

The problem being solved in this directory is:

$$\ddot{\theta} = -\frac{g}{l} \cos \theta$$

integrating both sides with respect to θ yields:

$$\begin{aligned} \frac{\dot{\theta}^2}{2} &= -\frac{g}{l} \sin \theta + C \\ \dot{\theta} &= \pm \sqrt{-\frac{2g}{l} \sin \theta + C} \\ \implies C &= \dot{\theta}_0^2 \\ \therefore \dot{\theta} &= \pm \sqrt{\dot{\theta}_0^2 - \frac{2g}{l} \sin \theta} \end{aligned}$$

where $\dot{\theta}_0$ is $\dot{\theta}$ when $\sin \theta = 0$ (therefore $\theta = n\pi$ where $n \in \mathbb{Z}$). t can therefore be computed as:

$$t = \pm \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{\dot{\theta}_0^2 - \frac{2g}{l} \sin \theta}}.$$

In this repository, the initial conditions are:

$$\theta(t=0) = \dot{\theta}(t=0) = 0.$$

In other words, the pendulum bob starts at the positive x axis with zero velocity and moves solely under the influence of gravity. If we imagine a pendulum subject to these conditions, it becomes clear that θ will range from $-\pi$ (the bob being right on the negative x -axis) to 0. If we wish to determine the period of θ (i.e. the value of χ such that $\theta(t+\chi) = \theta(t) \forall t$), we must set $\theta_0 = 0$, $\theta_1 = -\pi$ and multiply our final result by two (as our result will only reflect how long it takes to go from the positive x axis to the

negative x axis, not how long it will take to make the return trip). Namely:

$$\begin{aligned}\chi &= -2 \int_0^{-\pi} \frac{d\theta}{\sqrt{-\frac{2g}{l} \sin \theta}} \\ &= 2 \int_{-\pi}^0 \frac{d\theta}{\sqrt{-\frac{2g}{l} \sin \theta}}.\end{aligned}$$

Above we chose the negative on the \pm sign because otherwise we will get a negative value for t , and we are choosing to keep time positive. It is impossible to solve this integral analytically and use it to express θ in terms of t , therefore we are reduced to using numerical methods to approximate θ . The three numerical methods used in this directory are:

- `ode78` from the `ODE.jl` Julia module.
- Runge-Kutta 4th order method.
- The Newton-Kantorovich method to linearize the problem, and then a Chebyshev spectral method to approximate the solution to the linearized version of the problem.

Out of these, only the Newton-Kantorovich method likely needs further explanation. To linearize the problem, we used $\theta_{i+1} = \theta_i + \Delta_i$, where θ_i is our i th approximation of θ and Δ_i is our i th correction to θ . Substituting θ_{i+1} into our original equation yields:

$$\begin{aligned}\ddot{\theta}_i + \ddot{\Delta}_i &= -\frac{g}{l} \cos(\theta_i + \Delta_i) \\ &\approx -\frac{g}{l} (\cos \theta_i - \sin \theta_i \Delta_i)\end{aligned}$$

Rearranging we get the following linear ordinary differential equation for Δ_i :

$$\ddot{\Delta}_i - \frac{g}{l} \sin \theta_i \Delta_i = -\ddot{\theta}_i - \frac{g}{l} \cos \theta_i$$

where $\Delta_i(0) = \dot{\Delta}_i(0) = 0$. Alternatively, Δ_i 's initial conditions can be expressed in terms of the initial values of θ_i , such as:

$$\begin{aligned}\Delta_i(0) &= -\theta_i(0) \\ \dot{\Delta}_i(0) &= -\dot{\theta}_i(0).\end{aligned}$$

which can be useful if the matrices are ill-conditioned and the initial values start to markedly deviate from what they are supposed to as a result. Naturally, as is the case with Newton's method for approximating the solution to a nonlinear algebraic equation, one needs an initial guess as to the solution in order to apply the Newton-Kantorovich method. As we know the solution will be periodic, have a minimum of $\theta = -\pi$ and a maximum of $\theta = 0$, we can use the following as our first guess:

$$\theta_0 = \frac{\pi}{2} \left(\cos \left(\frac{2\pi t}{\chi} \right) - 1 \right).$$

With this first guess, our initial conditions are met (as $\theta_0(t=0) = 0$ and $\dot{\theta}_0(t=0) = 0$), our solution is periodic with a period of χ , and at $t = \frac{\chi}{2}$, $\theta_0 = -\pi$, which is what we would also expect from our solution.