Middle East Technical University Department of Computer Engineering

CENG 223

Discrete Computational Structures

Fall '2018-2019

Assignment 1

ANSWER SHEET

Question 1

1. Determine if the following compound propositions are a **tautology**, a **contradiction** or **neither one of them**. Construct a truth table for each proposition.

   (a) All of the outputs are T, so this is a tautology.

p	q	r	$p \to q$	$\neg r$	$p \wedge \neg r$	$(p \to q) \leftrightarrow (p \wedge \neg r)$	$\neg(q \wedge r)$	$((p \to q) \leftrightarrow (p \wedge \neg r)) \to \neg(q \wedge r)$
F	F	F	T	T	F	F	T	T
T	F	F	F	T	T	F	T	T
F	T	F	T	T	F	F	T	T
T	T	F	T	T	T	T	T	T
F	F	T	T	F	F	F	T	T
T	F	T	F	F	F	T	T	T
F	T	T	T	F	F	F	F	T
T	T	T	T	F	F	F	F	T

   (b) All of the outputs are F, so this is a contradiction.

p	q	$p \to q$	$p \vee q$	$q \to \neg p$	$(p \vee q) \wedge (p \to q)$	$(p \vee q) \wedge (p \to q) \vee (q \to \neg p)$	$\neg((p \vee q) \wedge (p \to q) \vee (q \to \neg p))$
F	F	T	F	T	F	T	F
T	F	F	T	T	F	T	F
F	T	T	T	T	T	T	F
T	T	T	T	F	T	T	F

2. Determine if the following predicate logic arguments are **valid** or **invalid**. Explain why you think the argument is **valid** or **invalid**. You do not need to make a formal proof for these questions. (**Hint:** Using counterexamples might be beneficial.)

   (a) $$\exists x P(x) \wedge \exists x Q(x) \to \exists x (P(x) \wedge Q(x))$$

   This argument is **invalid**. To show this, we will use counterexample.
   Let our domain be $D = \{1, 2\}$ and P becomes true only with the value 1 and Q becomes true only with the value 2.
   In this case $d[1/x]$ satisfies $P(x)$, so $\exists x P(x)$ is true.
   Similarly $d[2/x]$ satisfies $Q(x)$, so $\exists x Q(x)$ is also true.
   But there is no value in the domain that satisfies both $P(x)$ and $Q(x)$.
   So, we can see that $\exists x (P(x) \wedge Q(x))$ is not true, despite the premises are true.

(b) $\forall x P(x) \rightarrow \exists x P(x)$

Let **I** be an arbitrary interpretation and **d** an arbitrary variable assignment.
Suppose $\forall x P(x)$ is true in **I**.
Then for all **u** in the domain **D** of **I**, d[u/x] satisfies $\forall x P(x)$.
Then for an arbitrary **u** in the domain **D**, $P(u)$ is true.
If there is a $P(u)$ exists, we can say that $\exists x(P(x))$ is true.
If we can conclude this from any random interpretation, then we can conclude that $\forall x P(x) \rightarrow \exists x P(x)$ is a **valid** argument.

Question 2

Show that $(\neg p \lor p) \rightarrow ((p \land \neg q) \rightarrow r)$ and $(q \lor r) \lor \neg p$ are logically equivalent.

1. $(\neg p \lor p) \rightarrow ((p \land \neg q) \rightarrow r) \equiv \neg(\neg p \lor p) \lor ((p \land \neg q) \rightarrow r)$ - Table 7
2. $\equiv (p \land \neg p) \lor ((p \land \neg q) \rightarrow r)$ - De Morgan's Law
3. $\equiv F \lor ((p \land \neg q) \rightarrow r)$ - Negation Laws
4. $\equiv (p \land \neg q) \rightarrow r$ - Identity Laws
5. $\equiv \neg(p \land \neg q) \lor r$ - Table 7
6. $\equiv (\neg p \lor q) \lor r$ - De Morgan's Law
7. $\equiv \neg p \lor (q \lor r)$ - Associative Laws
8. $\equiv (q \lor r) \lor \neg p$ - Commutative Laws

Question 3

Let $W(x)$ be "x works in the lab", $Older(x, y)$ be "x is older than y", $Phd(x)$ be "x is a Phd. student", $Has\_CS\_Degree(x)$ be "x has a CS degree", $Knows(x, y)$ be "x knows y".

Use these predicates to express the following statements using quantifiers $\forall$ and $\exists$.

For all of the below sentences domains x, y and z are all the people in the world.

Note that, there are more than one correct answers for all of the questions. There can be other alternatives than the provided ones.

1. Everybody works in the lab has a CS degree.
   $\forall x(W(x) \rightarrow Has\_CS\_Degree(x))$
2. All Phd. students working in the lab knows each other.
   Alternative Solution1: $\forall x \forall y(((W(x) \land Phd(x)) \land (W(y) \land Phd(y))) \rightarrow (Knows(x, y) \land Knows(y, x)))$
   Alternative Solution2: $\forall x \forall y(((W(x) \land Phd(x)) \land (W(y) \land Phd(y))) \rightarrow (Knows(x, y)))$
3. Cenk is the oldest person working in the lab.
   Alternative Solution1: $W(Cenk) \land \forall x((W(x) \land x \neq Cenk) \rightarrow Older(Cenk, x))$
   Alternative Solution2: $\forall x((W(Cenk) \land W(x) \land x \neq Cenk) \rightarrow Older(Cenk, x))$
   Alternative Solution3: $\forall x((W(Cenk) \land W(x)) \rightarrow (x \neq Cenk \leftrightarrow Older(Cenk, x)))$

4. Everyone working in the lab is a Phd. student except Selen.
   $\forall x((W(x) \land x \neq Selen) \to Phd(x))$

5. Not all the people working in the lab knows everyone working in the lab.
   Alternative Solution1: $\exists x \exists y(W(x) \land W(y) \land (x \neq y) \land \neg Knows(x, y))$
   Alternative Solution2: $\neg(\forall x \forall y(W(x) \land W(y) \to Knows(x, y)))$

6. There are at most two Phd. students.
   $\forall x \forall y \forall z((Phd(x) \land Phd(y) \land Phd(z)) \to ((x = y) \lor (y = z) \lor (x = z)))$

7. There are at least three people older than Gizem.
   $\exists x \exists y \exists z((Older(x, Gizem) \land Older(y, Gizem) \land Older(z, Gizem)) \land (x \neq y) \land (y \neq z) \land (x \neq z) \land (x \neq Gizem) \land (y \neq Gizem) \land (z \neq Gizem))$

8. There is exactly one person who is doing Phd and working in the lab.
   $\exists x(W(x) \land Phd(x)) \land \forall x \forall y((W(x) \land Phd(x) \land W(y) \land Phd(y)) \to x = y)$

Question 4

Prove the following by using only the natural deduction rules for $\lor, \land, \to$, and $\neg$ introduction and elimination.
Any other rules/lemmas used should be proven by natural deduction as well.

$$(p \to r) \lor (q \to r) \vdash (p \land q) \to r$$

1.	$(p \to r) \lor (q \to r)$	premise
2.	$p \land q$	assumed
3.	$p \to r$	assumed
4.	p	$\land_{e_1}$, 2
5.	r	$\to_e$,3, 4
6.	$q \to r$	assumed
7.	q	$\land_{e_2}$, 2
8.	r	$\to_e$, 6, 7
9.	r	$\lor e$, 1, 3-5, 6-8
10.	$(p \land q) \to r$	$\to_i$, 2-9

Question 5

Prove the following by using only the natural deduction rules for $\lor, \land, \to$, and $\neg$ introduction and elimination.
Any other rules/lemmas used should be proven by natural deduction as well.

$$(\neg p \lor \neg q) \vdash (p \land q) \to r$$

1. $\neg p \vee \neg q$ premise
2. $p \wedge q$ assumed
3. $\neg p$ assumed
4. p $\wedge_{e_1}$, 2
5. $\perp$ $\neg_e$, 3, 4
6. r $\perp_e$, 5
7. $\neg q$ assumed
8. q $\wedge_{e_2}$, 2
9. $\perp$ $\neg_e$, 7, 8
10. r $\perp_e$, 9
11. r $\vee_e$, 1, 3-6, 7-10
12. $p \wedge q \to r$ $\to_i$, 2-11

Question 6

Prove the following by using only the natural deduction rules for $\vee, \wedge, \to, \neg, \forall$, and $\exists$ introduction and elimination. Any other rules/lemmas used should be proven by natural deduction as well.

$$\forall x(P(x) \to (Q(x) \to R(x))), \exists x(P(x)), \forall x(\neg R(x)) \vdash \exists x(\neg Q(x))$$

1. $\forall x(P(x) \to (Q(x) \to R(x)))$ premise
2. $\exists x(P(x))$ premise
3. $\forall x(\neg R(x))$ premise
4. x_0 $P(x_0)$ assumed
5. $P(x_0) \to (Q(x_0) \to R(x_0))$ $\forall_{x_e}$, 1
6. $Q(x_0) \to R(x_0)$ $\to_e$, 5, 4
7. $Q(x_0)$ assumed
8. $R(x_0)$ $\to_e$, 6, 7
9. $\neg R(x_0)$ $\forall_{x_e}$, 3
10. $\perp$ $\neg_e$, 8,9
11. $\neg Q(x_0)$ $\neg_i$, 7-10
12. $\exists x(\neg Q(x))$ $\exists_i$, 11
13. $\exists x(\neg Q(x))$ $\exists_{x_e}$, 2, 4-12