Smaller Discs to Cover the Unit Circle

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Abstract

Using a simple geometric arrangement, it is possible to find solutions $r(n)$ for Zahn's disk-covering problem that are better than the previously published best solutions.

Background

In 1962, Zahn stated the following problem: "Given a unit disk, find the smallest radius $r(n)$ required for n equal disks to completely cover the unit disk." He calculated a set or values for $r(n)$ for $1 \leq n \leq 10$. This paper presents a class of solutions to this problem which are better than the previously published best solutions $r(n)$, for two values $f(n)$. (For more details on the problem, see [1].)

Method

The arrangement of discs is a simple pattern with one disc placed at the origin and the remaining discs "petals" placed around the center disc such that they intersect twice, with an arc of θ. This pattern exhibits radial symmetry, as shown in Figure 1 with:

$$\theta = \frac{2\pi}{n-1}$$

We choose r such that the intersection of adjecent petals covers exactly from r to 1 units from the origin. Because of the radial symmetry and the fact that all circular segments are non-negative, this arrangement covers the unit circle.

Construction

The distance from the origin to each petal's radius is:

$$2r \cos \frac{\theta}{2}$$

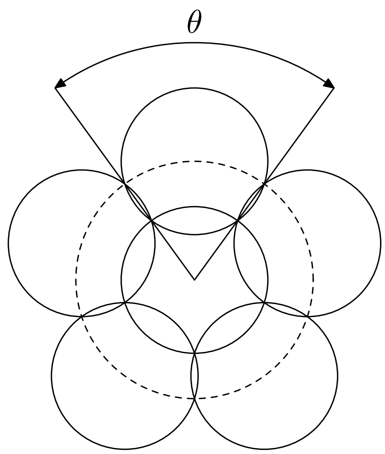

Figure 1: Disc arrangement

The distance between the centers of adjacent petals is given by the chord of the circle of radius $2r\cos\frac{\theta}{2}$:

$$2r\cos\frac{\theta}{2}2\sin\frac{\theta}{2}$$

The chord of the intersection of adjaent petals is, by design:

$$1 - r$$

Since the chord of adjacent petals is perpendicular to the distance between their centers, we can use a distance equation for the patel's radius as shown in Figure 2:

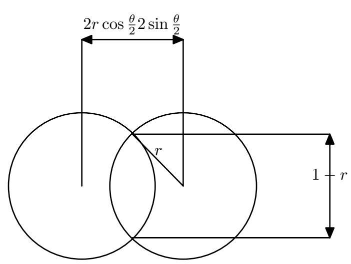

Figure 2: Adjacent petals

$$r^2 = (\frac{1-r}{2})^2 + (\frac{2\sin\frac{\theta}{2}2r\cos\frac{\theta}{2}}{2})^2$$

This simplifies nicely:

$$
\begin{aligned}
r^2 &= (\frac{1-r}{2})^2 + (r\sin\theta)^2 \\
r^2 - r^2\sin^2\theta &= (\frac{1-r}{2})^2 \\
2r\cos\theta &= 1-r \\
r &= \frac{1}{2\cos\theta + 1} \\
r &= \frac{1}{2\cos\frac{2\pi}{n-1} + 1}\,. \qquad (1)
\end{aligned}
$$

Results

By plugging values of n into our equation (1), we find two values that are better than the previously published best values.

Values of n	Zahn's $r(n)$	Our $r(n)$
1	1	n/a
2	1	n/a
3	$\frac{1}{2}\sqrt{3}$	n/a
4	$\frac{1}{2}\sqrt{2}$	n/a
5	0.609382864...	1
6	0.555	0.61803... $(\frac{\sqrt{5}-1}{2})$
7	$\frac{1}{2}$	$\frac{1}{2}$
8	0.437	0.44504...
9	0.422	**0.41421...** $(\sqrt{2}-1)$
10	0.398	**0.38196...**

Acknowledgements

I would like to thank Dr. Robert Styer, for bringing this problem to my attention and Kevin Penderghest, for introducing the graphical representation that this solution is based on.

References

[1] Weisstein, Eric W. "Disk Covering Problem." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/DiskCoveringProblem.html