The problem being solved in this directory is:

$$\ddot{\theta} = -\frac{g}{l}\cos\theta$$

integrating both sides with respect to θ yields:

$$\begin{aligned}
\frac{\dot{\theta}^2}{2} &= -\frac{g}{l}\sin\theta + C \\
\dot{\theta} &= \pm\sqrt{-\frac{2g}{l}\sin\theta + C} \\
\implies C &= \dot{\theta}_0^2 \\
\therefore \dot{\theta} &= \pm\sqrt{\dot{\theta}_0^2 - \frac{2g}{l}\sin\theta}
\end{aligned}$$

where $\dot{\theta}_0$ is $\dot{\theta}$ when $\sin\theta = 0$ (therefore $\theta = n\pi$ where $n \in \mathbb{Z}$). t can therefore be computed as:

$$t = \pm\int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{\dot{\theta}_0^2 - \frac{2g}{l}\sin\theta}}.$$

In this repository, the initial conditions are:

$$\theta(t=0) = \dot{\theta}(t=0) = 0.$$

In other words, the pendulum bob starts at the positive x axis with zero velocity and moves solely under the influence of gravity. If we imagine a pendulum subject to these conditions, it becomes clear that theta will range from $-\pi$ (the bob being right on the negative x-axis) to 0. If we wish to determine the period of θ (i.e. the value of χ such that $\theta(t+\chi) = \theta(t)\ \forall t$), we must set $\theta_0 = 0$, $\theta_1 = -\pi$ and multiply our final result by two (as our result will only reflect how long it takes to go from the positive x axis to the negative x axis, not how long it will take to make the return trip). Namely:

$$\begin{aligned}\chi &= -2\int_0^{-\pi} \frac{d\theta}{\sqrt{-\frac{2g}{l}\sin\theta}} \\ &= 2\int_{-\pi}^{0} \frac{d\theta}{\sqrt{-\frac{2g}{l}\sin\theta}}.\end{aligned}$$

Above we chose the negative on the $\pm$ sign because otherwise we will get a negative value for t, and we are choosing to keep time positive. It is impossible to solve this integral analytically and use it to express θ in terms of t, therefore we are reduced to using numerical methods to approximate θ. The three numerical methods used in this directory are:

- `ode78` from the ODE.jl Julia module.
- Runge-Kutta 4th order method.
- The Newton-Kantorovich method to linearize the problem, and then a Chebyshev spectral method to approximate the solution to the linearized version of the problem.

Out of these, only the Newton-Kantorovich method likely needs further explanation. To linearize the problem, we used $\theta_{i+1} = \theta_i + \Delta_i$, where θ_i is our ith approximation of θ and Δ_i is our ith correction to θ. Substuting θ_{i+1} into our original equation yields:

$$\begin{aligned}\ddot{\theta}_i + \ddot{\Delta}_i &= -\frac{g}{l}\cos\left(\theta_i + \Delta_i\right) \\ &\approx -\frac{g}{l}\left(\cos\theta_i - \sin\theta_i\Delta_i\right)\end{aligned}$$

Rearranging we get the following linear ordinary differential equation for Δ_i:

$$\ddot{\Delta}_i - \frac{g}{l}\sin\theta_i\Delta_i = -\ddot{\theta}_i - \frac{g}{l}\cos\theta_i$$

where $\Delta_i(0) = \dot{\Delta}_i(0) = 0$. Alternatively, Δ_i's initial conditions can be expressed in terms of the initial values of θ_i, such as:

$$\Delta_i(0) = -\theta_i(0)$$
$$\dot{\Delta}_i(0) = -\dot{\theta}_i(0).$$

which can be useful if the matrices are ill-conditioned and the initial values start to markedly deviate from what they are supposed to as a result. Naturally, as is the case with Newton's method for approximating the solution to a nonlinear algebraic equation, one needs an initial guess as to the solution in order to apply the Newton-Kantorovich method. As we know the solution will be periodic, have a minimum of $\theta = -\pi$ and a maximum of $\theta = 0$, we can use the following as our first guess:

$$\theta_0 = \frac{\pi}{2}\left(\cos\left(\frac{2\pi t}{\chi}\right) - 1\right)$$

With this first guess, our initial conditions are met (as $\theta_0(t = 0) = 0$ and $\dot{\theta}_0(t = 0) = 0$), and our solution is periodic with a period of χ.