TOBB ETÜ, BİLGİSAYAR MÜHENDİSLİĞİ BÖLÜMÜ

BİL 133 Combinatorics and Graph Theory

HOMEWORK 6 (30 Points)

Due Date: July 17, 2018

1 [2 Points] Same Sized Sets

We say that $|A| = |B|$ if and only if there is a bijection (one-to-one and onto function) from A to B.

Give two sets A and B such that $A \subsetneq B$, and $|A| = |B|$.

2 [4 Points] Another Induction Example

In this question, you will prove an identity you have probably encountered in calculus classes, by using mathematical induction.
Prove that for any real number $x > -1$, and any positive integer n; we have $(1+x)^n \geq 1 + nx$.

3 [4 Points] Making Use of Your Calculus Skills

Harmonic series ($H_n = \sum_{i=1}^{n} \frac{1}{i}$) pop up quite often in algorithm analysis so they pop up often in BIL 133 assignments as well. You are asked to prove an identity by using the integration techniques you learned in MAT 101.

Prove that $1 + \ln n \geq H_n \geq 1 + \ln(n+1) - \ln 2$ **for all positive integers** n**.**

4 [5 Points] Comparing hardness of Similar Looking Computational Problems

Let $\preceq$ be *is a restriction of* relation defined over the set of computational problems. We say that $p \preceq q$ if the problem p is a restriction(special case) of the problem q, or the problems p and q are equivalent. You are given a set of covering problems below. **List all pairs of the given problems related with respect to** $\preceq$. Justify your answer.

- *Set Cover:* Given a set of m items $U = \{1, 2, \ldots, m\}$ (called the universe), and a set $S = \{S_1, S_2, \ldots, S_n\}$ of n sets whose union equals the universe ($\bigcup_{i=1}^{n} S_i = U$), identify the smallest subset of S whose union equals the universe.
- *My Cover:* Given a set of m items $U = \{1, 2, \ldots, m\}$ (called the universe); and a set $S = \{S_1, S_2, \ldots, S_n\}$ of n sets such that $\bigcup_{i=1}^{n} S_i = U$, and each element of U is an element of *exactly* two sets of S. Identify the smallest subset of S whose union equals the universe.
- *Vertex Cover:* Given an undirected graph $G = (V, E)$, identify the smallest subset $S \subseteq V$ of vertices such that each edge $e \in E$ is incident upon at least one vertex of S.

5 [4 Points] Solving an Optimization Problem via Calls to the Decision Problem Algorithm

You want to solve the optimization problem version of the Traveling Salesman Problem. In this problem, you are given a distance matrix $D_{n \times n}$, where D_{ij} denotes the distance from city i to city j. You want to **find the distance of the shortest cycle** visiting all of the n cities.

You look at the distance matrix and see that there are exactly 1000 cities (i.e., $n = 1000$). Furthermore, you notice that all the entries of D are positive integers less than or equal to 1000.

You also have an algorithm that solves the decision version of this problem at your disposal. To illustrate, the algorithm takes a distance matrix $D_{n \times n}$ and an additional integer t as parameters, and returns true if and only if there is a cycle visiting all of the n cities with a total distance of at most t.

You want to solve the optimization problem given above and the only operation you are allowed is calling the algorithm for the decision version of the problem. Describe how you could solve this problem? How many calls you need to make to the algorithm that solves the decision version of the problem?

6 [11 Points] Shooting the Damned Rabbit

In this question, we investigate a game played by a hunter and a rabbit on the integer line where the rabbit hops along the integer line and the hunter tries to shoot the rabbit. The hunter and the rabbit alternate turns, where the hunter shoots, then the rabbit jumps, (then the hunter shoots, then the rabbits jumps, and so on...) until the hunter hits the rabbit. Rules are as following:

- The rabbit starts at a position $s \in \mathbb{Z}$ on the number line. Then, the hunter takes its first shot.
- Every time the hunter takes its shot, then the rabbit leaps $z \in \mathbb{Z}$ spaces, i.e., z may be positive or negative. If z is positive, then the rabbit jumps to the right. If z is negative, then the rabbit jumps to the left. Note that it always jumps the same amount in the same direction.
- The tricky part is that the values of s and z are <u>unknown</u> to the hunter!

  EXAMPLE: The rabbit may start at 6 and always jump +4 spaces so that its location changes as $6, 10, 14\ldots$ However, the hunter cannot observe this locations. The hunter only knows that the rabbit moves according to above rules.

Imagine that you are the shooter in this game. It may seem as if you are helpless at first. After all you never know where the rabbit is during the game. *So, how can you shoot it without knowing where it is?*
Indeed, this is what we ask in the question: Devise an algorithm to determine where you shoot at each turn so that you are guaranteed to hit the rabbit eventually. It might take an extraordinary long time for your algorithm to hit the rabbit, but it has to guarantee that it will in a <u>finite</u> amount of time.

HINT: Recall the proof we give in the lectures to show that the set of rational numbers are countable.