My advanced calculus exercises

Evgeny Markin

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Chapter 1

Starting points

1.1

Evaluate

$$\int_0^\infty \frac{dx}{1+x^2}$$

and

$$\int_{-\infty}^1 \frac{dx}{1+x^2}$$

$$\int_0^\infty \frac{dx}{1+x^2} = [\tan^{-1}]_0^\infty = \pi/2 - 0 = \pi/2$$

$$\int_{-\infty}^1 \frac{dx}{1+x^2} = [\tan^{-1}]_{-\infty}^1 = [\pi/4 + \pi/2] = \frac{3}{4}\pi$$

1.2

Determine

$$\int \frac{xdx}{1+x^2}$$

Which type of substitution did you use?

$$\int \frac{xdx}{1+x^2} = \frac{1}{2}\int \frac{2xdx}{1+x^2}$$

let $u(x) = 1 + x^2$. Then $u'(x) = 2x$ and therefore

$$\frac{1}{2}\int \frac{2xdx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(1+x^2)$$

I've used push-forward substitution here.

1.3

Carry out a change of variables to evaluate the integral and determite the type of substitution used.

$$\int_{-R}^{R} \sqrt{R^2 - x^2}dx$$

Let's try $x = R\sin(s)$ (the idea is to use the identity $\sin^2(x) + \cos^2(x) = 1$ here somewhere). It follows that $s = \arcsin(\frac{x}{R})$

$$\int_{-R}^{R} \sqrt{R^2 - x^2}dx = \int_{-R}^{R} \sqrt{R^2 - R^2\sin^2(s)}R\cos(x)ds = \int_{-R}^{R} R^2\sqrt{1 - \sin^2(s)}\cos(s)ds =$$

$$= R^2\int_{-R}^{R} \cos^2(s)ds = \frac{R^2}{2}\int_{-R}^{R} 1 + \cos(2s)ds = \frac{R^2}{2}\left[s + \frac{1}{2}\sin(2s)\right]_{-R}^{R} =$$

$$= \frac{R^2}{2}\left[\arcsin(\frac{x}{R}) + \frac{1}{2}\sin(2\arcsin(\frac{x}{R}))\right]_{-R}^{R} =$$

$$= \frac{R^2}{2}\left[\arcsin(1) + \frac{1}{2}\sin(2\arcsin(1)) - \arcsin(-1) - \frac{1}{2}\sin(2\arcsin(-1))\right] =$$

$$= \frac{R^2}{2}\left[\pi/2 + \frac{1}{2}\sin(\pi) + \pi/2 - \frac{1}{2}\sin(-\pi)\right] = \frac{R^2}{2}\pi = \frac{\pi R^2}{2}$$

Which is goot enough for me. We used the pullback approach here

1.4

Determine

$$\int \frac{\arctan(x)}{1 + x^2}dx$$

and show

$$\int_0^\infty \frac{\arctan(x)}{1 + x^2}dx = \pi^2/8$$

$$\int \frac{\arctan(x)}{1 + x^2}dx$$

let $u = \arctan(x)$. Then

$$\int \frac{\arctan(x)}{1 + x^2}dx = \int u = u^2/2 = \arctan(x)^2/2$$

thus

$$\int_0^\infty \frac{\arctan(x)}{1 + x^2}dx = \pi^2/8$$

as desired.

1.5

(a) Detrmine

$$\int \frac{1}{w(\ln(w))^p} dw$$

. *Which type of substitution did you use?*

Let $u = \ln(w)$. It follows that $du = \frac{1}{w} dw$. Thus

$$\int \frac{1}{w(\ln(w))^p} dw = \int \frac{1}{w} (\ln(w))^{-p} dw = \int (u)^{-p} du = \frac{u^{-p-1}}{-p-1} = \frac{(\ln(w))^{-p-1}}{-p-1}$$

. Given that $p \neq -1$.

Othersise it is $\ln(\ln(w))$.

(b) Evaluate

$$I = \int_2^\infty \frac{1}{w(\ln(w))^p} dw$$

. *for which values of p is I finite?*

For $p = 1$ we've got that

$$\lim \ln(\ln(w)) = \infty$$

thus it diverges

For $p \neq -1$ we've got

$$I = \int_2^\infty \frac{1}{w(\ln(w))^p} dw = \lim_{w \to \infty} \left[\frac{(\ln(w))^{-p-1}}{-p-1} \right] - \frac{(\ln(2))^{-p-1}}{-p-1}$$

The only thing that bothers us is that whether

$$(\ln(w))^{-p-1} = (\frac{1}{(\ln(w))})^{p-1}$$

converges. This happens whenever $p > 1$ (in that case we got that $\ln(w) \to \infty$). Otherwise it diverges.

1.6

State a condition that guarantees a function $x = \phi(s)$ has an inverse. Then use your condition to decide whether each of the following functions is invertible. When possible, find a formula for the inverse of each function that is invertible.

Such a condition is bijectivity on a given domain and codomain.

(a)

$$x = 1/s$$

Is bijective on a domain $R \setminus \{0\}$. Inverse is $s = 1/x$.

(b)

$$x = s + s^3$$

This one is bijective (because it is strictly increasing and unbounded below and above).

$$x = s + s^3$$

$$s^3 + s - x = 0$$

Maxima gives some god-awful result for an inverse function, but it can be obtained by solving the cubic polinomial.

(c)

$$x = \frac{s}{1 + s^2}$$

It is not surjective on R, and in addition, it is not injective. Thus it cannot be used without some heavy restrictions on the domain and codomain.

(d)

$$x = \sinh s = \frac{e^s - e^{-s}}{2}$$

Looks solid to me.

$$s = \text{asinh}(s)$$

is a desired inverse function;

(e)

$$x = \frac{s}{\sqrt{1 - s^2}}$$

Is bijective on restricted domain. I'm not sure if we can have an analytical inverse of this function.

(f)

$$x = ms + b$$

Is a standart linear function. If $m \neq 0$, then we've got inverse on whole R.

(g)

$$x = cosh(s)$$

Is bijective on restricted domain. Reverse is $s = \text{acosh}(x)$.

(h)

$$x = s - s^3$$

Is bijective on restricted domain. Inverse is terrible.

(i)

$$x = tanh(s)$$

Also bijective on restricted domain. Inverse is

$$s = \text{atanh}(x)$$

(j)

$$x = \frac{1-s}{1+s}$$

BIjective on restricted domain.

1.7

(a) Obtain formulas for $f(s) = \cos(\arcsin(s))$ and $g(s) = \tan(\arcsin(s))$ directly as funtions of s that involve neither trigonometric nor inverse trigonometric functions. Your answer will involve the square root function and polynomical expressions in s.

$$f(s) = \cos(\arcsin(s))$$

$$\cos(\arcsin(s)) = \sin(\pi/2 - \arcsin(s)) = \sin(\pi/2)\cos(\arcsin(s)) - \sin(\arcsin(s))\cos(\pi/2) =$$
$$= \sin(\pi/2)\cos(\arcsin(s)) = \cos(\arcsin(s))$$

$$\cos(\arcsin(s)) = \sqrt{1 - \sin^2(\arcsin(s))} = \sqrt{1-s^2}$$

$$g(s) = \tan(\arcsin(s))$$

$$\tan(\arcsin(s)) = \frac{\sin(\arcsin(s))}{\cos(\arcsin(s))} = \frac{s}{\sqrt{1-s^2}}$$

(b) Compute teh derivative of $\cos(\arcsin(s))$ using the chain rule and the derivatives of $\cos u$ and $\arcsin s$. Then compute the derivative of $f(s)$ using your expressions in part (a). Compare the two derivatives. Do the same for $\tan(\arcsin(s))$ and $g(s)$

$$f'(s) = -\sin(\arcsin(s))\frac{1}{\sqrt{1-s^2}} = -\frac{s}{\sqrt{1-s^2}}$$

$$f'(s) = -2s\frac{1}{2\sqrt{1-s^2}} = -\frac{s}{\sqrt{1-s^2}}$$

They are the same.

$$g'(s) = \sec^2(\arcsin(s))\frac{1}{\sqrt{1-s^2}} = \frac{1}{\cos^2(\arcsin(s))}\frac{1}{\sqrt{1-s^2}} = \frac{1}{(1-s^2)\sqrt{1-s^2}}$$

$$g'(s) = \frac{\sqrt{1-s^2} - s(-\frac{s}{\sqrt{1-s^2}})}{1-s^2} = \frac{\sqrt{1-s^2} + \frac{s^2}{\sqrt{1-s^2}}}{1-s^2} = \frac{\frac{1-s^2+s^2}{\sqrt{1-s^2}}}{1-s^2} = \frac{\frac{1}{\sqrt{1-s^2}}}{1-s^2} = \frac{1}{(1-s^2)\sqrt{1-s^2}}$$

They are also the same.

1.8

Use $x = \arcsin s$ *to show* $\int \cos^3 x dx = \sin(x) - \frac{\sin^3 x}{3}$.

$$s = \sin(x)$$

$$dx = \frac{1}{\sqrt{1-x^2}} ds$$

$$\int \cos^3(x) dx = \int \cos^3(\arcsin(s)) \frac{1}{\sqrt{1-s^2}} ds = \int (\sqrt{1-s^2})^3 \frac{1}{\sqrt{1-s^2}} ds =$$

$$\int 1 - s^2 ds = s - \frac{s^3}{3} = \sin(x) - \frac{\sin^3(x)}{3}$$

as desired.

1.9

(a) Write the microscope equation (i.e. the linear approximation) for $\phi(s) = \sqrt{s}$ *at* $s = 100$.

$$\phi'(s) = \frac{1}{2\sqrt{s}}$$

$$\phi'(100) = \frac{1}{2\sqrt{100}} = 1/20 = 0.05$$

Thus

$$\Delta\phi = 0.05\Delta s$$

(b) Use the microscope equation from part (a) to estimate $\sqrt{102}$ *and* $\sqrt{99.4}$

For the first one we've got that

$$\Delta s = |102 - 100| = 2$$

thus

$$\Delta\phi = 0.05 * 2 = 0.1$$

Therefore

$$\phi(102) \approx \phi(100) + 0.1 = 10 + 0.1 = 10.1$$

For the second one we've got

$$\Delta s = |100 - 99.4| = 0.06$$

thus

$$\Delta\phi = 0.06 * 0.05 = 0.03$$

and since $99.4 < 100$ we follow that

$$\phi(99.4) \approx \phi(100) - \Delta\phi = 10 - 0.03 = 9.97$$

(c) How far are your estimate from those given by a calculator?

$$\sqrt{102} \approx 10.0995049$$
$$\sqrt{99.4} \approx 9.969955$$

thus we've got error of approximately 10^{-3} in the first case and 10^{-4} in the second.

(d) Your estimates should be greater than the calculator values; use the graph of $x = \phi(s)$ to explain why this is so.

This is because derivative of this function is a decreasing function around 100.