Student Information

Full Name : Mustafa Ozan ALPAY
Id Number : 2309615

Answer 1

1. By writing the recurrences from 1 to n we get,

$$a_1 = 1$$

$$a_2 = a_1 + 2^2$$
$$a_3 = a_2 + 3^2$$
$$...$$
$$a_{n-1} = a_{n-2} + (n-1)^2$$
$$a_n = a_{n-1} + n^2$$

By summing the equations, we get,

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n-1} a_j + \sum_{k=1}^{n} k^2$$

By rewriting the equation,

$$a_n + \sum_{i=1}^{n-1} a_i = \sum_{j=1}^{n-1} a_j + \sum_{k=1}^{n} k^2$$

$$a_n = \sum_{k=1}^{n} k^2$$

By using the formula

$$\sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

we get a_n as,

$$a_n = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

2. To solve this recurrence, first we need to solve the associated linear homogenous equation $a_n = 2a_{n-1}$. The solutions for this equation are $a_n^{(h)} = \alpha 2^n$ where α is a constant. Since $F(n) = 2^n$ a reasonable trial solution is $a_n^{(p)} = C \cdot n \cdot 2^n$ where C is a constant. Substituting this into the recurrence solution, we get

$$C \cdot n \cdot 2^n = 2 \cdot C \cdot (n-1) \cdot 2^{n-1} + 2^n$$

Factoring out 2^n, we get

$$C \cdot n = C \cdot (n-1) + 1$$

The solution for this equation is $C = 1$, therefore $a_n^{(p)} = n \cdot 2^n$ for our trial solution.
By Theorem 5 of the section *8.2. Solving Linear Recurrence Relations* of the book, all solutions are of the form

$$a_n = 2 \cdot \alpha \cdot 2^{n-1} + n \cdot 2^n$$

Since we already know that $a_0 = 1$, by substituting $n = 0$ we can find α.

$$a_0 = 2 \cdot \alpha \cdot 2^{0-1} + 0 \cdot 2^0$$

$$a_0 = \alpha = 1$$

Therefore our recurrence is

$$a_n = 2^n \cdot (1+n)$$

Answer 2

1. **Basis Step**
$f(1) \leq g(1)$ holds, because

$$f(1) = 1^2 + 15 \cdot 1 + 5 = 21$$
$$g(1) = 21 \cdot 1^2 = 21$$

therefore $f(1) \leq g(1)$.

2. **Inductive Step**
Let's assume that $f(k) \leq g(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$k^2 + 15k + 5 \leq 21k^2$$

Under this assumption, we must show that it holds for $k+1$ as well, namely

$$(k+1)^2 + 15(k+1) + 5 \leq 21(k+1)^2$$

By expanding the $(k+1)^2$ parts, we get

$$k^2 + 2k + 1 + 15k + 15 + 5 \leq 21k^2 + 42k + 21$$

By arranging the inequality,

$$k^2 + 15k + 5 \leq 21k^2 + (40k + 5)$$

Since k is a positive integer, adding $40k + 5$ to the right hand side of the inequality $k^2 + 15k + 5 \leq 21k^2$ would not affect its truth value.

3. Since we have shown that the inequality holds for 1, and if it holds for an arbitrary integer k it also holds for $k+1$, we can conclude that $f(n) \leq g(n)$ is a correct statement by mathematical induction.

Answer 3

Answer 4

1. **a)** Since the initial value of a is 0, and since a is incremented 2 with every *for* loop with the iterator j, and since the *for* loop with the iterator j is traversed with a sequence of integers i, j such that

$$1 \leq j \leq i \leq n$$

we can say that the number of such sequences of integers is the number of ways to choose 2 integers with repetition allowed. Thus, from Theorem 2 of the section *6.5. Generalized Permutations and Combinations* of the book, we can say that it follows the following equation:

$$a = 2 \cdot C(n+1, 2)$$

Similarly, since the initial value of b is 0, and since b is incremented 1 with every *for* loop with the iterator k, and since the *for* loop with the iterator k is traversed with a sequence of integers i, j, k such that

$$1 \leq k \leq j \leq i \leq n$$

we can say that the number of such sequences of integers is the number of ways to choose 3 integers with repetition allowed. Thus, from Theorem 2 of the section *6.5. Generalized Permutations and Combinations* of the book, we can say that it follows the following equation:

$$b = C(n+2, 3)$$

By rearranging the equations, we can get the following values;

$$a = n \cdot (n+1)$$

$$b = \frac{n \cdot (n+1) \cdot (n+2)}{6}$$

b) If $a = b$ after the execution of the pseudocode, we can use the values that we obtained from *part a* to find the value of n.

$$a = b$$

$$n \cdot (n+1) = \frac{n \cdot (n+1) \cdot (n+2)}{6}$$

The solution has 3 distinct values, which are $\{-1, 0, 4\}$.
Since the loop starts from $i = 1$, the value of n must be greater than or equal to 1. Therefore,

$$n = 4$$

2. **a)** Distributing 10 different fruits into 3 distinguishable plates with each plate having exactly 2 fruits is of problem type *Distinguishable Objects and Distinguishable Boxes.* By using the principles regarding this type from the textbook, we can say that there are

$$C(10,2) \cdot C(8,2) \cdot C(6,2)$$

which gives us 18900 ways to distribute the given items.

b) Distributing 10 different fruits into 4 distinguishable plates while the plates having 1,2,3,4 fruits respectively is of problem type *Distinguishable Objects and Distinguishable Boxes.* By using *Theorem 4* from the section *6.5. Generalized Permutations and Combinations* of the textbook, we can say that there are exactly

$$\frac{10!}{1! \cdot 2! \cdot 3! \cdot 4!}$$

which gives us 12600 ways to distribute the given items.

c) Distributing 6 different fruits into 4 indistinguishable plates while all of the fruits being distributed is of problem type *Distinguishable Objects and Indistinguishable Boxes.* By using formula below of the section *6.5. Generalized Permutations and Combinations* of the textbook, the total number of ways of distributing n distinguishable objects into k indistinguishable boxes equals to

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

where $S(n,j)$ are called the *Stirling numbers of the second kind.* Therefore, we can say that there are exactly

$$\sum_{j=1}^{4} S(6,j) = \sum_{j=1}^{4} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^6$$

which gives us 187 ways to distribute the given items.

d) Distributing 6 indistinguishable fruits into 4 distinguishable plates while without having the requirement of all the fruits being distributed is of problem type *Indistinguishable Objects and Distinguishable Boxes.* By using formula from the section *6.5. Generalized Permutations and Combinations* of the textbook, there are $C(n+k-1,k)$ ways of distributing k objects into n boxes. To find out the total number of ways to distribute the dragon fruits into the boxes, we must consider the following situations;

- Distributing 6 dragon fruits into 4 plates ($k = 6, n = 4$)
- Distributing 5 dragon fruits into 4 plates ($k = 5, n = 4$)
- Distributing 4 dragon fruits into 4 plates ($k = 4, n = 4$)
- Distributing 3 dragon fruits into 4 plates ($k = 3, n = 4$)
- Distributing 2 dragon fruits into 4 plates ($k = 2, n = 4$)

- Distributing 1 dragon fruits into 4 plates ($k = 1, n = 4$)
- Distributing 0 dragon fruits into 4 plates ($k = 0, n = 4$)

Therefore the total number of ways to distribute the given items can be formulated as

$$C(9, 6) + C(8, 5) + C(7, 4) + C(6, 3) + C(5, 2) + C(4, 1) + C(3, 0)$$

which sums to 210 different ways of distribution.