Graver Basis

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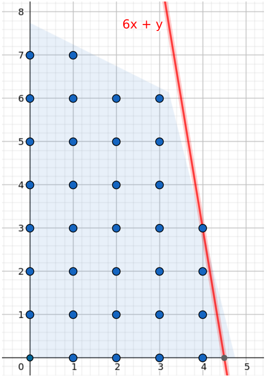
Integer Linear Programming

$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

$A \in \mathbb{Z}^{mxn}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, l and u lower and upper bounds for x

- NP-Hard
- Cutting plane methods
- Lattice-basis reduction
- Dynamic programming
- Graver basis techniques



Graver Basis

- **Definition:** Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, ..., n\}$
- **Definition:** A vector $u \in ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $ker(A)$.

Graver Basis $\equiv Gr(A)$

The Graver Basis of a given matrix A is defined as the set of integral indecomposable elements in the kernel of A.

(Initially defined as *universal integral test set* in [Graver 1975])

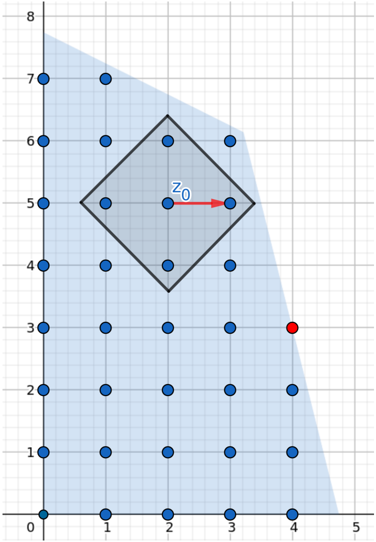
Graver Basis properties

- **Spanning:** Every integral element in ker(A) can be expressed as positive integral linear combination of elements in $Gr(A)$.
- **Optimality:** Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in Gr(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$
- **Bounds:** Given $A \in \mathbb{Z}^{mxn}$ and Δ an upper bound for the absolute value of each component of A, for every $g \in Gr(A)$:
  - $||g||_1 \leq m^{m/2}\Delta^m \cdot (n-m)$ [Onn 2010]
  - $||g||_1 \leq (2m\Delta + 1)^m$ [Eisenbrand,Hunkenschröder,Klein 2018]

Graver Basis properties

$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & x, y \in \mathbb{Z} \end{array} \right)$$

- If not optimal, an element in Graver basis is an improvement direction.
- If Graver basis bounded, we can restrict our improvement direction search.



Augmentation algorithm

General IP algorithm using Graver basis norm bound

1. From a feasible solution z_i
2. Find g^* optimum for the sub-problem:

   $max\{c^t g : Ag = 0, l - z_i \leq g \leq u - z_i, g \in \mathbb{Z}^n, ||g||_1 \leq ||Gr(A)||\}$

   - $g^* = 0 \implies z_i$ optimal solution.
   - $g^* \neq 0 \implies g^*$ improvement direction, loop back to 1 with $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

[Hemmecke, Onn, Romanchuk 2013]

N-Fold, a success example

A N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{rxt}, B_i \in \mathbb{Z}^{sxt}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

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N-Fold Graver basis bound

For all $g \in Gr(N)$ $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ where $L_B = (2s\Delta + 1)^s$

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N-Fold augmentation algorithm complexity

The N-Fold IP can be solved in time $(nt)^2 log^2(nt) \cdot \varphi(rs\Delta)^{O(r^2 s + rs^2)} + LP$

[Eisenbrand, Hunkenschröder, Klein 2018]

N-Fold, improving even more

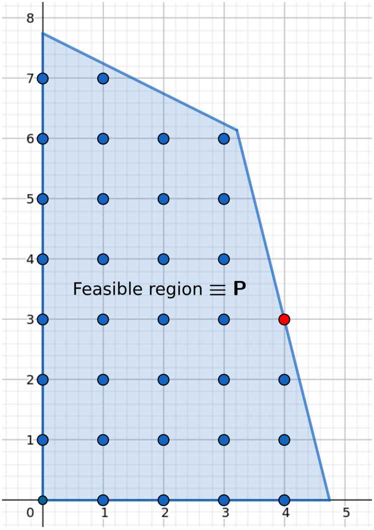

Figure 1: Linear Relaxation (LR)

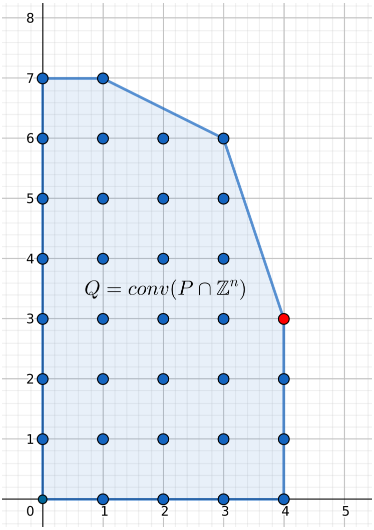

Figure 2: Restricted LR (RLR)

N-Fold, resolution steps

N-Fold RLR complexity

The N-Fold IP restricted linear relaxation problem can be solved in time

$$O(nt \cdot log^2(nt) \cdot \varphi p(r)(s\Delta)^{O(s^2)})$$

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N-Fold RLR complexity

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N-Fold RLR to optimum complexity

Given an optimal vertex of an N-Fold RLR, the N-Fold IP can be solved in time

$$O(nt \cdot (rs\Delta)^{O(r^2 s + s^2)})$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

N-Fold, from RLR to optimum

N-Fold proximity to RLR

Let x^* be an optimal vertex solution of a N-Fold RLR, then there exists an optimal solution z^* for the N-Fold IP verifying:

$$||z^* - x^*||_1 \leq (rs\Delta)^{O(rs)}$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

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Let $\gamma := (rs\Delta)^{O(rs)}$. For every $1 \leq \ell \leq n$:

$$||x^* - z||_1 \leq \gamma \implies \sum_{i=1}^{\ell} ||x_i^* - z_i||_1 \leq \gamma$$

$$\implies \sum_{i=1}^{\ell} ||A_i(x_i^* - z_i)||_\infty \leq \Delta\gamma = \gamma$$

N-Fold, from RLR to optimum

Therefore, for every feasible point z verifying the proximity bound we have:

$$\sum_{i=1}^{\ell} A_i x_i^* - \gamma \leq \sum_{i=1}^{\ell} A_i z_i \leq \sum_{i=1}^{\ell} A_i x_i^* + \gamma$$

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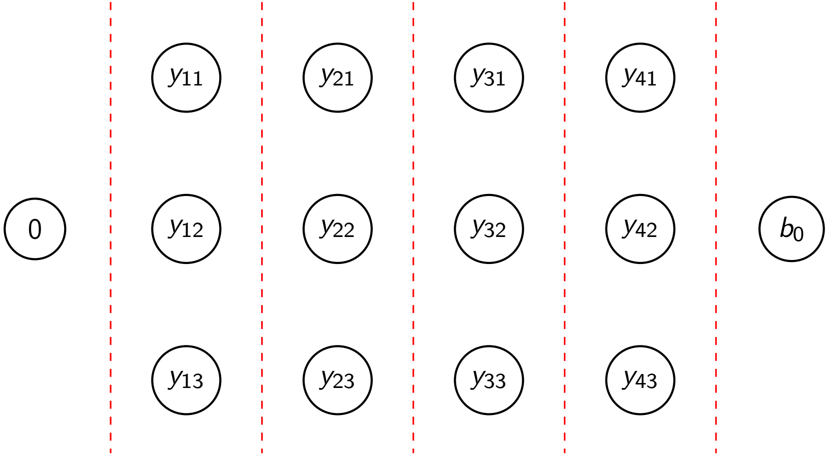
We now define the set S_ℓ as the set of $y \in \mathbb{Z}^r$ such that:

$$\sum_{i=1}^{\ell} A_i x_i^* - \gamma \leq y \leq \sum_{i=1}^{\ell} A_i x_i^* + \gamma$$

N-Fold, from RLR to optimum

We can construct a weighted directed acyclic graph G(V,E) with vertices:

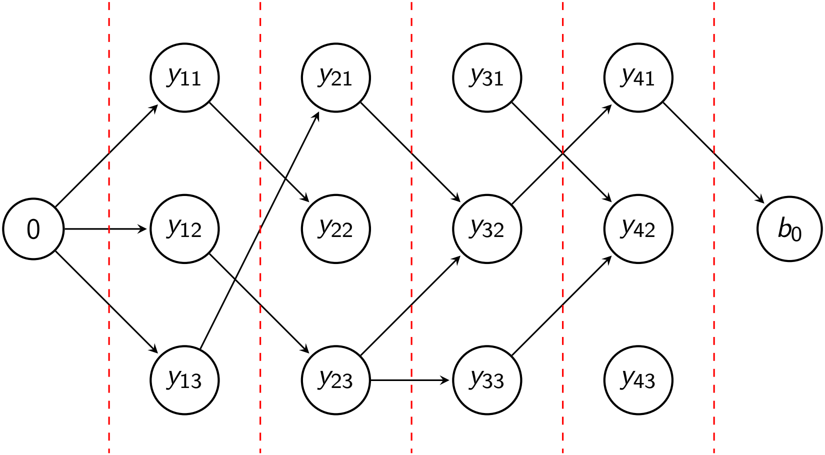
$$V = \{(\ell, y)/y \in S_\ell\} \cup \{(0,0),(n,b_0)\}$$



N-Fold, from RLR to optimum

Weighted edges between $(\ell - 1, y)$ and (ℓ, y') if the problem is feasible:

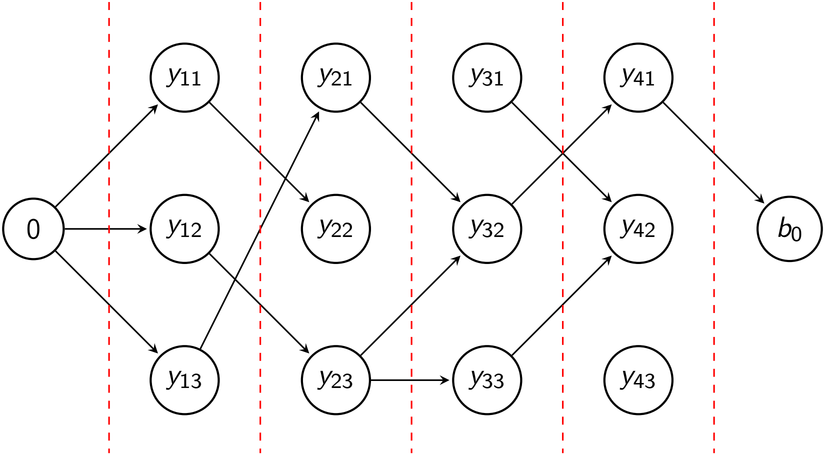
$$max\{c_\ell^t x : A_\ell x = (y' - y), B_\ell x = b_\ell, l_\ell \leq x \leq u_\ell\}$$



N-Fold, from RLR to optimum

Correspondence

A longest path from $(0,0)$ to (n, b_0) in G(V,E) corresponds to an optimal solution of the N-Fold integer program.



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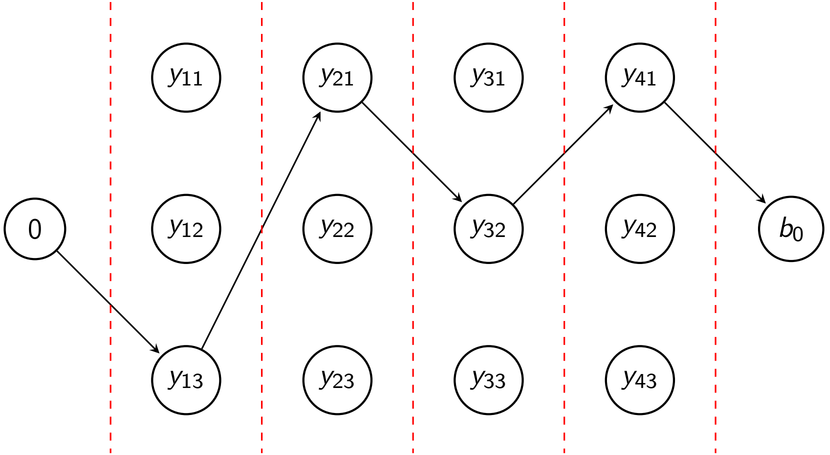

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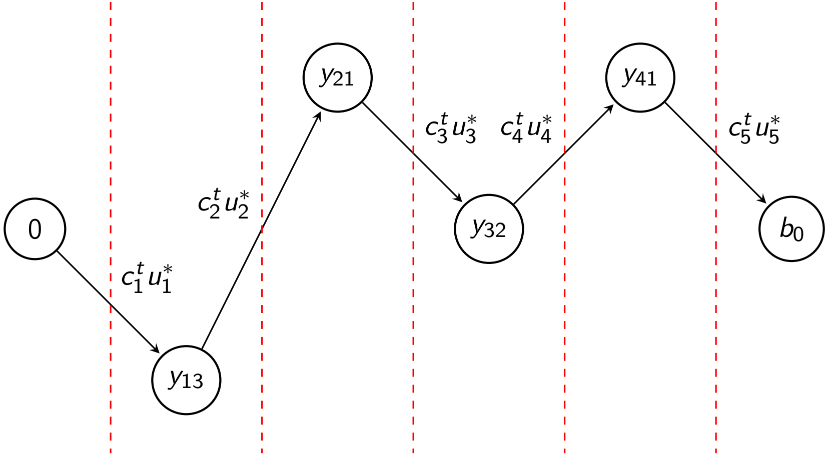

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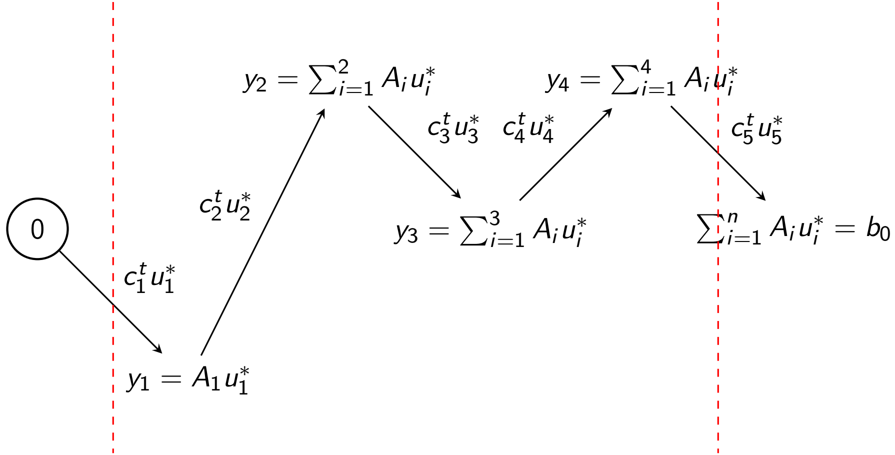

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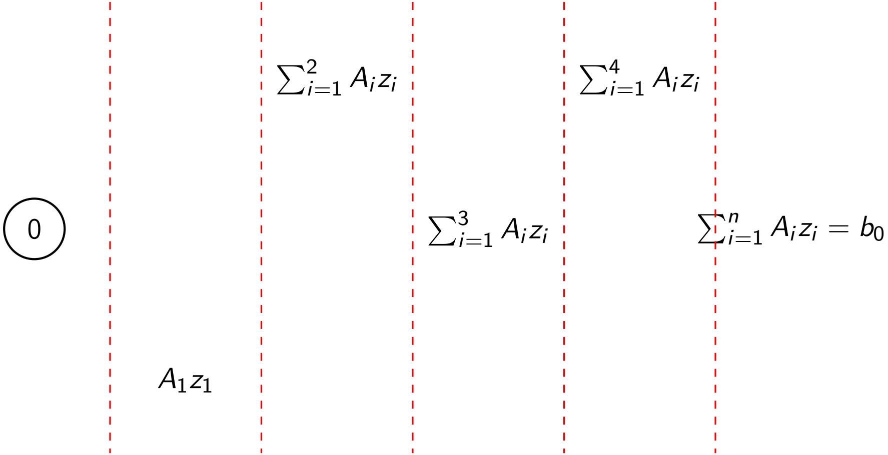

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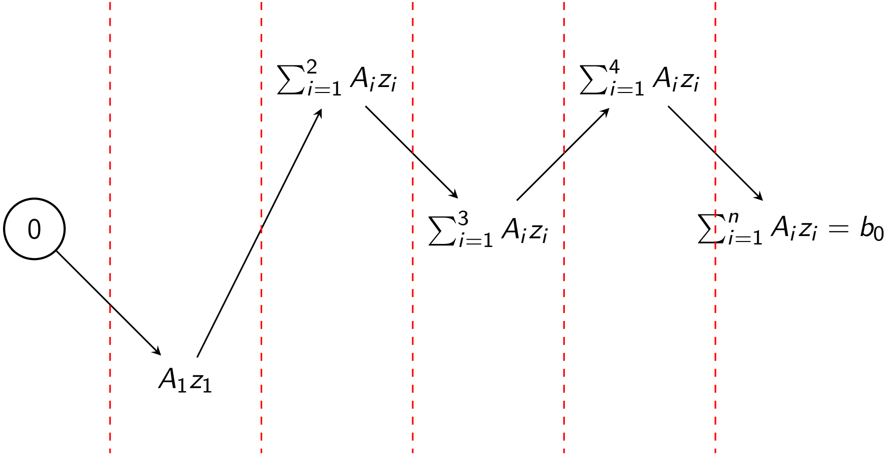

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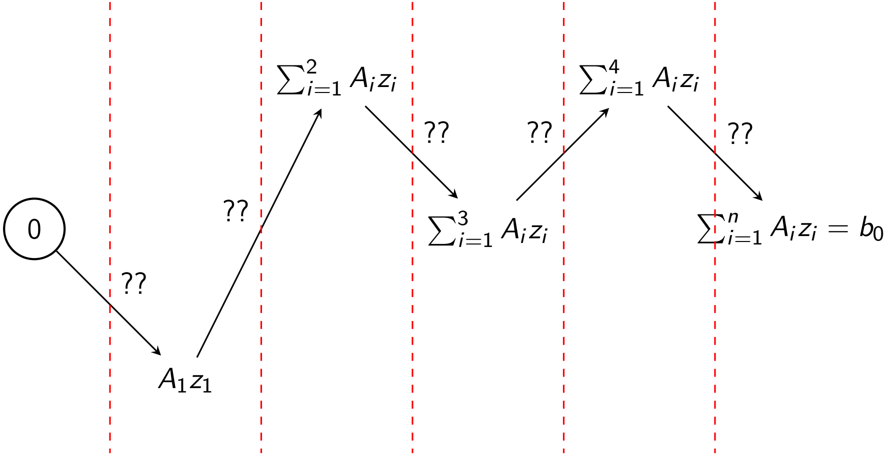

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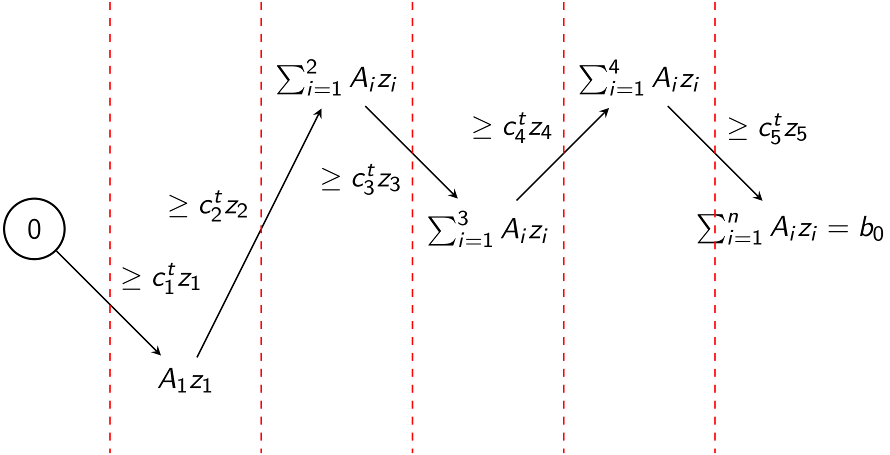

N-Fold, from RLR to optimum

- $|S_I| \leq (rs\Delta)^{O(r^2 s)}$
- $|V| + |E| \leq O(n(rs\Delta)^{O(r^2 s)})$
- The edge IP can be computed in time $t((r+s)\Delta)^{O(r+s)^2}$
- Longest path problem in a acyclic digraph can be solved in linear time.

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N-Fold complexity

The N-Fold IP can be solved in time $nt(rs\Delta)^{O(r^2 s + s^2)} + RLR$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

References

[1] Jack E. Graver. "On the foundations of linear and integer linear programming I". In: Syracuse University, New York, U.S.A., 1975.

[2] Kim-Manuel Klein Friedrich Eisenbrand Christoph Hunkenschröder. "Faster Algorithms for Integer Programs with Block Structure". In: École polytechnique fédérale de Lausanne, Switzerland, 2018.

[3] Robert Weismantel Jana Cslovjecsek Friedrich Eisenbrand. "N-fold integer programming via LP rounding". In: École polytechnique fédérale de Lausanne, Switzerland, 2020.