Graver Basis

Francisco Javier Blázquez Martínez

Department of discrete optimization

November 2020

Integer Linear Programming

$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

$A \in \mathbb{Z}^{mxn}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, l and u lower and upper bounds for x

- NP-Hard

Integer Linear Programming

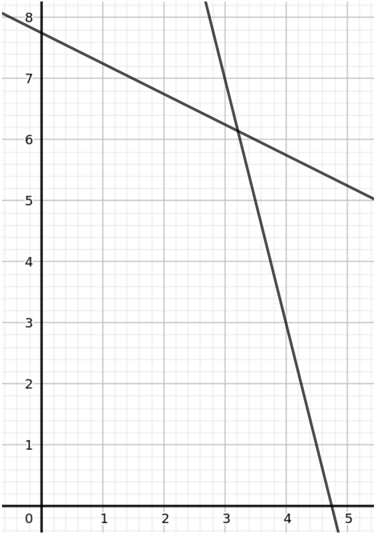
$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

$A \in \mathbb{Z}^{mxn}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, l and u lower and upper bounds for x

- NP-Hard
- Cutting plane methods
- Lattice-basis reduction
- Dynamic programming
- Graver basis techniques

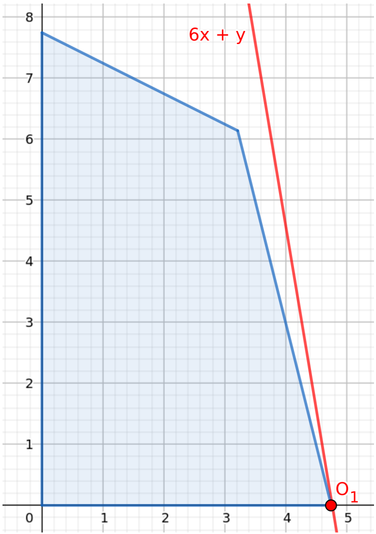
Integer Linear Programming

$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & x, y \in \mathbb{Z} \end{array} \right)$$



$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & \cancel{x, y \in \mathbb{Z}} \end{array} \right)$$

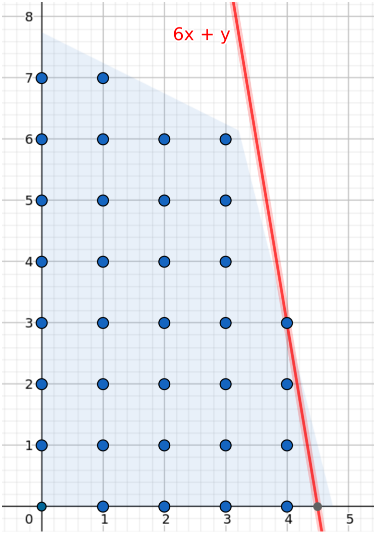
- Feasible region
- Extreme points
- Optimum in extreme points



Integer Linear Programming

$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & x, y \in \mathbb{Z} \end{array} \right)$$

- Feasible points
- Optimum can be far from the linear relaxation solution.

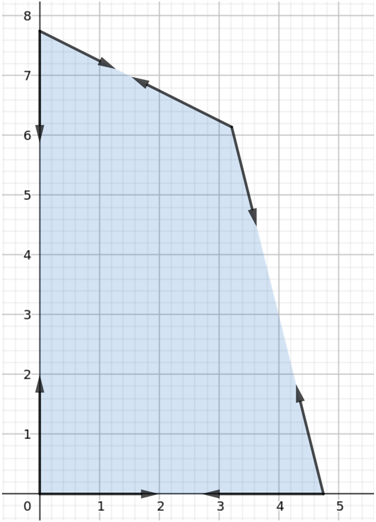


Integer Linear Programming

$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & \cancel{x, y \in \mathbb{Z}} \end{array} \right)$$

- Extreme directions:
  Allow Simplex fast optimality test and finding a direction of improvement in LP.

  Can we do the same for IP?



Graver Basis

- **Definition:** Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, ..., n\}$
- **Definition:** A vector $u \in ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $ker(A)$.

Graver Basis

- **Definition:** Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, ..., n\}$
- **Definition:** A vector $u \in ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $ker(A)$.

Graver Basis $\equiv Gr(A)$

The Graver Basis of a given matrix A is defined as the set of integral indecomposable elements in the kernel of A.

(Initially defined as *universal integral test set* in [Graver 1975])

Graver Basis properties

- **Spanning:** Every integral element in ker(A) can be expressed as positive integral linear combination of elements in $Gr(A)$.
- **Optimality:** Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in Gr(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$

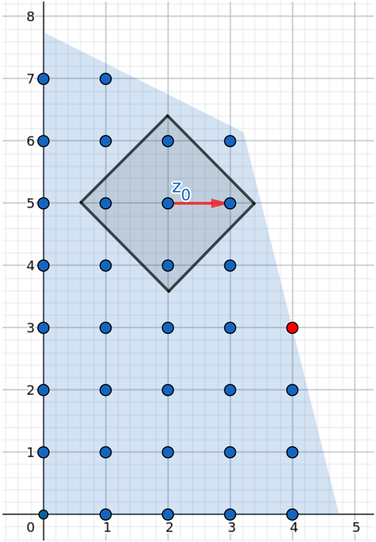
Graver Basis properties

- **Spanning:** Every integral element in ker(A) can be expressed as positive integral linear combination of elements in $Gr(A)$.
- **Optimality:** Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in Gr(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$
- **Bounds:** Given $A \in \mathbb{Z}^{mxn}$ and Δ an upper bound for the absolute value of each component of A, for every $g \in Gr(A)$:
  - $||g||_1 \leq m^{m/2}\Delta^m \cdot (n-m)$ [Onn 2010]
  - $||g||_1 \leq (2m\Delta + 1)^m$ [Eisenbrand,Hunkenschröder,Klein 2018]

Graver Basis properties

$$(P1) \equiv \left(\begin{array}{rl} & \max\ 6x + y \\ s.t: & 4x + y \leq 19 \\ & x + 2y \leq 31 \\ & x, y \geq 0 \\ & x, y \in \mathbb{Z} \end{array} \right)$$

- If not optimal, an element in Graver basis is an improvement direction.
- If Graver basis bounded, we can restrict our improvement direction search.



Augmentation algorithm

General IP algorithm using Graver basis norm bound

1. From a feasible solution z_i
2. Find g^* optimum for the sub-problem:
   $max\{c^t g : Ag = 0, l - z_i \leq g \leq u - z_i, g \in \mathbb{Z}^n, ||g||_1 \leq ||Gr(A)||\}$
   - $g^* = 0 \implies z_i$ optimal solution.
   - $g^* \neq 0 \implies g^*$ improvement direction, loop back to 1 with $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

[Hemmecke, Onn, Romanchuk 2013]

N-Fold, a success example

A N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{rxt}, B_i \in \mathbb{Z}^{sxt}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

N-Fold, a success example

A N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{rxt}$, $B_i \in \mathbb{Z}^{sxt}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

N-Fold Graver basis bound

For all $g \in Gr(N)$ $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ where $L_B = (2s\Delta + 1)^s$

N-Fold, a success example

A N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{rxt}$, $B_i \in \mathbb{Z}^{sxt}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

N-Fold Graver basis bound

For all $g \in Gr(N)$ $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ where $L_B = (2s\Delta + 1)^s$

N-Fold augmentation algorithm complexity

The N-Fold IP can be solved in time $n^2 t^2 \varphi log^2 nt \cdot (rs\Delta)^{O(r^2 s + rs^2)} + LP$

[Eisenbrand, Hunkenschröder, Klein 2018]

N-Fold, improving even more

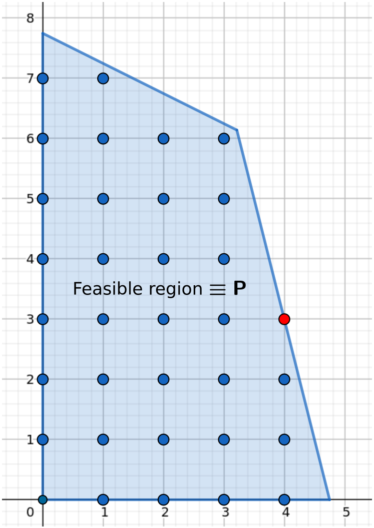


Figure 1: Linear Relaxation (LR)

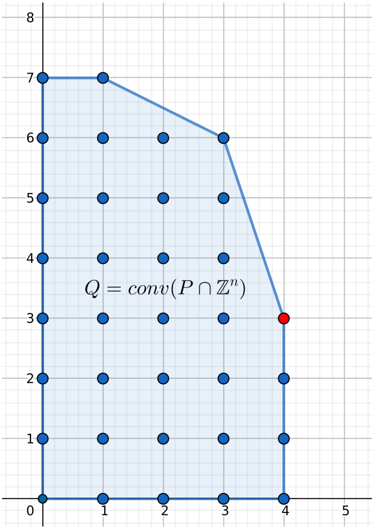


Figure 2: Restricted LR

N-Fold, improving even more

N-Fold proximity to restricted LR

Let x^* be an optimal vertex solution of a N-Fold IP restricted LR, then there exists an optimal solution for the N-Fold IP verifying:

$$||z^* - x^*|| \leq (rs\Delta)^{O(rs)}$$

N-Fold, improving even more

N-Fold proximity to restricted LR

Let x^* be an optimal vertex solution of a N-Fold IP restricted LR, then there exists an optimal solution for the N-Fold IP verifying:

$$||z^* - x^*|| \leq (rs\Delta)^{O(rs)}$$

N-Fold complexity

The N-Fold IP can be solved in time $nt(rs\Delta)^{O(r^2 s + s^2)} + LP$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

References

[1] Jack E. Graver. "On the foundations of linear and integer linear programming I". In: Syracuse University, New York, U.S.A., 1975.

[2] Kim-Manuel Klein Friedrich Eisenbrand Christoph Hunkenschröder. "Faster Algorithms for Integer Programs with Block Structure". In: École polytechnique fédérale de Lausanne, Switzerland, 2018.

[3] Robert Weismantel Jana Cslovjecsek Friedrich Eisenbrand. "N-fold integer programming via LP rounding". In: École polytechnique fédérale de Lausanne, Switzerland, 2020.