

LINEAR REGRESSION MODELS W4315

HOMEWORK 1 ANSWERS

September 12, 2010

Professor: Frank Wood

1. (20 points) Let $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ be a linear regression model with distribution of error terms unspecified (but with mean $E(\epsilon) = 0$ and variance $V(\epsilon_i) = \sigma^2$ (σ^2 finite) known). Show that $s^2 = MSE = \frac{\sum(Y_i - \hat{Y}_i)^2}{n-2}$ is an unbiased estimator for σ^2 . $\hat{Y}_i = b_0 + b_1 X_i$ where $b_0 = \bar{Y} - b_1 \bar{X}$ and $b_1 = \frac{\sum_i((X_i - \bar{X})(Y_i - \bar{Y}))}{\sum_i(X_i - \bar{X})^2}$

Answer:

First, let's denote the followings:

$$\hat{e}_i = y_i - \hat{y}_i$$

$$SXX = \sum_{i=1}^n (x_i - \bar{x})^2$$

$$SYY = \sum_{i=1}^n (y_i - \bar{y})^2$$

$$SXY = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

Now we set out to prove the following equation which accomplishes essentially the final result:

$$Var \hat{e}_i = E \hat{e}_i^2 = \left(\frac{n-2}{n} + \frac{1}{SXX} \left(\frac{1}{n} \sum_{j=1}^n x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i \bar{x} \right) \right) \sigma^2$$

To prove the above, realize that:

$$\begin{aligned} Var(\hat{e}_i) &= Var(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\ &= Var((y_i - \beta_0 - \beta_1 x_i) - (\hat{\beta}_0 - \beta_0) - x_i(\hat{\beta}_1 - \beta_1)) \\ &= Var(y_i) + Var(\hat{\beta}_0) + x_i^2 Var(\hat{\beta}_1) - 2Cov(y_i, \hat{\beta}_0) - 2x_i Cov(y_i, \hat{\beta}_1) + 2x_i Cov(\hat{\beta}_0, \hat{\beta}_1) \end{aligned}$$

The last equation holds because the covariance between any random variable and a constant is zero, and all the y_i 's are independent entailing that the $Cov(y_i, y_j) = 0, i \neq j$

$$Var(y_i) = \sigma^2$$

Notice that(some algebras needed here, and the following tricks are crucial in reducing the amount of calculation):

$$\sum (x_i - \bar{x}) = 0$$

$$\beta_1 = \frac{\sum x_i - \bar{x} y_i}{SXX}$$

So now we have:

$$\begin{aligned} Var(\beta_1) &= Var\left(\frac{SXY}{SXX}\right) \\ &= Var\left(\frac{\sum (x_i - \bar{x}) y_i}{SXX}\right) \\ &= \frac{1}{SXX^2} \sum x_i - \bar{x}^2 Var(y_i) \\ &= \frac{\sigma^2}{SXX} \end{aligned}$$

And:

$$\begin{aligned} Var(\beta_0) &= Var(\bar{y} - \hat{\beta}_1 \bar{x}) \\ &= Var\left(\sum \left(\frac{1}{n} - \frac{(x_i - \bar{x}) \bar{x}}{SXX}\right) y_i\right) \\ &= \sum \left(\frac{1}{n} - \frac{x_i - \bar{x}}{SXX} \bar{x}\right)^2 \sigma^2 \\ &= \sum \left[\frac{1}{n^2} + \frac{SXX * \bar{x}^2}{SXX^2} - \frac{2 \bar{x} (x_i - \bar{x})}{n * XSS}\right] \sigma^2 \\ &= \left[\frac{1}{n} + \frac{n \bar{x}^2}{SXX}\right] \sigma^2 \\ &= \frac{\sum x_i^2}{n * SXX} \sigma^2 \end{aligned}$$

For the other terms in the decomposition of $Var(\hat{e}_i)$, we have:

$$\begin{aligned} Cov(y_i, \hat{\beta}_1) &= Cov\left(y_i, \frac{\sum x_i - \bar{x} y_i}{SXX}\right) \\ &= \frac{x_i - \bar{x}}{SXX} Var(y_i) \\ &= \frac{x_i - \bar{x}}{SXX} \sigma^2 \end{aligned}$$

and:

$$\begin{aligned}
Cov(y_i, \hat{\beta}_0) &= Cov(y_i, \bar{y} - \hat{\beta}_1 \bar{x}) \\
&= Cov(y_i, \frac{\sum y_i}{n} - \sum (x_i - \bar{x}) y_i SXX \bar{x}) \\
&= \frac{\sigma^2}{n} + \bar{x} \frac{x_i - \bar{x}}{SXX} \sigma^2
\end{aligned}$$

At last, we have:

$$\begin{aligned}
Cov(\hat{\beta}_0, \hat{\beta}_1) &= Cov(\bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1) \\
&= Cov(\frac{\sum y_i}{n} - \sum \frac{(x_i - \bar{x}) \bar{x}}{SXX} y_i, \sum \frac{(x_i - \bar{x}) y_i}{SXX}) \\
&= \sum_{i=1}^n (\frac{1}{n} - \frac{x_i - \bar{x}}{SXX} \bar{x}) \frac{x_i - \bar{x}}{SXX} \sigma^2 \\
&= -\frac{\bar{x}}{SXX} \sigma^2
\end{aligned}$$

Then plug in all the parts back to the decomposition of $Var(\hat{e}_i)$, we have:

$$Var(\hat{e}_i) = (\frac{n-1}{n} + \frac{1}{SXX} (\frac{1}{n} \sum_{j=1}^n x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i \bar{x})) \sigma^2$$

Thus,

$$\begin{aligned}
E\hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n E\hat{e}_i^2 \\
&= \frac{1}{n} \sum_{i=1}^n [\frac{n-2}{n} + \frac{1}{SXX} (\frac{1}{n} \sum_{j=1}^n x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i \bar{x})] \sigma^2 \\
&= [\frac{n-2}{n} + \frac{1}{nS_{xx}} \{ \sum_{j=1}^n x_j^2 + \sum_{i=1}^n x_i^2 - 2SXX - 2\frac{1}{n} (\sum_{i=1}^n x_i)^2 \}] \sigma^2 \\
&= (\frac{n-2}{n} + 0) \sigma^2 \\
&= \frac{n-2}{n} \sigma^2
\end{aligned}$$

where the third equation holds because: $\sum x_i \bar{x} = \frac{1}{n} (\sum x_i)^2$

and the second to last equation holds since $\sum x_i^2 - \frac{1}{n} (\sum x_i)^2 = SXX$

From the above equation, the result flows.

2. (20 points) Derive the maximum likelihood estimators $\hat{\beta}_0, \hat{\beta}_1$, and $\hat{\sigma}^2$ for parameters β_0, β_1 , and σ^2 for the normal linear regression model (i.e. $\epsilon_i \sim_{iid} N(0, \sigma^2)$).

Answer:

To figure the MLE of the parameters, we need to first write down the likelihood function of the data, so under normal assumption, we have the log-likelihood function as follows:

$$\log L(\beta_0, \beta_1, \sigma^2 | x, y) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}.$$

For any fixed value of σ^2 , $\log L$ is maximized as a function of β_0 and β_1 , that minimize

$$\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \quad (1)$$

But to minimize this function is just to principle behind LSE, so it's apparent that the MLE of β_0 and β_1 are the same as their LSE's. Now, substituting in the log-likelihood, to find the MLE of σ^2 we need to maximize

$$-\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{2\sigma^2}$$

This maximization problem is nothing but MLE of σ^2 in ordinary normal sampling problems, which is easily given as

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

If you are not familiar with the MLE in normal sampling setting, you can take derivative with respect to σ^2 (N.B. not σ), and then set the derivative to be zero. The solution of the equation is just the MLE of σ^2 .

3. (20 points) $X_1, X_2, X_3, \dots, X_{100}$ are iid normal random variables with mean μ and variance σ^2 , we want to estimate the mean μ . Consider two estimators, X_1 and $\bar{X} = \frac{1}{100} \sum_{i=1}^{100} X_i$:

a. Show these two estimators are both unbiased. Also derive the distribution of each estimator. Which estimator do you think is better? Why?

b. When $\mu = 0$ and $\sigma^2 = 100$, generate $X_1, X_2, X_3, \dots, X_{100}$. Calculate the estimate \bar{X} and denote it as $\hat{\mu}_1$. Repeat this 100 times and plot the density histogram of $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \dots, \hat{\mu}_{100}$.

Overlay the probability density of \bar{X} on the plot (see matlab function “normpdf”).

Answer:

a. X_1 is unbiased is pretty obvious, for \bar{X} , using linearity of expectation, we have

$$E(\bar{X}) = E\left(\frac{1}{100} \sum_{i=1}^{100} X_i\right) = \frac{1}{100} \sum_{i=1}^{100} EX_i = \frac{1}{100} \sum_{i=1}^{100} \mu = \mu$$

Apprantly, $X_1 \sim N(\mu, \sigma^2)$. \bar{X} is a linear combination of normal random variables, so it is still a normal random variable. Thanks to indepenence, its variance is given by

$$Var(\bar{X}) = \frac{1}{100^2} Var\left(\sum_{i=1}^{100} X_i\right) = \frac{1}{100^2} \sum_{i=1}^{100} Var(X_i) = \frac{\sigma^2}{100}$$

so $\bar{X} \sim N(\mu, \frac{1}{100}\sigma^2)$. Obviously, \bar{X} is better since it has smaller variance.

4. (40 points) Copier maintenance.¹ The Tri-City Office Equipment Corporation sells an imported copier on a franchise basis and performs preventive maintenance and repair services on this copier. The data below have been collected from 45 recent calls on users to perform routine preventive number of minutes spent by the service person. Assume that first-order regression model($Y_i = b_0 + b_1X_i + \epsilon_i$) is appropriate.

i:	1	2	3	...	43	44	45
X_i	2	4	3	...	2	4	5
Y_i	20	60	46	...	27	61	77

- Obtain estimated regression function.
- Plot the estimated regression function and the data. How well does the estimated regression function fit the data?
- interpret b_0 in your estimated regression function. Does b_0 provide any relevant information here? Explain.
- Obtain a point estimate of the mean service time when $X = 5$ copiers are serviced.

Notice: You can get data for this problem on www.mhhe.com/KutnerALRM4e. Use

¹This is problem 1.20 in “Applied Linear Regression Models(4th edition)” by Kutner etc.)

MATLAB, do not use any other programming language. Only basic MATLAB operators are allowed, do not use any built-in functions to do the regression, i.e. the function “regress” cannot be used except, perhaps, to verify that your answer is correct before submitting your own implementation.

Answer: