbner fan* of the matrix A . This is an important invariant which allows us study how the solution of (1.1) changes as both b and c are allowed to vary. See [ST] for the basic theory.

The Gröbner fan of a matrix A can be efficiently calculated using the algorithm of Huber and Thomas [HT]. A highly efficient implementation was recently given by Anders Jensen in his program **CaTS**. This piece of software can currently be found at the web page <http://www.soopadoopa.dk/anders/cats/cats.html>.

3. Hilbert bases and Graver bases

Gröbner bases are closely related to other natural notions of test sets arising in the theory of integer programming. A classical such notion is that of a *Hilbert basis*. Consider the problem of solving a homogeneous system of linear equations over the non-negative integers. As before, we assume that the defining matrix A has d rows and n columns. Then our solution set is the following semigroup:

$$(3.1) \quad \ker_{\mathbb{N}}(A) = \{ u \in \mathbb{N}^n : A \cdot u = 0 \}.$$

Consider the subset of non-zero minimal elements of the semigroup:

$$(3.2) \quad \mathcal{H}_A = \{ u \in \ker_{\mathbb{N}}(A) \setminus \{0\} : \text{no element } v \in \ker_{\mathbb{N}}(A) \setminus \{0, u\} \text{ satisfies } v \leq u \}.$$

The following result is due to the 19th century invariant theorist Paul Gordan:

PROPOSITION 3.1. *The set \mathcal{H}_A is finite. It is the unique minimal set such that every vector in $\ker_{\mathbb{N}}(A)$ is an \mathbb{N} -linear combination of elements in \mathcal{H}_A .*

The finite set \mathcal{H}_A is called the *Hilbert basis* of the matrix A . Hilbert bases can also be computed using the program `4ti2`. We consider the same matrix as in (2.2) but we now alternate the sign pattern of its columns in the input file `example`:

```
7 3
  2 -3  5 -7 11 -13 17
43 -41 37 -31 29 -23 19
47 -53 59 -61 67 -71 73
```

After typing `hilbert example` and hitting the return key, about six seconds later, the Hilbert basis appears on a new file called `example.hil`:

```
7 1305
4 34 62 38 3 0 1
4 35 64 38 2 1 2
4 60 123 77 1 0 5
4 36 66 38 1 2 3
... ..
0 673 980 0 2 647 324
0 674 982 0 1 648 325
0 675 984 0 0 649 326
```

The Hilbert basis consists of 1,305 vectors, and it has a lot of internal structure.

Hilbert bases play an important role in the recent work of Robert Weismantel and his collaborators on “primal methods in integer programming”. The paper [HKW] introduces the notion of *integral basis* which is a slight generalization of Hilbert bases, and it presents a simplex-like *integral basis algorithm* which is shown to perform very well on standard benchmark problems in integer programming.

A larger test set associated with an integer matrix A is the Graver basis, which can be defined as follows. For any sign pattern $\sigma \in \{-1, +1\}^n$ let D_σ be the $n \times n$ -diagonal matrix with i -th entry σ_i . The *Graver basis* of A is the finite set

$$(3.3) \quad \mathcal{GR}_A := \bigcup_{\sigma \in \{-1, +1\}^n} D_\sigma \cdot \mathcal{H}_{AD_\sigma}$$

In this definition, we are taking the union over the 2^n Hilbert bases for the various matrices $A \cdot D_\sigma$. The signs are adjusted so that each Hilbert basis lies in the kernel of the original matrix A . Proposition 3.1 ensures that the Graver basis \mathcal{GR}_A is a finite subset of $\ker_{\mathbb{Z}}(A)$. The following result is proved in [Stu, §7].

PROPOSITION 3.2. *The Graver basis is a universal Gröbner basis. It contains, up to negating vectors, the Gröbner bases of A for all cost functions. In symbols,*

$$(3.4) \quad \bigcup_{c \in \mathbb{Z}^n} \mathcal{G}_{A,c} \subseteq \mathcal{GR}_A.$$

The Graver basis is the ultimate test set one can compute for a given integer matrix A . It provides a parametric solution to the integer programming problem (1.1) when both the right hand side b and the cost function c are allowed to vary.

EXAMPLE 3.3. If A is the 2×4 -matrix (1.2) in our coin problem, then

$$\mathcal{GR}_A = \{ (0, 3, -4, 1), (-5, 6, 0, 1), (-5, 3, 4, -2), (5, 0, -8, 3), (-5, 9, -4, 0) \}.$$

This Graver basis has only one more element than the Gröbner basis (1.4). The advantage of the Graver basis over the Gröbner basis is that we can now use (2.1) to solve the coin problem with respect to an arbitrary cost vector c . \square

The Graver basis has another natural interpretation in integer programming. Consider our original problem (1.1) but now add the requirement that the coordinates u_i of the solution u are bounded above by some quantities a_i .

$$(3.5) \quad \text{Minimize } c \cdot u \text{ subject to } A \cdot u = b, \ u \in \mathbb{N}^n, \text{ and } u \leq a.$$

Here we regard A and c as fixed and $(a, b) \in \mathbb{Z}^{n+d}$ as unspecified. It turns out that the Graver basis is the unique minimal test set for this family of integer programs.

THEOREM 3.4. *Let u be a feasible solution of (3.5). Then u is non-optimal if and only if there exists $g \in \mathcal{GR}_A$ with $g^+ \leq u$ and $g^- \leq a - u$, and in this case $u - g$ improves u . There is no smaller set than \mathcal{GR}_A which has this property.*

PROOF. We must prove the only if direction. Suppose u is non-optimal for (3.5) and let v be the corresponding optimal solution. Pick $\sigma \in \{-1, +1\}^n$ so that $D_\sigma(v - u)$ is a nonnegative vector. There exist elements h_1, \dots, h_r in the Hilbert basis $\mathcal{H}_{D_\sigma A}$ such that $D_\sigma(v - u) = h_1 + \dots + h_r$. and hence

$$v - u = D_\sigma h_1 + \dots + D_\sigma h_r,$$

where each summand lies in \mathcal{GR}_A . Since $c \cdot (v - u) < 0$, there exists at least one index i such that $D_\sigma \cdot h_i \cdot c < 0$. The vector $g = -D_\sigma \cdot h_i$ lies in \mathcal{GR}_A . The construction implies that it satisfies $g^+ \leq u$ and $g^- = (D_\sigma \cdot h_i)^+ \leq (v - u)^+ \leq a - u$.

We now show that every element $g = g^+ - g^-$ of \mathcal{GR}_A is needed in a test set for our problem. Suppose that $c \cdot g < 0$ and define $a = g^+ + g^-$ and $b = Ag^+ = Ag^-$. With these choices of a and b , the vectors g^+ and g^- are the only two feasible solutions for (3.5). Hence the move from g^+ to g^- must be in the test set. \square

In light of Proposition 3.2 and Theorem 3.4, it is highly desirable to be able to precompute the Graver basis of a given integer matrix. An algorithm for this computation is available in `4ti2`. But the reader should be warned that Example 3.3 is somewhat misleading: the Graver basis is often much larger than the Gröbner basis and it takes much longer to compute it. Consider again our `example` matrix,

```
7 3
 2  3  5  7 11 13 17
43 41 37 31 29 23 19
47 53 59 61 67 71 73
```

The command `graver example` produces the Graver basis in a file `example.gra`:

```
7 29417
0 1 2 0 -1 1 1
0 24 57 39 0 -2 2
0 25 59 39 -1 -1 3
... ..
14 -9 10 81 -6 -89 -37
64 86 11 1 -6 -48 -28
114 229 126 -1 -6 -11 -15
124 268 161 -7 -6 10 -6
... ..
```

This Graver basis has 29,417 elements and it took a couple of hours to compute. One nice feature of the Graver basis computation in `4ti2` is that the program allows the exploitation of symmetry. In many applications (e.g. in statistics) there is a group of symmetries acting on the columns of the matrix A , and the Graver basis

\mathcal{GR}_A is invariant under these symmetries. This feature allows the computation of some interesting Graver bases whose cardinalities are in the range of one million.

4. The integer programming gap

A commonly used first step towards solving a hard integer programming problem (1.1) is to begin by solving its *linear programming relaxation*:

$$(4.1) \quad \text{Minimize } c \cdot u \text{ subject to } A \cdot u = b \text{ and } u \in \mathbb{R}_{\geq 0}^n.$$

Linear programming problems are much easier both in practise and in theory. They can be solved in polynomial time using interior-point methods, and the simplex algorithm performs well in practise. The purpose of this section is to offer algebraic tools for comparing the hard problem (1.1) with the easier problem (4.1). For an algebraic perspective on the linear programming relaxation see [HT1].

As before, we fix $A \in \mathbb{Z}^{d \times n}$ and $c \in \mathbb{Z}^n$ and regard $b \in \mathbb{Z}^d$ as unspecified. We write $\text{IPopt}_{A,c}(b)$ for the optimal value of the integer program (1.1) and we write $\text{LPopt}_{A,c}(b)$ for the optimal value of the corresponding linear program (4.1). The difference of these quantities is a non-negative **rational** number

$$(4.2) \quad \text{IPopt}_{A,c}(b) - \text{LPopt}_{A,c}(b) \geq 0.$$

The *integer programming gap* is defined as the maximum of the differences (4.2) as b ranges over all right hand sides such that (1.1) is feasible:

$$(4.3) \quad \text{gap}(A, c) = \max \{ \text{IPopt}_{A,c}(b) - \text{LPopt}_{A,c}(b) : b \in \mathbb{Z}^d \text{ feasible for (1.1)} \}.$$

It appears as if we are taking the maximum over infinitely many different values, one for each feasible b , but actually there are only finitely many possible values for (4.2) if A and c are fixed, so the maximum is attained.

EXAMPLE 4.1. The integer programming gap of the coin problem (1.2) equals

$$\text{gap}(A, c) = 76/15 = 5.0666666\dots$$

This is the maximum advantage to be gained if we allow our coins to be cut into fractional pieces. The gap is attained for the right hand side $b = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$ in Example 1.1, where $\text{IPopt}_{A,c}(b) = 6$ is realized by $u = (4, 2, 0, 4)$. The optimal value of the linear program (4.1) is $14/15 = 0.93333\dots$ and is attained by $u' = (0, 0, 136/15, 14/15)$. Thus the best way to make one dollar and fourteen cents with ten fractional coins is to take $136/15$ dimes and $14/15$ quarters. \square

We now give a recipe for computing the gap by solving several auxiliary linear programming problems. For any optimal vector $u \in \mathbb{N}^n$ we define the *increase set*

$$\text{incr}(u) := \{ i \in \{1, 2, \dots, n\} : u + e_i \text{ optimal} \}.$$

A vector $u \in \mathbb{N}^n$ is said to be *maximally optimal* for (1.1) if $u + a$ is optimal for all vectors $a \in \mathbb{N}^n$ whose support $\{i : a_i > 0\}$ is a subset of the increase set $\text{incr}(u)$. For any fixed maximally optimal $u \in \mathbb{N}^n$, we consider the following linear program:

$$(4.4) \quad \text{Maximize } c \cdot (u - v) \text{ subject to } A \cdot (u - v) = 0 \text{ and } v_i \geq 0 \text{ for all } i \notin \text{incr}(u).$$

Here the decision variables are the coordinates of $v = (v_1, \dots, v_n)$.

THEOREM 4.2. (Hosten and Sturmfels [HS2]) *The maximum of the optimal values of the auxiliary linear programs (4.4), as u ranges over all maximally optimal solutions to (1.1), coincides with the integer programming gap, $\text{gap}(A, c)$.*

EXAMPLE 4.3. Our coin problem (1.2) has three maximally optimal solutions:

$$(4.5) \quad (4, 2, \underline{0}, \underline{0}), (\underline{0}, 2, \underline{0}, 2), (\underline{0}, 5, 3, 0).$$

In each case the increase set $\text{incr}(u)$ is indicated by the underlined coordinates. The vectors in (4.5) are easily derived from the Gröbner basis (1.3). For instance, the last portfolio (consisting of five nickels and three dimes) is maximally optimal because adding one nickel, dime or quarter makes that portfolio non-optimal but adding any number of pennies is fine. The program (4.4) for that portfolio equals

$$\begin{aligned} &\text{Maximize } 4 - v_2 - v_4 \text{ subject to } v_1 + v_2 + v_3 + v_4 = 4 \\ &v_1 + 5v_2 + 10v_3 + 25v_4 = 60, \quad v_2 \geq 0 \text{ and } v_4 \geq 0. \end{aligned}$$

The optimal value is 4. The optimal values of (4.4) for $(4, 2, \underline{0}, \underline{0})$ and $(\underline{0}, 2, \underline{0}, 2)$ are $76/15$ and 5 respectively, and hence $\text{gap}(A, c) = \max\{4, 5, 76/15\} = 76/15$. \square

Theorem 4.2 furnishes an algorithm for computing $\text{gap}(A, c)$ because the set of maximally optimal solutions to (1.1) is always finite and can be computed from the Gröbner basis $\mathcal{G}_{A,c}$ by the algebraic process of *irreducible decomposition of monomial ideals*. A highly efficient implementation of this process was developed by Alex Milowski in his Master's thesis project at San Francisco State University. The non-trivial gap computations in Examples 4.4 and 6.3 were done by Hosten and Milowski using `4ti2` (to derive the Gröbner basis), Milowski's software (to get the maximally optimal solutions) and `maple` (to solve the linear programs (4.4)).

EXAMPLE 4.4. Let $d = 3$, $n = 7$ and consider the instance discussed in (2.2):

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 \\ 43 & 41 & 37 & 31 & 29 & 23 & 19 \\ 47 & 53 & 59 & 61 & 67 & 71 & 73 \end{pmatrix}, \quad c = (1, 4, 9, 16, 25, 36, 49).$$

There are 553 maximally optimal solutions, and the gap is

$$(4.6) \quad \text{gap}(A, c) = 43771/183 = 239.1857923$$

The gap is attained by the right hand side $b = (661, 1710, 3994)^T$. For this choice of b , the optimal value of (1.1) equals 1,757 and is given by the optimal solution $u = (7, 4, 0, 22, 0, 3, 26)$, while the optimal value of (4.1) is a little less than 1,518 and is given by the optimal solution $u = (0, 0, 0, 0, 14029/244, 463/366, 521/732)$. \square

5. Short rational generating functions

The importance of rational generating functions for lattice point problems has been known to combinatorialists for a long time. Their role as an efficient tool in integer programming, however, has been recognized only quite recently, in response to the polynomial time algorithms of Barvinok [Bar] and Barvinok-Woods [BW]. This work was further extended by De Loera et.al. [DHHHSY], [DHHHY]. This section reports on these methods and their implementations in the software `LattE`.

As a point of entry consider the following variant of our problem: *List all optimal solutions to the integer program* (1.1). For a concrete example take $d = 1$, $n = 4$, $A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$, $c = (0, 0, 0, 1)$, and suppose $b \gg 0$. Here (1.1) equals

$$(5.1) \quad \text{Minimize } u_4 \text{ subject to } u_1 + u_2 + u_3 + u_4 = b \text{ and } u_1, u_2, u_3, u_4 \in \mathbb{N}.$$

The set of optimal solutions is the set of all lattice points $(u_1, u_2, u_3, 0)$ in a large triangle. We can write them all down as the terms of the generating function

$$(5.2) \quad \sum_{u \text{ optimal for (5.1)}} x_1^{u_1} x_2^{u_2} x_3^{u_3} x_4^{u_4} = \sum_{u_1=0}^b \sum_{u_2=0}^{b-u_1} x_1^{u_1} x_2^{u_2} x_3^{b-u_1-u_2}.$$

The number of terms in this series equals $(b+1)(b+2)/2 = O(b^2)$. This quantity is exponential in the size of the input, which is $O(\log(b))$. Indeed, the number of bits needed to write down the line (5.1) grows like the logarithm of the integer b , while the number of terms on the right hand side of (5.2) is exponential in $\log(b)$. It appears to be impossible to “list” all feasible solutions to (5.1) in polynomial time, given that their number grows exponentially in the input size. Nonetheless, it can be done, namely, by rewriting (5.2) as the *short rational generating function*

$$x_1^b \cdot (1 - \frac{x_2}{x_1})^{-1} (1 - \frac{x_3}{x_1})^{-1} + x_2^b \cdot (1 - \frac{x_1}{x_2})^{-1} (1 - \frac{x_3}{x_2})^{-1} + x_3^b \cdot (1 - \frac{x_1}{x_3})^{-1} (1 - \frac{x_2}{x_3})^{-1}.$$

The reader is invited to check that this rational function equals the series (5.2). The rational function can be computed in time $O(\log(b))$ and it represents the “list” of all optimal solutions to (5.1). This approach works for any integer program:

THEOREM 5.1. *Suppose that d and n are fixed. Then the number of optimal solutions to (1.1) and the rational generating function $\sum \{x^u : u \text{ optimal for (1.1)}\}$, which encodes the set of optimal solutions, can be computed in polynomial time.*

PROOF. The optimal value c^* of (1.1) can be computed in polynomial time using Lenstra’s algorithm [Len]. Now apply Barvinok’s lattice point algorithm [Bar] to the polytope $\{u \in \mathbb{R}_{\geq 0}^n : Au = b, c \cdot u = c^*\}$. It computes the desired generating function and its evaluation at $(1, 1, \dots, 1)$ in polynomial time. \square

The techniques underlying Barvinok’s algorithm were developed substantially further by Barvinok and Woods [BW]. Using their Projection Theorem, one can derive polynomial-time algorithms based on rational generating functions for essentially all of the algorithmic questions we have encountered so far. We refer to [BW] and [DHHHSY] for proofs of various parts of the following theorem.

THEOREM 5.2. *Consider a matrix $A \in \mathbb{Z}^{d \times n}$ and a vector $c \in \mathbb{Z}^d$ whose dimensions d and n are fixed. Then the rational generating functions which encode the following sets can be computed in time polynomial in the bit complexity of A and c :*

- (1) the Gröbner basis $\mathcal{G}_{A,c}$,
- (2) the set $\text{Opt}_{A,c}$ of all optimal solutions,
- (3) the set $\text{Min}(\mathbb{N}^n \setminus \text{Opt}_{A,c})$ of minimally non-optimal points,
- (4) the Hilbert basis \mathcal{H}_A ,
- (5) the Graver basis \mathcal{GR}_A ,
- (6) the set of maximally optimal solutions, and
- (7) the integer programming gap $\text{gap}(A, c)$.

The result (7) about the gap appears in [HS2]. The objects in (1)-(6) are highly structured subsets of \mathbb{Z}^n . It is this special structure which allows for a short encoding. For encoding the Gröbner basis $\mathcal{G}_{A,c}$, the paper [DHHHSY] uses a generating function in $2n$ variables as in (1.5). But all the sets in (1)-(6) can also be coded as formal sums of Laurent monomials (representing vectors in \mathbb{Z}^n), and the Barvinok-Woods method will give short rational functions for these encodings.

A magnificent computer program for solving lattice point problems by means of short rational generating functions has been developed by the group of Jesus De Loera at UC Davis. It is called **LattE** and can be obtained at the web site <http://www.math.ucdavis.edu/~latte/>. This program can be used to count the number of feasible solutions to an integer program (1.1) as follows.

Consider our coin problem in (1.2) with $b = \binom{999}{5000}$, so we wish to arrange 999 coins to be worth fifty dollars. In order to determine in how many ways this can be accomplished, we create the following **LattE** input file which we call **coins**:

```
6 5
999 -1 -1 -1 -1
5000 -1 -5 -10 -25
0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
2 1 2
```

The command `latte equ coins` will count the number of feasible solutions to (1.1). The output which appears on the screen reveals that the answer is 9,352:

```
This is LattE v1.0 beta.    (September 17, 2002)
Revised version.          (Aug      1, 2003)
The polytope has 4 vertices.
```

```
....
Creating generating function.
Starting final computation.
```

```
**** THE GRAND TOTAL IS: 9352 ****
```

```
Computation done.
Time: 0.01 sec
```

This run of **LattE** has created the following output on a new file called **coins.maple**:

```
gF:=x[0]^4999170*x[1]^(-8000506)*x[2]^1000*x[3]^166/((1-x[0]^(-4995)
*x[1]^7993*x[2]^(-1)*x[3]^(-1))*(1-x[0]^5*x[1]^(-9)*x[3]^(-1)))+
x[0]^4999170*x[1]^(-8000506)*x[2]^1000*x[3]^166/((1-x[0]^(-9985)
....
... *x[1]^7993*x[2]^(-1)*x[3]^(-1))*(1-x[0]^(-5)*x[1]^9*x[3]));
```

This is the short rational generating function representing the formal sum of 9,352 monomials, one for each feasible solution. You can get the expanded form of this generating function reading the file **coins.maple** into the computer algebra system **maple**. After you have done this, please type the **maple** command `simplify(gF);`.

The program **LattE** can also be used to solve the minimization problem (1.1). To this end we need to add the cost vector `0 1 0 1` in an extra line at the end of the input file **coins**. Typing now the **LattE** command sequence `./latte equ min coins2`, we obtain the following output on the screen

```
....
All cones have been decomposed.
6 cones in total.
Computing the points in the Parallelepiped of the unimodular Cones.
```

An optimal solution for [0 1 0 1] is: [555 2 441 1].

The optimal value is: 3.

The gap is: 3

Computation done.

We conclude from this **LattE** session that the best way of making fifty dollars with 999 coins is to take one **quarter**, two nickels, 441 dimes and 555 quarters.

I tried **LattE** on considerably bigger problems and I found that it performs quite well. The speed is particularly impressive for knapsack problems ($d = 1$) with large integer coefficients. For this class of problems, **LattE** is faster than the current version of the commercial software **CPLEX** on some instances. This parallels the observation, already made in [HS1], that programs like **CPLEX** are not always the best choice for low-dimensional problems with large integers, given that they are designed for highly structured 0/1 problems with many variables.

The authors of **LattE** informed me that they intend to incorporate all of the tasks listed in Theorem 5.2 into a future version of their program. The lesson to be learned here is that algebraic software like **4ti2** and **LattE** can definitely play a useful role in the box of tools available to practitioners of integer programming.

6. Some integer programs arising in statistics

We present an application to the statistical theory of *disclosure limitation*. See [CG] and [DF] and the references therein. Suppose we are given data in the form of an n -dimensional table of nonnegative integers. The aim is to release some marginals of the table but not the table's entries themselves. If the range of possible values that a particular entry can attain in any table satisfying the released marginals is too narrow then this entry may be exposed. This shows the importance of determining tight upper and lower bounds for each entry in a given table.

A choice of marginals corresponds to fixing subsets F_1, \dots, F_k of $\{1, \dots, n\}$. It can be represented by a zero-one matrix A , as described in [HSu, §1]. In statistical language, the matrix A specifies a *hierarchical model* for a *contingency table* with n *factors*. Suppose v is a table with nonnegative integer entries, where the marginals are computed according to a fixed hierarchical model A and let $v_{i_1 i_2 \dots i_n}$ be a particular cell of the table v . What we are interested in is the following **table entry security problem**: *Compute optimal lower and upper bounds L and U such that $L \leq u_{i_1 i_2 \dots i_n} \leq U$ for all tables u which have the same marginals as v .*

The table entry security problem is an integer program: minimize (or maximize) $u_{i_1 i_2 \dots i_n}$ over all tables with nonnegative *integer* entries subject to fixing the marginals. In order to write this integer program in the standard form (1.1), we need to give the precise definition of the relevant matrices A . Consider $d_1 \times \dots \times d_n$ -tables with entries $u_{i_1 i_2 \dots i_n}$ where $1 \leq i_j \leq d_j$. We fix a hierarchical model by specifying F_1, \dots, F_k . The marginals of our table are computed with respect to these subsets. If $F_i = \{j_1, \dots, j_s\}$ then the F_i -marginal is a $d_{j_1} \times \dots \times d_{j_s}$ table b with entries

$$(6.1) \quad b_{k_1 \dots k_s} = \sum_{i_{j_1}=k_1, \dots, i_{j_s}=k_s} u_{i_1 \dots i_n}.$$

We define A to be the zero-one matrix with $d_1 d_2 \dots d_n$ columns representing the linear map that computes the marginals of tables. We let u be the vector of variables representing the cell entries. Then $A \cdot u$ represents the k lower-dimensional

tables computed as in (6.1). The **table entry security problem** is

$$(6.2) \quad \text{Minimize (Maximize) } u_{11\dots 1} \text{ subject to } A \cdot u = b, u \geq 0, u \text{ integral.}$$

Here we only consider the cell entry $u_{11\dots 1}$ (corresponding to the first column of A) because there is a transitive symmetry group acting on the columns of A .

EXAMPLE 6.1. The classical *transportation problem* [Sch, p. 221] corresponds to $d_1 \times d_2$ -tables where the marginals are computed with respect to $F_1 = \{1\}$ and $F_2 = \{2\}$. The *three-dimensional transportation problem* [Vla] concerns $d_1 \times d_2 \times d_3$ -tables with $F_1 = \{1, 2\}$, $F_2 = \{1, 3\}$, and $F_3 = \{2, 3\}$. The marginals are

$$b_{ij} = \sum_k u_{ijk}, \quad b_{ik} = \sum_j u_{ijk}, \quad b_{jk} = \sum_i u_{ijk}.$$

For a discussion from the Gröbner basis perspective see [Stu, §14.C]. □

EXAMPLE 6.2. Consider the *four-cycle model* for binary random variables. Here $n = 4$, $d_1 = d_2 = d_3 = d_4 = 2$, $F_1 = \{1, 2\}$, $F_2 = \{2, 3\}$, $F_3 = \{3, 4\}$, and $F_4 = \{1, 4\}$. The matrix A has $d_1 d_2 + d_2 d_3 + d_3 d_4 + d_1 d_4 = 16$ rows and it has $d_1 d_2 d_3 d_4 = 16$ columns. We write it in 4ti2 format on a file name `fourcycle`:

```
16
16
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0
0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0
0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0
0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0
0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
```

The Graver basis of this matrix A consists of 106 vectors. The 4ti2 command `graver fourcycle` delivers the Graver basis on a new file `fourcycle.gra`:

```
16 106
0 0 0 0 0 1 0 -1 1 -1 0 0 -1 0 0 1
-1 0 1 0 1 0 -1 0 1 0 0 -1 -1 0 0 1
0 0 0 0 0 1 0 -1 0 0 0 0 0 -1 0 1
0 0 0 0 1 0 -1 0 0 0 0 0 -1 0 1 0
1 0 -1 0 0 0 0 0 -1 0 1 0 0 0 0 0
0 0 -1 1 0 0 1 -1 0 0 0 0 0 0 0 0
-1 1 0 0 1 -1 0 0 0 0 0 0 0 0 0 0
0 -1 1 0 0 1 -1 0 1 0 0 -1 -1 0 0 1
... ..
```

In light of Proposition 3.2, we can use this Graver basis with the Algorithm (2.1) to solve (1.1) for any cost function. In particular, we can use it solve (6.2). \square

As the parameters d_1, \dots, d_n increase, it becomes harder to solve the integer program (6.2) exactly. Researchers in disclosure limitation have resorted to solving the linear programming relaxation (4.1) instead: minimize (or maximize) $u_{i_1 i_2 \dots i_n}$ over all tables with nonnegative *real* entries subject to fixing the marginals. This relaxation is tractable, but it usually fails to deliver the exact integers L and U . One faces the problem of finding the integer programming gap for the table entry security problem. This application was the original motivation for the paper [HS2].

EXAMPLE 6.3. What follows may serve as a test case for future software for computing the gap. We consider the K_5 -model for five binary random variables. Here $n = 5$, $k = 10$, $d_1 = \dots = d_5 = 2$ and the F_i are the ten two-element subsets of $\{1, 2, 3, 4, 5\}$. The cost function is $c = -e_{11111}$, corresponding to maximizing in (6.2). The matrix A has 40 rows and 32 columns, and it has rank 16. We found

$$(6.3) \quad \text{gap}(A, c) = 3.$$

The gap is attained by the following $2 \times 2 \times 2 \times 2 \times 2$ -table:

$$(6.4) \quad u = e_{11112} + e_{11121} + 2 \cdot e_{11211} + 2 \cdot e_{12111} + 2 \cdot e_{21111} + e_{22222}.$$

This table is optimal for the maximization problem in (6.2). The 40 entries of the right hand side vector b are the entries in the ten marginal 2×2 -tables:

$$\begin{pmatrix} u_{11***} & u_{12***} \\ u_{21***} & u_{22***} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \dots, \begin{pmatrix} u_{***11} & u_{***12} \\ u_{***21} & u_{***22} \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}$$

Since the unit table e_{11111} does not appear in (6.4), we have $\text{IPopt}_{A,c}(b) = 0$.

The optimal value of the linear programming relaxation equals $\text{LPopt}_{A,c}(b) = 3$. This value is attained by the following fractional $2 \times 2 \times 2 \times 2 \times 2$ -table

$$v = 3 \cdot e_{00000} + \frac{2}{3}e_{11211} + \frac{1}{3}e_{11221} + \frac{2}{3}e_{12111} + \frac{1}{3}e_{12122} + \frac{1}{3}e_{12211} + \frac{1}{3}e_{12221} + \frac{1}{3}e_{12222} + \frac{2}{3}e_{21112} + \frac{1}{3}e_{21121} + \frac{2}{3}e_{21211} + \frac{1}{3}e_{21222} + \frac{2}{3}e_{22111} + \frac{1}{3}e_{22121}.$$

We invite the reader to check that the tables u and v have the same marginals. \square

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