GRAVER BASES

Francisco Javier Blázquez Martínez

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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Chair of discrete optimization

Project director: Dr. Friedrich Eisenbrand

Project supervisor: Jana Cslovjecsek

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To my brother Joaquín, for showing me what mathematics were during dinners at home.

To my parents, for showing me beyond the scope of mathematics.

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Chapter 1

Introduction

Hereafter, the underlying problem is the classical *Integer Program* (IP), that we formulate in the following way:

$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$
$$A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m, c \in \mathbb{Z}^n, l \text{ and } u \text{ lower and upper bounds for } x$$

Despite the simplicity of its formulation, allowing only linear constraints and a linear objective function, the importance of IP is well known . A large number of problems in diverse fields of the mathematics and algorithms with an infinity of applications admit an IP formulation. Unfortunately, it's also well known that integer programming is NP-Complete, what means that no efficient algorithm is likely to exist for solving an IP in the general case. This explains the great interest in restricted formulations of the problem and in certain resolution techniques even when they can't be applied to the general IP. In the following sections we present the latest techniques based on the **Graver bases** and its bounds as well as their application to the **N-Fold IP**, a restricted formulation of the IP which has won relevance in the last decades given its theoretical properties and its wide applications.

For this purpose, in chapter 2 we introduce the Graver basis of a given matrix, explore its properties, bounds, and how these can be applied for solving the general IP. We then study in chapter 3 the N-Fold case and show, with the help of Graver bases, that the *N-Fold IP* can be solved in polynomial time, delving into the best known algorithm for this case.

Chapter 2

Graver bases

In this chapter we first define formally the Graver basis of a matrix, presenting and proving its main properties. After this, we present two algorithms for the general IP based on these properties, studying their complexity and main limitations.

Before introducing the concept of Graver basis of a matrix, we define the partial order $\sqsubseteq$ in $\mathbb{R}^n$ by $u \sqsubseteq v$ if $u_i \cdot v_i \geq 0$ and $|u_i| \leq |v_i|$ for all i. Note that the condition $u_i \cdot v_i \geq 0$ means that $\sqsubseteq$ can only compare *sign compatible* vectors, i.e., vectors with the same sign componentwise. We also introduce the notation $\mathcal{L}(A)$ for the integral kernel of a matrix excluding zero, i.e., $\mathcal{L}(A) := \{z \in \mathbb{Z}^n : Az = 0, z \neq 0\}$.

Definition 2.1 (Graver basis). The Graver basis $\mathcal{G}(A)$ of a given matrix $A \in \mathbb{Z}^{m \times n}$ is defined as the set of $\sqsubseteq$-minimal elements in $\mathcal{L}(A)$.

Graver bases were initially defined as *universal integral test set* in [1] by Jack. E. Graver in 1975. They often appear defined in an equivalent way as the nonzero integral indecomposable elements in $ker(A)$. *Indecomposable* in the sense that they can not be expressed as the sum of two sign compatible vectors. We now study their properties:

Proposition 2.2. *For every matrix A, $\mathcal{G}(A)$ is a finite set.*

Proof. Dickson's lemma states that every subset of $\mathbb{N}^n$ has a finite number of minimal elements with the order $\leq$ componentwise. It's easy to see that this implies that $\mathcal{L}(A)$ has a finite number of $\sqsubseteq$-minimal elements in every orthant. As the elements in different orthants are not comparable we have that $\mathcal{G}(A)$ is the union of 2^n finite sets, concluding the proof. $\square$

Unfortunately, the cardinality of $\mathcal{G}(A)$ may be exponential in n, the number of columns of A. This limits the explicit computation and usage of the Graver basis to only certain cases but doesn't limit its theoretical properties. The most important of these properties is expressed in the following proposition:

Proposition 2.3. *Every integral element in $ker(A)$ can be expressed as positive integral linear combination of sign compatible elements in $\mathcal{G}(A)$.*

Proof. The proof is by induction on the well partial order $\sqsubseteq$. For the base case we see that given $u \in \mathcal{L}(A)$ $\sqsubseteq$-minimal then u belongs to $\mathcal{G}(A)$ and the result holds.

For the induction case lets take $u \in \mathcal{L}(A)$ not $\sqsubseteq$-minimal. Therefore there exists $u_1 \in \mathcal{L}(A)$ such that $u_1 \sqsubseteq u, u_1 \neq u$. We take $u_2 = u - u_1$. Note that thanks to the definition of $\sqsubseteq$ u, u_1 and u_2 are sign compatible and thanks to $u \neq u_1$ necessarily $u_1, u_2 \sqsubset u$. With this appreciations, the proof concludes after applying the induction hypothesis:

$$u = u_1 + u_2 = \sum \alpha_{1i} g_{1i} + \sum \alpha_{2i} g_{2i} = \sum \alpha_j g_j$$

□

This proposition is the reason why Graver bases were introduced as *universal integral test set.* It ensures that, given any feasible point, the whole feasible region can be expressed in terms of elements in $\mathcal{G}(A)$. Note that thanks to requiring positive coefficients and sign compatible elements we avoid cancellations in every component.

In the next proposition we see how, thanks to this property, we can do an optimality test for any feasible point using only elements in the Graver basis.

Proposition 2.4. *Given a feasible point z of the IP, z is not optimum if and only if there exists $g \in \mathcal{G}(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$.*

Proof. If there exists $g \in \mathcal{G}(A)$ such that $c^t g > 0$ is clear that $z + g$ is a feasible point which strictly improves the objective function, so z is not an optimum.

For the other implication, if z is not an optimum we can take a feasible point y improving z. Thanks to the previous proposition there exist $g_i \in \mathcal{G}(A)$, $\alpha_i \geq 0$ s. t. $(y - z) = \sum \alpha_i g_i$. Then $0 < c^t(y - z) = \sum \alpha_i c^t g_i$ so at least one $g_i \in \mathcal{G}(A)$ verifies $c^t g_i > 0$. Finally, $z + g_i$ is feasible because thanks to $\alpha_i \geq 0$ and g_i being sign compatible with $y - z$ for all i we have: $l \leq z \leq z + g_i \leq z + \sum \alpha_i g_i = y \leq u$.

□

2.1 Graver basis greedy augmentation algorithm

We now consider how to solve the general IP with the help of Graver bases. Note that proposition 2.4 doesn't only give us an optimality test but also provides us an improvement direction if the feasible point is not optimal. We can follow that improvement direction to get a better feasible point and then repeat this process. That is the idea of the following procedure (introduced in [1]):

General IP algorithm using Graver basis

1. From a feasible solution z_i
2. Find g^* optimum for the augmentation step problem:
   $max\{c^t g : g \in \mathcal{G}(A), l \leq z_i + g \leq u\}$
   - $c^t g^* \leq 0 \implies z_i$ optimal solution.
   - $c^t g^* > 0 \implies g^*$ improvement direction, loop back to 1 with:
     $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

Note that the algorithm finishes in a finite number of steps thanks to the lower and upper bounds l and u. We can assume the bounds are finite and, this way, the objective function is also bounded. Since every iteration we are strictly increasing the objective function, no infinite loop is possible. As stated in [2], it is always possible to add suitable polynomial upper and lower bounds without excluding some optimal solution if any, so assuming l and u to be finite is no loss of generality.

The question that arises now is the complexity of this algorithm. It was analyzed in [3, Theorem 3.3] showing that it's polynomial. This of course doesn't mean we have a polynomial algorithm for the general IP, it means that, given an IP along with its Graver basis, we have a polynomial algorithm in this input size. The complexity of the problem remains in computing the Graver basis which, as we announced, may be exponential. This makes the algorithm non-viable but for small matrices.

Another way to estimate the complexity of the algorithm is using [4, Theorem 2.b], which precises that the number of augmentation steps is polynomial. Assuming the Graver basis is given as part of the input, we could solve any augmentation step by iterating over the elements in $\mathcal{G}(A)$, checking the constraints and the objective function, which has complexity $n \cdot |\mathcal{G}(A)|$. Therefore, if the Graver basis is given the algorithm is polynomial in $n \cdot |\mathcal{G}(A)|$.

2.2 Graver basis bound augmentation algorithm

Up to this point we have seen how Graver bases allow a straightforward algorithm for the general IP. However, we have also seen that its main drawback is that it requires the explicit computation of the Graver basis. In this section we show how we can avoid computing the Graver basis thanks to bounds for the ℓ_1-norm of its elements.

Proposition 2.5 (Graver basis bounds). *Given $A \in \mathbb{Z}^{mxn}$ and Δ an upper bound for the absolute value of each component of A, for every $g \in \mathcal{G}(A)$:*

- $||g||_1 \leq m^{m/2}\Delta^m \cdot (n-m)$ *[Onn 2010]*
- $||g||_1 \leq (2m\Delta + 1)^m$ *[Eisenbrand,Hunkenschröder,Klein 2018]*

We refer to [5] and [6] for the proofs. Note that both bounds are exponential in the number of rows of A but the second one has the advantage of being independent in the number of columns. This will be a key fact in the next chapter.

Why should bounds to the Graver bases help? Because thanks to the proposition 2.4, the search of an improvement direction can be restricted to the elements in the Graver basis and, thanks to the bounds, we can restrict our search space without excluding any element of the Graver basis. This is the idea of the following algorithm [2]:

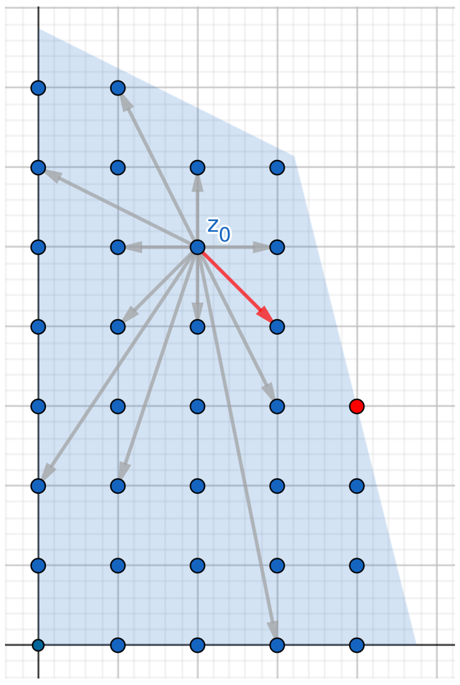


FIGURE 2.1: Feasible region

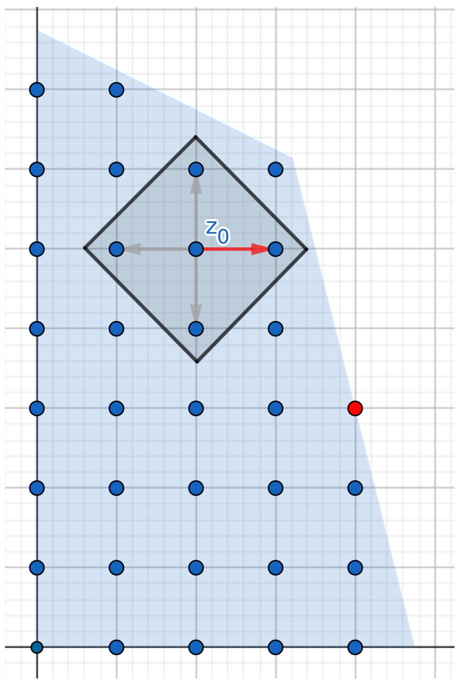


FIGURE 2.2: Bounded region

General IP algorithm using Graver basis norm bound

1. From a feasible solution z_i
2. Find g^* optimum for the augmentation step problem:
   $max\{c^t g : Ag = 0, l - z_i \leq g \leq u - z_i, g \in \mathbb{Z}^n, ||g||_1 \leq ||\mathcal{G}(A)||\}$
   - $g^* = 0 \implies z_i$ optimal solution.
   - $g^* \neq 0 \implies g^*$ improvement direction, loop back to 1 with:
     $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

As we advanced, the main advantage of this algorithm is that it doesn't require the explicit computation of the Graver basis. However, the complexity is totally dependent on the added restriction $||g||_1 \leq ||\mathcal{G}(A)||$ and, as we saw in the proposition 2.2, the only bounds we have for the general case are exponential. This means that for the general IP the lower and upper bounds are much more restrictive than the Graver basis bound and therefore we have to explore the whole feasible region.

In certain cases we can get a much tighter bound for the Graver basis elements and this can help us to get a faster algorithm. The *N-Fold IP* is an iconic example.

Chapter 3

N-Fold IP

A generalized *N-Fold IP* has constraint matrix A of the form ($A_i \in \mathbb{Z}^{rxt}, B_i \in \mathbb{Z}^{sxt}$):

$$\mathcal{N} = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

It was presented in [3] in 2006 in a simplified version in which $\forall i, j\ A_i = A_j, B_i = B_j$. This simplified N-Fold matrix is totally determined given $A \in \mathbb{Z}^{r\times t}$, $B \in \mathbb{Z}^{s\times t}$ and the order n and we denote it by $N^{(n)}_{A,B}$. Hereafter, we'll refer to the generalized formulation as *N-Fold IP*.

The *N-Fold IP* has a wide range of applications. In [3] it's applied to the multiway transportation and cutting-stock problems and in [2] to privacy and disclosure control in statistical databases just to name a few examples beyond the typical in operations research. In fact, the *N-Fold IP* is universal. Every IP can be expressed as an *N-Fold IP*. This is because, as shown in [7], every IP can be modeled as a slim 3-way transportation program, which can be expressed as an *N-Fold IP*. However, this is more a theoretical achievement which shows the expressiveness power of the N-Fold rather than a useful practical approach for solving any IP.

In any case, the *N-Fold IP* is interesting by itself since it has good theoretical properties. In the following sections we will study it with the help of Graver bases to finally obtain a roughly linear algorithm for its resolution. We start with the following proposition:

Theorem 3.1. *Fixed any $A \in \mathbb{Z}^{r\times t}$ and $B \in \mathbb{Z}^{s\times t}$, there is an algorithm polynomial in n that computes the Graver basis of the N-Fold matrix $N^{(n)}_{A,B}$.*

We won't go into the details of the proof of this proposition, they can be seen in [3, Theorem 4.2]. However, we consider important to remark that this proposition implies that, fixed A and B, the cardinality of $\mathcal{G}(N^{(n)}_{A,B})$ is bounded by a polynomial function of n.

This has an important consequence, the greedy Graver basis augmentation algorithm presented before has, in this case, polynomial complexity for every augmentation step and therefore polynomial complexity. However it's still remaining the problem of obtaining an initial feasible solution. That is precisely what solves the next proposition:

Lemma 3.2. *Fixed any $A \in \mathbb{Z}^{r\times t}$ and $B \in \mathbb{Z}^{s\times t}$, there is an algorithm polynomial in n that, given a demand vector $b \in \mathbb{Z}^{s+nr}$, either finds a feasible point $x \in \mathbb{Z}^{nt}$ to the simplified* N-Fold IP *of order n, or asserts that no feasible solution exists.*

Proof. This is a result proved in [3, Lemma 5.2]. The idea is that we can add $2n(s+r)$ artificial variables to construct an *N-Fold IP* for which an initial feasible solution is trivial for any right side b. The constraint matrix would result as below:

$$N = \begin{pmatrix} A & I_s & -I_s & 0 & 0 & A & I_s & -I_s & 0 & 0 & \cdots & A & I_s & -I_s & 0 & 0 \\ B & 0 & 0 & -I_r & I_r & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & -I_r & I_r & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & B & 0 & 0 & -I_r & I_r \end{pmatrix}$$

This way the result is obtained applying the *two phase method* which, thanks to the previous proposition, runs in polynomial time. Note that we need to add I and $-I$ terms because the artificial variables have to be positive. □

Theorem 3.3 (Simplified N-Fold IP is polynomially solvable). *Fix any pair of integer matrices $A \in \mathbb{Z}^{r\times t}$ and $B \in \mathbb{Z}^{s\times t}$. Then there is a polynomial time algorithm that solves the simplified N-Fold integer programming problem on any input n, b, c.*

Proof. Propositions 3.1 and 3.2 ensure that we can compute the Graver basis of $N^{(n)}_{A,B}$ and a initial feasible point in polynomial time for any n, b and c. Applying the algorithm 2.1 after this solves the problem in global polynomial time. □

3.1 N-Fold augmentation algorithm

In the previous section we have seen that the simplified *N-Fold IP* is polynomially solvable. This is an important theoretical result but we haven't fully specified the algorithm and its complexity. Polynomial doesn't mean efficient and this algorithm could be useless in practice.

In this section we present a quadratic algorithm for the generalized *N-Fold IP*. The key to this algorithm is a bound for the ℓ_1-norm of the N-Fold matrix Graver basis elements. To obtain it, we need first to introduce the *Steinitz lemma* [8].

Lemma 3.4 (Steinitz Lemma). *Let $v_1, ..., v_n$ be vectors with $||v_i|| \leq \Delta$ for $i = 1, ..., n$. If $\sum_{i=1}^{n} v_i = 0$, then there is a reordering $\pi \in S_n$ such that for each $k \in \{1, ..., n\}$ the partial sum $p_k := \sum_{i=1}^{k} v_{\pi(i)}$ satisfies $||p_k|| \leq n\Delta$.*

The Steinitz lemma has proven to be a very useful tool when it comes to the *N-Fold IP*. It's one of the main components behind the following results. Before moving to the next one, we clarify the notation used. Hereafter, $\mathcal{N}$ will denote the *generalized N-Fold* matrix as in 3, n will denote its order and $\Delta := ||\mathcal{N}||_\infty$.

Lemma 3.5 (N-Fold Graver basis bound). $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ *where* $L_B = (2s\Delta + 1)^s$ *for all* $g \in \mathcal{G}(\mathcal{N})$

Proof. Let y be a Graver basis element of $\mathcal{N}$, in coherence with the block decomposition of $\mathcal{N}$ we split it in blocks $y = ((y^{(1)}), ..., (y^{(n)}))$, where $y^{(i)} \in \mathbb{Z}^t$. Since every $y^{(i)} \in ker(B^{(i)})$ we decompose it as the sum of Graver basis elements $y^{(i)} = y_1^{(i)} + ... + y_{N_i}^{(i)}$. We have:

$$0 = A_1 y^{(1)} + ... + A_n y^{(n)} = A_1 y_1^{(1)} + ... + A_1 y_{N_1}^{(1)} + ... + A_n y_1^{(n)} + ... + A_n y_{N_n}^{(n)} =: v_1 + ... + v_N$$

Where $N = \sum_{i=1}^{n} N_i$. Note now that $y_j^{(i)} \leq L_B$ and then $||v_i||_\infty = ||A_k y_j^{(k)}||_\infty \leq \Delta L_B$ for all $i \in \{1, ..., N\}$. This bound for the infinity norm implies that $N \leq (2r\Delta L_B + 1)^r$ because all the v_i have to be distinct or y wouldn't be minimal. At this point we can apply the Steinitz Lemma to reorder the v_i to bound each partial sum by $r\Delta L_B$ in the l_∞ norm. Finally, as $v_i = A_k y_j^{(k)}$ and $||y_j^{(k)}||_1 \leq L_B$ each v_i is the sum of at most L_B columns of some A_k and we have:

$$||y||_1 \leq L_B(2r\Delta L_B + 1)^r = (2s\Delta + 1)^s(2r\Delta(2s\Delta + 1)^s + 1)^r = L_A$$

□

Note that, unlike the bounds for the general case which are exponential in the number of rows, this doesn't depend on n. It's therefore very restrictive and that makes the algorithm 2.2 efficient. For knowing the complexity the algorithm in the N-Fold case we need to analyze the number of augmentation steps needed for reaching a solution as well as the complexity of each of these. This is done in [6, Lemmas 4, 5 and 6], we enunciate here the main results:

Lemma 3.6. *The augmentation step of the algorithm 2.2 for the N-Fold IP can be solved in time* $nt(rs\Delta)^{\mathcal{O}(r^2s+rs^2)}$.

We'll see a dynamic program for solving the augmentation step in the next section, as part of the best algorithm known for the *N-Fold IP*. We postpone the proof of this lemma until then.

It's only remaining to bound the number of augmentation steps needed by the algorithm to finish to have it's complexity. This is provided by the following lemma.

Lemma 3.7. *For an N-Fold IP let* $\Gamma := max_i(u_i - l_i)$. *Given an initial feasible point the algorithm 2.2 reaches an optimum solution by solving the augmentation step at most* $\mathcal{O}(ntlog(\Gamma)(log(nt\Gamma) + \varphi)$ *times, where* φ *is the maximum encoding length of the input.*

This complexity depends on the lower and upper bounds since, at the end of the day, they are restrictions to the feasible region and therefore to the problem's complexity. In total, ignoring the encoding lengths, the algorithm 2.2 has a running time in $\mathcal{O}(n^2t^2log(\Gamma)log(nt\Gamma)(rs\Delta)^{\mathcal{O}(r^2s+rs^2)})$, roughly quadratic in the number of variables.

We can avoid the dependency with the lower and upper bounds by starting the algorithm 2.2 not from an arbitrary feasible point, but from one "close" to the optimum. We first solve the linear relaxation problem and then start from the closest integral feasible point to this solution. The following proximity bound allows us to create artificial l and u constraints which are independent of the ones given.

Lemma 3.8. *Given an N-Fold IP* $\mathcal{N}$, *let* x^* *be an optimum solution to its linear relaxation. Then, there exists an optimum integral solution* z^* *to* $\mathcal{N}$ *with* $||x^* - z^*||_1 \leq ntL_A$.

The complexity of this variant of the algorithm 2.2 is given in the following theorem, main result of this section and one of the best algorithms known for the *N-Fold IP*.

Theorem 3.9 (N-Fold augmentation algorithm complexity). *The N-Fold IP can be solved in time* $(nt)^2log^2(nt) \cdot \varphi(rs\Delta)^{O(r^2s+rs^2)} + LP$

3.2 N-Fold via LP rounding

The main drawback of the augmentation algorithm is that after solving the linear relaxation, we can be arbitrarily far from the optimal solution of the IP, what means that we might need many augmentation steps. In this section we follow a different approach introduced in [9]. Intuitively, we first solve a more restricted linear relaxation which optimum is closer to the optimum of the IP. After solving this restricted linear relaxation, we can find an optimum with only one augmentation step. We first introduce this restricted linear relaxation (RLR):

3.2.1 Restricted linear relaxation

We can express the *N-Fold IP* with the following equivalent block decomposition:

$$(\mathcal{N} - IP) \equiv max\{\sum (c^{(i)})^t y^{(i)} : \sum A^{(i)} y^{(i)} = b_0, y^{(i)} \in P_i, y \in \mathbb{Z}^{nt}\}$$
$$P_i = \{x \in \mathbb{R}^t : B^{(i)} x = b_i, l_i \leq x \leq u_i\}, A \in \mathbb{Z}^{m \times n}, b_0 \in \mathbb{Z}^r, c_i, l_i, u_i \in \mathbb{Z}^t$$

The classical linear relaxation (LR) is obtained by only removing the $y \in \mathbb{Z}^{nt}$ constraint, the RLR restricts also each P_i to $Q_i = conv(P_i \cap \mathbb{Z}^t)$.

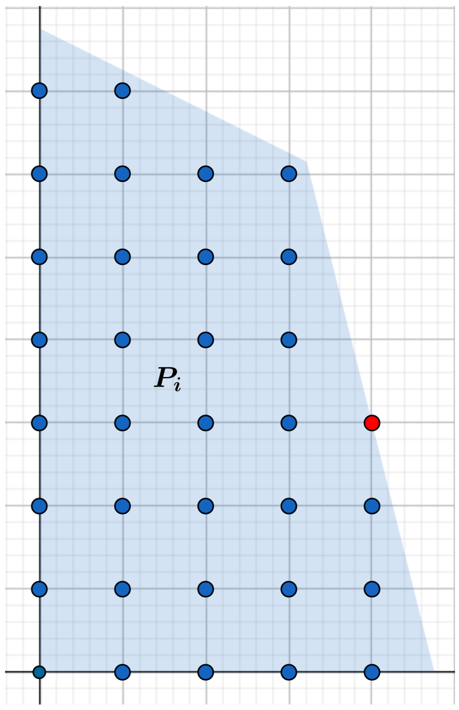


FIGURE 3.1: Linear relaxation

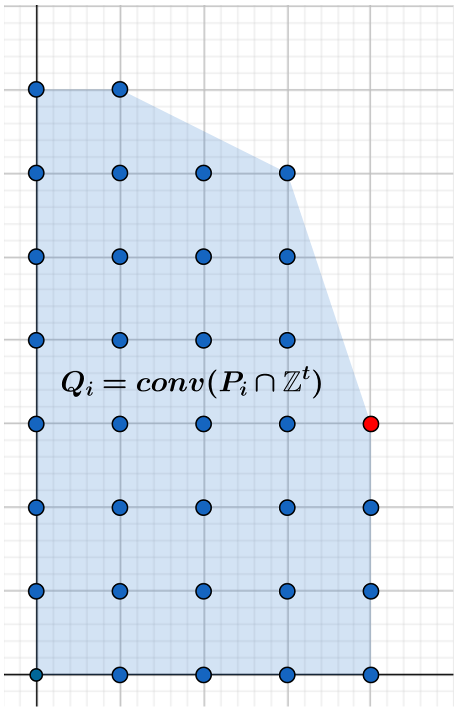


FIGURE 3.2: Restricted LR

Solving the restricted linear relaxation problem is not an easy task in practice. It was proved in [9] by applying the *ellipsoid method* that it can be solved in linear time (ignoring logarithmic factors). We won't go into details of the proof here. In the following lemma and in the next ones, φ denotes the maximum encoding length of the input and p a polynomial.

Lemma 3.10 (N-Fold RLR complexity). *The N-Fold IP restricted linear relaxation problem can be solved in time*

$$O(nt \cdot log^2(nt) \cdot \varphi p(r)(s\Delta)^{O(s^2)})$$

As in the previous algorithm, solving this RLR will be the first step. We can see that the complexity of this first step has slightly increased. However, this algorithm will be more efficient thanks to the following result [9, Section 3]:

Lemma 3.11 (N-Fold proximity to RLR). *Let x^* be an optimal vertex solution of a N-Fold RLR, then there exists an optimal solution z^* for the N-Fold IP verifying:*

$$||z^* - x^*||_1 \leq (rs\Delta)^{O(rs)}$$

Note that, unlike the proximity bound to the LR, which was linear in the number of variables, once fixed the order of the matrices A_i and B_i (parameters r and s) the bound is constant. This, together with the dynamic program we present below, is the reason of success of our algorithm.

3.2.2 Dynamic program

We present in this section the second and last step of our algorithm. A program which takes an optimal solution of the restricted linear relaxation and finds an optimal solution of the *N-Fold IP*. The idea behind this program [9] is creating a weighted directed acyclic graph in which a longest path between two specific vertices corresponds to an optimal solution. Then, using dynamic programming we can compute a longest path in a directed acyclic graph in linear time in the number of nodes and edges. This is also the program used for proving the lemma 3.6.

Proposition 3.12 (N-Fold RLR to optimum complexity). *Given an optimal vertex of an N-Fold RLR, the N-Fold IP can be solved in time*

$$O(nt \cdot (rs\Delta)^{O(r^2s+s^2)})$$

Proof. For avoiding heavy notation, we denote $\gamma := (rs\Delta)^{O(rs)}$. First we define:

$$S_\ell := \{y \in \mathbb{Z}^r : ||\sum_{i=1}^{\ell} A_i x_i^* - y||_\infty \le \gamma\}$$

We construct now a weighted directed acyclic graph G(V,E) with vertices

$$V = \{(\ell, y)/y \in S_\ell\} \cup \{(0,0),(n,b_0)\}$$

and weighted edges between $(\ell-1, y)$ and (ℓ, y') if the following problem is feasible and, if so, we take the weight as the maximum:

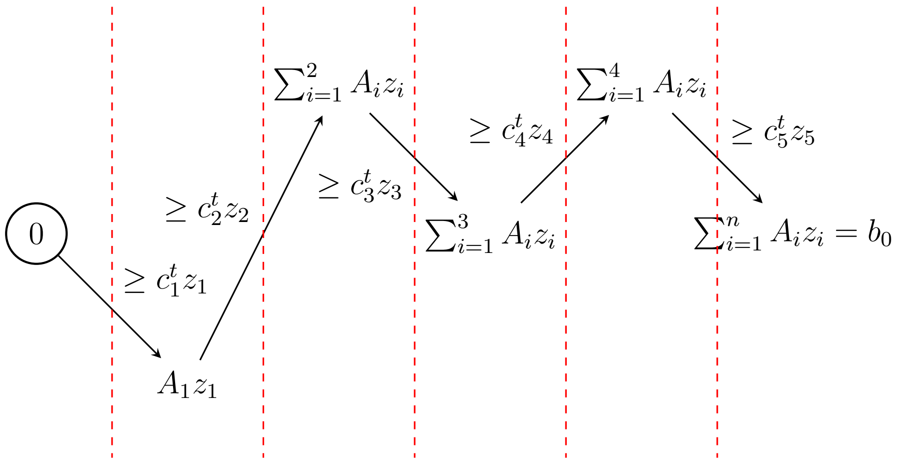
$$EP_\ell \equiv max\{c_\ell^t x : A_\ell x = (y'-y), B_\ell x = b_\ell, l_\ell \le x \le u_\ell\}$$

Now the graph is constructed, we proof that a longest path from $(0,0)$ to (n, b_0) in G(V,E) corresponds to an optimal solution of the *N-Fold integer program.*

First we prove that every feasible point z verifying the proximity bound corresponds to a path from $(0,0)$ to (n, b_0) and that the value of the objective function at this point is lower or equal than the path cost. Note that z fulfils:

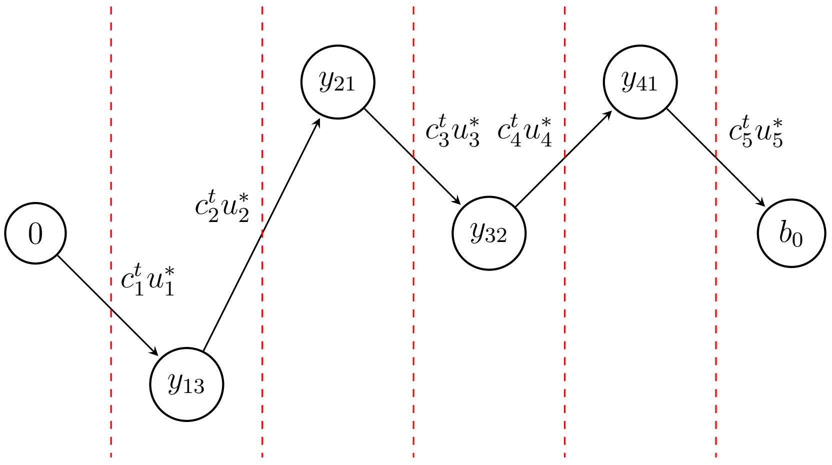
$$\begin{aligned} ||x^* - z||_1 \le \gamma &\implies \sum_{i=1}^{\ell} ||x_i^* - z_i||_1 \le \gamma \\ &\implies \sum_{i=1}^{\ell} ||A_i(x_i^* - z_i)||_\infty \le \Delta\gamma = \gamma \end{aligned}$$

Then for every $\ell \in \{1, ..., n\}$ $(\ell, \sum_{i=1}^{\ell} A_i z_i)$ is a vertex and z_ℓ is a feasible point of EP_ℓ.



The result is then clear. Note also that the lemma 3.11 ensures that at least one optimal solution of the *N-Fold IP* verifies the proximity bound. We therefore have that the longest path from $(0,0)$ to (n, b_0) has cost greater or equal than the optimal solution.

Analogously we prove that given any path path from $(0,0)$ to (n,b_0) there exists a feasible point with objective function value equal to the path cost. This can be seen because, given a path, the edges can be associated with an optimal solution of EP_ℓ. Since the path starts in 0 and finishes in b_0 we can construct a feasible point y simply by taking $y^{(i)}$ as the optimal solution of EP_i. It's easy to see that y respects the constraints and clearly the path cost is the value of the objective function in this point. Note that this implies that every path cost is bounded by the optimal solution of the *N-Fold IP*.



Having both inequalities, it's clear that the longest path cost is equal to the optimal solution of the *N-Fold IP*.

Finally, the complexity of the algorithm is the complexity of first constructing the graph and then applying the longest path dynamic program. Note that, as S_ℓ is defined from a bound for the ∞-norm, $|S_l| \leq (rs\Delta)^{O(r^2s)}$. This easily implies, given the structure of the graph, that $|V|+|E| \leq O(n(rs\Delta)^{O(r^2s)})$. Note also that EP_ℓ can be solved in time $t((r+s)\Delta)^{O(r+s)^2}$ since it's an IP with t variables and $(r+s)$ constraints.

Putting all these together we get that we can build the graph in $O(nt(rs\Delta)^{O(r^2s+s^2)})$ by solving EP_ℓ for each edge and then apply the dynamic program for finding a longest path in $O(n(rs\Delta)^{O(r^2s)})$, finishing the proof. □

We end up with the following result which summarizes the complexity of the algorithm presented to the *N-Fold IP* based on the proximity bound 3.11.

Theorem 3.13 (N-Fold complexity). *The N-Fold IP can be solved in time*

$$nt(rs\Delta)^{O(r^2s+s^2)} + O(nt \cdot log^2(nt) \cdot \varphi p(r)(s\Delta)^{O(s^2)})$$

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