Student Information

Full Name : Mustafa Ozan ALPAY
Id Number : 2309615

Answer 1

a

We can say that

- the set of possible green candy counts are $\{0, 2, 4\}$
- the set of possible red candy counts are $\{4, 5, 6, 7, 8, 9, 10\}$
- the set of possible blue candy counts are $\{1, 3, 5, 7, 9\}$

The generating functions for the aforementioned possible selections are as follows;

- $(x^0 + x^2 + x^4)$
- $(x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$
- $(x^1 + x^3 + x^5 + x^7 + x^9)$

Since the total sum must be equal to 10, in the product of the generating functions, the coefficient of the term with x^{10} would give us the number of ways.

$$= (x^0 + x^2 + x^4) \cdot (x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}) \cdot (x^1 + x^3 + x^5 + x^7 + x^9)$$

$$= x^{23} + \cdots + 9x^{11} + 6x^{10} + 6x^9 + \cdots + x^5$$

Since the coefficient of the term with x^{10} is 6, there are 6 ways to select 10 candies in such ways.

b

We can say that

- the set of possible green candy counts are $\{0, 2, 4\}$
- the set of possible red candy counts are $\{4, 5\}$
- the set of possible blue candy counts are $\{1, 3, 5\}$

The generating functions for the aforementioned possible selections are as follows;

- $(x^0 + x^2 + x^4)$
- $(x^4 + x^5)$

- $(x^1 + x^3 + x^5)$

Since the total sum must be equal to 10, in the product of the generating functions, the coefficient of the term with x^{10} would give us the number of ways.

$$= (x^0 + x^2 + x^4) \cdot (x^4 + x^5) \cdot (x^1 + x^3 + x^5)$$

$$= x^{14} + x^{13} + 2x^{12} + 2x^{11} + 3x^{10} + 3x^9 + 2x^8 + 2x^7 + x^6 + x^5$$

Since the coefficient of the term with x^{10} is 3, there are 3 ways to select 10 candies in such ways.

c

Let us take an arbitrary $G(x)$ where $G(x) = \dfrac{1}{(1-2x)(1+3x)}$. By applying the partial fractions method, we can get the following,

$$G(x) = \frac{1}{5} \cdot \left(\frac{3}{1+3x} + \frac{2}{1-2x} \right)$$

Since $F(x) = 7x^2 \cdot G(x)$, we can say that,

$$F(x) = \frac{7}{5} \cdot x^2 \cdot \left(\frac{3}{1+3x} + \frac{2}{1-2x} \right)$$

By using the following equation,

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$$

we can rewrite $F(x)$ as the following,

$$F(x) = \frac{7}{5} \cdot x^2 \cdot \left(3 \cdot \sum_{k=0}^{\infty} -3^k x^k + 2 \cdot \sum_{k=0}^{\infty} 2^k x^k \right)$$

By rearranging the terms,

$$F(x) = \frac{7}{5} \cdot \left(3 \cdot \sum_{k=2}^{\infty} (-3^{k-2} \cdot x^k) + 2 \cdot \sum_{k=2}^{\infty} (2^{k-2} \cdot x^k) \right)$$

Since the upper and the lower limits of the summations are the same, we can gather them together.

$$F(x) = \frac{7}{5} \cdot \sum_{k=2}^{\infty} (-3^{k-1} \cdot x^k + 2^{k-1} \cdot x^k)$$

$$F(x) = \frac{7}{5} \cdot \sum_{k=2}^{\infty} \left(x^k \cdot (-3^{k-1} + 2^{k-1}) \right)$$

d

First, let's try to find the value of s_0. By putting $n = 1$ in the recurrence, we get,

$$s_1 = 8s_0 + 10^0$$

Since we know that s_1 is 9, we can conclude that s_0 is 1.

If we multiply both sides of the recurrence relation by x^n, we obtain,

$$s_n x^n = 8s_{n-1}x^n + 10^{n-1}x^n$$

Let $G(x)$ be the generating function of the sequence $s_0, s_1, s_2, \cdots$ with the following equation,

$$G(x) = \sum_{n=0}^{\infty} s_n x^n$$

By summing both sides of the multiplied recurrence relation starting with $n = 1$, we get,

$$\begin{aligned}
G(x) - 1 = \sum_{n=1}^{\infty} s_n x^n &= \sum_{n=1}^{\infty}(8s_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} s_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=1}^{\infty} s_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} s_n x^n + x\sum_{n=0}^{\infty}(10x)^n \\
&= 8x \cdot G(x) + \frac{x}{1-10x} \\
G(x) - 1 &= 8x \cdot G(x) + \frac{x}{1-10x}
\end{aligned}$$

By solving the equation for $G(x)$, we get,

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

By using partial fractions, we get,

$$G(x) = \frac{1}{2} \cdot \left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

By using the following equation (for $|ax| < 1$) ,

$$\sum_{n=0}^{\infty}(ax)^n = \frac{1}{1-ax}$$

$$\begin{aligned}
G(x) &= \frac{1}{2} \cdot \left(\sum_{n=0}^{\infty}(8x)^n + \sum_{n=0}^{\infty}(10x)^n\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{2}\left((8x)^n + (10x)^n\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n
\end{aligned}$$

Therefore, we can conclude that

$$s_n = \frac{1}{2}\left(8^n + 10^n\right)$$

Answer 2

a

- Let us pick $n = 20$, $m = 4$, and $k = 8$.
- $C_{20} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$
- $A_4 = \{4, 8, 12, 16, 20\}$
- $A_8 = \{8, 16\}$
- Since $4|8$, $A_8 \subseteq A_4$ must hold.
- Since $\{8, 16\} \subseteq \{4, 8, 12, 16, 20\}$ is a valid statement, we can say that we verified that if $m|k$, then $A_k \subseteq A_m$.

b

- We need to prove that the prime roots of a composite number is always less than or equal to the square-root of the number.
  - Since n is a positive composite number, we can rewrite n as $x \cdot y$, where $x, y \in Z$ and $1 < x, y < n$.
  - Let us suppose $x \leq y$.
  - In the case that $x > \sqrt{n}$, $y \geq x > \sqrt{n}$ also must hold.
  - However, if $y \geq x > \sqrt{n}$ is a correct statement, then $n = xy > \sqrt{n} \cdot \sqrt{n} > n$, which is a contradiction. Therefore our claim is incorrect.

- $x \leq \sqrt{n}$ is a correct claim.
- In that case, the x value can be a composite or a prime number, which does not affect the correctness of "the prime roots of a composite number is always less than or equal to the square-root of the number".

- Since the A_i sets of non-prime numbers will be the subsets of their prime roots' sets, the union of the prime A_i sets will be equal to the union of all A_i sets.
- Since we already proved that the "prime roots of a composite number is always less than or equal to the square-root of the number", we can conclude that the following equation is correct:

$$\bigcup_{i=2}^{n-1} A_i = \bigcup_{primes\, p \leq \sqrt{n}} A_p$$

c

- To explain this phenomena better, let us visualize it by giving $n = 45$, and $m = 6$. In that case, $A_6 = \{12, 18, 24, 30, 36, 42\}$. We can clearly see that $|A_6| = 6$. (Which also can be calculated by $|A_6| = \left\lfloor \frac{45}{6} \right\rfloor - 1 = 6$
- In any number n, there are $\left\lfloor \frac{n}{m} \right\rfloor$ times m exist. Since while counting the A_m count we ignore the first n, $|A_m| = \left\lfloor \frac{n}{m} \right\rfloor - 1$ must hold for any $m \geq 2$.

d

- To understand this question easily, let us visualize it with picking $a = 3, b = 4, n = 30$.
- Since 3 and 4 are relatively prime, and since they both are less than 30, the first statement holds.
- $A_3 = \{6, 9, 12, 15, 18, 21, 24, 27, 30\}$
- $A_4 = \{8, 12, 16, 20, 24, 28\}$
- $A_{12} = \{24\}$
- $(A_3 \cap A_4) - A_{12} = 12$
- From the example above we can see that the only element existing in the $(A_a \cap A_b) - A_{ab}$ is the ab element itself; since due to the definition of A, ab will exist in both A_a and A_b, but it will not exist in A_{ab}.

e

- We can generalize the A_p sets as the following;
$$A_p = \{x | x = p \cdot n, n \in N, n > 1\}$$
- Since for each $p \in P$, the p values will be relatively prime as well, the only common elements in the sets will be their least common multiple, and its multiples.
- For relatively prime numbers, the least common multiple will be the multiplication of the numbers. Therefore, the least common multiple value of every $p \in P$ is basically the multiplication of each number.
- The least common multiple can be denoted as the following;
$$lcm(\forall p \in P) = \prod_{p \in P} p$$
- To find the cardinality of this set, we can simply apply the floor function. Therefore we can formulate it as follows;
$$\left| \bigcap_{p \in P} A_p \right| = \left\lfloor \frac{n}{\prod_{p \in P} p} \right\rfloor$$

f

- Let us write C_{45}, A_2, A_3, and A_5.
- $C_{45} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45\}$
- $A_2 = \{4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44\}$
- $A_3 = \{6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45\}$
- $A_5 = \{10, 15, 20, 25, 30, 35, 40, 45\}$
- From the sets above, we can see that each set has common elements with each other. By using the Inclusion-Exclusion Principle, we can say the following;
$$|C_{45}| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$$

g

- Since $|C_{45}| = 30$, by the Inclusion-Exclusion Principle, the total number of primes up to 45 should be,
$$= 45 - |C_{45}| - 1$$
- Since there are 45 numbers between $[1, 45]$, subtracting C_{45} from this would give us the number of primes +1 (since we included 1 in our counting). Therefore we must subtract 1 from this calculation, which yields to 14 prime numbers.

Answer 3

a

- Let us pick the ordered sets $x = (a, b)$, $y = (c, d)$, and $z = (e, f)$; where $\{a, b, c, d, e, f\} \in Z$. By the definition of *transitivity*, if $(x, y) \in Z$ and $(y, z) \in Z$, then $(x, z) \in Z$ for all $x, y, z \in Z$.

- For x and y, we can write the following logical statement:

$$(a, b) \ll (c, d) \leftrightarrow (a < c) \vee ((a = c) \wedge (b \leq d))$$

- For y and z, we can write the following logical statement:

$$(c, d) \ll (e, f) \leftrightarrow (c < e) \vee ((c = e) \wedge (d \leq f))$$

- If the logical statement is a *tautology* for x and y, and if the logical statement is also a *tautology* for y and z, we can merge the two statements if the $\ll$ relation is transitive.

$$(a, b) \ll (c, d) \ll (e, f) \leftrightarrow (((a < c) \vee ((a = c) \wedge (b \leq d))) \wedge ((c < e) \vee ((c = e) \wedge (d \leq f))))$$

- Let us denote

  1. $p \equiv a < c$
  2. $q \equiv c < e$
  3. $r \equiv (a = c) \wedge (b \leq d)$
  4. $s \equiv (c = e) \wedge (d \leq f)$

- The right hand side of the logical statement becomes as the following;

$$(p \vee r) \wedge (q \vee s)$$

- By using **Distributive laws**, we get

$$(p \vee r) \wedge (q \vee s) \equiv (p \wedge q) \vee (p \wedge s) \vee (r \wedge q) \vee (r \wedge s)$$

- If we rewrite the p, q, r, s values into the statement back, we get,

  1. $p \wedge q \equiv (a < c) \wedge (c < e) \equiv a < e$
  2. $p \wedge s \equiv (a < c) \wedge ((c = e) \wedge (d \leq f)) \equiv (a < c = e) \wedge (d \leq f)$
  3. $r \wedge q \equiv ((a = c) \wedge (b \leq d)) \wedge (c < e) \equiv (a = c < e) \wedge (b \leq d)$
  4. $r \wedge s \equiv ((a = c) \wedge (b \leq d)) \wedge ((c = e) \wedge (d \leq f)) \equiv (a = c = e) \wedge (b \leq d \leq f)$

- If the given $\ll$ relation is *transitive*, it should hold if $(a < e) \vee ((a = e) \wedge (b \leq f))$

- Since we are interested in the *transitive* property, the 2^{nd} and 3^{rd} statements do not say much since they have terms with c and d. In order to continue with our investigation, we are going to take a look at the statements with a, b, e, f.

  1. $p \wedge q \equiv a < e$
  2. $p \wedge s \equiv a < e$ *(since the original statement had a $\wedge$ relation, both sides of the statement must hold.)*
  3. $r \wedge q \equiv a < e$ *(since the original statement had a $\wedge$ relation, both sides of the statement must hold.)*
  4. $r \wedge s \equiv (a = e) \wedge (b \leq f)$

- Since the initial statement had 4 different logical statements conjoined using a logical or statement, in the case when only one of them is true, the given statement would yield a True value. By conjoining the aforementioned 4 statements, we get;

$$(a < e) \vee ((a = e) \wedge (b \leq f))$$

- Since we reached our initial statement, the given relation is **transitive** by logic rules.

b

- To show that the $\propto$ relation is an equivalence relation, we need to show that it is reflexive, symmetric, and transitive.

  1. **Reflexivity**
     - Let us pick a real number that satisfies $x \geq k$ for every x.
     - We can select any k and the equations $f(x) = f(x)$ and $g(x) = g(x)$ will be satisfied.
     - If $f(x) = g(x)$ for $\propto$ relation, then we can say that the $\propto$ relation is reflexive for any x that is $x \geq k$.
  2. **Symmetricity**
     - Let us pick a real number that satisfies $x \geq k$ for every x.
     - We can select any k and the equation $f(x) = g(x)$ will be satisfied for the relation $f \propto g$.
     - Similarly, if $f(x) = g(x)$, then $g(x) = f(x)$ must hold.
     - Therefore, we can say that for any $x \geq k$, the relation $g \propto f$ will be satisfied.
     - Since both $f \propto g$ and $g \propto f$ holds for any k that satisfies $x \geq k$, we can conclude that the $\propto$ relation is symmetric for any x that is $x \geq k$.
  3. **Transitivity**
     - Let us pick 3 functions f, g, h; respectively.
     - To prove transivity, we need to show that if $f \propto g$ and $g \propto h$, then $f \propto h$ must hold for any x that is $x \geq k$.

- For an arbitrary k such that $x \geq k$, $f(x) = g(x)$ and $g(x) = h(x)$ holds by the definition of the relation.
- By this equalities, we can say that $f(x) = h(x)$ for any x that is $x \geq k$.
- Therefore we can say that the given relation is transitive.

- Since we have shown that the given $\propto$ relation is **reflexive**, **symmetric**, and **transitive**, we can conclude that $\propto$ is an **equivalence relation**.