N-fold integer programming via LP rounding

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September 28, 2020

Abstract

We consider *integer programming* problems $\max\{c^T x : \mathscr{A} x = b, l \leqslant x \leqslant u, x \in \mathbb{Z}^{nt}\}$ where $\mathscr{A}$ has the following block-structure $\mathscr{A} = \begin{pmatrix} A_1 & \dots & A_n \\ B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}$, $A_i \in \mathbb{Z}^{r \times t}$ and $B_i \in \mathbb{Z}^{s \times t}$ for $1 \leqslant i \leqslant n$. These are *N-fold* integer programming problems. This class has received considerable attention in the recent literature. N-fold integer programming is *fixed parameter tracktable* with parameters r, s and Δ, where Δ is an upper bound on the largest absolute value of a coefficient of the constraint matrix $\mathscr{A}$. After a decade of continuous progress, the currently fastest algorithm for N-fold integer programming by Jansen et al. (2019) has a running time of $(rs\Delta)^{O(r^2s+s^2)}\phi^2 \cdot nt \log^{O(1)}(nt)$. Here ϕ is the largest binary encoding length of a number in the input. This algorithm, like its predecessors are based on the *augmentation framework*, a tailored integer programming variant of local search.

In this paper we propose a different approach that is not based on augmentation. Our algorithm relies on a stronger LP-relaxation of the N-fold integer program instead. This relaxation can be solved in polynomial time with parameter dependence $(s\Delta)^{O(s^2)}$ by resorting to standard techniques from convex optimization. We show that, for any given optimal vertex solution x^* of this relaxation, there exists an optimal integer solution z^* that is within short ℓ_1-distance, namely $\|x^* - z^*\|_1 \leqslant (rs\Delta)^{O(rs)}$. With dynamic programming one can then find an optimal integer solution of the N-fold IP in time $(rs\Delta)^{O(r^2s+s^2)}\,nt$. This, together with an off-the-shelf-method from convex optimization, results in the currently fastest algorithm for N-fold integer programming.

1 Introduction

An *integer program (IP)* is an optimization problem of the form

$$\max\{c^T x : Ax = b,\, l \leqslant x \leqslant u,\, x \in \mathbb{Z}^n\},$$

described by a *constraint matrix* $A \in \mathbb{Z}^{m\times n}$, an *objective function* vector $c \in \mathbb{Z}^n$ a *right-hand-side* vector $b \in \mathbb{Z}^m$ and *lower* and *upper bounds* on the variables $l \leqslant x \leqslant u$. Many algorithmic problems, most notably problems from *combinatorial optimization* and the *geometry of numbers* can be formulated as an *integer linear program*, see, e.g. [**?**, **?**]. Although integer programming is NP-complete [**?**], commercial solvers are nowadays capable of solving large instances efficiently in practice. An important technique that is used in integer programming solvers, stems from leveraging the sparsity of the constraint matrix by decomposing the matrix into block structure, see [**?**, **?**]. These techniques are very important in practice and an essential component in competitive solvers.

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De Loera et al. [?] proposed the following model of a sparse integer program in block structure. An *N-fold integer program* is of the form

$$\begin{aligned} \max c^T x \\ \mathscr{A} x = b \\ 0 \leqslant x \leqslant u \\ x \in \mathbb{Z}^{nt} \end{aligned}, \qquad \text{with} \quad \mathscr{A} = \begin{pmatrix} A_1 & A_2 & \dots & A_n \\ B_1 & 0 & \dots & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & B_n \end{pmatrix} \tag{1}$$

where, for $i \in \{1,\dots,n\}$ $A_i \in \mathbb{Z}^{r\times t}$, $B_i \in \mathbb{Z}^{s\times t}$, $\|A_i\|_\infty, \|B_i\|_\infty \leqslant \Delta$, and $r,s,t,n,\Delta \in \mathbb{Z}_+$. This model provides a measure of the extent to which the integer program is in block structure. In particular, if the first r constraints are deleted, then the integer program decomposes into n independent smaller problems.

The N-fold setting has gained strong momentum in the last years, especially in the fields of *parameterized complexity* and *approximation algorithms*. An algorithm is *fixed parameter tracktable (fpt)* with respect to a parameter k derived from the input, if its running time is of the form $f(k)\cdot n^{O(1)}$ for some computable function f. Hemmecke et al. [?] have shown that N-fold integer programming is fixed parameter tracktable with respect to Δ, s, r and t. Their running time is $O(n^3\phi)\Delta^{O(t(rs+st))}$ where ϕ is the logarithm of the largest absolute value of a component of the input. This has been used, for example in [?, ?, ?] to derive novel fpt-results in scheduling. N-fold IPs have also been successfully applied to derive fpt-results for string and social choice problems [?, ?]. The exponential dependence on t was removed by Eisenbrand et at. [?] and Koutecký et at. [?]. The currently fastest algorithm for N-fold integer programming was provided by Jansen, Lassota and Rohwedder [?]. Ignoring logarithmic factors, the running time of their algorithm is $(rs\Delta)^{O(r^2s+s^2)}\phi^2\cdot nt$, see also [?].

Contributions of our paper

N-fold integer programming has been a success story for augmentation algorithms. Those are primal algorithms that start at a feasible integer point and incrementally move to integer points with better objective function value. All previous fpt-algorithms including the fastest algorithm for N-fold integer programming prior to this paper [?] are based on such augmentation schemes. Our main contributions are the following.

i) We propose a linear relaxation of the N-fold integer programming for which any optimal vertex solution is very close to some optimal integral solution. More precisely, we show that for each optimal vertex x^* of this relaxation, there exists an optimal integer solution z^* with

$$\|x^* - z^*\|_1 \leqslant (rs\Delta)^{O(rs)}.$$

This relaxation can be solved in time that is polynomial in the input-length and $(s\Delta)^{O(s^2)}$ by resorting to the ellipsoid method in a straightforward way.

ii) We describe a dynamic program that, given an optimal vertex solution of this relaxation, computes an optimal integer solution of the N-fold IP in time

$$(rs\Delta)^{O(r^2s+s^2)}nt.$$

iii) We show that the standard LP-relaxation of the N-fold IP does not feature such a proximity. There exists a family of N-fold integer programming problems such that the ℓ_∞-distance of an optimal vertex solution x^* of the standard LP-relaxation to each integer optimal solution z^* is bounded from below by

$$\|x^* - z^*\|_\infty = \Omega(n).$$

iv) Finally we sketch an analysis of the ellipsoid method if applied to the here discussed relaxation. An optimal vertex solution can be found in time (ignoring logarithmic factors)

$$(s\Delta)^{O(s^2)}\phi^2\, n\, t \cdot p(r)$$

for some polynomial $p(\cdot)$.

All together we obtain the currently fastest algorithm for N-fold integer programming problems.

Related work

Integer programming can be solved in polynomial time, if the dimension is fixed [**?**, **?**]. Closely related to the results that are presented here are dynamic programming approaches to integer programming [**?**]. In [**?**] it was shown that an integer program $\max\{c^T x : Ax = b,\, l \leqslant x \leqslant u,\, x \in \mathbb{Z}^n\}$ with $A \in \mathbb{Z}^{s\times n}$ can be solved in time $(s\Delta)^{O(s^2)} n$ and in time $(s\Delta)^{O(s)}$ if there are no upper bounds on the variables. Jansen and Rohwedder [**?**] obtained better constants in the exponent of the running time of integer programs without upper bounds. A tight lower bound was presented by Knop et al. [**?**].

2 Relaxations of N-fold integer programs

We re-write the N-fold integer program in a form that will be more convenient for us. Following standard notation, see, e.g. [**?**], we partition the variables $x \in \mathbb{R}^{n\cdot t}$ into groups of t components respectively, i.e., we will write $x = (x^{(1)},\dots,x^{(n)})$ where each $x^{(i)} \in \mathbb{R}^t$ is called the *i-th block* of the variables. Similarly, we partition the right hand side into blocks $b = (b_0, b_1, \dots, b_n)$ where $b_0 \in \mathbb{Z}^r$ and $b_i \in \mathbb{Z}^s$ for $i = 1,\dots,n$. We furthermore define the polyhedra $P_i = \{x^{(i)} \in \mathbb{R}^t \colon B_i x^{(i)} = b_i,\, 0 \leqslant x^{(i)} \leqslant u_i\} \subseteq \mathbb{R}^t$. With this notation, the N-fold IP can be described as

$$\begin{aligned}
\max \sum_{i=1}^n c_i^T x^{(i)} & \\
\sum_{i=1}^n A_i x^{(i)} = b_0 & \qquad (2)\\
x^{(i)} \in P_i \quad i = 1,\dots,n & \qquad (3)\\
x \in \mathbb{Z}^{n\cdot t}. &
\end{aligned}$$

The constraints (2) are the *linking constraints* whereas the constraints (3) are the *block constraints*. Without the linking constraints, the N-fold IP consists of n individual small-dimensional integer programming problems $\max\{c_i^T x^{(i)} \colon x^{(i)} \in P_i \cap \mathbb{Z}^t\}$ which can be solved independently from each other. The standard relaxation of the N-fold IP is the linear programming problem, that is obtained by relaxing the integrality constraint $x \in \mathbb{Z}^{n\cdot t}$ to $x \in \mathbb{R}^{n\cdot t}$.

The quality of the standard relaxation

Our goal is to describe a relaxation, that can be solved in polynomial time, and that has LP/IP proximity of $(rs\Delta)^{O(rs)}$. We first provide an example that demonstrates that the standard relaxation does not feature such a proximity bound. In this example the number n of blocks is odd, the number of variables per block is two. We denote them by $x_1^{(i)}$ and $x_2^{(i)}$, $1 \leqslant i \leqslant n$ and they are constrained to be non-negative. We maximize

$$\sum_{i=1}^n (2+\varepsilon)x_1^{(i)} + (3-\varepsilon)x_2^{(i)}$$

under the block constraints

$$2x_1^{(i)} + 3x_2^{(i)} = 3, i = 1, \dots, n-1$$

and

$$2x_1^{(n)} + 3x_2^{(n)} = 6n.$$

Notice that, for these constraints, the unique optimal LP-solution is $x_1^{*(i)} = 3/2$ and $x_2^{*(i)} = 0$ for $i = 1, \dots, n-1$ and $x_1^{*(n)} = 3n$ and $x_2^{*(n)} = 0$ respectively. We add the linking constraint

$$\sum_{i=1}^{n-1} (x_1^{(i)} + x_2^{(i)}) - (x_1^{(n)} + x_2^{(n)}) = 3/2(n-1) - 3n$$

which is satisfied by the optimal LP solution x^*. It may be checked that the optimal IP-solution is defined by setting $z_1^{*(i)} = 0$, $z_2^{*(i)} = 1$, $i = 1, \dots, n-1$, $z_1^{*(n)} = 3n - 3(n-1)/2$ and $z_2^{*(n)} = n-1$. We obtain that $z_2^{*(n)} - x_2^{*(n)} = n-1$ and we have the following lower bound on the ℓ_∞-distance.

Proposition 1. *There exists a family of N-fold integer programming problems such that the ℓ_∞-distance of an optimal solution x^* of the standard LP-relaxation to each integer optimal solution z^* is bounded from below by*

$$\|x^* - z^*\|_\infty = \Omega(n).$$

Replacing the block polyhedra by their integer hull

In the above example, if we replaced the block constraints by the constraints describing their respective integer hull, i.e., the convex hull of integer solutions, then since these consist of one point, namely $(0,1)$ only, the optimal fractional solution would be integral already. We do exactly this. Define $Q_i = \mathrm{conv}(P_i \cap \mathbb{Z}^t)$ for $i = 1, \dots, n$ and consider

$$\begin{aligned} &\max \sum_{i=1}^{n} c_i^T x^{(i)} \\ &\textstyle\sum_{i=1}^{n} A_i x^{(i)} = b_0 \\ &\qquad x^{(i)} \in Q_i \quad i = 1, \dots, n \\ &\qquad x \in \mathbb{Z}^{n \cdot t}. \end{aligned} \tag{4}$$

The only difference to the standard formulation is the replacement of the block constraints in (3) by the integer hull of the P_i. This stronger relaxation can be solved by a fixed parameter tracktable algorithm. This is an immediate consequence of the polynomial-time equivalence of separation and optimization, see [**?**]. We recall that the *separation problem* for a given $y^* \in \mathbb{R}^k$ and implicitly given convex set $K \subseteq \mathbb{R}^k$ requires to determine whether y^* is an element of K and if not, to find a hyperplane which separates y^* and K. Whether x^* satisfies the linking constraints can be checked easily. After that we need to check for each block $x^{*(i)}$, whether it is in the integer hull Q_i of P_i. Optimizing over $Q_i \in \mathbb{R}^t$ can be done in time polynomial in t and $(s\Delta)^{O(s^2)}$, see [**?**]. Again, by the equivalence of separation and optimization, this separation problem can be done efficiently as well. A separating hyperplane for $x^{*(i)}$ and Q_i is also a separating hyperplane for x^*.

Proposition 2. *Given an N-fold integer program by its constraint matrix $\mathscr{A} \in \mathbb{Z}^{r+ns \times nt}$, objective function vector $c \in \mathbb{Z}^{nt}$ and upper bounds $u \in \mathbb{Z}^{nt}$, one can find an optimal vertex solution of its strengthened relaxation of* (4) *in time polynomial in $(s\Delta)^{O(s^2)}$, $n \cdot t$ and ϕ, where ϕ is the largest encoding length of a number in the input.*

We postpone a more precise analysis of the ellipsoid method to Section 5 and instead begin to investigate the quality of the relaxation of (4).

Lemma 3. *Let $(x^{*(1)}, \dots, x^{*(n)})$ be a vertex solution of the relaxation of* (4)*. All but r of the $x^{*(i)}$ are vertices of the Q_i respectively and thus all but r of the $x^{*(i)}$ are integral.*

Proof. Suppose by contradiction that $x^{*(1)}, \dots, x^{*(r+1)}$ are not vertices of the respective Q_i. Then there exists nonzero vectors $d_1, \dots, d_{r+1} \in \mathbb{R}^t$ such that $x^{*(i)} \pm d_i \in Q_i$ for all $i = 1, \dots, r+1$. Consider the $r+1$ vectors of the form $A_i d_i \in \mathbb{R}^r$. They have to be linearly dependent and thus, there exist $\lambda_1, \dots, \lambda_{r+1} \in \mathbb{R}$ not all zero such that

$$\sum_{i=1}^{r+1} \lambda_i A_i d_i = 0$$

By re scaling the λ_i we can suppose that $x^{*(i)} \pm \lambda_i d_i \in P_i$. Consider

$$d = (\lambda_1 d_1, \dots, \lambda_{r+1} d_{r+1}, 0, \dots, 0) \in \mathbb{R}^n$$

Then $x^* + d, x^* - d \in P$ and $x^* = \frac{1}{2}(x^* + d) + \frac{1}{2}(x^* - d)$. So x^* is a convex combination of two feasible points and thus not a vertex. □

3 Proximity

We now come to our main result and show that an optimal vertex solution of the linear relaxation of (4) is close to an optimal integer solution of the N-fold integer program. More precisely, we prove the following result.

Theorem 4. *Let $x^* \in \mathbb{R}^{nt}$ be an optimal vertex solution of the relaxation of* (4)*. There exists an optimal integer solution $z^* \in \mathbb{Z}^{nt}$ of* (1) *with*

$$\|x^* - z^*\|_1 \leqslant (rs\Delta)^{O(rs)}.$$

Theorem 4 shows that the proximity bound for $\|x^* - z^*\|_1$ does neither depend on the number of blocks n nor on the number t that is the dimension of the block polyhedra P_i.

Next we present an **overview of the proof**, see Figure 1. Throughout this section, we assume that x^* is an optimal vertex solution of the linear relaxation of (4) that we partition into its blocks $x^{*(i)} \in \mathbb{R}^t$ for $i = 1, \dots, n$.

- We let $y^{*(i)}$ be a nearest integer point to $x^{*(i)}$ with respect to the ℓ_1-norm that lies on the minimal face of Q_i containing $x^{*(i)}$. In Theorem 5 we show the bound $\|x^{*(i)} - y^{*(i)}\|_1 \leqslant r^2(s\Delta)^{O(s)}$.
- We let z^* be an optimal integer solution such that $\|y^* - z^*\|_1$ is minimal. In Theorem 9 we show $\|y^* - z^*\|_1 \leqslant (rs\Delta)^{O(rs)}$.
- Since at most r of the $x^{*(i)}$ are non-integral and in particular not equal to $y^{*(i)}$ (Lemma 3), the theorem follows by applying the above bounds

$$\begin{aligned} \|x^* - z^*\|_1 &\leqslant \|x^* - y^*\|_1 + \|y^* - z^*\|_1 \\ &\leqslant r \cdot r^2(s\Delta)^{O(s)} + (rs\Delta)^{O(rs)} \\ &\leqslant (rs\Delta)^{O(rs)}. \end{aligned}$$

Figure 1: The proof of Theorem 4.

We need to review some basic terminology at this point. Let $A \in \mathbb{Z}^{m\times n}$ be an integer matrix. An integer vector $z \in \ker(A)$ is called a *cycle* of A. Two vectors $u, v \in \mathbb{R}^n$ are said to be *sign compatible* if $u_i \cdot v_i \geqslant 0$ for each i. A cycle $u \in \ker(A)$ is *indecomposable* if it is not the sum of two sign-compatible and non-zero cycles of A. The set of indecomposable and integral elements from the kernel of A is called the *Graver basis* of A [**?**], see also [**?**, **?**]. The ℓ_1-norm of an element of the Graver basis of a given matrix $A \in \mathbb{Z}^{m\times n}$ is bounded by $(2\,m\cdot\Delta+1)^m$, where Δ is an upper bound on the absolute value of each entry of A, see [**?**].
Recall that Q_i is the integer hull of the polyhedron $P_i = \{x^{(i)} \in \mathbb{R}^t : B_i x = b_i, 0 \leqslant x^{(i)} \leqslant u^{(i)}\}$.

Proposition 5. *Let F_i be the minimal face of Q_i containing $x^{*(i)}$. Then there exists an integer point $y^{*(i)} \in F_i \cap \mathbb{Z}^t$ with*

$$\|x^{*(i)} - y^{*(i)}\|_1 \leqslant r^2(2s\Delta+1)^s.$$

The proof of this proposition uses standard arguments of polyhedral theory, see, e.g. [**?**].

Proof. Let $w \in F_i$ be an arbitrary vertex of F_i and let C be the cone

$$C = \{\lambda(f-w) : f \in F_i, \lambda \geqslant 0\}.$$

The extreme rays of this cone are elements of the Graver basis $\mathcal{G}_i$ of B_i, see, e.g. [**?**, Lemma 3.15]. Each element of $\mathcal{G}_i$ has ℓ_1 norm bounded by $(2s\Delta+1)^s$, see [**?**]. The affine dimension of F_i is bounded by r. By Carathéodory's Theorem [**?**, p. 94] there exist r elements $g_1, \dots, g_r \in \mathcal{G}_i$ such that

$$x^{*(i)} = w + \sum_{j=1}^{r} \lambda_j g_j.$$

We consider the integer point $\zeta = w + \sum_{j=1}^{r} \lfloor \lambda_j \rfloor g_j \in \mathbb{Z}^t$. There are two cases. If $\zeta \in F_i$, then, by the triangle inequality, the distance in ℓ_1-norm of $x^{*(i)}$ to the nearest integer point in F_i is bounded by $r(2s\Delta+1)^s$. Otherwise, the line segment between $x^{*(i)}$ and ζ exits F_i in a lower dimensional face of Q_i contained in F_i. Call $\tilde{x}^{(i)}$ the intersection point. We apply an inductive argument now: there is an integer point on this lower dimensional face that has ℓ_1-distance at most $(r-1)^2(2s\Delta+1)^s$ from $\tilde{x}^{(i)}$. The bound $r^2(2s\Delta+1)^s$ follows by applying the triangle inequality. □

In the following, we keep the notation of Proposition 5 and denote $(y^{*(1)}, \dots, y^{*(n)}) \in \mathbb{Z}^{nt}$ by y^*. If $z^* \in \mathbb{Z}^{nt}$ is a feasible integer solution of the N-fold IP (1), then each $z^{*(i)}$ is in Q_i, in particular

$$B_i(y^{*(i)} - z^{*(i)}) = 0. \tag{5}$$

Therefore, there exists a multiset $\mathcal{L}_i$ of Graver basis elements of the matrix B_i that are sign-compatible with $y^{*(i)} - z^{*(i)}$ such that $y^{*(i)} - z^{*(i)}$ decomposes into the sum of the elements of $\mathcal{L}_i$

$$y^{*(i)} - z^{*(i)} = \sum_{g\in\mathcal{L}_i} g.$$

Lemma 6. *For each $i = 1, \dots, n$ and each $\mathcal{H}_i \subseteq \mathcal{L}_i$ one has*

i)

$$\begin{aligned} z^{*(i)} + \sum_{h\in\mathcal{H}_i} h &\in Q_i \\ y^{*(i)} - \sum_{h\in\mathcal{H}_i} h &\in Q_i. \end{aligned}$$

ii) There exists an $\varepsilon_{\mathcal{H}_i} > 0$ *such that*

$$x^{*(i)} - \varepsilon_{\mathcal{H}_i} \sum_{h \in \mathcal{H}_i} h \quad \in \quad Q_i.$$

Proof. The assertion i) follows standard arguments, see, e.g. [**?**]. One has

$$\begin{aligned} B_i \left(z^{*(i)} + \sum_{h \in \mathcal{H}_i} h \right) &= b_i, \text{ and} \\ B_i \left(y^{*(i)} - \sum_{h \in \mathcal{H}_i} h \right) &= b_i. \end{aligned}$$

and both $z^{*(i)} + \sum_{h \in \mathcal{H}_i} h$ and $y^{*(i)} - \sum_{h \in \mathcal{H}_i} h$ are integer. Since the Graver basis elements of $\mathcal{H}_i$ are sign-compatible with $y^{*(i)} - z^{*(i)}$ the bounds $0 \leqslant x^{(i)} \leqslant u_i$ are satisfied by both points as well. Thus both points are feasible integer points of the system $B_i x = b_i$, $0 \leqslant x^{(i)} \leqslant u_i$ which implies that they lie in Q_i.

The proof of the assertion ii) requires some polyhedral theory, see [**?**]. Let the polyhedron Q_i be described by the inequalities

$$Q_i = \{x \in \mathbb{R}^t : D^{(i)} x \leqslant p^{(i)}\}$$

for some integer matrix $D^{(i)} \in \mathbb{Z}^{m_i \times t}$ and integer vector $p^{(i)} \in \mathbb{Z}^{m_i}$. What is the inequality description of the minimal face $F_i \subseteq Q_i$ containing $x^{*(i)}$? Let $I_i \subseteq \{1, \ldots, m_i\}$ be the index set corresponding to the inequalities of $D^{(i)} x \leqslant p^{(i)}$ that are satisfied by $x^{*(i)}$ with equality. The inequality description of F_i is obtained from $D^{(i)} x \leqslant p^{(i)}$ by setting the inequalities indexed by I_i to equality.

The important fact now is the following: Since $y^{*(i)} \in F_i$, all the inequalities indexed by I_i and possibly more are also tight at $y^{*(i)}$. However, subtracting $\sum_{h \in \mathcal{H}_i} h$ from $y^{*(i)}$, one obtains a point of Q_i. Therefore, we can move, starting at $x^{*(i)}$, in the direction of $-\sum_{h \in \mathcal{H}_i} h$ some positive amount, without leaving Q_i. This means that assertion ii) holds. □

Lemma 7. *Suppose that* z^* *is an IP optimal solution of the N-fold integer program* (1) *that is closest to* y^* *w.r.t. the* ℓ_1*-distance and suppose each* $y^{*(i)} - z^{*(i)}$ *decomposes into*

$$y^{*(i)} - z^{*(i)} = \sum_{h \in \mathcal{L}_i} h, \quad i = 1, \ldots, n.$$

where the $\mathcal{L}_i$ *are multisets of Graver basis elements of* B_i*, sign-compatible with* $y^{*(i)} - z^{*(i)}$*. For each selection of sub-multisets*

$$\mathcal{H}_1 \subseteq \mathcal{L}_1, \mathcal{H}_2 \subseteq \mathcal{L}_2, \cdots, \mathcal{H}_n \subseteq \mathcal{L}_n$$

one has

$$\sum_{i=1}^{n} \sum_{h \in \mathcal{H}_i} A_i h \neq 0.$$

Proof. For the sake of contradiction, let $\mathcal{H}_1 \subseteq \mathcal{L}_1, \cdots, \mathcal{H}_n \subseteq \mathcal{L}_n$ be a selection of sub-multisets such that

$$\sum_{i=1}^{n} \sum_{h \in \mathcal{H}_i} A_i h = 0 \tag{6}$$

holds. By Lemma 6 i) one has $z^{*(i)} + \sum_{h \in \mathcal{H}_i} h \in Q_i$ for each i. Since, in addition (6) holds, this means that

$$z^* + \left(\sum_{h \in \mathcal{H}_1} h, \ldots, \sum_{h \in \mathcal{H}_n} h \right) \tag{7}$$

is an integer feasible solution of the N-fold IP (1). Similarly, Lemma 6 ii) implies that there exists an $\varepsilon > 0$ such that

$$x^* - \varepsilon\left(\sum_{h\in\mathcal{H}_1} h, \ldots, \sum_{h\in\mathcal{H}_n} h\right)$$

is a feasible solution of the strengthened relaxation of (4). Since z^* and x^* were optimal solutions of the IP (1) and the relaxation of (4) respectively, this implies that

$$\sum_{i=1}^{n} c_i^T \sum_{h\in\mathcal{H}_i} h = 0,$$

and thus that the objective values of z^* and (7) are the same. This however is a contradiction to the minimality of the distance of z^* to y^* in the ℓ_1-norm. As the optimal integer solution (7) is closer to y^*. □

As in the proximity result presented in [**?**], we make use of the so-called Steinitz lemma.

Theorem 8 (Steinitz (1913))**.** *Let $x_1, \ldots, x_n \in \mathbb{R}^r$ and such that*

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \|x_i\| \leqslant 1 \text{ for each } i.$$

There exists a permutation $\pi \in S_n$ such that all partial sums satisfy

$$\|\sum_{j=1}^{k} x_{\pi(j)}\| \leqslant c(r) \text{ for all } k = 1, \ldots, n.$$

Here $c(r)$ is a constant depending on r only.

We now bound $\|y^* - z^*\|_1$ by using Theorem 8 with $c(r) = r$, see [**?**].

Theorem 9. *Suppose that z^* is an IP optimal solution of the N-fold integer program* (1) *that is closest to y^* in ℓ_1-norm, then*

$$\|y^* - z^*\|_1 \leqslant (rs\Delta)^{O(rs)}.$$

Proof. We use the notation of the statement of Lemma 7. Denote the matrix $(A_1|A_2|\ldots|A_n) \in \mathbb{Z}^{r\times nt}$ by A. We have

$$\begin{aligned} 0 \quad = \quad A(x^* - z^*) \quad &= \quad A(x^* - y^*) + A(y^* - z^*) \\ &= \quad A(x^* - y^*) + \sum_{i=1}^{n}\sum_{h\in\mathcal{L}_i} A_i h. \end{aligned}$$

Since the ℓ_1-norm of each Graver basis element $h \in \mathcal{L}_i$ is bounded by $(2s\Delta+1)^s$, the ℓ_∞ norm of each $A_i h$ is bounded by $\Delta(2s\Delta+1)^s$. The Steinitz Lemma (Theorem 8) implies that the integral vectors

$$A_i h \in \mathbb{Z}^r, 1 \leqslant i \leqslant n, h \in \mathcal{L}_i \tag{8}$$

can be permuted in such a way such that the distance in the ℓ_∞-norm of each prefix sum to the line-segment spanned by 0 and $A(x^* - y^*)$ is bounded by $\Delta(2s\Delta+1)^s$ times the dimension r, i.e., by $R := r\Delta(2s\Delta+1)^s$. Note that each prefix sum is an integer point. The number of integer points that are within distance R to the line segment spanned by 0 and $A(x^* - y^*)$ is at most

$$\|A(x^* - y^*)\|_1 \cdot R^r. \tag{9}$$

But

$$\begin{aligned}\|A(x^*-y^*)\|_1 &\leqslant \sum_{i=1}^{n}\|A_i(x^{*(i)}-y^{*(i)})\|_1\\ &\leqslant r\cdot\Delta r^2(2s\Delta+1)^s,\end{aligned}$$

where the last inequality follows from Proposition 2 and Proposition 5. Thus this number of integer points is bounded by

$$r^{r+3}\Delta^{r+1}(2s\Delta+1)^{s(r+1)}.$$

If $\sum_{i=1}^{n}|\mathscr{L}_i|$ is larger than this bound, then, in the Steinitz re-arrangement of the vectors there exist two prefix sums, that are equal. This yields sub-multisets

$$\mathscr{H}_1\subseteq\mathscr{L}_1,\mathscr{H}_2\subseteq\mathscr{L}_2,\cdots,\mathscr{H}_n\subseteq\mathscr{L}_n$$

for which one has

$$\sum_{i=1}^{n}\sum_{h\in\mathscr{H}_i}A_i h=0.$$

By Lemma 7, this is not possible. This implies that

$$\|y^*-z^*\|_1\leqslant r^{r+3}\Delta^{r+1}(2s\Delta+1)^{s(r+1)+1}=(rs\Delta)^{O(rs)}.$$

□

We conclude now with the proof of Theorem 4.

Proof of Theorem 4. By the triangle inequality one has

$$\|x^*-z^*\|_1\leqslant\|x^*-y^*\|_1+\|y^*-z^*\|_1.$$

We have seen that the first term on the right is bounded by $r^3(2s\Delta+1)^2$ and that the second term is bounded by $(rs\Delta)^{O(rs)}$. □

4 A dynamic program

Let x^* be an optimal vertex solution of the strengthened relaxation. We now describe a dynamic programming approach that computes an optimal integer solution of the N-fold integer program. We can restrict our search to an optimal integer solution z with $\|x^*-z\|_1\leqslant(rs\Delta)^{O(rs)}$. For such a $z\in\mathbb{Z}^{n\cdot t}$ one has in the linking constraints for each $1\leqslant\ell\leqslant n$

$$\|\sum_{i=1}^{\ell}A_i(x^{*(i)}-z^{(i)})\|_\infty\leqslant(rs\Delta)^{O(rs)},\tag{10}$$

which implies

$$\sum_{i=1}^{\ell}A_i x^{*(i)}-(rs\Delta)^{O(rs)}\leqslant\sum_{i=1}^{\ell}A_i z^{(i)}\leqslant\sum_{i=1}^{\ell}A_i x^{*(i)}+(rs\Delta)^{O(rs)}.\tag{11}$$

Let $S_\ell\subseteq\mathbb{Z}^r$ be the set of integer vectors $y\in\mathbb{Z}^r$ that satisfy

$$\sum_{i=1}^{\ell}A_i x^{*(i)}-(rs\Delta)^{O(rs)}\leqslant y\leqslant\sum_{i=1}^{\ell}A_i x^{*(i)}+(rs\Delta)^{O(rs)}.\tag{12}$$

We generate all these sets S_ℓ. Clearly, the cardinality of each S_ℓ satisfies

$$|S_\ell| \leqslant (rs\Delta)^{O(r^2 s)}. \tag{13}$$

The problem of finding an optimal solution to the N-fold integer program is a a longest path problem in the following weighted directed acyclic graph $D = (V, A)$. The vertices are

$$V = \{(\ell, y) : 1 \leqslant \ell \leqslant n, \, y \in S_\ell\}.$$

There exists an arc from a vertex (ℓ, y) to a vertex $(\ell + 1, y')$ if the integer program

$$\max\left\{c_i^T x^{(i)} : A_i x^{(i)} = y' - y, B_i x^{(i)} = b_i, 0 \leqslant x^{(i)} \leqslant u_i, \, x^{(i)} \in \mathbb{Z}^t\right\} \tag{14}$$

is feasible. The weight of this arc being the value of the optimal solution of (14). The integer program can be solved in time $((r+s)\Delta)^{O(r+s)^2} t$, see [**?**]. A longest path from $(0,0)$ to (n, b_0) in this graph corresponds to an optimal solution of the N-fold integer program. The number of nodes and arcs is bounded by $O(n(rs\Delta)^{O(r^2 s)})$. Since a longest path in a directed acyclic graph can be computed in linear time, we have the following theorem.

Theorem 10. *Given an optimal vertex solution of the strengthened relaxation of* (4) *of an N-fold integer program, one can compute an optimal solution of the N-fold integer program in time*

$$(rs\Delta)^{O(r^2 s + s^2)} n t.$$

5 Solving the relaxation

In this section, we sketch an analysis of the running time of the ellipsoid method if it is applied to the relaxation of (4). The analysis is most likely far from tight but still, we obtain the fastest algorithm for N-fold integer programming if standard arguments are used. We proceed by the technique of Lagrangean relaxation, see, e.g. [**?**, Chapter 24.3].

We dualize the linking constraints of (4) and obtain for $\lambda \in \mathbb{R}^r$ the linear program defining the function

$$\begin{aligned} L(\lambda) = \quad & \max \textstyle\sum_{i=1}^n c_i^T x_i + \lambda^T \left(\sum_{i=1}^n A_i x_i - b_0\right) \\ & x_i \in Q_i, \, i = 0, \ldots, n \end{aligned} \tag{15}$$

Notice that, for a given $\lambda \in \mathbb{R}^r$, $L(\lambda)$ can be computed in time $n \cdot t \cdot (s\Delta)^{O(s^2)}$, since (15) requires the solution of n individual integer programming problems in dimension t, see [**?**]. The value $L(\lambda)$ is an upper bound on the optimal value of the relaxation of (4) for each $\lambda \in \mathbb{R}^r$ and

$$\min_{\lambda \in \mathbb{R}^r} L(\lambda) \tag{16}$$

is equal to the optimal value of the relaxation of (4). The function $L(\lambda)$ is a piece-wise linear convex function. The optimization problem (16) is a convex optimization problem in dimension r. If an optimal solution λ^* is found, then it can be completed to an optimal dual solution of the linear relaxation of (4) and from there one can determine an optimal solution of the relaxation of (4) again. These conversions require a running time that is linear in n and polynomial in t.

We want to solve $\min L(\lambda)$, which is equivalent to minimizing β over the polyhedron $\mathscr{P} = \{(\lambda, \beta) : \beta \geqslant L(\lambda)\}$ which is the epigraph of the piece-wise linear convex function $L(\lambda)$ and this can be done with the ellipsoid method. To solve the feasibility problem of a (non explicitly given) polyhedron in $\mathbb{R}^k$, the ellipsoid method requires $O(\Phi p(k))$ calls to a separation oracle. Here Φ is the largest encoding length of a coefficient in an inequality description of the polyhedron and $p(\cdot)$ is a fixed polynomial. If a linear optimization problem has to be solved, then, by using binary search, this requires an additional factor of Φ. So all-together, the ellipsoid method finds an optimal solution of a linear programming problem by relying on $O(\Phi^2 p(k))$ calls to a separation oracle, see [**?**, **?**]. To understand the complexity of the ellipsoid method to solve the problem 16, we need to understand the binary encoding length of a coefficient of the inequality description of $\mathscr{P}$. To understand the order of magnitude of Φ we now consider the separation problem for a given point (λ^*, β^*) and $\mathscr{P}$. Clearly (λ^*, β^*) is an element of $\mathscr{P}$ if and only if the optimal value of the following integer optimization problem is at most β^*:

$$\max\left\{\sum_i (c_i + \lambda^{*T} A_i) x_i - \lambda^{*T} b_0 : x_i \in Q_i,\, i = 1, \ldots, n\right\}. \tag{17}$$

There is an optimal solution $\hat{x_1}, \ldots, \hat{x_n}$ of (17) where the $\hat{x_i}$ are vertices of the Q_i respectively. This means that $\mathscr{L}$ is described by the inequalities

$$\beta \geqslant \sum_i (c_i + \lambda^T A_i)\hat{x}_i - \lambda^T b_0,\, \hat{x}_i \in V_i,\, i = 1, \ldots, n,$$

where V_i is the set of vertices of Q_i. These inequalities can be re-written as

$$\beta \geqslant \sum_i c_i^T \hat{x}_i + \left(\sum_i A_i \hat{x}_i - b_0\right)^T \lambda,\, \hat{x}_i \in V_i,\, i = 1, \ldots, n. \tag{18}$$

The logarithm of the largest absolute value of an integer in the inequalities (18) is $O(\log(ntM^2)$, where M is an upper bound on the absolute values of any integer in the input of the N-fold integer program. The number of oracle calls to a separation algorithm for $\mathscr{P}$ is therefore

$$O(p(r) \cdot \log^2(ntM^2)). \tag{19}$$

All-together we have a total running time of

$$O(n \cdot (s\Delta)^{O(s^2)} \cdot t \cdot p(r) \log^2(nt)\phi^2)$$

to find an optimal solution of the strengthened LP relaxation of (4).