LINEAR REGRESSION MODELS W4315

HOMEWORK 5 ANSWERS

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Instructor: Frank Wood

1. (20 points) [1] Bonferroni inequality (4.2a) which is given as

$$P(\bar{A}_1 \bigcap \bar{A}_2) \geq 1 - \alpha - \alpha = 1 - 2\alpha$$

deals with the case of two statements, A_1 and A_2. Extend the inequality to the case of n statements, namely, $A_1, A_2 \ldots, A_n$, each with statement confidence coefficient $1 - \alpha$.

Answer:

Following the thread on Page 155 in the textbook, we have:

$$P(A_1) = P(A_2) = \ldots = P(A_n) = \alpha$$

then

$$P(\cap_{i=1}^{n} \bar{A}_i) = P(\overline{\cup_{i=1}^{n} A_i}) = 1 - P(\cup_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i)$$

So we have $P(\cap_{i=1}^{n} \bar{A}_i) \geq 1 - n\alpha$

2. (40 points) [3] In a small-scale regression study, the following data were obtained: Assume

i:	**1**	**2**	**3**	**4**	**5**	**6**
X_{i1}	7	4	16	3	21	8
X_{i2}	33	41	7	49	5	31
Y_i	42	33	75	28	91	55

that regression model (1) which is:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta X_{i2} + \epsilon_i \tag{1}$$

[1] This is problem 4.22 in 'Applied Linear Regression Models(4th edition)' by Kutner etc.
[3] This is problem 6.27 in 'Applied Linear Regression Models(4th edition)' by Kutner etc.

with independent normal error terms is appropriate. Using matrix methods, obtain (a) $\mathbf{b}$; (b) $\mathbf{e}$; (c) $\mathbf{H}$; (d) SSR; (e) $s^2\{\mathbf{b}\}$; (f) $\hat{Y}_h$ when $X_{h1} = 10$, $X_{h2} = 30$; (g) $s^2\{\hat{Y}_h\}$ when $X_{h1} = 10$, $X_{h2} = 30$. For the notations, please refer to section 6.4.

Answer:

3. (40 points) Consider the classical regression setup

$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

We want to find the maximum likelihood estimate of the parameters.
a. if $\epsilon \sim \mathbf{N}(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{I})$. Give the maximum likelihood estimate of β and σ^2.
b. if $\epsilon \sim \mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$ and Σ is known. Give the maximum likelihood estimate of β.

Answer:

a. The log likelihood is proportional to

$$-n\log(\sigma) - \frac{(y - X\beta)^T\Sigma^{-1}(y - X\beta)}{2\sigma^2}$$

take derivatives with respect to β and σ, we have

$$\begin{cases} -n/\sigma + \frac{(y-X\beta)^T\Sigma^{-1}(y-X\beta)}{\sigma^3} = 0 \\ X^TX\beta - X^Ty = 0 \end{cases}$$

The second equation is becasue

$$\frac{\partial\beta^TA\beta}{\partial\beta} = 2A\beta$$
$$\frac{\partial\alpha\beta}{\partial\beta} = \alpha^T$$

So

$$\hat{\beta} = (X^TX)^{-1}X^Ty$$
$$\hat{\sigma^2} = \frac{1}{n}y^T(I - X(X^TX)^{-1}X^T)y = \frac{\text{SSE}}{n}$$

b. Very similarly, we have when $\epsilon \sim \mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$, the MLE of β is

$$\hat{\beta} = (X^T\Sigma^{-1}X)X^T\Sigma^{-1}y$$