My real analysis exercises

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Exercises are from UTM-040 Understanding analysis by Stephen Abbott. Edition is unknown, but the date in the preface is August 2000. I don't own any rights for the book and I am not planning to aquire them anytime soon.

Also it wouldn't hurt to mention, that some (or maybe even most) of exercises were not proof-read after they were written, and that none of them were evaluated by anyone who matters. There are a lot of mistakes out there, both in mathematical and logical parts, and in the language they were written as well. I am doing this kind of stuff for fun, so cut me some slack.

1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a simular argument work to show $\sqrt{6}$ is irrational?
    Suppose that $\sqrt{3}$ is a rational number; then it is true that

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{3}$$

where m and n are at their lowest possible terms. Then

$$\sqrt{3}n = m$$

$$3n^2 = m^2$$

Therefore we can state, that $m\%3 = 0$. Therefore $\exists k : 3k = m$. Thus we can reformulate formula as

$$3n^2 = (3k)^2$$

$$n^2 = 3k^2$$

Therefore $n\%3 = 0$ as well. Therefore n and k are not in their possible terms, which conradicts our initial assumtions. Therefore we can state that $\sqrt{3} \notin \mathbf{Q}$.
    Let's try the same argument for $\sqrt{6}$.

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{6}$$

$$\sqrt{6}n = m$$

$$6n^2 = m^2$$

then m has as their dividers both 2 and 3. Therefore $m\%2 = 0$ and $m\%3 = 0$. Therefore we can proceed with the same argument as earlier

$$6n^2 = (6k)^2$$

$$n^2 = 6k^2$$

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Therefore n is divided by 6, etc., etc., $\sqrt{6} \notin \mathbf{Q}$.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Suppose that $\sqrt{4}$ is a rational number; then it is true that

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{2}$$

where m and n are at their lowest possible terms. Then

$$\sqrt{4}n = m$$

$$4n^2 = m^2$$

n can still be odd and m can still be even. In other words, m is divisible by a prime, and the number under the radical consists of two primes. Therefore if a number decomposes to two equal sets of primes, then its square root is a rational number. Otherwise it isn't.

1.2.2

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific exapmple where the statement in question does not hold.

(a) if $A_1 \supseteq A_2 \supseteq A_3 \supseteq$ ... are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well. -

$\cap_{n=1}^{\infty} A_n = (0, 1/n)$ has no numbers in it.

Proof is easy -

$$\forall x \in \mathbf{R} > 0 : \exists n \in N : 1/n < x$$

(b) if $A_1 \supseteq A_2 \supseteq A_3 \supseteq$ ... are all finite, nonempty sets of real numbers, then the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty. -

True.

There is no need for the proof, but I'll supply one anyways. If all A_n are finite and nonempty, then $\exists j \in \mathbf{N} : |A_1| = j$. Therefore, because of the same reasons, there are only $j - 1$ times when

$$A_k \supset A_{k+1}$$

can happen, because after $j - 1$ times the set will be empty. Therefore, because it is finite, their intersection will have finite number of elements and will be non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

False: let

$$x \notin A, x \notin B, x \in C$$

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Then
$$x \in A \cap (B \cup C); x \notin = (A \cap B) \cup C$$

(c) $A \cap (B \cap C) = (A \cap B) \cap C$
True. Kinda goes without a proof; if you imagine a Vien diagram, then it's obvious.
(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
True. For the same reason as before.

I'm sure that there exist more concrete versions of those proofs, but I'm not required to provide any. My suspition on why is it so, is because it's a little more complicated and requires more knowlege in set theory and/or logic.

1.2.3 (De Morgan's Laws).

Let A and B be subsets of $\mathbf{R}$
(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq (A \cap B)^c$.
If we have two sets A and B, then $\mathbf{R}$ desintegrates into 4 different sets: A, B, A^c, B^c. Therefore there must exists sets

$$S_1 = A \cap B$$

$$S_2 = A^c \cap B$$

$$S_3 = A \cap B^c$$

$$S_4 = A^c \cap B^c$$

An element cannot be in the set and not in the set at the same time. Therefore, there does not exist an element, which is in two of S_n's.

For any $x \in \mathbf{R} \to x \in A$ or $x \notin A$. Therefore an element of $\mathbf{R}$ needs to be in at least one of those sets. It is easily seen by

$$A \cap \mathbf{R} = A$$

$$A \cap (B \cup B^c) = A$$

$$(A \cap B) \cup (A \cap B^c)) = A$$

Therefore $\cup_{n=1}^{4} S_n = \mathbf{R}$ and $\cap_{n=1}^{4} S_n = \emptyset$.
Suppose $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Therefore $x \in S_2 \cup S_3 \cup S_4$.
Suppose that $x \in A^c \cup B^c$. Then $x \in S_2 \cup S_3 \cup S_4$.
Therefore $(A \cap B)^c \subseteq A^c \cup B^c$.
(b) Prove the reverse inclusion
As seen in part (a)

$$(A \cap B)^c = S_2 \cup S_3 \cup S_4 = A^c \cup B^c$$

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(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.
No need to do both ways.

$$(A \cup B)^c = S_4 = A^c \cap B^c$$

1.2.4

Verify the triangle inequality in the special cases where
  (a) a and b have the same sign
Suppose $a \geq 0$, $b \geq 0$. Then $|a| = a$ and $|b| = b$. Therefore

$$|a + b| = a + b = |a| + |b| \leq |a| + |b|$$

Suppose $a < 0$, $b < 0$. Then $|a| = -a$ and $|b| = -b$; also $a + b < 0 \rightarrow |a + b| = -(a+b) = -a - b$. Therefore

$$|a + b| = -a + (-b) = |a| + |b| \leq |a| + |b|$$

.

  (b) $a \geq 0$, $b < 0$ and $a + b \geq 0$.

$$a + b \geq 0 \rightarrow a + b = |a + b|$$

Also, $|a| = a$ and $|b| = -b$. Therefore

$$a + b \geq 0 \rightarrow a \geq -b \rightarrow a \geq |b| \rightarrow |a| \geq |b|$$

$$b < 0$$
$$b \leq 0$$
$$2b \leq 0$$
$$b + b \leq 0$$
$$b \leq (-b)$$
$$a + b \leq a + (-b)$$
$$|a + b| \leq |a| + |b|$$

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1.2.5

Use the triangle inequality to establish the inequalities
   (a) $|a - b| \leq |a| + |b|$;

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

   (b) $||a| - |b|| \leq |a - b|$;
let $a = a + b - b$. Then

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|a| - |b| \leq |a - b|$$

$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$$

$$|b| - |a| \leq |a - b|$$

$$|a| - |b| \geq -|a - b|$$

$$-|a - b| \leq |a| - |b| \leq |a - b| \rightarrow ||a| - |b|| \leq |a - b|$$

1.2.6

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

   (a) Let $f(x) = x^2$. if $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

First things first: $f(A) = [0, 4]$; $f(B) = [1, 16]$ (without any proof because if we don't go with axiomatic stuff, then it is obvious).

$$f(A \cap B) = f([1, 2]) = [1, 4]$$

$$f(A) \cap f(B) = [1, 4]$$

Therefore in this case $f(A) \cap f(B) = f(A \cap B)$.

$$f(A \cup B) = f([0, 4]) = [0, 16] = f(A) \cup f(B)$$

   (b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
Let $A = [-1, 0]$ and $B = [0, 1]$. Then

$$f(A \cap B) = f(\{0\}) = \{0\}$$

$$f(A) \cap f(B) = [0, 1] \cap [0, 1] = [0, 1] \neq f(A \cap B)$$

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

$$x \in g(A \cap B) \to x \in g(A)$$
$$x \in g(A \cap B) \to x \in g(B)$$

Therefore
$$x \in g(A \cap B) \to x \in g(A) \cap g(B)$$

Thus
$$g(A \cap B) \subseteq g(A) \cap g(B)$$

(d) Form and prove a conjecture aout the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

$$x \in g(A) \to x \in g(A \cup B)$$
$$x \in g(B) \to x \in g(A \cup B)$$

Therefore
$$x \in g(A) \cup g(B) \to x \in g(A \cup B)$$

Thus
$$g(A) \cup g(B) \subseteq g(A \cup B)$$

Suppose that

$$\exists y \in \mathbf{R} : y \in g(A \cup B); y \notin g(A) \cup g(B)$$

Then $\exists q_1 \in A \cup B : g(q_1) = y$ but
$$\forall q_2 \in A, q_3 \in B : g(q_2) \neq y; g(q_3) \neq y$$

Therefore $q_1 \notin A$ and $q_1 \notin B$. Therefore $q_1 \in A^c \cap B^c$. Using De Morgan rule
$$q_1 \in A^c \cap B^c \to q_1 \in (A \cup B)^c$$

therefore
$$q_1 \notin g(A \cup B)$$

which is a contradiction. Therefore

$$y \in g(A \cup B) \to g(A) \cup g(B)$$

Thus
$$g(A \cup B) \subseteq g(A) \cup g(B)$$

Therefore if we take into account previous conclusion

$$g(A \cup B) = g(A) \cup g(B)$$

for any g.

1.2.7

Given a function $f : D \to \textbf{R}$ and a subset $B \subseteq \textbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This is called the preimage of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

$$f^{-1}(A) = [-2, 2]$$
$$f^{-1}(B) = [-1, 1]$$
$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$$

(b) The good behaviour of preimages demonstated in (a) is completely general. Show that for an arbitrary function $g : \textbf{R} \to \textbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \textbf{R}$.

By definition
$$x \in g^{-1}(A \cap B) \to \exists y \in A \cap B : y = g(x)$$

Therefore if we we use the fact $y \in A \cap B \to y \in A$ and $y \in A \cap B \to y \in B$

$$x \in g^{-1}(A \cap B) \to \exists y \in A : y = g(x) \to x \in g^{-1}(A)$$

$$x \in g^{-1}(A \cap B) \to \exists y \in B : y = g(x) \to x \in g^{-1}(B)$$

therefore $x \in g^{-1}(A \cap B)$ implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, or in other words

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$$

In other direction:
$$x \in g^{-1}(A) \to \exists y_1 \in A : y_1 = g(x)$$
$$x \in g^{-1}(B) \to \exists y_2 \in B : y_2 = g(x)$$

$x \in g^{-1}(A) \cap g^{-1}(B)$ implies that $\exists y_1 \in A : g(x) = y_1$ and $\exists y_2 \in B : y_2 = g(x)$. Because g is a function we know, that for every x there exists only one $y = g(x)$. Therefore $y_1 = y_2 = g(x)$. Thus we can state that $y \in A \cap B$.

thus

$$x \in g^{-1}(A) \cap g^{-1}(B) \to \exists y \in A \cap B : y = g(x) \to x \in g^{-1}(A \cap B)$$

Therefore

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$$

If we take previous conclusion into account, then it follows that

$$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$$

as desired.

Now let's prove that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$:

If $x \in g^{-1}(A) \cup g^{-1}(B))$ then $\exists y \in A : y = g(x)$ or $\exists y \in B : y = g(x)$. If we take into account that $y \in A \rightarrow y \in A \cup B$ then we can conclude that

$$x \in g^{-1}(A) \cup g^{-1}(B)) \rightarrow \exists y \in A \cup B : y = g(x) \rightarrow x \in g^{-1}(A \cup B)$$

Thus

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B)$$

In other direction:

$$x \in g^{-1}(A \cup B) \rightarrow \exists y \in A \cup B : y = g(x) \rightarrow y = g(x)$$

As proven before $g(A \cup B) = g(A) \cup g(B)$ and therefore $y \in g(A \cup B)$ implies that either $y \in g(A)$ or $y \in g(B)$. Therefore

$$x \in g^{-1}(A \cup B) \rightarrow x \in g^{-1}(A) \cup g^{-1}(B)$$

$$g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B)$$

And if we combine this fact with previous conclusion:

$$g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$$

as desired.

1.2.8

Form the logical negation of each claim. One way to do this is to simply add "It is bot the case that ... " in front of each assertion, but for each statement, try to embed the word "not" as deeply into the resulting sentence as possible (or avoid using it altogether).

(a) For all real numbers satisfying $a < b$, there exist an $n \in \mathbf{N}$ such that $a + 1/n < b$.
There exist real numbers $a < b$ such that $a + 1/n \geq b$ for all $n \in \mathbf{N}$.

(b) Between every two distinct real numbers, there is a rational number
There exist two real numbers, such that there are only irrational numbers between them.

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(c) For all natural numbers $n \in \boldsymbol{N}$, $\sqrt{n}$ is either a natural number or an irrational number

There exist a natural number $n \in \mathbf{N}$, such that $\sqrt{n}$ is a rational number, that is not a natural number. (This one is a little bit weird if we try to negate this, but "not" is stuffed as deep as possible)

(d) Given any real number $x \in \boldsymbol{R}$, there exist $n \in \boldsymbol{N}$ satistying $n > x$

There exist a real number $x \in \mathbf{R}$, such that for all $n \in \mathbf{N}$ it is true that $n \leq x$.

1.2.9

Show that the sequence $(x_1, x_2, x_3, ...)$ defined in Example 1.2.7 is bounded above by 2; that is, prove that $x_n \leq 2$ for every $n \in \boldsymbol{N}$.

The mentioned sequence is defined by

$$x_1 = 1$$

$$x_{n+1} = (1/2)x_n + 1$$

A great advice about induction states, that when you hear words "prove" and "sequence" in the same sentence, then the word "induction" should pop up in your head. So here we go

Base case: $x_1 \leq 2$.

Inductive proposition: $x_n \leq 2$

Inductive step:

$$x_n \leq 2$$

$$(1/2)x_n \leq 1$$

$$(1/2)x_n + 1 \leq 2$$

$$x_{n+1} = (1/2)x_n + 1 \leq 2$$

$$x_{n+1} \leq 2$$

as desired.

1.2.10

Let $y_1 = 1$, and for each $n \in \boldsymbol{N}$ define $y_{n+1} = (3 * y_n + 4)/4$.

(a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \boldsymbol{N}$.

Base case: $y_1 = 1 < 4$

Inductive proposition: $y_n < 4$

Inductive step:

$$y_n < 4$$

$$3 * y_n < 12$$

$$3 * y_n + 4 < 16$$

$$(3 * y_n + 4)/4 < 4$$

$$y_{n+1} = (3 * y_n + 4)/4 < 4$$

$$y_{n+1} < 4$$

as desired.

(b) Use another induction to show the sequence $(y_1, y_2, y_3, ...)$ is increasing
We need to show that $y_{n+1} - y_n \geq 0$
As shown earlier

$$y_n < 4$$

therefore

$$y_n < 4$$

$$\frac{y_n}{4} < 1$$

$$1 > \frac{y_n}{4}$$

$$1 - \frac{y_n}{4} > 0$$

$$1 + (\frac{3y_n}{4} - y_n) > 0$$

$$\frac{3y_n}{4} + 1 - y_n > 0$$

$$(3 * y_n + 4)/4 - y_n > 0$$

$$y_{n+1} - y_n > 0$$

as desired.

1.2.11

If a set A contains n elements, prove that the number of different subsets of A is equal to 2^n. (Keep in mind that the empty set $\emptyset$ is considered to be a subset of every set.)

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary system: 0 for excluded, 1 for included. Therefore there exist 2^n possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of $\emptyset$ are $\emptyset$ itseft ($2^0 = 1$ in total). Subsets of set with one element are $\emptyset$ and set itself ($2^1 = 1$ in total).

Proposition is that set with n elements has 2^n subsets.

Inductive step is that for set with $n+1$ elements can either have or hot have the $n+1$'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

1.2.12

For this exerice, assume Exercise 1.2.3 has been successfully completed (as it was)
(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup ... \cup A_n) = A_1^c \cap A_2^c \cap ... \cap A_n^c$$

First of all, base case

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

Proposition

$$(A_1 \cup A_2 \cup ... \cup A_n) = A_1^c \cap A_2^c \cap ... \cap A_n^c$$

Step

$$(A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1}) = A_1^c \cap A_2^c \cap ... \cap A_n^c \cap A_{n+1}^c$$

Let us denote $Q = A_1^c \cap A_2^c \cap ... \cap A_n^c$. Then by inductive proposition

$$A_1^c \cap A_2^c \cap ... \cap A_n^c \cap A_{n+1}^c = Q \cap A_{n+1}^c = (Q^c \cup A_{n+1}) = (A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1})$$

as desired.

(b) Explain why induction cannot be used to conclude

$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c$$

It might be useful to consider part (a) of Exercise 1.2.2.

Induction cannot be used on this one, because for induction we need finite set of elements. This stands on the fact, that if induction works for some case, then it works for

$n-1$'th element, and for $n-2$'th element and this way all the way down to the base case. As an example of why it doesn't work, we can take into accound exercise 1.2.2, where all of the elements have infinite number of elements, and their intersection would have infinite amount of elements for any finite number of elements, but it is not true for the infinite amount.

(c) Is the statement in part (b) valid? If so, write a proof that does not use induction. Suppose that

$$x \in (U_{n=1}^{\infty} A_n)^c$$

Then $x \notin A_n$ for every $n \in \mathbf{N}$. Therefore $x \in A_n^c$ for every $n \in \mathbf{N}$. Therefore $x \in \cap_{n=1}^{\infty} A_n^c$. Thus

$$(\cup_{n=1}^{\infty} A_n)^c \subseteq \cap_{n=1}^{\infty} A_n^c$$

Suppose that $x \in \cap_{n=1}^{\infty} A_n^c$. Then $x \notin A_n$ for every $n \in \mathbf{N}$. Therefore $x \notin (U_{n=1}^{\infty} A_n)$ for every $n \in \mathbf{N}$. Therefore $x in \cup_{n=1}^{\infty} A_n)^c$ for every $n \in \mathbf{N}$. Therefore

$$\cap_{n=1}^{\infty} A_n^c \subseteq (\cup_{n=1}^{\infty} A_n)^c$$

Thus if we combine two statements

$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c$$

as desired.

1.3.1

Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4, 5\}$ and define addition and multiplication modulo 5. In other words, compute the integer remainder when $a+b$ and ab are divided by 5, and use this as the value for the sum and product, respectively.

(a) Show that, given any element $z \neq 0$ in $\mathbf{Z}_5$, there existsan element y such that $z + y = 0$. The element y is called the additive inverse of z.

It is true, that for every of those elements we can set $y = 5 - z$, for which it is true that $z + y = 5$ and $(z + y)\%5 = 0$ as desired.

(b) Show that, given any element $z \neq 0$ in $\mathbf{Z}_5$, there existsan element x such that $zx = 1$. The element x is called the multiplicative inverse of z.

$$(1 * 1)\%5 = 1$$
$$(2 * 3)\%5 = 1$$
$$(3 * 2)\%5 = 1$$
$$(4 * 4)\%5 = 1$$

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as desired.

(c) The existence of additive and multiplicative inverses is part of the definition of a field. Investigate the set $\mathbf{Z}_4 = \{0,1,2,3\}$ (where addition and multiplication are defined modulo 4) for the existense of additive and multiplicative inverses. Make a conjecture about the values of n for which additive inverses exist in $\mathbf{Z}_n$, and then form another conjecture about the existence of multiplicative inverses.

For $\mathbf{Z}_4$ we define additive inverse the same way we defined it in the part (a) $(4 - z = y)$. For multiplicative inverse we have the way with 1, but any for 2 we don't have multiplicative inverse.

Therefore the conjecture about the additive inserse is that every Z_n with addition defined as addition modulo n we have additive inverse.

Proof of it is that every element of $\mathbf{Z}_n$ is less than n, and that because of this there exists $s = n - x \in \mathbf{Z}_n$

For multiplicative inverse the conjecture is that it exists only when n is a prime number.

1.3.2

(a) Write a formal definition in the style of Definition 1.3.2 for the infinum or greatest lower bound of a set

A real number s is the *greatest lower bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) s is a lower bound

(ii) if b is any lower bound for A, then $s \geq b$.

(b) Now, state and prove a version of Lemma 1.3.7 for greatest lower bounds.

Lemma for greatest lower bounds

Assume $s \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then, $s = inf(A)$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s + \epsilon > a$.

Proof:

In one direction: Suppose $s = inf(A)$. Then, by definition of greatest lower bound, there does not exist a lower bound, greater than s. In other words, suppose that there exist $\epsilon > 0$ such that there is no element $a \in A$ such that $a < s + \epsilon$. Then $s + \epsilon$ is a lower bound, which is greater than s, therefore s is not a greatest lower bound, which is a contradiction. Therefore there does not exist $\epsilon > 0$ for which there exist no $a \in A$ such that $a < s + \epsilon$. Therefore for every $\epsilon > 0$ there exist an $a \in A$ such that $a < s + \epsilon$, as desired.

In other direction: suppose that s is a lower bound and for every $\epsilon > 0$ there exist $a \in A$ such that $a < s + \epsilon$. Then any number $s + \epsilon$ is not a lower bound. Therefore any number, which is greater than s is not a lower bound. Therefore any lower bound is less or equal to s. Therefore s is a greatest lower bound.

Maybe this proof is a little bit more coplicated, than it should be, but at least every step is followed properly.

1.3.3

(a) Let A be bounded below, and define $B = \{b \in \mathbf{R} : b$ is a lower bound for $A\}$. Show that $sup(B) = inf(A)$.

B is a set, therefore Axiom of Completeness states that there exist real number $k = sup(B)$. Therefore all lower bounds are less or equal to k.

In order to prove that k is infinum, we need to show that it is a lower bound.

We do it by contradiction: suppose that k is not a lower bound. Therefore there exists $a \in A$ such that $k > a$. Let $\epsilon = k - a > 0$. Then, because $k = sup(B)$ there exist $b \in B$ such that $k - \epsilon < b$. Therefore $a < b$. Therefore there exist an element of A, that is less than lower bound of A. Therefore b is not a lower bound. Therefore we have a contradiction. Thus k is a lower bound.

Because k is a lower bound, $k \in B$ by definition of B. Therefore it is a lower bound, that is greater or equal to any other lower bounds, because it is a supremum of B. Therefore it is an infinum of A by definition of infinum.

(b) Use (a) to explain why there is no need to assert that greatest upper bound exist as part of the Axiom of Completeness.

Proof of part (a) does not take into account the fact, that A (that is bounded below) has an infinum. We prove its existence of the infinum by the fact, that we have a set of lower bounds (which is in its turn has a supremum by Axiom of Completeness), and setting the fact, that its supremum is lower bound itself and therefore proving that it is in the set of lower bounds and therefore setting the fact, that it exists.

(c) Propose another way to use the Axiom of Completeness to prove that sets bounded below have greatest lower bounds.

The only idea, that goes into my mind is to create set $B = \{-a : a \in A\}$. Then it'll have a supremum, for which the inverse will be the infinum of the set. We can polish this idea with some theorems and axioms, but I'm satisfied with the current proof already, and nobody is requiring it.

1.3.4

Assume that A abd B are nonempty, bounded above, and satisfy $B \subseteq A$. Show $sup(B) \leq sup(A)$.

We prove it by contradiction: let $B \subseteq A$ and

$$sup(B) > sup(A)$$

Then let $\epsilon = sup(B) - sup(A) > 0$. Then by lemma we have

$$b \in B : b > sup(B) - \epsilon$$

$$b \in B : b > sup(B) - sup(B) + sup(A)$$

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$$b \in B : b > sup(A)$$

Therefore $b > sup(A)$, which is an upper bound for A and by extension $\forall a \in A : b > a$. Therefore $b \notin A$ and $b \in B$. Therefore $B \not\subseteq A$, which is a contradiction. Therefore $sup(B) \le sup(A)$.

1.3.5

Let $A \subseteq \boldsymbol{R}$ be bounded above, and let $c \in \boldsymbol{R}$. Define the sets $c + A$ and cA by $c + A = \{c + a : a \in A\}$ and $cA = \{ca : a \in A\}$

(a) Show that $sup(c + A) = c + sup(A)$.

We gonna prove it by contradiction

Suppose $c + sup(A)$ is not an upper bound of $c + A$. Then

$$\exists n \in c + A : n > c + sup(A)$$

let us call such element l; also, by the definition of $c + A$

$$\forall n \in c + A : \exists a \in A : c + a = n$$

therefore

$$\exists a \in A : c + a = l$$

because $l > c + sup(A)$

$$c + a > c + sup(A)$$

$$a > sup(A)$$

Which is a contradiction. Therefore $c + sup(A)$ is an upper bound for $c + A$.

Suppose $c + sup(A) \ne sup(c + A)$. Then $sup(c + A)$ is less than $c + sup(A)$. Let $\epsilon = c + sup(A) - sup(c + A)$. Then

$$\exists k \in A : k > sup(A) - \epsilon$$

$$k > sup(A) - c - sup(A) + sup(c + A)$$

$$k > -c + sup(c + A)$$

Therefore

$$\exists h \in c + A : h = k + c$$

$$h - c = k$$

$$h - c > -c + sup(c + A)$$

$$h > sup(c + A)$$

therefore

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$$\exists h \in c + A : h > sup(c + A)$$

which is a contradiction. Therefore $c + sup(A) = sup(c + A)$, as desired.

(b) If $c \geq 0$, show that $sup(cA) = c * sup(A)$

If $c = 0$, then it $cA = \{0\}$, and the case is trivial. Therefore let's discuss further case when $c > 0$.

Suppose $c * sup(A)$ is not an upper bound for cA.

Then

$$\exists q \in cA : q > c * sup(A)$$

by the definition of cA

$$\exists j \in A : q = cj$$

therefore

$$q > c * sup(A)$$

$$cj > c * sup(A)$$

$$j > sup(A)$$

Which is a contradiction, because $j \in A$. Therefore $c * sup(A)$ is an upper bound for cA.

Suppose $c * sup(A) \neq sup(cA)$. Then $sup(cA)$ is less than $c * sup(A)$.

$$c * sup(A) > sup(cA)$$

$$c * sup(A) - sup(cA) > 0$$

$$\frac{c * sup(A) - sup(cA)}{c} > 0$$

Let $\epsilon = \frac{c*sup(A)-sup(cA)}{c}$. Then

$$\exists k \in A : k > sup(A) - \epsilon$$

$$k > sup(A) - \frac{c * sup(A) - sup(cA)}{c}$$

$$k > sup(A) - sup(A) - sup(cA)/c$$

$$k > sup(cA)/c$$

Therefore

$$\exists h \in cA : h = ck$$

$$h/c = k$$

$$h/c > sup(cA)/c$$

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$$h > sup(cA)$$

therefore

$$\exists h \in cA : h > sup(cA)$$

Which is a contradiction. Therefore $sup(cA) = c * sup(A)$ as desired.

(c) Postulate a simular type of statement for $sup(cA)$ for the case $c < 0$

Proposition: suppose that A is bounded below; if $c < 0$ then $sup(cA) = c * inf(A)$

Suppose $c * inf(A)$ is not an upper bound for cA.

Then

$$\exists q \in cA : q > c * inf(A)$$

by the definition of cA

$$\exists j \in A : q = cj$$

therefore

$$q > c * inf(A)$$

$$cj > c * inf(A)$$

$$j < inf(A)$$

Which is a contradiction, because $j \in A$. Therefore $c * sup(A)$ is an upper bound for cA.

Suppose $c * inf(A) \neq sup(cA)$. Then $sup(cA)$ is less than $c * inf(A)$.

$$sup(cA) < c * inf(A)$$

$$sup(cA) - c * inf(A) < 0$$

$$\frac{sup(cA) - c * inf(A)}{c} > 0$$

Let $\epsilon = \frac{sup(cA) - c*inf(A)}{c} > 0$. Then

$$\exists k \in A : k < inf(A) + \epsilon$$

$$k < inf(A) + \epsilon$$

$$k < inf(A) + \frac{sup(cA) - c * inf(A)}{c}$$

$$k < inf(A) + sup(cA)/c - inf(A)$$

$$k < sup(cA)/c$$

Therefore

$$\exists h \in cA : h = ck$$

$$h/c = k$$
$$h/c = k < sup(cA)/c$$
$$h/c < sup(cA)/c$$
$$h > sup(cA)$$

therefore

$$\exists h \in cA : h > sup(cA)$$

Which is a contradiction. Therefore $sup(cA) = c * inf(A)$ as desired.

1.3.6

Compute, without proofs, the suprema and infima of the following sets:
  (a) $\{n \in \mathbf{N} : n^2 < 10\}$
  $sup = 3,\ inf = 1.$
  (b) $\{n/(m + n) : m, n \in \mathbf{N}\}$
  $sup = 1/2,\ inf = 0.$
  (c) $\{n/(2n + 1) : n \in \mathbf{N}\}$
  $sup = 1/2,\ inf = 1/3.$
  (d) $\{n/m : m, n \in \mathbf{N} \text{ with } m + n \leq 10\}$
  $sup = 9,\ inf = 1/9.$

1.3.7

Prove that if a is an upper bound for A, and if a is also an element A, then it must be that $a = sup(A)$
   Let's prove this one by contradiction
   Suppose that $a \neq sup(A)$. Because a is still an upper bound, $sup(A) < a$, Therefore $a \in A$, but $sup(A) < a$, which is a contradiction. Therefore $a = sup(A)$.

1.3.8

If $sup(A) < sup(B)$, then show that there exists an element $b \in B$, that is an upper bound for A.
   Let $\epsilon = sup(B) - sup(A)$. Then

$$\exists b \in B : b > sup(B) - \epsilon$$
$$b > sup(B) - \epsilon$$

$$b > sup(B) - sup(B) + sup(A)$$
$$b > sup(A)$$

therefore b is an upper bound for A.

1.3.9

Without worryong about formal proofs for the moment, decide if the following statements about suprema and infima are true or false. For any that are false, supply an example where the claim in question does not appear to hold.

(a) A finite, nonempty set always contains its supremum

True

(b) If $a < L$ for every element a in the set A, then $sup(A) < L$. False. $sup((0,1)) = 1$; $\forall a \in (0,1) : a < 1$, therefore $sup(A) = L$.

(c) If A and B are sets with the property that $a < b$ for every $a \in A$ and $b \in B$, then it follows that $sup(a) < inf(B)$

False. $sup((0,1)) = inf((1,2))$

(d) If $sup(A) = s$ and $sup(B) = t$, then $sup(A + B) = s + t$. The set $A + B$ is defined as $A + B = \{a + b : a \in A \text{ and } b \in B\}$.

True

(e) If $sup(A) \leq sup(B)$, then there exists an element $b \in B$ that is an upper bound for A.

False. $sup([1,2]) = sup((1,2))$

1.4.1

Without doing too much work, show how to prove Theorem 1.4.3 in the case where $a < 0$ by converting this case into the already proven.

First of all, let's state the theorem itself.

Theorem 1.4.3 (Density of Q in R). *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Then let's talk about the possible cases for $a < 0$. Then $b > a$ by the assumtions of the theorem. Therefore $b \geq 0$ or $b < 0$. If $b > 0$ then there exist 0 between them. If $b = 0$, then there exist a rational number $b = 0 < 1/n < -a$, and by extention $a < 1/n < b$ as desired. Therefore we will have some work to do only with case $b < 0$.

It is proven, that $a_1 < r < b_1$ if $0 \leq a_1 < b_1$. Therefore for $a < b < 0$. Therefore $-a > -b > 0$. Therefore if we set $a_1 = -b$ and $b_1 = -a$ then by previously stated theorem, there exist

$$a_1 < r < b_1$$

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$$-b < r < -a$$
$$b > r > a$$
$$a < r < b$$

as desired.

1.4.2

Recall that $\boldsymbol{I}$ stands for the set of irrational numbers.
   (a) Show that if $a, b \in \boldsymbol{Q}$, then ab and $a + b$ are elements of $\boldsymbol{Q}$ as well.
   Because $a, b \in Q$ $\exists m_1, m_2 \in \mathbf{Z}$, $\exists n_1, n_2 \in \mathbf{N}$ such that

$$a = \frac{m_1}{n_1}$$

$$b = \frac{m_2}{n_2}$$

therefore
$$a + b = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2}$$

$\mathbf{Z}$ is presumed closed under addition and $\mathbf{N}$ is presumed closed under $\mathbf{N}$, therefore

$$a + b = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2} \in \mathbf{Q}$$

also $\mathbf{Z}$ is closed under multiplication, and therefore

$$ab = \frac{m_1 m_2}{n_1 n_2} \in \mathbf{Q}$$

   (b) Show that if $a \in \boldsymbol{Q}$ and $t \in \boldsymbol{I}$, then $a + t \in \boldsymbol{I}$ and $at \in \boldsymbol{I}$ as long as $a \neq 0$.
   We prove both things by contradiction.
   Suppose $a + t \in \mathbf{Q}$. Then $\exists b \in Q : a + t = b$. Also, $a \in Q \rightarrow -a \in Q$. Therfore

$$a + t = b$$
$$t = b - a$$

Therfore, because Q is closed under addition (as we discussed previously), $t \in Q$, which is a contradiction. Therefore $a + t \in I$.
   Suppose $at \in Q$ for $a \neq 0$. Then

$$\exists b \in Q : at = b$$

therefore if $a = m/n \in Q$, then $1/a = n/m \ inQ$. Therefore

$$t = b/a$$

Therefore $t \in Q$, which is a contradiction. Therefore $at \in I$ for $a \neq 0$.

(c) Part (a) can besummarized by saing that Q is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st.

I is not closed under addition, nor under multiplication. Proof is $\sqrt{2} + 1$ and $0 - \sqrt{2}$ are both irrational, but

$$\sqrt{2} + 1 - \sqrt{2} = 1 \in Q$$

and

$$\sqrt{2} * \sqrt{2} = 2 \in Q$$

1.4.3

Using Exercise 1.4.2, supply a proof for Corollary 1.4.4 by applying Theorem 1.4.3 to the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

First, let's state Corollary 1.4.4.

Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.

We know, that between two numbers $a_1 < b_1$ there exists a rational number r, for which it is true

$$a_1 < r < b_1$$

Let $a_1 = a + \sqrt{2}$ and $b_1 = b + \sqrt{2}$. Then

$$a_1 < r < b_1$$

$$a + \sqrt{2} < r < b + \sqrt{2}$$

$$a < r - \sqrt{2} < b$$

As we know, $r - \sqrt{2}$ is an irrational number, therefore between a and b there exists an irrational number.

1.4.4

Use the Archimedean Property of R to rigorously prove that $inf\{1/n : n \in N\} = 0$

It is true, that $\forall n \in N : n > 0$. Therefore

$$n > 0$$

$$1/n > 0$$

Therfore 0 is a lower bound for $1/n$. Also, because of Archimededean Property, if we take any $\forall \epsilon > 0 : \exists 1/n : 1/n < \epsilon$

$$\forall \epsilon > 0 : \exists 1/n : 1/n < \epsilon$$

$$\forall \epsilon > 0 : \exists 1/n : 0 + \epsilon > 1/n$$

Therfore for every ϵ there exists an element of a set such that $0 + \epsilon$ is greater than this set. Therefore 0 is an infinum of this set, as desired.

1.4.5

Prove that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the Nested Interval Property must be closed for the conclusion of the theorem to hold.

First of all, $1/n > 0$ implies, that if $y \leq 0$ then $y \notin (0, 1/n)$ for any $n \in N$.

Because of the Archimedean Property, for every $y \in R > 0$ there exists $n \in N$ such that $1/n < y$. Therefore there does not exist $y > 0$ such that $y \geq 1/n$ for every $n \in N$. Therefore for every $y \geq 0$ there exist $n \in N$ such that $y \notin (0, 1/n)$.

Therefore there are no real numbers in $\cap_{n=1}^{\infty}(0, 1/n)$. Therefore

$$\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$$

as desired.

This conclusion proves, that if we have use an open interval for nested interval property, then we'll have a problem.

1.4.6

(a) Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = sup(T)$.

First, Theorem 1.4.5 states that there exists a real number $\alpha \in R$ satisfying $\alpha^2 = 2$.

Our assumpion is that

$$T = \{t \in R : t^2 < 2\} \text{ and } \alpha = sup(T)$$

Our strategy is to state that if $a^2 > 2$, then a is not a least upper bound.

If a is an least upper bound for T, then it is true, that

$$\forall \epsilon > 0 : \exists t \in T : t > \alpha - \epsilon$$

Therefore for every $n \in N$
$$\exists t \in T : t > \alpha - 1/n$$

Now let us follow with the proof. For all $n \in N$.

$$t > \alpha - 1/n$$

$$\alpha - 1/n < t$$
$$(\alpha - 1/n)^2 < t^2$$

$$(\alpha - 1/n)^2 < t^2$$
$$\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < t^2$$
$$\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < t^2$$

Let's jusfity something now. $1^2 = 1 < 2$, therefore $1 < \alpha$. Therefore $2\alpha - 1 > 0$. Also, because $\alpha^2 > 2$, $\alpha^2 - 2 > 0$. Therefore

$$\frac{\alpha^2 - 2}{2\alpha - 1} > 0 \in R$$

Now, let us pick $n \in N$ such that

$$1/n < \frac{\alpha^2 - 2}{2\alpha - 1}$$

then

$$1/n < \frac{\alpha^2 - 2}{2\alpha - 1}$$
$$1/n^2 < \frac{\alpha^2 - 2}{2\alpha - 1}$$
$$n^2 > \frac{2\alpha - 1}{\alpha^2 - 2}$$
$$\frac{2\alpha - 1}{n^2} < \alpha^2 - 2$$
$$\frac{2\alpha}{n^2} - \frac{1}{n^2} < \alpha^2 - 2$$
$$-\alpha^2 + \frac{2\alpha}{n^2} - \frac{1}{n^2} < -2$$
$$\alpha^2 - \frac{2\alpha}{n^2} + \frac{1}{n^2} > 2$$
$$\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2$$

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but
$$\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < t^2 < 2$$

Therefore we have a contradiction. Therefore $\alpha \leq 2$. Therefore $\alpha = 2$, as desired. Phew.

(b) Modify the argument to prove the existence of $\sqrt{B}$ for any real number $b \geq 0$.

Let's discuss the case $a^2 < b$.

$$(\alpha + 1/n)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha + \frac{2\alpha + 1}{n}$$

Let
$$\frac{1}{n} < \frac{b - \alpha}{2\alpha + 1}$$

then
$$(\alpha + \frac{1}{n})^2 = \alpha^2 + (b - \alpha^2) = b$$

Therefore $\alpha + 1/n \in T$. Therefore $a^2 \geq b$.

Now let
$$1/n < \frac{\alpha^2 - b}{2\alpha - 1}$$

. Then by reasoning in the last part of exercise we can state, that $\alpha^2 \leq b$. Therefore $\alpha^2 = b$, as desired.

1.4.7

Finish the following proof for Theorem 1.4.12.

First of all, let us state Theorem 1.4.12

If $A \subseteq B$ and B is countable, then A is either contable, finite or empty.

Assume B is a countable set. Thus, there exists $f : N \to B$, which 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = min\{n \in N : f(n) \in A\}$. As a start to a definition of $g : N \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from N onto A.

Proposition: $g(n) = f(n_n)$.

Step: let $n_{n+1} = min\{k > n \in N : f(k) \in A\}$. Then $f(n) < n_{n+1} \notin n_{n+1}$. Therefore $g(n + 1) = f(n_{n+1})$. Therefore

$$\forall n_1 \neq n_2, k_1, k_2 \in N \exists g(n_1) = f(k_1) \neq g(n_2) = f(k_2)$$

and

$$\forall l \in g(N) \exists k \in N : g(k) = f(n_k) = l$$

Therefore there exist a bijective function $g : N \to A$. Therefore A is countable, as desired.

If A is not infinite, then it's finite (duh).

Same with empty.

1.4.8

Use the following outline to supply for the statements in Theorem 1.4.13.

First of all, let's state Theorem 1.4.13

(i) If $A_1, A_2, ...A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup ... \cup A_m$ is countable.

(ii) If A_n is countable set for each $n \in N$, then $\cup_{n=1}^{\infty} A_n$ is countable.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.4.8 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2, with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Let us first set $B_2 = A_2 \setminus A_1$. We will do it in order to have two useful properties:

$$a \in B_2 \to a \in A_1^c \cup A_2 \to a \notin A_1$$

$$A_1 \cup B_2 = A_1 \cup (A_1^c \cap B_2) = (A_1 \cap A_1^c) \cup (A_1 \cap B_2) = A_1 \cap B_2$$

Let's finally begin with the proof. $B_2 \subseteq A_2$. By using previous theorem we can state, that B_2 is either countable, finite, or empty. If it is empty, then $A_1 \cup A_2 = A_1$, and therefore is countable.

For the finite case we'll need function, that returns the n'th smallest element of the set. We can argue, that this function does not need the rigorous definition, but we'll give it anyways:

$$s_F(1) = \{x \in F : \forall y \in F : x \leq y\}$$

$$s_F(n) = \{x \in F \setminus \{s(1), ..., s(n-1)\}\} : \forall y \in F \setminus \{s(1), ..., s(n-1)\}\} : x \leq y\}$$

for the finite set $F \subseteq R$.

$$\forall n_1 > n_2 \in N : s_F(n_2) \notin F \setminus \{s(1), ..., s(n_1 - 1)\} \to s_F(n_1) \neq s_F(n_2)$$

therefore the function is injecttive

We remove one element at every iteration from F every iteration, therefore $\{s_F(1), ...s_F(|F|)\}$ spans the whole set F. Therefore the function is surjective. Therefore the function is bijective.

Now let us define $q : N \to A_1 \cup B_2$. Let $g_1 : N \to A_1$ be a bijective function, that exists, because the set A_1 is countable. Then

$$q(x) = \begin{cases} s_{B_2}(x) & \text{if } x \leq |B_2| \\ g_1(x - |B_2|) & \text{if } x > |B_2| \end{cases} \tag{1}$$

Let's analyse this function. If $x_1 < x_2 \leq |B_2|$ or $x_1 > x_2 > |B_2|$, then $q(x_1) \neq q(x_2)$ by injectivity of s_{B_2} or $g_1(x)$ respectively. If $x_1 \leq |B_2| < x_2$, then $q(x_1) \in B_2$ and $q(x_2) \in A_1$. Those two sets are disjoint, and therefore $q(x_1) \neq q(x_2)$. Therefore for all possible $x_1, x_2 \in N \rightarrow q(x_1) \neq q(x_2)$. Therefore the function is injective.

Let $b \in N$ and $N_b = \{x \in N > b\}$. Also let $N_a = \{x \in N_b : x - b\}$. Then

$$\forall c \in N_a : \exists v \in N_b : c = v - b$$

$$\forall v \in N_b : \exists n \in N : v = n + b$$

$$\forall c \in N_a : \exists n \in N : c = n$$

therfore $N \subseteq N_a$.

$$\forall n \in N : \exists v \in N_b : v = n + b$$

$$\forall v \in N_b : \exists c \in N_a : c = v - b$$

$$\forall n \in N : \exists c \in N_a : c = n$$

therfore $N_a \subseteq N$. Thus $N = N_a$.

Therefore

$$\forall x > |B_2| : \exists n \in N : g_1(x - |B_2|) = g_1(n)$$

Therefore g_1 spans the whole set A_1. s_{B_2} spans the whole B_2 by surjectivity of s. Therefore q spans $A_1 \cup B_2$. Therefore it is surjective.

Therefore $A_1 \cup B_2$ is bijective, therefore $A_1 \cup B_2$ is countable if B_2 is finite. One last thing, that I want to add before finishing this case is to acknowledge the fact, that this theorem is true not for real numbers only. This fact throws our minimal element part out of the proof. This misfortune can be avoided through the usage of lists and by converting sets into them. I did not use this fact here because of the need to axiomaticly define lists, which I can, but don't want to do.

Now let us proceed with the case when B_2 is countable. Let $g_1 : N \rightarrow A_1$ be a bijective funciton for A_1 and $g_2 : N \rightarrow B_2$ be a bijective funciton for B_2 (both of those exist because of the fact, that both of the sets are countable). Let $q : N \rightarrow A_1 \cup B_2$ be defined as

$$q(x) = \begin{cases} g_1((x+1)/2) \text{ if x is odd} \\ g_2(x/2) \text{if x is even} \end{cases} \tag{2}$$

Then $\forall n_1 \neq n_2 \in N \rightarrow g(n_1) \neq g(n_2)$ because of either injectivity of both functions, or because of the fact, that both sets are disjoint. Therefore q is injective. Also

$$x \text{ is odd} : (x+1)/2 = x \text{is even} : x/2 = N$$

Therefore q spans $A_1 \cup B_2$. Therefore the function is surjective.

Thus q is bijective and $A_1 \cup B_2 = A_1 \cup B_2$ is countable if B_2 is countable.

Therefore for any countable sets A_1 and A_2 it is true, that their union is countable as well.

Now, explain how the more general statement in (i) follows

We prove the first part of the teorem by induction.

Base: $A_1 \cup A_2$ is countable.

Proposition: $\cup_{n=1}^m A_n$ is countable for $m \in N$.

Step:

$$\cup_{n=1}^{m+1} A_n = \cup_{n=1}^m A_n \cup A_{m+1}$$

$\cup_{n=1}^m A_n$ is a countable set because of the proposition. A_{m+1} is countable by assumtion of the theorem. Therefore $\cup_{n=1}^{m+1} A_n$ is a union of two countable sets, and therefore is itself contable, as desired.

(b) Explain why induction connot be used to prove part (ii) of Theorem 1.4.13 from part (i)

Induction cannot be used to prove part (ii) because of the fact, that induction relies of finality of the given set.

(c) Show how arranging N into the two-dimentional array

$$1 \; 3 \; 6 \; 10 \; 15 \; ...$$

$$2 \; 5 \; 9 \; 14 \; ...$$

$$4 \; 8 \; 13 \; ...$$

$$7 \; 12 \; ...$$

$$11 \; ...$$

$$...$$

leads to a proof of a Theorem 1.4.13 (ii)

This proof will not be as rigorous as the ones before that, but i'll try them anyways. Let us convert all of the given sets onto lists. Then let us construct lists

$$l_1 = A_{1_1}$$

$$l_2 = A_{2_1}, A_{1_2}$$

$$l_3 = A_{3_1}, A_{2_2}, A_{3_1}$$

$$...$$

$$l_n = A_{n_1}, A_{n-1_2}, A_{n-2_31}...$$

if A_{n_n} already has been included into the lists $l_{1..n-1}$, then we don't include it in the list. Then if we concatenate all of those lists together, then we will have function $q : N \rightarrow \cup_{n=1}^\infty A_n$ which is defined as

$$q(x) = \text{x'th element of the final list}$$

This function will be injective, because we threw out already included elements, and will be surjective, because all of the elements of $\cup_{n=1}^{\infty} A_n$ are eventually in the list. Thus, we have a bijective function between N and $\cup_{n=1}^{\infty} A_n$. Therefore $\cup_{n=1}^{\infty} A_n$ is countable, as desired.

1.4.9

(a) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Short answer: bijectivity implies inversability.

Medius answer: injectivity leads us to the fact, that for every element of codomain there exists only one element of the domain and by extension lets us define inverse function in the first place; surjectiviry guarantees, that the inverse function is defined for every element of the domain.

Long answer:

$A \sim B$ implies, that there exists a bijective function $g : A \to B$. Bijective means that

$$\forall a_1 \neq a_2 \in A \to g(a) \neq g(b)$$

$$\forall b \in B : \exists a \in A : g(a) = b$$

This implies that for all $b \in B$ there exist only one $a \in A$ such that $g(a) = b$. This fact lets us define $g^{-1} : B \to A$ as

$$g^{-1}(b) = a \text{ such that } g(a) = b$$

Then

$$\forall b_1 \neq b_2 \in B \to g^{-1}(b_1) \neq g^{-1}(b_2)$$

because g is a function and therefore $g^{-1}(b_1) \neq g^{-1}(b_2) \to g(a_1) \neq g(a_2) \to a_1 \neq a_2$. Therefore g^{-1} is injective.

$$\forall a \in A : \exists g(a) \to \exists b \in B : g^{-1}(b) = a$$

and therefore

$$\forall a \in A : \exists b \in B : g^{-1}(b) = a$$

therefore function is surjective.

Therefore the function is bijective and therefore $B \sim A$.

Therefore $A \sim B \iff B \sim A$, as desired.

(b) For three sets A, B and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These two properties are what is meant by saying that $\sim$ is an equivalence relation.

Because $A \sim B$ and $B \sim C$ there exist two bijective functions $g_1 : A \to B$ and $g_2 : B \to C$. Therefore

$$\forall a_1 \neq a_2 \in A \to g_1(a_1) \neq g_1(a_2)$$

$$\forall b_1 \neq b_2 \in B \rightarrow g_2(b_1) \neq g_2(b_2)$$

therefore

$$\forall a_1 \neq a_2 \in A \rightarrow g_1(a_1) \neq g_1(a_2) \rightarrow g_2(g_1(a_1)) \neq g_2(g_1(a_2))$$

Also

$$\forall a \in A : \exists b \in B : g_1(a) = b$$

$$\forall b \in B : \exists c \in C : g_2(b) = c$$

and therefore

$$\forall a \in A : \exists b \in B : g_1(a) = b \rightarrow \exists c \in C : g_2(g_1(a)) \in C$$

Therefore $g_2 \circ g_1 : A \rightarrow C$ is a bijective function. Therefore $A \sim C$, as desired.

1.4.10

Show that the set of all finite subsets of N is a countable set. (It turns out that the set of all subsets of N is not a countable set. This is a title of Section 1.5)

Our strategy here will be to show that for every $n \in N$ set of sets of length n is countable, therefore their union is countable.

Maybe I should proceed with induction. Base: First of all, set N in countable. Therefore the set of sets of length 1 is countable.

Proposition: Suppose that set of sets of length $n \in N$ is countable.

Step: Because set of sets of length N is countable, let us take a set S_m. Then, let us add a number into in , which is not already in this set. To be more precise, let us define

$$N_m = N \setminus S_m$$

$$S_{m_k} = S_m \cup \{\text{k'th number of} N_m\}$$

Then each of S_{m_k} is a union of countable sets, and therefore countable. Therefore $\cup_{k=1}^{\infty} S_{m_k} = S_{m+1}$ is countable.

Therefore each set of sets of length $n \in N$ is countable. Therefore their union is countable. Therefore set of finite subsets of N is countable, as desired.

1.4.11

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 \leq x, y \leq 1\}$

(a) Find a 1-1 function, that maps (0, 1) into, but not necessarily onto, S. (This is easy.)

Yeah, it is. Let $g(x) = (x, 0.5)$.

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into (0, 1). Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999999999).

Let $g(x, y) : S \to (0, 1)$ be such a function, that maps digits of x into the odd digits of the result, and y into the event digits of the result. As an example

$$g(.235, .746) = .273456$$

This function is into, because we are essentially writing two dirrerent numbers in an odd way. Therefore $\forall s_1, s_2 \in S \to g(s_1) \neq g(s_2)$

The problem arises with surjectivity. Suppose that we have a number $x = 0.2$. If this function would be the output of our function, then y would need to be 0; Therefore the function is not surjective.

The Schroder-Bernstein Theorem discussed in Exercise 1.4.13 to follow can now be applised to conclude that $(0, 1) \sim S$

1.4.12

A real number $x \in R$ is called algebraic if there exists integers $a_0, a_1, a_2, ..., a_n \in Z$, not all zero, such that

$$a_n x^n + a_{n-1} x^{x-1} + ... + a_1 x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called transcendental numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$ and $\sqrt{3} + \sqrt{2}$ are algebraic.

Let $a_0 = -2$, $a_1 = 0$ and $a_2 = 1$. Then

$$a_2 x^2 + a_1 x + a_0 = 0$$
$$x^2 + 0 - 2 = 0$$
$$x^2 = 2$$
$$x = \sqrt{2} \text{ or } x = -\sqrt{2}$$

Therefore $\sqrt{2}$ is algebraic.

Let $a_0 = -2$, $a_1 = 0$, $a_2 = 0$ and $a_3 = 1$. Then

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$$

$$x^3 + 0 - 2 = 0$$
$$x^3 = 2$$
$$x = \sqrt[3]{2}$$

Therefore $\sqrt[3]{2}$ is algebraic.

Let $a_0 = 1$, $a_1 = 0$, $a_2 = -10$, $a_3 = 0$ and $a_4 = 1$. Then

$$x^4 - 10x^2 + 1 = 0$$
$$x^4 - 10x^2 + 25 = 24$$
$$(x^2 - 5)^2 = 4 * 6$$
$$x^2 - 5 = 2\sqrt{6}$$
$$x^2 = 5 + 2\sqrt{6}$$
$$x^2 = 2 + 3 + 2\sqrt{2}\sqrt{3}$$
$$x = \sqrt{2} + \sqrt{3}$$

Therefore $\sqrt{2} + \sqrt{3}$ is algebraic.

(b) Fix $n \in N$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable. (For each $m \in N$, consider polynomicals $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ that satisfy $|a_n| + |a_{n-1}| + ... + |a_1| + |a_0| \leq m$.)

Let us fix $m \in M$, then there exist a finite number of $a_n, ..., a_1, a_0 \in Z$ such that $|a_n| + |a_{n-1}| + ... + |a_1| + |a_0| \leq 0$. Therefore for each $m \in M$ there are finitely many combinations of coefficients and thus finitely many numbers of roots. Therefore let us correspont each root, with a number $n \in N$ is ascending order (keeping in mind, we should check, that this number is not already in the function). Then we'll get a bijective function from N to all the possible roots of polynomials with integral coefficiens. Therefore A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Because A_n is countable for each $n \in N$ we know, that $\cup_{n=1}^{\infty} A_n$ is countable as well. By definition, algebraic number is a root of one of such polynomials, and therefore algebraic numbers are a countable set.

The fact, that the set of algebraic numbers is countable presents us with a fact, that the set of transcedental numbers is not countable (because if it was, then the set of real numbers would be a union of two countable sets and therefore countable as well).

1.4.13 (Schroder-Bernstein Theorem)

Assume that there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

(a) The range of f is defined by $f(X) = \{y \in Y : y = f(x)$ for some $x \in X$. Let $y \in f(X)$. (Because f is not neccesarily onto, the range $f(X)$ may not be all of Y.) Explain why there exists a unique $x \in X$ such that $f(x) = y$. Now define $f^{-1}(y) = x$, and show that f^{-1} is a 1-1 function from $f(X)$ onto X.

Because f is given to be injective, it is true that

$$\forall x_1 \neq x_2 \in X \to f(x_1) \neq f(x_2)$$

f is also a function. By definition of a function we know, that for every element of domain there exists only one element of codomain. Therefore

$$\forall f(x_1) \neq f(x_2) \in f(X) \to x_1 \neq x_2$$

By plugging y into the equation we have

$$\forall y_1 \neq y_2 \in f(X) \to x_1 \neq x_2$$

Thus we can define $f^{-1} : f(X) \to X$:

$$f^{-1}(y) = \{x \in X : f(x) = y\}$$

This function will be injective by the fact that

$$\forall y_1 \neq y_2 \in f(X) \to x_1 \neq x_2 \to f^{-1}(x_1) \neq f^{-1}(x_2)$$

In a similat way, we can also defin the 1-1 function $g^{-1} : g(X) \to Y$
Same logic applies to $g(x)$. Therefore we have

$$g^{-1}(x) = \{y \in Y : g(y) = x\}$$

which by the same logic is injective.

(b) Let $x \in X$ be arbitrary. Let the chain C_x be the set consisting of all elements of the form

$$(1) \ ..., f^{-1}(g^{-1}(x)), g^{-1}(x), x, f(x), g(f(x)), f(g(f(x))), ...$$

Explain why the number of elements to the left of x in the above chain may be zero, finite, or infinite

The number of elements on the left is 0, if there does not exists $y \in Y$ such that $g(y) = x$.

The number of elements on the left is finite number, if after some element l there does not exist an element $y \in Y$ or $x \in X$, such that $g(y) = l$ or $f(x) = l$.

The number of elements on the left is infinite, if there exist both of those numbers for every element (for example, if both functions are onto).

(c) Show that any two chains are either identical, or completely disjoint

Suppose that $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ by injectivity of f. Also, $f(x) \in Y$, and therefore $g(f(x_1)) \neq g(f(x_2))$ and so on.

f^{-1} and g^{-1} are injective as well. Therefore the same logic applies. Thus we can state that

$$\forall x_1 \neq x_2 \to C_{x_1} \cap C_{x_2} = \emptyset$$

and

$$\forall x_1 = x_2 \to C_{x_1} \cap C_{x_2} = C_{x_1} = C_{x_2}$$

(d) Note that the terms of the chain in (1) alternate between elements of X and elements of Y. Given a chain C_x, we want to focus on $C_x \cap Y$, which is just a part of the chain that sits in Y.

Define the set A to be the union of all chains C_x satisfying $C_x \cap Y \subseteq f(X)$. Let B constst if the union of the remaining chains not in A. Show that any chain contained in B must be of the form

$$y, g(y), f(g(y)), g(f(g(y))), ...$$

where y is an element of Y that is not in $f(X)$

Let A and B be defined as in the exercise text. Then

$$\forall C_x \in B : \exists c \in C_x : c \in Y \text{ and } c \notin f(X)$$

. Thus

$$\nexists x \in X : f(x) = c$$

at the same time

$$c \in Y \to \exists x \in X : g(y) = x$$

.

Becaus $g : Y \to g(Y) \subseteq X$ and $f : X \to f(X) \subseteq Y$ we can write the chain in the form

$$c, g(c), f(g(c)), ...$$

Substituting c with y we get

$$y, g(y), f(g(y)), ...$$

where $y \notin f(X)$, as desired.

(e) Let $X_1 = A \cap X$, $X_2 = B \cap X$, $Y_1 = A \cap Y$, and $Y_2 = B \cap Y$. Show that f maps X_1 onto Y_1 and that g maps Y_2 onto X_2. Use this infortation to prove $X \sim Y$.

Let X_1, X_2, Y_1 and Y_2 be defined as in exercise text. Then let us discuss $f(X_1)$

$$A = \{C_x : C_x \cap Y \subseteq f(X)\}$$

$$X_1 = \{C_x \cap X : C_x \cap Y \subseteq f(X)\}$$
$$Y_1 = \{C_x \cap Y : C_x \cap X \subseteq g(Y)\}$$

therefore
$$\forall y \in Y_1 : \exists x \in X_1 : f(x) = y$$

and
$$\forall x \in X_2 : \exists y \in Y_2 : g(y) = x$$

Or in other words, $f : X_1 \to Y_1$ is surjective(onto).
By the same logic, $g : Y_2 \to X_2$ is also surjective.
Let us define function $h : X \to Y$

$$h(x) = \begin{cases} f(x) \text{ if } x \in X_1 \\ g^{-1}(x) \text{ if } x \in X_2 \end{cases} \tag{3}$$

Functions f and g^{-1} are both injective. $X_1 \cap X_2 = \emptyset$. Therefore h is injective as well. Also, $X_1 \cup X_2 = X$, and both f and g are surjective. Therefore h is surjective as well.

Therefore this function is bijective. Therefore we have a bijective function from X to Y. Therefore $X \sim Y$, as desired.

1.5.1

Show that $(0,1) = \{x \in R : 0 < x < 1\}$ is uncountable if and only if R is uncountable.
Let

$$f(x) = \frac{2x - 2)}{x^2 - 2x}$$

Calsulus shows that this function maps $(0,1) \to R$. Also, it shows, that ist is increasing and therefore it is bijective. Thus there exist a bijective. Thus $(0,1) \sim R$. Therefore it is uncountable if and only if R is uncountable.

1.5.2

(a) Explain why the real numnder $x = b_1 b_2 b_3...$ cannot be f(1).
If $f(1) = b$ then
$$b_1 = 2 \to b_1 \neq 2$$
$$b_1 \neq 2 \to b_1 = 2$$

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Therefore we have a contradiction.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in N$.

If $f(1) = b$ then

$$b_n = 2 \rightarrow b_n \neq 2$$

$$b_n \neq 2 \rightarrow b_n = 2$$

Therefore we have a contradiction for $n \in N$.

(c) Point ount the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

If $(0,1)$ is countable, then $b \in (0,1)$, but it cannot be correspondent to $n \in N$ for any $f : N \rightarrow (0,1)$. Therefore either $b \notin (0,1)$ or $(0,1)$ is not countable. Because $b \in (0,1)$ we conclude that $(0,1)$ is uncountable.

1.5.3

Supply rebuttals to the following complaints about the proof of Theorem 1.5.1

(a) Every rational number has a decimal expansion so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of Q must me countable, the proof of Theorem 1.5.1 must be flawed

We can try to apply the same argument to Q, but now we have a problem with the fact, that every rational number in decimal expansion repeats itself after some point or another. Therefore $b \notin Q$, therefore we cannot conclude anything.

(b) A few numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can be written as .5 or as .4999.... Doesn't this cause some problems?

No, it doesn't. And the reason on why it doesn't cause any problems is because our argument stems on the fact, that given $b \in R$ is not in the set, if the set is countable. Therefore different representations problem is irrelevant.

1.5.4

Let S be the set consisting of all sequences of 0's and 1's. Observee that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, ...) : a_n = 0 \, or \, 1\}$$

As an example, the sequence (1, 0, 1, 0, 1, 0, ...) is an element of S, as the sequence (1, 1, 1, 1, ...).

Give a rigorous argument, that S is uncountable

I don't know, if it counts, as rigorous, but here we go.

Each $s \in S$ corresponds to a binary reresentation of a number in $(0,1)$. Therefore $S \sim (0,1) \sim R$, therefore it is uncountable.

We can work around sets and NIP to show the same thing if we want to, but I dont want to.

1.5.5

(a) Let $A = \{a,b,c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

$$\emptyset, \{a\}, \{b\}, \{b,a\}, \{c\}, \{c,a\}, \{c,b\}, \{c,b,a\}$$

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements. (Constructing a particular subset of A can be interpreted as making a series of decisions about whether or not toinclude each element of A.

Repeat of 1.2.11

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary number: 0 for excluded, 1 for included. Therefore there exist 2^n possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of $\emptyset$ are $\emptyset$ itseft ($2^0 = 1$ in total). Subsets of set with one element are $\emptyset$ and set itself ($2^1 = 1$ in total).

Proposition is that set with n elements has 2^n subsets.

Inductive step is that for set with $n+1$ elements can either have or hot have the $n+1$'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

1.5.6

(a) Using the particular set $A = \{a,b,c\}$, exhibit two different 1-1 mappings from A to $P(A)$

$$f_1(x) = \begin{cases} a \to \{a\} \\ b \to \{b\} \\ c \to \{c\} \end{cases} \tag{4}$$

$$f_2(x) = \begin{cases} a \to \emptyset \\ b \to \{b\} \\ c \to \{c\} \end{cases} \tag{5}$$

(b) Letting $B = \{1,2,3,4\}$, produce an example of a 1-1 map $g : B \to P(B)$.

$$f_3(x) = \begin{cases} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 4 \end{cases} \qquad (6)$$

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings, that are onto

$$|A| < |P(A)|$$

1.5.7

Return to the particular functions constructed in Exercise 1.5.6 and construct subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

For f_1:
$$B = \{\emptyset\}$$

For f_2:
$$B = \{a\}$$

For f_3:
$$B = \{\emptyset\}$$

1.5.8

(a) First, show that the case $a' \in B$ leads to a contradiction.

Suppose that $a' \in B$. By definition of B, $a \notin f(a')$. Therefore we have a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Suppose that $a' \notin B$. Therfore, by definition of B, $a' \notin B \to a' \in B$. Therefore we have a contradiction. Thus, we cannot construct a surjective map from A to $P(A)$. Therefore $A \not\sim P(A)$.

1.5.9

As a final exercise, answer each of the following be establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all function from $\{0,1\}$ to N countable or uncountable?

Examples of such functions

$$f_1 = \begin{cases} \{0\} \to 5 \\ \{1\} \to 123 \end{cases} \qquad (7)$$

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$$f_2 = \begin{cases} \{0\} \to 7 \\ \{1\} \to 7 \end{cases} \tag{8}$$

Each of those functions we can correspond to a set of $Q^+ = q \in Q : q > 0$, which is countable (infinite subset of a countable set $\to$ countable) .

To clarify my result, think of $f(0)$ as of numerator, and $f(1)$ as denumenator.

(b) Is the set of all function from N to $\{0,1\}$ countable or uncountable.

Uncountable. Each of those function we can correspond to a sequence of 0's and 1's from Exercise 1.5.4.

(c)Given a set B, a subset A of $P(B)$ is called an antichain, if no element of A is a subset of any other element of A. Does $P(N)$ contain an uncountable antichain?

Yes, it does. We can correspond a function from N to $\{0,1\}$ to an antichain by adding element $2n$ to the set, if the n'th element of a sequence is 0, and adding $2n - 1$ to the set, if the n'th position is 1. Therefore, for 2 different chains we will have two different sets, each of which will be different by al least two numbers, and therefore not a subset of each other.

4.4.1

a

Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

In order to show, that f is continous we need to show, that $\forall \epsilon \in \mathbf{R} \; \exists \delta$ s.t.

$$|x - c| < \delta \to |f(x) - f(c)| < \epsilon$$

Let's rewrite the first formula

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + cx + c^2)| = |x - c||x^2 + cx + c^2|$$

We can put $|x - c|$ can be as small as we want it to be. Therefore we need an upper bound for $|x^2 + cx + c^2|$.

$$|x^2 + cx + c^2| \leq |x^2| + |cx| + |c^2| \leq (|c| + 1)^2 + |c|(|c| + 1) + |c|^2$$

Therefore if we take $\delta = min\{1, \epsilon/((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)\}$ then

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2| \leq \epsilon \frac{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)}{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)}) = \epsilon$$

Therefore $f(x) = x^3$ is continous on $\mathbf{R}$.

(b)

Argue, using Theorem 4.4.6, that f is not uniformly continuous on $\mathbf{R}$

Theorem 4.4.6 (Sequential Criterion for Nonuniform Continuity). A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if $\exists \epsilon > 0$ and two sequences (x_n) and (y_n) in A satisfying

$|x_n - y_n| \to 0$ *but* $|f(x_n) - f(y_n) \leq \epsilon_0$

In order to show that $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$ let us use sequences

$$x_n = n$$

$$y_n = (n + 1/n)$$

Firstly

$$|x_n - y_n| = |n - n - 1/n| = |-1/n| = 1/n \to 0$$

on the other hand

$$|f(x_n) - f(y_n)| = |n^3 - (n + 1/n)^3| = |n^3 - (n^3 + 3\frac{n^2}{n} + 3\frac{n}{n^2} + \frac{1}{n^3})| =$$

$$= |-3n - \frac{3}{n} - \frac{1}{n^3}| \leq |3n| \to \infty$$

rmaxima seems to eraborate this statement, therefore $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n) \to \infty$

Therefore $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$.

(c)

Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Suppose that $A \subset \mathbf{R}$ and $\exists M \in \mathbf{R}$ s.t. $\forall x \in A$ $x \leq M$ (i.e. A is bounded M)

Then, $\forall c \in A$ and $\forall \epsilon \in \mathbf{R}$

$$\frac{\epsilon}{((|M| + 1)^2 + |M|(|M| + 1) + |M|^2} \leq \frac{\epsilon}{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2}$$

Therefore if we take

$$\delta = min\{1, \frac{\epsilon}{((|M| + 1)^2 + |M|(|M| + 1) + |M|^2}\}$$

then $|x - c| < \delta$ implies, that $|f(x) - f(c)| < \epsilon$, therefore making $f(x)$ uniformly continous by definition

4.4.2

Show that $f(x) = 1/x^3$ is uniformly continous on the set $[1, \infty)$, but is not on the set $(0, 1]$
In order to show, that $f(x)$ is continous on the set $[1, \infty)$ let us first prove that it is just continous, with the hope that δ is not depentant on x

$$|\frac{1}{x^3} - \frac{1}{c^3}| = |\frac{c^3 - x^3}{x^3 c^3}| = |\frac{(c-x)(x^2 + cx + c^2)}{x^3 c^3}| = |(c-x)\frac{x^2 + cx + c^2}{x^3 c^3}| = |c-x||\frac{x^2 + cx + c^2}{x^3 c^3}| =$$

$$= |x - c||\frac{x^2 + cx + c^2}{x^3 c^3}|$$

Therefore we need to show that if δ is bounded above at 1, then $|\frac{x^2 + cx + c^2}{x^3 c^3}|$ is bounded above at $[1, \infty)$ by some constant, but $(0, 1]$ isn't.

$$|\frac{x^2 + cx + c^2}{x^3 c^3}| = |\frac{1}{c^3 x} + \frac{1}{c^2 x^2} + \frac{1}{cx^3}| \le |\frac{1}{c^3 x}| + |\frac{1}{c^2 x^2}| + |\frac{1}{cx^3}|$$

for $x \in [1, \infty)$ each of those fractions are bounded above by 1, therefore for $x \in [1, \infty)$

$$|\frac{x^2 + cx + c^2}{x^3 c^3}| \le 3$$

therefore if we pick $\delta < \epsilon/3$ then it follows, that $|f(x) - f(c)| < \epsilon$ for $x \in [1, \infty)$
on the other hand,

$$\lim_{x \to 0}(|\frac{x^2 + cx + c^2}{x^3 c^3}|) \to \infty$$

Therefore we will need smaller deltas as we approach 0; to put it more concretely let's use the theorem for **Sequential Criterion for Nonuniform Continuity**.
Let us pick

$$x_n = 1/n$$

$$y_n = 1/(n+1)$$

then

$$|x_n - y_n| = |\frac{1}{n} - \frac{1}{n+1}| = |\frac{n+1-n}{n(n+1)}| = |\frac{1}{n^2+1}| \to 0$$

but

$$|f(x_n) - f(y_n)| = |1/(\frac{1}{n})^3 - 1/(\frac{1}{n+1})^3| = |1/(\frac{1}{n^3}) - 1/(\frac{1}{(n+1)^3})| = |n^3 - (n+1)^3| =$$

$$= |n^3 - (n^3 + 3n^2 + 3n + 1)| = |3n^2 + 3n + 1| \to \infty$$

therefore by **4.4.6** $f(x)$ is not uniformly continous on $(0, 1]$, as desired

4.4.3

Furnish the details (including an argument for Exercise 3.3.1 if it is not already done) for the proof of the Extreme Value Theorem (Theorem 4.4.3).

Let us first complete 3.3.1

Exercise 3.3.1. Show that if K is compact, then sup K and inf K both exist and are elements of K.

Because K is compact, it is both closed and bound; therefore, because it is bounded,

$$\exists M \in \mathbf{R} > 0 : \forall x \in K$$

$$|x| \leq M$$

Therefore there exist lower and upper bound of K. Therefore, by axiom of completenss, there exist both $sup(K)$ and $inf(K)$ (i.e. both least upper bound and greatest lower bound)

Now let's prove that there exists a sequence that converges to either $sup(k)$ or $inf(k)$.

To be continued...

4.2.1

Use Definition 4.2.1 to supply a proof for the following limit statements.

(a) $\lim_{x \to 2}(2x + 4) = 8$.

(b) $\lim_{x \to 0} x^3 = 0$.

(c) $\lim_{x \to 2} x^3 = 8$.

(d) $lim_{x \to \pi}[[x]] = 3$, where [[x]] denotes the greatest integer less than or equal to x.

Let's first state Definition 4.2.1

Definition 4.2.1. Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(a):

$$|f(x) - L| = |2x + 4 - 8| = |2x - 4| = 2|x - 2| < \epsilon$$

$$|x - 2| < \frac{\epsilon}{2}$$

$$\delta = \frac{\epsilon}{2} \to |2x + 4 - 8| < \epsilon$$

as desired.

(b):

$$|f(x) - L| = |x^3 - 0| = |x^3| = |x|^3 < \epsilon$$

$$|x| < \sqrt[3]{\epsilon}$$

$$\delta = \sqrt[3]{\epsilon} \to |x^3| < \epsilon$$

as desired.

(c):

$$|f(x) - L| = |x^3 - 8| = |(x-2)(x^2 + 2x + 4)| = |x-2||x^2 + 2x + 4| < \epsilon$$

$$|x-2| < \frac{\epsilon}{|x^2 + 2x + 4|}$$

Suppose that we set the maximum delta at 1; then upper bound for $|x^2 + 2x + 4|$ is:

$$|x^2 + 2x + 4| \leq |x^2| + |2x| + 4 = |x|^2 + 2|x| + 4 \leq (|c| + 1)^2 + 2(|c| + 1) + 4 =$$

$$= (2 + 1)^2 + 2(2 + 1) + 4 = 9 + 6 + 4 = 19$$

Therefore

$$\delta = min\{1, \epsilon/19\} \rightarrow |x^3 - 8| = |x-2||x^2 + 2x + 4| < \frac{\epsilon}{19} * 19 = \epsilon$$

as desired.

(d):

$$|[[x]] - 3| = [[0.1415926...]] = 0 < \epsilon$$

Suppose that we pick $\delta = 0.1$, then any $x \in V_\delta$ will satisfy $|[[x]] - 3| = 0 < \epsilon$ for any $\epsilon > 0$ as desired.

4.2.2

Assume a particular $\delta > 0$ has been constructed as a suitable response to a particular ϵ challenge. Then, any larger/smaller (pick one) δ will also suffice.

Smaller. This follows from the fact, that

$$\delta_1 < \delta_2 \rightarrow V_{\delta_1} \subset V_{\delta_2}$$

4.2.3

Use Corollary 4.2.5 to show that each of the following limits does not exist.

(a) $\lim_{x \to 0} |x|/x$

(b) $\lim_{x \to 1} g(x)$ where g is Dirichlet's function from Section 4.1.

I'll not state corollary 4.2.5 function here, because it's tedious, but it'll be obvious which corollary I'm talking about by the context.

(a): let

$$(x_n) = 1/n$$

$$(y_n) = -1/n$$

then

$$(x_n) \to 0; (y_n) \to 0$$

but

$$|x_n|/x_n = 1$$
$$|y_n|/y_n = -1$$

therefore the limit does not exist.

(b):

The Dirichlet function is

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases} \tag{9}$$

let

$$(x_n) = 2/n + 1$$
$$(y_n) = \sqrt{2}/n + 1$$

then

$$(x_n) \to 1; (y_n) \to 1$$

but

$$(x_n) = 2/n + 1 \in \mathbf{Q}$$
$$(y_n) = \sqrt{2}/n + 1 \notin \mathbf{Q}$$

therefore

$$D(x_n) = 1$$
$$D(y_n) = 0$$

thus the function is not continous at 1

4.2.4

Review the definition of Thomae's function $t(x)$ from Section 4.1.

(a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.

(c) Make an educated conjecture for $\lim_{x \to 1} t(x)$, and use Definition 4.2.1B to verify the claim. Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x)\epsilon\}$. Argue that all the points in this set are isolated.

The definition of Thomae function is

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases} \tag{10}$$

(a): Let our three sequences be

$$(x_n) = n/(n+1)$$

$$(y_n) = (n+1)/n$$

$$(z_n) = \sum_{i=1}^{n} \frac{1}{2^n}$$

(b):

$$t(x_n) = \{1/2, 1/3, 1/4, 1/5, 1/6, 1/7...\}$$

$$t(y_n) = \{1, 1/2, 1/3, 1/4, 1/5, 1/6...\}$$

$$t(z_n) = \{1/2, 1/4, 1/8, 1/16...\}$$

(c): The educated conjecture here is that $\lim_{x \to 1} t(x) = 0$

In order to prove that conjecture author suggests, that we use $\epsilon - \delta$ definition. Let's try it;

$$|t(x)| < \epsilon$$

For all $\epsilon \in \mathbf{R} > 0$

Therefore by archimedes property there exists a number $n \in \mathbf{N}$ s.t. $\frac{1}{n} < \epsilon$. Thus suppose that we have $\delta = 1/n$. Then our proposition is that

$$\forall b \in (1 - 1/n; 1 + 1/n) \to |t(b)| < \epsilon$$

If $b \notin \mathbf{Q}$ then $t(b) = 0$ and therefore $|t(b)| < \epsilon$; therefore we need to prove, that any number $b = m_1/n_1 \in (1 - 1/n; 1 + 1/n) \cap \mathbf{Q}$ is such, that $|t(b)| = 1/n_1 < 1/n$. Also suppose $m_1 = n_1 + k$ (it's worth noting that in this case $k \in \mathbf{Z}$); then

$$1 - \frac{1}{n} < \frac{m_1}{n_1} < 1 + \frac{1}{n}$$

$$1 - \frac{1}{n} < \frac{n_1 + k}{n_1} < 1 + \frac{1}{n}$$

$$1 - \frac{1}{n} < 1 + \frac{k}{n_1} < 1 + \frac{1}{n}$$

$$-\frac{1}{n} < \frac{k}{n_1} < \frac{1}{n}$$

$$|\frac{k}{n_1}| < \frac{1}{n}$$

$$|k||\frac{1}{n_1}| = |k||t(\frac{1}{n_1})| < \frac{1}{n}$$

therefore because $k \in \mathbf{Z}$

$$|t(\frac{1}{n_1})| = |\frac{1}{n_1}| < \frac{1}{n|k|} < \frac{1}{n}$$

thus for each $\epsilon > 0$ we can find a corresponding $\delta > 0$ as desired.

4.2.5

Suppose that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$
(*ii*) $\lim_{x \to c}[f(x) + g(x)] = L + M$
(*iii*) $\lim_{x \to c}[f(x)g(x)] = LM$
(a) *Supply the details for how Corollary 4.2.4 part (ii) follows from the sequential criterion for functional limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.*
From the algebraic limit theorem we know, that if $(a_n) \to a$ and $(b_n) \to b$ then

$$(a_n) + (b_n) = a + b$$

We also know, that for any sequence $(c_n) \to c$ it is true, that $f(c_n) \to L$ and $g(c_n) \to M$; therefore by the algebraic limit theorem

$$f(c_n) + g(c_n) = L + M$$

for any sequence $(c_n) \to c$. Therefore we can state that

$$\lim_{x \to c}(f(x) + g(x)) = L + M$$

as desired
(b) *Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.*
$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$; therefore for any $\epsilon_1 > 0$ we can find $\delta_1 > 0$ s.t.

$$|x - c| < \delta_1 \to |f(x) - L| < \epsilon_1$$

Also for the same ϵ_1 there exist $\delta_2 > 0$ s.t.

$$|x - c| < \delta_2 \to |g(x) - M| < \epsilon_1$$

let $\delta_3 = min\{\delta_1, \delta_2\}$; then it is true that

$$|x - c| < \delta_3 \to |f(x) - L| < \epsilon_1$$
$$|x - c| < \delta_3 \to |g(x) - M| < \epsilon_1$$

because $V_{\delta_1} \subseteq V_{\delta_3}$ and $V_{\delta_2} \subseteq V_{\delta_3}$
therefore

$$|f(x) - L| + |g(x) - M| < 2\epsilon_1$$

Therefore

$$|f(x) + g(x) - L - M| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M| < 2\epsilon_1$$

Thus for any $\epsilon > 0$ there exist corresponding $\epsilon_1 = \frac{\epsilon}{2}$ for which there exist corresponding $\delta = min\{\delta_1, \delta_2\}$ (where δ_1 is a delta for $f(x)$ and δ_2 is a delta for $g(x)$) which satisfies

$$|x - c| < \delta \to |f(x) + g(x) - (L + M)| < \epsilon$$

therefore $\lim_{x \to c}(f(x) + g(x)) = L + M$ as desired.
(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).
(a):
From the algebraic limit theorem we know, that if $(a_n) \to a$ and $(b_n) \to b$ then

$$(a_n)(b_n) = ab$$

We also know, that for any sequence $(c_n) \to c$ it is true, that $f(c_n) \to L$ and $g(c_n) \to M$; therefore by the algebraic limit theorem

$$f(c_n)g(c_n) = LM$$

for any sequence $(c_n) \to c$. Therefore we can state that

$$\lim_{x \to c}(f(x)g(x)) = LM$$

as desired
(b):
$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$;

In order to prove the needed limit let's first use some algebra

$$|f(x)g(x) - LM| =$$

$$|f(x)g(x) + f(x)M - f(x)M - LM| =$$

$$|f(x)(g(x) - M) + M(f(x) - L)| \leq |f(x)(g(x) - M)| + |M(f(x) - L)| =$$

$$|f(x)||g(x) - M| + |M||f(x) - L|$$

our strategy is to show that both elements of the last sum are less or equal to $\epsilon/2$
Let $\epsilon > 0$.

$$|M||f(x) - L| < \frac{\epsilon}{2}$$

If $M = 0$ then the abovementioned inequality always holds and we are free to choose any δ_1;

Otherwise tet us pick δ_1 such that inequality

$$|f(x) - L| < \frac{\epsilon}{2|M|}$$

holds.

The next step is a little bit more complicated because we need to work with $f(x)$; let us pick y = 1; then because $\lim_{x \to c} f(x) = L$ we know that there exists δ_2 s.t. $|x - c| < \delta_2 \to |f(x) - L| < 1$.

Therefore

$$|f(x) - L| < 1$$

Little sidenote: let's prove that

$$|a - b| < c \to |a| < |b| + c$$

Firstly some preliminary stuff

$$|a - b| \geq 0 \to c > |a - b| > 0 \to c > 0$$

$$|a - b| < c \to -c < a - b < c$$

$$b - c < a < b + c$$

Now let's see all the cases for $a, b \in \mathbf{R}$
if $a \geq 0$ and $b \geq 0$ then

$$a < b + c$$

$$|a| < |b| + c$$

if $a < 0$ and $b \geq 0$ then
$$b + c \geq 0 > a$$
$$a < b + c$$
$$|a| < |b| + c$$

if $a \geq 0$ and $b < 0$ then
$$b - c < a < b + c$$
$$-b + c > -a > -b - c$$
$$|b| + c > -a > |b| - c$$
$$-|b| - c < a < c - |b|$$
$$|a| < c - |b| \leq c + |b|$$
$$|a| < c + |b|$$

if $a < 0$ and $b < 0$ then
$$b - c < a < b + c$$
$$-b + c > -a > -b - c$$
$$|b| + c > |a| > |b| - c$$
$$|b| + c > |a|$$
$$|a| < |b| + c$$

Therefore $\forall a, b \in \mathbf{R}$
$$|a - b| < c \rightarrow |a| < |b| + c$$

as desired.

Back to our exercise:

$$|f(x) - L| < 1$$
$$|f(x)| < |L| + 1$$

Therefore we can state that upper bound for our $|f(x)|$ with $\epsilon = 1$ is $|L| + 1$
Thus if we pick δ_2 sufficient for

$$|g(x) - M| < \frac{\epsilon}{2(|L| + 1)}$$

therefore if we pick $\delta = min\{\delta_1, \delta_2\}$ then

$$|x - c| < \delta \rightarrow$$
$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

therefore $\lim_{x \to c}[f(x)g(x)] = LM$ as desired

4.2.6

Let $g : A \to \mathbf{R}$ and assume that f is bounded function on $A \subseteq \mathbf{R}$ (i.e. there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$). Show that if $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} g(x)f(x) = 0$ as well.

Here we can't use an intuitive approach of just using algebraic limit theorem because $f(x)$ may not hav limit at c. Anyways we proceed by $\epsilon - \delta$ approach.

Therefore we need to show that

$$\exists \delta : |f(x)g(x)| < \epsilon$$

First of all,

$$|f(x)g(x)| = |f(x)||g(x)|$$

Then we notice, that because $f(x)$ is bounded

$$\exists M \in \mathbf{R} > 0 : |f(x)| \leq M$$

therefore

$$|f(x)||g(x)| < |M||g(x)| = M|g(x)|$$

therefore if we pick δ sufficient for $|g(x)| < \frac{\epsilon}{M}$ then it follows that

$$|f(x)g(x)| \leq M|g(x)| < \epsilon$$

therefore

$$\forall \epsilon \in \mathbf{R} \exists \delta : |x - c| < \delta \to |f(x)g(x)| < \epsilon$$

therefore

$$\lim_{x \to c}[f(x)g(x)] = 0$$

as desired.

4.2.7

(a) The statement $\lim_{x \to 0} 1/x^2 = \infty$ certainly makes intuitive sense. Construct a rigirius definition in the "challenge-response" style of Definition 4.2.1 for a limit statement of the form $\lim_{x \to c} f(x) = \infty$ and use it to prove the previous statement

Definition of limit to infinity Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = \infty$ provided that, for all $\epsilon \in \mathbf{R}$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

Now we need to show that for $f(x) = 1/x^2$

$$\lim_{x \to 0} f(x) = \infty$$

First

$$f(x) > \epsilon$$

$$\frac{1}{x^2} > \epsilon$$

$$x^2 < \frac{1}{\epsilon}$$

$$x < \sqrt{\frac{1}{\epsilon}}$$

therefore if we pick $\delta = \sqrt{\frac{1}{\epsilon}}$, then it follows that

$$f(x) > \epsilon$$

as desired.

Quick (and insufficient) test in Python seems to corraborate this statement

(b) Now construct a definition for the statement $\lim_{x \to \infty} f(x) = L$. *Show* $\lim_{x \to \infty} 1/x = 0$

Definition of infinite limit Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to \infty} f(x) = L$ provided that, for all $\epsilon \in \mathbf{R} > 0$, there exists a δ such that whenever $x > \delta$ (and $x \in A$) it follows that $|f(x) - c| < \epsilon$.

We start as ususal at the ϵ

$$|f(x) - 0| < \epsilon$$

$$|1/x| < \epsilon$$

Given that we can pick any δ as we want, we can pick it at the very least at 0 to get rid of the absolute value

$$1/x < \epsilon$$

$$x > 1/\epsilon$$

therefore $\delta = 1/\epsilon$ then it follows that

$$|f(x) - 0| < \epsilon$$

as desired.

(c) What would a rigorous definition for $\lim_{x \to \infty} f(x) = \infty$ *would look like? Give an example of such a limit*

Definition of infinite limit to infinity Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to \infty} f(x) = \infty$ provided that, for all $\epsilon \in \mathbf{R}$, there exists a δ such that whenever $x > \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

The corresponding example of such a limit is $f(x) = x$.

4.2.8

Assume $f(x) \geq g(x)$ for all x in some set A on which f and g are defined. Show that for any limit point c of A we must have

$$\lim_{x \to c} f(x) \geq \lim_{x \to c} g(x)$$

I'm gonna do it by using contradiction; suppose that $f(x)$ and $g(x)$ are defined as in the exercise, but

$$\lim_{x \to c} f(x) < \lim_{x \to c} g(x)$$

then it follows that there exist a sequence $(a_n) \to c$ such that $f(a_n) \geq g(a_n)$ for all $n \in \mathbf{N}$; Therefore $\lim(f(a_n)) \geq \lim(g(a_n))$ and but it contradicts our initial assumption.

4.2.9 (Squeeze Theorem)

Let f, g and h satisfy $f(x) \geq g(x) \geq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c of A, show $\lim_{x \to c} g(x) = L$ as well

As proven in the previous exercise

$$\forall x \in A : f(x) > g(x) \to \lim_{x \to c} f(x) \geq \lim_{x \to c} g(x)$$

therefore

$$\lim_{x \to c} f(x) = L \geq \lim_{x \to c} g(x)$$

and

$$\lim_{x \to c} g(x) \geq \lim_{x \to c} h(x) = L$$

Thus

$$L \geq \lim_{x \to c} g(x) \geq L$$

therefore

$$\lim_{x \to c} g(x) = L$$

as desired.

4.3.1

Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continous at $c = 0$

We're gonna use $\epsilon - \delta$ definition. First of all, let's state that $g(0) = 0$. Therefore

$$|f(x) - f(c)| = |\sqrt[3]{x} - 0| < \epsilon$$

$$|\sqrt[3]{x}| < \epsilon$$

Here I would like to proof that $\forall x \in \mathbf{R} : |\sqrt[3]{x}| = \sqrt[3]{|x|}$: if $x \geq 0$ then $|\sqrt[3]{x}| = \sqrt[3]{x} = \sqrt[3]{|x|}$; if $x < 0$ then $|\sqrt[3]{x}| = \sqrt[3]{-x} = \sqrt[3]{|x|}$. Therefore

$$|\sqrt[3]{x}| = \sqrt[3]{|x|} =< \epsilon$$

is justified.

Therefore we can state that

$$|x| =< \epsilon^3$$

Thus if we pick $\delta = \epsilon^3$ then

$$|x - c| = |x| < \delta \to |f(x) - f(c)| = |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\epsilon^3} = \epsilon$$

Therefore g is continous at 0

(b) Prove that g is continous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ will be helpful)

We're gonna use $\epsilon - \delta$ definition once again.

$$|f(x) - f(c)| = |\sqrt[3]{x} - \sqrt[3]{c}| < \epsilon$$

First, let's use a little algebra

$$|\sqrt[3]{x} - \sqrt[3]{c}| = |\sqrt[3]{x} - \sqrt[3]{c}| * 1 = |\sqrt[3]{x} - \sqrt[3]{c}| \frac{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})} = \frac{|\sqrt[3]{x} - \sqrt[3]{c}|(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})}$$

Let's look now at the sum $\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2}$: $\sqrt[3]{x^2} \geq 0$ because it is a square. For $\sqrt[3]{x} + \sqrt[3]{x^2}$ we need to be able to articulate δ so that both x and c are the same sign; if we fo that then it becomes nonnegative. $\sqrt[3]{c^2} \geq 0$ because it is a square

Therefore if right now we pinky-promise that we will account for unusual delta in the future, then we are able to say that

$$\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2} \geq 0$$

And therefore

$$\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2} = |\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2}|$$

Continuing with our initial algebra

$$\frac{|\sqrt[3]{x} - \sqrt[3]{c}|(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})} = \frac{|\sqrt[3]{x} - \sqrt[3]{c}||\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2}|}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})} = \frac{|x - c|}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})} =$$

$$|x - c|\frac{1}{(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})} < \epsilon$$

As we disussed earlier $(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2}) \geq 0$ and therefore

$$|x - c| < \epsilon(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2})$$

Thus, if we pick $\delta = min\{\epsilon(\sqrt[3]{x^2} + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c^2}), |x - 0|\}$ (we need the second value because we need the sum to be equal to its absolute value;) then

$$|x - c| < \delta \rightarrow |f(x) - f(c)| < \epsilon$$

Therefore $f(x) = \sqrt[3]{x}$ is continous on **R**.

4.3.2

(a) Supply a proof for Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.
First, let's state the theorem

Theorem 4.3.9 (Composition of Continuous Functions). *Given $f : A \rightarrow R$ and $g : B \rightarrow R$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f = g(f(x))$ is well-defined on A.*

If f is continous ac $c \in A$, and if g is continous at $f(c) \in B$, then $g \circ f$ is continous at c.

Firstly, the fact that both f and g are continous tells that

$$\forall \epsilon_1 \in \mathbf{R} : \exists \delta_1 : |x - c| < \delta_1 \rightarrow |f(x) - f(c)| < \epsilon_1$$

$$\forall \epsilon_2 \in \mathbf{R} : \exists \delta_2 : |x - c| < \delta_2 \rightarrow |g(x) - g(c)| < \epsilon_2$$

And we need to prove that

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - c| < \delta \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

The main strategy for this one is to plug some delta into some epsilon (or vice versa), and get some use out of it.

Firstly, let's get some things out of the way: let us fix particular $c \in A$ and $\epsilon \in \mathbf{R} > 0$. Then, let's plug this ϵ at $f(c)$ into the continuity of $g(x)$ so we can get a $\delta_g > 0$. Therefore we will have

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |x - f(c)| < \delta_g \rightarrow |g(x) - g(f(c))| < \epsilon$$

which is kinda close to the thing, that we're trying to prove.

We also know that

$$\forall \epsilon_f \in \mathbf{R} : \exists \delta_f : |x - c| < \delta_f : |f(x) - f(c)| < \epsilon_f$$

therefore it is true that

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |y - f(c)| < \delta_g \rightarrow |g(y) - g(f(c))| < \epsilon$$

$$\exists \delta_f : |x - c| < \delta_f \rightarrow |f(x) - f(c)| < \delta_g$$

From this we can state that

$$\forall \epsilon \in \mathbf{R} > 0 : \exists \delta_f : |x - c| < \delta_f \rightarrow |f(x) - f(c)| < \delta_g \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

This doesn't sound too persuasive for me, so I probably need to explore it a little but more.

Suppose that with all the present assumptions, we get the given ϵ. If we plug it into definition of continuity for $g(x)$ at $g(f(c))$, then we'll get the neccesary δ_g. If we plug δ_g as an ϵ for the definition of continuity of $f(x)$, then we'll get δ_f.

We can probably prove it with a little bit more clarity. We need to prove that

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - f(c)| < \delta \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

Firstly, definition of contonuity of $g(x)$ gives us the fact, that

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |x - f(c)| < \delta_g \rightarrow |g(x) - g(f(c))| < \epsilon$$

then if $x \in f(A)$ then $\exists y \in A$ s.t. $f(y) = x$ therefore

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |f(y) - f(c)| < \delta_g \rightarrow |g(f(y)) - g(f(c))| < \epsilon$$

From the definition of continuity of f we know that

$$\forall \epsilon_f \in \mathbf{R} : \exists \delta_f : |x - c| < \delta_f : |f(x) - f(c)| < \epsilon_f$$

Therefore

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - f(c)| < \delta \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

as desired.

(b) Give another proof of this theorem using the sequantial characterization of continuity (from Theorem 4.3.2 (iv))

Theorem 4.3.2 (iv) states that if $(x_n) \rightarrow c$ (with $x_n \in A$), then $f(x_n) \rightarrow f(c)$.

Because $f(x)$ is continous we can state that for every sequence $(x_n) \rightarrow c$ it is true that $f(x_n) \rightarrow f(c)$. Therefore because $f(x_n)$ is a sequence itself, we can state that $g(f(x_n)) \rightarrow g(f(c))$. Therefore it is true, that for every sequence $(x_n) \rightarrow c$ it follows, that $g(f(x_n)) \rightarrow g(f(c))$. Therefore $g(f(x))$ is continous, as desired.

4.3.3

Using the $\epsilon - \delta$ characteriation of continuity (and tus using no previous results anbout the sequences), show that the linear function $f(x) = ax + b$ is continous at every poinnt of $\mathbf{R}$.

Let's start with our usual stuff

$$|f(x) - f(c)| < \epsilon$$

$$|ax + b - (ac + b)| = |ax + b - ac - b| = |a||x - c| < \epsilon$$

$$|x - c| < \epsilon/a$$

Therefore if we pick $\delta = \epsilon/a$ then it follows that $|f(x) - f(c)| < \epsilon$, as desired.

4.3.4

(a) Show using Definition 4.3.1 that any function f with domain $\mathbf{Z}$ with necessarily be continous at every point in its domain.

Suppose that $f : Z \to R$. We need to prove that

$$\forall \epsilon : \exists \delta : |x - c| < \delta \to |f(x) - f(c)| < \epsilon$$

Suppose that we pick $\delta = 0.1$ (or any other value, such that the only one of the domain values will be in the needed neighborhood). Then there will be only one number in the domain neighhborhood, and because of that we can state that

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

Therefore the fucntion is continous, as desired.

(b) Show in general that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continous at c.

In this particular case we can't just set δ at some number, so we gotta be a little more creative. To be distract myself from getting any unproductive ideas, I should state here that $\mathbf{Q}$ is a set of isolated points.

To be continued

3.2.1

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite get used

Theorem 3.2.3 states that

(ii) The intersection of a finite collection of open sets is open.

The assumption of the finality of the set is used in the fact, that we need the minimum of the epsilons.

(b) Give an example of an infinite collection of nested open sets

$$O_1 \supseteq O_2 \supseteq O_3 \supseteq O_4 \supseteq \dots$$

whose intersection $\cap_{n=1}^{\infty} O_n$ is closed and nonempty.

First of all, we should state that open is not an opposite of closed in this context. We can get $O_n = (-\infty; \infty)$. Then this definition (technically) fits into the requrement of qs

Let $O_n = (1 - 1/n, 2 + 1/n)$. Let us also define

$$A = \cap_{n=1}^{\infty} O_n$$

Suppose that $x \in A$. To be continued...

2.2.1

Verify, using the definition of convergence of a sequence, that the following sequenceces converge to the proposed limit.

(a) $\lim \frac{1}{(6n^2+1)} = 0$

Let ϵ be arbitrary. Choose $N \in \mathbf{N} : N > \sqrt{\frac{1}{6\epsilon} - \frac{()}{1})(6)}$. Let $n \in N \geq N$. Then

$$n > \sqrt{\frac{1}{6\epsilon} - \frac{1}{6}}$$

$$n^2 > \frac{1}{6\epsilon} - \frac{1}{6}$$

$$6n^2 > 1/\epsilon - 1$$

$$6n^2 + 1 > 1/\epsilon$$

$$\frac{1}{(6n^2+1)} < \epsilon$$

$$|\frac{1}{(6n^2+1)}| < \epsilon$$

as desired.

(b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$

Let ϵ be arbitrary. Choose $N \in \mathbf{N} : N > \frac{13}{4\epsilon} - 10/4$. Let $n \in N \geq N$. Then

$$n > \frac{13}{4\epsilon} - 10/4$$

$$4n > \frac{13}{\epsilon} - 10$$

$$4n + 10 > \frac{13}{\epsilon}$$

$$\frac{13}{4n + 10} < \epsilon$$

$$\left|\frac{13}{4n + 10}\right| < \epsilon$$

$$\left|\frac{-13}{4n + 10}\right| < \epsilon$$

$$\left|\frac{6n + 2 - 6n - 15}{4n + 10}\right| < \epsilon$$

$$\left|\frac{2(3n + 1) - 3(2n + 5)}{2(2n + 5)}\right| < \epsilon$$

$$\left|\frac{3n + 1}{2n + 5} - \frac{3}{2}\right| < \epsilon$$

as desired.

(c) $\lim \frac{2}{\sqrt{n+3}} = 0$

Let ϵ be arbitrary. Choose $N \in \mathbf{N} : N > \frac{2^2}{\epsilon} - 3$. Let $n \in N \geq N$. Then

$$n > \frac{2^2}{\epsilon} - 3$$

$$n + 3 > \frac{2^2}{\epsilon}$$

$$\sqrt{n + 3} > \frac{2}{\epsilon}$$

$$\frac{2}{\sqrt{n + 3}} < \epsilon$$

$$\left|\frac{2}{\sqrt{n + 3}}\right| < \epsilon$$

as desired.

2.2.2

What happens if we reverse the order of the quantifiers on Definition 2.2.3?

Definition: A sequence (x_n) verconges to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Can you give an example of a vercongent sequence, that is divirgent? What exactly is being described in this strange definition?

An example of a vercongent sequence:

$$(x_n) = 5$$

.

An example of a vercongent sequence, that is divergent:

$$(x_n) = (-1)^n$$

.

Here described a bounded sequence (i.e. $|(x_n)| < M$ for some $M > 0 \in R$

2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.
(a) At every college in the United States, there is a student who is at least seven feet tall
There exist a college, where every student is shorter than 7 feet.
(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B
There exist a college, where every professor gives C or less every time.
(c) There exist a college in the United States where every student is at least six feet tall
In all colleges across US there exists a student, who is shorter than 6 feet.

2.2.4

Argue that the sequence

$$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, (5 \text{ zeroes}), 1...$$

does not converge to zero. For what values of $\epsilon > 0$ does there exist a response N? For which values of $\epsilon > 0$ is there no suitable response?
If we set $\epsilon = 0.5$, then from time to time elements will get out of the desired range.
For values $\epsilon > 1$ there always exist a suitable N.
For values $0 < \epsilon \leq 0$ there exists no responce.

2.2.5

Let $[[x]]$ be the greatest integer less than of equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. Find $\lim a_n$ and supply proofs for each conclusion if

(a) $a_n = [[1/n]]$

$$\forall n \in N \to 0 < 1/n \leq 1$$

$$n \in N \geq 2 \to 0 < 1/n < 1$$

Therefore for $n > 2$

$$\forall \epsilon > 0 \in R : |[[1/n]]| = [[1/n]] = 0 < \epsilon$$

as desired.

(b) $a_n = [[(10+n)/2n]]$

Let n ¿ 10. Then

$$n > 10$$

$$2n > 20$$

$$\frac{10}{2n} < \frac{1}{2}$$

$$0 < \frac{10}{2n} + \frac{1}{2} < 1$$

$$[[\frac{10}{2n} + \frac{1}{2}]] = 0$$

Thus for every $\epsilon > 0$ we can pick $N = 10$ and it follows that

$$|[[(10+n)/2n]]| < 0$$

Therefore $(a_n) \to 0$, as desired.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may need to be".

The key word here is "may". It may have to be larger, it may not. In some cases one value works for all of the ϵ's. I bet my money on the fact, that it can be rigorously proven, that this is the case only for the sequences, that are constant after some term, but I'll skip that

2.2.6

Suppose that for a particular $\epsilon > 0$ we have found a suitable value of N that "works" for a given sequence in the sence of Definition 2.2.3.

(a) Then, any larger/smaller (pick one) N will also work for the same $\epsilon > 0$.

Larger. This fact follows from definition.

(b) Then, this same N will also work for any larger/smaller value of ϵ.

Larger. $x \in V_\epsilon \to x \in V_{\epsilon+s}$

2.2.7

Informally speaking, the sequence $\sqrt{n}$ "converges to infinity".

   (a) Imitate the logical structure of Definition 2.2.3 to create a rigorous definition for the mathematical statement $\lim x_n = \infty$. Use this definition to prove $\lim \sqrt{n} = \infty$.

   Definition of convergence to infinity A sequence (a_n) converges for infinity if, for every $\epsilon \in R$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $a_n > \epsilon$.

   I relaxed a bit statement about the ϵ, namely substituted $\epsilon > 0$ to $\epsilon \in R$. In case you are wondering, Both cases are the equivalent.

   Let $\epsilon \in R$ be arbitrary. Then we can pick $N \in \mathbf{N} : N > \epsilon^2$. Then for $n \in \mathbf{N} > N$:

$$n > \epsilon^2$$

$$\sqrt{n} > \epsilon$$

   Therefore the sequence converges to infinity

   (b) What does your definition in (a) say about the particular sequence $(1, 0, 2, 0, 3, 0, 4, 0, 5, 0, ...)$

   It says, that it doesn't converge to infinity (specifically for the case $\epsilon > 0$)

2.2.8

Here are two useful definitions:

   (i) A sequence (a_n) is eventually in a set $A \subseteq R$ if there exists $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

   (ii) A sequence (a_n) is frequently in a set $A \subseteq R$ if for every $N \in \mathbf{N}$ there exists $n \geq N$ such that $a_n \in A$.

   (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

   Frequently

   (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

   Eventually is stronger. Eventually implies frequently.

   (c) Give an alternative rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

   We want to use eventually

   A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, sequence is eventually in $V_\epsilon(a)$.

   (d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

   It is not necessarily eventually in $(1.9, 2.1)$. Example:

$$(1, 2, 2, 2, 3, 2, 4, 2, 5, 2, 6, 2, ...)$$

It is indeed frequently in $(1.9, 2.1)$. This stems from the fact, that before a_n, there exist only a finite number of elements before it. Therefore 2 is bound to be met again at some time.

2.3.1

Show that the constant sequence $(a, a, a, ...)$ converges to a.

$$|a_n - a| = |a - a| = 0$$

therefore for every $n \in N$ it is true, that $|a_n - a| < \epsilon$. Therefore $(a_n) \to a$, as desired.

2.3.2

Let $x_n >\geq 0$ for all $n \in N$.
   (a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$
   Suppose that $(x_n) \to 0$. Then

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \in> N : |x_n - 0| < \epsilon$$

.

$$|x_n - 0| < \epsilon$$
$$|x_n| < \epsilon$$
$$x_n < \epsilon$$
$$\sqrt{x_n} < \sqrt{\epsilon}$$

therefore
$$\forall \epsilon \in R > 0 : \exists \epsilon = \sqrt{\epsilon} > 0 : \exists N \in \mathbf{N} : \forall n \in> N : |\sqrt{x_n} - 0| < \epsilon$$

. therefore
$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \in> N : |\sqrt{x_n} - 0| < \epsilon$$

. From which it follows that $(\sqrt{x_n}) \to 0$ by definition of a limit.
   (b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$
   Suppose that $(x_n) \to x$. Then

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \in> N : |x_n - x| < \epsilon$$

.

$$|x_n - x| < \epsilon$$
$$|(\sqrt{x_n} + \sqrt{x})(\sqrt{x_n} - \sqrt{x})| < \epsilon$$

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$$|\sqrt{x_n} - \sqrt{x}| < \frac{\epsilon}{|\sqrt{x_n} + \sqrt{x}|}$$

$$|\sqrt{x_n} + \sqrt{x}| \leq |\sqrt{x_n}| + |\sqrt{x}| = \sqrt{x_n} + \sqrt{x} \to \frac{\epsilon}{|\sqrt{x_n} + \sqrt{x}|} < \frac{\epsilon}{\sqrt{x_n} + \sqrt{x}}$$

$$|\sqrt{x_n} - \sqrt{x}| < \frac{\epsilon}{\sqrt{x_n} + \sqrt{x}}$$

(x_n) is convergent and therefore bounded. Therefore there exists $M \in R > 0 : x_n < M$. Therefore $\sqrt{x_n} < \sqrt{M}$. Thus $\sqrt{x_n} + \sqrt{x} < \sqrt{M} + \sqrt{x}$. Therefore

$$|\sqrt{x_n} - \sqrt{x}| < \frac{\epsilon}{\sqrt{M} + \sqrt{x}}$$

therefore

$$\forall \epsilon \in R > 0 : \exists \epsilon_1 = \frac{\epsilon}{\sqrt{M} + \sqrt{x}} > 0 : \exists N \in \mathbf{N} : \forall n \in > N : |\sqrt{x_n} - \sqrt{x}| < \epsilon$$

. therefore

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \in > N : |\sqrt{x_n} - \sqrt{x}| < \epsilon$$

. From which it follows that $(\sqrt{x_n}) \to x$ by definition of a limit.

2.3.3 (Squeezze Theorem).

Show that if $x_n \leq y_n \leq z_n$ for all $n \in N$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

$$\forall n \in N : x_n \leq y_n \to \lim x_n \leq \lim y_n$$

$$\forall n \in N : y_n \leq z_n \to \lim y_n \leq \lim z_n$$

therefore

$$\forall n \in N : x_n \leq y_n \leq z_n \to \lim x_n \leq \lim y_n \leq \lim z_n$$

Therefore

$$\lim x_n = \lim z_n = l \to \lim y_n = l$$

as desired.

2.3.4

Show that limits, if they exist, must be unique. In other words, assume $\lim a_n = l_1$ and $\lim a_n = l_2$, and prove that $l_1 = l_2$.

We will procede with a proof by contradiction.

Suppose $l_1 \neq l_2$. Then

$$\forall \epsilon > 0 : \exists N \in N : n \geq N \to |a_n - l_1| \leq \epsilon$$

$$\forall \epsilon > 0 : \exists N \in N : n \geq N \to |a_n - l_2| \leq \epsilon$$

Let $\epsilon = |l_1 - l_2|/2$. Then $\exists N_1 \in N$ such that $n_1 \geq N$ implies that

$$|a_{n_1} - l_1| < |l_1 - l_2|/2$$

also there exists $N_2 \in N$ such that $n_2 \geq N_2$ implies that

$$|a_{n_2} - l_2| < |l_1 - l_2|$$

Let $n = max\{n_1, n_2\}$. Then

$$|a_n - l_1| < |l_1 - l_2|/2$$

$$|a_n - l_2| < |l_1 - l_2|/2$$

$$|a_n - l_1| + |a_n - l_2| < |l_1 - l_2|$$

$$|a_n - l_1| + |l_2 - a_n| < |l_1 - l_2|$$

then by triangular inequality

$$|a_n - l_1 + l_2 - a_n| \leq |a_n - l_1| + |l_2 - a_n| < |l_1 - l_2|$$

$$|a_n - l_1 + l_2 - a_n| < |l_1 - l_2|$$

$$|-l_1 + l_2| < |l_1 - l_2|$$

$$|l_1 - l_2| < |l_1 - l_2|$$

which is a contradiction. Therefore

$$\lim a_n = l_1 \text{ and } \lim a_n \to l_2 \to l_1 = l_2$$

as desired.

2.3.5

Let (x_n) and (y_n) be given, and deefine (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ...)$. Prove that (z_n) is convergent it and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

In the forward direction:

Suppose that (z_n) is convergent to some $l \in R$. Then

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : n \geq N \to |z_n - l| < \epsilon$$

Because (z_n) is "shuffled" we can follow that

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : (n+1)/2 \geq N \to |x_{(n+1)/2} - l| < \epsilon$$

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : n/2 \geq N \to |y_{n/2} - l| < \epsilon$$

Therefore if we let $m_1 = (n+1)/2$ and $m_2 = n/2$

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : m_1 \geq N \to |x_{m_1} - l| < \epsilon$$

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : m_2 \geq N \to |y_{m_2} - l| < \epsilon$$

Therefore $\lim x_n = \lim y_n = l$, as desired.

In the backward direction:

Suppose that both (x_n) and (y_n) are convergent to some l. Then

$$\forall \epsilon > 0 : \exists N_1 \in \mathbf{N} : m_1 \geq N_1 \to |x_{m_1} - l| < \epsilon$$

$$\forall \epsilon > 0 : \exists N_2 \in \mathbf{N} : m_2 \geq N_2 \to |y_{m_2} - l| < \epsilon$$

If we pick $N = (max\{N_1, N_2\} + 1) * 2$, then

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : n \geq N \to |x_{(n-1)/2} - l| < \epsilon$$

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : m_2 \geq N \to |y_{n/2} - l| < \epsilon$$

If n is odd then $z_n = x_{(n-1)/2}$, and if n is even then $z_n = y_{n/2}$. Therefore

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : n \geq N \to |z_n - l| < \epsilon$$

Thus $(z_n) \to l$ as well, as desired.

2.3.6

Show that if $(b_n) \to b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.

As proven in 1.2.5

$$||a| - |b|| \leq |a - b|$$

Therefore $||b_n| - |b|| \leq |b_n - b|$ Thus

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : n \geq N \to ||b_n| - |b|| < \epsilon$$

Thus $(|b_n|) \to |b|$, as desired.

(b) Is the converse of part (a) true? If we know that $|b_n| \to |b|$, can we deduce that $(b_n) \to b$?

No. Glaring example is

$$a_n = (-1)^n$$

2.3.7

(a) Let (a_n) be a boundeed (non necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are not allowed to use Algebraic Limit Theorem to prove that?

Let $M > |a_n|$ for all $n \in N$ (it exists because of the boundness of a_n). Then

$$|a_n b_n| = |a_n||b_n| \leq M|b_n|$$

$$|a_n b_n| \leq M|b_n|$$

$$-M|b_n| \leq a_n b_n \leq M|b_n|$$

Both $(M|b_n|)$ and $(-M|b_n|)$ converge to 0. Therefore $(a_n b_n)$ converges to 0 as well by Squeeze theorem.

A little sidenote: this statement can be proven with the standart definition of limit as well, but that proof is longer and I wanted to use brand-new, self-proven theorem on this one.

We can't use Algebraic Limit Theorem on that one beacause it prerequsites that both sequences are convergent, and here it is not the case.

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

We can probably conclude that if (a_n) is divergent, then $(a_n b_n)$ is divergent as well, but we can't conclude nothing definitive about just a bounded sequence.

(c) Use (a) for prove Theoremt 2.3.3, part (iii), for the case when $a = 0$.

If (b_n) is convergent (possibly to zero), then it is bounded. Therefore, by (a), we can state that

$$(a_n b_n) \to ab = 0$$

But wait a second, we used something from the algebraic property in part (a)! Doesn't in mean, that we proved a theorem by assuming, that it is true? No. In part (a) we concluded, that $M|b_n|$ is convergent to 0 by using part (i) of algebraic limit theorem. We didn't even use the case when $M = 0$, and therefore we can sleep well.

2.3.8

Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):
(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges

$$(x_n) = n$$

$$(y_n) = -n$$

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges

Impossible.

I want to say, that the algebraic limit theorem prevents this statement to be true, but I don't know if we can apply it here.

Suppose that $(x_n + y_n)$ converges to l. Let $(ln) = l$ be a constant sequence. Then

$$(x_n + y_n - l) \to 0$$

Also, because both (x_n) and (l) converge, $(l - x_n)$ converges as well. therefore

$$(x_n + y_n - l + l - x_n)$$

converges as well. Therefore (y) converges, which is a contradiction.

Turns out that yeah, it applies.

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges

$$(b_n) = 1/n$$

.

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

Impossible.

Same algebraic limit theorem with a bit of convergence to infinity

(e) Two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge, but (b_n) diverge.

$$(a_n) = 1/n$$
$$(b_n) = (-1)^n$$

2.3.9

Does Theorem 2.3.4 remain true, if all of the inequalities are assumed to be strict? If we assume, for instance, that a convergent sequence (x_n) satisfies $x_n > 0$ for all $n \in N$, what we may conclude about the limit?

If we swap all inequalities to strict ineqaulities, then the theorem is false. Example: $(b_n) = 1/n \to 0$, but $b_n > 0$ for all n's.

2.3.10

If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, then show that $(b_n) \to b$.
$(a_n) \to 0$ implies, that for every $\epsilon \in R > 0$ we can get $|a_n| < \epsilon$. Therefore

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \geq N \to |b_n - b| < a_n \leq |a_n| < \epsilon$$

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \geq N \to |b_n - b| < \epsilon$$

Therefore $(b_n) \to b$, as desired.

2.3.11 (Cesaro Means).

Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + ... + x_n}{n}$$

also converges to the same limit.

The strategy for the proof for this theorem is to firtsly show, that this statement is true for all of the sequences, that converge to 0. Then we'll show that this case is equivalent to the theorem in general.

First, let's prove that this is true for the case when $(a_n) \to 0$

Strategy here is to show, that

Suppose that we have a given $\epsilon \in R > 0$. Then there exists $q \in N$ such that $|a_q| < \epsilon$. Also, because of the fact, that $(a_n) \to 0$ there exist $k \in N \geq q$ such that $|a_k| \geq |a_{q_1}|$ (i.e. maximum element of this set) for all $q_1 \in N \geq q$.

Then, let

$$N > \frac{|a_1| + |a_2| + ... + |a_{q-1}| - q|a_k|}{\epsilon - |a_k|}$$

If, for some god fosaken reason, $q \geq N$, then we can set N to be more than q, and it wouldn't make the differenct. Same applies to k, if it matters (it doesn't).

Therefore if we let $n > N$, then we can concur that

$$n > \frac{|a_1| + |a_2| + ... + |a_{q-1}| - q|a_k|}{\epsilon - |a_k|}$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}| - q|a_k|}{\epsilon - |a_k|} < n$$

$$|a_1| + |a_2| + ... + |a_{q-1}| - q|a_k| < \epsilon n - |a_k|n$$

$$|a_1| + |a_2| + ... + |a_{q-1}| + |a_k|n - q|a_k| < \epsilon n$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + |a_k| - \frac{q|a_k|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{n|a_k|}{n} - \frac{q|a_k|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{n|a_k| - q|a_k|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{(n - q)|a_k|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{|a_k| + ... + |a_k|}{n} < \epsilon$$

because $|a_{q+}| \leq |a_k|$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{|a_q| + ... + |a_n|}{n} \leq \frac{|a_1| + |a_2| + ... + |a_{q-1}|}{n} + \frac{|a_k| + ... + |a_k|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_{q-1}| + |a_q| + ... + |a_n|}{n} < \epsilon$$

$$\frac{|a_1| + |a_2| + ... + |a_n|}{n} < \epsilon$$

$$|\frac{a_1 + a_2 + ... + a_n}{n}| = \frac{|a_1 + a_2 + ... + a_n|}{n} \leq \frac{|a_1| + |a_2| + ... + |a_n|}{n} < \epsilon$$

$$|\frac{a_1 + a_2 + ... + a_n}{n}| < \epsilon$$

$$|\frac{a_1 + a_2 + ... + a_n}{n} - 0| < \epsilon$$

Therefore

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n > N \to |\frac{a_1 + a_2 + ... + a_n}{n} - 0| < \epsilon$$

Therefore $(a_n) \to 0$ implies that $(\frac{a_1 + a_2 + ... + a_n}{n}) \to 0$ as well.

Now, proud of our achievement, we can proceed with the initial argument. Suppose that $(x_n) \to l$. Then if we set $l_n = l$, then $(l_n) \to l$. Therefore $(x_n - l_n) \to 0$. And because of it, we can use our initial conclusion.

$$(\frac{x_1 - l_n + x_2 - l_n + ... + x_n - l_n}{n}) \to 0$$

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$$\left(\frac{x_1 + x_2 + ... + x_n - l_n n}{n}\right) \to 0$$

$$\left(\frac{x_1 + x_2 + ... + x_n}{n} - \frac{l_n n}{n}\right) \to 0$$

$$\left(\frac{x_1 + x_2 + ... + x_n}{n} - l_n\right) \to 0$$

Therefore

$$\left(\frac{x_1 + x_2 + ... + x_n}{n}\right) \to l$$

When push comes to shove, we can prove it axiomatically throuh

$$\left|\frac{x_1 - l_n + x_2 - l_n + ... + x_n - l_n}{n}\right| < \epsilon$$

$$\left|\frac{x_1 + x_2 + ... + x_n - nl_n}{n}\right| < \epsilon$$

$$\left|\frac{x_1 + x_2 + ... + x_n}{n} - \frac{nl_n}{n}\right| < \epsilon$$

$$\left|\frac{x_1 + x_2 + ... + x_n}{n} - l_n\right| < \epsilon$$

$$\left|\frac{x_1 + x_2 + ... + x_n}{n} - l\right| < \epsilon$$

Which comes to the same conclusion. Therefore now we can affirmatively state that

$$(x_n) \to l \text{ implies } \frac{x_1 + x_2 + ... + x_n}{n} \to l$$

as desired.

Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

$$x_n = ([[n/2]])(-1)^n$$

Where $[[]]$ is a floor function. Therefore

$$x_n = (0, 0, 1, -1, 2, -2, ...)$$

2.3.12

Consider the doubly indexed array $a_{m,n} = m/(m + n))$.
(a) Intuitively speaking, what should $\lim_{m,n\to\infty} a_{m,n}$ represent? Compute the "iterated" limits

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} \text{ and } \lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}$$

Intuitively speaking $\lim_{m,n\to\infty} a_{m,n}$ should mean that the larger the m and n get, the closer $a_{m,n}$ becomes.

$$\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty} m/(m+n) = \lim_{n\to\infty} m/m + m/n = 1$$

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty} 1 = 1$$

$$\lim_{m\to\infty} a_{m,n} = \lim_{m\to\infty} m/(m+n) = \lim_{m\to\infty} = m/m + m/n = \infty$$

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = \lim_{n\to\infty} \infty = \infty$$

(b) Formulate a rigorous definition in the style of Definition 2.2.3 for the statement

$$\lim_{m,n\to\infty} a_{m,n} = l$$

A list is a set of length n in form

$$\{a_1, \{a_1, a_2\}, \{a_1, a_2, a_3\}, ..., a_{n-1}, a_n\}\}\}\}...$$

It is denoted by

$$(a_1, a_2, a_3, ..., a_n)$$

A double indexed sequence $a_{m,n}$ is a function from the list (j, k), where $j, k \in N$ to R.

A sequence $(a_{m,n})$ converges to a real number a if, for every positive number ϵ, there exist $N_1, N_2 \in \mathbf{N}$ such that whenever $m > N_1$ and $n > N_2$ it follows that $|a_{m,n} - a| < \epsilon$.

2.4.1

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.
First of all, let's state Theorem 2.4.6

Theorem 2.4.6 (Caucht condensation Test) *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + ...$$

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converges.

Suppose that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Then a partial sum

$$s_{2^k} = b_1+2b_2+4b_4+... = b_1+b_2+b_2+b_4+b_4+b_4+b_4... \leq b_1+b_2+b_3+b_4+b_5+b_6+b_7+... = s_n$$

Therefore, partial sum for $\sum_{n=0}^{\infty} 2^n b_{2^n}$ is less than $\sum_{n=1}^{\infty} b_n$. Therefore $\sum_{n=1}^{\infty} b_n$ diverges, as desired.

2.4.2

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

Firstly, let us write a couple of elements of this sequence to see what's going on:

$$(x_n) = 3, 1, 1/3, 3/11, ...$$

Those elements of the sequence tell us, that the sequence is probably decreasing and also bounded below by some constant around 0.25, and therefore bounded below by 2. Also, the proof of the fact that the sequence is bounded above by 4 might come in handy.

To prove the proposition, that this sequence is convergent we will use property, that if a sequence is bounded and monotone, then it is convergent. Let's start with proposition that the this sequence is decreasing.

We'll use induction for this one, because it seems right;

Base: $x_1 < 4$.

Proposition: $x_n < 4$

Step:

$$x_{n+1} = \frac{1}{4 - x_n} < 4$$

$$4 - x_n > \frac{1}{4}$$

$$-x_n > \frac{1}{4} - 4$$

$$x_n < 3\frac{1}{4}$$

Therefore $x_n < 4$. From this we can conclude that

$$x_n < 4$$

$$4 - x_n > 0$$

$$\frac{1}{4 - x_n} > 0$$

$$x_{n+1} > 0$$

Now let us prove, that the sequence is decreasing. We'll also do it with induction.

Base: $x_1 - x_2 = 3 - 1 = 2 > 0$

Proposition: $x_{n-1} - x_n > 0$.

Step: We had proved already, that $x_n > 0$, and $x_n < 4$ Therefore

$$x_n - x_{n+1} > 0$$

$$x_n - \frac{1}{4 - x_n} > 0$$

$$(4 - x_n)x_n - 1 > 0$$

$$4x_n - x_n^2 - 1 > 0$$

$$x_n^2 - 4x_n + 1 < 0$$

$$(x_n - (2 - \sqrt{3}))(x_n - (2 + \sqrt{3})) < 0$$

It can be shown, that our sequence bounded by $2 - \sqrt{3}$ and $2 + \sqrt{3}$.

Thus the sequence is decrasing and bounded, and therefore convergent.

(b) Now thatwe know $\lim x_n$ *exists, explain why* $\lim x_{n+1}$ *must also exist and equal the same value.*

By the definition of limit of a sequence, for every ϵ there exist an N such that for all $n \geq N$ $|x_n - l| < \epsilon$. Therefore, because every $(x_{n+1}) \subset (x_n)$ we can conclude, that (x_{n+1}) converges to the same value.

(c) Take the limit of each side of the recursive eqation in part (a) of this exercise to explicitly compute $\lim x_n$.

$$\lim x_{n+1} = \frac{1}{4 - \lim x_n}$$

$$\lim x_n = \frac{1}{4 - \lim x_n}$$

$$\lim x_n = \frac{1}{4 - \lim x_n}$$

$$(4 - \lim x_n)\lim x_n = 1$$

$$4\lim x_n - (\lim x_n)^2 - 1 = 0$$

$$(\lim x_n)^2 - 4\lim x_n + 1 = 0$$

From my 5th grade class I remember that.

$$\lim x_n = 2 - \sqrt{3} \text{ or } \lim x_n = 2 + \sqrt{3}$$

Given that $2 + \sqrt{3} > 3$ and that the sequence is decreasing and first element of it is 3, we can state that $x_n > 3$ for all n. Therefore

$$\lim x_n = 2 - \sqrt{3}$$

2.4.3

Following the model of Exercise 2.4.2, show that the sequence defined by $y_1 = 1$ and $y_{n+1} = 4 - 1/y_n$.

Let's write down a few elements of this sequence:

$$y_1 = 1, 3, 11/3, 41/11, ...$$

This one is probably increasing and bounded above by 4 and below by 1

Base: $1 \leq y_1 = 1 \leq 4$

Proposition: $1 \leq y_n \leq 4$

Step:

$$1 \leq y_n \leq 4$$
$$1 \geq 1/y_n \geq 1/4$$
$$-1 \leq -1/y_n \leq -1/4$$
$$3 \leq 4 - 1/y_n \leq 3\frac{3}{4}$$
$$1 \leq 4 - 1/y_n \leq 4$$
$$1 \leq y_{n+1} \leq 4$$

$$1 \leq y_n \leq 2 + \sqrt{3}$$
$$1 \geq 1/y_n \geq \frac{1}{2 + \sqrt{3}}$$
$$-1 \leq -1/y_n \leq -\frac{1}{2 + \sqrt{3}}$$
$$3 \leq 4 - 1/y_n \leq 4 - \frac{1}{2 + \sqrt{3}}$$

$$3 \leq 4 - 1/y_n \leq \frac{8 + 4\sqrt{3} - 1}{2 + \sqrt{3}}$$

$$3 \leq 4 - 1/y_n \leq \frac{7 + 4\sqrt{3}}{2 + \sqrt{3}}$$

$$3 \leq 4 - 1/y_n \leq \frac{(2 + \sqrt{3})^2}{2 + \sqrt{3}}$$

$$3 \leq 4 - 1/y_n \leq 2 + \sqrt{3}$$

as desired.

Therefore $1 \leq y_n \leq 4$ for all $n \in N$.

Now, let us try to prove that the sequence is increasing

$$y_{n+1} \geq y_n$$

$$y_{n+1} - y_n \geq 0$$

$$4 - 1/y_n - y_n \geq 0$$

$$4y_n - 1 - y_n^2 \geq 0$$

$$y_n^2 - 4y_n + 1 \leq 0$$

$$(y - (2 + \sqrt{3}))(y - (2 - \sqrt{3})) \leq 0$$

Therefore if we prove that the sequence is bounded by $2 + \sqrt{3}$ and $2 + \sqrt{3}$, then we'll have our proof

Base: $2 - \sqrt{3} \leq y_1 = 1 \leq 2 + \sqrt{3}$

Proposition: $2 - \sqrt{3} \leq y_n \leq 2 + \sqrt{3}$

Step:

$$2 - \sqrt{3} \leq y_n \leq 2 + \sqrt{3}$$

$$\frac{1}{2 - \sqrt{3}} \geq 1/y_n \geq \frac{1}{2 + \sqrt{3}}$$

$$-\frac{1}{2 - \sqrt{3}} \leq -1/y_n \leq -\frac{1}{2 + \sqrt{3}}$$

$$4 - \frac{1}{2 - \sqrt{3}} \leq 4 - 1/y_n \leq 4 - \frac{1}{2 + \sqrt{3}}$$

$$\frac{7 - 4\sqrt{3}}{2 - \sqrt{3}} \leq y_{n+1} \leq \frac{7 + 4\sqrt{3}}{2 + \sqrt{3}}$$

Given that $7 + 4\sqrt{3} = (2 + \sqrt{3})^2$ and $7 - 4\sqrt{3} = (2 - \sqrt{3})^2$

$$2 - \sqrt{3} \leq y_{n+1} \leq 2 + \sqrt{3}$$

Therefore all of elements of our sequence are bounded by $2 - \sqrt{3}$ and $2 + \sqrt{3}$. Thus $y_{n+1} \geq y_n$, and therefore the sequence is increasing.

Now we have the proof that the sequence is both bounded and increasing, and thus converges to some number. After some calculation (and by using common sense) we can also conslude, that the sequence converges to $2 + \sqrt{3}$.

2.4.4

Show that

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

converges and find the limit.

If we write the same expression as in exerice but using powers instead of radicals we'll get

$$2^{\frac{1}{2}}, 2^{\frac{3}{4}}, 2^{\frac{7}{8}}, ...$$

Therefore the sequence can be written as

$$x_n = 2^{\sum_{m=1}^{n} 2^{-m}}$$

$2^m > 0$ for all $m \in N$, therefore the sum is increasing. Thus, the power, in which we put 2 is also increasing, therefore the whole sequence is increasing.

Because each element of this sequence is a power of a positive number, we can conclude, that all of the elements are bound below by 0.

Therefore we just need to prove that

$$\sum_{m=1}^{\infty} 2^{-m} = 1$$

and we'll have our proof.

We'll do it by using the properties of this sum. We can see that

$$\sum_{m=1}^{n} 2^{-m} + 2^{-n} = 1$$

$2^{-n} > 0$, and therefore $1 - \sum_{m=1}^{n} 2^{-m} = 2^{-n} > 0$ for all $m \in N$.

Thus, for every $\epsilon > 0$ we can conclude that $\exists M \in \mathbf{N}$ such that for all $m \geq M$.

$$1/m < \epsilon$$

therefore

$$\sum_{m=1}^{\infty} 2^{-m} = 1$$

and thus

$$(x_n) = 2^{\sum_{m=1}^{n} 2^{-m}} \to 2^1 = 2$$

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2.4.5 (Calculating Square Roots)

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$$

a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

We'll use induction to show that x_n^2 is always greated than 2.

Base: $2^2 = 4 > 2$.

Proposition: $x_n^2 > 2$

Step:

$$x_n^2 > 2$$
$$x_n^2 - 2 > 0$$
$$(x^2 - 2)^2 > 0$$
$$x_n^4 - 4x_n^2 + 4 > 0$$
$$x_n^2 - 4 + \frac{4}{x_n^2} > 0$$
$$(x_n^2 + 4 + \frac{4}{x_n^2}) > 8$$
$$\frac{1}{4}(x_n^2 + 4 + \frac{4}{x_n^2}) > 2$$
$$x_{n+1}^2 > 2$$

Thus $x_n^2 > 2$ for all $n \in \mathbf{N}$.

Thus

$$x_n^2 > 2$$
$$x_n^2 - 2 > 0$$
$$\frac{1}{2}x_n - \frac{2}{2x_n} > 0$$
$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{2}{x_n}) = \frac{1}{2}x_n - \frac{2}{2x_n} > 0$$

Therefore the sequence is decreasing. Now let us prove, that every element of the sequence is positive.

Base: $x_1 > 0$

Step:

$$x_n > 0$$
$$x_n^2 > 0$$

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$$x_n^2 > -2$$

$$x_n^2 + 2 > 0$$

$$x_n + \frac{2}{x_n} > 0$$

$$(x_n + \frac{2}{x_n}) > 0$$

$$\frac{1}{2}(x_n + \frac{2}{x_n}) > 0$$

$$x_{n+1} > 0$$

Thus every element of the sequence is positive and decreasing. Therefore the sequence is convergent (because it is bounded above by the first element and below by 0).

Thus this sequence converges to a positive limit. Let $l = \lim x_n$. Then also $l = \lim x_{n+1}$. Thus

$$l = \frac{1}{2}(l + 2/l)$$

$$2l = l + 2/l$$

$$l = 2/l$$

$$l^2 = 2$$

$$l = \sqrt{2}$$

as desired.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

$$x_1 = c$$

.

$$x_n = \frac{1}{2}(x_n + \frac{c}{x_n})$$

The proof that this sequence works is obtained through the same logic, as in part (a).

2.4.6 (Limit Superior)

Let (a_n) be a bounded sequence.
   (a) Prove that the sequence defined by $y_n = sup\{a_k : k \geq n\}$ converges
   In order to prove that we need to show that this seqeunce is both bounded and monotone (in this case, decreasing).
   Suppose that $n_1 > n_2 \in \mathbf{N}$. Then it follows that

$$\{a_k : k \geq n_1\} \subseteq \{a_k : k \geq n_2\}$$

   Therefore

$$\forall l \in \{a_k : k \geq n_2\} \rightarrow l \leq sup\{a_k : k \geq n_2\}$$

and thus

$$\forall l \in \{a_k : k \geq n_1\} \rightarrow l \leq sup\{a_k : k \geq n_2\}$$

(i.e. if a number o is a supremum for $\{a_k : k \geq n_1\}$, then it is an upper bound for $\{a_k : k \geq n_2\}$). And thus $y_n \geq y_{n+1}$. Therefore sequence (y_n) is decreasing.
   Also, because the sequence is bounded, $\exists M \in R > 0$ such that $|a_n| \leq M$ for all $n \in N$. Thus, y_n is bounded below by $-M$. Therefore the sequence is decreasing and bounded. Therefore it converges.
   (b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Proveide a reasonable definition for $\liminf a_n$ and briefly explay why it always exists for any bounded sequence.
   For a bounded sequence (a_n), $\liminf a_n$ is a limit of a sequence, defined by

$$y_n = \inf\{a_k : k \geq n\}$$

.

   Because for any $n_1 > n_2 \in \mathbf{N}$ it follows that $\{a_k : k \geq n_1\} \subseteq \{a_k : k \geq n_2\}$, we will get that the sequence y_n is bounded, and because of the boundness of (a_n) we will get that the sequence is increasing.
   (c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence, for which this ineqality is strict.
   We'll use a proof by contradiction on this one.
   Suppose that $\liminf a_n > \limsup a_n$. Then let $l = \liminf a_n - \limsup a_n$.
   Thus there exists $n \in N$ for which

$$|\sup\{a_k : k \geq n\} - \limsup a_n| < l/2$$

$$-l/2 < \sup\{a_k : k \geq n\} - \limsup a_n < l/2$$

$$\liminf a_n - l/2 < \sup\{a_k : k \geq n\} + \liminf a_n - \limsup a_n < \liminf a_n + l/2$$

$$\limsup a_n + l/2 < \sup\{a_k : k \geq n\} + l < \liminf a_n + l/2$$

$$\sup\{a_k : k \geq n\} + l/2 < \liminf a_n$$

$$\sup\{a_k : k \geq n\} < \liminf a_n - l/2$$

Therefore a_n

At the same time there exists $m \in M$ for which

$$|\inf\{a_k : k \geq m\} - \liminf a_n| < l/2$$

$$-l/2 < \inf\{a_k : k \geq m\} - \liminf a_n < l/2$$

$$\liminf a_n - l/2 < \inf\{a_k : k \geq m\} < l/2 + \liminf a_n$$

$$\liminf a_n - l/2 < \inf\{a_k : k \geq m\} < l/2 + \liminf a_n$$

Thus

$$\sup\{a_k : k \geq n\} < \inf\{a_k : k \geq m\}$$

Pick $j = \max\{m, n\}$. Thus

$$\{a_k : k \geq n\} \subseteq \{a_k : k \geq j\}$$

$$\{a_k : k \geq m\} \subseteq \{a_k : k \geq j\}$$

Therefore

$$\sup\{a_k : k \geq n\} \geq \sup\{a_k : k \geq j\}$$

$$\inf\{a_k : k \geq n\} \leq \inf\{a_k : k \geq j\}$$

Thus

$$\sup\{a_k : k \geq j\} \leq \sup\{a_k : k \geq n\} < \inf\{a_k : k \geq m\} \leq \inf\{a_k : k \geq j\}$$

$$\sup\{a_k : k \geq j\} < \inf\{a_k : k \geq j\}$$

Therefore supremum of the set is less than an infinum of a set, which is a contradiction. Therefore $\liminf a_n \leq \limsup a_n$.

Example of a sequence, for which this inequality is strict is

$$a_n = -1^n$$

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

In forward direction:

Suppose that $\liminf a_n = \limsup a_n = l$. Then for every $\epsilon > 0$ there exists $n \in N$ such that

$$|\sup\{a_k : k \geq n\} - l| < \epsilon$$

Also there exists $m \in N$ such that

$$|\inf\{a_k : k \geq m\} - l| < \epsilon$$

for the same ϵ.

Let us set $j = \max\{n, m\}$. Then

$$-\epsilon < \sup\{a_k : k \geq j\} - l < \epsilon$$

$$l - \epsilon < \sup\{a_k : k \geq j\} < \epsilon + l$$

Also

$$-\epsilon < \inf\{a_k : k \geq j\} - l < \epsilon$$

$$l - \epsilon < \inf\{a_k : k \geq j\} < l + \epsilon$$

For any $j \in N \geq k$

$$a_j \leq \sup\{a_k : k \geq j\} < \epsilon + l$$

$$l - \epsilon < \inf\{a_k : k \geq j\} \leq a_j$$

therefore

$$l - \epsilon < a_j < l + \epsilon$$

$$-\epsilon < a_j - l < \epsilon$$

$$|a_j - l| < \epsilon$$

Therefore

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : \forall n \geq N \rightarrow |a_n - l| < \epsilon$$

or in other words, $\lim a_n = l = \liminf a_n = \limsup a_n$, as desired.

In other direction:

Suppose that $\lim a_n = l$. From this we can state that

$$|a_n - l| < \epsilon$$

$$\epsilon < a_n - l < \epsilon$$

$$l - \epsilon < a_n < \epsilon + l$$

Thus both $\sup\{a_k : k \geq n\}$ and $\inf\{a_k : k \geq n\}$ are bounded by $l - \epsilon$ and $l + \epsilon$. In other words

$$l - \epsilon \leq \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \leq \epsilon + l$$

This is true for all $\epsilon > 0$. Thus if we set $\epsilon = 1/j$ where $j \in N$, there exists appropriate $i \in N$ for which

$$l - 1/j \leq \inf\{a_k : k \geq i\} \leq \sup\{a_k : k \geq i\} \leq l + 1/j$$

Therefore if we take a limit of all sides of this inequality we can state that

$$\lim(l - 1/j) \leq \liminf\{a_k : k \geq i\} \leq \limsup\{a_k : k \geq i\} \leq \lim(l + 1/j)$$

$$l \leq \liminf a_n \leq \limsup a_n \leq l$$

Here, the ability to take limits of all sides is justified by the fact, that $\forall n \in N > k : a_n \geq b_n \to \lim a_n \geq \lim b_n$.

Thus $\liminf a_n = \limsup a_n = l$, as desired.

2.5.1

Prove Theorem 2.5.2

First of all, let us state the theorem itself.

Theorem 2.5.2 Subsequences of a convergent sequence converge to the same limit.

Suppose that a_n is a convergent sequence, b_n is a subsequence of a_n, and l is a limit of a_n. Because a_n is a convergent sequence we can state that

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \forall n \geq N \to |a_n - l| < \epsilon$$

For each $N \in \mathbf{N}$ there exists $M \in \mathbf{N}$, such that $M \geq N$ and $b_M = a_N$. Also, for each $m > M \in \mathbf{N}$ there exists $n > N \in \mathbf{N}$ such that $b_m = a_n$ Thus we can state that

$$\forall \epsilon \in R > 0 : \exists N \in \mathbf{N} : \exists M \in \mathbf{N} > N : \forall m \geq M \to |b_m - l| < \epsilon$$

$$\forall \epsilon \in R > 0 : \exists M \in \mathbf{N} : \forall m \geq M \to |b_m - l| < \epsilon$$

or in other words, $(b_n) \to l$, as desired.

2.5.2

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + ...$ converges to a limit L (i.e. the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + ... + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + ... + a_{n_2}) + (a_{n_2+1} + a_{n_2+2} + ... + a_{n_3}) + ...$$

leads to a series that also converges to L.

Suppose that we regrouped terms of the above-given sequence. Then each of the terms will be defined by sequence of partial sums

$$s_1 = (a_1 + a_2 + ... + a_{n_1})$$

$$s_2 = (a_1 + a_2 + ... + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + ...a_{n_2})$$

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$$s_n = a_1 + ... + a_{n_n}$$

Therefore, s_n is a subsequence of an original sequence. Therefore, because the original sequence converges to the same limit, s_n will be convergent to the same limit, as desired.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Our proof doesn't apply to any of the sequences, discussed in the 2.1 because each one of them was not convergent , which is a prerequisite for part (a).

2.5.3

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that does not contain 0 or 1 as a term but contains subsequences , converging to each of those values

$$a_n = \begin{cases} 1/n \text{ if n is even} \\ 1 - 1/n \text{ if n is odd} \end{cases} \tag{11}$$

(b) A monotone sequence that diverges but has a convergent subseqence

Suppose that we have such a sequqnce. Then, because subseqeuence is convergent, it is bounded. Thus, there exist $M > 0$ for which $|a_{n_j}| \geq M \to -M \leq a_{n_j} \leq M$. For each of elements of the original sequemce, there exists an element of subsequence, that is "further down the line" (i.e. $\forall n \in N : \exists m, j \in N : n < m$ and $a_m = a_{n_j}$). Then

$$a_n \leq a_{m_j} \leq M$$

in case of the increasing sequence

$$a_n \geq a_{m_j} \geq -M$$

in case of the decreasing sequence.

Thus, the original sequence is bounded. Thus the original sequence is bounded and monotone. Therefore the original sequence is convergent. Therefore we have a contradiction.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, ...\}$

Let

$$a_n = [1], [1, 1/2], [1, 1/2, 1/3], [1, 1/2, 1/3, 1/4], [1, 1/2, 1/3, 1/4, 1/5], ...$$

where [and] are added just for the visual clue.

Therefore we have in it a subsequence, that converges to an arbitrary number $1/n$ for $n \in N$.

(d) An unbounded sequence with a convergent subsequence.

$$a_n = \begin{cases} n \text{ if n is odd} \\ 0 \text{ if n is even} \end{cases} \tag{12}$$

(e) A sequence that has a subsequence that is bounded but contains no subsequence that converges

Impossible. Because there exists bounded subsequence, which is in and of itself a sequence, there exists a convergent subsequence of a subsequence (by Bolzano-Weierstrass Theorem). Subsequence of a subsequence is a subsequence of original sequence. Thus an original sequence contains a convergent subsequence.

2.5.4

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in R$. Show that (a_n) must converge to a.

Because (a_n) is bounded, there exists $M > 0$ such that $|a_n| < M$.

Suppose that we have an ϵ-neighborhood around a. Then $[-M, M] \setminus V_\epsilon(a)$ could contait elements of (a_n). Suppose that $[-M, M] \ V_\epsilon(a)$ contains infinite amount of elements of (a_n). Then there exists a subsequence of (a_n), that it is outside of V_ϵ. Thus, there exists a convergent subsequence of sequence, that converges to some number, other, than a, which is a contradiction. Therefore there exists only finite amount of elements in $[-M, M] \setminus V_\epsilon(a)$.

Therefore, for any ϵ, there exists an N (which is a maximum index of a finite elements of elements outside the neighborhood, or 1 in case that there are no elements outside of the neighborhood), for which it is true, that $n > N \in \mathbf{N} \rightarrow a_n \in V_\epsilon(a)$. Therefore, $(a_n) \rightarrow a$ by topological version of definition of convergence.

2.5.5

Extend the result proved in Example 2.5.3 to the case $|b| < 1$. Show that $\lim(b^n) = 0$ whenever $-1 < b < 1$.

Let $-1 < b < 1$. Then we need to prove that

$$\forall \epsilon : \exists N \in \mathbf{N} : \forall n \geq N \rightarrow |b^n - 0| < \epsilon$$

$$\forall \epsilon : \exists N \in \mathbf{N} : \forall n \geq N \rightarrow |b^n| < \epsilon$$

We already know, that for $0 < b_1 < 1$ it is true, that $\lim(b_1^n) = 0$. Thus, for $b = 0$, the case is triival, and for $b < 0$ it defaults to $|b| = b_1$. Therefore $\lim(b^n) = 0$ for $-1 < b < 1$.

2.5.6

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in R : x < a_n \text{ for indefinetly many terms } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using Axiom of Completeness.)

The wording of this exercise doesn't make it easy to understand what exactly are we trying to prove. After some pondering, I concluded, that what this particular exercise is trying to say, is that there exists a subsequence, that is convergent to a limit supremum of this sequence. If it does, then some rewording of 2.4.6 part(a) will give you the desired result. If it doesn't, then I don't know what do I need to prove, and therefore declare this particular exerice as unfinished.

This exercise already took too much time which was spent on anything but math, therefore instead of pondering further, I will rather spend time on other exerices from this book.

After some research on this topic, I found out that there exist another proof of Bolzano-Weierstrass Theorem, that doesn't use NIP, and my idea now is that this particular exercise tried to ask us to give that proof. That proof can be (kinda) derived from the limit superior theorem, by the fact, that if a limit superior exists, then there exists a subsequence in the original sequence, that converges to limit superior.

2.6.1

Give an example of each of the following, or argue that such a request is impossible
(a) A Cauchy sequence that is not monotone

$$a_n = (-0.5)^n$$

It convergent (as proven in previous exercises), and is a Cauchy sequence (because it is convergent)
(b) A monotone sequence that is not Cauchy

$$a_n = n$$

It is not convergent, therefore not a Caucy sequence.
(c) A Cauchy sequence with a divergent subsequence.

Impossible, because a Cauchy sequence is a convergent sequence, and all of the subsequences of convergent sequence are convergent to the same number.
(d) An unbounded sequence containing a subsequence that is Cauchy

$$a_n = \begin{cases} n \text{ if n is odd} \\ 0 \text{ if n is even} \end{cases} \tag{13}$$

2.6.2

Supply a proof for Theorem 2.6.2

Firstly, let us state the theorem itself

Theorem 2.6.2 *Every convergent sequence is a Cauchy sequence*

Suppose that (x_n) converges to x. Thus,

$$\forall \epsilon/2 > 0 : \exists N \in \mathbf{N} : \forall n \geq N \to |x_n - x| < \epsilon/2$$

Let $m > N$. Then

$$|x_n - x| < \epsilon/2$$
$$|x_m - x| < \epsilon/2$$

thus

$$|x_n - x| + |x_m - x| < \epsilon$$
$$|x_n - x| + |x - x_m| < \epsilon$$
$$|x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon$$
$$|x_n - x + x - x_m| < \epsilon$$
$$|x_n - x_m| < \epsilon$$

Thus any convergent sequence is a Cauchy sequence, as desired.

2.6.3

(a) Explain how the following pseudo-Cauchy property differs from the proper definition of a Cauchy sequence: A sequence (s_n) is pseude-Cauchy if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

It obviously differs in the fact, that in given definition we are only onsidering the element, that goes after one element, instead of all of elements, that are after.

(b) If possible, give an example of a divirgent sequence (s_n) that is pseudo-Cauchy

Harmonic series.

2.6.4

Assume (a_n) and (b_n) are Cauchy sequences. Use a triangle inequality argument to prove $c_n = |a_n - b_n|$ is Cauchy.

Therefore for each ϵ there exists

$$\forall \epsilon/2 > 0 : \exists N_1 \in \mathbf{N} : \forall n, m \geq N_1 \to |a_n - a_m| < \epsilon/2$$

$$\forall \epsilon/2 > 0 : \exists N_2 \in \mathbf{N} : \forall n, m \geq N_2 \to |b_n - b_m| < \epsilon/2$$

Let us pick $N = \max\{N_1, N_2\}$. Then

$$|a_n - a_m| + |b_n - b_m| < \epsilon$$

$$|c_n - c_m| = ||a_n - b_n| - |a_m - b_m|| \leq |a_n - b_n - a_m + b_m| \leq |a_n - a_m| + |b_n - b_m| < \epsilon$$

$$|c_n - c_m| < \epsilon$$

Thus (c_n) is also Cauchy, as desired.

2.6.5

If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

Suppose that (x_n) and (y_n) are Cauchy sequences. Therefore

$$\forall \epsilon/2 > 0 : \exists N_1 \in \mathbf{N} : \forall m, n \geq N_1 \to |x_n - x_m| < \epsilon/2$$

and

$$\forall \epsilon > 0 : \exists N_2 \in \mathbf{N} : \forall m, n \geq N_2 \to |y_n - y_m| < \epsilon/2$$

Therefore let $N = \max\{N_1, N_2\}$. Then

$$\forall \epsilon/2 > 0 : \exists N \in \mathbf{N} : \forall m, n \geq N \to |x_n - x_m| < \epsilon/2$$

and

$$\forall \epsilon > 0 : \exists N \in \mathbf{N} : \forall m, n \geq N \to |y_n - y_m| < \epsilon/2$$

Thus

$$|x_n - x_m| + |y_n + y_m| < \epsilon$$

$$|x_n - x_m + y_n - y_m| = |x_n - x_m + y_n - y_m| \leq |x_n - x_m| + |y_n + y_m| < \epsilon$$

therefore $(x_n + y_n)$ is a Cauchy sequence as well, as desired.

(b) Do the same for the product $(x_n y_n)$.

Suppose that (x_n) and (y_n) are Cauchy sequences. Therefore they are bounded by $|x_n| < M_1$ and $|y_n| < M_2$. Let us pick the greatest $M = \max\{M_1, M_2\}$. Then

$$\forall \epsilon/2 > 0 : \exists N_1 \in \mathbf{N} : \forall m, n \geq N_1 \to |x_n - x_m| < \frac{\epsilon}{2M}$$

and
$$\forall \epsilon > 0 : \exists N_2 \in \mathbf{N} : \forall m, n \geq N_2 \rightarrow |y_n - y_m| < \frac{\epsilon}{2M}$$

Thus

$$|x_n y_n - x_m y_m| = |x_n y_n - x_n y_m + x_n y_m - x_m y_m| = |x_n(y_n - y_m) + y_m(x_n - x_m)| \leq$$

$$\leq |x_n||y_n - y_m| + |y_m||x_n - x_m| \leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(x_n y_n)$ is a Cauchy sequence as well.

2.6.6

(a) Assume the Nested Interval Property (Theorem 1.4.1) is true and use a technique similar to the one employed in the proof of the Bolzano-Weierstrass Theorem to give a proof for the Axiom of Completeness (The reverse implication was given in Chapter 1. This shows, that AoC is equivalent to NIP

Suppose that we have a bounded set A. Then, because it is bounded, there exists an interval $I_1 = [a, b]$, where $a \in A$, and $b \in \mathbf{R} \setminus A$ - upper bound of A. Then, let us divide this set (and in general set I_n) into two sets of equal length $[a, l]$ and $[l, b]$. Then set I_{n+1} to the set, $[l, b]$, if it has any elements of A; otherwise set it to $[a, l]$. Because of the NIP

$$\cap_{n=1}^{\infty} I_n = B \neq \emptyset$$

Suppose now, that there exist two elements $c < d \in B$. Then let us look at the interval $[c, d]$ with length h. Because of the Archimedian property, there exists a $n \in N$, such that $1/n < |d - c|$, and thus there must exists $1/2^n < |d - c|$. Therefore there exists a set I_j, which length is less than distance between c and d. Therefore we have a contradiction. (The absolute value here is redundant, but it doesn't hurt). Therefore there exists only one number in B. Let us henceforth call it l.

Suppose, that there exists $q \in A$, such that $q > l$. Then it follows, that there exists an interval $[q, l]$. By the same logic, as in the previous paragraph, we can conclude, that such element does not exist. Therefore, for all $q \in A$ it is true, that $q \leq l$. Thus, l is an upper bound.

Suppose now, that r is an upper bound of A with the property, that $r < l$. Then $[r, l]$ contains no elements of A. Therefore there exists $I_g \subseteq [r, l]$ such that it contains no elements of A, which is a contradiction. Therefore, for any upper bound r it is true, that $r \geq l$.

Therefore l is an upper bound with the property, that any upper bound is either greater or equal to l. Thus, l is a lowest upper bound (or supremum) of A. Therefore, by existence of such a bound, any bounded set of R has a least upper bound. In other words,

$$NIP \iff AoC$$

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, as desired.

(b) Use the Monotone Convergence Theorem to give a proof of the Nested Interval Property. (This extablishes the equivalence of AoC, NIP, and MCT

Suppose, that we have nested intervals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4... \subseteq I_n \subseteq ...$$

Then, let $(a_n) = $ lower bound of I_n. Then it follows, that

$$a_1 \leq a_2 \leq a_3... \leq a_n \leq \text{ upper bound of any of } I_n$$

Therefore, this sequence is increasing and bounded above (and below by a_1). Thus, it is convergent to some number $l_1 \in R$.

Let us also define $(b_n) = $ upper bound of I_n

$$b_1 \geq b_2 \geq b_3... \geq b_n \geq \text{ lower bound of any of } I_n$$

Therefore, this sequence is decreasing and bounded below (and above by b_1). Thus, it is convergent to some number $l_2 \in R$.

Now look at l_1. Let us pick I_j with lower bound a_1 and upper bound b_1. Suppose, that $l_1 < a_1$. Then let $\epsilon = a_1 - l_1$. Then it follows, that all of the elements of the sequence after g (i.e. a_n for $n \geq g$) are not in $V_\epsilon(l_1)$. Thus, we have a contradition. Thus, $l \geq a_1$. Wit hthe same logic and opposite sign we can prove that $l \geq b_1$. Thus, $l \in I_g$ for any $g \in N$. Thus,

$$l \in \cap_{n=1}^{\infty} I_n$$

Thus, the intersection of any number of nested sets is non-empty. Thus,

$$MCT \iff NIP$$

, as desired.

(c) This time, start with the Bolzano-Weierstrass Theorem and use it to construct a proof of the Nested Interval Property. (Thus, BW is equivalent to NIP, and hence to AoC and MCT as well.)

Suppose that we have nested intervals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4... \subseteq I_n \subseteq ...$$

Now let us define a sequence, such that every member of a sequence is a number is a corresponding interval I_n.

$$(a_n) = \{x : x \in I_n\}$$

First of all, this sequence is going to be bounded by I_1. Therfore, by BW, it has a convergent subsequence. Let us call this subsequence (a_{n_j}), and call a limit of such a sequence l. Then by the same logic as in part (b), $l \in I_g$ for any $g \in \mathbf{N}$, and therefore

$$l \in \cap_{n=1}^{\infty} I_n$$

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Thus,
$$BW \iff NIP$$

(d) Finally, use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem. This is a final link in the equivlence of the five characterizations of completeness discussed at the end of Section 2.6

Suppose that we have a bounded sequence (a_n). Then, it follows, that there exists $M > 0$ such that $a_n \in [-M, M]$ for any $n \in N$. We are going to decribe how to get from I_j to I_{j+1}, and by extension from a_{n_j} to $a_{n_{j+1}}$. For the complete costruction let $I_1 = [-M, M]$ and proceed from there. Now let's start constructing sequence.

Pick a_{n_j} from interval $I_j = [a, b]$. Then divide I_j into two equal-length intervals $[a, l]$ and $[b, l]$. As discussed in previous exercises, one of them will hav infinite amount of elements. Let I_{j+1} be interval with infinite amount of elements of (a_n).

Therefore for each of $\epsilon > 0$ there will exists $j \in N$ such that it will be true, that $I_j \subseteq V_\epsilon(a_j)$. And thus will contain every element of subsequence $\{a_{n_k} : k \geq j\}$ (the proof for validity of this statement was already discussed in original proof of BW). Thus, (a_{n_j}) will be a Cauchy sequence, and therefore convergent to some value. Therefore
$$CC \iff BW$$

2.7.1

Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums
$$s_n = a_1 - a_2 + a_3 - ... \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

Let us state the theorem itself first:

Theorem 2.7.7 (Alternating Series Test) *Let (a_n) be a sequence satisfying*

(i) $a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq ...$ and

(ii) $(a_n) \to 0$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

Firstly, let us draw some conclusions from the theorem. If $a_n < 0$ for some $n \in N$, then all of the a_j for $j > n$ are $a_j \leq a_n < 0$, and thus it does not converge to zero, which is a contradiction. Thus we can state that $a_n \geq 0$ for all $n \in N$.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

Let $\epsilon > 0$. Then, because (a_n) converges, there exist $N \in \mathbf{N}$ such that for all $n > N$ it follows that $|a_n| < \epsilon$.

Let $s_n = a_1 - a_2 + a_3 - ... \pm a_n$. Now let $s_m = a_1 - a_2 + a_3 - ... \pm a_m$ for $m \geq n$. Then

$$|s_m - s_n| = |a_{n+1} - a_{n+2} + ... \pm a_m|$$

Let us examine isnterval $[0, a_n]$ with regards to elemets, that follow a_n:

$$a_n \geq a_{n+1} \rightarrow a_n - a_{n+1} \in [0, a_n]$$

Also, because $a_n \geq a_{n+1} \geq a_{n+2}$ we can state that $a_n - a_{n+1} + a_{n+2} \in [0, a_n]$. Thus

$$[a_n - a_{n+1} + a_{n+2}, a_{n+2}] \subseteq [a_n - a_{n+1}, a_n] \subseteq [0, a_n]$$

Thus

$$a_n \geq a_{n+1} - a_{n+2} + a_{n+3} - ... \pm a_m$$

for any $m \geq n$. Therefore

$$|s_m - s_n| = |a_{n+1} - a_{n+2} + ... \pm a_m| \leq a_n < \epsilon$$

Therefore we can state that for any $\epsilon > 0$ there exist N, such that for all $n, m \geq N$ it follows that

$$|s_n - s_m| \leq a_n < \epsilon$$

Therefore (s_n) is a Cauchy sequence. Thus it converges, as desired.

(b) *Supply another proof for this result using the NIP (Theorem 1.4.1).*

Let us define interval $[a_n - a_{n+1}; a_n]$ for some $n \in N$. We can see that a_n and $a_n - a_{n-1}$ are in this interval. Because $a_{n+2} \geq 0$ and $a_{n+2} \leq a_{n+1}$ we can state that

$$a_n - a_{n+1} \leq a_{n+2} + a_n - a_{n+1} \leq a_n$$

It also doesn't hurt to state here that

$$a_{n+2} + a_n - a_{n+1} \leq a_n \rightarrow [a_n - a_{n+1}, a_{n+2} + a_n - a_{n+1}] \subseteq [a_n - a_{n+1}; a_n]$$

Now let us define

$$I_n = [a_{2n} - a_{2n+1}, a_{2n}]$$

from our discussion in previous paragraph we can state that

$$a_{2n+m} \in I_n$$

where $m \in N \cup \{0\}$. Also

$$I_n \subseteq I_{n+1}$$

For the future use let us also note the fact, that for every $j \in N$ there exists an interval I_n, whose length is less or equal to a_j.

Thus we have a collection of nested intervals. NIP states that

$$\cap_{n=1}^{\infty} I_n = B \neq \emptyset$$

Suppose that we have $a \neq b \in B$. Then there exist $\epsilon = |a - b|$, for which by convergence of (a_n) there exists N for which it is true that

$$\forall n \geq N \to |a_n| \leq \epsilon$$

Therefore there exists an interval, whose length is less than the distance betweeen a and b. Therefore both a and b cannot be at some I_n at the same time. Therefore we have a contradiction, and we can state that we cannot have two distinct elements in B. Thus, there exists only one element in B.

Note that for every partial sum s_n we can find corresponding I_m such that

$$s_n \in I_m$$

Thus we can state that

$$(s_n) \to l$$

or in other words,

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \lim s_n = l$$

where l is the sole element of B.

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test

Let us note that

$$s_{2n} = a_1 - a_2 + ... + a_{2n-1} - a_{2n}$$

Thus

$$s_{2n} - s_{2(n+1)} = a_{2n+1} - a_{2n+2} \geq 0$$

thus series (s_{2n}) is increasing. Also

$$s_{2n+1} - s_{2(n+1)+1} = -a_{2n+2} + a_{2n+3} \leq 0$$

thus series (s_{2n+1}) is decreasing.

Because (s_{2n+1}) is decreasing, there exists $a = \sup(s_{2n+1})$. Because (s_{2n}) is increasing, there exists $b = \inf(s_{2n})$. Every s_j is either equal to some s_{2n} if j is even or to some s_{2n+1} if j is odd. Thus (s_n) is bounded above by a and bounded below by b.

By MCT we can state that

$$\lim s_{2n} = l_1$$

$$\lim s_{2n+1} = l_2$$

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Now let note that for all $n \in N$

$$s_{2n} - s_{2n+1} = a_{2n}$$

Because all subsequences of convergent sequence converge to the same number we can state that

$$\lim(s_{2n} - s_{2n+1}) = \lim(a_{2n}) = 0$$

thus

$$\lim s_{2n} = \lim s_{2n+1} = l_1 = l_2 = l$$

Because for every n there exists $s_{2n} \leq s_j$ and $s_{2n+1} \leq s_j$ we can conclude that by Squeeze theorem

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \lim s_n = l$$

as desired.

2.7.2

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

Let us firstly state the theorem itself

Theorem 2.7.4 (Comparison Test) *Assume (a_k) and (b_k) are sequences satisfying* $0 \leq a_k \leq b_k$ *for all $k \in N$.*

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges

(i)

Let us pick some $\epsilon > 0$. Then, by Cauchy Criterion for sequences, there exists $m > n \geq N \in textbfN$. such that

$$|s_n - s_m| = |b_{m+1} + ... + b_n| = b_{m+1} + ... + b_n < \epsilon$$

for s_n - partial sum for $\sum_{k=1}^{\infty} b_k$.

Let l_n be a partial sum for $\sum_{k=1}^{\infty} a_k$. Then

$$|l_n - l_m| = |a_{m+1} + ... + a_n| = a_{m+1} + ... + a_n \leq b_{m+1} + ... + b_n = |s_n - s_m| \leq \epsilon$$

Thus, by Cauchy Criterion for Series, $\sum_{k=1}^{\infty} a_k$ converges.

(ii)

By Cauchy Criterion for sequences (or rather by negation of it), there exists $\epsilon > 0$ such that for all $m > n \in N$ it is true that

$$|l_n - l_m| = |a_{m+1} + ... + a_n| = a_{m+1} + ... + a_n > \epsilon$$

for l_n - partial sum for $\sum_{k=1}^{\infty} a_k$.

Let s_n be a partial sum for $\sum_{k=1}^{\infty} b_k$. Then

$$\epsilon < |l_n - l_m| = |a_{m+1} + ... + a_n| = a_{m+1} + ... + a_n \leq b_{m+1} + ... + b_n = |s_n - s_m|$$

For all $m > n \geq N$

Thus, by Cauchy Criterion for Series, $\sum_{k=1}^{\infty} b_k$ diverges.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem

Firstly, let us look closer at $\sum_{k=1}^{\infty}(b_k)$: first thing that we could notice, is that every term of the sum is non-negative, and therefore we can state that the sequence of partial sums for it is increasing. Same can be said about $\sum_{k=1}^{\infty}(a_k)$.

(i)

Let us define s_k to be a partial sum for $\sum_{k=1}^{\infty}(b_k)$ and l_k to be a partial sum for $\sum_{k=1}^{\infty}(a_k)$. Because s_k is increasing and convergent, it is true that

$$s_k \leq l$$

for any $k \in N$ where $l = \lim s_k$. Thus

$$l_k \leq s_k \leq l$$

for any $k \in N$. Therefore l_k is bounded above by l as well. Bringing up to our attention again that a_k is non-negative, we can state that

$$0 \leq l_k \leq l$$

thus $\sum_{k=1}^{\infty}(a_k)$ is bounded and monotone and therefore by MCT, convergent.

(ii)

Here we will not neccesarily use MCT itself, but the fact, that any convergent function is bounded.

Partial sums for $\sum_{k=1}^{\infty}(a_k)$ are not convergent and increasing, therefore we can state that they are unbounded, precisely that it is convergent to infinity. Because $l_k \leq s_k$, the same can be said about partial sums for $\sum_{k=1}^{\infty}(b_k)$.

2.7.3

Let $\sum a_n$ be given. For each $n \in N$, let $p_n = a_n$ if a_n os positive and assign $p_n = 0$ if a_n is negative. In a similar manner, let $q_n = a_n$ if a_n is negative and $q_n = 0$ if a_n is positive.

(a) Argue that if $\sum a_n$ diverges, then at least one of $\sum p_n$ or $\sum q_n$ diverges.

We are going to proceed with a proof by contradiction.

Suppose that both $\sum p_n$ and $\sum q_n$ converge. Let s_n be a parital sum for $\sum a_n$, $s1_n$ a parial sum for p_n and $s2_n$ a partial sum for q_n. Then

$$s1_n + s2_n = p_1 + p_2 + ... + p_n + q_1 + q_2 + ... + q_n = a_1 + a_2 + a_3 + ... + a_n = s_n$$

Thus $\sum a_n$ converges, which is a contradiction. Therefore both sums $\sum q_n$ and $\sum p_n$ cannot be convergent. Therefore at least one of $\sum q_n$ or $\sum p_n$ must diverge.

(b) Show that if $\sum a_n$ converges conditionaly, then both $\sum p_n$ and $\sum q_n$ diverge.

We are going to proceed with a proof by contradiction as well. Suppose that p_n converges (we are only looking at the case with p_n here, but the same logic can be applied for q_n as well).

Then, we can state that $\sum -p_n$ converges as well. Thus $\sum a_n - p_n$ converges. Therefore $\sum q_n$ also converges. Therefore $\sum p_n - q_n$ converges as well. Therefore $\sum a_n$ converges absolutely, which is a contradiction.

Therefore we can state, that in orger to $\sum a_n$ to converge absolutely, both $\sum p_n$ and $\sum q_n$ must diverge, as desired.

2.7.4

Give an example to show that it is possible for both $\sum x_n$ and $\sum y_n$ diverge, but for $\sum x_n y_n$ converge.

$$x_n = y_n = 1/n \to x_n y_n = 1/n^2$$

2.7.5

(a) Show that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely. Does this proposition hold without absolute convergence?

Let s_n be a partial sum for $\sum |a_n|$. Then, because $\sum |a_n|$ converges, there exists $M > 0$ such that $s_n \leq M$. Therefore each $|a_n| \leq M$. Therefore

$$s_n^2 = |a_1^2| + |a_2^2| + ... + |a_n^2| \leq M|a_1| + M|a_2| + ... + M|a_n| = M(|a_1| + |a_2| + ... + |a_n|) = Ms_n$$

Threefore, because of the convergence $\sum |a_n|$ we can state, that $M \sum |a_n| = \sum M|a_n|$ converges as well. Thus, $\sum |a_n|^2$ also converges. Thus $\sum a_n^2$ converges absolutely, as desired.

If we substitute a_n with some conditionally convergent series, then this proof collapses on the moment, when we state that $|a_n| \leq M$. That doesn't neccesarily disprove the proposition, only shows where does absolute convergence plays the role in our proof. In

order to disprove this proposition we need an example of a convergent sequence, that contradicts our proposition. One such sequence is

$$\sum a_n = \sum \frac{(-1)^{n+1}}{\sqrt{n}}$$

$$\sum a_n^2 = \sum \frac{1}{n}$$

where the first sum is convergent by Alternating Series Test, and the last is divergent, because is is a harmonic series.

(b) If $\sum a_n$ converges and $a_n \geq 0$, can we conclude anything about $\sum \sqrt{a_n}$?

Not really.

$a_n = \frac{1}{n^2}$ implies that $\sum a_n$ conveges, but $\sum \sqrt{a_n}$ is a Harmonic Series. For $a_n = \frac{1}{n^2}$ both $\sum a_n$ and $\sum \sqrt{a_n}$ converge (both conclusions are justified by the fact, that we had drawn some time ago that the series $\sum 1/n^p$ converges if and only if $p > 1$).

2.7.6

(a) Show that if $\sum x_n$ converges absolutely, and the sequence (y_n) is bounded. then the sum $\sum x_n y_n$ converges.

Let s_n be a partial sum for $\sum |x_n|$ and l_n be a partial sum for $\sum |y_n x_n|$. Then, because (y_n) is bounded, there exists $M > 0$ such that $|y_n| \leq M$. Therefore

$$l_n = |x_1 y_1| + |x_2 y_2| + ... + |x_n y_n| \leq M|x_1| + M|x_2| + ... + M|x_n| = M(|x_1| + |x_2| + ... + |x_n|) = M s_n$$

Therefore $\sum x_n$ being absolutely convergent implies that $\sum M x_n$ is also absolutely convergent, which implies that $\sum x_n y_n$ is also absolutely convergent.

(b) Find a counterexample that demonstrates that part (a) does not always hold if the convergence of $\sum x_n$ is conditional

Same as in exercise 2.7.5

$$\sum x_n = \sum \frac{(-1)^{n+1}}{\sqrt{n}}$$

$$(y_n) = \frac{(-1)^{n+1}}{\sqrt{n}}$$

$$\sum y_n x_n = \sum \frac{1}{n}$$

2.7.7

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7

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Corollary 2.4.7 states that the series

$$\sum_{n=1}^{\infty} 1/n^p$$

converges if and only if $p > 1$

Let us employ Cauchy Condensation Test here. Let us first play with the term of the sequence for it.

$$2^n a_{2^n} = 2^n \frac{1}{(2^n)^p} = \frac{2^n}{2^{np}} = 2^{n-np} = 2^{n(1-p)} = (2^{1-p})^n$$

We know, that geometric series converges if and only if for its term ar^k, $|r| < 1$. Thus, we can state, that by Cause Condensation Test, the original series converges if and only if

$$|2^{1-p}| < 1$$

$$2^{1-p} < 1$$

$$1 - p < \log_2(1)$$

$$1 - p < 0$$

$$p > 1$$

as desired.

2.7.8

Prove Theorem 2.7.1 part (ii)

Theorem 2.7.1 part (ii) states that

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ then $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Let s_n be a partial sum for the series $\sum_{k=1}^{\infty} a_k = A$, t_n be a partial sum for $\sum_{k=1}^{\infty} b_k = B$ and p_n be a parial sum for $\sum_{k=1}^{\infty}(a_k + b_k)$. Then

$$s_n + t_n = a_1 + a_2 + ... + a_n + b_1 + b_2 + ... + b_n = a_1 + b_1 + a_2 + b_2 + ... + a_n + b_n = p_n$$

for every $n \in N$.

Thus we can state that $\lim(a_n + b_n) = A + B$ by Algebraic Limit Theorem. Thus

$$\lim(a_n + b_n) = \lim p_n = \sum(a_n + b_n) = A + B$$

as desired.

2.7.9 (Ratio Test)

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim |\frac{a_{n+1}}{a_n}| = r < 1$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. (Why must such an r' exist?) Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n|r'$

First of all,

$$|\frac{a_{n+1}}{a_n}| = |a_{n+1}\frac{1}{a_n}| = |a_{n+1}||\frac{1}{a_n}|$$

.

If $a_n \geq 0$, then $|\frac{1}{a_n}| = \frac{1}{a_n} = \frac{1}{|a_n|}$. If $a_n < 0$ then $|\frac{1}{a_n}| = -\frac{1}{a_n} = \frac{1}{-a_n} = \frac{1}{|a_n|}$. Therefore $|\frac{1}{a_n}| = \frac{1}{|a_n|}$. Thus

$$|\frac{a_{n+1}}{a_n}| = |a_{n+1}||\frac{1}{a_n}| = \frac{|a_{n+1}|}{|a_n|}$$

Thus

$$\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{|a_{n+1}|}{|a_n|} = r < 1$$

r' exists, beacause for any $\epsilon > 0$ there exists $n \in N$ such that $1/n < \epsilon$. Therefore if we set $\epsilon = 1 - r$, then there exists $1/n_1 < \epsilon$. Therefore

$$0 < 1/n_1 < \epsilon$$

$$0 < 1/n_1 < 1 - r$$

$$r < r + 1/n_1 < 1$$

Suppose that we set $\epsilon = r' - r$. Then, by definition of limit, there exists $N \in \mathbf{N}$, such that for any $n \geq N$ it follows that

$$|\frac{|a_{n+1}|}{|a_n|} - r| < \epsilon$$

$$|\frac{|a_{n+1}|}{|a_n|} - r| < r' - r$$

$$-(r' - r) < \frac{|a_{n+1}|}{|a_n|} - r < r' - r$$

$$-r' + r < \frac{|a_{n+1}|}{|a_n|} - r < r' - r$$

$$-r' + 2r < \frac{|a_{n+1}|}{|a_n|} < r'$$

$$\frac{|a_{n+1}|}{|a_n|} < r'$$

$$|a_{n+1}| < r'|a_n|$$

as desired.

(b) *Why does $|a_N| \sum (r')^n$ necessarily converge?*

Because we have r is a limit of non-negative terms, we can conclude, that $r \geq 0$. Thus, $0 \leq r' < 1$. Thus, $|r'| < 1$. Therefore, $\sum (r')^n$ is a geometric series with $|r| < 1$. Therefore it converges. Therefore $|a_N| \sum (r')^n$ is convergent. Also, $\sum |a_N|(r')^n$ is convergent as well.

(c) *Now, show that $\sum |a_n|$ converges.*

Let s_n be a partial sum for $\sum a_n$ and let q_n be a partial sum for $\sum |a_N| r'^n$.

Let ϵ be arbitrary. Then, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1 \to |a_{n+1}| \leq |a_n| r'$. $\sum a_n$. There also exists $N_2 \in \mathbf{N}$ such that $m > n \geq N_2$ implies than $|q_m - q_n| < \epsilon$ (this comes from Cauchy Criterion for Series). Then let $N = \max\{N_1, N_2\}$.

Then for $m > n \geq N$.

$$|s_m - s_n| = ||a_{n+1}| + |a_{n+2}| + ...|a_m|| = |a_{n+1}| + |a_{n+2}| + ...|a_m| \leq$$

$$\leq |a_n|r' + |a_{n+1}|r' + ... + |a_{m-1}|r' \leq |a_{n-1}|r'^2 + |a_n|r'^2 + ... + |a_{m-2}|r'^2 \leq$$

$$\leq |a_N|r'^{N-n} + |a_N|r'^{N-n+1} + ... + |a_N|r'^{N-m} = |q_m - q_n| < \epsilon$$

Therefore there exists $N \in \mathbf{N}$ such that for every $m > n \geq N$

$$|s_m - s_n| < \epsilon$$

Thus, by Cauchy criterion for series, the series $\sum |a_n|$ is convergent. Thus $\sum a_n$ is absolutely convergent.

2.7.10

(a) *Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.*

I think that for this one the strategy is to somehow compare element of a given sequence, and then conclude, that the partial sum of a desired series is more than a partial sum for harmonic series, and therefore is divergent.

First of all, $n \in N \to n \geq 0$, therfore $a_n n \geq 0$. Therefore we can state that $l \geq 0$.

Let (s_n) be a sequence of partial sums for $\sum a_n$. Then let $\epsilon = l$. Then there exists $N \in \mathbf{N}$ such that for all $n \in \mathbf{N} \geq N$ it follows that

$$|na_n - l| > \epsilon$$

$$na_n - l > \epsilon$$

$$na_n - l > l$$

$$na_n > 2l$$

$$a_n > \frac{2l}{n}$$

Let us now look at the Harmonic Series. Define q_n to be a partial sum for the Harmonic Series. We know, that it is divergent. Thus, there exists an ϵ, such that

$$|q_m - q_n| \geq \epsilon$$

for any $m > n \in \mathbf{N}$. Thus

$$\left| \frac{1}{m} + \frac{1}{m-1} + ... + \frac{1}{n+1} \right| \geq \epsilon$$

$$\frac{1}{m} + \frac{1}{m-1} + ... + \frac{1}{n+1} \geq \epsilon$$

$$\frac{2l}{m} + \frac{2l}{m-1} + ... + \frac{2l}{n+1} \geq 2l\epsilon$$

$$a_m + a_{m-1} + ... + a_{n+1} \geq 2l\epsilon$$

$$|s_m - s_n| \geq 2l\epsilon$$

Thus, there exists $2l\epsilon \in R$ such that for every $m > n \in \mathbf{N}$ it follows that

$$|s_m - s_n| \geq 2l\epsilon$$

Thus, by Cauchy Criterion for series, the series $\sum a_n$ is divergent.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Suppose that all assumption in the exercise are true. Then, by defintion of limit, for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that if $n \geq N$, then it follows that

$$|n^2 a_n - l| < \epsilon$$

$$-\epsilon < n^2 a_n - l < \epsilon$$

$$-\epsilon + l < n^2 a_n < l + \epsilon$$

$$\frac{l - \epsilon}{n^2} < a_n < \frac{l + \epsilon}{n^2}$$

$$0 < a_n < \frac{l + \epsilon}{n^2}$$

We know, that the sum $\sum 1/n^2$ is convergent. Therefore $\sum \frac{l+\epsilon}{n^2}$ is convegent as well for any $l + \epsilon \in R$. Thus, by Comparison Test, $\sum a_n$ is convergent as well, as desired.

2.7.11

Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge, but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challanging, produce exampless where (a_n) and (b_n) are positive and decreasing.

If the relax the exercise and try to make just normal versions of given things, then we'll get

$$a_x = \begin{cases} 1 \text{ if n is even} \\ 1/n^2 \text{ if n is odd} \end{cases} \tag{14}$$

and

$$b_x = \begin{cases} 1 \text{ if n is odd} \\ 1/n^2 \text{ if n is even} \end{cases} \tag{15}$$

Then the obvious minimum will converge, as desired. In this case we even have every element of every sequence to be positive.

The idea behind this one is probably to construct the desired case from "barely" divergent sequences and series (In this case we probably need something along the lines of Harmonic Series)

It (probably) can be shown, that the desired sequences are convergent to zero (or at the very least one of them).

On this particular exercise we are going to assume some things without any hope for proof or concrete base, and we are going to pull'em out of thin air. First of all: the required seiries are most likely to be "barely" divergent. Secondly, required sequences (not series) are probably convergent to 0. It's actually hard to belive, that such sequences exist.

I looked this exercise up on the internet for some hints, and I got some. Suppose that: a_n - equal to $1/n^2$ for "odd" turns (i.e. for [1], [3, 4], [7, 8, 9], ...), and maximum of itself for other terms b_n - equal to $1/n^2$ for "even" turns (i.e. for [2], [5, 6], [10, 11, 12], ...), and maximum of itself for other terms

Then I think, that both of them diverge, but cannot prove it. There exists a more robust example, but I think, that it is pretty cursed anyways. I'm done with this exercise anyways, not gonna spend more time on it.

2.7.12 (Summation by Parts)

Let (x_n) and (y_n) be sequences, and let $s_n = x_1 + x_2 + ... + x_n$. Use the observation that $x_j = s_j - s_{j-1}$ to verify formula

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1})$$

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$$s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) =$$

$$= s_n y_{n+1} - s_m y_{m+1} + s_{m+1}(y_{m+1} - y_{m+2}) + s_{m+2}(y_{m+2} - y_{m+3}) + \ldots + s_n(y_n - y_{n+1}) =$$

$$= s_n y_{n+1} - s_m y_{m+1} + s_{m+1} y_{m+1} - s_{m+1} y_{m+2} + s_{m+2} y_{m+2} - s_{m+2} y_{m+3} + \ldots + s_n y_n - s_n y_{n+1} =$$

$$= -s_m y_{m+1} + s_{m+1} y_{m+1} - s_{m+1} y_{m+2} + s_{m+2} y_{m+2} - s_{m+2} y_{m+3} + \ldots + s_{n-1} y_{n-1} - s_{n-1} y_n + s_n y_n =$$

$$= y_{m+1}(s_{m+1} - s_m) + y_{m+2}(s_{m+2} - s_{m+1}) + \ldots y_n(s_n - s_{n-1}) =$$

$$= y_{m+1} x_{m+1} + y_{m+2} x_{m+2} + \ldots + y_n x_n = \sum_{j=m+1}^{\infty} x_j y_j$$

as desired.

2.7.13 (Dirichlet's Test)

Dirichlet's Test for convergence states that if the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded (but not necessarily convergent), and if (y_n) is a sequence satisfyong $y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0$ with $\lim y_n = 0$, then the series $\sum_{n=1}^{\infty} x_n y_n$ converges.
(a) Let $M > 0$ be an upper bound for the partial sums of $\sum_{n=1}^{\infty} x_n$. Use Exercise 2.7.12 to show that

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M |y_{m+1}|$$

Suppose that everything in exercise is true. Then it follows that $|s_j| < M$. Then

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| = \left| s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) \right| \leq$$

$$\leq |s_n y_{n+1}| + |s_m y_{m+1}| + \left| \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) \right| =$$

$$\leq |s_n||y_{n+1}| + |s_m||y_{m+1}| + |M| \left| \sum_{j=m+1}^{n} (y_j - y_{j+1}) \right| \leq$$

$$\leq M|y_{n+1}| + M|y_{m+1}| + M \left| \sum_{j=m+1}^{n} (y_j - y_{j+1}) \right| =$$

$$= M \left(|y_{n+1}| + |y_{m+1}| + \left| \sum_{j=m+1}^{n} (y_j - y_{j+1}) \right| \right) \leq$$

$$\leq M \left(|y_{n+1}| + |y_{m+1}| + |y_{m+1} - y_{m+2} + y_{m+2} - y_{m+3} + ... + y_n - y_{n+1}| \right) =$$

$$= M \left(|y_{n+1}| + |y_{m+1}| + |y_{m+1} - y_{n+1}| \right) =$$

$$= M \left(y_{n+1} + y_{m+1} + y_{m+1} - y_{n+1} \right) = 2M|y_{m+1}|$$

Therefore

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M|y_{m+1}|$$

as desired

(b) Prove Dirichlet's Test just stated.

Suppose that we are given some $\epsilon > 0$. By the convergence of (y_n), there exists N, such that for $\epsilon_1 = \epsilon/2M$ and $m + 1 \geq N$ it follows that

$$|y_{m+1}| < \epsilon_1$$

$$|y_{m+1}| < \epsilon/2M$$

$$2M|y_{m+1}| < \epsilon$$

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M|y_{m+1}| < \epsilon$$

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| < \epsilon$$

Thus, by Cauchy Criterion for Series, it follows that $\sum_{n=1}^{\infty} x_j y_j$ converges, as desired.

(c) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Firstly, 2.7.7 states that

Theorem 2.7.7 (Alternating Series Test) *Let (a_n) be a sequence satisfying*

$$\text{(i) } a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq ... \text{ and}$$

$$\text{(ii) } (a_n) \to 0$$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

We cas set $x_n = (-1)^{n+1}$, which is gonna be bounded by $[-1, 1]$, and $y_n = a_n$, which converges to 0 and satisfies all of the necessary prerequisites for Dirichlet's Test to be applied.

2.7.14 (Abel's Test).

Abel's Test for convergence states that if the series $\sum_{n=1}^{\infty} x_n$ converges, and if y_n is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq ... \geq 0$$

then the series $\sum_{n=1}^{\infty} x_n y_n$ converges.

   (a) Carefully point out how the hypothesis of Abel's Test differs from that of Dirichlet's Test in Exercise 2.7.13.

First of all, Abel's Test requires $\sum x_n$ to be not only bounded, but convergent as well. Secondly, (y_n) is not required to be convergent.

   (b) Assume that $\sum a_n$ has partials sums, that are bounded by a constant $A > 0$, and assume $b_1 \geq b_2 \geq b_3 \geq ... \geq 0$. Use Exercise 2.7.12 to show that

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq 2Ab_1$$

let s_n be a sequence of partial sums for $\sum a_n$. Then

$$\left| \sum_{j=1}^{n} a_j b_j \right| = \left| s_n b_{n+1} - s_1 b_2 + \sum_{j=1}^{n} s_j (b_j - b_{j+1}) \right| \leq$$

$$\leq |s_n b_{n+1}| + |s_0 b_1| + \left| \sum_{j=1}^{n} s_j (b_j - b_{j+1}) \right|$$

$$\leq |s_n b_{n+1}| + |s_0 b_1| + \sum_{j=1}^{n} |s_j (b_j - b_{j+1})|$$

$$\leq |s_n b_{n+1}| + |s_0 b_1| + \sum_{j=1}^{n} |s_j (b_j - b_{j+1})|$$

$$\leq |s_n||b_{n+1}| + |s_0||b_1| + \sum_{j=1}^{n} |s_j||(b_j - b_{j+1})|$$

$$\leq A|b_{n+1}| + A|b_1| + \sum_{j=1}^{n} A\,|(b_j - b_{j+1})|$$

$$\leq A(|b_{n+1}| + |b_1| + \sum_{j=1}^{n} |(b_j - b_{j+1})|)$$

$$\leq A(b_{n+1} + b_1 + \sum_{j=1}^{n}(b_j - b_{j+1}))$$

$$\leq A(b_{n+1} + b_1 + b_1 - b_2 + b_2 - b_3 + ... + b_n - b_{n+1})$$

$$\leq A(b_1 + b_1) = 2Ab_1$$

as desired.

(c) Prove Abel's Test via the following strategy. For a fixed $m \in N$, apply part (b) to $\sum_{j=m+1}^{n} x_j y_j$ by setting $a_n = x_{m+n}$ and $b_n = y_{m+n}$. (Argue that an upper bound on the partial sums of $\sum_{n=1}^{\infty} a_n$ can be made arbitrarily small by taking m to be large.)

Let us fix $m \in N$. Then set $a_n = x_{m+n}$ and $b_n = y_{m+n}$. Then $\sum_{j=1}^{n} a_j$ is equal to $\sum_{m+1}^{m+n} x_n$. Thus, for every $\epsilon > 0$ we can find $m \in N$, such that $\sum_{j=1}^{n} a_j < \epsilon$ by convergence of $\sum x_j$ and by using Cauchy Criterion. Therefore we always have $A = \epsilon$. Thus

$$| \sum_{j=m+1}^{n} x_j y_j | = | \sum_{j=1}^{n} a_n b_n | < 2Ab_1$$

Therefore by Cauchy Criterion for Series, $\sum x_j y_j$ is convergent.

2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n \to \infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Firstly, let us state the requested array: $\{a_{ij} : i, j \in \mathbf{N}\}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$ (I'd gladly portrait the grid here if I knew how to do it).

I don't think, that we need concrete proofs and/or some other rigorous stuff, so it'll suffice to show that

$$s_{1,1} = -1$$

$$s_{2,2} = -3/2$$

$$s_{3,3} = -7/4$$

Or in general, $(s_{n,n}) \to -2$, or in other words,

$$\lim_{n \to \infty} s_{n,n} = -2$$

as desired.

The resulting value is equal to one of the earlier computed results.

2.8.2

Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \boldsymbol{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty}b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges.

Suppose that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges. This implies that for every $i \in N$ it is true that $\sum_{j=1}^{\infty}|a_{ij}|$ converges. This implies that $\sum_{j=1}^{\infty}a_{ij}$ converges. It is also true, tha $\sum_{i=1}^{\infty}b_i$ converges. Triangular inequality implies that

$$b_i = \sum_{j=1}^{\infty}|a_{ij}| \geq \left|\sum_{j=1}^{\infty}a_{ij}\right|$$

$$0 \leq \left|\sum_{j=1}^{\infty}a_{ij}\right| \leq b_i$$

Thus, by Comparison Test

$$\sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}a_{ij}\right|$$

converges. Thus, by Absolute Convergence Test

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges too, as desired.

2.8.3

Theprem 2.8.1. *Let $\{a_{ij} : i, j \in N\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ converge to the same value. Moreover,

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}$

(a) *Prove that the set $\{t_{mn} : m, n \in \boldsymbol{N}\}$ is bounded above, and use this fact to conclude that the sequence (t_{nn}) converges.*

Because

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| = l$$

is convergent, we can state that t_{mn} is less than l. Also, $t_{m,m} - t_{n,n} \geq 0$ for $m \geq n$. Thus, the sequence is bounded and monotone, and therefore convergent.

(b) *Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) is a Cauchy sequence and hence converges.*

Let $\epsilon > 0$. Then there exists $N \in \mathbf{N}$ such that for $m > n \in N$ it follows that

$$|t_{mm} - t_{nn}| < \epsilon$$

$$||a_{m,m}| + |a_{m,m-1}| + ... + |a_{n+1,n+1}|| < \epsilon$$

$$|a_{m,m}| + |a_{m,m-1}| + ... + |a_{n+1,n+1}| < \epsilon$$

$$|a_{m,m} + a_{m,m-1} + ... + a_{n+1,n+1}| \leq |a_{m,m}| + |a_{m,m-1}| + ... + |a_{n+1,n+1}| < \epsilon$$

$$|a_{m,m} + a_{m,m-1} + ... + a_{n+1,n+1}| < \epsilon$$

$$|s_{mm} - s_{nn}| < \epsilon$$

Therefore (s_{nn}) is a Cauchy sequence, and therefore it is convergent, as desired.

2.8.4

(a) *Argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.*
There exist $m_0, n_0 \in N$ such that

$$B - \frac{\epsilon}{2} < t_{m_0,n_0} \leq B$$

We know, that $t_{m_1,n_0} - t_{m_0,n_0} \geq 0$ for $m_1 \geq m_0$. Thus, if we pick $N_1 = \max\{m_0, n_0\}$, then $m, n \geq N_1$ implies that
$$B - \frac{\epsilon}{2} < t_{m,n} \leq B$$

as desired

(b) *Now, show that there exists an N such that*

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.
We know, that $(t_{n,n})$ is a Cauchy sequence, and thus, for the given ϵ there exists N, such that $m > n \geq N$ implies that

$$|t_{m,m} - t_{n,n}| < \epsilon/2$$

$$||a_{m,m}| + |a_{m-1,m}| + ... + |a_{n+1,n+1}|| < \epsilon/2$$

$$|a_{m,m}| + |a_{m-1,m}| + ... + |a_{n+1,n+1}| < \epsilon/2$$

$$|a_{m,m}| + |a_{m-1,m}| + ... + |a_{n+1,m}| < \epsilon/2$$

$$|a_{m,m} + a_{m-1,m} + ... + a_{n+1,m}| < \epsilon/2$$

$$|a_{m,m} + a_{m-1,m} + ... + a_{n+1,m}| < \epsilon/2$$

$$|s_{nn} - S| < \epsilon/2$$

$$|s_{nn} - S| + |a_{m,m} + a_{m-1,m} + ... + a_{n+1,m}| < \epsilon$$

$$|s_{nn} + a_{m,n} + ... + a_{n+1,n} - S| \leq |s_{nn} - S| + |a_{m,m} + a_{m-1,m} + ... + a_{n+1,m}| < \epsilon$$

$$|s_{nn} + a_{m,n} + ... + a_{n+1,n} - S| < \epsilon$$

$$|s_{mn} - S| < \epsilon$$

Thus, there exists $N \in \mathbf{N}$ such that $m, n \geq N$ implies

$$|s_{mn} - S| < \epsilon$$

as desired