Graver Basis

Francisco Javier Blázquez Martínez

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Integer Linear Programming

The underlying question is how to solve the integer linear problem (IP).

$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

Where $A \in \mathbb{Z}^{mxn}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, l and u lower and upper bounds for x.

IP is NP-Hard. There are algorithms with polynomial complexity for certain IPs and there are algorithms for the general IP based on cutting plane methods, dinamic programming, lattice-basis reduction... Thanks to the study of Graver basis new algorithms have appeared improving classic techniques in certain cases.

Graver Basis

- **Definition 1:** Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, ..., n\}$
- **Definition 2:** A vector $u \in ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $ker(A)$.

Graver Basis

The Graver Basis of a given matrix A is defined as the set of integral indecomposable elements in the kernel of A.

Graver Basis properties

- **Property 1:** Every integral element in ker(A) can be expressed as positive integral linear combination of elements in GR(A).
- **Property 2:** Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in G(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$

The **Property 2** we have seen is an optimality test which has the advantage that only needs to consider the Graver basis elements. An important

Algorithm

1. Given a feasible solution z_0
2. Solve the subproblem:
   $max\{c^t g : Ag = 0, l - z_0 \leq g \leq u - z_0, g \in \mathbb{Z}^n, ||g||_1 \leq ||G(A)||\}$
   - $g^* = 0 \implies z_0$ optimal solution.
   - $g^* \neq 0 \implies$ g* improvement direction, loop back to 1 with $z_1 = z_0 + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

One example of success, N-Fold

A N-Fold IP has constriction matrix A of the form $(A_i \in \mathbb{Z}^{rxt}, B_i \in \mathbb{Z}^{sxt})$:

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

N-Fold Graver basis bound

For all $g \in G(A)$ $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$

References

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