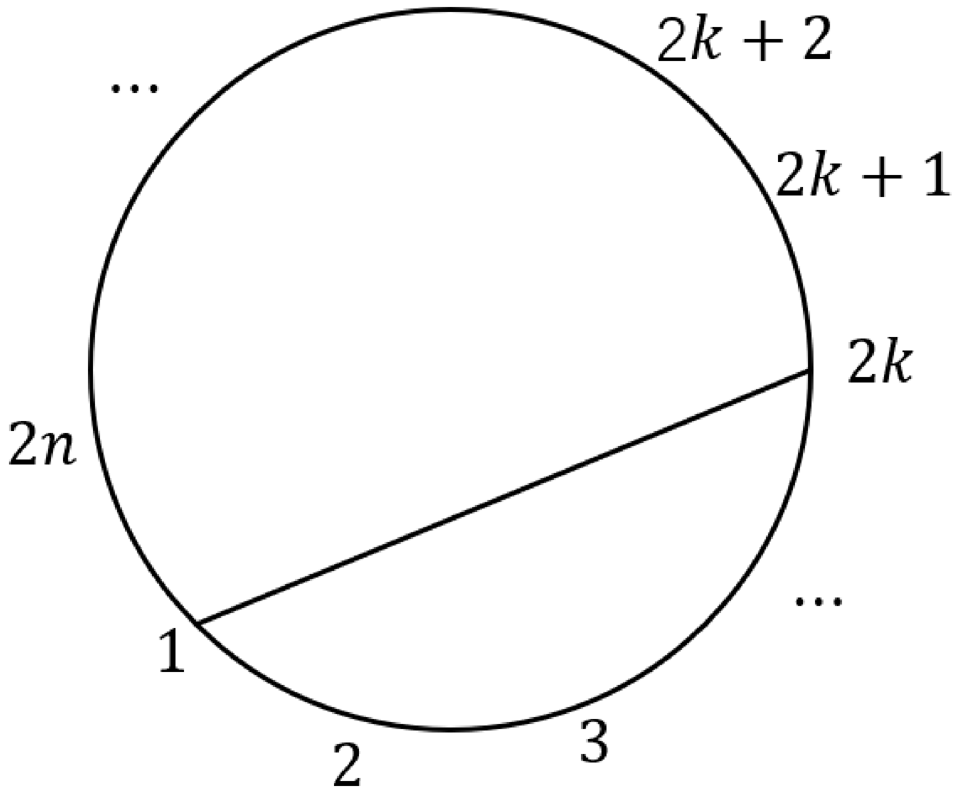
第8章 特殊计数序列

EX1

> Let 2n (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.



记该问题的解为h_n，选择一端固定在1上的线段为基线，另一端指向2k，圆上的2n个点被分为两组，一组有2k-2个，另一组有2n-2k个，同时问题h_n被划分为h_{k-1}和h_{n-k}。所以有，

$$h_n = \sum_{k=1}^{n} h_{k-1} h_{n-k}, \quad n \geq 1, h_0 = h_1 = 1$$

显然h_n与卡特兰数C_n有相同的递推关系和初始项，因此，

$$h_n = \frac{1}{n+1}\binom{2n}{n}$$

EX1注

本题与第7章EX41是类似的问题

EX2

> Prove that the number of 2-by-n arrays
>
> $$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}$$
>
> that can be made from the numbers 1,2, ..., 2n such that
>
> $$x_{11} \leq x_{12} \leq \cdots \leq x_{1n}$$
> $$x_{21} \leq x_{22} \leq \cdots \leq x_{2n}$$
>
> $$x_{11} \leq x_{21}, x_{12} x_{22}, ..., x_{1n} \leq x_{2n}$$
>
> equals the nth Catalan number, C_n.

将数组第一行的元素标记为+1，第二行元素标记为-1。
问题可以转化为：将+1和-1按照从左到右的顺序排列，并且保证第i个+1在第i个-1前面，即$x_{1i} \leq x_{2i}$（$1 \leq i \leq n$）。

这与前k项和满足

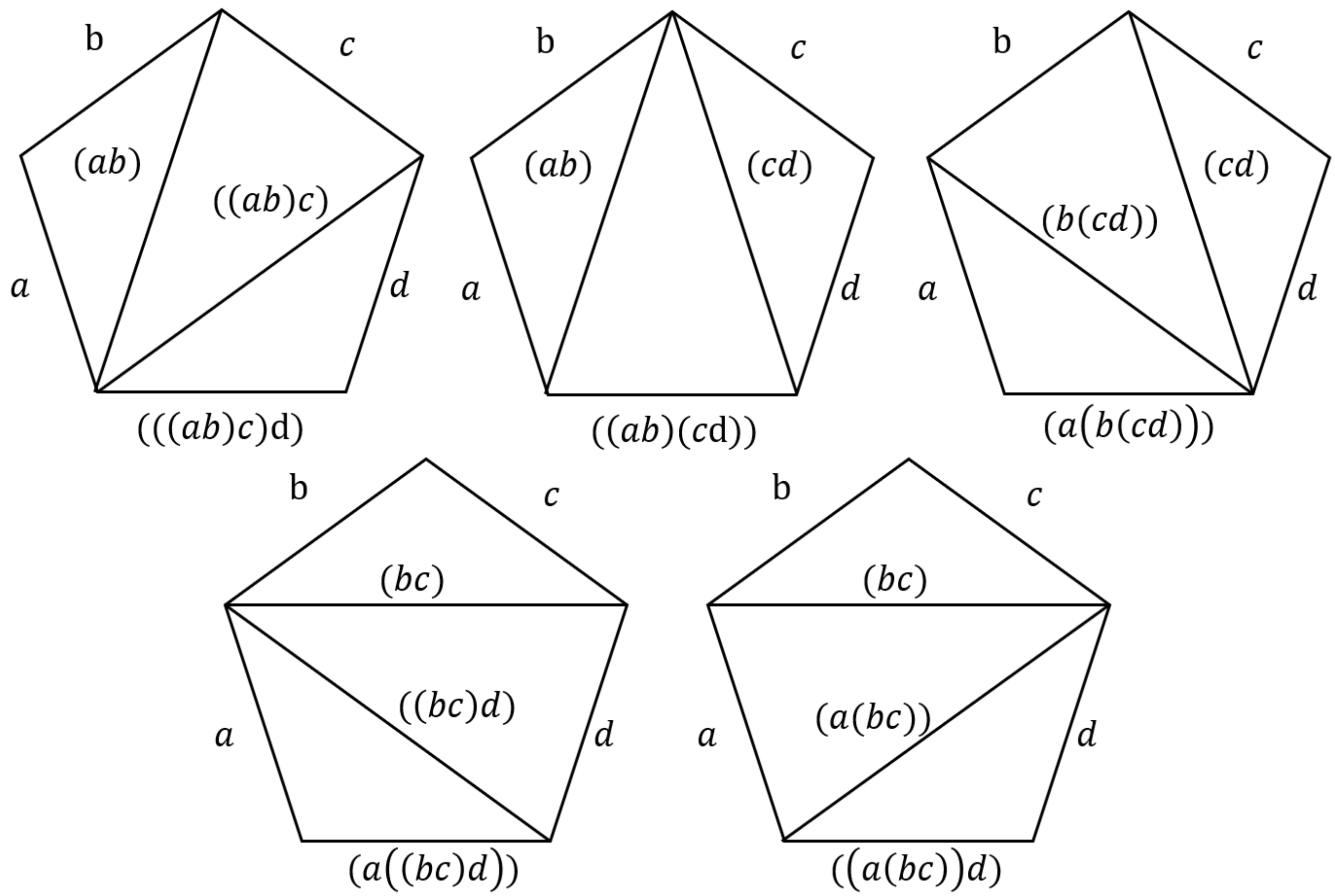
$$a_1 + a_2 + \cdots + a_k \geq 0$$

等价，该问题与卡特兰数的组合意义相同，解即为第n个卡特兰数。

EX3

> Write out all of the multiplication schemes for four numbers and the triangularization of a convex polygonal region of five sides corresponding to them.

考虑固定顺序的乘法，因此一共有$C_{n-1} = C_3 = \frac{1}{4}\binom{6}{3} = 5$种方案，与之对应的三角形划分如图所示。

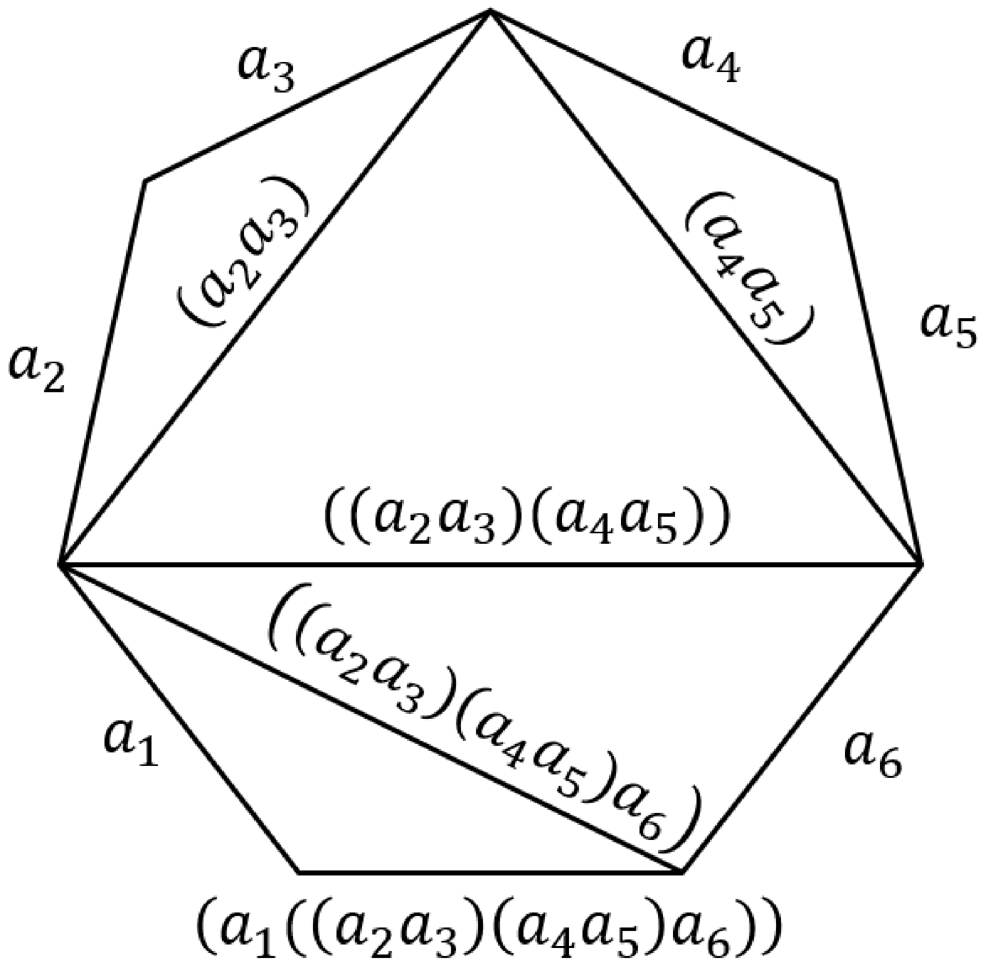


EX4

> Determine the triangularization of a convex polygonal region corresponding to the following multiplication schemes:
>
> (a) $(a_1 \times (((a_2 \times a_3) \times (a_4 \times a_5)) \times a_6))$
>
> (b) $(((a_1 \times a_2) \times (a_3 \times (a_4 \times a_5))) \times ((a_6 \times a_7) \times a_8))$

EX4(a)

以EX4(a)为例，步骤同上一题，



EX5

加星题，略。

EX6

> Let the sequence $h_0, h_1, ..., h_n, ...$ be defined by $h_n = 2n^2 - n + 3, (n \geq 0)$. Determine the difference table, and find a formula for $\sum_{k=0}^{n} h_k$.

h_n是2次多项式，因此有$\Delta^3 h_n = 0$，

计算$h_0 = 3, h_1 = 4, h_2 = 9$，一阶差分$\Delta^1 h_0 = 1, \Delta^1 h_1 = 5$, 二阶差分$\Delta^2 h_0 = 4$，即得到第0条对角线。

$$\begin{array}{cccc} 3 & 4 & 9 & \cdots \\ 1 & 5 & 9 & \cdots \\ 4 & 4 & 4 & \cdots \\ 0 & 0 & 0 & \cdots \end{array}$$

所以$h_n = 3\binom{n}{0} + \binom{n}{1} + 4\binom{n}{2}$，进而

$$
\begin{aligned}
\sum_{k=0}^{n} h_k =& 3\sum_{k=0}^{n}\binom{k}{0}+\sum_{k=0}^{n}\binom{k}{1}+4\sum_{k=0}^{n}\binom{k}{2}\\
=& 3\binom{n+1}{1}+\binom{n+1}{2}+4\binom{n+1}{3} \quad n\geq 0
\end{aligned}
$$

EX7

> The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the Oth row of its difference table are 1, -1, 3, 10, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

由题意，h_n是3次多项式，那么$\Delta^4 h_n=0$，求出差分表第0条对角线，

$$
\begin{array}{ccccc}
1 & -1 & 3 & 10 & \cdots \\
-2 & 4 & 7 & \cdots & \\
6 & 3 & \cdots & & \\
-3 & \cdots & & & \\
0 & \cdots & & &
\end{array}
$$

因此$h_n=\binom{n}{0}-2\binom{n}{1}+6\binom{n}{2}-3\binom{n}{3}$，进而

$$
\begin{aligned}
\sum_{k=0}^{n} h_k &= \sum_{k=0}^{n}\binom{k}{0}-2\sum_{k=0}^{n}\binom{k}{1}+6\sum_{k=0}^{n}\binom{k}{2}-3\sum_{k=0}^{n}\binom{k}{3}\\
&=\binom{n+1}{1}-2\binom{n+1}{2}+6\binom{n+1}{3}-3\binom{n+1}{4} \quad n\geq 0k
\end{aligned}
$$

EX8

> Find the sum of the fifth powers of the first n positive integers.

设$h_n=n^5$，那么它的六阶差分为0，求出差分表，

$$
\begin{array}{cccccc}
0 & 1 & 32 & 243 & 1024 & 3125\cdots \\
1 & 31 & 211 & 781 & 2101\cdots & \\
30 & 180 & 570 & 1320 & \cdots & \\
150 & 390 & 750 & \cdots & & \\
240 & 360 & \cdots & & & \\
120 & \cdots & & & & \\
0 & \cdots & & & &
\end{array}
$$

$$
k^5=\binom{k}{1}+30\binom{k}{2}+150\binom{k}{3}+240\binom{k}{4}+120\binom{k}{5}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} k^5 &= \sum_{k=1}^{n}\binom{k}{1}+30\sum_{k=1}^{n}\binom{k}{2}+150\sum_{k=0}^{n}\binom{k}{3}+240\sum_{k=0}^{n}\binom{k}{4}+120\sum_{k=0}^{n}\binom{k}{5}\\
&=\binom{n+1}{2}+30\binom{n+1}{3}+150\binom{n+1}{4}+240\binom{n+1}{5}+120\binom{n+1}{6}
\end{aligned}
$$

EX9

> Prove that the following formula holds for the kth-order differences of a sequence $h_0, h_1, \cdots, h_n, \cdots$:
>
> $$\Delta^k h_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} h_{n+j}$$

采用数学归纳法证明，当k=0时，有

$$\Delta h_n = (-1)^0 \binom{0}{0} h_0 = h_0$$

成立，假设当$k=m$时结论成立，即有

$$\Delta^m h_n = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} h_{n+j}$$

当$k=m+1$时，

$$\begin{aligned}
\Delta^{m+1} h_n =& \Delta^m h_{n+1} - \Delta^m h_n \\
=& \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} h_{n+1+j} - \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} h_{n+j} \\
=& \sum_{j=1}^{m+1} (-1)^{m-j+1} \binom{m}{j-1} h_{n+j} - \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} h_{n+j} \\
=& (h_{n+(m+1)} - (-1)^m h_n) + \sum_{j=1}^{m} ((-1)^{m-j+1} \binom{m}{j-1} - (-1)^{m-j} \binom{m}{j}) h_{n+j} \\
=& (-1)^{(m+1)-(m+1)} h_{n+(m+1)} + (-1)^{(m+1)-0} h_{n+0} + \sum_{j=1}^{m} ((-1)^{m+1-j} \binom{m}{j-1} + (-1)^{m+1-j} \binom{m}{j}) h_{n+j} \\
=& (-1)^{(m+1)-(m+1)} h_{n+(m+1)} + (-1)^{(m+1)-0} h_{n+0} + \sum_{j=1}^{m} (-1)^{m+1-j} \binom{m+1}{j}) h_{n+j} \\
=& \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} h_{n+j}
\end{aligned}$$

综上，证毕。

EX10

> If h_n is a polynomial in n of degree m, prove that the constants $c_0, c_1, \cdots, c_m$ such that
>
> $$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \cdots + c_m \binom{n}{m}$$

> are uniquely determined. (Cf. Theorem 8.2.2.)

本题主要证明**唯一性**，假设存在不同的序列，$\{c_i\}_{i=0}^m$和$\{d_i\}_{i=0}^m$使得存在i满足$c_i \neq d_i, 0 \leq i \leq m$，

$$\begin{aligned} h_n = & c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_m\binom{n}{m} \\ = & d_0\binom{n}{0} + d_1\binom{n}{1} + \cdots + d_m\binom{n}{m} \end{aligned}$$

$$\sum_{k=0}^{m}(c_k - d_k)\binom{n}{k} = 0$$

显然$\binom{n}{k} > 0$，那么只能是$c_k - d_k = 0, 0 \leq k \leq m$，这与假设矛盾，因此假设不成立。

EX11

> Compute the Stirling numbers of the second kind S(8, k), (k = 0, 1, ..., 8).

第二类Stirling数的性质，

1. $S(p,0) = 0, p \geq 1$
2. $S(p,p) = 1, p \geq 0$
3. $S(p,k) = kS(p-1,k) + S(p-1,k-1)$

进行打表，

k	0	1	2	3	4	5	6	7	8
$S(8,k)$	0	1	127	966	1701	1050	266	28	1

EX11 验证程序

```cpp
#include <iostream>
#include <vector>

using namespace std;
int main() {
    auto stirList = vector<vector<int>>(10, vector<int>(10, 0));
    for(int p = 1; p < 10; ++ p) {
        stirList[p][p] = 1;
    }
    for(int p = 2; p < 10; ++ p) {
        for(int k = 1; k < p; ++ k) {
            stirList[p][k] = stirList[p-1][k] * k + stirList[p-1][k-1];
        }
    }
    for(int p = 0; p < 10; ++ p) {
        for(int k = 0; k <= p; ++ k) {
            printf("%d\t", stirList[p][k]);
        }
        printf("\n");
    }

    printf("\n S(8, k), k = 0, 1, 2, ..., 8\n");
    for(int k = 0; k <= 8; ++ k) {
        printf("%d\t", stirList[8][k]);
    }
    return 0;
}
```

EX12

> Prove that the Stirling numbers of the second kind satisfy the following relations:
>
> (a) $S(n, 1) = 1, \quad (n \geq 1)$
>
> (b) $S(n, 2) = 2^{n-1} - 1, \quad (n \geq 2)$
>
> (c) $S(n, n-1) = \binom{n}{n}, \quad (n \geq 1)$
>
> (d) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4} \quad (n \geq 2)$

EX12(a)

由定理8.2.5知$S(p, k)$是把p个元素集合划分到k个不可区分的盒子且没有空盒子的划分个数。
因此，$S(p, 1)$是把p个元素划分到1个盒子且没有空盒子的划分个数，显然只有1种。

EX12(b)

$$S(p, p) = 1, S(p, 1) = 1$$

$$
\begin{aligned}
S(n,2) =&2S(n-1,2)+S(n-1,1)\\
=&2S(n-1,2)+1\\
=&2(2S(n-2,2)+S(n-2,1))+1\\
=&2^2S(n-2,2)+(1+2)\\
=&2^3S(n-3,2)+(1+2+2^2)\\
=&\cdots\\
=&2^{n-2}S(n-(n-2),2)+(1+2+2^2+\cdots+2^{n-3})\\
=&\frac{1-2^{n-1}}{1-2}\\
=&2^{n-1}-1
\end{aligned}
$$

EX12(c)

使用数学归纳法证明，当n=1时，$S(1,0)=0=\binom{1}{2}$，显然成立。假设当$n=k$时有$S(k,k-1)=\binom{k}{2}$，当$n=k+1$时有，

$$
\begin{aligned}
S(k+1,k) =&kS(k,k)+S(k,k-1)\\
&=k+\binom{k}{2}\\
=k+\frac{k(k-1)}{2}\\
&=\frac{k(k+1)}{2}\\
&=\binom{k+1}{2}
\end{aligned}
$$

综上，证毕。

EX12(d)

考虑问题：将n个元素划分到n-2个不可区分的盒子且没有空盒子的个数S(n, n-2)。

如果有一个盒子中有三个元素，有$\binom{n}{3}$种情况；如果有两个盒子各有两个元素，先从n个元素中选出2个，再从剩余n-2个元素中选出2个，两种情况对称，因此是$\frac{\binom{n}{2}\binom{n-2}{2}}{2!}=3\binom{n}{4}$。

因此有$S(n,n-2)=\binom{n}{3}+3\binom{n}{4}$。

EX13

> Let X be a p-element set and let Y be a k-element set. Prove that the number of functions $f: X\rightarrow Y$ which map X onto Y equals
>
> $$k!S(p,k)=S^{\#}(p,k)$$

X映射到Y是满射，映射函数等价于把p个元素放入到k个**可区分**的盒子中，即有$S^{\#}(p,k)$个；同时由可区分盒子与不可区分盒子划分的关系，有$S^{\#}(p,k)=k!S(p,k)$，因此映射函数的个数也等于$k!S(p,k)$。

EX13吐槽

到上函数是什么鬼？没想到**onto**竟然是满射的意思，学到了。

EX14

加星题，略。

EX15

> The number of partitions of a set of n elements into k **distinguishable** boxes (some of which may be empty) is k_n. By counting in a different way, prove that
>
> $$k^n=\binom{k}{1}1!S(n,1)+\binom{k}{2}2!S(n,2)+\cdots+\binom{k}{n}n!S(n,n)$$
>
> If $k\geq n$, define $S(n,k)$ to be 0.

方法一：考虑把n个元素分别放在k个盒子中，每个元素有k种放置放法，因此共k^n种方法。

方法二：先区分盒子是否非空，从k个盒子中选出i个非空盒子，问题变为把n个元素放入i个可区分盒子且盒子非空中的方法数，即为$S^{\#}(n,i)$，i可能的取值为$i=1,2,\cdots,k$，

$$\sum_{i=1}^{k}\binom{k}{i}S^{\#}(n,i)=\sum_{i=1}^{k}\binom{k}{i}i!S(n,i)$$

方法一和方法二是同一问题的两种解决方法，因此等价，所以有，

$$k^n=\binom{k}{1}1!S(n,1)+\binom{k}{2}2!S(n,2)+\cdots+\binom{k}{n}n!S(n,n)$$

EX15注

本题中文书中有翻译错误，把可区分写成了不可区分。
EX13应该也是翻译错误（onto）。

EX16

> Compute the Bell number B_8. (Cf. Exercise 11.)

Bell数B_p是第p行的第二类Stirling数$S(p,k)$之和。

$$0+1+127+966+1701+1050+266+28+1=4140$$

EX16验证程序

```
#include <iostream>
#include <vector>

using namespace std;
int main() {
    auto stirList = vector<vector<int>>(10, vector<int>(10, 0));
    for(int p = 1; p < 10; ++ p) {
        stirList[p][p] = 1;
    }
    for(int p = 2; p < 10; ++ p) {
        for(int k = 1; k < p; ++ k) {
            stirList[p][k] = stirList[p-1][k] * k + stirList[p-1][k-1];
        }
    }
    for(int p = 0; p < 10; ++ p) {
        for(int k = 0; k <= p; ++ k) {
            printf("%d\t", stirList[p][k]);
        }
        printf("\n");
    }

    printf("\n S(8, k), k = 0, 1, 2, ..., 8\n");
    int bellNum = 0;
    for(int k = 0; k <= 8; ++ k) {
        printf("%d\t", stirList[8][k]);
        bellNum += stirList[8][k];
    }
    printf("\n Bell number B8 is %d\n", bellNum);
    return 0;
}
```

EX17

> Compute the triangle of Stirling numbers of the first kind s(n, k) up to n = 7.

第一类Stirling数的递推关系为，

$$s(p,k) = (p-1)s(p-1,k) + s(p-1,k-1)$$

初始条件与第二类Stirling数相同，$s(p,p) = 1, s(p,0) = 0, p \geq 1, s(0,0) = 1$。

EX17验证程序

```cpp
#include <iostream>
#include <vector>

using namespace std;
int main() {
    auto firstStirList = vector<vector<int>>(10, vector<int>(10, 0));
    for(int p = 1; p < 10; ++ p) {
        firstStirList[p][p] = 1;
    }
    for(int p = 2; p < 10; ++ p) {
        for(int k = 1; k < p; ++ k) {
            firstStirList[p][k] = firstStirList[p-1][k] * (p-1)
                                + firstStirList[p-1][k-1];
        }
    }
    for(int p = 0; p < 10; ++ p) {
        for(int k = 0; k <= p; ++ k) {
            printf("%d\t", firstStirList[p][k]);
        }
        printf("\n");
    }

    printf("\n s(7, k), k = 0, 1, 2, ..., 7\n");
    for(int k = 0; k <= 7; ++ k) {
        printf("%d\t", firstStirList[7][k]);
    }
    return 0;
}
```

EX17考试说明

第一类Stirling数本次考试不考，当然是在我做完第一类Stirling数的题目后才通知的。~~感觉亏了一个亿~~。

总体来说不是很难，也是通过递推式做计算。

EX18

> Write $[n]_k$ as a polynomial in n for k = 5,6, and 7.

由定义可以求出$[n]_5$，

$$\begin{aligned}[n]_5 =& n(n-1)(n-2)(n-3)(n-4)\\ =& n^5-10n^4+35n^3-50n^2+24n\end{aligned}$$

也可以通过查表写$[n]_7$，

$$\begin{aligned}[n]_7 =& \sum_{k=0}^{7}(-1)^{7-k}s(7,k)n^k\\ =& n^7-21n^6+175n^5-735n^4+1624n^3-1764n^2+720n\end{aligned}$$

EX19

> Prove that the Stirling numbers of the first kind satisfy the following formulas:
>
> (a) $s(n,1)=(n-1)!, \quad (n\ge 1)$
>
> (b) $s(n,n-1)=\binom{n}{2}, \quad (n\ge 1)$

结合递归式，易证。

EX20

> VerifY that $[n]_n$ = n!, and write n! as a polynomial in n using the Stirling numbers of the first kind. Do this explicitly for n = 6.

$[n]_p$的定义形式$[n]_p=n(n-1)(n-2)\cdots(n-(p-1)), p\ge 1$，当n=0时，$[n]_0=1$。

带入$p=n$显然有$[n]_n=n!, n\ge 0$。

$[n]_p$与第一类Stirling数的关系，$[n]_p=\sum_{k=0}^{p}(-1)^{k-p}s(p,k)n^k$，

带入$p=n$有，

$$n!=[n]_n=\sum_{k=0}^{n}(-1)^{n-k}s(p,k)n^k$$

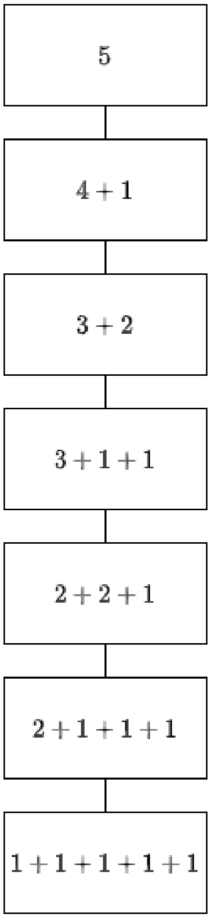
当$n=6$时，结合s(p, k)三角形有，

$$6!=6^6-15\times 6^5+85\times 6^4-225\times 6^3+274\times 6^2-120\times 6^1+0\times 6^0$$

EX21

> For each integer n = 1,2,3,4,5, construct the diagram of the set $\mathcal{P}_n$ of partitions of n, partially ordered by majorization.

~~这里的图（diagram）指的是Ferrers图，这是很容易画出的，以$5=4+1$为例，~~



EX21注

本题应该是优超（majorize）关系的Hasse图。

EX22

> (a) Calculate the partition number p_6 and construct the diagram of the set $\mathcal{P}_6$, partially ordered by majorization.
>
> (b) Calculate the partition number p_7 and construct the diagram of the set $\mathcal{P}_7$, partially ordered by majorization.

EX22(a)

以(a)为例，6对应的分拆为，

$$6,\ 51,\ 42,\ 411,\ 33,\ 321,\ 3111,\ 222,\ 2211,\ 21111,\ 111111$$

因此$p_6=11$。

EX23

> A total order on a finite set has a unique maximal element (a largest element) and a unique minimal element (a smallest element). What are the largest partition and smallest partition in the lexicographic order on $\mathcal{P}_n$ (a total order)?

最大分拆为n，最小分拆为$n=1+1+\cdots+1$。

EX24

> A partial order on a finite set may have many maximal elements and minimal elements. In the set $\mathcal{P}_n$ of partitions of n partially ordered by majorization, prove that there is a unique maximal element and a unique minimal element.

EX24说明

看了几份答案，似乎都不是严格的证明，只是稍微解释了n比其它都大，$\underbrace{1+1+\cdots+1}_{n个}$比其它都小。

EX25

> Let $t_1,t_2,\cdots,t_m$ be distinct positive integers, and let
>
> $$q_n=q_n(t_1,t_2,\cdots,t_m)$$
>
> equal the number of partitions of n in which all parts are taken from $t_1,t_2,\cdots,t_m$. Define $q_0=1$. Show that the generating function for $q_0,q_1,\cdots,q_n,\cdots$ is
>
> $$\prod_{k=1}^{m}(1-x^{t_k})^{-1}$$

$$\begin{aligned}\prod_{k=1}^{m}(1-x^{t_k})^{-1}=&\frac{1}{1-x^{t_1}}\frac{1}{1-x^{t_2}}\cdots\frac{1}{1-x^{t_m}}\\=&(\sum_{n_1=0}^{\infty}x^{t_1n_1})(\sum_{n_2=0}^{\infty}x^{t_2n_2})\cdots(\sum_{n_m=0}^{\infty}x^{t_mn_m})\\=&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_m=0}^{\infty}x^{n_1t_1+n_2t_2\cdots+n_mt_m}\\=&\sum_{n=0}^{\infty}q_nx^n\end{aligned}$$

由分拆数的性质，q_n等于方程$n_1t_1+n_2t_2+\cdots+n_mt_m=n$非负整数解$n_1,n_2,\cdots,n_m$的个数，所以$q_0,q_1,\cdots,q_n,\cdots$的生成函数为

$$\prod_{k=1}^{m}(1-x^{t_k})^k$$

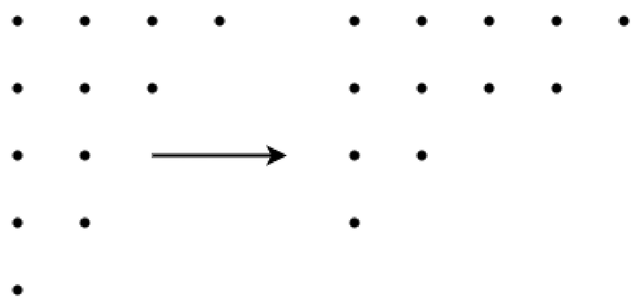
EX26

> Determine the conjugate of each of the following partitions:
>
> (a) $12=5+4+2+1$

> (b) $15 = 6 + 4 + 3 + 1 + 1$
>
> (c) $20 = 6 + 6 + 4 + 4$
>
> (d) $21 = 6 + 5 + 4 + 3 + 2 + 1$
>
> (e) $29 = 8 + 6 + 6 + 4 + 3 + 2$

EX26(a)

以(a)为例，先画出Ferrrers图，再画出共轭分拆的图，



因此共轭分拆为$12 = 4 + 3 + 2 + 2 + 1$。

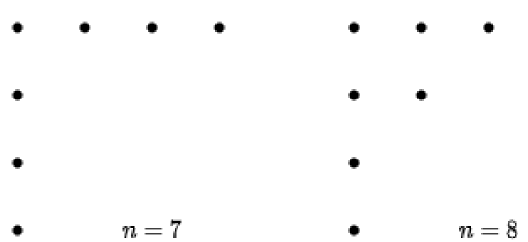
EX27

> For each integer n > 2, determine a self-conjugate partition of n that has at least two parts.

设λ是n的分拆$n_1 + n_2 + \cdots + n_k$，当n为奇数时，取$n_1 = \frac{(n+1)}{2}, n_2 = n_3 = \cdots = n_k = 1, k = \frac{n+1}{2}$。

当n为偶数时，取$n_1 = \frac{n}{2}, n_2 = 2, n_3 = n_4 = \cdots = n_k = 1, k = \frac{n}{2}$。

以n=7和n=8分别为奇数和偶数的例子，如图，



EX28

> Prove that conjugation reverses the order of majorization; that is, if λ and μ are partitions of n and λ is majorized by μ, then μ^* is majorized by λ^*.

由题意，当λ被μ优超时，有

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \mu_i, \quad 1 \leq i \leq k \tag{1}$$

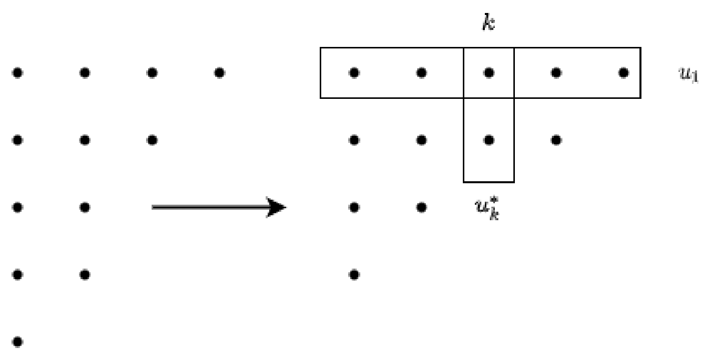
假设$\mu^* \not\leq \lambda^*$，即存在k使，

$$\mu_1^* + \mu_2^* + \cdots + \mu_i^* \leq \lambda_1^* + \lambda_2^* + \cdots + \lambda_i^*, \quad 1 \leq i < k$$
$$\mu_1^* + \mu_2^* + \cdots + \mu_k^* > \lambda_1^* + \lambda_2^* + \cdots + \lambda_k^*$$

即有$\mu_k^* > \lambda_k^*$，记$u = \mu_k^*, v = \lambda_k^*$。又因为$\mu^*$和$\lambda^*$都是n的分拆，所以有

$$\mu_{k+1}^* + \mu_{k+2}^* + \cdots \leq \lambda_{k+1}^* + \lambda_{k+2}^* + \cdots$$

如图，由互换行列前后的关系可得，



$$\mu_{k+1}^* + \mu_{k+2}^* \cdots = \sum_1^u (u_i - k),\ \lambda_{k+1}^* + \lambda_{k+2}^* + \cdots = \sum_{i=1}^v (\lambda_i - k)$$

有放缩，

$$\sum_{i=1}^v (\mu_i - k) < \sum_{i=1}^u \leq \sum_{i=1}^v (\lambda_i - k)$$

可得

$$\mu_1 + \mu_2 + \cdots + \mu_v < \lambda_1 + \lambda_2 + \cdots + \lambda_v \tag{2}$$

其中(1)式与(2)式矛盾，因此假设不成立，综上，证毕。

EX29

> Prove that the number of partitions of the positive integer n into parts each of which is at most 2 equals $\lfloor n/2 \rfloor + 1$. (Remark: There is a formula, namely the nearest integer to $\frac{(n+3)^2}{12}, for the number of partitions of n into parts each of which is at most 3 but it is much more difficult to prove. There is also one for partitions with no part more than 4, but it is even more complicated and difficult to prove.)

当$n = 2r$时，每一部分至多是2的分拆为

$$1^n,\ 2^1 1^{n-2},\ 2^2 1^{n-4},\ \cdots,\ 2^r$$

当$n=2r+1$时，每一部分最多是2的分拆为

$$1^n,\ 2^1 1^{n-2},\ 2^2 1^{n-4},\ \cdots,\ 2^r 1^1$$

不论奇偶，都是分拆为r+1个部分，当r为偶数时，$r+1=\frac{n}{2}+1=\lfloor n/2\rfloor+1$，当r为奇数时，$r+1=\frac{n-1}{2}+1=\lfloor (n+1)/2\rfloor=\lfloor n/2\rfloor+1$。

综上，分拆成每一部分至多是2的分拆数等于$\lfloor n/2\rfloor$。

EX30

> Prove that the partition function satisfies
>
> $$p_n > p_{n-1} \quad (n \geq 2)$$

考虑n-1的分拆数和n的分拆数，显然所有n-1的分拆数+1都是n的分拆数，此外n还有它自己作为分拆数，因此一定有$p_n > p_{n-1}$。

EX31

> Evaluate $h_{k-1}^{(k)}$ the number of regions into which k-dimensional space is partitioned by k - 1 hyperplanes in general position.

$$h_{k-1}^{(k)}=\binom{k-1}{0}+\binom{k-1}{1}+\cdots+\binom{k-1}{k-1}+\binom{k-1}{k}=2^{k-1}$$

EX31注

超平面是8.4节的内容，本次考试不涉及。

EX32

> Use the recurrence relation (8.31) to compute the small Schroder numbers s_8 and s_9.

小Schroder数的性质：

1. $s_1=s_2=1$
2. $(n+2)s_{n+2}-3(2n+1)x_{n+1}+(n-1)s_n=0, n\geq 1$

由递推关系和初始项，可以计算出$s_3=3, s_4=11, s_5=45, s_6=197, s_7=903$，

$$8s_8-3\times 13\times 903+5\times 197=0$$

求出$s_8=4279$，同理可以求出$9s_9-3\times 15\times 4279+6\times 903=0$，得$s_9=20793$。

EX33

> Use the recurrence relation (8.32) to compute the large Schroder numbers R_7 and R_8. Verify that $R_7 = 2s_8$ and $R_8 = 2s_9$, as stated in Corollary 8.5.8.

大Schroder数的性质：

$$R_n = \sum_{r=0}^{n} \frac{1}{n-r+1} \frac{(2n-r)!}{r![(n-r)!]^2}$$

带入$n = 7$计算，

$$\begin{aligned}
R_7 =& \sum_{r=0}^{7} \frac{1}{8-r} \frac{(14-r)!}{r![(7-r)!]^2} \\
=& \frac{1}{8}\frac{14!}{0!(7!)^2} + \frac{1}{7}\frac{13!}{1!(6!)^2} + \frac{1}{6}\frac{12!}{2!(5!)^2} + \frac{1}{5}\frac{11!}{3!(4!)^2} + \frac{1}{4}\frac{10!}{4!(3!)^2} + \frac{1}{3}\frac{9!}{5!(2!)^2} + \frac{1}{2}\frac{8!}{6!(1!)^2} + \frac{1}{1}\frac{7!}{7!(0!)^2} \\
=& 8558 = 2 \times 4279 = 2s_8
\end{aligned}$$

EX33注

题目要求使用**递推关系**计算，大Schroder数的递推关系为，

$$R_n = R_{n-1} + \sum_{k=1}^{n} R_{k-1} R_{n-k}$$

注意与Catalan数进行区分。

EX34

> Use the generating function for the large Schroder numbers to compute the first few large Schroder numbers.

大Schroder数序列的生成函数为

$$\sum_{n=0}^{\infty} R_n x^n = \frac{1}{2x}(-(x-1) - \sqrt{x^2 - 6x + 1})$$

$\sqrt{x^2 - 6x + 1}$在x=0处的泰勒级数为$1 - 3x - 4x^2 - 12x^3 - 44x^4 + \cdots$，

因此，

$$\begin{aligned}
\sum_{k=0}^{\infty} R_n x^n =& \frac{1}{2x}(-(x-1) - (1 - 3x - 4x^2 - 12x^3 - 44x^4 + \cdots)) \\
=& \frac{2x + 4x^2 + 12x^3 + 44x^4 + \cdots}{2x} \\
=& 1 + 2x + 6x^2 + 22x^3 + \cdots
\end{aligned}$$

综上，有$R_0 = 1, R_1 = 2, R_2 = 6, R_3 = 22$。

EX35

> Use the generating function for the small Schroder numbers to compute the first few small Schroder numbers.

小Schroder数序列的生成函数为

$$\sum_{n=1}^{\infty} s_n x^n = \frac{1}{4}(1 + x - \sqrt{x^2 - 6x + 1})$$

同上，带入$\sqrt{x^2 - 6x + 1}$的泰勒级数，

$$\begin{aligned}\sum_{k=1}^{\infty} s_n x^n =& \frac{4x + 4x^2 + 12x^3 + 44x^4 + \cdots}{4} \\ =& x + x^2 + 3x^3 + 11x^4 + \cdots\end{aligned}$$

综上，有$r_1 = 1, r_2 = 1, r_3 = 3, r_4 = 11$。

EX36

> Prove that the Catalan number C_n equals the number of lattice paths from (0,0) to (2n, 0) using only upsteps (1, 1) and downsteps (1, -1) that never go above the horizontal axis (so there are as many up steps as there are downsteps). (These are sometimes called *Dyck paths*.)

记上行步(1,1)为-1，下行步(1,-1)为+1，步行序列为$a_1, a_2, \cdots, a_{2n}$。

因为起点y坐标与终点y坐标相同，那么一定有n个上行步（+1）和n个下行步（-1），并且从不经过水平轴上方的格路径，即前k项和$a_1 + a_2 + \cdots + a_k \geq 0, 1 \leq k \leq 2n$，该问题与第n个Catalan数的组合意义相同，因此等于C_n。

EX37

加星题，略。