Part I

Appendix: Mathematical Background

Chapter 1

Summations

A.1 Summation formulas and properties

A.1-1

Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$

Short answer:

$$\sum cg(x) = c \sum g(x)$$

Long answer:

Suppose that $g \in O(f_k(i))$. It follows that there exists n_i and c_i such that $0 \leq g(n) \leq cf_i(n)$. Thus we can pick $n = \max\{n_0, n_1, ...\}$ and $c = \max\{c_0, c_1, ...\}$. We know that both n and c will work all of functions f_k. Therefore by linearity of summations

$$\sum_{k=1}^{n} O(f_k(i)) = \sum_{k=1}^{n} cf_k(i) == c \sum_{k=1}^{n} f_k(i) == O(\sum_{k=1}^{n} f_k(i))$$

(notation is a little abused and there is nothing is rigorously proven, but it'll do).

A.1-2

Find a simple formula for $\sum_{k=1}^{n} (2k-1)$.

$$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} (2k) - \sum_{k=1}^{n} (1) = 2 \sum_{k=1}^{n} (k) - n = 2\frac{n(n+1)}{2} - n = n(n+1) - n = n^2$$

A.1-3

Interpret the decimal number $111,111,111$ *in light of equation A.6*

$$111,111,111 = \sum_{k=0}^{9} 10^k = \frac{10^{10}-1}{10-1}$$

A.1-4

Evaluate the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ...$

The series converges absolutely to 2, so we are free to do anything with it.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ... = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \sum_{k=0}^{\infty} \frac{1}{2}^{1+2k} = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2}^{2k} = \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{2}^{2k} =$$

$$= \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{4}^{k} = \left(1 - \frac{1}{2}\right)\frac{1}{1-\frac{1}{4}} = \frac{1}{2} * \frac{4}{3} = \frac{2}{3}$$

A.1-5

Let $c \geq 0$ *be a constant. Show that* $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$

We can follow that $\sum_{k=1}^{n} k^c \in O(n^{c+1})$ by the fact that $(\forall k \leq n \in Z^+, c \in (0,\infty))(k \leq n^c)$ and thus

$$\sum_{k=1}^{n} k^c \leq \sum_{k=1}^{n} n^c = n * n^c = n^{c+1}$$

thus

$$\sum_{k=1}^{n} k^c \in O(n^{c+1})$$

$$\sum_{k=1}^{n} k^c = \sum_{k=1}^{n-1} k^c + n^c = n^c \sum_{k=1}^{n} \frac{k^c}{n^c} =$$

Let $f(n) = n^c$. It can be seen from the graph that

$$\int_0^n f(x)dx \leq \sum_{k=1}^{n} k^c \leq \int_0^n f(x+1)dx$$

Thus

$$\int_0^n f(x)dx = \int_0^n x^c = \frac{n^{c+1}}{c+1} \in$$

$$\int_0^n f(x+1)dx = \int_0^n (x+1)^c = \frac{(n+1)^{c+1}}{c+1}$$

Thus we can state that $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$ (I'm not good enough yet to show that $\frac{(n+1)^{c+1}}{c+1} \in \Theta(n^{c+1})$, but I'm pretty sure that it's true TODO).

A.1-6

Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ *for* $|x| < 1$

We know that for $|x| < 1$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

thus if we differentiate both sides we get

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

and then if we multiply all of it by x we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

thus if we factor all of this jazz we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = -\frac{x(x+1)}{(x-1)^3}$$

and if we tuck this minus into denominator we'll get (which we can do because the power is odd)

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(x+1)}{(1-x)^3}$$

as desired.

A.1-7

Prove that $\sum_{k=1}^{n} \sqrt{k \lg k} = \Theta(n^{3/2} \lg^{1/2} n)$

$$\int \sqrt{k \lg k} =$$

TODO

A.1-8

Show that

$$\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$$

by manipulating the harmonic series

In the book we're reassured that

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

which is also pretty straightforward to follow if we think of the desired sum as the chopped integral of $\ln(n)$

We want to find the sum of reciprocals of odd numebers. Since $n \in Z_+$ is either odd or even, but not both, we follow that

$$\sum_{k=1}^{n} 1/(2k-1) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2}\frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}$$

and since

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

we follow that

$$\sum_{k=1}^{n} 1/(2k-1) = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2}(\ln(n) + O(1)) = \ln(n^{1/2}) + 1/2O(1) = \ln(\sqrt{n}) + O(1)$$

as desired (justification that $1/2O(1) = O(1)$ follows directly from the definition of O).

A.1-9

Show that

$$\sum_{k=0}^{\infty} (k-1)/2^k = 0$$

We can use standard series manipulations to get

$$\sum_{k=0}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} (k-1)/2^k = -1 + \sum_{k=2}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} k/2^{k+1} =$$

$$= -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k$$

We can also manipulate it differently to get

$$\sum_{k=0}^{\infty} (k-1)/2^k = \sum_{k=0}^{\infty} k/2^k - 1/2^k = \sum_{k=0}^{\infty} k/2^k - \sum_{k=0}^{\infty} 1/2^k = \sum_{k=0}^{\infty} k/2^k - 2 = \sum_{k=1}^{\infty} k/2^k - 2$$

Now assuming that the original sum converges we get an equation

$$\sum_{k=1}^{\infty} k/2^k - 2 = -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k$$

$$\sum_{k=1}^{\infty} k/2^k - \frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\sum_{k=1}^{\infty} k/2^k = 2$$

and by substituting the result into any of the previous results (I'll take the first) we get that

$$\sum_{k=0}^{\infty} (k-1)/2^k = -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k = -1 + 1 = 0$$

as desired.

A.1-11

Evaluate the product

$$\prod_{k=2}^{n} 1 - \frac{1}{k^2}$$

$$\prod_{k=2}^{n} 1 - \frac{1}{k^2} = \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \prod_{k=2}^{n} \frac{(k+1)(k-1)}{k^2} = \frac{\prod_{k=2}^{n}(k+1)\prod_{k=2}^{n}(k-1)}{\prod_{k=2}^{n}(k^2)} =$$

$$= \frac{\prod_{k=3}^{n+1} k \prod_{k=1}^{n-1} k}{(\prod_{k=2}^{n} k)^2} = \frac{\frac{1}{2} * 1 * 2 * \prod_{k=3}^{n} k * (n+1) * \frac{1}{n} * n * \prod_{k=1}^{n-1} k}{(1 * \prod_{k=2}^{n} k)^2} = \frac{\frac{1}{2} * \prod_{k=1}^{n} k * (n+1) * \frac{1}{n} * \prod_{k=1}^{n} k}{(1 * \prod_{k=2}^{n} k)^2} =$$

$$= \frac{\frac{1}{2} * (n+1) * \frac{1}{n} * (\prod_{k=1}^{n} k)^2}{(\prod_{k=1}^{n} k)^2} = \frac{1}{2} * (n+1) * \frac{1}{n} = \frac{1}{2n} + \frac{1}{2}$$

as desired.

A.2 Bounding summations

Chapter 2

Sets, Etc.

1-1

Draw Venn diagrams that illustrate the first of the distributive laws (B.1)

TODO, add picture here

1-2

Prove the generalization of DeMorgan's laws to any finite collection of sets

Copy from real analysis exercises

Suppose that $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. It follows, that x is not in the union of given sets. Therefore there is no set E_n such that $x \in E_n$ (because if there would be such a set, then x wouldn't be in $(\cup_{\lambda\in\Lambda} E_\lambda)^c$). Therefore $x \in \cap_{\lambda\in\Lambda} E^c_\lambda$. Therefore

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cap_{\lambda\in\Lambda} E^c_\lambda$$

The proof of reverse inclusion is the same as with the forward, but in reverse order.

$x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c$ implies that x is not in every E_n. Therefore there exists $x \in E^c_n$ for some E_n. therefore it is in $\cup_{\lambda\in\Lambda} E^c_\lambda$. The proof of reverse inclusion uses the same argument, but in other direction.

1-3

TODO

1-4

Show that the set of odd natural numbers is countable.

Let us set a function $f : A \to N$, where A denotes the set of odd natural numbers

$$f(n) = (n+1)/2$$

for this function we've got

$$f^{-1}(n) = 2n - 1$$

Both funcitons are injective and therefore f is bijective. Therefore we've got a bijective function betweeen A and N, therefore $A \sim N$, therefore it is conuntable, as desired.

1-5

Show that for any finite set S, the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

Another copy from real analysis

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary system: 0 for excluded, 1 for included. Therefore there exist 2^n possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of $\emptyset$ are $\emptyset$ itseft ($2^0 = 1$ in total). Subsets of set with one element are $\emptyset$ and set itself ($2^1 = 1$ in total).

Proposition is that set with n elements has 2^n subsets.

Inductive step is that for set with $n+1$ elements can either have or hot have the $n+1$'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

1-6

Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.

The tuple is actually just a re-writing of particular set

$$(a_1, a_2, ..., a_n) = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}...\{a_1, a_2, a_3, ..., a_n\}\}$$

Chapter 3

Counting and Probability

C.1 Counting

C.1-15

Show that for all integers $n \geq 0$

$$\sum_{k=0}^{n} C(n,k)k = n2^{n-1}$$

We can use the Gauss' argument for the sum of triangle numbers. Basically that

$$\sum_{k=1}^{n} = 1 + 2 + ... + n$$

implies that

$$2\sum_{k=1}^{n} = (1 + 2 + ... + n) + (n + (n-1) + ...2 + 1) = (n+1) + (n-1+2) + ... = n(n+1)$$

and thus

$$\sum_{k=1}^{n} = n(n+1)/2$$

We follow that

$$\sum_{k=0}^{n} C(n,k)k = 0C(n,0) + C(n,1) + 2C(n,2)... + nC(n,n)$$

thus

$$2\sum_{k=0}^{n} C(n,k)k = (0C(n,0) + C(n,1) + 2C(n,2)... + nC(n,n)) +$$

$$+(nC(n,n) + (n-1)C(n,n-1) + ... + 2C(n,2) + C(n,1) + 0C(n,0))$$

We know from properties of binomials that $C(n,k) = C(n,n-k)$ (which rigorously can be proven by the explicit function), and thus

$$2\sum_{k=0}^{n} C(n,k)k = (0C(n,0) + C(n,1) + 2C(n,2)... + nC(n,n))+$$

$$+(nC(n,0) + (n-1)C(n,1) + ... + 2C(n,n-2) + C(n,n-1) + 0C(n,0))$$

thus

$$2\sum_{k=0}^{n} C(n,k)k = nC(n,0) + nC(n,1) + nC(n,2)... + nC(n,n) =$$

$$n(C(n,0) + C(n,1) + C(n,2)... + C(n,n)) =$$

$$n(\sum_{k=0}^{n} C(n,k) = n2^n$$

therefore we can compress the whole shebang to get

$$2\sum_{k=0}^{n} C(n,k)k = n2^n$$

and thus

$$\sum_{k=0}^{n} C(n,k)k = n2^{n-1}$$

as desired.