LINEAR REGRESSION MODELS W4315

HOMEWORK 1 ANSWERS

September 21, 2010

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1. (20 points) Let $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ be a linear regression model with distribution of error terms unspecified (but with mean $E(\epsilon) = 0$ and variance $V(\epsilon_i) = \sigma^2$ (σ^2 finite) known). Show that $s^2 = MSE = \frac{\sum(Y_i - \hat{Y}_i)^2}{n-2}$ is an unbiased estimator for σ^2. $\hat{Y}_i = b_0 + b_1 X_i$ where $b_0 = \bar{Y} - b_1 \bar{X}$ and $b_1 = \frac{\sum_i ((X_i - \bar{X})(Y_i - \bar{Y}))}{\sum_i (X_i - \bar{X})^2}$.

Answer:

First, let's denote the followings:

$$\begin{aligned}
\hat{e_i} &= y_i - \hat{y_i} \\
S_{XX} &= \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \\
S_{YY} &= \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \\
S_{XY} &= \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i
\end{aligned}$$

Consequently, $b_1 = S_{XY}/S_{XX}$.

Now we set out to prove the following equation which essentially accomplishes the final result:

$$Var\hat{e_i} = E\hat{e_i}^2 = (\frac{n-2}{n} + \frac{1}{S_{XX}}(\frac{1}{n}\sum_{j=1}^{n} x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i\bar{x}))\sigma^2$$

To prove the above display, realize that:

$$\begin{aligned}
Var(\hat{e_i}) &= Var(y_i - b_0 - b_1 x_i) \\
&= Var((y_i - \beta_0 - \beta_1 x_i) - (b_0 - \beta_0) - x_i(b_1 - \beta_1)) \\
&= Var(y_i) + Var(b_0) + x_i^2 Var(b_1) - 2Cov(y_i, b_0) - 2x_i Cov(y_i, b_1) + 2x_i Cov(b_0, b_1)
\end{aligned}$$

The last equation holds because the covariance between any random variable and a constant is zero, and all the y_i's are independent entailing that the $Cov(y_i, y_j) = 0, i \neq j$

Then we need to calculate each term of the above display

$$\begin{aligned} Var(b_1) &= Var(\frac{S_{XY}}{S_{XX}}) \\ &= Var(\frac{\sum (x_i - \bar{x})y_i}{S_{XX}}) \\ &= \frac{1}{S_{XX}^2} \sum (x_i - \bar{x})^2 Var(y_i) \\ &= \frac{\sigma^2}{S_{XX}} \end{aligned}$$

And:

$$\begin{aligned} Var(b_0) &= Var(\bar{y} - b_1\bar{x}) \\ &= Var(\sum (\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}})y_i) \\ &= \sum (\frac{1}{n} - \frac{x_i - \bar{x}}{S_{XX}}\bar{x})^2\sigma^2 \\ &= \sum [\frac{1}{n^2} + \frac{(x_i - \bar{x})^2 * \bar{x}^2}{S_{XX}^2} - \frac{2}{n}\frac{\bar{x}(x_i - \bar{x})}{S_{XX}}]\sigma^2 \\ &= [\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}}]\sigma^2 \\ &= \frac{\sum {x_i}^2}{n * S_{XX}}\sigma^2 \end{aligned}$$

For the other terms in the decomposition of $Var(\hat{e_i})$, we have:

$$\begin{aligned} Cov(y_i, b_1) &= Cov(y_i, \frac{\sum (x_i - \bar{x})y_i}{S_{XX}}) \\ &= \frac{x_i - \bar{x}}{S_{XX}} Var(y_i) \\ &= \frac{x_i - \bar{x}}{S_{XX}}\sigma^2 \end{aligned}$$

and:

$$
\begin{aligned}
Cov(y_i, b_0) &= Cov(y_i, \bar{y} - b_1\bar{x}) \\
&= Cov(y_i, \frac{\sum y_i}{n} - \frac{\sum (x_i - \bar{x})y_i}{S_{XX}}\bar{x}) \\
&= \frac{\sigma^2}{n} + \bar{x}\frac{x_i - \bar{x}}{S_{XX}}\sigma^2
\end{aligned}
$$

At last, we have:

$$
\begin{aligned}
Cov(b_0, b_1) &= Cov(\bar{y} - b_1\bar{x}, b_1) \\
&= Cov(\frac{\sum y_i}{n} - \sum \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}y_i, \sum \frac{(x_i - \bar{x})y_i}{S_{XX}}) \\
&= \sum_{i=i}^{n}(\frac{1}{n} - \frac{x_i - \bar{x}}{S_{XX}}\bar{x})\frac{x_i - \bar{x}}{S_{XX}}\sigma^2 \\
&= -\frac{\bar{x}}{S_{XX}}\sigma^2
\end{aligned}
$$

Then plug in all the parts back to the decomposition of $Var(\hat{e_i})$, we have:
$Var(\hat{e_i}) = (\frac{n-2}{n} + \frac{1}{S_{XX}}(\frac{1}{n}\sum_{j=1}^{n} {x_j}^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i\bar{x}))\sigma^2$
Thus,

$$
\begin{aligned}
E\hat{\sigma}^2 &= \frac{1}{n-2}\sum_{i=1}^{n} E{\hat{e_i}}^2 \\
&= \frac{1}{n-2}\sum_{i=1}^{n}[\frac{n-2}{n} + \frac{1}{S_{XX}}(\frac{1}{n}\sum_{j=1}^{n} x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i\bar{x})]\sigma^2 \\
&= [1 + \frac{1}{nS_{xx}}\{\sum_{j=1}^{n} x_j^2 + \sum_{i=1}^{n} x_i^2 - 2S_{XX} - 2\frac{1}{n}(\sum_{i=1}^{n} x_i)^2\}]\sigma^2 \\
&= (1 + 0)\sigma^2 \\
&= \sigma^2
\end{aligned}
$$

where the third equation holds because: $\sum x_i\bar{x} = \frac{1}{n}(\sum x_i)^2$
and the second to last equation holds since $\sum x_i^2 - \frac{1}{n}(\sum x_i)^2 = S_{XX}$
From the above equation, the result flows.

2. (20 points) Derive the maximimum likelihood estimators $\hat{\beta}_0, \hat{\beta}_1$, and $\hat{\sigma}^2$ for parameters β_0, β_1, and σ^2 for the normal linear regression model (i.e. $\epsilon_i \sim_{iid} N(0, \sigma^2)$).

Answer:

To figure the MLE of the parameters, we need to first write down the likelihood function of the data, so under normal assumption, we have the log-likelihood function as follows:

$logL(\beta_0, \beta_1, \sigma^2|x, y) = -\frac{n}{2}log(2\pi) - \frac{n}{2}log\sigma^2 - \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}$.

For any fixed value of σ^2, $logL$ is maximized as a function of β_0 and β_1, that minimize

$$\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 \tag{1}$$

But to minimize this function is just to principle behind LSE, so it's apparent that the MLE of β_0 and β_1 are the same as their LSE's. Now, substituting in the log-likelihood, to find the MLE of σ^2 we need to maximize

$$-\frac{n}{2}log(2\pi) - \frac{n}{2}log\sigma^2 - \frac{\sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2}{2\sigma^2}$$

This maximization problem is nothing but MLE of σ^2 in ordinary normal sampling problems, which is easily given as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2$$

If you are not familiar with the MLE in normal sampling setting, you can take derivative with respect to σ^2 (N.B. not σ), and then set the derivative to be zero. The solution of the equation is just the MLE of σ^2.

3. (20 points) $X_1, X_2, X_3, \ldots, X_{100}$ are iid normal random variables with mean μ and variance σ^2, we want to estimate the mean μ. Consider two estimators, X_1 and $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$:
a. Show these two estimators are both unbiased. Also derive the distribution of each estimator. Which estimator do you think is better? Why?
b. When $\mu = 0$ and $\sigma^2 = 100$, generate $X_1, X_2, X_3, \ldots, X_{100}$. Calculate the estimate $\bar{X}$ and denote it as $\hat{\mu}_1$. Repeat this 100 times and plot the density histogram of $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \ldots, \hat{\mu}_{100}$.

Overlay the probability density of $\bar{X}$ on the plot (see matlab function "normpdf").

Answer:

a. X_1 is unbiased is pretty obvious, for $\bar{X}$, using linearity of expectation, we have

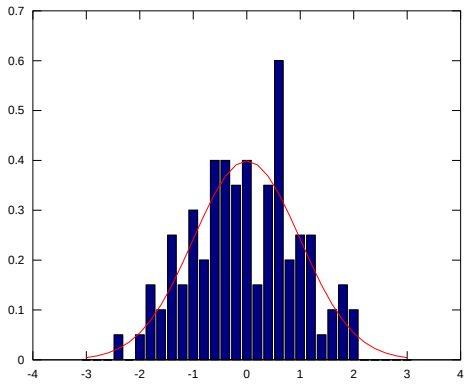
$$E(\bar{X}) = E(\frac{1}{100}\sum_{i=1}^{100} X_i) = \frac{1}{100}\sum_{i=1}^{100} EX_i = \frac{1}{100}\sum_{i=1}^{100} \mu = \mu$$

Apprantly, $X_1 \sim N(\mu, \sigma^2)$. $\bar{X}$ is a linear combination of normal random variables, so it is still a normal random variable. Thanks to indepedence, its variance is given by

$$Var(\bar{X}) = \frac{1}{100^2} Var(\sum_{i=1}^{100} X_i) = \frac{1}{100^2}\sum_{i=1}^{100} Var(X_i) = \frac{\sigma^2}{100}$$

so $\bar{X} \sim N(\mu, \frac{1}{100}\sigma^2)$. Obviously, $\bar{X}$ is better since it has smaller variance.

b.



4. (40 points) Copier maintenance.[1] The Tri-City Office Equipment Corporation sells an imported copier on a franchise basis and performs preventive maintenance and repair services on this copier. The data below have been collected from 45 recent calls on users to perform routine preventive number of minutes spent by the service person. Assume that first-order regression model($Y_i = b_0 + b_1 X_i + \epsilon_i$) is appropriate.

i:	**1**	**2**	**3**	...	**43**	**44**	**45**
X_i	2	4	3	...	2	4	5
Y_i	20	60	46	...	27	61	77

a. Obtain estimated regression function.
b. Plot the estimated regression function and the data. How well does the estimated regression function fit the data?
c. interpret b_0 in your estimated regression function. Does b_0 provide any relevant information here? Explain.
d. Obtain a point estimate of the mean service time when $X = 5$ copiers are serviced.

Notice: You can get data for this problem on www.mhhe.com/KutnerALRM4e. Use MATLAB, do not use any other programming language. Only basic MATLAB operators are allowed, do not use any built-in functions to do the regression, i.e. the function "regress" cannot be used except, perhaps, to verify that your answer is correct before submitting your own implementation.

Answer:

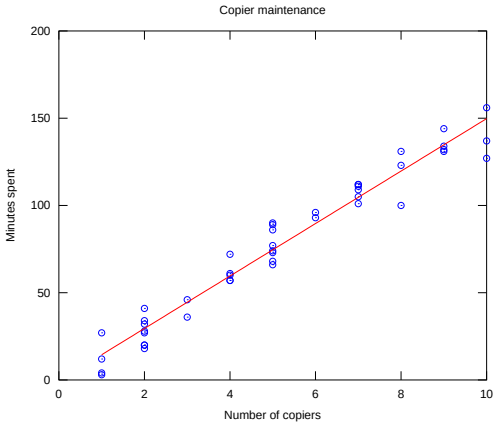
a. $y = -0.58 + 15.04 * x + \epsilon$;

b. Based on the graph, the model fit quite well.

c. $b_0 = -0.58$ means number of minutes spent to servie 0 copier, which doesn't make any practical sense.

d. $5 * b_1 + b_0 = 74.596$.

[1] This is problem 1.20 in "Applied Linear Regression Models(4th edition)" by Kutner etc.)


Copier maintenance
Minutes spent
Number of copiers
0
50
100
150
200
0
2
4
6
8
10