My linear algebra exercises

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Contents

Preface

Exercises are from ”Linear algebra done right” by Sheldon Axler, 3rd ed. I've already read this book before and completed some exercises from it. Right now I want to brush up the material once again, put all the proofs on a more durable material than paper and to prepare myself to what's gonna happen afterwards.

Chapter 1

1.A R^n and C^n

1

Suppose a and b are real numbers, not both 0. Find real nuber c and d such that

$$1/(a+bi) = c+di$$

$$\frac{1}{a+bi} = c+di$$
$$\frac{1}{a+bi} - c - di = 0$$
$$\frac{a-bi}{(a+bi)(a-bi)} = c+di$$
$$\frac{a-bi}{(a^2+b^2)} = c+di$$
$$\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = c+di$$

Thus $c = \frac{a}{a^2+b^2}$ and $d = -\frac{b}{a^2+b^2}$

2

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 *(meaning that its cube equals 1)*

$$(\frac{-1+\sqrt{3}i}{2})^3 = \frac{(-1+\sqrt{3}i)^3}{8} = \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)^2}{8} = \frac{(-1+\sqrt{3}i)(1-2\sqrt{3}i-3)}{8} =$$

$$= \frac{(-1+\sqrt{3}i)(-2-2\sqrt{3}i)}{8} = \frac{2+2\sqrt{3}i-2\sqrt{3}i+6}{8} = \frac{8}{8} = 1$$

as desired.

3

Find two distinct square roots of i

Square root of i, I assume, is a number, whose square is equal to i. Suppose that $(a+bi)^2 = i$. It follows that

$$(a+bi)^2 = a^2 + 2abi - b^2$$

So if we set

$$a = b = 1/\sqrt{2}$$

this equation holds. Also it holds for

$$a = b = -1/\sqrt{2}$$

maxima seems to agree with me on this one

4

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$ and $\beta = a_2 + b_2 i$. It follows

$$\alpha + \beta = a_1 + b_1 i + a_2 + b_2 i = a_2 + b_2 i + a_1 + b_1 i = \beta + \alpha$$

as desired.

5

Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$, $\beta = a_2 + b_2 i$, $\lambda = a_3 + b_3 i$. It follows that

$$\alpha+(\beta+\lambda) = a_1+b_1i+(a_2+b_2i+a_3+b_3i) = (a_1+b_1i+a_2+b_2i)+a_3+b_3i = (\alpha+\beta)+\lambda$$

6

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

$$\alpha+(\beta+\lambda) = (a_1+b_1i)((a_2+b_2i)+(a_3+b_3i)) = ((a_1+b_1i)(a_2+b_2i))+(a_3+b_3i) = (\alpha\beta)\lambda$$

7

Show that for every $\alpha \in \boldsymbol{C}$ there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha + \beta = 0$

Suppose that there exist two different $\beta_1 \neq \beta_2$ such that $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$. It follows that

$$\beta_1 = \beta_1 + 0 = \beta_1 + \alpha + \beta_2 = \alpha + \beta_1 + \beta_2 = 0 + \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

8

Show that for every $\alpha \in \boldsymbol{C}$ with $\alpha \neq 0$ there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha\beta = 1$

Suppose that it is not true and there exist two different $\beta_1 \neq \beta_2$ such that

$$\alpha\beta_1 = 1 \textit{ and } \alpha\beta_2 = 1$$

it follows then that

$$\beta_1 = 1 * \beta_1 = \alpha\beta_2\beta_1 = \alpha\beta_1\beta_2 = 1 * \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

etc

The rest of the section is the repetition of this kind of stuff. That is a lot of writing, and not a lot of thinking, so I'll skip it. I don't ususally like to skip sections, but I have aa feeling, that I've completed this thing on paper somewhere, and there is not much reason to rewrite it here.

Chapter 2

1.B Definition of Vector Space

1

Prove that $-(-v) = v$ *for every* $v \in V$.

For v there exists only one $-v$. For $-v$ there exists only one $-(-v))$.

Thus

$$v = v + 0 = v + (-v) + (-(-v)) = 0 + (-(-v)) = -(-v)$$

as desired (idk if it's true, I'm not good at axioms and stuff)

2

Suppose $a \in F, v \in V$, *and* $av = 0$. *Prove that* $a = 0$ *or* $v = 0$.

Suppose that $a \neq 0$, $v \neq 0$ but $av = 0$. It follows that there exist $1/a$ - multiplicative inverse of a. It follows that

$$1/a * av = 1/a * 0$$

$$1v = 0$$

$$v = 0$$

which is a contradiction. Thus either $a = 0$ or $v = 0$.

3

Suppose $v, w \in V$. *Explain why there exists a unique* $x \in V$ *such that* $v + 3x = w$.

Suppose that there exists $x_1 \neq x_2$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$. Thus

$$3x_1 = w - v = 3x_2$$

$$x_1 = \frac{1}{3}(w - v) = x_2$$

which is a contradiction.

Same can be stated from the fact that x is a unique additive inverse of $\frac{1}{3}(v - w)$.

4

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Additive indentity. Empty set does not have zero element in it. BTW $\{0\}$ is a vector space.

5

Show that n the defintition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

$$0v = 0$$
$$(1 - 1)v = 0$$
$$1v - 1v = 0$$
$$v - v = 0$$
$$v + (-v) = 0$$

6

Let ∞ and $-\infty$ denote two distinct object, neither of which is in R. Define an addition and multiplication on $R \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and the product of two real numbers is as usual, and for $t \in R$ define

$$t\infty = \begin{cases} -\infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ \infty \text{ if } t > 0 \end{cases}$$

$$t(-\infty) = \begin{cases} \infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ -\infty \text{ if } t > 0 \end{cases}$$

$$t + \infty = \infty + t = \infty$$
$$t + (-\infty) = (-\infty) + t = (-\infty)$$
$$\infty + \infty = \infty$$
$$(-\infty) + (-\infty) = (-\infty)$$
$$\infty + (-\infty) = 0$$

Is $R \cup \{\infty\} \cup \{-\infty\}$ a vector space over R? Explain.
I don't think that it is.

$$(t + \infty) - \infty = \infty - \infty = 0$$
$$t + (\infty - \infty) = t + 0 = t$$

thus

$$t + (\infty - \infty) \neq (t + \infty) - \infty$$

thus $R \cup \{\infty\} \cup \{-\infty\}$ is not associative, therefore it is not a vector space.

Chapter 3

1.C Subspaces

1

For each of the following subsets of F^3, determine whether it is a subspace of F^3:

(a) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$

Yes, it is. 0 is contained within it.

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

therefore

$$x_1 + y_1 + 2(x_2 + y_2) + 3(x_3 + y_3) = x_1 + 2x_2 + 3x_3 + y_1 + 2y_2 + 3y_3 = 0 + 0 = 0$$

therefore it is closed under addition

$$n(x_1, x_2, x_3) = (nx_1, nx_2, nx_3)$$

$$nx_1 + 2nx_2 + 3nx_3 = n(x_1 + 2x_2 + 3x_3) = 0n = 0$$

therefore it is closed under multiplication.

(b) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$

It's not a subspace, because it does not contain zero.

(c) $\{(x_1, x_2, x_3) \in F^3 : x_1 x_2 x_3 = 0\}$

It's not a subspace, because

$$(0, 1, 1) + (1, 0, 0) = (1, 1, 1)$$

therefore it's not closed under addition.

(d) $\{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}$

It's a subspace, proof is the same as in (a), can be seen more clearly when we rewrite constraint as

$$x_1 = 5x_3 \rightarrow x_1 + 0x_2 - 5x_3 = 0$$

2

Verify all the assertions in Example 1.35