Middle East Technical University Department of Computer Engineering

CENG 223

Discrete Computational Structures

Fall 2018-2019

Homework 3

Due date: December 4 2018, Tuesday, 23:55

Question 1

1.1

Solve the recurrence:

$$a_n = a_{n-1} + n^2, \quad a_1 = 1$$

1.2

Solve the recurrence:

$$a_n = 2a_{n-1} + 2^n, \quad a_0 = 1$$

Question 2

Let $f(n) = n^2 + 15n + 5$ and
$g(n) = 21n^2$
Use mathematical induction to show that $f(n) \leq g(n)$ for all n where n is a positive integer. Note: Answers that do not use mathematical induction will not be evaluated.

Question 3

For the following questions, you may only use recursive definitions and structural induction in your answers. Otherwise, you will get a grade of zero.

3.1

Given the statement

$$(p_n \wedge (p_{n-1} \wedge (p_{n-2} \wedge ... \wedge (p_2 \wedge (p_1))...))) \rightarrow q \vdash (p_n \rightarrow (p_{n-1} \rightarrow (p_{n-2} \rightarrow ...(p_2 \rightarrow (p_1 \rightarrow q))...)))$$

where $n \in \mathbb{N}^+$, q and for i in $[1..n]$ each p_i are **unique** valid atomic propositions in propositional logic, and $\vdash$ is the entailment relation used in natural deduction, answer the following questions.

a. Give recursive definitions of functions $\Phi : \mathbb{N}^+ \rightarrow \Sigma^*$ and $\Psi : \mathbb{N}^+ \rightarrow \Sigma^*$ such that $\Phi(i)$ yields strings of the form $(p_i \wedge (p_{i-1} \wedge (p_{i-2} \wedge ... \wedge (p_2 \wedge (p_1))...))) \rightarrow q$ that constitute the LHS of the $\vdash$ relation containing i atomic propositions p_i in order, and, likewise, $\Psi(i)$ generates strings of the form $(p_i \rightarrow (p_{i-1} \rightarrow (p_{i-2} \rightarrow ...(p_2 \rightarrow (p_1 \rightarrow q))...)))$ forming the RHS of the $\vdash$ relation. You can assume that the string alphabet $\Sigma = \{(,), \wedge, \rightarrow\} \cup \{p_i : i \in [1..n]\} \cup \{q\}$ and make strings using successive applications of concatenation operation.

b. Use structural induction to prove

$$\Phi(n) \vdash \Psi(n)$$

for $n \geq 1$. Clearly show all steps.

3.2

Using the definition of binary trees in Section 5.3 of the textbook, answer the following.

a. Recursively define the height of an arbitrary binary tree using the conventions that the height of the empty tree is -1 and the height of a binary tree with a single vertex is 0 as the base cases.

b. Define 223-tree as a binary tree with the property that for every vertex the absolute value of the difference of heights of its left subtree and right subtree is at most 2. Recursively define functions $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ that map a 223-tree of height h to the maximum and minimum number of vertices it can have, respectively.

c. Use structural induction to prove that the functions f and g correctly produce the maximum and minimum number of vertices for all 223-trees of varying heights.

(*Hint:* Use trees with smaller heights as the left and the right subtrees of a new binary tree which, in turn, has a greater height.)

Question 4

Explain your reasoning and show your work clearly for the following questions. Use permutations and combinations in your answers.

4.1

a. Determine the values of a and b in terms of n, after the execution of the pseudocode given below.

b. Determine n, if $a = b$ after the execution of the pseudocode.

```
a = 0;
b = 0;
for i = 1 to n do
    for j = 1 to i do
        a = a + 2;
        for k = 1 to j do
            b = b + 1;
        end
    end
end
```

4.2

How many ways are there to distribute

a. 10 different fruits into 3 distinguishable plates, if each plate will have exactly 2 fruits?

b. 10 different fruits into 4 distinguishable plates so that the plates have one, two, three and four objects in them, respectively.

c. 6 different fruits into 4 indistinguishable plates? (All of the fruits will be distributed.)

d. 6 indistinguishable dragon fruits into 4 distinguishable plates if not all of the dragon fruits have to be used?

Regulations

1. You have to write your answers to the provided sections of the template answer file given.
2. **Late Submission:** Not allowed.
3. **Cheating: We have zero tolerance policy for cheating**. People involved in cheating will be punished according to the university regulations.
4. **Updates & Announces:** You must follow the newsgroup (news.ceng.metu.edu.tr) for discussions and possible updates.
5. **Evaluation:**Your latex file will be converted to pdf and evaluated by course assistants. The .tex file will be checked for plagiarism automatically using "black-box" technique and manually by assistants, so make sure to obey the specifications.

Submission

Submission will be done via COW. Download the given template answer file "hw3.tex". When you finish your exam upload the .tex file with the same name to COW.
Note: You cannot submit any other files. Don't forget to make sure your .tex file is successfully compiled in Inek machines using the command below.

```
$ pdflatex hw3.tex
```