CENG 223

Discrete Computational Structures

Fall 2018-2019

Homework 3 Answers

Question 1

Q1.1

$a_n = a_n^h + a_n^p$
$a_n^h = a_n - a_{n-1} = 0$
Let write the characteristic equation of the homogenous equation
$r - 1 = 0 \rightarrow r = 1$
$a_n^h = \beta 1^n$
$a_n^p = a_{n-1} + n^2$ Guess $a_n^p = n(Bn^2 + Cn + D)$ we multiply by n because homogeneous system contains the root 1^n.
$a_n - a_{n-1} = n^2$
$n(Bn^2 + Cn + D) - (n-1)(B(n-1)^2 + C(n-1) + D) = n^2$
$(Bn^3 + Cn^2 + Dn) - (B(n-1)^3 + C(n-1)^2 + D(n-1)) = n^2$ When we solve the equation we get
$B = \frac{1}{3}, C = \frac{1}{2}, D = \frac{1}{6}$
$a_n^p = n(n^2\frac{1}{3} + n\frac{1}{2} + \frac{1}{6})$
$a_n = a_n^h + a_n^p$
$a_n = \beta 1^n + n(n^2\frac{1}{3} + n\frac{1}{2} + \frac{1}{6})$
$a_1 = \beta + 1 = 1 \Rightarrow \beta = 0$
$a_n = n(n^2\frac{1}{3} + n\frac{1}{2} + \frac{1}{6})$

Q1.2

$a_n = a_n^h + a_n^p$
$a_n^h = a_n - 2a_{n-1} = 0$
Let write the characteristic equation of the homogenous equation
$r - 2 = 0 \rightarrow r = 2$
$a_n^h = \beta 2^n$
$a_n^p = 2a_{n-1} + 2^n$ Guess $a_n^p = An2^n$ we multiply by n because homogeneous system contains the root 2^n.
$An2^n - 2A(n-1)2^{n-1} = 2^n$ divide by 2^n two side the equation
$An - A(n-1) = 1 \rightarrow A = 1$ So
$a_n^p = n2^n$
$a_n = a_n^h + a_n^p$

$a_n = \beta 2^n + n2^n$ And $a_0 = 1$ is given So
$a_0 = \beta = 1$ and Finally
$a_n = 2^n + n2^n \to 2^n(n+1)$

Question 2

Let $f(n) = n^2 + 15n + 5$ and
$g(n) = 21n^2$
Use mathematical induction to show that $f(n) \leq g(n)$ for all n where n is a positive integer. Note: Answers that do not use mathematical induction will not be evaluated.

BASE CASE:
Showing it is true for $n = 1$
$f(1) = 1 + 15 + 5 = 21$ and $g(1) = 21$ so $f(1) = g(1)$.
Hence $f(n) \leq g(n)$ holds for $n = 1$.

INDUCTIVE HYPOTHESIS:
Assume it is true for $n = k$
$f(k) \leq g(k)$
$k^2 + 15k + 5 \leq 21k^2$

INDUCTION:
Proving that it is true for $n = k + 1$
$(k+1)^2 + 15(k+1) + 5 = k^2 + 2k + 1 + 15k + 15 + 5 = k^2 + 17k + 21$
$21(k+1)^2 = 21(k^2 + 2k + 1) = 21k^2 + 42k + 21$
From inductive hypothesis we know that $k^2 + 15k + 5 \leq 21k^2$ is true
we can say $k^2 + 15k + 5 + 2k + 16 \leq 21k^2 + 42k + 21$ because of $2k + 16 \leq 42k + 21$ where $n > 0$.
Hence by mathematical induction, $f(n) \leq g(n)$ for all $n > 0$.

Question 3

Question 4

More detailed explanations can be found in *Section 6.5*, Generalized Permutations and Combinations, of the book.

4.1

a. *A solution is given in Example 6 (p. 427) for the more general case of this problem. The reader is advised to check that solution. The solution given below is* ***NOT*** *the most gentle solution, yet it avoids the repetition.*
For each valid (i, j) tuple, the value of a is incremented by 2, and similarly the value of b is incremented by 1 for each valid (i, j, k) tuple. Validness conditions are:

1. each element of the tuple is an integer between $[1, n]$

2. the tuple is in non-increasing order

As such, we can pick any two elements from the closed interval $[1, n]$ and put them in a non-increasing order to acquire a valid 2-tuple corresponding to (i, j). Since the numbers picked up become valid after ordering them, the order in which they are picked is not relevant to us.
The number of choices for the first number is n, and for the second number, still n. So we can choose n^2 such numbers. However this counts some choices twice, i.e. picking $(1, 2)$ and $(2, 1)$ is the same for our purposes. Then maybe we can divide by 2! to disregard the order the numbers are picked in. This gives us $n^2/2$ many such tuples. There is a problem with this argument, can you spot it?
The problem is, when we divide by 2! we assume each such tuple comes in pairs. But when we consider numbers of the form (i, i) it is easy to see that we picked them only once to begin with. There are exactly n such tuples, and we have discounted $n/2$ of them. Adding this to $n^2/2$ that we have found before completes the number of valid (i, j)s:

$$\frac{n^2 + n}{2}$$

a is incremented twice for each such tuple, hence, $a = n^2 + n$ after the execution of the pseudocode.
By similar arguments:
\# of ways to pick 3 numbers: n^3
\# of ways to arrange 3 numbers: 3!
\# of under-counted 3-tuples of the form (i, j, j): $n^2/2$ *(the reader should verify)*
\# of under-counted 3-tuples of the form (i, i, i): $n/3$ *(the reader should verify)*
Arranging these terms, we have:

$$\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} = \frac{n^3 + 3n^2 + 2n}{6} = \frac{n(n+1)(n+2)}{6}$$

So b is incremented by this amount, and we have $b = \frac{n(n+1)(n+2)}{6}$.

b. We just equate a and b given above to get:

$$n(n + 1) = n(n + 1)(n + 2)/6 \to n + 2 = 6 \to n = 4$$

4.2

a. *The problem is of type 'distinguishable objects into distinguishable boxes'. See p.429, Example 8 of the textbook.*
Since we have 10 different fruits at hand, and each plate will have exactly 2 fruits, let's pick 6 fruits, counting the number of ways to order them. We have 10 choices for the first fruit, 9 for the second, ..., 5 for the sixth. This corresponds to (unsurprisingly) $P(10, 6) = 10!/4!$. Now, thinking of the first as selected for the first plate, and so on, we have to disregard the order of the fruits in a single plate. Fruits in each of the plates can be ordered in 2! different ways. Then the number we are looking for is:

$$\frac{10!}{4!2!2!2!}$$

b. Similar problem, with a non-symmetric nature.
Selecting 10 fruits out of 10, counting the number of ways to pick them: $P(10, 10)$. As before, think of the first as if it is served to the first plate, the second and third to the second plate, and so on. Then we have to discard the order of placement of this fruits in the plates. The solution

then becomes $\frac{10!}{1!2!3!4!}$. We are not done yet. The problem is now not symmetric, so it **does** matter which plate gets how many fruits. There are 4! ways to rearrange 4 distinguishable boxes. Hence, the solution is:

$$\frac{10!}{1!2!3!4!}4! = \frac{10!}{1!2!3!}$$

c. *Problem type: distinguishable objects into indistinguishable boxes. See p. 430, Example 10 of the textbook.*
Stirling numbers of the second kind will be used. We simply sum the appropriate Stirling numbers:

$$S(6,1) + S(6,2) + S(6,3) + S(6,4) = 1 + 31 + 90 + 65 = 187$$

The reader is advised to check this for an intuitive interpretation of the Stirling numbers of the second kind, or this for a more detailed discussion.

d. *Problem type: indistinguishable objects into distinguishable boxes. See p. 425, Theorem 2 of the textbook.*
Since the objects are indistinguishable we can convert this problem into the following:

$$x_1 + x_2 + x_3 + x_4 = n \qquad n \in \{0,1,2,3,4,5,6\},\ x_i \geq 0$$

Here, plates serves as distinct variables, as they are distinguishable, and each fruit serves as the number 1 since they are indistinguishable. In this vein, we sum the number of ways acquired for different ns:

$$\sum_{n=0}^{6} C(n+4-1,3) = 1 + 4 + 10 + 20 + 35 + 56 + 84 = 210$$

Answers disregarding $n = 0$ is also fine, for there is somewhat of an ambiguity in the word *distribute* in the way it is used.