My probability and statistics exercises

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Contents

Chapter 1

Introduction to Probability

1.1 The History of Probability

1.2 Interpretations of Probability

1.3 Experiments and Events

1.4 Set Theory

Exercises in this section (or exercises similar to them) are handled in the set theory course

1.5 The Definition of Probability

1	2/5
2	0.7
3a	1/2
3b	1/6
3c	3/8
4	0.6
5	0.4
6	0.5
8	30
11a	1 - $\pi/4$
11b	0.75
11c	2/3
11d	0
14a	0.38, 0.16
14b	0.04

A little notation, related to 6:

$$Pr(A) = 0.5$$

$$Pr(B) = 0.2$$

$$Pr(A \cap B) = 0.1$$

$$Pr(A \cup B) = 0.6$$

$$Pr((A \cup B) \cap (A \cap B)^c) = P(A \cup B) - P((A \cup B) \cap (A \cap B)) = P(A \cup B) - P(A \cap B) = 0.5$$

1.5.7

If $Pr(A) = 0.4$ and $Pr(B) = 0.7$, then we follow that the maxium $Pr(A \cap B)$ is attained if $A \subset B$, in which case $Pr(A \cap B) = Pr(A) = 0.4$. The minimum is obtained if $A \cup B = S$, in which case $Pr(A \cap B) = 0.1$

1.5.9

The event that exaclty one of the events occurs can be expressed as

$$(A \cap B^c) \cup (A^c \cap B)$$

which comes from either the definition of xor, common sense or something else, depending on your preferences. Thus we follow that

$$Pr((A \cap B^c) \cup (A^c \cap B)) = Pr(A \cap B^c) + Pr(A^c \cap B) - Pr((A \cap B^c) \cap (A^c \cap B)) =$$

$$= Pr(A \cap B^c) + Pr(A^c \cap B) - Pr((A \cap A^c) \cap (B^c \cap B)) =$$

$$= Pr(A \cap B^c) + Pr(A^c \cap B) = Pr(A) - Pr(A \cap B) + Pr(B) - Pr(B \cap A) =$$

$$= Pr(A) - Pr(A \cap B) + Pr(B) - Pr(A \cap B) = Pr(A) + Pr(B) - 2Pr(A \cap B)$$

as desired (rules used in this derivitation: association of unions, $A \cap A^c = \emptyset$ and other trivial stuff)

1.5.10

$$Pr(A \cap B^c) = Pr(A) - Pr(A \cap B)$$

$$Pr(A \cap B^c) + Pr(A \cap B) = Pr(A)$$

as desired.

1.5.12

Suppose that $n > m \in N$. Then we follow that by definition

$$B_m \subseteq A_m$$

and

$$B_n \subseteq A_m^c$$

thus we follow that

$$B_m \cap B_n \subseteq A_m \cap A_m^c = \emptyset$$

thus

$$B_m \cap B_n = \emptyset$$

therefore we conclude that $B_1, B_2...$ are disjoint sets. Thus we follow that

$$Pr(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} Pr(B_i)$$

For $n = 2$ we've got that

$$B_1 \cup B_2 = A_1 \cup (A_1^c \cap A_2) = (A_1 \cup A_1^c) \cap (A_1 \cup A_2) = A_1 \cup A_2$$

and by induction we can follow that

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

thus

$$Pr(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} Pr(B_i)$$

implies that

$$Pr(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} Pr(B_i)$$

for $n \in N$. Given that n is arbirtary, we can follow that

$$Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(B_i)$$

as desired.

1.5.13

First equation follow from induction on the result that

$$Pr(A \cup B) \leq Pr(A) + Pr(B)$$

the second equation follows from the first equation, DeMorgan laws and induction on the form

$$Pr(A \cap B) = Pr((A^c \cup B^c)^c) = 1 - Pr(A^c \cup B^c) \geq 1 - (Pr(A^c) + Pr(B^c))$$

1.5.14

$$Pr(A) = 0.34$$
$$Pr(B) = 0.12$$
$$Pr(O) = 0.5$$
$$Pr(AB) = 1 - 0.34 - 0.12 - 0.5 = 0.04$$
$$Pr(a - A) = 0.34 + 0.04 = 0.38$$
$$Pr(a - B) = 0.12 + 0.04 = 0.16$$

1.6 Finite Sample Spaces

1	1/2
2	1/2
3	2/3
4	1/7
5	4/7
6	1/4
8b	1/4

1.6.7

The possinble genotypes are Aa and aa with probabilities 1/2 and 1/2 respectively

1.6.8a

The sample space of the experiment is $\{heads, tails\} \times \{1, 2, 3, 4, 5, 6\}$,

1.7 Counting Methods

1	14
2	9000
3	120
4	24
5	5/18
6	5/324
7	0.014731
8	360 / 2401
9	1 / 20
10a	r/100
10b	r/100
10c	r/100

1.7.11

$$s(n) = \frac{1}{2}\log(2\pi) + (n+\frac{1}{2})\log n - n \approx \log n!$$

$$\log n! - \log{(n-m)!} = \log\frac{n!}{(n-m)!}$$

$$s(n)-s(n-m) = \frac{1}{2}\log(2\pi)+(n+\frac{1}{2})\log n - n - (\frac{1}{2}\log(2\pi)+((n-m)+\frac{1}{2})\log n - m - (n-m)) =$$

$$= (n+\frac{1}{2})\log n - n - ((n-m)+\frac{1}{2})\log{(n-m)} + (n-m) =$$

$$= (n+\frac{1}{2})\log n - ((n-m)+\frac{1}{2})\log{(n-m)} - m \approx \log\frac{n!}{(n-m)!}$$

$P(n,m) = \frac{n!}{(n-m)!} = \exp(s(n)-s(n-m))$

1.8 Combinatorial Methods

1	184756
2	latter
3	equal
4	1 / 10626
5	-
6	2/n
7	(n - k - 1)/C(n, k)
8	(n - k)/C(n, k)
9	(n + 1)/C(2n, n)
10	15/92 $\approx$ 0.16304
11	1/75 $\approx$ 0.01333
12	69/119 $\approx$ 0.57983
13	173/1518 $\approx$ 0.114
14	-
15	-
16a	48/175 $\approx$ 0.27429
16b	$2^{50}/C(100,50) \approx 0$
17	$4C(13,4)/C(52,4) = 44/4165 \approx 0.0105$
18	$C(20,2)^5/C(100,10) \approx 0.0143$
19	-
20	-
21	C(365 + 7 - 1, 7)
22	-

1.8.5

Prove that

$$\frac{\prod_{4155 \le i \le 4251} i}{\prod_{2 \le i \le 97} i}$$

is an integer

$$\frac{\prod_{4155 \le i \le 4251} i}{\prod_{2 \le i \le 97} i} = \frac{\prod_{4155 \le i \le 4251} i}{\prod_{1 \le i \le 97} i} =$$

$$= \frac{\prod_{4155 \le i \le 4251} i}{97!} = \frac{4251!}{4154!97!} = \frac{4251!}{4154!(4251-4174)!} = C(4251, 4154)$$

and binomial coefficients are integers (pretty sure that we can follow that by induction in some more advanced course).

1.8.10

There are total of $C(24,10)$ possible subsets of length 10 in the space of 24. We follow that there are $C(22,8)$ ways to pick 8 normal bulbs, which is what required to pick 2 defective bulbs. Therefore the probability is

$$\frac{C(22,8)}{C(24,10)} = 15/92 \approx 0.16304...$$

1.8.12

Using the same logic as in 1.8.10, there is a possibility $\frac{C(33,8)}{C(35,10)}$ that same two guys will be in the first team, and probability of $\frac{C(33,23)}{C(35,10)}$ that they'll be in the other team. Thus the total probability is the sum of two.

1.8.14

Prove that for all positive integers n, k such that $n \geq k$

$$C(n,k) + C(n,k-1) = C(n+1,k)$$

$$\begin{aligned}
C(n,k) + C(n,k-1) &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} = \\
&= \frac{n!}{k(n-k)!(k-1)!} + \frac{n!}{(n-k+1)(n-k)!(k-1)!} = \\
&= \frac{(n-k+1)n!}{k(n-k+1)(n-k)!(k-1)!} + \frac{kn!}{k(n-k+1)(n-k)!(k-1)!} = \\
&= \frac{(n-k+1)n! + kn!}{k(n-k+1)(n-k)!(k-1)!} = \frac{n!((n-k+1)+k)}{k(n-k+1)(n-k)!(k-1)!} = \\
&= \frac{n!(n+1)}{k(n-k+1)(n-k)!(k-1)!} = \frac{(n+1)!}{((n+1)-k)!k!} = C(n+1,k)
\end{aligned}$$

as desired.

1.8.15

(a) Prove that

$$\sum_{i=0}^{n} C(n,i) = 2^n$$

We can follow that from the fact that there are 2^n subsets of any given finite set, which means that the number of subsets of different lengths sums up to 2^n.

Another way to do this is to use binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^k y^{n-k}$$

thus if we subisitute x and y for 1, we get

$$(1+1)^n = \sum_{i=0}^{n} C(n,i)1^k 1^{n-k}$$

$$2^n = \sum_{i=0}^{n} C(n,i)$$

(b) Prove that

$$\sum_{i=0}^{n} (-1)^i C(n,i) = 0$$

I'm sure that there is a neat explanation for this one as well, but using the binomial theorem once again, but now substituting 1 for x and -1 for y we get

$$(1-1)^n = \sum_{i=0}^{n} C(n,i)1^i(-1)^{n-i}$$

$$\sum_{i=0}^{n} C(n,i)1^i(-1)^{n-i} = 0$$

we can follow through the even-odd argument that $1^i(-1)^{n-i} = (-1)^i$, but I'll skip it.

1.8.19

(rewording) Prove the formula for unordered sampling with replacement.

This thing is ought to be covered rigorously in a course for discrete maths, combinatorics or something of sorts. Currentry there is a better proof at Belcastro's "Discrete mathematics with ducks".

1.8.20

Prove the binomial theorem 1.8.2

1.8.2 states that

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^i y^{n-i}$$

Let

$$I = \{n \in \omega : (x+y)^n = \sum_{i=0}^{n} C(n,i)x^i y^{n-i}\}$$

We follow that

$$(x+y)^0 = C(0,0)x^0y^0 = 1$$

Thus $0 \in I$. (we can start with a base case of 1 as well for a more clear example, but I like this one more, and it suffices as well).

Now suppose that $n \in I$. We follow that

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^i y^{n-i}$$

thus we follow that

$$(x+y)(x+y)^n = (x+y)\left[\sum_{i=0}^{n} C(n,i)x^i y^{n-i}\right]$$

Left-hand side is reduced to

$$(x+y)(x+y)^n = (x+y)^{n+1}$$

Right-hand side is obviously a bit trickier, but we can follow

$$(x+y)\sum_{i=0}^{n} C(n,i)x^i y^{n-i} =$$

$$= x\sum_{i=0}^{n} C(n,i)x^i y^{n-i} + y\sum_{i=0}^{n} C(n,i)x^i y^{n-i} =$$

$$= \sum_{i=0}^{n} C(n,i)x^{i+1} y^{n-i} + \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} =$$

$$= \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} + \sum_{i=0}^{n} C(n,i)x^{i+1} y^{n-i} =$$

$$= C(n,n)x^{n+1}y^0 + \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} + \sum_{i=0}^{n-1} C(n,i)x^{i+1} y^{n-i} =$$

$$= x^{n+1} + \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} + \sum_{i=0}^{n-1} C(n,i)x^{i+1} y^{n-i} =$$

$$= x^{n+1} + \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} + x\sum_{i=0}^{n-1} C(n,i)x^i y^{n-i} =$$

$$= x^{n+1} + \sum_{i=0}^{n} C(n,i)x^i y^{n+1-i} + x\sum_{i=1}^{n} C(n,i-1)x^{i-1} y^{n-(i-1)} =$$

$$= x^{n+1} + C(n,0)x^0 y^{n+1} + \sum_{i=1}^{n} C(n,i)x^i y^{n+1-i} + \sum_{i=1}^{n} C(n,i-1)x^i y^{n+1-i} =$$

$$= x^{n+1} + y^{n+1} + \sum_{i=1}^{n} C(n,i)x^i y^{n+1-i} + \sum_{i=1}^{n} C(n,i-1)x^i y^{n+1-i} =$$

$$= x^{n+1} + y^{n+1} + \sum_{i=1}^{n} (C(n,i) + C(n,i-1))x^i y^{n+1-i} =$$

$$= x^{n+1} + y^{n+1} + \sum_{i=1}^{n} C(n+1,i)x^i y^{n+1-i} = x^{n+1} + C(n+1,0)x^0 y^{n+1-0} + \sum_{i=1}^{n} C(n+1,i)x^i y^{n+1-i} =$$

$$= x^{n+1} + \sum_{i=0}^{n} C(n+1,i)x^i y^{n+1-i} = x^{n+1}y^0 + \sum_{i=0}^{n} C(n+1,i)x^i y^{n+1-i} =$$

$$= C(n+1,n+1)x^{n+1}y^{n+1-(n+1)} + \sum_{i=0}^{n} C(n+1,i)x^i y^{n+1-i} = \sum_{i=0}^{n+1} C(n+1,i)x^i y^{n+1-i}$$

Thus we follow

$$(x+y)^{n+1} = \sum_{i=0}^{n+1} C(n+1,i)x^i y^{n+1-i}$$

or

$$(x+y)^{n^+} = \sum_{i=0}^{n^+} C(n^+,i)x^i y^{n^+-i}$$

which means that $n \in I \Rightarrow n^+ \in I$, from which we conclude that $I = \omega$, and thus

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^i y^{n-i}$$

for all $n \in \omega$, as desired.

1.8.22

Skip

1.9 Multinomial Coefficients

1	$(21!)/(7!*7!*7!)$
2	$50!/(18!*12!*12!*8!)$
3	$300!/(5!*8!*287!)$
4	$(3!3!2!)/10! = 1/50400$
5	$M(n,(n_1,...,n_6))/6^n$
6	$(7!)/(2*6^7)$
7	$M(12,(6,2,4))*M(13,(4,6,3))/M(25,(10,8,7))$
8	$M(12,(3,3,3,3)*M(40,(10,10,10,10))/M(52,(13,13,13,13)$
9	$4!/M(52,(13,13,13,13)$
10	$(2!*3!*4!)/9!$

1.10 The Probability of a Union of Events

1	≈ 0.11913
2	85
3	45

1.10.1

$$Pr(A_1) = Pr(A_2) = Pr(A_3) = C(4,2)*C(48,3)/C(52,5)$$
$$Pr(A_1 \cup A_2) = Pr(A_1 \cup A_3) = Pr(A_2 \cup A_3) = C(4,2)*C(48,3)*C(45,3)/C(52,5)^2$$
$$Pr(A_1 \cup A_2 \cup A_3) = 0$$

$$Pr(A_1 \cup A_2 \cup A_3) = 3*C(4,2)*C(49,3)/C(52,5) - 3C(4,2)*C(49,3)*C(46,3)/C(52,5)^2$$

TODO later (probably never).