Regression Introduction and Estimation Review

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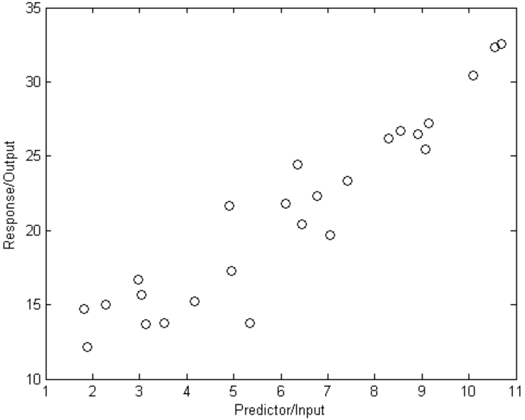
Quick Example - Scatter Plot



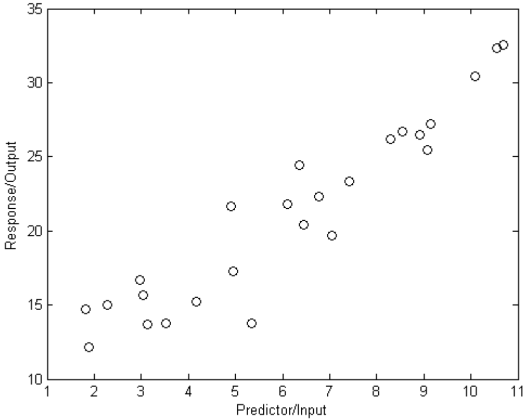
Linear Regression

- Want to find parameters for a function of the form

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Distribution of error random variable not specified

Quick Example - Scatter Plot



Formal Statement of Model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Y_i value of the response variable in the i^{th} trial
- β_0 and β_1 are parameters
- X_i is a known constant, the value of the predictor variable in the i^{th} trial
- ϵ_i is a random error term with mean $\mathbb{E}(\epsilon_i)$ and variance $\text{Var}(\epsilon_i) = \sigma^2$
- $i = 1, \ldots, n$

Properties

- The response Y_i is the sum of two components
  - Constant term $\beta_0 + \beta_1 X_i$
  - Random term ϵ_i
- The expected response is

$$
\begin{aligned}
\mathbb{E}(Y_i) &= \mathbb{E}(\beta_0 + \beta_1 X_i + \epsilon_i) \\
&= \beta_0 + \beta_1 X_i + \mathbb{E}(\epsilon_i) \\
&= \beta_0 + \beta_1 X_i
\end{aligned}
$$

Expectation Review

- Definition

$$\mathbb{E}(X) = \mathbb{E}(X) = \int XP(X)dX,\; X \in \mathcal{R}$$

- Linearity property

$$\begin{aligned} \mathbb{E}(aX) &= a\,\mathbb{E}(X) \\ \mathbb{E}(aX + bY) &= a\,\mathbb{E}(X) + b\,\mathbb{E}(Y) \end{aligned}$$

- Obvious from definition

Example Expectation Derivation

$$P(X) = 2X, 0 \leq X \leq 1$$

Expectation

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^1 XP(X)dX \\
&= \int_0^1 2X^3 dX \\
&= \frac{2X^2}{3}|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Expectation of a Product of Random Variables

If X,Y are random variables with joint distribution $P(X, Y)$ then the expectation of the product is given by

$$\mathbb{E}(XY) = \int_{XY} XYP(X, Y)dXdY.$$

Expectation of a product of random variables

What if X and Y are independent? If X and Y are independent with density functions f and g respectively then

$$
\begin{aligned}
\mathbb{E}(XY) &= \int_{XY} XYf(X)g(Y)dXdY \\
&= \int_X \int_Y XYf(X)g(Y)dXdY \\
&= \int_X Xf(X)[\int_Y Yg(Y)dY]dX \\
&= \int_X Xf(X)\,\mathbb{E}(Y)dX \\
&= \mathbb{E}(X)\,\mathbb{E}(Y)
\end{aligned}
$$

Regression Function

- The response Y_i comes from a probability distribution with mean

$$\mathbb{E}(Y_i) = \beta_0 + \beta_1 X_i$$

- This means the regression function is

$$\mathbb{E}(Y) = \beta_0 + \beta_1 X$$

Since the regression function relates the means of the probability distributions of Y for a given X to the level of X

Error Terms

- The response Y_i in the i^{th} trial exceeds or falls short of the value of the regression function by the error term amount ϵ_i
- The error terms ϵ_i are assumed to have constant variance σ^2

Response Variance

Responses Y_i have the same constant variance

$$\begin{aligned}
\mathrm{Var}(Y_i) &= \mathrm{Var}(\beta_0 + \beta_1 X_i + \epsilon_i) \\
&= \mathrm{Var}(\epsilon_i) \\
&= \sigma^2
\end{aligned}$$

Variance (2^{nd} central moment) Review

- Continuous distribution

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \int (X - \mathbb{E}(X))^2 P(X) dX, \; X \in \mathcal{R}$$

- Discrete distribution

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \sum_i (X_i - \mathbb{E}(X))^2 P(X_i), \; X \in \mathcal{Z}$$

Alternative Form for Variance

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) \\
&= \mathbb{E}((X^2 - 2X\,\mathbb{E}(X) + \mathbb{E}(X)^2)) \\
&= \mathbb{E}(X^2) - 2\,\mathbb{E}(X)\,\mathbb{E}(X) + \mathbb{E}(X)^2 \\
&= \mathbb{E}(X^2) - 2\,\mathbb{E}(X)^2 + \mathbb{E}(X)^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X)^2.
\end{aligned}
$$

Example Variance Derivation

$$P(X) = 2X, 0 \leq X \leq 1$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \int_0^1 2XX^2 dX - (\frac{2}{3})^2 \\
&= \frac{2X^4}{4}|_0^1 - \frac{4}{9} \\
&= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}
\end{aligned}$$

Variance Properties

$$\begin{aligned}
\mathrm{Var}(aX) &= a^2\,\mathrm{Var}(X) \\
\mathrm{Var}(aX + bY) &= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) \,\text{if}\, X \perp\!\!\!\perp Y \\
\mathrm{Var}(a + cX) &= c^2\,\mathrm{Var}(X) \,\text{if}\, a, c \text{ both constant}
\end{aligned}$$

More generally

$$\mathrm{Var}(\sum a_i X_i) = \sum_i \sum_j a_i a_j \mathrm{Cov}(X_i, X_j)$$

Covariance

- The covariance between two real-valued random variables X and Y, with expected values $\mathbb{E}(X) = \mu$ and $\mathbb{E}(Y) = \nu$ is defined as

$$Cov(X, Y) = \mathbb{E}((X - \mu)(Y - \nu))$$

- Which can be rewritten as

$$\begin{aligned} Cov(X, Y) &= \mathbb{E}(XY - \nu X - \mu Y + \mu\nu), \\ Cov(X, Y) &= \mathbb{E}(XY) - \nu\,\mathbb{E}(X) - \mu\,\mathbb{E}(Y) + \mu\nu, \\ Cov(X, Y) &= \mathbb{E}(XY) - \mu\nu. \end{aligned}$$

Covariance of Independent Variables

If X and Y are independent, then their covariance is zero. This follows because under independence

$$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y) = \mu\nu.$$

and then

$$Cov(XY) = \mu\nu - \mu\nu = 0.$$

Least Squares Linear Regression

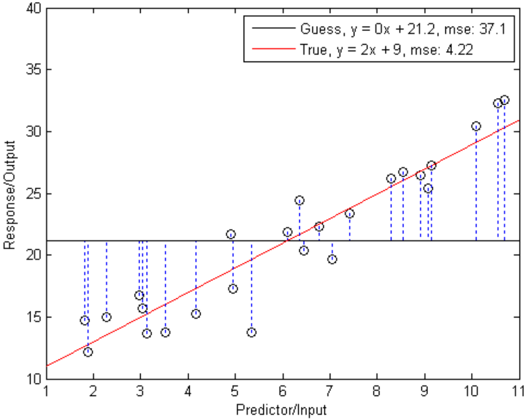
- Seek to minimize

$$Q = \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_i))^2$$

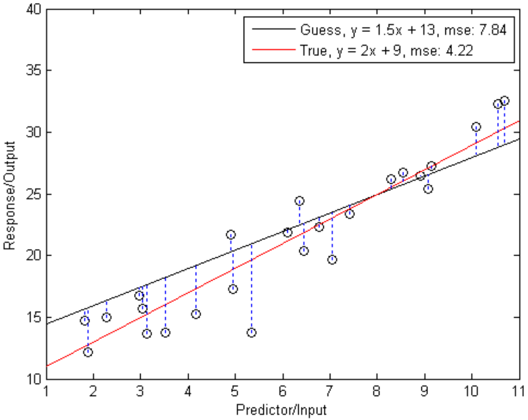
- By careful choice of b_0 and b_1 where b_0 is a point estimator for β_0 and b_1 is the same for β_1

  How?

Guess #1


Guess, y = 0x + 21.2, mse: 37.1
True, y = 2x + 9, mse: 4.22
Response/Output
Predictor/Input
40
35
30
25
20
15
10
1
2
3
4
5
6
7
8
9
10
11

Guess #2


Guess, y = 1.5x + 13, mse: 7.84
True, y = 2x + 9, mse: 4.22
Response/Output
Predictor/Input
40
35
30
25
20
15
10
1
2
3
4
5
6
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8
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10
11

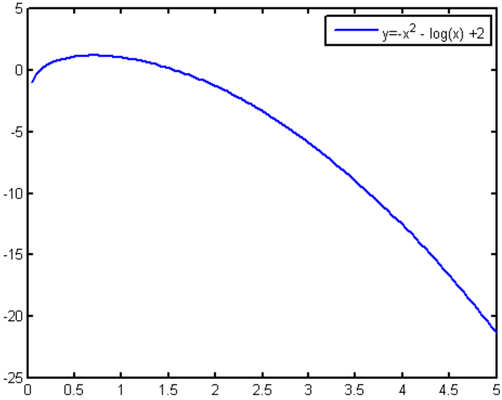
Function maximization

- Important technique to remember!
  - Take derivative
  - Set result equal to zero and solve
  - Test second derivative at that point
- Question: does this always give you the maximum?
- Going further: multiple variables, convex optimization

Function Maximization

Find

$$\underset{x}{\text{argmax}} \quad -x^2 + ln(x)$$



Least Squares Max(min)imization

- Function to minimize w.r.t. b_0 and b_1, b_0 and b_1 are called point estimators of β_0 and β_1 respectively

$$Q = \sum_{i=1}^{n} (Y_i - (b_0 + b_1 X_i))^2$$

- Minimize this by maximizing -Q
- Either way, find partials and set both equal to zero

$$\begin{aligned} \frac{dQ}{db_0} &= 0 \\ \frac{dQ}{db_1} &= 0 \end{aligned}$$

Normal Equations

- The result of this maximization step are called the normal equations.

$$\begin{aligned}\sum Y_i &= nb_0 + b_1 \sum X_i \\ \sum X_i Y_i &= b_0 \sum X_i + b_1 \sum X_i^2\end{aligned}$$

- This is a system of two equations and two unknowns. The solution is given by...

Solution to Normal Equations

After a lot of algebra one arrives at

$$
\begin{aligned}
b_1 &= \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} \\
b_0 &= \bar{Y} - b_1\bar{X} \\
\bar{X} &= \frac{\sum X_i}{n} \\
\bar{Y} &= \frac{\sum Y_i}{n}
\end{aligned}
$$