My Measure, Integration, & Real Analysis exercises

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2025

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Chapter 1

Riemann Integration

1.1 Riemann Integral

1.1.1

Suppose $f : [a, b] \to R$ *is a bounded function such that*

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

for some partition P *of* $[a, b]$*. Prove that* f *is a constant function on* $[a, b]$*.*

Suppose that f is not contant. We want to follow that there is a section of the partition P whose elements are not all equal.

Assume that all images of elements of any subinterval of a partition are equal. This means that for a given x_j, x_{j+1} we have that if $x_1, x_2 \in [x_j, x_{j+1}$, then $f(x_1) = f(x_2)$. We then follow that $f(x_1) = f(x_2)$, $f(x_2) = f(x_3)$, and so on, which implies that images of all elemets of the partition are equal. By our assumption we have that all the elements in between elements of the partition are also equal, which implies that f is a constant function.

Thus if f is not a constant function, then there is a subinterval of the partition $[x_j, x_{j+1}]$ such that there are $q_1, q_2 \in [x_j, x_{j+1}]$ for which $f(q_1) \neq f(q_2)$. Now relabel q_1, q_2 so that $f(q_1) > f(q_2)$. We follow that

$$\sup_{[x_j, x_{j+1}]} f \geq f(q_1) > f(q_2) \geq \inf_{[x_j, x_{j+1}]} f$$

and thus

$$\sup_{[x_j, x_{j+1}]} f > \inf_{[x_j, x_{j+1}]} f$$

$$(x_{j+1} - x_j) \sup_{[x_j, x_{j+1}]} f > (x_{j+1} - x_j) \inf_{[x_j, x_{j+1}]} f$$

We then can state that for any x_k, x_{k+1} we have that

$$(x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f \leq (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

which further implies that

$$\sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f \leq \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

by then adding the partition in question to both sides we have that

$$(x_{j+1} - x_j) \inf_{[x_j, x_{j+1}]} f + \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f <$$

$$< (x_{j+1} - x_j) \sup_{[x_j, x_{j+1}]} f + \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

thus

$$\sum_{k \in P} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f < \sum_{k \in P} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

and when we apply definitions we get that

$$L(f, P, [a, b]) < U(f, P, [a, b])$$

as desired.

1.1.2

Suppose $a \leq s < t \leq b$. Define $f : [a, b] \to R$ by

$$f(x) = \begin{cases} s < x < t \to 1 \\ 0 \ \textit{otherwise} \end{cases}$$

Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f = t - s$

We can follow that every part of the sum in the definition

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

is nonnegative since $f(x) \geq 0$ for all $x \in [a, b]$. Same goes for $U(f, P, [a, b])$. We then follow that there is $n \in N$ such that

$$s < s + 1/n < t - 1/n < t$$

and for all $m > n$ we have

$$s < s + 1/m < t - 1/m < t < b$$

We then can create a set of partitions

$$P_m = a, s, s + 1/m, t - 1/m, t, b$$

for which we have

$$L(f, P_m, [a, b]) =$$

$$= (s - a) * 0 + (s + 1/m - s) * 0 + (t - 1/m - s - 1/m) * 1 + (t - t + 1/m) * 0 + (b - t) * 0 =$$

$$= t - 1/m - s - 1/m = t - s - 2/m$$

and

$$U(f, P_m, [a, b]) =$$

$$= (s - a) * 0 + (s + 1/m - s) * 1 + (t - 1/m - s - 1/m) * 1 + (t - t + 1/m) * 1 + (b - t) * 0 =$$

$$= s + 1/m - s + t - 1/m - s - 1/m + t - t + 1/m =$$

$$= t - s =$$

thus we have

$$U(f, P_m, [a, b]) - L(f, P_m, [a, b]) = 2/m$$

and since for all $\epsilon > 0$ we have $m \in N$ such that $n \geq m \Rightarrow 2/m < \epsilon$ we follow that for all $\epsilon > 0$ there is $m \in N$ such that

$$U(f, [a, b]) - L(f, [a, b]) \leq U(f, P_m, [a, b]) - L(f, P_m, [a, b]) < \epsilon$$

thus proving that f is indeed Riemann integrable on $[a, b]$. We then follow that

$$L(f, P_m, [a, b]) \leq \int_a^b f \leq U(f, P_m, [a, b])$$

for all m, and thus

$$t - s - 2/m \leq \int_a^b f \leq t - s$$

for all m. This in turn implies that

$$\int_a^b f = t - s$$

as desired.

1.1.3

Suppose $f : [a, b] \to R$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

We can follow that

$$U(f, P, [a, b]) \geq U(f, [a, b])$$

$$L(f, [a, b]) \geq L(f, [a, b])$$

by defnition. Thus we have that

$$U(f, P, [a, b]) - L(f, P, [a, b]) \geq U(f, [a, b]) - L(f, P, [a, b]) \geq U(f, [a, b]) - L(f, [a, b])$$

and thus for every $\epsilon > 0$ we have that

$$U(f, [a, b]) - L(f, [a, b]) < \epsilon$$

Since $U(f, [a, b]) \geq L(f, [a, b])$ we follow then that $U(f, [a, b]) - L(f, [a, b]) \geq 0$ in general, and in particular we've got that

$$U(f, [a, b]) - L(f, [a, b]) = 0 \Leftrightarrow U(f, [a, b]) = L(f, [a, b])$$

thus implying the desired result by definition.

Assume now that f is Riemann integrable. Let $\epsilon > 0$. We follow that there are partions P_1, P_2 such that

$$U(f, P_1, [a, b]) - U(f, [a, b]) < \epsilon/2$$

$$L(f, [a, b]) - L(f, P_2, [a, b]) < \epsilon/2$$

we then can sum up the previous inequalities to get

$$U(f, P_1, [a, b]) - U(f, [a, b]) + L(f, [a, b]) - L(f, P_2, [a, b]) < \epsilon$$

Since $U(f, [a, b]) = L(f, [a, b])$ by the fact that f is Riemann integrable on $[a, b]$, which implies that

$$U(f, P_1, [a, b]) - L(f, P_2, [a, b]) < \epsilon$$

We then follow that there is $P_3 = P_1 \cup P_2$ for which we have

$$U(f, P_3, [a, b]) - L(f, P_3, [a, b]) < \epsilon$$

by 1.5, as desired.

1.1.4

Suppose $f, g : [a, b] \to R$ are Riemann integrable. Prove that $f + g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

We can follow that

$$L(f + g, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f + g$$

$$\inf_{[x_{j-1}, x_j]} f + g = \inf_{[x_{j-1}, x_j]} f + \inf_{[x_{j-1}, x_j]} g$$

and similar for sup and $U(f, P, [a, b])$. Don't know whether or not we've got the latter equality proven, but it's trivial to prove if not. Similar case holds for U.

We then follow that for a given $\epsilon/2 > 0$ we have P_1, P_2 such that

$$U(f, P_1, [a, b]) - L(f, P_1, [a, b]) < \epsilon/2$$

$$U(g, P_2, [a, b]) - L(g, P_2, [a, b]) < \epsilon/2$$

we thus have that by $P_3 = P_1 \cup P_2$ we have

$$U(f, P_3, [a, b]) - L(f, P_3, [a, b]) < \epsilon/2$$

$$U(g, P_3, [a, b]) - L(g, P_3, [a, b]) < \epsilon/2$$

and thus we can add those up to get

$$U(f, P_3, [a, b]) - L(f, P_3, [a, b]) + U(g, P_3, [a, b]) - L(g, P_3, [a, b]) < \epsilon$$

thus by the first paragraph we've got

$$U(f + g, P_3, [a, b]) - L(f + g, P_3, [a, b]) < \epsilon$$

which by previous exercise proves that $f + g$ is Riemann integrable, as desired.

By the first paragraph as well we can follow that

$$\int_a^b f + g = L(f+g, [a, b]) = \sup_P L(f + g, P, [a, b]) = \sup_P L(f, P, [a, b]) + L(g, P, [a, b]) = \sup_P L(f, P, [a, b]) + \sup_P$$

as desired.

The rest was handled in my real analysis course

1.2 Riemann Integral Is Not Good Enough

1.2.1

Define $f : [0,1] \to R$ as follows:

$$f(a) = \begin{cases} 0 \textit{ if } a \textit{ is irrational} \\ 1/n \textit{ if } a \textit{ is rational and } n \textit{ is the smallest positive integer such that } a = m/n \end{cases}$$

Show that f is Riemann integrable and comute $\int_0^1 f$. We firstly follow that for any given $0 \leq x_1 < x_2 \leq 1$ we have that there is $x_1 < x_3 < x_2$ such that

$$f(x_3) = 0$$

and thus for any partition P we've got that

$$L(f, P, [a, b]) = 0$$

thus

$$L(f, [a, b]) = 0$$

We then follow that for any given $\epsilon > 0$ there is $n \in N$ such that $1/n < \epsilon$, and thus we can create a set

$$R = \{i/j : j \leq n, 0 \leq i \leq n\}$$

that is finite. Thus we can follow that

$$\sup_{[0,1] \setminus R} f = 1/(n + 1) < \epsilon$$

and thus we have that for every $\epsilon > 0$ there is P such that

$$U(f, P, [a, b]) < \epsilon$$

thus implying that

$$U(f, [a, b]) = 0$$

therefore we conclude that

$$\int_0^1 f = 0$$

Chapter 2

Measures

2.1 Outer Measure on R

2.1.1

Prove that if A and B are subsets of R and $|B| = 0$, then $|A \cup B| = |A|$

2.8 implies that

$$|A \cup B| \leq |A| + |B| = |A|$$

and since $A \subseteq A \cup B$ we follow by 2.5 that

$$|A| \leq |A \cup B|$$

which gives us the desired result