# GRAVER BASIS

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Chair of discrete optimization

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*To my brother Joaquín, for showing me what mathematics were during dinners at home.*

*To my parents, for teaching me beyond the scope of mathematics.*

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# Chapter 1

# Introduction

The underlying problem we try to solve is the clasical *Integer program* (IP) that we formulate in the following way:

$$(IP) \equiv max\{c^t x : Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$$

$A \in \mathbb{Z}^{mxn}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, $l$ and $u$ lower and upper bounds for x

Despite its simplicity, it's well known the importance of IP. Several problems in diverse fields of the mathematics and algorithms admit an IP equivalent formulation (examples?). Unfortunately, IP is NP-Complete. This means that there is no efficient (polynomial) algorithm for solving IP in the general case (say general techniques and complexity?) and, therefore, knowing their importance and the lack of a general efficient algorithm for their resolution, there has been a great interest in restricted formulations of the problem and their resolution techniques.

In this project we present the concept of **Graver Basis** and its applications for solving the IP with, of course, the theoretical justification of this based on its properties. We apply this to a concrete IP formulation, the N-Fold IP and prove that it leads to a polynomial and efficient algorithm for this case.

# Chapter 2

# Graver Basis

**Definition 2.1.** Two vectors $u, v \in \mathbb{R}^n$ are said to be **sign compatible** if $u_i \cdot v_i \geq 0$ for all $i \in \{1, ..., n\}$.

**Definition 2.2.** A vector $u \in ker(A)$ is **indecomposable** if it is not the sum of two sign compatible and non zero elements in $ker(A)$.

**Definition 2.3 (Graver basis).** The Graver Basis of a given matrix $A \in \mathbb{Z}^{mxn}$ is defined as the set of integral indecomposable elements in the kernel of A.
(Initially defined as *universal integral test set* in [Graver 1975])

**Proposition 2.4.** *For every matrix A, $Gr(A)$ is a finite set.*

**Proposition 2.5.** *Every integral element in ker(A) can be expressed as positive integral linear combination of elements in $Gr(A)$.*

**Proposition 2.6.** *Given z in the feasible region of an IP, z is not optimum if and only if there exists $g \in Gr(A)$ s.t. $c^t g > 0$ and $l \leq z + g \leq u$*

**Proposition 2.7 (Graver basis bounds).** *Given $A \in \mathbb{Z}^{mxn}$ and $\Delta$ an upper bound for the absolute value of each component of A, for every $g \in Gr(A)$:*

- $||g||_1 \leq m^{m/2}\Delta^m \cdot (n - m)$ *[Onn 2010]*
- $||g||_1 \leq (2m\Delta + 1)^m$ *[Eisenbrand,Hunkenschröder,Klein 2018]*

**Bases of augmentation algorithm**

- If not optimal, an element in Graver basis is an improvement direction.

- If Graver basis bounded, we can restrict our improvement direction search.

**General IP algorithm using Graver basis norm bound**

1. From a feasible solution $z_i$
2. Find $g^*$ optimum for the sub-problem:
   $max\{c^t g : Ag = 0, l - z_i \leq g \leq u - z_i, g \in \mathbb{Z}^n, ||g||_1 \leq ||Gr(A)||\}$
   - $g^* = 0 \implies z_i$ optimal solution.
   - $g^* \neq 0 \implies g^*$ improvement direction, loop back to 1 with $z_{i+1} = z_i + \lambda \cdot g^*$ with the biggest $\lambda$ respecting the bounds.

[Hemmecke, Onn, Romanchuk 2013]

# Chapter 3

# N-Fold

A generalized N-Fold IP has constriction matrix A of the form ($A_i \in \mathbb{Z}^{rxt}$, $B_i \in \mathbb{Z}^{sxt}$):

$$N = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

# Chapter 4

# N-Fold augmentation algorithm

- **Steinitz Lemma**
  Let $v_1, ..., v_n$ be vectors with $||v_i|| \leq \Delta$ for $i = 1, ..., n$. If $\sum_{i=1}^{n} v_i = 0$, then there is a reordering $\pi \in S_n$ such that for each $k \in \{1, ..., n\}$ the partial sum $p_k := \sum_{i=1}^{k} v_{\pi(i)}$ satisfies $||p_k|| \leq n\Delta$.

It's possible (using Steinitz Lemma) to obtain a much tighter bound for the norm of the elements in the Graver basis than the ones mentioned before. This implies a restriction in the space of search for the improvement direction in the augmentation algorithm making it much faster.

- **N-Fold Graver basis bound**
  For all $g \in Gr(N)$ $||g||_1 \leq L_B(2r\Delta L_B + 1)^r =: L_A$ where $L_B = (2s\Delta + 1)^s$

- **N-Fold augmentation algorithm complexity**
  The N-Fold IP can be solved in time $(nt)^2 log^2(nt) \cdot \varphi(rs\Delta)^{O(r^2 s + rs^2)} + LP$

[Eisenbrand, Hunkenschröder, Klein 2018]

# Chapter 5

# N-Fold via LP rounding

**N-Fold resolution via RLR**

- **N-Fold RLR complexity**
  The N-Fold IP restricted linear relaxation problem can be solved in time

$$O(nt \cdot log^2(nt) \cdot \varphi p(r)(s\Delta)^{O(s^2)})$$

- **N-Fold RLR to optimum complexity**
  Given an optimal vertex of an N-Fold RLR, the N-Fold IP can be solved in time

$$O(nt \cdot (rs\Delta)^{O(r^2 s+s^2)})$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

**N-Fold from RLR to optimum**

- **N-Fold proximity to RLR**
  Let $x^*$ be an optimal vertex solution of a N-Fold RLR, then there exists an optimal solution $z^*$ for the N-Fold IP verifying:

$$||z^* - x^*||_1 \leq (rs\Delta)^{O(rs)}$$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

### Facts for N-Fold complexity

- $|S_l| \leq (rs\Delta)^{O(r^2 s)}$
- $|V| + |E| \leq O(n(rs\Delta)^{O(r^2 s)})$
- The edge IP can be computed in time $t((r+s)\Delta)^{O(r+s)^2}$
- Longest path problem in a acyclic digraph can be solved in linear time.

### N-Fold complexity

- **N-Fold complexity**
  The N-Fold IP can be solved in time $nt(rs\Delta)^{O(r^2 s + s^2)} + RLR$

[Cslovjecsek, Eisenbrand, Weismantel 2020]

# Appendix A

# Graver basis computation with 4ti2

# Appendix B

# IP resolution with Graver Basis example

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