My advanced calculus exercises

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Contents

Chapter 1

Starting points

1.1

Evaluate

$$\int_0^\infty \frac{dx}{1+x^2}$$

and

$$\int_{-\infty}^1 \frac{dx}{1+x^2}$$

$$\int_0^\infty \frac{dx}{1+x^2} = [\tan^{-1}]_0^\infty = \pi/2 - 0 = \pi/2$$

$$\int_{-\infty}^1 \frac{dx}{1+x^2} = [\tan^{-1}]_{-\infty}^1 = [\pi/4 + \pi/2] = \frac{3}{4}\pi$$

1.2

Determine

$$\int \frac{xdx}{1+x^2}$$

Which type of substitution did you use?

$$\int \frac{xdx}{1+x^2} = \frac{1}{2}\int \frac{2xdx}{1+x^2}$$

let $u(x) = 1 + x^2$. Then $u'(x) = 2x$ and therefore

$$\frac{1}{2}\int \frac{2xdx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(1+x^2)$$

I've used push-forward substitution here.

1.3

Carry out a change of variables to evaluate the integral and determite the type of substitution used.

$$\int_{-R}^{R} \sqrt{R^2 - x^2} dx$$

Let's try $x = R\sin(s)$ (the idea is to use the identity $\sin^2(x) + \cos^2(x) = 1$ here somewhere)

$$\int_{-R}^{R} \sqrt{R^2 - x^2} dx = \int_{-R}^{R} \sqrt{R^2 - R^2\sin^2(s)} R\cos(x) ds = \int_{-R}^{R} |R| R \sqrt{1 - \sin^2(s)} \cos(s) ds =$$

$$= |R| R \int_{-R}^{R} |\cos(s)| \cos(s) ds$$

Although this one didn't give us the desired result, given that expression under the intergral is pretty awful, this gave us a pretty interesting idea. $\cos(x) = |\cos(x)|$ for $-\pi/2 < x < \pi/2$. Thus

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$-1 < \frac{2x}{\pi} < 1$$

$$-R < \frac{2Rx}{\pi} < R$$

therefore for

$$-R < \frac{2R}{\pi} x < R$$

, we've got that $\cos(\frac{2R}{\pi}x) = \left|\cos(\frac{2R}{\pi}x)\right|$. Therefore let's try to plug in this god-awful thing into our intergral

Let $x = R\sin(\frac{2Rs}{\pi})$. Then we get that $dx = \frac{2R^2}{\pi}\cos(\frac{2Rs}{\pi})ds$. Thus

$$\int_{-R}^{R} \sqrt{R^2 - x^2} dx = \int_{-R}^{R} \sqrt{R^2 - R^2\sin^2(\frac{2Rs}{\pi})} \frac{2R^2}{\pi} \cos(\frac{2Rs}{\pi}) ds =$$

$$= \int_{-R}^{R} |R| |\cos(\frac{2Rs}{\pi})| \frac{2R^2}{\pi} \cos(\frac{2Rs}{\pi}) ds = \frac{2R^2|R|}{\pi} \int_{-R}^{R} |\cos(\frac{2Rs}{\pi})| \cos(\frac{2Rs}{\pi}) ds =$$

$$= \frac{2R^2|R|}{\pi} \int_{-R}^{R} \cos^2(\frac{2Rs}{\pi}) ds = R|R| \frac{1}{2} [\sin(2$$