Student Information

Full Name : Ali Doğan
Id Number : 2237261

Answer 1

a

We can denote the problem as follows :

$$(1 + x^2 + x^4) \times (x^4 + x^5 + x^6 + x^7 + x^8 + x^9) \times (x + x^3 + x^5)$$

n in x^n, represents the number of candies to be selected from that group. Therefore the coefficient of x^{10} is the answer such that $x^g \times x^r \times x^b = x^{10}$ where g,r and b represents number of green, red and blue candies to be selected.
The first one is the green candies, second one is red candies, and the last one is blue candies. Note that although the constraint says blue candies are in odd numbers, we can not take more than 6 blue candies since red candies should be more than three. Also note that we chould choose ten many candies which is even and in the constraint number of green's should even and number of blue's should be odd. In this case, we can observe that choosing red candies in even number is not possible. So the equation could be simplified as :

$$(1 + x^2 + x^4) \times (x^5 + x^7 + x^9) \times (x + x^3 + x^5)$$

Total solutions are :

$$x^0x^5x^5, x^0x^7x^3, x^0x^9x^1, x^2x^5x^3, x^2x^7x^1, x^4x^5x^1$$

Thus, there are six ways to select 10 candies in such constraints.

b

This time, we can not take more than five red candies as well as the other candies.Thus, the problem changes slightly as follows :

$$(1 + x^2 + x^4) \times (x^4 + x^5) \times (x + x^3 + x^5)$$

again the functions represents Grenn , Red , and Blue candies respectively. Also we can find the solution by eliminating the unwanted possibilities from a) such that remainings will be the ways to select 10 candies without taking any candy more than 5 times :

$$x^0x^5x^5, x^2x^5x^3, x^4x^5x^1$$

So, there are three ways to select 10 candies in such constraints.

c

First, we can simplify F(X) as follows :

$$F(x) = \frac{7x^2}{(1-2x)(1+3x)} = \frac{Ax+B}{1-2x} + \frac{Cx+D}{1+3x} \quad \textit{Where A,B,C,D are some integers}$$

After some algebraic operations, without changing F(x) we can find $A = \frac{7}{5}, \quad C = \frac{-7}{5}, \quad B = 0, \quad D = 0$ So,

$$F(x) = \frac{7}{5}\frac{x}{(1-2x)} - \frac{7}{5}\frac{x}{(1+3x)}$$

Using the property $\frac{1}{1-ax} = \sum_{n=0}^{\infty}(a^n x^n)$:

$$\frac{7}{5}\frac{x}{(1-2x)} = \frac{7x}{5}\sum_{n=0}^{\infty}(2^n x^n) \quad (1)$$

$$\frac{-7}{5}\frac{x}{(1+3x)} = \frac{-7x}{5}\sum_{n=0}^{\infty}((-3)^n x^n) \quad (2)$$

$$\frac{7x}{5}\sum_{n=0}^{\infty}(2^n x^n) = \frac{7}{5}\sum_{n=1}^{\infty}(2^{n-1} x^n) \quad (1) \quad \textit{(shifting)}$$

$$\frac{-7x}{5}\sum_{n=0}^{\infty}((-3)^n x^n) = \frac{-7}{5}\sum_{n=1}^{\infty}((-3)^{n-1} x^n) \quad (2) \quad \textit{(shifting)}$$

Thus,

$$F(x) = \frac{7}{5}\sum_{n=1}^{\infty}(2^{n-1} - (-3)^{n-1})x^n$$

d

Let $A(x)$ be a generating function such that $A(x) = \sum_{n=0}^{\infty}(s_n x^n)$. If we can manage to find A(x) in a closed form, we can derive it to the sequence form so that we can find s_n. For the sake of simplicity, we can use $s_0 = 1$ as another initial condition with s_1.

$$s_n = 8s_{n-1} + 10^{n-1}$$

$$\implies \sum_{n=1}^{\infty} s_n x^n = 8\sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\implies \sum_{n=0}^{\infty} s_n x^n - s_0 = 8\sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\implies A(x) - s_0 = 8\sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\implies A(x) - s_0 = 8x\sum_{n=0}^{\infty} s_n x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\implies A(x) - s_0 = 8xA(x) + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\implies A(x) - s_0 = 8xA(x) + \sum_{n=0}^{\infty} 10^{n}x^{n+1}$$

$$\implies A(x) - s_0 = 8xA(x) + x\sum_{n=0}^{\infty} 10^{n}x^{n}$$

$$\implies A(x) - 1 = 8xA(x) + x\frac{1}{1-10x}$$

$$\implies A(x) - 8xA(x) = 1 + x\frac{1}{1-10x}$$

$$\implies A(x)(1-8x) = \frac{1-9x}{1-10x}$$

$$\implies A(x) = \frac{1-9x}{(1-10x)(1-8x)}$$

We can simplify A(x) so that we can use some useful generating functions to solve A(x) :

$$\frac{1-9x}{(1-10x)(1-8x)} = \frac{C}{1-8x} + \frac{B}{1-10x}$$

$$C + B = 1 \quad and \quad -10C - 8B = -9$$

$$\implies C = \frac{1}{2}, B = \frac{1}{2}$$

$$\implies A(x) = \frac{1}{2}(\frac{1}{1-8x} + \frac{1}{1-10x}) = \frac{1}{2}(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n)$$

$$A(x) = \sum_{n=0}^{\infty} \frac{(8^n + 10^n)}{2} x^n$$

Thus $s_n = \frac{8^n+10^n}{2}$

Answer 2

a

Take any $b \in A_k$ such that $k|b$. Since $m|k$, from the transitive property of '|' relation, $m|k \wedge k|b \implies m|b$, thus $\forall b \in A_k \implies b \in A_m$.
Hence, $A_k \subseteq A_m$.

b

From the fundamental theorem of arithmetic, we can represent any integer greater than 1 as a product of prime numbers. Since any composite number can be written as a product of two or more prime numbers, instead of looking all the numbers from 2 to n-1 and finding the numbers who are divisible by that number, we can look only prime numbers so that we can find numbers that are divisible by smaller numbers, numbers which are prime. Part a) actually helps to understand this part if we can observe that instead of any A_k, we can look A_p where p is a prime number of k since $p|k$. Since from the definition of A_m , we can guarantee that the numbers in the set A_m can be writen as a product of at least two prime numbers because $m \geq 2$ divides a greater number, hence that number is composite number. Therefore, we can conclude that :

$$\bigcup_{i=2}^{n-1} A_i = \bigcup_{primes p<n} A_p$$

Also from the trial division, we know every composite number has a prime divisor less than its square root, so we can conclude that it is meaningless to look prime numbers greater than $\sqrt{n}$:

$$\bigcup_{i=2}^{n-1} A_i = \bigcup_{primes p\leq\sqrt{n}^1} A_p$$

Summary : From fundamental theorem of arithmetic, we know every composite number can be written as a product of two or more prime numbers. Also from trial division, every composite number has a prime number less than or equal to its square root. Morever, from part a), we can say that $A_m \subseteq A_p$ where p is a prime number which divides m which is also less than or equal to square root of m. Thus, without adding any prime number we improved the right hand side of equation (1).

c

We can define A_m as follows:

$$A_m = \{x| \quad m|x, x \in \{m+1, m+2, ..., n\}\} \quad \forall m \geq 2$$

It means that A_m consists all the numbers divisible by m starting from m+1 to n where m is greater than or equal to 2 . From the property of modulo arithmetic, $m|x$ means $\exists q \in N, \quad x = mq$. Also from the constraint of A_m we know that $x \leq n$ and $q \neq 1$ since $m \neq x$. Thus q can be between 2 and $\lfloor \frac{n}{m} \rfloor$ (integer division). So we can modify A_m as :

$$A_m = \{q_i m \quad | \quad i \in \{2, 3, ...\lfloor n/m \rfloor\}\} \quad \forall m \geq 2$$

Then, the number of elements in A_m is :

$$|A_m| = \lfloor n/m \rfloor - 1 \quad \forall m \geq 2$$

d

Recall the definition of A_m :

$$A_m = \{x| \quad m|x, x \in \{m+1, m+2, ..., n\}\} \quad \forall m \geq 2$$

Note that the lower bound of the set A_m is $m+1$.
$A_a \cap A_b$ means all the numbers divisible by a and also by b. If the numbers are relatively prime, then the element in this intersection should be divisible by ab. The lower bound of the set $A_a \cap A_b$ is $max(a,b)+1$ (also note that $ab > max(a,b)+1$ since they are relatively prime) . However, the set A_{ab} does not contain the element $a \times b$, because its lower bound is $ab+1$. Thus, except the element ab, all other elements are the same.
Hence, $(A_a \cap A_b) - A_{ab} = ab$

e

From part d) we observed that any set A_m, A_n with the constraint m and n are relatively prime numbers, $A_m \cap A_n$ is equal to $A_{mn} + \{mn\}$. The reason beyond this is because of the definition of A_m, we can not take m in the set. We can expand this idea with more components. $\bigcap\limits_{p \in P} A_p$ means the intersection of the set A of all prime numbers in the set P. This intersection contains all the numbers divisible by all the prime numbers in the set P. Thus, all the elements in $\bigcap\limits_{p \in P} A_p$ divisible by $p_1 \times p_2 \times ......p_n$ where $p_i \in P, i = 1, 2, 3, ..., n$. Let the number p' represents the multiple of all prime numbers in the set P such that , $p' = p_1 \times p_2 \times ......p_n$. Thus except p' itself, every other elements in the set $\bigcap\limits_{p \in P} A_p$ is in the set $A_{p'}$. Thus,

$$\bigcap_{p \in P} A_p = A_{p'} + \{p'\}$$

$$\implies |\bigcap_{p \in P} A_p| = |A_{p'}| + 1$$

From part c),

$$|A_{p'}| = \lfloor n/p' \rfloor - 1$$

Thus $|\bigcap\limits_{p \in P} A_p| = \lfloor n/p' \rfloor$ where $p' = p_1 \times p_2 \times ......p_n$, $p_i \in P, i = 1, 2, ...n$.

f

From part b) :

$$C_{45} = \bigcup_{primes \quad p \leq \sqrt{45}} A_p$$

$$\implies \quad C_{45} = A_2 \cup A_3 \cup A_5$$

Thus,

$$|C_{45}| = |A_2 \cup A_3 \cup A_5|$$

By Inclusion-Exclusion principle,

$$|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$$

$$|C_{45}| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$$

g

With the help of part c),d) and e) , we can find $|A_2|, |A_3|, |A_5|, |A_2 \cap A_3|, \ldots.., |A_2 \cap A_3 \cap A_5|$:

$$|A_2| = \lfloor 45/2 \rfloor - 1 = 21$$

$$|A_3| = \lfloor 45/3 \rfloor - 1 = 14$$

$$|A_5| = \lfloor 45/5 \rfloor - 1 = 8$$

$$|A_2 \cap A_3| = |A_6| + 1 = \lfloor 45/6 \rfloor = 7$$

$$|A_2 \cap A_5| = |A_{10}| + 1 = \lfloor 45/10 \rfloor = 4$$

$$|A_3 \cap A_5| = |A_{15}| + 1 = \lfloor 45/15 \rfloor = 3$$

$$|A_2 \cap A_3 \cap A_5| = |A_{30}| + 1 = \lfloor 45/30 \rfloor = 1$$

Then number of composite numbers up to 45 are :

$$|C_{45}| = 21 + 14 + 8 - 7 - 4 - 3 + 1 = 30$$

C_{45} denotes the composite numbers starting from 2 to 45. If we include the number 1, then the number of nonprime numbers up to 45 are 31. By inclusion-exclusion principle, There are $45 - 31 = 14$ many prime numbers in between $(1, 45]$.

Answer 3

a

First, we can write the definition of transitive suitable for Z^2:
Def:*Relation $\ll$ is called transitive if whenever (a,b)$\ll$(c,d) and (c,d)$\ll$(e,f), then (a,b)$\ll$(e,f), for all (a,b),(c,d),(e,f) $\in Z^2$*
PROOF: Assume (a,b) $\ll$ (c,d) and (c,d) $\ll$ (e,f). Then, from the definition of this relation there are four possibilities:
1) a < c and c < e :
We can see that a < c < e, thus a < e , so from the definition $(a, b) \ll (e, f)$

2) a < c and c = e and d$\leq$f :
$c = e \implies a < e$, Thus $(a, b) \ll (e, f)$

3) $a = c, b \leq d, c = e, d \leq f$:
$(a = c = e) \wedge (b \leq d) \wedge (d \leq f) \implies (a = e \wedge b \leq f)$, Thus $(a, b) \ll (e, f)$

4) $a = c, b \leq d, c < e$:
$(a = c) \wedge (c < e) \implies a < e$, Thus $(a, b) \ll (e, f)$.

For all possibilities, we used the transitive property of "$<$" or "$\leq$" relations.

b

We can show the relation is equivalence relation by proving it is reflexive, symmetric and transitive. Since $f(x) = f(x)$ for all x's in the domain, the relation is reflexive. Next, suppose that $\exists k, \quad f(x) = g(x)$, for every $x \geq k$, then $g(x) = f(x)$ with the same k. Thus, the relation is symmetric. Finally, suppose that $\exists k, \exists l \; f(x) = g(x) \quad \forall x \geq k$ and $g(x) = h(x) \quad \forall x \geq l$. Then $f(x) = h(x), \quad \forall x \geq max(k, l)$. Hence, the relation is transitive.