Middle East Technical University Department of Computer Engineering

CENG 223

Discrete Computational Structures

Fall 2018-2019

Homework 2

Due date: November 21 2018, Wednesday, 23:55

Question 1

Let $f : A \to B$ and $g : B \to C$. Prove the following:

a) If $C_0 \subseteq C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.

b) If $g \circ f$ is injective, what can be said about the injectivity of f and g?

c) If $g \circ f$ is surjective, what can be said about the surjectivity of f and g?

Question 2

Let $\iota_C : C \to C$ such that $\iota_C(x) = x, \forall x \in C$ for a set C. Given $f : A \to B$, we say that a function $g : B \to A$ is a **left inverse for f** if $g \circ f = \iota_A$, and a function $h : B \to A$ is a **right inverse for f** if $f \circ h = \iota_B$.

a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

b) Can a function have more than one left inverse? What about right inverses?

c) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

Question 3

The product $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite.
We can define a function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to A$, where $A = \{(x, y) | y \leq x \wedge x, y \in \mathbb{Z}^+\}$, by the equation

$$f(x, y) = (x + y - 1, y)$$

Then we can construct a function $g : A \to \mathbb{Z}^+$ by

$$g(x, y) = \frac{1}{2}(x - 1)x + y.$$

Show that f and g defined above are bijections.

Question 4

a) $x \in \mathbb{R}$ is **algebraic** if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

b) A real number is said to be **transcendental** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable.

Question 5

If $n \ln n = \Theta(k)$, show that

$$n = \Theta\Big(\frac{k}{\ln k}\Big).$$

Question 6

We call a positive integer **perfect** if it equals the sum of its positive divisors other than itself.

a) Show that 6 and 28 are perfect.

b) Show that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is prime.

Question 7

a) Given $x \equiv c_1 \pmod{m}$ and $x \equiv c_2 \pmod{n}$ where c_1, c_2, m, n are integers with $m > 0, n > 0$ show that the solution x exists if and only if $\gcd(m, n) | c_1 - c_2$.

b) If the solution exists to the above system, show that it is unique in the interval $[\ 0,\ \mathrm{lcm}(m, n)\ )$

Regulations

1. You have to write your answers to the provided sections of the template answer file given.
2. **Late Submission:** Not allowed.
3. **Cheating: We have zero tolerance policy for cheating**. People involved in cheating will be punished according to the university regulations.
4. **Updates & Announces:** You must follow the newsgroup (news.ceng.metu.edu.tr) for discussions and possible updates.
5. **Evaluation:**Your latex file will be converted to pdf and evaluated by course assistants. The .tex file will be checked for plagiarism automatically using "black-box" technique and manually by assistants, so make sure to obey the specifications.

Submission

Submission will be done via COW. Download the given template answer file "hw2.tex". When you finish your exam upload the .tex file with the same name to COW.
Note: You cannot submit any other files. Don't forget to make sure your .tex file is successfully compiled in Inek machines using the command below.

```
$ pdflatex hw2.tex
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