

Graver basis

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Contents

Acknowledgements	i
1 Introduction	1
2 Graver Basis	2
3 N-Fold	4
4 N-Fold augmentation algorithm	5
5 N-Fold via LP rounding	6
A Graver basis computation with 4ti2	8
B IP resolution with Graver Basis example	9
Bibliography	10

Chapter 1

Introduction

The underlying problem we try to solve is the classical *Integer program* (IP) that we formulate in the following way:

$$(IP) \equiv \max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

$A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{Z}^n, l$ and u lower and upper bounds for x

Despite its simplicity, it's well known the importance of IP. Several problems in diverse fields of the mathematics and algorithms admit an IP equivalent formulation (examples?). Unfortunately, IP is NP-Complete. This means that there is no efficient (polynomial) algorithm for solving IP in the general case (say general techniques and complexity?) and, therefore, knowing their importance and the lack of a general efficient algorithm for their resolution, there has been a great interest in restricted formulations of the problem and their resolution techniques.

In this project we present the concept of **Graver Basis** and its applications for solving the IP with, of course, the theoretical justification of this based on its properties. We apply this to a concrete IP formulation, the N-Fold IP and prove that it leads to a polynomial and efficient algorithm for this case.

Chapter 2

Graver Basis

Definition 2.1. Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, \dots, n\}$.

Definition 2.2. A vector $u \in \ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $\ker(A)$.

Definition 2.3 (Graver basis). The Graver Basis of a given matrix $A \in \mathbb{Z}^{m \times n}$ is defined as the set of integral indecomposable elements in the kernel of A .
(Initially defined as *universal integral test set* in [Graver 1975])

Proposition 2.4. For every matrix A , $Gr(A)$ is a finite set.

Proposition 2.5. Every integral element in $\ker(A)$ can be expressed as positive integral linear combination of elements in $Gr(A)$.

Proposition 2.6. Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in Gr(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$

Proposition 2.7 (Graver basis bounds). Given $A \in \mathbb{Z}^{m \times n}$ and Δ an upper bound for the absolute value of each component of A , for every $g \in Gr(A)$:

- $\|g\|_1 \leq m^{m/2} \Delta^m \cdot (n - m)$ [Onn 2010]
- $\|g\|_1 \leq (2m\Delta + 1)^m$ [Eisenbrand, Hunkenschröder, Klein 2018]

Bases of augmentation algorithm

- If not optimal, an element in Graver basis is an improvement direction.

- If Graver basis bounded, we can restrict our improvement direction search.

General IP algorithm using Graver basis norm bound

1. From a feasible solution z_i
2. Find g^* optimum for the sub-problem:

$$\max\{c^t g : Ag = 0, l - z_i \leq g \leq u - z_i, g \in \mathbb{Z}^n, \|g\|_1 \leq \|Gr(A)\|\}$$

- $g^* = 0 \implies z_i$ optimal solution.
- $g^* \neq 0 \implies g^*$ improvement direction, loop back to 1 with $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest λ respecting the bounds.

[Hemmecke, Onn, Romanchuk 2013]

Chapter 3

N-Fold

A generalized N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{xt}, B_i \in \mathbb{Z}^{st}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

Chapter 4

N-Fold augmentation algorithm

- **Steinitz Lemma**

Let v_1, \dots, v_n be vectors with $\|v_i\| \leq \Delta$ for $i = 1, \dots, n$. If $\sum_{i=1}^n v_i = 0$, then there is a reordering $\pi \in S_n$ such that for each $k \in \{1, \dots, n\}$ the partial sum $p_k := \sum_{i=1}^k v_{\pi(i)}$ satisfies $\|p_k\| \leq n\Delta$.

It's possible (using Steinitz Lemma) to obtain a much tighter bound for the norm of the elements in the Graver basis than the ones mentioned before. This implies a restriction in the space of search for the improvement direction in the augmentation algorithm making it much faster.

- **N-Fold Graver basis bound**

For all $g \in Gr(N)$ $\|g\|_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ where $L_B = (2s\Delta + 1)^s$

- **N-Fold augmentation algorithm complexity**

The N-Fold IP can be solved in time $(nt)^2 \log^2(nt) \cdot \varphi(rs\Delta)^{O(r^2s+rs^2)} + LP$

[Eisenbrand, Hunkenschröder, Klein 2018]

Chapter 5

N-Fold via LP rounding

N-Fold resolution via RLR

- **N-Fold RLR complexity**

The N-Fold IP restricted linear relaxation problem can be solved in time

$$O(nt \cdot \log^2(nt) \cdot \varphi p(r)(s\Delta)^{O(s^2)})$$

- **N-Fold RLR to optimum complexity**

Given an optimal vertex of an N-Fold RLR, the N-Fold IP can be solved in time

$$O(nt \cdot (rs\Delta)^{O(r^2s+s^2)})$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

N-Fold from RLR to optimum

- **N-Fold proximity to RLR**

Let x^* be an optimal vertex solution of a N-Fold RLR, then there exists an optimal solution z^* for the N-Fold IP verifying:

$$\|z^* - x^*\|_1 \leq (rs\Delta)^{O(rs)}$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

Facts for N-Fold complexity

- $|S_l| \leq (rs\Delta)^{O(r^2s)}$
- $|V| + |E| \leq O(n(rs\Delta)^{O(r^2s)})$
- The edge IP can be computed in time $t((r+s)\Delta)^{O(r+s)^2}$
- Longest path problem in a acyclic digraph can be solved in linear time.

N-Fold complexity

- **N-Fold complexity**
The N-Fold IP can be solved in time $nt(rs\Delta)^{O(r^2s+s^2)} + RLR$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

Appendix A

Graver basis computation with 4ti2

Appendix B

IP resolution with Graver Basis example

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