HW#3

Q.3)

3.1.

a. Let $\phi' : \mathbb{N}^+ \to \Sigma^*$ be defined as

$$\phi'(1) = \bullet\big((, \bullet(p_1,)\big).$$

$$\phi'(n) = \bullet\Big((, \bullet\big(p_n, \bullet(\wedge, \bullet(\phi'(n-1),))\big)\Big), \quad n \in \mathbb{N} \text{ and } n \geq 2.$$

(infix not^{n} is OK)

Then

$$\phi(n) = \bullet\big(\phi'(n), \bullet(\to, q)\big), \quad n \in \mathbb{N}^+.$$

$$\psi(1) = \bullet\Big((, \bullet\big(p_1, \bullet(\to, \bullet(q,))\big)\Big).$$

$$\psi(n) = \bullet\Big((, \bullet\big(p_n, \bullet(\to, \bullet(\psi(n-1),))\big)\Big), \quad n \in \mathbb{N} \text{ and } n \geq 2.$$

b. <u>base case</u> $n = 1$

$$p_1 \to q \vdash p_1 \to q$$

(minor technicality: Ignore paranthesis)

can be proven as

1. $p_1 \to q$ premise
2. $p_1 \to q$ copy, 1

using N.D. rules.

<u>inductive hypothesis</u> Assume that $\phi(n) \vdash \psi(n)$ holds for arbitrary n, i.e. there is a N.D. proof of finite length j that produces $\psi(n)$ as its conclusion when $\phi(n)$ is provided as a premise.

inductive step We want to show $\phi(n+1) \vdash \psi(n+1)$.

Rewriting

$$\phi(n+1) = (p_{n+1} \wedge \phi'(n)) \to q$$

$$\psi(n+1) = (p_{n+1} \to \psi(n))$$

(dots omitted)

Via inductive hypothesis, we have a proof $\mathcal{E}$ of finite length $j \geq 2$, of the following form:

1.	$\phi'(n) \to q$	premise
$\vdots$	$\vdots$	
(j).	$\psi(n)$	$\mathcal{E}$, 1 – (j-1).

To prove the goal, we initiate the following N.D. proof:

1.	$(p_{n+1} \wedge \phi'(n)) \to q$	premise
2.	p_{n+1}	assumed
3.	$\phi'(n)$	assumed
4.	$(p_{n+1} \wedge \phi'(n))$	$\wedge$i on 2,3
5.	q	$\to$e on 1,4
6.	$\phi'(n) \to q$	$\to$i on 3-5
7.	$\psi(n)$	$\mathcal{E}$ on 6
8.	$(p_{n+1} \to \psi(n))$	$\to$i on 2-7

$\Rightarrow$ use of inductive hypothesis

Thus, $\phi(n+1) \vdash \psi(n+1)$ is proven.

HW#3

Q.3.)

3.2.

a. Let T denote the set of binary trees defined Section 5.3 of the textbook.

Then, $h : T \to \mathbb{N} \cup \{-1\}$ be a function computing height of a tree as follows.

Let $T_0 \in T$ be the empty tree.

$$h(T_0) = -1.$$

If $T_1, T_2 \in T$ then the binary tree $T' = T_1 \cdot T_2$ has height $(T' \neq T_0)$

$$h(T') = 1 + \max(h(T_1), h(T_2)).$$

b. Def'n of 2-3-tree is given.

$$\begin{cases} f(0) = 1. \\ f(h) = f(h-1) + 2^h, \; h \in \mathbb{N}^+ \end{cases}$$

or $$\begin{cases} f(0) = 1. \\ f(h) = 2f(h-1) + 1, \; h \in \mathbb{N}^+. \end{cases}$$

$$\begin{cases} g(0) = 1. \\ g(1) = 2. \\ g(2) = 3. \\ g(h) = g(h-1) + g(h-3) + 1, \; h \in \mathbb{N}, h > 2. \end{cases}$$

c. Proof for $\begin{cases} f(0)=1. \\ f(h)=f(h-1)+2^h, \ h\in\mathbb{N}^+. \end{cases}$

either

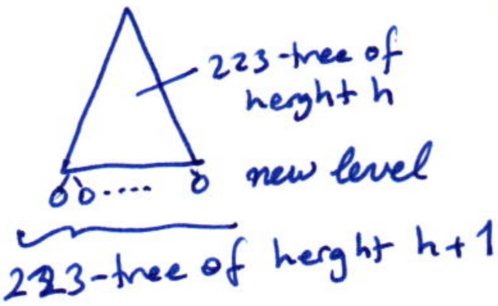
<u>base case</u>: $h=0$

A binary tree of height 0 comprises of a single vertex. Thus $f(0)=0$.

<u>inductive hypothesis</u>: Assume that $f(h)=f(h-1)+2^h$ for arbitrary $h\geq 0$.

<u>inductive step</u>: We want to prove that $f(h+1)=f(h)+2^{h+1}$.

Max number of nodes in a 223-tree of height $(h+1)$ can be achieved by adding a full-level to a 223-tree of height h, and the number of vertices in the introduced level is 2^{h+1}



(which can also be proved via mathematical induction.)

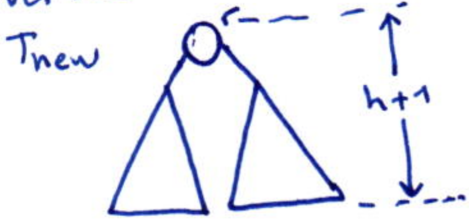
Thus, $f(h+1)=f(h)+2^{h+1}$!

or Proof for $\begin{cases} f(0)=1. \\ f(h)=2f(h-1)+1, \ h\in\mathbb{N}^+. \end{cases}$

<u>base case</u>: $h=0$ (same reason)

<u>inductive hypothesis</u>: Assume that $f(h)=2f(h-1)+1$ for arbitrary $h\geq 0$.

<u>inductive step</u>: Max number of nodes in a 223-tree of height $(h+1)$ can be attained by building a new 223-tree with root r whose left and right subtrees are 223-trees of height h with $f(h)$ vertices.



$h(T_{new})=1+\max(h,h)=h+1.$

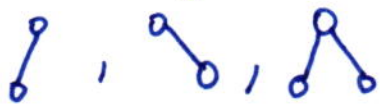
$$f(h+1)=f(h)+f(h)+1$$
$$=2f(h)+1.$$

Proof for $\begin{cases} g(0) = 1. \\ g(1) = 2. \\ g(2) = 3. \\ g(h) = g(h-1) + g(h-3) + 1,\ h \in \mathbb{N},\ h > 2. \end{cases}$

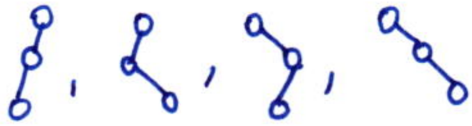
basis

case $h = 0$: As number of vertices of a binary tree of height 0 is always 1 and such a tree is a 223-tree, $g(0) = 1$.

case $h = 1$: Following 223-trees of height 1 may exist:

{  } and consequently $g(1) = 2$.

case $h = 2$: The least number of vertices in a binary tree can be attained if the tree has a 'linear' structure. Following 223-trees of height 2 have the minimum number of vertices:

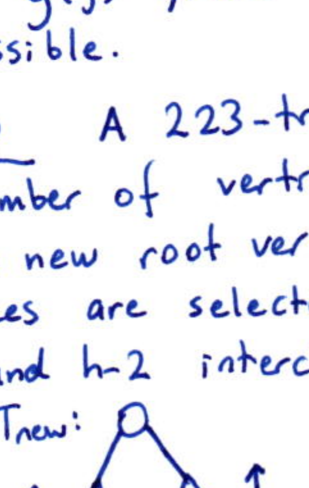
{  } and we have $g(2) = 3$.

inductive hypothesis

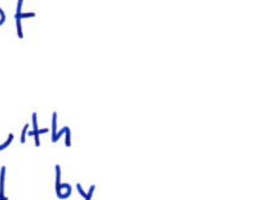
Assume that for all 223-trees of height $1 \le j \le h$ $g(j)$ yields the minimum number of vertices possible.

inductive step A 223-tree of height $(h+1)$ with minimum number of vertices can be obtained by introducing a new root vertex for which left and right subtrees are selected as 223-trees having heights h and $h-2$ interchangeably:

subcase i. T_{new}:



subcase ii. T_{new}:



For both cases, we have

$$h(T_{new}) = 1 + \max(h, h-2) = h+1$$

and

subtrees are 223-trees via inductive hypothesis

and for the root vertex $\text{abs}(h-(h-2)) = 2$,

thus T_{new} is also a 223-tree

and

minimum number of vertices exist for both subtrees via inductive hypothesis

and to increase the height of a binary tree by 1, the least number of elements that can be added is 1,

consequently T_{new} has the least number of vertices for height $(h+1)$ and

$$\begin{aligned} g(h+1) &= g(h) + g(h-2) + 1 \\ &= g(h-2) + g(h) + 1 \quad \blacksquare. \end{aligned}$$