

# My abstract algebra exercises

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# Preface

This is another yet another attempt at making any progress with abstract algebra, this time with 'Abstract Algebra: An Integrated Approach' by Joseph H. Silverman. I really hope that it works out this time.

So far, it's been most pleasurable journey. This book embodies all the things, that I like in the mathematics books: lots of rigor and a bit of lightheartedness, that really lights it up.

After some time with this book my opinion changed a bit in a worse direction due to an increasing number of typos that I have to deal with.  $\subseteq$  and  $\subset$  are often mixed up, I've encountered an exercise (2.30(c)), whose whole text is one big typo, and some others. On a brighter note, nothing seems to be as messed up as an unprovable exercise from the Lovett's book, and every typo is kinda handled by the context.

Also, the exercise numbering in this book is a tad bit messed up: it enumerates all the exercises for a given chapter in one sequence without splitting on the section of a chapter. I don't bother translating them to the normal format, and by extension some references in the text of the exercise is a bit messed up.

Some of the notation migrated from my previous endeavours in the maths. Notably for permutation groups I often use a cyclic representation, where I do not skip identities. Some of the notation comes from the book, notable examples of this notation are:

$\langle g \rangle$ - a cyclic group that is generated by  $g$

also it is not explicitly defined, but  $\cong$  is presumed to mean isomorphism

I might mix up isometry and isomorphy here and there, but it's pretty clear from context what exactly do I mean.

Since it's an algebra book, and algebra has rings, that are denoted by  $R$ , and those are used in similar context with reals, that are also denoted by me by  $R$ , we can see that there's a need to differentiate between the two. Thus approximately halfway through the chapter about rings I've started using  $\mathbb{R}$  for reals.

## Chapter 1

# A Potpourri of Preliminary Topics

*All of the topics, discussed in this chapter I know already; skip*

## Chapter 2

# Groups - Part 1

### Notes

I'm gonna use the symbol  $*$  as the generic group function and  $e$  as an identity until stated otherwise since it's most convenient to me. Sometimes I'll omit  $*$  whenever it's clear what's going on. Also, sometimes I omit parenthesis, but given that we've got associativity, we can omit them without problems. For rigorousness' sake, I'll define it to mean left-associative (i.e.  $a * b * c = (a * b) * c$ ).

## 2.1 Introduction to Groups

### 2.1.1

By substituting shapes for numbers we get a trivial exercise

### 2.1.2

*Let  $n$  be a positive integer, and let  $S_n$  be the group of permutations of the set  $\{1, 2, \dots, n\}$  as described in Example 2.19. Prove that  $S_n$  is a finite group, and give a formula for the order of  $S_n$ .*

From the combinatorics we know that the number of permutations is exactly the factorial of the cardinality of the underlying set.

### 2.1.3

*(a) Let  $S$  be a finite set, and let  $\phi : S \rightarrow S$  be a function. Prove that the injectivity, surjectivity, and bijectivity of this function are equivalent*

I'm pretty sure that we've proven that rigorously in the set theory course. If not, then the proof comes from contradiction and the cardinality of the codomain of, which proves

that injectivity and surjectivity are equivalent, and bijectivity comes from definition.

(b) Give an example of an infinite set  $S$  and a function  $\phi : S \rightarrow S$  such that  $\phi$  is injective but not surjective

We can let  $S = \omega$ ,  $\phi(x) = 2x$ , which gives us range of even numbers.

(c) Give an example of an infinite set  $S$  and a function  $\phi : S \rightarrow S$  such that  $\phi$  is surjective but not bijective

We can set  $S = \omega$  and

$$\phi(x) = \begin{cases} x = 0 \rightarrow 1 \\ x - 1 \text{ otherwise} \end{cases}$$

#### 2.1.4

*This one involves drawing and is pretty trivial; skip*

## 2.2 Abstract Groups

### 2.2.1

Let  $G$  be a group. In this exercise you will prove the remaining parts of Proposition 2.9. Be sure to justify each step using the group axioms or by reference to a previously proven fact

(a)  $G$  has exactly one identity element.

Suppose that  $e_1$  and  $e_2$  both satisfy identity axiom. We follow that both of them are in  $G$  and thus

$$e_1 = e_1 e_2 = e_2 e_1 = e_2$$

which comes directly from the Identity Axiom.

(b)  $g, h \in G \Rightarrow (g * h)^{-1} = h^{-1} g^{-1}$

We follow that

$$(g * h)^{-1} (g * h) = e$$

by definition of identity. Thus

$$(g * h)^{-1} (g * h) * h^{-1} = e h^{-1}$$

$$(g * h)^{-1} (g * h) * h^{-1} g^{-1} = e h^{-1} g^{-1}$$

by the fact that  $*$  is a binary function. Thus

$$(g * h)^{-1} g * (h * h^{-1}) * g^{-1} = e h^{-1} g^{-1}$$

$$(g * h)^{-1} g * g^{-1} = e h^{-1} g^{-1}$$

$$(g * h)^{-1} (g * g^{-1}) = e h^{-1} g^{-1}$$

$$(g * h)^{-1} = eh^{-1}g^{-1}$$

by associative laws, and then we've got that

$$(g * h)^{-1} = h^{-1}g^{-1}$$

by the identity axiom, as desired

$$(c) \ g \in G \Rightarrow (g^{-1})^{-1} = g$$

We follow that

$$(g^{-1})^{-1} * (g^{-1}) = e$$

by the inverse axiom. Thus

$$(g^{-1})^{-1} * (g^{-1}) * g = e * g$$

$$(g^{-1})^{-1} * (g^{-1}) * g = g$$

by identity and properties of functions. Thus

$$(g^{-1})^{-1} * ((g^{-1}) * g) = g$$

$$(g^{-1})^{-1} * e = g$$

$$(g^{-1})^{-1} = g$$

by associativity and so on, as desired.

### 2.2.2

Let  $G$  be a group, let  $g, h \in G$ , and suppose that  $g$  has order  $n$  and that  $h$  has order  $m$ .

(a) If  $G$  is an abelian group and if  $\gcd(m, n) = 1$ , prove that the order of  $gh$  is  $mn$ .

Firstly, I want to follow a couple of things:

If order of  $g$  is  $m$ , and order of  $g^{-1}$  is not  $n < m$ , then

$$e = e * e = g^m(g^{-1})^n = g^{m-n} = e$$

which is a contradiction. Similar case holds for  $n > m$ . Thus if order of  $g$  is  $n$ , then order of  $g^{-1}$  is also  $n$ .

We also follow that if  $g$  has finite order  $n$ , then  $(g^k)^n = (g^n)^k = e$ , and thus  $g^k$  has finite order for any  $k \in \mathbb{Z}$ . Moreover, since  $(g^k)^n = e$  we follow that  $g^k$ 's order divides  $n$ .

We also follow that if  $g$  has finite order  $n$ , then

$$g * g^{n-1} = e$$

$$g^{n-1} = g^{-1}$$

Now back to our exercise. We follow that

$$(gh)^{mn} = g^{mn}h^{mn} = (g^n)^m(h^m)^n = e^m e^n = e$$

where we can split it this way since  $G$  is abelian. Thus we follow that order of  $gh$  is finite and divides  $mn$ .

Suppose that the order of  $gh$  is  $k$ . We follow that  $k \leq mn$  since it divides  $mn$ .

$$(gh)^k = e$$

thus

$$(gh)^k = g^k h^k = e$$

if  $g^k \neq e$ , then we follow that  $h^k = (g^k)^{-1} \neq e$ , thus  $h^k = (g^{-1})^k = (g^{n-1})^k$ . Thus we follow that  $h^k$  is a multiple of  $g$ , and thus its order divides order of  $g$   $m$ . Since  $h^k$  is both multiple of  $g$  and  $h$ , we follow that its order divides both  $m$  and  $n$ , and since  $\gcd(m, n) = 1$ , we follow that its order is 1. Thus  $h^k = e$ , which is a contradiction.

Thus we conclude that  $g^k = e$ . For  $h^k$  we've got a similar case. Thus we follow that  $k$  divides both  $m$  and  $n$ , and thus it's either 1 or  $mn$ . If  $k = 1$ , then we follow that  $g = h^{-1}$ , and thus  $\gcd(m, n) = 1$  implies that the order of both  $g$  and  $h$  cannot be anything other than 1, and thus  $k = mn = 1$ . If  $k \neq 1$ , then we follow that  $k = mn$ , as desired.

(b) Give an example showing that (a) need not be true of we allow  $\gcd(m, n) > 1$

We can have some group where order of  $g$  is  $n > 1$  (for example  $g = 1$  in  $G = \mathbb{Z}/5$ ) and set  $h = g^{-1}$ , for which we'll have that order of  $gh$  is 1.

(c) Give an example of a nonabelian group showing that (a) need not be true even if we retain the requirement that  $\gcd(m, n) = 1$ .

A dihedral group of a triangle with  $g = r$  and  $h = f$  will do. We'll have that order of  $rf$  is 2:

$$(rf)^2 = (rf)(rf) = f^{-1} r r f = e$$

with order of  $g$  being 2 and order of  $h$  being 3

(d) Again assume that  $G$  is an abelian group, and let  $l = mn/\gcd(m, n)$  (i.e.  $l = \text{lcm}(m, n)$ ). Prove that  $G$  has an element of order  $l$ .

We follow that  $n$  divides order of  $g^m$ . We also follow that  $m$  divides order of  $h^n$ . Since  $l = \text{lcm}(m, n)$  is divided by both  $m$  and  $n$  we follow that  $g^m h^n$ 's order divides  $l$ .

If there's  $k \leq \text{lcm}(m, n)$  such that

$$(g^m h^n)^k = e$$

then we follow that

$$g^{mk} h^{nk} = e$$

Here we're gonna employ a more generalized version of an argument in part (a): If  $g^{mk} \neq e$ , then  $k < \text{lcm}(m, n)$ , thus  $h^{nk}$ 's order is a multiple of both  $m$  and  $n$ .  $h^{nk}$  cannot be 1, and its order must be then  $\text{lcm}(m, n)$  or 1, and it can be neither, thus we've got a contradiction. Thus we conclude that  $g^{mk} = h^{nk} = 1$ , which implies that  $k$  divides both  $m$  and  $n$ , which implies that  $k = \text{lcm}(m, n)$ , as desired.

## 2.2.3

Definition 2.6 says that a group is a set  $G$  with a composition law satisfying three axioms. In particular, it says that there's an identity element  $e \in G$  that works on both sides and that every element  $g \in G$  has an inverse that works on both sides. Suppose that we weaken the requirements to specify that the identity and inverse work only on one side. In other words, we suppose that  $G$  is a set with a composition law satisfying the following weaker axioms:

- (a) (Right-Identity Axiom) There is an element  $e \in G$  so that  $g * e = g$  for all  $g \in G$
- (b) (Right-Inverse Axiom) For all  $g \in G$  there is an element  $h \in G$  so that  $g * h = e$
- (c) (Associative Law)  $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$  for all  $g_1, g_2, g_3 \in G$ .

Prove that  $G$  is a group.

We're gonna start with establishing the inverse axiom for  $G$ , as suggested in the hint to the exercise. Suppose that for  $g \in G$  there's  $h \in G$  such that

$$g * h = e$$

we also follow that for  $h$  there's an element  $k$  such that  $h * k = e$  by the same axiom. We thus follow that

$$h * g = h * g * e = h * g * h * k = h * (g * h) * k = h * e * k = (h * e) * k = h * k = e$$

where we use justification of

$$a \rightarrow (h * k = e) \rightarrow c \rightarrow b \rightarrow c \rightarrow a \rightarrow (h * k = e)$$

in our equalities (given axioms are presented by letters). Thus we've got that  $h * g = e = g * h$  (i.e. normal inverse axiom)

Now suppose that  $g \in G$ . We follow that

$$e * g = g * g^{-1} * g = g * (g^{-1} * g) = g * e = g$$

which practically establishes the Identity axiom. The associative law is unchanged from the standard definition of the group, and thus we're following that  $G$  is indeed a group, as desired.

## 2.2.4

There are other sorts of algebraic structures that are similar to groups in that they are sets  $S$  that have a composition law

$$S \times S \rightarrow S, (s_1, s_2) \rightarrow s_1 * s_2$$

but they have fewer or different axioms than a group. In this exercise we explore two of these structures.

*The set  $S$  with its composition law is a monoid if it has an identity element  $e \in S$  and satisfies the associative law, but elements are not required to have inverses.*

*The set  $S$  with its composition law is a semigroup if its composition law is associative, but it need not have an identity element or inverses.*

i.e. monoid satisfies associative and identity, semigroup satisfies associative, and group satisfies all of them. Hence we've got nested classes of structures:

semigroup  $\subseteq$  monoid  $\subseteq$  group

*For each of the following sets  $S$  and composition laws  $*$  determine if  $(S, *)$  is a group, a monoid, or a semigroup.*

(a) *The set of natural numbers  $N = \{1, 2, 3, \dots\}$  with the composition law being addition.*

I usually do include 0 into the set of naturals, but in this particular case it seems that we don't do it.

In this particular case we follow that  $N$  has associativity with  $+$ , rigorous proof of which comes from the construction of the naturals. In the course of the set theory I've gone through that proof, but I'm pretty sure that it's not required here.

If we don't include 0 into  $N$ , then we don't have an identity in  $N$ , which can be proven by defining order  $>$  on  $N$  and proceeding from there (which was also handled in the course on set theory). Hence it is only a semigroup. If we include it, however, then we get an identity, and thus this set becomes a monoid.

We can also state that it's not got identities.

(b) *The set of extended natural numbers  $N_0 = \{0, 1, 2, 3, \dots\}$  with the composition law being addition.*

handled in part (a)

(c)  $(\mathbb{Z}, +)$

We can follow that it's a superset of  $N$ , which gets its inverses ( $a^{-1} = -a$ ) and hence becomes a full-blown group.

(d)  $(N, *)$

We follow that it's associative, has the identity 1, and hence is at least a monoid. It's not got multiplicative inverses (because that would be  $\mathbb{Q}_+$ ), and hence is not a group.

(e)  $(N_0, *)$

Same as previous, 0 does not change associativity and identities. Is not a group for the same reason as the previous case.

(f)  $(\mathbb{Z}, *)$

Same as previous for the same reason.

(g) *The set of integers  $\mathbb{Z}$  with the composition law  $m * n = \max\{m, n\}$*

If there's  $m \in \mathbb{Z}$ , then there's  $n_0, n_1 \in \mathbb{Z}$  such that

$$n_0 < m < n_1$$

and hence

$$n_0 * m = n_0, n_1 * m = m$$



and hence we follow that there's no identity.

We follow that

$$(n * m) * k = \max\{\max\{n, m\}, k\} = \max\{n, m, k\} = \max\{n, \max\{m, k\}\} = n * (m * k)$$

hence we've got associativity. Thus the given structure is a semigroup.

(h) *The set of naturals  $N$  with the composition law  $m * n = \max\{m, n\}$*

It's got associativity and hence this thing is a semigroup. We can follow that  $1 \leq m$  for all  $m \in N$ , and thus  $\max\{1, m\} = m$ . Thus we follow that we also have a monoid. Inverses are not present, and thus it's not a group.

(i) *The set of naturals  $N$  with the composition law  $m * n = \min\{m, n\}$*

We've got associativity. If  $m \in N$ , then there's  $n \in N$  such that  $n > m$ , and thus  $\min\{m, n\} = m$ , which means that there's no identity, and hence this thing is only a semigroup.

(j) *The set of naturals  $N$  with the composition law  $m * n = mn^2$*

We follow that

$$(m * n) * k = (mn^2) * k = mn^2k^2$$

$$m * (n * k) = m * nk^2 = mn^2k^4$$

setting  $m, n, k$  to primes we can follow that  $(m * n) * k \neq m * (n * k)$ , which implies that this thing is neither a group, a monoid, nor a semigroup.

## 2.2.5

*Look up magmas, Moufang loops, quandles, and matroids*

Magma is just a set with a binary function. There are some discussions about closure, but as long as the thing provided with a set is a binary function, it's a magma.

TODO: look up the rest

## 2.3 Interesting Examples of Groups

### Notes

Although I'm pretty sure that the collection of groups is a proper class (i.e. not a set), I don't have a proof for that. I'll try to change it now.

Empty sets cannot be groups since they've gotta have an element (identity).

Although we can try to do something with cardinals and whatnot, we know that for any set  $S$  there's an injection to the set  $S^S$ . We then follow that there's a subset of bijections in  $S^S$ . We then follow that this subset of bijections is a group (permutations), and hence we conclude that for each set there's a group, and thus a collection of groups is a proper class, as desired.

We can also skip all this stuff, and state singleton of any set  $S$  is a trivial group under projection. Trivial group (if I'm not mistaken) is a group that's got only an identity in it.

Permutation Group practically states that a set of bijections over a set constitutes a group under composition. There are no further restrictions, which is pretty neat.

Matrix group leads us to an interesting idea, which is also bleeding into dihedral groups: as long as a subset of the set of bijections is closed under composition, includes inverses, and includes an identity, given subset is a group (associativity is a property of composition). Can we lose some restrictions though?

Suppose that there's a set of bijections over a set and it is closed under composition. If a set is finite, then we can follow that the set of pairs of bijections is larger than the set of bijections, hence there's got to be a pair

$$S \circ C = S$$

or something like that, which would imply that  $C$  is an identity. If the set is infinite, then this proof will not do. Finality of the set does not imply the existence of inverses:  $\{C, e\}$  will be closed under composition, will have an identity, and will not have inverses.

Given a set of nonzero naturals  $N$ , for each  $n \in N$  we can have  $S_n : \omega \rightarrow \omega$

$$S_n(x) = x + n$$

For each  $n, m \in N$  we've got that

$$(S_n \circ S_m)(x) = S_n(x + m) = x + m + n = S_{m+n}(x)$$

which gives us a set of bijections, that is closed under a composition, does not include an identity ( $0 \notin N$  by our restriction), and no element has an inverse.

Thus we conclude that if a set of bijections is finite and is a singleton, then it's a group. If it's not a singleton and is finite, then it's a monoid (i.e. associativity and identity). If it's not finite, then it's just a semigroup (i.e. associativity exclusively).

We can't have a group of functions, where the set is not a singleton and domain and range of the function are distinct, since two functions are supposed to compose. We can't have non-bijections be present, since we've got to have both left-hand side and right-hand side inverses of any given element.

### 2.3.1

*Let  $G$  be a finite cyclic group of order  $n$ , and let  $g$  be a generator of  $G$ . Prove that  $g^k$  is a generator of  $G$  if and only if  $\gcd(k, n) = 1$ .*

Suppose that  $g^k$  is a generator of  $G$ . If  $k = 1$ , then we follow that  $\gcd(k, n) = \gcd(1, n) = 1$ , thus assume that  $k \neq 1$ .

We follow that there's  $m \in \omega$  such that  $m \neq 0$  and

$$(g^k)^m = g^{km} = g$$

thus

$$g^{km-1} = e$$

and hence we follow that  $km - 1$  is multiple of order of  $G$ . Thus there exists  $j \in Z$  such that  $km - 1 = jn$ . Thus  $km - jn = 1$ , which implies that  $\gcd(k, n) = 1$ , as desired.

Every implication in the forward direction is pretty much a biconditional (not exactly though, we need to add some quantifiers to the mix), so it works in the reverse direction as well.

Also, we haven't gone into a proof that  $\gcd(m, n) = l$  if and only if  $l$  is the lowest positive number such that there exist  $i, j \in Z$  such that

$$mi + jn = l$$

but we've essentially proven the result for  $\gcd(m, n) = 1$  in 1.35

### 2.3.2

*Skip*

### 2.3.3

*Prove that the Dihedral group  $D_n$ , as described in Example 2.22, has exactly  $2n$  elements*

We can follow that there are  $n$  vertices, and each of them gotta go to one of the other  $n$  spaces. There are  $n$  ways to put the first vertex into any of those places. Whenever we put the first vertex  $n$  into some place, there are 2 places where the second vertex can go, so now we're down to  $2n$  possible positions. Whenever we put the second vertex down, positions of the rest are determined, and hence we can follow that there are total of  $2n$  possible positions, as desired.

### 2.3.4

*(a) Let  $Q^*$  be the set of non-zero rational numbers, with the group law being multiplication. Prove that  $Q^*$  is a group.*

We follow that  $1 * q = q$ , and hence we've got an identity. If  $q \in Q^* \Rightarrow q \neq 0 \wedge q \in Q$ , then we follow that  $1/q$  exists, and thus there's  $q^{-1}$  such that  $qq^{-1} = 1$ . Multiplication's associativity does not require a proof (or can be provided, but in this course it is assumed), and thus we conclude that  $Q^*$  is a group (moreover, an abelian group), as desired.

*the rest is skipped*

### 2.3.5

*(b) Let  $p$  be a prime number. Prove that the set of non-zero elements of  $Z/pZ$  is a group using multiplication as the group law*

We follow that for each  $i \in \mathbb{Z}/p\mathbb{Z}$ ,  $i * 1 = i$ , and thus  $i$  is our identity. Associativity of multiplication under  $\mathbb{Z}$  implies our associativity.

From group theory we know that there's  $x$  such that  $ax \equiv 1 \pmod{m}$  if and only if  $\gcd(a, m) = 1$ . We then follow that for any given  $i \in \mathbb{Z}/p\mathbb{Z}$  we've got that  $\gcd(i, p) = 1$  since  $p$  is prime, and thus there's  $a \in \mathbb{Z}/p\mathbb{Z}$  such that  $a * i = 1$ . Thus we've got the inverse, and  $\mathbb{Z}/p\mathbb{Z}$  is a group under multiplication, as desired.

We can probably also prove the other direction if we want. If  $p$  is not 1 and is not a prime, then there's a divisor  $q$  of  $p$  in  $\mathbb{Z}/p\mathbb{Z}$ , which implies that only multiple of  $q$  is its multiple, or 0, which implies that it's got no inverse and hence  $\mathbb{Z}/p\mathbb{Z}$  is not a group, as desired. If  $p$  is not a prime, however, then we've got that  $\mathbb{Z}/p\mathbb{Z}$  is a trivial group. We can probably reword this thing so it becomes a theorem, but I'm too lazy to do this.

(c) Heck, I've proven this already and I didn't even read this section

(d) blah blah blah

Got pretty much the same proof as part (a). The only thing of note is that we've got new notation:

$$(\mathbb{Z}/p\mathbb{Z})^*$$

which is a group under multiplication of numbers relatively prime to  $p$ .

### 2.3.6

Let  $C$  be the set of complex numbers, that is, the (... a crude definition of complex numbers)

(a) We make  $C$  into ...

It's a group under addition, yes. Proof is trivial

(b)  $C^* = C \setminus \{0\}$  is a group under multiplication.

Yes it is. Multiplicative inverse is kinda whacky in complex numbers, let me find it:

$$\frac{1}{a+bi} = \frac{(a-bi)}{(a+bi)(a-bi)} = \frac{(a-bi)}{a^2-b^2i^2} = \frac{(a-bi)}{a^2-b^2*(-1)} = \frac{(a-bi)}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

which I would never ever find on my own, if I wouldn't start a complex analysis course at some time in the past (probably should look into it not, this thing looks pretty fun)

### 2.3.7

(a) Let

$$GL_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc \neq 0 \right\}$$

be the indicated set of 2-by-2 matrices, with composition law being matrix multiplication. Prove that  $GL_2(R)$  is a group.

It's a set of bijections, that is closed under composition (see linear algebra course), each element is invertible by the restriction, and has identity, thus it's a group. We can also

scale this thing to complex numbers, and we can also increase dimensions of the underlying sets and substitute restriction from the determinant-based to just being invertible (which are equivalent).

(b) Let

$$SL_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc = 1 \right\}$$

be the indicated set of 2-by-2 matrices, with composition law being matrix multiplication. Prove that  $SL_2(R)$  is a group.

Same logic as before. We can scale with determinants here as well.

This group does not include all the isometries though, which is kinda interesting. Some isometries (e.g. 1, -1 on diagonal) will not be included here. TODO: research this thing a bit more

(c) ...

Pretty much is solved by my discussion in previous points. A thing to remember though:

**General** linear group is a set of invertible linear functions.

**Special** linear group is a set of matrices with 1 for the determinant.

### 2.3.8

Let  $GL_2(R)$  be the general linear group. Prove or disprove that each of the following subsets of  $GL_2(R)$  is a group. In the case of non-groups, indicate which of the group conditions fail.

(a)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : a, b, c, d \in R, ad - bc = 2 \right\}$$

Not a group, identity is not in there, and composition is not closed ( $|AB| = |A||B| = 2 * 2 = 4$ )

(b)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : a, b, c, d \in R, ad - bc \in \{1, -1\} \right\}$$

Is a group, pretty sure that those describe the isometries.

(c)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : a, b, c, d \in R, c = 0 \right\}$$

Is a set of upper-triangulars, which are closed under composition, and contain the identity. Inverses of upper-triangular are also upper-triangular, thus we've got a group.

(d)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : a, b, c, d \in R, d = 0 \right\}$$

Does not contain the identity, and inverses are not present

(e)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : a, b, c, d \in R, a = d = 1, c = 0 \right\}$$

Pretty sure that this once is a group as well.

### 2.3.9

Let  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$  be the group of quaternions. We claimed that the group law for  $Q$  is determined by

$$i^2 = j^2 = k^2 = ijk = -1$$

Use these formulas to prove following formulas, which completely determine the group operations on  $Q$ :

We first follow that  $(-1)^{-1} = -1$ . Thus

$$i * i = -1$$

$$(i * i)^{-1} = (-1)^{-1}$$

$$i^{-1} * i^{-1} = -1$$

$$i^{-1} = -1 * i$$

$$i^{-1} = -i$$

where the last part comes from the note that  $-1$  commutes with everything. We can follow that  $k^{-1} = -k$  and  $j^{-1} = -j$  by the same logic.

We follow now that

$$i * j = i * j * 1 = i * j * (k * -k) = (i * j * k) * -k = -1 * -k = k$$

$$j * k = 1 * j * k = (-i * i) * j * k = -i * -1 = i$$

$$k * i = 1 * k * i = -j * j * k * i = -j * (j * k) * i = -j * i * i = -j * -1 = j$$

we then follow that

$$-k = k^{-1} = (i * j)^{-1} = -j * -i = j * i$$

and so on for all the inverses.

### 2.3.10

Skip

**2.3.11**

*Practically continue with the functions in the notes. Part (a) is handled*

*(b) If  $X$  is a finite set with  $n$  elements, Prove that  $\mathcal{E}_X$  is a finite monoid and compute how many elements it has.*

We follow that the number of functions is  $n^n$ , the rest is handled in the notes

*(c) If  $|X| \geq 3$ , prove that  $\mathcal{E}_X$  is not commutative, i.e. show that there are elements  $\phi, \psi \in \mathcal{E}_X$  satisfying  $\phi \circ \psi \neq \psi \circ \phi$ .*

We can map a portion of this set to 1, 2, 3, and then get functions

$$\phi(x) = \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 2 \\ 3 \rightarrow 2 \end{cases}$$

$$\psi(x) = \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3 \\ 3 \rightarrow 3 \end{cases}$$

which will suffice.

**2.4 Group Homomorphism****2.4.1**

*Recall that two groups  $G_1, G_2$  are said to be isomorphic if there is a bijective homomorphism*

$$\phi : G_1 \rightarrow G_2$$

*The fact that  $\phi$  is bijective means that the inverse map  $\phi^{-1} : G_2 \rightarrow G_1$  exists. Prove that  $\phi^{-1}$  is a homomorphism from  $G_2$  to  $G_1$ .*

We essentially need to prove that

$$\phi^{-1}(h_1 * h_2) = \phi^{-1}(h_1) * \phi^{-1}(h_2)$$

for all  $h_1, h_2 \in G_2$ .

$\phi$  is a bijection, and thus for  $h_1, h_2 \in G_2$  there are  $g_1, g_2 \in G_1$  such that  $\phi(g_1) = h_1, \phi(g_2) = h_2$ . By the fact that  $\phi$  is a bijection we also follow that  $g_1 = \phi^{-1}(h_1), g_2 = \phi^{-1}(h_2)$ . Thus we've got that

$$\phi^{-1}(h_1 * h_2) = \phi^{-1}(\phi(g_1) * \phi(g_2)) = \phi^{-1}(\phi(g_1 * g_2)) = g_1 * g_2 = \phi^{-1}(h_1) * \phi^{-1}(h_2)$$

as desired

**2.4.2**

Let  $G$  be a group, and consider the function

$$\phi : G \rightarrow G, \phi(g) = g^{-1}$$

(a) Prove that  $\phi(\phi(g)) = g$  for all  $g \in G$

$$\phi(\phi(g)) = \phi(g^{-1}) = (g^{-1})^{-1} = g$$

(b) Prove that  $\phi$  is a bijection.

We follow that for each  $g \in G$  there's  $g^{-1} \in G$ , thus  $\phi(g^{-1}) = g$ , which implies that the range of  $\phi$  is  $G$ .

We follow that if  $g_1 \neg g_2$ , then  $g_1^{-1} \neq g_2^{-1}$ , and thus  $\phi$  is injective.

(c) Prove that  $\phi$  is a group homomorphism if and only if  $G$  is an abelian group.

**Forward direction:** We know that  $\phi$  is bijective, and thus for all  $h_1, h_2 \in G$  there are  $g_1, g_2 \in G$  such that  $\phi(g_1) = h_1, \phi(g_2) = h_2$ . If  $G$  is a group homomorphism, then we follow that

$$h_1 * h_2 = \phi(g_1)\phi(g_2) = \phi(g_1g_2) = (g_1g_2)^{-1} = g_2^{-1}g_1^{-1} = h_2 * h_1$$

thus the group is abelian.

**Reverse direction:**

Let us keep assumptions about our variables. If  $G$  is abelian, then we follow that

$$h_2 * h_1 = g_2^{-1}g_1^{-1} = (g_1g_2)^{-1} = \phi(g_1g_2)$$

$$h_1 * h_2 = g_1^{-1}g_2^{-1} = \phi(g_1) * \phi(g_2)$$

Since  $G$  is abelian we follow that  $h_1h_2 = h_2h_1$ , and thus we can equate given equalities to get the desired result.

**2.4.3**

Skip

**2.4.4**

Let  $G_1$  and  $G_2$  be groups, and suppose that  $\phi : G_1 \rightarrow G_2$  is an isomorphism

(a) Prove that if  $G_1$  is finite, then  $G_2$  is also finite, and that they satisfy  $|G_1| =_c |G_2|$

Isomorphism is a bijection, thus sets have equal cardinality.

(b) Suppose that  $G_1$  is abelian. Prove that  $G_2$  is abelian

Let  $g_1, g_2 \in G_2$ . Since  $\phi$  is a bijection we follow that there are  $h_1, h_2 \in G_1$  such that  $\phi(h_1) = g_1$  and  $\phi(h_2) = g_2$ . We follow that

$$g_1g_2 = \phi(h_1)\phi(h_2) = \phi(h_1h_2) = \phi(h_2h_1) = \phi(h_2)\phi(h_1) = g_2g_1$$



as desired.

*The solution for the rest of this exercise practically copies the structure of part (a)*

### 2.4.5

*In this exercise,  $C_n$  is a cyclic group of order  $n$ ,  $D_n$  is the  $n$ th dihedral group, and  $S_n$  is the  $n$ th symmetric group.*

*(a) Prove that  $C_2$  and  $S_2$  are isomorphic.*

We follow that both  $C_2$  and  $S_2$  have 2 and  $2! = 2$  elements respectively. Non-identity elements can interact with identity in a predictable way, or with themselves, where they've got an order of 2. This pretty much concludes the proof.

*(b) Prove that  $D_3$  is isomorphic to  $S_3$ .*

We know that  $D_3$  is a subgroup (although we haven't defined at this point what a subgroup is, it doesn't take a genius to guess) of  $S_3$ , with the same cardinality.

*(c) Let  $m \geq 3$ . Prove that for every  $n$ , the groups  $C_m$  and  $S_n$  are not isomorphic.*

If  $m \neq n!$ , then we follow that groups have different cardinalities, thus there's no bijection between the two, and thus no isomorphisms.

If  $m = n!$ , then we follow that since  $m \geq 3$  that  $n \geq 3$  as well. We follow that there are elements of  $S_n$  that map  $(1, 2, 3)$  to  $(1, 3, 2)$  and  $(2, 1, 3)$  and the rest to identity. Composition of those elements is not commutative ( $(2, 3, 1)$  and  $(3, 1, 2)$  in forward and reverse respectively), and thus  $S_n$  is not abelian.  $C_n$  on the other hand is commutative, and thus we follow that they aren't isomorphic. This also proves that  $S_n$  is non-abelian for any  $n \geq 3$ .

*(d) Prove that for every  $n \geq 4$  the groups  $D_n$  and  $S_n$  aren't isomorphic*

They don't have the same cardinalities.

*(e) More generally, let  $m \geq 4$ , and let  $n \geq 4$ . Prove that the groups  $D_m$  and  $S_n$  aren't isomorphic.*

We once again look at the case when  $2m = n!$ . We probably want to find some irregularities with orders of the elements here. We follow that if  $m$  is odd, then there is only 1 element of order 2 ( $f$ ). If  $m$  is even, then we've got 3 such elements ( $f, r^{m/2}, fr^{m/2}$ ). For  $S_n$  for  $n \geq 4$  we follow that there are at least 6 of them (bijections that swap elements, there are  $C(4, 2) = 6$  of them at least).

*(f) The cyclic group  $C_8$ , the dihedral group  $D_4$ , and the quaternion group  $Q$  are non-abelian groups of order 8. Prove that they aren't isomorphic.*

First of all,  $C_8$  is abelian (as is any cyclic group for that matter). If we ignore that, then we can follow that there is one element of order 4 (2) in  $C_8$ , and there are 6 ( $\pm i, \pm j, \pm k$ ) in  $Q$ . There are 3 elements of order 2 in  $D_4$ , but only one in  $C_8$ .  $Q$  has 1 element of order 2.

## 2.4.6

Skip

## 2.5 Subgroups, Cosets, and Lagrange's Theorem

## Notes

Important definition from the exercises: Let  $G$  be a group, and let  $S \subseteq G$  be a subset of  $G$ . The subgroup of  $G$  generated by  $S$ , which we denote by  $\langle S \rangle$ , is the intersection of all of the subgroups of  $G$  that contains  $S$ ; i.e.

$$\langle S \rangle = \bigcap_{S \subseteq H \subseteq G: H \text{ is a subgroup of } G} H$$

Exercise proves that  $\langle S \rangle$  is essentially a subgroup, where every element is a multiple of some combination (i.e. ordered list in non-abelian and multiset in abelian) of elements of  $S$  or their inverses, and also the smallest subgroup, that contains the entirety of  $S$ .

Another important definition is the definition of centralizer. The centralizer of the subset  $(Z_G(S))$  is the most general of them, they are provided below

We also follow that if  $H_1, H_2$  are subgroups of  $G$ , then  $H_1 \cap H_2$  is a subgroup of  $G$ , which is sort of trivial to prove. Important to note that union of two subgroups is not necessarily a group.

## 2.5.1

Let  $G$  be a group and let  $g \in G$  be an element of order  $n$ , and let  $k \geq 1$ .

(a) Prove that  $g^k$  has order  $n/\gcd(n, k)$ .

We follow that  $g$  is a generator for a cyclic group, in which multiples of  $g^k$  comprises a subgroup. Since  $g$  has order  $n$  we follow that  $\langle g \rangle$  is a cyclic group of order  $n$ .

We know that  $\gcd(n, k)$  is the lowest number that is a positive sum of multiples of  $n$  and  $k$  respectively. Thus there are integers  $a, v$  such that

$$an + vk = \gcd(n, k)$$

We then follow that

$$(g^n)^a * (g^k)^v = g^{\gcd(n, k)}$$

order of  $g$  is  $n$  and thus

$$(g^n)^a * (g^k)^v = (e)^a * (g^k)^v = e * (g^k)^v = (g^k)^v = g^{\gcd(n, k)}$$

We thus follow that multiples of  $g^{\gcd(n, k)}$  are in cyclic subgroup of  $g^k$ . There are exactly  $n/\gcd(n, k)$  such elements.

Suppose that  $v$  is not a multiple of  $\gcd(n, k)$  and suppose that there's  $j \in Z_+$  such that  $(g^k)^j = g^{k*j} = g^v$ . Since  $v$  is not a multiple of  $\gcd(n, k)$ , it is not a multiple of  $k$ . Thus we follow that  $k * j > n$ . We thus conclude that there's a maximal  $h \in Z_+$  such that

$$kj - hn = v$$

$$kj + (-h)n = v$$

thus we conclude that  $v$  is a multiple of  $\gcd(n, k)$ , which is a contradiction.

(b) Use (a) to give a quick proof of Exercise 2.10, which says that  $G = \langle g \rangle$  is a cyclic group of order  $n$ , then  $g^k$  generates  $G$  if and only if  $\gcd(n, k) = 1$

We follow that if  $g^k$  generates  $G$ , then  $|\langle g^k \rangle| = n/\gcd(n, k) = n$ . Thus  $\gcd(n, k) = 1$ . Reverse case is pretty much the same.

### 2.5.2

Let  $G$  be a cyclic group of order  $n$ , and let  $d \geq 1$  be an integer.

(a) Prove that every subgroup of  $G$  is cyclic.

Since  $G$  is cyclic we follow that there's  $g \in G$  that generates  $G$ .

Suppose that  $H$  is a subgroup of  $G$ . If  $g \in H$ , then  $G = H$ , and thus we're done. Thus suppose that  $g \notin H$ . We then follow that there is a lowest possible  $i \in Z_+$  such that  $g^i \in H$ . We follow that every multiple of  $i$  is in  $H$ . Suppose that  $k \in Z_+$  is not a multiple of  $i$  and such that  $g^k \in H$ . We follow that  $\gcd(k, i) < i$  since  $k$  is not a multiple of  $i$ . Thus we follow that there are  $a, v \in Z$  such that

$$ak + vi = \gcd(k, i)$$

and thus

$$g^{ak} + g^{vi} = g^{\gcd(k, i)}$$

thus  $g^{\gcd(k, i)} \in H$ , which implies that  $i$  is not the lowest element of  $Z_+$  such that  $g^i \in H$ , which is a contradiction. Thus we conclude that multiples of  $i$  are the only elements of  $H$ , which implies that  $H$  is cyclic, as desired.

Important note: this argument also works for groups  $G$ , and so we can follow that any subgroup of any cyclic group is cyclic.

(b) If  $d$  divides  $n$ , prove that  $G$  has a unique subgroup of order  $d$ .

Let  $H_1$  and  $H_2$  be two subgroups of order  $d$ . We follow that for both of there are least elements  $i_1, i_2$  of  $Z_+$  such that  $g^{i_1} \in H_1$   $g^{i_2} \in H_2$ . Suppose that  $i_1 \neq i_2$  and assume that  $i_1 < i_2$ . Previous arguments in (a) imply that  $g^{i_1}$  and  $g^{i_2}$  generate  $H_1$  and  $H_2$ . We know also that  $H_1$  is comprised entirely of powers of  $g^{i_1}$ . If  $\gcd(i_1, d) \neq i$ , we follow that  $\gcd(i_1, d) < i$ , and thus there are  $a, v \in Z$  such that

$$(g^{i_1})^a * (g^d)^v = g^{\gcd(i_1, d)}$$

$$\begin{aligned}(g^{i_1})^a * (e)^v &= g^{gcd(i_1, d)} \\ (g^{i_1})^a &= g^{gcd(i_1, d)}\end{aligned}$$

We thus follow that  $g^{gcd(i_1, d)} \in H_1$ , and thus  $i_1$  is not the lowest positive integer such that  $g^{i_1} \in H$ , which is a contradiction. We thus conclude that  $gcd(i_1, d) = i_1$ , and thus  $i_1 | d$ . This implies that  $|H_1| = |\langle g^{i_1} \rangle| = d/i_1$ . Thus if  $i_1 \neq i_2$  we follow that  $|H_1| \neq |H_2|$ , which is a contradiction. Thus we conclude that  $i_1 = i_2$ , and thus  $H_1 = H_2$ , as desired.

(c) If  $d$  does not divide  $n$ , prove that  $G$  does not have a subgroup of order  $d$ .

We follow that number of cosets of  $G$  of order  $d$  is a natural number, and thus Lagrange's Theorem implies that order of any subgroup of  $G$  divides  $n$ , as desired.

### 2.5.3

Let  $G$  be a group, and let  $H \subseteq G$ . Prove that  $H$  is a subgroup if and only if it has the following properties:

- (1)  $H \neq \emptyset$
- (2) For every  $h_1, h_2 \in H$ ,  $h_1 * h_2^{-1} \in H$

Forward direction is covered by axioms.

Suppose that  $H \neq \emptyset$  and for every  $h_1, h_2 \in H$ ,  $h_1 * h_2^{-1} \in H$ . We follow that there's  $h \in H$ , and thus  $h * h^{-1} = e \in H$ . Thus we follow the identity axiom.

Suppose that  $k \in H$ . We follow that  $k \in H$  and  $e \in H$ , thus  $e * k^{-1} = k^{-1} \in H$ . Thus we follow the inverse axiom.

Let  $h_1, h_2 \in H$ . We follow that  $h_2^{-1} \in H$ , and thus  $h_1 * h_2^{-1} \in H$ , thus giving us closure, which completes the definition of the subgroup, as desired.

### 2.5.4

Let  $G$  be a group, and let  $H \subseteq G$  be a nonempty subset of  $G$ . Prove that  $H$  is a subgroup if and only if it has the following property:

For every  $a \in H$ , we have  $H = \{ah : h \in H\}$

If  $H$  is a subgroup, then we follow that  $a \in H$  implies that  $a^{-1} \in H$ , and thus for every  $h \in H$  we follow that  $a^{-1}h \in H$ , and thus  $h = e * h = aa^{-1}h \in \{ah : h \in H\}$ , thus  $H \subseteq \{ah : h \in H\}$ . Reverse subset argument is trivial, and thus those two sets are equal, as desired.

Suppose that for every  $a \in H$  we have  $H = \{ah : h \in H\}$ . Let  $b \in H$  be an arbitrary element. We follow that  $ab \in \{ah : h \in H\} = H$ . We thus follow that all  $H = \{ab * h : h \in H\}$ . Thus there's an element  $k$  of  $h$  such that  $abk = a$ , which implies that  $bk = e$ , and thus  $k = b^{-1}$ . Thus we conclude that  $b^{-1} \in H$ . Since  $b$  is arbitrary, we conclude that  $H$  contains inverses of its elements. Since  $b^{-1} \in H$ , we follow that

$$H = \{ah : h \in H\}$$

and thus  $ab^{-1} \in H$ , which implies that  $H$  is a subgroup, as desired.

## 2.5.5

This exercise generalizes the notion of the cyclic subgroup generated by an element of a group as described in Example 2.37. Let  $G$  be a group, and let  $S \subseteq G$  be a subset of  $G$ . The subgroup of  $G$  generated by  $S$ , which we denote by  $\langle S \rangle$ , is the intersection of all of the subgroups of  $G$  that contains  $S$ ; i.e.

$$\langle S \rangle = \bigcap_{S \subseteq H \subseteq G: H \text{ is a subgroup of } G} H$$

We also note that the set of subgroups of  $G$  is a subset of  $\mathcal{P}(G)$ , and thus let us define

$$K = \{H \in \mathcal{P}(G) : H \text{ is a subgroup of } G \text{ and } S \subseteq H \subseteq G\}$$

which is a set and our original definition can be rewritten as

$$\langle S \rangle = \bigcap_{H \in K} H$$

(a) Prove that  $S$  is not an empty set.

We follow that  $S$  is a subset of  $G$ ,  $G \subseteq G$ , and  $G$  is a subgroup of  $G$ , and thus

$$g \in K$$

Thus  $K$  is nonempty, which is also important, since there are no intersections of empty sets. We follow that every subgroup of  $G$  contains  $e$ , and thus  $e \in \langle S \rangle$ , as desired.

(b) Prove that  $\langle S \rangle$  is a subgroup of  $G$

Let  $a, b \in \langle S \rangle$ . We follow that every element of  $K$  contains  $a, b$ , and since every element of  $K$  is a subgroup that  $b^{-1}$  is also in every element of  $K$ . Thus  $ab^{-1}$  is also in every element of  $K$ , and thus  $ab^{-1} \in S$ , which implies that  $S$  is a subgroup, as desired.

(c) Suppose that  $L$  is a subgroup of  $G$  and that  $S \subseteq L$ . Prove that  $\langle S \rangle \subseteq L$ . (The whole text of this part of the exercise is one big typo)

We follow that  $L \in K$ , and thus  $\langle S \rangle = \bigcap_{H \in K} H \subseteq L$  by definition.

(d) Let  $T$  be the set of inverses of the elements in  $S$ ; i.e.

$$T = \{g^{-1} : g \in S\}$$

Prove that  $\langle S \rangle$  is equal to the following set of products

$$\langle S \rangle = \{e * g_1 g_2 \dots g_n : g_{1..n} \in S \cup T\}$$

(I've deviated a tad bit from the definition in the book since we gotta take care of the  $S = \emptyset$  case))

We firstly follow that  $S \subseteq \langle S \rangle$ , and since  $\langle S \rangle$  is a subgroup of  $G$  we conclude that  $T \subseteq \langle S \rangle$  by definition of  $T$ . We thus follow that for any

$$q \in \{e * g_1 g_2 \dots g_n : g_{1..n} \in S \cup T\}$$

$q \in \langle S \rangle$  by closure of composition in the subgroup, thus we've got the  $\supseteq$  case.

We now follow that  $rhs$  is nonempty by definition (in case when  $S = \emptyset$  we follow that  $e \in rhs$ ). If  $a, b \in rhs$ , then there are finite lists  $l_1, l_2$  over elements of  $S \cup T$  such that

$$\prod l_1 = a; \prod l_2 = b$$

we thus follow that  $b^{-1}$  is also in  $rhs$  since inverses of  $S \cup T$  is closed under inverses by definition, and thus we've gotta invert elements of  $l_2$  and order them in reverse to get  $b^{-1}$ . Thus we conclude that  $ab^{-1} \in rhs$  since  $ab^{-1}$  will be just a longer list of elements of  $S \cup T$ , and thus we follow that  $rhs$  is a subgroup of  $G$  that contains  $S$ . This implies that  $\langle S \rangle \subseteq rhs$  by definition, which in tandem with previous paragraph gives us the desired equality.

### 2.5.6

Let  $G$  be a finite group, let  $m \geq 1$  be an integer, and let

$$G[m] = \{g \in G : g^m = e\}$$

be the set of elements of  $G$  whose order divides  $m$ .

(a) If  $G$  is an abelian group, prove that  $G[m]$  is a subgroup of  $G$ .

We follow that  $e^m = e$ , thus  $e \in G[m]$ , and thus  $G[m] \neq \emptyset$ .

We follow that if  $b \in G$ , then  $b^m = e$ , thus  $(b^{-1})^m = e$ , and thus  $b^{-1} \in G[m]$ . Let  $a, b \in G[m]$ . We follow that  $(ab^{-1})^m = a^m b^{-m} = e * e = e$  where we can use the first equality due to the fact that  $G$  is abelian, which implies that  $ab^{-1} \in G[m]$ , which implies that  $G[m]$  is a subgroup, as desired.

(b) Give an example of non-abelian group and  $m \geq 1$  such that  $G[m]$  is not a subgroup of  $G$ .

We are aware of several non-abelian groups: permutation group, special linear group, and dihedral group.

Let us look at  $D_3$  and get all orders of that group:

$$|e| = 1$$

$$|r| = 3$$

$$|r^2| = 3$$

$$|s| = 2$$

$$|sr| = 2$$

$$|sr^2| = 2$$

We thus follow that  $D_3[2] = \{e, s, sr, sr^2\}$ . We then follow that  $|ssr| = |r| = 3$ , and thus we conclude that this set is not closed under composition, and thus it is not a group, as desired.

### 2.5.7

*Skip*

### 2.5.8

Let  $G$  be a group, let  $A, B \subseteq G$  be subgroups of  $G$ , and let  $\phi$  be the map

$$\phi : A \times B \rightarrow G, \phi(a, b) = ab$$

(a) Prove that  $A \cap B = \{e\}$  if and only if the map  $\phi$  is an injection

Suppose that  $A \cap B = \{e\}$ . Let  $a_1, a_2 \in A$ ,  $b_1, b_2 \in B$ , and suppose that

$$\phi(a_1, b_1) = \phi(a_2, b_2)$$

we follow that

$$a_1 b_1 = a_2 b_2$$

$$a_1 = a_2 b_2 b_1^{-1}$$

$$a_2^{-1} a_1 = b_2 b_1^{-1}$$

We follow that  $a_2^{-1} a_1 \in A$  and  $b_2 b_1^{-1} \in B$ , and since they are equal we conclude that

$$a_2^{-1} a_1 = b_2 b_1^{-1} = e$$

which implies that  $a_1 = a_2$  and  $b_1 = b_2$ , which in turn implies that  $\phi$  is an injection, as desired.

Now suppose that  $\phi$  is an injection. Suppose that there's  $q \in A \cap B$  such that  $q \neq e$ . Intersection of any two subgroups is a subgroup, and thus  $q^{-1} \in A \cap B$ . And since  $q \neq e$  we follow that  $q^{-1} \neq e$ . Thus we follow that  $\phi(e, e) = e * e = e = qq^{-1} = \phi(q, q^{-1})$  while  $\langle e, e \rangle \neq \langle q, q^{-1} \rangle$ , which implies that  $\phi$  is not an injection, which is a contradiction.

(b) We can turn  $A \times B$  into a group by using the group operation

$$\langle a_1, b_1 \rangle * \langle a_2, b_2 \rangle = \langle a_1 * a_2, b_1 * b_2 \rangle$$

Prove that  $\phi$  is a homeomorphism of groups if and only if element of  $A$  commutes with every element of  $B$ .

Suppose that  $\phi$  is a homeomorphism. This implies that

$$\phi(\langle a_1, b_1 \rangle * \langle a_2, b_2 \rangle) = \phi(\langle a_1, b_1 \rangle) * \phi(\langle a_2, b_2 \rangle) = a_1 b_1 a_2 b_2$$

it also implies that

$$\phi(\langle a_1, b_1 \rangle * \langle a_2, b_2 \rangle) = \phi(\langle a_1, b_1 \rangle * \langle a_2, b_2 \rangle) = \phi(\langle a_1 * a_2, b_1 * b_2 \rangle) = a_1 a_2 b_1 b_2$$

thus

$$a_1 a_2 b_1 b_2 = a_1 b_1 a_2 b_2$$

for all  $a_1 a_2 \in A, b_1, b_2 \in B$ . By setting  $a_1 = b_2 = e$  we get the desired implication.

By not setting  $a_1 = b_1 = e$  and running the argument for the forward implication (with possible reorder of equalities) in reverse we can also get the reverse implication, which gives us the desired result.

### 2.5.9

Let  $g$  be a finite group, and let  $A, B \subseteq G$  be subgroup of  $G$ , and supposet that  $\gcd(|A|, |B|) = 1$ . Prove that  $A \cap B = e$

$A \cap B$  is a subgroup of both  $A$  and  $B$ , and thus its order gotta divide orders of both  $A$  and  $B$ . Since  $\gcd(|A|, |B|) = 1$  we conclude that order of  $A \cap B$  is 1, which gives us the desired result.

### 2.5.10

Let  $G$  be a group. The center of  $G$ , denoted by  $Z(G)$ , is the seet of elements of  $G$  that commute with every other element; i.e.

$$Z(G) = \{g \in G : (\forall h \in G)(gh = hg)\}$$

(a) Prove thtat  $Z(G)$  is a subgroup of  $G$ .

We follow that for all  $h \in H$  we've got that  $eh = h = he$ , thus  $e \in Z(G)$ , and hence  $Z(G) \neq \emptyset$

Let  $a, b \in Z(G)$ . Let  $h \in G$  be arbitrary. We follow that

$$bh = hb$$

$$(bh)b^{-1} = hbb^{-1}$$

$$(bh)b^{-1} = h$$

$$b^{-1}bhb^{-1} = b^{-1}h$$

$$hb^{-1} = b^{-1}h$$



we thus follow that since  $h, b$  are arbitrary that  $b \in Z(G)$  implies that  $b^{-1} \in Z(G)$ .

With the same assumptions we then follow that

$$(ab^{-1})h = a(b^{-1}h) = ahb^{-1} = (ah)b^{-1} = h(ab^{-1})$$

thus  $ab^{-1} \in Z(G)$ , and thus we conclude that  $a, b \in Z(G)$  implies that  $ab^{-1} \in Z(G)$ , which with nonemptiness of  $Z(G)$  gives us the desired result.

(b) *When does  $Z(G)$  equal  $G$ ?*

The obvious case of equality comes when  $G$  is abelian. Assume that  $G$  is non-abelian. This implies that there are  $a, b \in G$  such that  $ab \neq ba$ . We thus conclude that neither  $a$  nor  $b$  are in  $Z(G)$ , and thus  $Z(G) \neq G$ , which implies that  $Z(G) = G$  if and only if  $G$  is abelian.

(c) *Compute the center of the symmetric group  $S_n$ .*

We follow that for  $n \in \{1, 2\}$ ,  $S_n$  is abelian, and thus  $Z(S_2) = S_2$ . Let us enumerate elements of  $S_3$ :

$$S_3 = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$$

we follow that  $Z(S_3)$  includes at least  $e$ . Something tells me that  $Z(S_3)$  consists exclusively of  $e$ . Let us particularly look at the "shift" operator  $(2, 3, 1)$  (the one that maps one element to the next with mapping the last element to the first). We follow that any given permutation group contains one. For  $n \geq 3$  we can also find an element, that is distinct from the shift element, that swaps two elements in place (we're gonna take  $(2, 1, 3)$  in this case). We follow that

$$(2, 3, 1) \circ (2, 1, 3) = (1, 3, 2)$$

$$(2, 1, 3) \circ (2, 3, 1) = (3, 2, 1)$$

thus we follow that shift operator is not in the  $Z(S_3)$ .

Generally let  $a \in \{1, \dots, n\}$ , let  $q \in S_n$  be an element such that it maps  $a$  to itself, and maps  $(a + 1) \bmod n$  not to itself, and let  $s \in S_n$  be the shift operator. We follow that  $(q \circ s)(a) = q(s(a)) = q(a + 1)$  and  $s(q(a)) = s(a) = a + 1$ . By definition  $q(a + 1) \neq a + 1$ , and thus we conclude that neither shift operator, nor the element that has partial identity are in  $Z(S_n)$ .

With a given example we can generalize even more to create a proof for our conjecture:

Let  $n \geq 3$  and let  $q \in S_n$  be such that  $q \neq e$ . Since  $q \neq e$  we follow that there's an element  $a \in \{1, \dots, n\}$  such that  $q(a) \neq a$ . We then follow that there exists an element  $s \in S_n$  such that  $s(a) = a$ , and  $s(q(a)) \neq q(a)$  (we can do this because there are at least 3 elements in  $\{1, \dots, n\}$ : we take the  $q(a)$  and some element  $b \in \{1, \dots, n\} \setminus \{q(a), a\}$ , then map  $a$  to itself, and swap  $q(a)$  and  $b$ ; the fact that there's an element that is distinct from both  $q(a)$  and  $a$  is the restriction that fails for  $S_2$ ). We follow that

$$(s \circ q)(a) = s(q(a)) \neq q(a) = q(s(a)) = (q \circ s)(a)$$

and thus  $s \circ q \neq q \circ s$ . Therefore we conclude that  $q \notin Z(S_n)$ . Since the only restriction on  $q$  is that  $q \in S_n \setminus \{e\}$  we follow that  $Z(S_n) = \{e\}$  for all  $n \geq 3$ , as desired.

(d) Compute the center of the dihedral group  $D_n$ .

We follow that

$$s * r \neq r * s = s * r^{n-1}$$

thus neither  $r$  nor  $s$  are in  $Z(D_n)$  for  $n \geq 3$ . We follow that

$$sr^j * r = sr^{j+1}$$

$$rsr^j = sr^{-1}r^j = sr^{j-1}$$

thus for  $n \geq 3$  we follow that  $sr^{j+1} \neq sr^{j-1}$ , and thus for all  $j \in \{1, \dots, n-1\}$  we follow that  $sr^j \notin Z(D_n)$ . We also follow that

$$r^j * s = sr^{n-j}$$

$$s * r^j = sr^j$$

thus if  $j \neq n/2$  we follow that  $r^j \notin Z(D_n)$ . Thus we follow that if  $n$  is odd that  $Z(D_n) = \{e\}$ . We follow that

$$r^{n/2} * sr^k = sr^{n-n/2+k} = sr^{n/2+k}$$

$$sr^k * r^{n/2} = sr^{n/2+k}$$

for all  $k \in \{0, \dots, n\}$ . Case with  $r^k$  for arbitrary  $k$  is trivial, and thus we conclude that if  $n$  is even that  $Z(D_n) = \{e, r^{n/2}\}$ , which is kinda surprising.

(e) Compute the center of the quaternion group  $Q$ .

We follow that none of the  $i, j, k$  are commuting with each other (by definition), nor do their inverses, and thus none of them are in  $Z(Q)$ .  $\pm 1$  on the other hand commute with everything (once again, by definition), and thus  $Z(Q) = \{\pm 1\}$ .

### 2.5.11

Let  $G$  be a group, and let  $g \in G$ . The centralizer of  $g$ , denoted  $Z_G(g)$  is the set of elements of  $G$  that commute with  $g$ , i.e.

$$Z_G(g) = \{g' \in G : gg' = g'g\}$$

(a) Prove that  $Z_G(g)$  is a subgroup of  $G$ .

The proof is pretty much the same as with centralizer of the entire group. We can also note that the definition implies that  $Z(G)$  is a subgroup of  $Z_G(g)$  for any given  $g \in G$ . Also important to note that  $g \in Z_G(g)$ .

(b) Compute the centralizer  $Z_G(g)$  for the following groups:

(i)  $G = D_4$  and  $g = r$

4 is even, and thus  $\{e, r^2\}$  are in  $Z_G(g)$ . Any rotation is also there.  $sr \neq sr^3$ , and thus  $s$  is not included there.  $sr^j r = sr^j \neq sr^{j-1} = rsr^j$ , which implies that none of the  $sr^j$  are in there

(ii)  $G = D_4$  and  $g = sr^j$

$r$  and  $r^3$  aren't in there. If  $j = 2$ , then  $s$  is also in there, otherwise it isn't. If  $j = 2$ , then  $sr sr^2 = r^3$  and  $sr^2 sr = r$ , thus  $sr$  is not in there.  $sr^3$  is the inverse of  $sr$ , and thus it's not in there. The rest can be handled pretty easily.

(iii)  $G = GL_2(R)$  and  $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

We follow that this matrix is diagonal, and thus it commutes with other diagonal matrices. If matrix is not diagonal then

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} q & w \\ e & r \end{pmatrix} = \begin{pmatrix} qa & wa \\ ed & rd \end{pmatrix}$$

$$\begin{pmatrix} q & w \\ e & r \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} qa & wd \\ ea & rd \end{pmatrix}$$

thus we need  $wa = wd$  and  $ea = ed$ . Therefore if  $a \neq d$  we need  $w = e = 0$  and this would include only include other diagonal matrices, that we've already included. If  $a = d$ , then any matrix will do.

(c) Prove that  $Z_G(g) = G$  if and only if  $g \in Z(G)$ .

IF  $g \in Z(G)$ , then by definition every element of  $G$  commutes with  $G$ , and thus  $Z_G(g) = G$ .

Assume that  $Z_G(g) = G$ . We follow that every element of  $G$  commutes with  $g$ , and thus  $g \in Z(G)$ .

(d) More generally, if  $S \subseteq G$  is any subset of  $G$ , then the centralizer of  $S$  is the set

$$Z_G(S) = \{g \in G : (\forall s \in S)(sg = gs)\}$$

Prove that  $Z_G(S)$  is a subgroup of  $G$

Proof is borderline identical to the case of centralizer of group/element.

### 2.5.12

This exercise explains when two elements of  $G$  determine the same coset of  $H$ . Let  $G$  be a group, let  $H$  be a subgroup of  $G$ , and let  $g_1, g_2 \in G$ . Prove that the following three statements are equivalent:

- (1)  $g_1 H = g_2 H$ .
- (2) There are  $h \in H$  such that  $g_1 = g_2 h$ .
- (3)  $g_2^{-1} g_1 \in H$ .

Assume that  $g_1H = g_2H$ . We follow that  $e \in H$  and thus  $g_1e \in g_2H$ . Thus there's  $h \in H$  such that  $g_1e = g_1 = g_2h$  by definition. Thus

$$g_1 = g_2h$$

$$g_2^{-1}g_1 = h \in H$$

We can run this in reverse to establish equivalence of (2) and (3). We follow that since  $g_1 = g_2h$  that  $g_1H \cap g_2H \neq \emptyset$ , and thus they are equal.

### 2.5.13

*Let  $G$  be a finite group whose only subgroups are  $\{e\}$  and  $G$ . Prove that either  $G = \{e\}$  or else that  $G$  is a cyclic group whose order is prime.*

Set with  $G = \{e\}$  is trivial, thus assume that  $G \neq \{e\}$ . Let  $q \in G \neq e$ . We follow that since  $G$  is finite that  $q$  has an order, and thus its powers constitute a subgroup  $\langle q \rangle$ . Since order of  $G$  is a prime we follow that order of  $\langle q \rangle$  divides the order of  $G$  and thus order of  $\langle q \rangle$  is either 1 or order of  $G$ . Since  $q \neq e$  we conclude the latter, and thus  $\langle q \rangle = G$ , which implies that  $G$  is cyclical, as desired.

### 2.5.14

*Let  $G$  be a group, and let  $K \subseteq H \subseteq G$  be subgroups. We may thus view  $K$  as a subgroup of  $G$  or as a subgroup of  $H$ . We also recall that the index of a subgroup is its number of distinct cosets.*

*(a) If  $G$  is finite, prove the Index Multiplication Rule:*

$$(G : K) = (G : H)(H : K)$$

We follow that

$$|H| = |K|(H : K)$$

$$|G| = |H|(G : H)$$

and

$$|G| = |K|(G : K)$$

therefore

$$|G|/|K| = (G : K)$$

thus

$$|G| = |K|(H : K)(G : H)$$

$$|G|/|K| = (H : K)(G : H)$$

$$|G|/|K| = (G : H)(H : K)$$

$$(G : K) = (G : H)(H : K)$$

as desired.

(b) *Prove that the Index Multiplication Rule is true even if  $G, H, K$  are allowed to be infinite groups, provided that we assume that  $(G : K)$  is finite*

Let  $Q'$  be the set of distinct cosets of  $H$  in  $G$ . Let  $W'$  be the set of distinct cosets of  $K$  in  $H$ . For each  $q' \in Q'$  there's an element  $q$  of  $G$  such that  $qH = q'$ . By AoC we can define subset  $Q$  of  $G$  that correspond to  $Q'$ . For each  $w' \in W'$  there's an element  $w$  of  $H$  such that  $wK = w'$ . By AoC we can define subset  $W$  of  $H$  that correspond to  $W'$ .

Let  $j$  be a coset of  $K$  in  $G$ . We follow that there's  $g \in G$  such that  $gK = j$ . We follow that  $gH$  is a coset of  $H$ , and thus there's a unique  $q \in Q$  such that  $qH = gH$ . We follow then that  $g^{-1}q \in H$  (see a couple of exercises above), and thus  $(g^{-1}q)K$  is a coset of  $K$  in  $H$ . Thus there's a unique  $w \in W$  such that

$$wK = (g^{-1}q)K$$

thus

$$qwK = gK$$

therefore for each coset of  $K$  in  $G$  there's a pair  $\langle q, w \rangle \in Q \times W$ . By uniqueness of  $q$  and  $w$  in  $Q$  and  $W$ , and the fact that  $j$  is an arbitrary coset we conclude that there's an injection from a set of cosets of  $K$  in  $G$  to  $Q \times W$ .

Let  $\langle q_1, w_1 \rangle, \langle q_2, w_2 \rangle \in Q \times W$  are such that  $\langle q_1, w_1 \rangle \neq \langle q_2, w_2 \rangle$ . We follow that if  $q_1 \neq q_2$ , and thus  $q_1H \neq q_2H$  by definition.  $w_1w_2 \in H$ , and thus  $w_1w_2H = H$ . Since  $K \subseteq H$  we follow that

$$q_1w_1K \subseteq q_1H$$

$$q_2w_2K \subseteq q_2H$$

and thus

$$q_1w_1K \neq q_2w_2K$$

If  $q_1 = q_2$  we follow that  $w_1 \neq w_2$ . This implies that

$$w_1K \neq w_2K$$

and thus

$$q_1w_1K \neq q_1w_2K$$

and since  $q_1 = q_2$  we follow that

$$q_1w_1K \neq q_2w_2K$$

This implies that if  $\langle q_1, w_1 \rangle, \langle q_2, w_2 \rangle \in Q \times W$  then  $q_1w_1K \neq q_2w_2K$ , and thus we conclude that there's an injection from  $Q \times W$  to a set of cosets of  $K$ . Schroder-Bernstein now implies that the set of cosets of  $K$  in  $G$  is equal to  $|Q \times W|$ . If the latter is finite, then we follow

that  $(G : K) = |Q| * |W|$ . Definition of  $Q$  and  $W$  respectively imply that  $|Q| = (G : H)$  and  $|W| = (H : K)$ , which implies that

$$(G : K) = (G : H)(H : K)$$

as desired.

### 2.5.15

Let  $p$  be a prime and let  $a \in Z$  be an integer satisfying  $p \nmid a$ . Use Exercise 2.13(b) and Lagrange's theorem to prove

**Fermat's Little Theorem:**  $a^{p-1} \equiv 1 \pmod{p}$

We follow that if  $a \nmid p$ , then  $a \not\equiv 0 \pmod{p}$ . We thus follow that there's  $r \in Z_+$  such that  $r < p$  and there's an integer  $k$  such that  $a = kp + r$ . This implies that

$$a^n = (kp + r)^n = \sum_{i \in \{0, \dots, n\}} bc(n, i)(kp)^i r^{n-i}$$

by the binomial theorem ( $bc$  is the binomial coefficient). We then follow that the only element of the sum that does not have  $p$  as its multiple is the one where  $i = 0$ . Thus we follow that

$$a^n \equiv \sum_{i \in \{0, \dots, n\}} bc(n, i)(kp)^i r^{n-i} \equiv r^n \pmod{p}$$

thus we follow that the original theorem is equivalent to the statement that for all  $r \in Z_+$  such that  $r < p$  we've got that

$$r^{p-1} \equiv 1 \pmod{p}$$

We know that 1 is the identity in the multiplicative group  $(Z/pZ)^*$ . We also follow that order of  $(Z/pZ)^*$  is  $p - 1$ . We thus follow that any cyclic subgroup of  $(Z/pZ)^*$  has order that divides  $p - 1$ . Therefore we follow that for each provided  $r$  we've got that order of  $r$  is some divisor  $n$  of  $p - 1$ , and thus

$$r^{p-1} = r^{n*k} = e^k = e = 1$$

which gives us the desired result.

## 2.6 Products of groups

### 2.6.1

Let  $G_1, G_2, \dots, G_n$  be groups. Generalize definition 2.54 by explaining why

$$G_1 \times \dots \times G_n$$

with component-wise operation is a group.

We can follow this case by induction. Interesting thing is to ask that if  $K$  is an indexed set of groups, then is  $\prod K$  a group?

We can follow that the product of identities is an identity. Inverses are also handled. Component-wise association gives us also association for the product group.  $\prod K$  is abelian if and only if all the constructing groups are abelian.

### 2.6.2

Let  $G$  be a group, let  $A$  and  $B$  be subgroups of  $G$  and consider the map

$$\phi : A \times B \rightarrow G, \phi(a, b) = ab$$

(a) If  $G$  is an abelian group, prove that  $\phi$  is a homomorphism.

Follows from the exercise that we've already finished (there we've handled the specific case that  $B \subseteq Z_G(A)$ )

(b) If  $G$  is an abelian group, prove that

$$\ker(\phi) = \{\langle c, c^{-1} \rangle : c \in A \cap B\}$$

We can follow that application of  $\phi$  to the element of  $rhs$  will give us identity by definition, and thus we've got  $\supseteq$ .

If  $\phi(i, j) = e$ , then by definition  $ij = e$ , and thus  $i = j^{-1}$ . Thus we've got the  $\subseteq$ .

(c) Suppose that there are elements  $a \in A$ ,  $b \in B$  such that  $ab \neq ba$ . Prove that  $\phi$  is not a homeomorphism

Exercise that handled this premise before concludes with biconditional, which gives us this implication

*the rest of the exercises are skipped*

## Chapter 3

# Rings - Part 1

### 3.1 Abstract Rings and Ring Homomorphisms

#### Notes

Distributive law can be generalized through application of associativity, which will give us

$$a * (b + c + d) = ab + ac + ad$$

and the same for the other direction. We can also by induction do the same with a general sum.

$$b \sum a_i = \sum ba_i$$

The idea of ring homomorphism can be summed up to preserving functions upon application of the function, and mapping identities to identities.

#### 3.1.1

Let  $R$  be a ring, and let  $a, b \in R$ .

(a) Prove that

$$(-a) * (-b) = a * b$$

$$(-a) * (-b) = (0 + (-a)) * (-b) = (a + (-a) + (-a)) * (-b) = a * (-b) + (-a) * (-b) + (-a) * (-b)$$

(we've added 0 to the  $-a$  and then distributed over it) thus

$$(-a) * (-b) = a * (-b) + (-a) * (-b) + (-a) * (-b)$$

and cancelation law implies that

$$0 = a * (-b) + (-a) * (-b)$$



this in turn implies that

$$(-(a * (-b))) = (-a) * (-b)$$

we can do a similar thing for  $(-b)$ , which will result in

$$(-((-a) * b)) = (-a) * (-b)$$

Given that everything has additive inverse, we can follow that

$$a + b = (-((-a) * b)) = -(a * (-b))$$

which will imply that

$$\begin{aligned} (-a)*(-b) &= -(a*(-b)) = -((-(-a))*(-b)) = -((-(-a))*(-b)) = -(-(-(-a))*b) = \\ &= -(-(a*b)) = a*b \end{aligned}$$

as desired.

(b) Our proof of Proposition 3.2(a) used the multiplicative identity element  $1_R$  of  $R$ . Find a proof that works even if  $R$  does not have a multiplicative identity

We follow that for all  $b \in R$

$$ab = a * (b + 0) = ab + a0$$

thus by cancelation law of the group we follow that

$$-(ab) + ab = -(ab) + ab + a0$$

$$0 = 0 + a0$$

$$0 = a0$$

as desired.

### 3.1.2

Let  $R$  be a commutative ring

(a) Suppose that the map

$$f : R \rightarrow R, f(a) = a^2$$

is a ring homomorphism. Prove that  $1_R + 1_R = 0_R$ .

By definition of ring homeomorphism we follow that

$$1_R = f(1_R)$$

and thus we follow that

$$1_R + 1_R = f(1_R) + f(1_R) = f(1_R + 1_R) = (1_R + 1_R)^2 = (1_R + 1_R)(1_R + 1_R) =$$

$$= 1_R(1_R + 1_R) + 1_R(1_R + 1_R) = (1_R + 1_R) + (1_R + 1_R)$$

thus

$$(1_R + 1_R) = (1_R + 1_R) + (1_R + 1_R)$$

$+$  is a group, thus we're justified to use cancelation to get that

$$1_R + 1_R = 0$$

as desired

(b) Conversely, if  $2 = 0$  in the ring  $R$ , prove that  $f(a)^2$  is a homomorphism from  $R$  to  $R$

We follow that

$$0^2 = 0$$

$$1^2 = 1$$

so identities are there.

One thing to point out is that  $R$  is commutative, which is used in deriviations here

$$\begin{aligned} f(a+b) &= (a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = \\ &= a^2 + ab + ab + b^2 = f(a) + ab + ab + f(b) = f(a) + f(b) + ab(1+1) = \\ &= f(a) + f(b) + *0 = f(a) + f(b) \end{aligned}$$

thus we've got the addition

$$f(a * b) = (a * b)^2 = (a * b) * (a * b) = a * b * a * b = a^2 * b^2 = f(a) * f(b)$$

and multiplication, that now satisfies the definition of the homomorphism.

(c) Suppose that the map

$$f : R \rightarrow R, f(a) = a^3$$

is a ring homomorphism. Prove that  $6 = 0$  in the ring  $R$

$$\begin{aligned} f(1+1) &= (1+1)^3 = (1+1) * (1+1) * (1+1) = 8 \\ f(1+1) &= f(1) + f(1) = 1 + 1 = 2 \end{aligned}$$

thus

$$8 = 2$$

$$6 = 0$$

as desired. In general we follow that if  $f(a) = a^n$  is a homomorphism, then  $2^n - 2 = 0$

## 3.2 Interesting Examples of Rings

### Notes

When checking for subrings, we follow that associativity of both operations and their distributivity gets grandfathered by the subset, and thus the only things that we need to check for are the identities and closures of operations

When we're using generalized rings with polynomials, it is possible that different lists of coefficients might give the same functions. This is easily observed, when we remember that there exist finite rings, and thus the space of functions from this ring to itself is finite, but the list of possible coefficients is infinite.

When we're talking about polynomial equality, we mean that they've got the same list of coefficients. This essentially means that a polynomial is practically a finite list of elements of the ring (and thus the set of all polynomials is a set of all finite lists of coefficients), and not the function.

It's also important to remember, that coefficients of polynomials are the elements of the underlying ring, but powers are natural numbers by definition.

We can also theoretically have polynomials over polynomials, and so on, and so forth.

In the book, the ring of quaternions is defined over coefficients in  $R$ , but the subrings of  $R$  can be also used to define appropriate subrings.

Matrix ring is an important example of the fact that two non-zero elements when multiplied can produce a zero.

Although we haven't observed any product rings (i.e. rings that are made up by sticking two rings into cartesian product), it's pretty obvious that we can define them and everything will line up

We can rephrase the condition of uniqueness of homomorphism from  $ZZ$ : if  $f$  and  $g$  are homomorphisms from  $Z$  to some ring  $R$ , then  $f = g$ .

A little note about polynomials: whenever we've got a ring  $R$ , by polynomials in  $R$  we understand a list of coefficients of  $R$ , first one on whom is not zero. This implies that although we can have a finite ring  $R$ , its ring of polynomials  $R[x]$  is infinite. Cardinality of the ring  $R[x]$  is then the same, as the cardinality of the set of sequences over  $R$ .

This further implies that since the set of functions over a finite ring  $R$  is finite, but the set of polynomials is infinite, that there are two distinct polynomials, that correspond to the same function. In real numbers we know (or at the very least assume pretty heavily) that each polynomial represents a distinct function, so then the question is as follows: what are the conditions, that give us injectivity from the set of polynomials to the set of functions?

Assume that  $f, g \in R[x]$ , are such that  $f \neq g$  in terms of polynomials. Assume that  $f = g$  in terms of functions. We follow that for all  $x \in R$  we have that

$$f(x) = g(x)$$

thus we have that

$$\begin{aligned}\sum a_i x^i &= \sum b_i x^i \\ \sum (a_i - b_i) x^i\end{aligned}$$

since  $f \neq g$  in terms of polynomials we follow that there is  $k \in \omega$  such that  $a_k \neq b_k$ . Moreover, we follow that since  $f = g$  we follow that  $f(0) = g(0)$ , and thus  $a_0 = b_0$ , otherwise we're screwed. We thus conclude that there is polynomial  $h = f - g$  such that  $h$  is non-constant and nonzero, and such that for all  $r \in R$  we follow that

$$h(r) = 0$$

at the very least it gives us that if  $R$  is an infinite integral domain, then it's got at most  $\deg(h)$  roots, and thus the given case is impossible. Thus all of the  $Z$ ,  $Q$ ,  $R$ , and other normal things are covered.

### 3.2.1

Let  $m \geq 1$  be an integer, and define a map

$$\phi : Z \rightarrow Z/mZ, \phi(a) = a \pmod{m}$$

Prove that  $\phi$  is a ring homomorphism.

We firstly follow that  $\phi(0) = 0, \phi(1) = 1$  by definition, and thus identities are mapped properly. Fact that

$$a \equiv c \pmod{m}, b \equiv d \pmod{m} \Rightarrow a + b \equiv c + d \pmod{m}, a * b \equiv c * d \pmod{m}$$

implies that all the function properties are satisfied, and thus  $\phi$  is a homeomorphism, as desired.

### 3.2.2

(a) Let  $\alpha = 7$  and  $\beta = 11$  be elements of the ring  $Z/17Z$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

We follow that

$$7 + 11 = 18 \equiv 1 \pmod{17}$$

$$7 * 11 = 77 \equiv 9 \pmod{17}$$

(b) Let  $\alpha = 2 + 4x$  and  $\beta = 1 + 4x + 3x^2$  be elements of the ring  $(Z/7Z)[x]$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

$$(2 + 4x) + (1 + 4x + 3x^2) = 3 + x + 3x^2$$

$$(2 + 4x) * (1 + 4x + 3x^2) = (2 + 1x + 6x^2) + (4x + 2x^2 + 5x^3) = 2 + 5x + x^2 + 5x^3$$

(c) Let  $\alpha = 3 + 2i$  and  $\beta = 2 - 3i$  be elements of the ring  $Z[i]$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

$$3 + 2i + 2 - 3i = 5 - i$$

$$(3 + 2i) * (2 - 3i) = 6 - 9i + 4i + 6 = 12 - 5i$$

(d) Let  $\alpha = 3 + 2x - x^2$  and  $\beta = 2 - 3x + x^2$  be elements of the ring  $Z[x]$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

$$(3 + 2x - x^2) + (2 - 3x + x^2) = 5 - x$$

$$\begin{aligned} (3 + 2x - x^2) * (2 - 3x + x^2) &= (6 - 9x + 3x^2) + (4x - 6x^2 + 2x^3) + (-2x^2 + 3x^3 - x^4) = \\ &= 6 - 15x - 5x^2 + 5x^3 - x^4 \end{aligned}$$

(e) Let  $R = Z[i]$  and let  $\alpha = (1 + i) + (2 - i)x - x^2$  and  $\beta = (2 + i) + (1 + 3i)x$  be elements of the ring  $Z[x]$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

$$(1 + i) + (2 - i)x - x^2 + (2 + i) + (1 + 3i)x = (2 + 2i) + (3 + 2i)x - x^2$$

$$\begin{aligned} &((1 + i) + (2 - i)x - x^2) * ((2 + i) + (1 + 3i)x) = \\ &= (2 + 2i + i - 1) + (4 - 2i + 2i + 1)x + (-2 - i)x^2 + (1 + i + 3i - 3)x + (2 - i + 6i + 3)x^2 + (-1 - 3i)x^3 = \\ &= (1 + 3i) + 5x + (-2 - i)x^2 + (-2 + 4i)x + (5 + 5i)x^2 + (-1 - 3i)x^3 = \\ &= (1 + 3i) + (3 + 4i)x + (3 + 4i)x^2 + (-1 - 3i)x^3 \end{aligned}$$

(f) Let  $\alpha = 1 + 2i - j + k$  and  $\beta = 2 - i + 3j - k$  be elements of the ring  $H$ . Compute  $\alpha + \beta$  and  $\alpha * \beta$ .

$$1 + 2i - j + k + 2 - i + 3j - k = 3 + i + 2j$$

$$\begin{aligned} &(1 + 2i - j + k) * (2 - i + 3j - k) = \\ &= (2 - i + 3j - k) + 2i(2 - i + 3j - k) - j(2 - i + 3j - k) + k(2 - i + 3j - k) = \\ &= (2 - i + 3j - k) + (4i + 2 + 6k + 2j) - (2j + k - 3 - i) + (2k - j - 3i + 1) = \\ &= (2 - i + 3j - k) + (2 + 4i + 2j + 6k) + (3 + i - 2j - k) + (1 - 3i - j + 2k) = \\ &= 8 + i + 2j + 6k \end{aligned}$$

**3.2.3**

For any integer  $D$  that is not the square root of an integer, we can form a ring

$$Z[\sqrt{D}] = \{a + b\sqrt{D} : a, b \in Z\}$$

If  $D > 0$ , then  $Z[\sqrt{D}]$  is a subring of  $R$ , while if  $D < 0$ , then in any case it is a subring of  $C$ .

(a) Let  $\alpha = 2 + 3\sqrt{5}$  and let  $\beta = 1 - 2\sqrt{5}$ . Compute the quantities

$$a + b, a * b, a^2$$

We follow that

$$\begin{aligned} 2 + 3\sqrt{5} + 1 - 2\sqrt{5} &= 3 + \sqrt{5} \\ (2 + 3\sqrt{5}) * (1 - 2\sqrt{5}) &= 2 - 4\sqrt{5} + 3\sqrt{5} - 30 = -28 - \sqrt{5} \\ (2 + 3\sqrt{5})^2 &= 4 + 12\sqrt{5} + 45 = 49 + 12\sqrt{5} \end{aligned}$$

(b) Prove that the map

$$\phi : Z[\sqrt{D}] \rightarrow Z[\sqrt{D}], \phi(a + b\sqrt{D}) = a - b\sqrt{D}$$

is a ring homomorphism.

We follow that identities are golden.

$$\begin{aligned} \phi(a + b\sqrt{D} + c + d\sqrt{D}) &= \phi((a + c) + (b + d)\sqrt{D}) = \\ &= (a + c) - (b + d)\sqrt{D} = a - b\sqrt{D} + c - d\sqrt{D} = \phi(a + b\sqrt{D}) + \phi(c + d\sqrt{D}) \end{aligned}$$

$$\phi((a + b\sqrt{D}) * (c + d\sqrt{D})) = \phi(ac + ad\sqrt{D} + bc\sqrt{D} + bdD) = ac + bdD - ad\sqrt{D} - bc\sqrt{D}$$

$$\begin{aligned} \phi(a + b\sqrt{D}) * \phi(c + d\sqrt{D}) &= (a - b\sqrt{D}) * (c - d\sqrt{D}) = \\ &= ac + bdD - ad\sqrt{D} - bc\sqrt{D} \end{aligned}$$

thus everything checks out, as desired.

(c) Prove that for all  $\alpha \in Z[\sqrt{D}]$

$$\alpha * \bar{\alpha} \in Z$$

$$\begin{aligned} \alpha * \bar{\alpha} &= (a + b\sqrt{D}) * (a - b\sqrt{D}) = \\ &= a^2 - ab\sqrt{D} + ab\sqrt{D} + b^2D = a^2 + b^2D \end{aligned}$$

given that  $a, b, D$  are integers, we follow the desired conclusion.

**3.2.4**

Let  $\rho$  be the complex number  $\rho = \frac{-1+i\sqrt{3}}{2} \in C$ , and let

$$Z[\rho] = \{a + b\rho : a, b \in Z\}$$

(a) Prove that  $\rho^3 = 1$ . Thus  $\rho$  is a cube root of unity.

$$\begin{aligned} \left(\frac{-1+i\sqrt{3}}{2}\right)^3 &= \left(\frac{\sqrt{3}}{2}i - \frac{1}{2}\right)^3 = \left(\frac{\sqrt{3}}{2}i\right)^3 - 3\left(\frac{\sqrt{3}}{2}i\right)^2\left(\frac{1}{2}\right) + 3\left(\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 = \\ &= -\frac{3\sqrt{3}}{8}i + \frac{9}{8} + \frac{3\sqrt{3}i}{8} - \frac{1}{8} = 1 + 0 = 1 \end{aligned}$$

as desired.

(b) Prove that  $\rho^2 + \rho + 1 = 0$

$$\rho^3 - 1 = (\rho - 1)(\rho^2 + \rho + 1)$$

we follow that lhs is not zero, and thus rhs is zero, as desired.

(c) Prove that the polynomial  $X^3 - 1$  factors as

$$X^3 - 1 = (X - 1)(X - \rho)(X - \rho^2)$$

We follow that

$$\begin{aligned} (X - \rho)(X - \rho^2) &= X^2 - \rho^2X - \rho X + 1 = X^2 - \rho^2X - \rho X - X + X + 1 = \\ &= X^2 - X(\rho^2X + \rho - 1) + X + 1 = X^2 - 0X + X + 1 = X^2 + X + 1 \end{aligned}$$

thus

$$(X - 1)(X - \rho)(X - \rho^2) = (X - 1)(X^2 + X + 1)$$

and

$$X^3 - 1 = (X - 1)(X^2 + X + 1)$$

as desired.

(d) Prove that  $Z[\rho]$  is a subring of  $C$ .

By setting  $a, b$  appropriately we get that identities are present. Additive group is closed by general properties of  $C$ .

$$\begin{aligned} (a + b\rho) * (c + d\rho) &= ac + ad\rho + bc\rho + bd\rho^2 = ac + ad\rho + bc\rho + bd\rho^2 + bd\rho + bd - bd\rho - bd = \\ &= ac + ad\rho + bc\rho + bd(\rho^2 + \rho + 1) - bd\rho - bd = ac + ad\rho + bc\rho + bd0 - bd\rho - bd = \\ &= (ac - bd) + \rho(ad + bc - bd) \end{aligned}$$

thus this thing is also closed under multiplication, which makes  $Z[\rho]$  a subring, as desired.

**3.2.5**

*This one is reasoned in the notes, and is a bunch of arithmetic*

**3.2.6**

Let  $R$  be commutative ring, let  $c \in R$ , and let  $E_c : R[x] \rightarrow R$  be evaluation map  $E_c(f) = f(c)$ .

(a) Prove that  $E_c$  is a ring homomorphism.

We follow that there's a polynomial  $x - c$  and  $x - c + 1$  that maps identities. We follow that

$$E_c(f + g) = (f + g)(c) = f(c) + g(c) = E_c(f) + E_c(g)$$

$$E_c(f * g) = (f * g)(c) = f(c) * g(c) = E_c(f) * E_c(g)$$

by algebraic properties of functions.

(b) Prove that  $E_c(f) = 0$  if and only if there's a polynomial  $g(x) \in R[x]$  satisfying  $f(x) = (x - c)g(x)$

$$f(x) = f(x) - 0 = f(x) - f(c) = \sum a_i(x^j - c^j)$$

We know that

$$x^n - c^n = (x - c) \sum_{i=1}^n x^{n-i} c^{i-1}$$

Which gives us the desired result (needed to look this one up). Reverse is trivial

**3.2.7**

Let  $R$  be a non-commutative ring, with elements  $\alpha, \beta \in R$  such that  $\alpha \neq \beta$ . We can still turn  $R[x]$  into a ring by always writing polynomials in the form

$$p(x) = \sum a_i x^i$$

and using the usual rule that  $x$  commutes with all of the elements of  $R$ . We can also define an evaluation map  $E_\alpha : R[x] \rightarrow R$  by setting

$$E_\alpha(\sum a_i x^i) = \sum a_i \alpha^i$$

Prove that if  $R$  is non-commutative, then  $E_\alpha$  is not a homomorphism.

We follow that

$$E_\alpha(0) = 0$$

$$E_\alpha(1) = 1$$

thus we've got identities, and so there's nothing there.



We know that both  $\alpha$  and  $\beta$  are polynomials, but

$$E_\alpha(x * \beta) = E_\alpha(\beta x) = \beta \alpha$$

$$E_\alpha(x) * E_\alpha(\beta) = \alpha * \beta$$

which implies that  $E_\alpha(x) * E_\alpha(\beta) \neq E_\alpha(x * \beta)$ , which in turn implies that  $E_\alpha$  is not a homomorphism, as desired.

### 3.2.8

*Skip*

### 3.2.9

For any commutative ring  $R$ , let

$$M_2(R) = \left\{ \begin{pmatrix} a & d \\ c & d \end{pmatrix} : a, b, c, d \in R \right\}$$

be the set of 2-by-2 matrices with entries in  $R$ . Define addition and multiplication in an obvious way

(a) Prove that  $M_2(R)$  is a ring.

Additive identity is the matrix with zeros in it. Multiplicative identity is the one with multiplicative identities on the diagonal, and the rest being zeroes.

Properties that this thing is an abelian group under addition are following from the fact that  $R$  is an abelian group.

Closure under multiplication is derived from the definition of multiplication. Derivations of associativity and distributivity take a lot of paper, but are pretty uneventful.

(b) Prove that  $M_2(R)$  is non-commutative

This is where our assumptions that  $0 \neq 1$  are kicking in: if a trivial ring is a singleton, then this thing is commutative. But we don't assume that, and thus

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which given that  $1 \neq 0$  implies the desired result.

(c) Find non-zero elements  $A, B \in M_2(R)$  such that  $AB = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

as desired.

(d) Let  $A \in M_2(R)$ . Prove that there's  $B \in M_2(R)$  satisfying  $AB = I$  if and only if  $ad - bc$  has a multiplicative inverse in  $R$ .

Let's start with the forward direction, and assume the appropriate things. We're gonna implicitly derive the determinant here. We follow that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} pa + rb & qa + sb \\ pc + rd & qc + sd \end{pmatrix} = I$$

thus we follow that

$$\begin{cases} pa + rb = qc + sd = 1 \\ qa + sb = pc + rd = 0 \end{cases}$$

thus

$$\begin{cases} (pa + rb) * (qc + sd) = 1 \\ (qa + sb) * (pc + rd) = 0 \end{cases}$$

$$\begin{cases} paqc + pasd + rbqc + rbsd = 1 \\ qapc + qard + sbpc + sbrd = 0 \end{cases}$$

$$\begin{cases} paqc + pasd + rbqc + rbsd = 1 \\ paqc + qard + sbpc + rbsd = 0 \end{cases}$$

thus

$$\begin{aligned} paqc + pasd + rbqc + rbsd - (paqc + qard + sbpc + rbsd) &= 1 \\ pasd + rbqc - (qard + sbpc) &= 1 \\ psad + rqbc - rqad - psbc &= 1 \\ ps(ad - bc) + rq(bc - ad) &= 1 \\ ps(ad - bc) - rq(ad - bc) &= 1 \\ (ps - rq)(ad - bc) &= 1 \end{aligned}$$

which implies that there's  $ps - rq \in R$  that is a multiplicative inverse of  $(ad - bc)$ , as desired.

Assume that  $ad - bc$  has a multiplicative inverse. The following is just a simple formula for computing inverse of the matrix in  $R^2$ . We follow that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} da - cb & -ba + ba \\ -dc + cd & -bc + ad \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

We then follow that if we would've changed every entry  $e$  in the right matrix for  $e/(ad-bc)$ , then we would've gotten identity as the result. Thus we conclude that there's a matrix  $B$  such that  $AB = I$ , as desired

(e) Let  $A, B \in M_2(R)$ . Prove that  $AB = I$  if and only if  $BA = I$

Given previous point and lots of notation, we can conclude the desired result.

(f) More generally, prove all of the above for  $M_n(R)$  for  $n \geq 2$ .

I'm not gonna prove this thing rigorously, but I'll outline a proof because of the fact that most of it is tedious and not particularly interesting.

When it comes to a ring, we get group properties for free, closure of multiplication is somewhat trivial. Distributivity can be handled piecewise: for  $(A+B)*C$  let  $q_{i,j}$  be some entry in the resulting matrix

$$q_{i,j} = \sum_{k=1}^n (a_{i,k} + b_{i,k})c_{k,j} = \sum c_{k,j}a_{i,k} + \sum c_{k,j}b_{i,k}$$

and through lots of notation we can follow the rest pretty easily, thus making  $M_n(R)$  a ring.

By putting our original matrix for  $M_2(R)$  into the upper left corner of the matrix we get non-commutativity.

Proof that  $AA^{adj} = (\det A)I$  is heavily dependent on what we choose to be the definition of the determinant. If we start with the Laplace's formula as the definition of the determinant, we can follow that matrices with identical rows/columns are zero, and thus everything that is not on diagonal is zero. We would then follow that diagonals are determinants by definition, and thus we get desired result.

$AA^{adj} = (\det A)I$  implies that if  $(\det A)$  has a multiplicative inverse in  $R$ , then we would get that  $(\det A)A^{adj}$  is an inverse of  $A$  (we would need to weave through or define scalar multiplication for matrices), which gives the desired result. Proof that invertible  $A$  commutes with  $A^{-1}$  can be shown piecewise.

One important point to note is that here we don't get any benefits for going through determinant to define such things. We would get practically the same results, if we would've gone through the usual definitions of linear transformations and whatnot.

### 3.2.10

Let  $R_1$  and  $R_2$  be commutative rings, and let  $\phi : R_1 \rightarrow R_2$  be homomorphism. We define a map  $\Phi : R_1[x] \rightarrow R_2[x]$  by

$$\Phi(\sum a_i x^i) = \sum \phi(a_i) x^i$$

(a) Prove that  $\Phi$  is a ring homomorphism.

We get identities by the fact that  $\phi$  is a homomorphism. We follow that

$$\Phi(\sum a_i x^i) + \Phi(\sum b_i x^i) = \sum \phi(a_i) x^i + \sum \phi(b_i) x^i =$$

$$\begin{aligned}
&= \sum \phi(a_i)x^i + \phi(b_i)x^i = \sum (\phi(a_i) + \phi(b_i))x^i = \\
&= \sum (\phi(a_i + b_i))x^i = \sum (\phi(a_i + b_i)x^i) = \Phi(\sum (a_i + b_i)x^i) = \Phi(\sum a_i x^i + \sum b_i x^i) \\
&\Phi(\sum a_i x^i) + \Phi(\sum b_i x^i) = (\sum \phi(a_i)x^i) * (\sum \phi(b_i)x^i) = \sum \sum \phi(a_i)\phi(b_j)x^{i+j} = \\
&= \sum \sum \phi(a_i b_j)x^{i+j} = \Phi(\sum \sum a_i b_j x^{i+j}) = \Phi((\sum a_i x^i) * (\sum b_i x^i))
\end{aligned}$$

as desired.

(b) If  $\phi$  is a ring isomorphism, prove that  $\Phi$  is a ring isomorphism

$R_1[x]$  is a polynomial, which by definition is the list. There's a general theorem (if not proven explicitly, then derived pretty easily), that piecewise application of a bijection on elements of a given list gives us a bijection. This implies that if  $\phi$  is a bijection, then  $\Phi$  is a bijection, which given homomorphism implies isomorphism, as desired.

### 3.2.11

*Lots of notation, but the essence boils down to the fact that we can define  $R[x,y]$  - a set of polynomials with two variables. All the things that concern evaluation and whatnot are essentially the same in regards to this version. We can also define such a thing for a polynomial in an arbitrary number of variables.*

### 3.2.12

*Let  $R[x,y]$  be the ring of polynomials in two variables with coefficients in  $R$ , as described in Exercise 3.13. In this exercise we will look at polynomials that don't change if we swap  $x$  and  $y$ . For example, the polynomials*

$$x + y, xy, x^2 + y^2$$

*are invariant under  $x \leftrightarrow y$  swap. We observe that our third example can be expressed using the first two examples,*

$$x^2 + y^2 = (x + y)^2 - 2xy$$

*In other words, if we let  $g_2(u, v) = u^2 - 2v$ , then  $x^2 + y^2 = g_2(x + y, xy)$*

(a) *Do the same for  $x^3 + y^3$  and  $x^4 + y^4$*

We follow that generally speaking there's a binomial expansion

$$(x + y)^n = \sum_{i=0}^n bc(i, n) x^i y^{n-i}$$

from which we follow that

$$(x + y)^n = \sum_{i=1}^{n-1} bc(i, n)x^i y^{n-i} + x^n + y^n$$

which implies that

$$\begin{aligned} x^n + y^n &= (x + y)^n - \sum_{i=1}^{n-1} bc(i, n)x^i y^{n-i} = (x + y)^n - xy \sum_{i=1}^{n-1} bc(i, n)x^{i-1} y^{n-i-1} = \\ &= (x + y)^n - xy \sum_{i=0}^{n-2} bc(i, n)x^i y^{n-i} \end{aligned}$$

Let us look at the problem at hand:

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y)$$

We follow that for  $n = 3$  we've got that

$$\sum_{i=1}^{n-1} bc(n, i)x^i = 3x^2y + 3xy^2 = 3xy(x + y)$$

thus  $\sum_{i=1}^{n-1} bc(n, i)x^i = g(x + y, xy)$  for  $n = 3$ . Assume that for all  $j$  such that  $j < n$  we've got that  $\sum_{i=1}^{j-1} bc(j, i)x^i = g(x + y, xy)$  we then follow that for arbitrary  $n$

$$\begin{aligned} \sum_{i=1}^{n-1} bc(i, n)x^i y^{n-i} &= xy \sum_{i=1}^{n-1} bc(i, n)x^{i-1} y^{n-i-1} = \\ &= xy \sum_{i=0}^{n-2} bc(i, n)x^i y^{n-i} = xy * g(x + y, xy) \end{aligned}$$

thus we follow that we've got the inductive step, and thus for all  $n \geq 3$  we've got the IH. This implies that for all  $n \geq 3$

$$x^n + y^n = (x + y)^n - \sum_{i=1}^{n-1} bc(i, n)x^i y^{n-i} = (x + y)^n - g(x + y, xy) = g'(x + y, xy)$$

as desired.

(b) *Handled in the previous point.*

(c) *Even more generally, suppose that  $f(x, y) \in R[x, y]$  is any polynomial with the symmetry property*

$$f(x, y) = f(y, x)$$

Prove that there's a polynomial  $g$  such that

$$f(x, y) = g(x + y, xy)$$

If  $f(x, y) = f(y, x)$ , then we follow that for every element of the sum  $a_i x^i y^j$  there's an element  $a_i x^j y^i$ . This implies that  $f(x, y)$  can be partitioned into pairs  $a_i x^i y^j + a_i x^j y^i$ . This implies that for each such pair we can take

$$a_i x^i y^j + a_i x^j y^i = a x^{j-i} y^{j-i} (x^i + y^i)$$

latter part  $(x^i + y^i)$  of this product is a polynomial of form  $g(x + y, xy)$  by the previous point, which in turn implies that the whole polynomial is a sum of polynomials in form  $g(x + y, xy)$ , which is itself a polynomial in form  $g(x + y, xy)$ , which implies the desired conclusion.

### 3.2.13

*I've practically done this one before in the real analysis course. There might be some deviations, but I'm pretty sure that they are minor. If it isn't the case, then skip.*

### 3.2.14

For a quaternion  $\alpha = a + bi + cj + dk \in H$ , let  $\bar{\alpha} = a - bi - cj - dk$ .

(a) Prove that  $\alpha\bar{\alpha} \in \mathbb{R}$ .

$$\begin{aligned} & (a + bi + cj + dk) * (a - bi - cj - dk) = \\ &= a^2 - abi - acj - adk + abi + b^2 - bicj - bidk + acj - cjb i + c^2 - cjdk + adk - dkbi - dkci + d^2 = \\ &= a^2 - abi - acj - adk + abi + b^2 - bck + bdj + acj + bck + c^2 - cdi + adk - bdj + cdi + d^2 = \\ &= a^2 + b^2 + c^2 + d^2 \in \mathbb{R} \end{aligned}$$

as desired.

(b) Prove that  $\alpha\bar{\alpha} = 0$  if and only if  $\alpha = 0$

Follows from the fact that  $a^2 + b^2 + c^2 + d^2 = 0 \Leftrightarrow a = b = c = d = 0$  for any elements of  $\mathbb{R}$

(c) Suppose that  $\alpha, \beta \in H$  and that  $\alpha\beta = 0$ . Prove that either  $\alpha = 0$  or  $\beta = 0$ .

Assume that  $\alpha \neq 0$ . We follow that there's  $q \in \mathbb{R}$  such that  $\alpha\bar{\alpha} = q = a^2 + b^2 + c^2 + d^2$ . We then follow that

$$\bar{\alpha} * \frac{1}{q}$$

is a multiplicative inverses of  $\alpha$ . This in turn implies that

$$\alpha^{-1} \alpha \beta = \alpha^{-1} 0$$

$$1\beta = 0$$

$$\beta = 0$$

Same idea applies to  $\beta$ , but from the different side, which implies the desired result.

(d) Let  $\alpha, \beta \in H$ . Prove that

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$$

and

$$\overline{\alpha * \beta} = \overline{\alpha} * \overline{\beta},$$

$$\begin{aligned}\overline{\alpha} + \overline{\beta} &= (a - bi - cj - dk) + (a' - b'i - c'j - d'k) = \\ &= (a + a') - (b + b')i - (c + c')j - (d + d')k = \overline{\alpha + \beta}\end{aligned}$$

by distributivity and whatnot. Multiplicity is proven in the same way, but it's more tedious.

(e) Let  $\alpha \neq 0$ . Prove that there's  $\beta \in H$  such that  $\alpha\beta = \beta\alpha = 1$

handled in point (c)

### 3.2.15

let  $R$  be a possibly non-commutative ring. The **center** of  $R$  consists of the elements of  $R$  that commute with every other element of  $R$ .

$$R^{\text{center}} = \{a \in R : \alpha\beta = \beta\alpha \text{ for every } \beta \in R\}$$

(a) Prove that  $R^{\text{center}}$  is a commutative subring of  $R$

Proof goes along the same way as the center of group.

(b) What is the center of the ring of quaternions  $H$ ?

We follow that every real entry is commutative.

$$(a + bi + cj + dk) * k = ak + bik + cjk + dk^2 = ak - bj + ci - d$$

$$k * (a + bi + cj + dk) = ak + bki + ckj + dk^2 = ak + bj - ci - d$$

this implies that  $(a + bi + cj + dk)$  is not commutative with  $k$  if  $b \neq 0$  and  $c \neq 0$ . Multiplying with  $j$  will give the same result, but for  $c$  and  $k$ , which in sum implies that only reals are in the center.

*Skipped the rest*

### 3.3 Some Important Special Types of Rings

#### Notes

Important note: definition of the field requires a commutative ring. Thus we can follow that field is a couple of abelian groups, that are stich together, have distributivity, and thus one of them is not defined over anothers identity.

For cancelation property we require we require biconditional for some reason, but  $b = c$  or  $a = 0$  implies that  $ab = ac$  for any given ring, and thus I'm assumning that there's a typo there (in the text of the theorem we even see the word "implication")

Now let us prove a somewhat trivial, but useful theorem. Suppose now that  $f : R \rightarrow R'$  is an injective ring homomorphism, and  $R'$  is an integral domain. We can follow that if  $a, b$  are such that

$$a * b = 0$$

then

$$f(a) * f(b) = 0$$

and thus  $f(a) = 0$  or  $f(b) = 0$  since  $R'$  is an integral domain. Since  $f$  is injective we follow that  $f(a) = 0 \Rightarrow a = 0$  or  $f(b) \Rightarrow b = 0$ , which implies that  $R$  is also an integral domain. Therefore

**Theorem:** If  $f : R \rightarrow R'$  is an injective ring homomorphism, and  $R'$  is an integral domain, then  $R$  is an integral domain as well.

We don't have the same thing going on with fields, sinse for example identity for  $f : Z \rightarrow R$  is injective, but  $Z$  is not a field. If  $f$  is a bijection though, then we can follow that  $R$  is a field.

#### 3.3.1

Let  $m$  be a positive integet.

(a) Prove that  $Z/mZ$  is an integral domain if and ony if  $m$  is a prime

Suppose that  $Z/mZ$  is an ingeral domain. Assume that it's not a prime. if  $m = 1$ , then this thing is trivial, and thus assume that  $m > 1$ . This implies that there's a divisor  $n$  of  $m$  that is less than  $m$ . We follow that  $m/n$  is a nonzero element of  $Z/mZ$ , and thus

$$n * (m/n) = m = 0$$

which implies that  $Z/mZ$  is not an integral domain, which is a contradiction.

Suppose that  $m$  is a prime. We follow that  $(Z/mZ)^*$  is a group under multiplication, and thus product of two nonzeroes is closed there, and thus is nonzero. This implies that  $Z/mZ$  is an integral domain, as desired.

(b) Prove that  $Z/mZ$  is a field if and ony if  $m$  is a prime



If  $Z/mZ$  is a field, then it's an integral domain, and thus  $m$  is a prime. Implications from the fact that  $(Z/mZ)^*$  is a group give us the fact that all non-zeroes got inverses, and thus  $Z/mZ$  is a field, as desired.

### 3.3.2

Let  $R$  be a field. Prove that  $R$  is an integral domain.

Suppose that  $R$  is a field, but not integral domain. The fact that it's not an integral domain implies that there are nonzero  $\alpha, \beta \in R$  such that

$$\alpha\beta = 0$$

the fact that  $\alpha$  is a nonzero in a field implies that there exists  $\alpha^{-1}$ . This implies that

$$\alpha^{-1}\alpha\beta = \alpha^{-1}0$$

$$\beta = 0$$

which is a contradiction.

### 3.3.3

Prove that each of the following rings is not a field

(a)  $Z[i]$

We follow that  $2 * (a + bi) = 2a + 2bi$ , where the latter part is not identity.

(b)  $\mathbb{R}[x]$

We follow that  $x * 1 = x$  and  $x * \sum a_i x^i = \sum a_i x^{i+1}$ , which implies the desired.

(c)  $H$

This one took me by surprise a bit, but it's important to note that  $H$  **has inverses**, but since it's not multiplicative, we follow that it's not a field. Same is true for the next point, but it is also not an integral domain.

### 3.3.4

Let  $R$  be a commutative ring. Prove that  $R$  is an integral domain if and only if it's got cancellation property.

Let  $R$  be an integral domain. Let  $a, b, c \in R$  be such that  $ab = ac$  and  $a \neq 0$ .

$$ab = ac \Leftrightarrow ab - ac = 0 \Leftrightarrow a(b - c) = 0$$

Given that  $a \neq 0$  we follow that there are no nonzero elements  $q$  of  $R$  such that  $aq = 0$ , and thus we conclude that

$$b - c = 0$$

which implies the desired result. Given that every implication here is biconditional, we get the reverse case for free.

**3.3.5**

*Let  $R$  be a finite integral domain. Prove that  $R$  is a field.*

Let  $R$  be a finite integral domain, and let  $a \in R \neq 0$ . We follow that if  $ab = ac$ , then  $b = c$ , which implies that multiplication by  $a$  is an injection. Given that  $R$  is finite and multiplication by  $a$  is a function with domain equal to its codomain, we follow that multiplication by  $a$  is a bijection, and thus there's  $b \in R$  such that  $a * b = 1$ , which implies that  $R$  is a field, as desired.

**3.3.6**

*This one is a lot of notation, but not a lot of substance*

**3.3.7**

*Let  $R$  be a commutative ring. The degree of a non-zero polynomial  $f(x) \in R[x]$  is the highest power of  $x$  that appears in  $f(x)$ .*

*(a) Prove that for all  $f(x), g(x) \in R[x]$  we have*

$$\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$$

We know that sum of polynomials is the piecewise addition of the coefficients, that correspond to each degree, thus implying that the desired result

*(b) If  $\deg(f) \neq \deg(g)$ , prove that the previous equation is an equality*

Once again, follows directly from the definition

*(c) Suppose that  $R$  is an integral domain. Prove that for all non-zero  $f(x), g(x) \in R[x]$  we have*

$$\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$$

We follow that the largest powers of  $f(x)$  and  $g(x)$  are in the form  $a_i x^i$  and  $b_j x^j$ , and thus the largest power of the product is

$$a_i x^i b_j x^j = (a_i b_j) x^{i+j}$$

the fact that  $R$  is an integral domain implies that  $a_i b_j$  is not zero, thus implying the desired result.

*(d) Let  $R = \mathbb{Z}/6\mathbb{Z}$ . Find polynomials  $f(x), g(x) \in R[x]$  with  $\deg(f) = \deg(g) = 1$  such that previous equality does not hold.*

We follow that

$$3x * 2x = 6x = 0x = 0$$

as desired.

## 3.3.8

Let  $R$  be a commutative ring.

(a) Prove that there's exactly one integral domain  $R$  such that the map

$$f : R \rightarrow R, f(a) = a^6$$

is a ring homomorphism.

Firstly, we want to state what exactly what "exactly one integral domain" mean. We want to follow that firstly there exists an integral domain such that all the rules apply, and that if there's another ring that that satisfies those conditions, then it's isomorphic to the ring, that we've made firstly.

We know that if there's a ring homomorphism from  $f(a) = a^n$ , then  $2^n - 2 = 0$  (from the second exercise in this chapter). In this case we've got  $62 = 0$ . It is important to note that this statement is an implication, and not a biconditional.

We know that prime decomposition of 62 is

$$62 = 31 * 2$$

With  $Z/2Z$  we follow that  $f$  is an identity ( $f(0) = 0^6 = 0, f(1) = 1^6 = 1$ ), and this thing is a ring homomorphism. We also follow that  $Z/2Z$  is a field, and thus an integral domain. Thus we follow that there's a ring such that  $f$  is a homomorphism (and isomorphism also).

With  $Z/3Z$  we follow that  $f(0) = 0, f(1) = 1, f(2) = 1$ , which implies that

$$f(2) + f(1) = 1$$

$$f(2 + 1) = f(3) = 0$$

which means that  $Z/3Z$  does not satisfy the conditions.

For multiplication we follow that

$$f(a * b) = (ab)^6 = a^6 b^6 = f(a) * f(b)$$

which is true for all commutative rings. Identities are also pretty trivial.

Addition is a bit more interesting:

$$f(a + b) = (a + b)^6 = \sum bc(n, i) a^n b^{n-i}$$

$$f(a) + f(b) = a^6 + b^6$$

thus we need

$$\sum_{i=1}^{n-1} bc(n, i) a^n b^{n-i} = 0$$

list of binomial coefficients in this case is

$$6, 15, 20, 15, 6$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

We can also follow that

$$\sum_{i=1}^{n-1} bc(n, i) a^n b^{n-i} = ab \sum_{i=1}^{n-1} bc(n, i) a^{n-1} b^{n-i-2}$$

We'll then follow that both  $a, b$  are nonzero, then  $ab$  is also nonzero, and thus

$$\sum_{i=1}^{n-1} bc(n, i) a^{n-1} b^{n-i-2}$$

must be zero. We then follow that none of the products of powers of  $a$  and  $b$  are zeroes. We then can also follow that

We follow that

$$f(1+1) = 2^6 = f(1) + f(1) = 2$$

thus

$$2^6 = 2$$

we then follow by induction that

$$a^6 = a$$

thus by cancellation property  $a^5 = 1$  or  $a = 0$ .

Suppose that there's a ring, that's got distinct elements  $0, 1, k$ . We follow that

$$f(k+1) = (k+1)^6$$

$$f(k) + f(1) = f(k) + 1 = k^6 + 1$$

Let's start over:

Let  $0, 1, k$  be all distinct items of  $R$ . We follow that

$$f(k) + f(1) = k^6 + 1$$

$$\begin{aligned} f(k+1) + f(k-1) + f(1-k) &= f(k) + f(1) + f(k) + f(-1) + f(k) + f(-1) = \\ &= k^6 + 1 + k^6 + 1 + k^6 + 1 = 3k^6 + 3 \end{aligned}$$

$$f(k+1+k-1+1-k) = f(k+1) = k^6 + 1$$

thus we follow that

$$3k^6 + 3 = k^6 + 1$$

$$3(k^6 + 1) = k^6 + 1$$

thus

$$3 = 1$$

### 3.3.9

Let  $R$  be a ring. We define three properties that an element  $a \in R$  may possess.

$a$  is nilpotent if  $a^n = 0$  for some  $n \geq 1$ .

$a$  is unipotent if  $a - 1$  is nilpotent

$a$  is idempotent if  $a^2 = a$ .

(a) If  $R$  is an integral domain, describe all of the nilpotent, unipotent, and idempotent elements of  $R$ .

We follow that if  $a^n = 0$ , then  $a = 0$  by cancellation property, which implies that there's one nilpotent element.

If  $a - 1$  is nilpotent, then  $(a - 1)^n = 0$ , thus  $(a - 1) = 0$ , and therefore  $a = 1$ .

If  $a$  is idempotent, then  $a^2 = a$ , thus  $a = 0$  or  $a = 1$  by cancellation property.

(b) Let  $p \in \mathbb{Z}$  be a prime and let  $k \geq 1$ . Describe all of the nilpotent elements in  $\mathbb{Z}/p^k\mathbb{Z}$ .  
Skipped the rest of the exercises for later.

## 3.4 Unit groups and Product rings

### 3.4.1

(a) Compute the unit group  $\mathbb{Z}^*$

We follow that anything times zero is zero, so it's out of the question for all the subsequent things.

We know that  $-1, 1$  have themselves as inverses, and any other number fails to have one.

(b) Compute the unit group  $\mathbb{Q}^*$

All of the  $\mathbb{Q}^*$  is the unit group.

(c) Compute the unit group  $\mathbb{Z}[i]^*$

We follow that  $-1, 1$  are in the unit group.  $i^4 = 1$ , and thus  $i, -i$  are also in the group. Thus we follow that for the more general case

$$a + bi$$

that if one of the numbers  $a, b$  are equal to 1, while the other is zero, we've got it in the unit group. If  $a = b = 1$ , then we're out of luck though.

We know that

$$1/(a + bi) = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

and thus in order to  $a + bi$  to be in  $Z[i]$  we need both  $\frac{a}{a^2 + b^2}$  and  $\frac{b}{a^2 + b^2}$  to be integers. We follow that

$$|a| \leq a^2 \leq a^2 + b^2$$

and for  $a \notin \{-1, 0, 1\}$  we follow that

$$|a| < a^2 \leq a^2 + b^2$$

which implies that there for the case  $a \notin \{-1, 0, 1\}$  there are no inverses for  $a + bi$ . We've got the same conclusions for  $b$ , and thus we conclude that  $\{1, -1, i, -i\}$  is the unit group (which seems to be pretty straightforward).

(d) Consider the ring  $Z[\sqrt{2}]$ . Prove that  $1 + \sqrt{2} \in Z[\sqrt{2}]^*$ . Prove that the powers of  $1 + \sqrt{2}$  are all different, and use that fact to deduce that  $Z[\sqrt{2}]$  has infinitely many elements.

We follow that

$$\frac{1}{1 + \sqrt{2}} = \frac{(1 - \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{(1 - \sqrt{2})}{1 - 2} = -(1 - \sqrt{2}) = -1 + \sqrt{2}$$

by pretty much same derivation we can follow that  $(1 + \sqrt{2})^n$  (and by extension  $(1 - \sqrt{2})^n$ ) are in  $Z[\sqrt{2}]^*$ . We follow that power function is strictly increasing, and thus  $(1 + \sqrt{2})^n$  are all different for all  $n \in \omega$ , and thus  $Z[\sqrt{2}]^*$  has infinite amount of elements.

(e) Prove that  $\mathbb{R}[x]^* = \mathbb{R}^*$

We follow that by multiplying any nonzero polynomial in  $\mathbb{R}$  by other polynomial nonzero we get only polynomials of equal or greater degree, and thus follow that the only one order of polynomial that has an option of having multiplicative inverses is the polynomial of degree 1.

(f) Prove that  $1 + 2x$  is a unit in the ring  $(Z/4Z)[x]$

$$1 + 2 * 0 = 1$$

$$1 + 2 * 1 = 3$$

$$1 + 2 * 2 = 1$$

$$1 + 2 * 3 = 3$$

we then follow that in order to be constant, elements that result in 3 should be, when multiplied, be equal to 1. We follow that  $3 * 3 = 1$  in this case, and thus we follow that

$$(1 + 2x)^2 = 1$$

which implies that  $1 + 2x$  has itself for an inverse, as desired

*Maybe I'll tackle the last one later*

**3.4.2**

(a) Let  $R$  be commutative ring, and suppose that its unit group  $R^*$  is finite with  $|R^*| = n$ . Prove that every element of  $a \in R^*$  satisfies

$$a^n = 1$$

We follow that since  $R^*$  is finite that  $a$  has an order  $k$ . This order  $k$  divides order of the group  $n$ , and thus

$$a^n = a^{k*j} = 1^j = 1$$

as desired.

(b) - already taken care of in previous exercises

**3.4.3**

(a) Compute the unit group  $(\mathbb{Z}/p\mathbb{Z})^*$  for each of the primes 7, 11, 13. Which ones are cyclic?

Since they are all primes, we follow that groups are exactly  $(\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ .

In  $(\mathbb{Z}/7\mathbb{Z})^*$  we follow that order of 3 is 6. In  $(\mathbb{Z}/11\mathbb{Z})^*$  we follow that order of 2 is 10. For 13 we've got that 2 has order 12.

(b) Compute the unit group  $(\mathbb{Z}/m\mathbb{Z})^*$  for each of the composite numbers  $m = 8, 9, 15$ . Which ones are cyclic?

We follow that

$$(\mathbb{Z}/8\mathbb{Z})^* = \{1, 3, 5, 7\}$$

we then follow that  $3^2 = 5^2 = 7^2 = 1$ , and thus none of them are cyclic.

We follow that

$$(\mathbb{Z}/9\mathbb{Z})^* = \{1, 2, 4, 5, 7, 8\}$$

Order of 2 is 6, which means that this thing is cyclic.

We follow that

$$(\mathbb{Z}/15\mathbb{Z})^* = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

Orders of those elements get up to 4, so this group is not cyclic.

**3.4.4**

Let  $R_1, \dots, R_n$  be rings.

(a) Prove that the product  $R_1 \times \dots \times R_n$  is a ring.

Follows directly from piecewise application of standard rules of the respective rings.

(b) Yeah, projection is a homomorphism

Skip the rest

**3.4.5**

*Prove that the product ring  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  is isomorphic to the ring  $\mathbb{Z}/6\mathbb{Z}$  by writing down an explicit isomorphism  $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$*

Let  $f$  be a presumed isomorphism between the rings. We follow that

$$0 \rightarrow (0, 0)$$

$$1 \rightarrow (1, 1)$$

by properties of homomorphism. The fact that  $\mathbb{Z}/6\mathbb{Z}$  is cyclic and generated by 1 gives us an idea on how to proceed:

$$2 \rightarrow (0, 2)$$

$$3 \rightarrow (1, 0)$$

$$4 \rightarrow (0, 1)$$

$$5 \rightarrow (1, 2)$$

From this we can follow that the first index of value of our function in  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  is 0 on even numbers and 1 on odd, which kinda gives us an idea that it might have to do something with % operation. Same goes for the latter index, which gives us an explicit definition.

$$f(x) = (x \% 2, x \% 3)$$

Explicit enumeration here gives us the proof that the function is a bijection. Properties of homomorphism can be followed by either explicit enumeration, of perhaps operations are transferred by % function.

**3.4.6**

*Let  $R$  be a non-commutative ring. The group of units of  $R$  is*

$$R^* = \{a \in R : \exists b, c \in R : ab = ca = 1\}$$

(a) *If  $ab = ca = 1$ , prove that  $b = c$ .*

We follow that

$$b = eb = cab = ce = c$$

(b) *Prove that  $R^*$  is a group, where we use multiplication for the group law.*

Identity is an inverse for itself, and hence is present in the thing. Associativity is taken care of by the properties of the ring, and thus we need only to handle the closure on inverses. Let  $a \in R^*$ . We follow that there are  $b, c \in R$  such that

$$ab = ca = 1$$



Previous point implies that  $b = c$ , and thus

$$ab = ba = 1$$

and thus we conclude that  $b$  is the inverse of  $a$ . By the same point we follow that  $b \in R^*$ , and thus we conclude that  $a \in R^* \Rightarrow a^{-1} \in R^*$ , as desired.

### 3.4.7

*Skip*

### 3.4.8

Let  $R$  be a ring.  $e$  is idempotent if  $e^2 = e$ .

*Errata shows that there's an error in this exercise, either  $R$  should be commutative, or  $e$  should be in the center.*

(a) Let  $e \in R$  be idempotent. Prove that  $1 - e$  is idempotent and that the product of  $e$  and  $1 - e$  is 0.

We follow that

$$(1 - e)^2 = (1 - e)(1 - e) = 1 * 1 - 1 * e - e * 1 + e * e = 1 - e - e + e = 1 - e$$

thus  $1 - e$  is idempotent by definition.

We follow that

$$(1 - e)e = e - e * e = e - e = 0$$

once again by definitions.

(b) Let  $e \in R$  be idempotent with  $e \neq 0$ . Prove that we can turn  $eR = \{ea : a \in R\}$  into a ring by treating  $e$  as the multiplicative identity.

We follow that  $0 \in eR$ ,  $a \in R \Rightarrow -a \in R$ , and thus  $ea \in eR \Rightarrow -ea \in eR$ , thus we've got inverses, and associativity and commutativity is handled by properties of  $R$ , thus making  $R$  into an abelian group under  $+$ .

Let  $q \in eR$ . We follow that there's  $a \in R$  such that  $q = ea$ , and thus  $eq = eea = ea = q$ , thus making  $e$  into an identity over  $R$ . Since  $1 \in R$ , we follow also that  $e \in eR$ . Associativity and distributivity over  $eR$  is inherited by  $R$ , and thus we conclude that  $eR$  is indeed a ring, as desired.

(c) Let  $e \in R$  be idempotent with  $e \neq 0, e \neq 1$ . Prove that the map

$$R \rightarrow eR \times (1 - e)R$$

$$a \rightarrow (ea, (1 - e)a)$$

is a ring isomorphism.

Under assumption that  $R$  is commutative we follow that

$$e(a + b) = ea + eb$$

$$e(a * b) = e * e * a * b = e * a * e * b$$

for any idempotent  $e$ , which implies that given function is a product of homomorphisms, which means that the whole thing is a homomorphism.

Let  $a, b \in R$  be arbitrary. If  $f(a) = f(b)$  we follow that  $ea = eb$  and  $(1 - e)a = (1 - e)b$ . Since  $ea = eb$  we follow that

$$(1 - e)a = (1 - e)b$$

$$b - ea = a - eb$$

$$b - ea = a - ea$$

$$b = a$$

which gives us an implication  $f(a) = f(b) \Rightarrow a = b$ , which implies that  $f$  is injective.

Let  $q \in eR \times (1 - e)R$ . We follow that there are  $a, b \in R$  such that

$$q = (ea, (1 - e)b)$$

We follow that

$$f(ea) = (eea, (1 - e)ea) = (ea, 0a) = (ea, 0)$$

$$f((1 - e)b) = (e(1 - e)b, (1 - e)(1 - e)b) = (0b, b) = (0, b)$$

and thus

$$f(ea + (1 - e)b) = f(ea) + f((1 - e)b) = (ea, 0) + (0, (1 - e)b) = (ea, (1 - e)b) = q$$

by homomorphisms and whatnot, which implies that for arbitrary  $q$  there's an elements of  $R$  that is mapped into  $q$ , which means that  $f$  is surjective, which gives us isomorphism, as desired.

(d) Let  $R_1, R_2$  be rings. Find idempotents  $e_1, e_2 \in R_1 \times R_2$  that also satisfy  $e_1 e_2 = 0$ . Prove that if  $\alpha \in R_1 \times R_2$ , then there are unique elements  $a_1 \in R_1$  and  $a_2 \in R_2$  such that  $\alpha = a_1 e_1 + a_2 e_2$ .

Something tells me that we can use  $\langle 1, 0 \rangle$  and  $\langle 0, 1 \rangle$  as our  $e_1$  and  $e_2$  respectively. We then follow that if  $\alpha \in R_1 \times R_2$ , then it's in the form  $\langle a, b \rangle$ , which in turn implies that we can take

$$a * e_1 + b * e_2$$

to be our value. The fact that this factorization is unique is kinda trivial

## 3.4.9

We use a symbol  $L$  to define a ring

$$R = \{a + bL : a, b \in Z\}$$

where addition is defined in an obvious way and multiplication is defined by using the rules

$$L^2 = 1, La = aL$$

For the rest of this thing we're assuming that  $L \notin Z$ , since otherwise some things don't hold

(a) Find a formula for the product  $(a + bL) * (c + dL)$ .

We follow that

$$(a + bL) * (c + dL) = ac + adL + bLc + bLdL = ac + bd + (ad + bc)L$$

(b) Prove that the ring  $R$  has zero divisors

If  $L = 1$ , then it does not. But if we add an assumption that  $L \notin Z$ , then

$$(1 - L)(1 + L) = 1 + L - L - L^2 = 1 - 1 = 0$$

(c) Prove that the ring  $R$  is not isomorphic to the ring  $Z \times Z$

Assume that there's an isomorphism  $f$ . Identities are mapped pretty easily,

$$0 \rightarrow (0, 0)$$

$$1 \rightarrow (1, 1)$$

but we then need to map  $L$  somewhere. We follow that there is  $(a, b)$  such that  $f(L) = (a, b)$ . Since  $L$  is not in the  $Z$ , we conclude that  $a \neq b$ . We then follow that  $L * L = 1$ , and thus

$$a * a = 1$$

$$b * b = 1$$

and since there are no multiplicative inverses for any  $a, b \in Z$  apart from 1 and  $-1$  we follow that  $a, b \in \{1, -1\}$ . This in turn implies that either  $f(L) = (-1, 1)$  or  $f(L) = (1, -1)$ . Assume the latter. We then follow that

$$f(1 + L) = (2, 0)$$

$$f(1 - L) = (0, 2)$$

Now let's try to square some numbers:

$$f((1 + L)(1 - L)) = f(1 - L + L - L^2) = f(0) = 0$$

$$f((1+L)(1-L)) = (2, 0) * (0, 2) = 0$$

which is unhelpful.

$$\begin{aligned} f((1-L)(1-L)) &= f(1-L-L+L^2) = f(2-2L) = (0, 4) \\ f((1-L)(1-L)) &= (0, 2) * (0, 2) = (0, 4) \end{aligned}$$

which is also unhelpful.

I'm pretty sure that we somehow need to prove that the  $R$  has multiplicative inverses for some elements, but  $Z \times Z$  does not. We can follow that

$$(2-L) * (1+L) = 2 + 2L - L - 1 = 1 + L$$

while

$$((2, 2) - (1, -1)) * ((1, 1) + (1, -1)) = (1, 3) * (2, 0) = (2, 0)$$

which might be wrong again.

Let us try to prove the thing then, and maybe we'll have some problems then. All the identities are handled, additivity is trivial, and for multiplication we've got

$$\begin{aligned} f((a+bL) * (c+dL)) &= f((ac+bd) + (ad+bc)L) = (ac+bd+ad+bc, ac+bd-ad-bc) \\ f(a+bL) * f(c+dL) &= (a+b, a-b) * (c+d, c-d) = (ac+ad+bc+bd, ac-ad-bc+bd) \end{aligned}$$

so we're kinda shafted in this regard. Maybe the  $f$  is not a bijection then. It's certainly a function from  $R$  to  $Z \times Z$ , and is defined on all  $R$ . If there are  $a, b \in R$  such that  $a \neq b$  and  $f(a) = f(b)$ , then we conclude that  $f(a-b) = f(a) - f(b) = 0$ , and hence there's a nonzero element  $q \in R$  such that it maps to 0. Construction of  $f$  though prohibits such a thing, and thus we must conclude that it's an injection.

Okay, now I've got it. Let  $a+bL \in R$ . We follow that

$$f(a+bL) = (a+b, a-b)$$

we then conclude that the difference between those two

$$a+b-a+b=2b$$

is always even. Thus we conclude that there's no way for example to map the thing to  $(1, 0)$ , since the difference between indices is odd, and thus we conclude that  $f$  is not surjective, and thus we conclude that  $R$  and  $Z \times Z$  are not isomorphic.

(d) However, prove that the ring

$$S = \{a+bL : a, b \in Q\}$$

is isomorphic to the ring  $Q \times Q$ .

This case grants us on the other hand the ability to get surjectivity, that we can prove through a lot of notation, but it's easy to see why it is the case from the last paragraph of the previous point.

**3.4.10**

Let  $R_1, R_2, \dots$  be an infinite list of rings. blah-blah-blah, we can define rings from infinite list of rings by the same reason as for the multiple rings (i.e. piecewise application). Part (b) comes from the fact that the ring that is there's an injection from  $2^\omega$  to  $Z/2Z^\omega$

Skip the rest

**3.5 Ideals and Quotient Rings****Notes**

Firstly it's important to note that ideals are mostly not subrings due to the fact that the ones that aren't equal to the whole space  $R$  don't have multiplicative identities in them. They are subgroups under addition however, which is evident from their definition.

Given that ideals are subgroups under the addition, we can conclude from our endeavours in group theory that the set of distinct cosets of an ideal constitutes a partition of a ring  $R$ . A partition of a ring  $R$  with respect to an ideal  $I$  is denoted by  $R/I$ . Apart from the initial ideal  $I$  we can follow that no other element of  $R/I$  is an ideal due to the fact that any given ideal  $I$  has got to have 0 in it.

Suppose that we've got an  $R/I$ . Let  $q \in R/I$  and  $b \in q$ . Since  $q \in R/I$  we follow that by definition there is an element  $j \in R$  such that

$$q = j + I$$

(where sum of an element of  $R$  and a subset of  $R$  is defined in an obvious way).  $b \in q$ , and thus by definition of addition of a set to an element we conclude that there's an element  $k \in I$  such that

$$b = j + k$$

and thus

$$j = b - k$$

and therefore if  $w \in q$  is an arbitrary element, then there's  $e \in I$  such that

$$w = j + e$$

thus

$$w = b - k + e$$

since  $k, e \in I$  we follow that  $-k \in I$ , thus  $-k + e \in I$ , and therefore  $w \in b + I$ . Thus  $q \subseteq b + I$ . The case with  $\supseteq$  is kinda similar, which gives us an identity  $q = b + I$ , which together with the fact that  $b$  and  $q$  are arbitrary element gives us a biconditional

$$q \in R/I \wedge b \in q \Leftrightarrow q = b + I$$

thus any given element of  $R/I$  can be represented as a sum of any of its elements and  $I$ .

Formulas that were given in the book don't give out too much rigor, and thus let us clarify the whole shebang:

In the book we're given formulas

$$(a + I) + (b + I) = (a + b) + I$$

$$(a + I) * (b + I) = (a * b) + I$$

and without any context this thing does not make much sense. We know that for any given coset  $q$  there's an element  $a \in R$  such that  $q = a + I$  (which we're gonna call an *additive representation* of a coset). Additive representations are only representations, and from our previous point in the notes we can follow that any given coset can have several additive representations, and a single representation refers only to one coset. Thus this representation can be formulated as "one-to-many" from cosets to representations.

With our notion of representations we can now make sense of the formulas. Given two representations we follow that there exists a single representation, that conforms to the provided formulas. Given that there is a single coset that is represented by a representation, we conclude that formula essentially is a binary function from representations to cosets. In the book (and further in the exercises) we are proving the fact that those formulas are "well defined", i.e. that given two cosets, there exists only one coset, to which their respective representations are being sent, and thus that we can define a proper function from pairs of cosets to cosets, that conforms to the formula.

Although with proper maintenance we can define sums and products in a way, that they were defined in the book, I wonder if we could define the same thing in terms of ranges. Given  $a, b \in R/I$ , we can define sum and product functions  $+: R/I \times R/I \rightarrow \mathcal{P}(R)$   $*: R/I \times R/I \rightarrow \mathcal{P}(R)$  by

$$a + b = \{q + w \in R : q \in a \wedge w \in b\}$$

$$a * b = \{q * w \in R : q \in a \wedge w \in b\}$$

where it is essentially a range of underlying  $+$  and  $*$  respectively over a set of pairs between elements of  $a$  and  $b$ . Problem is that those functions are having a codomain of  $\mathcal{P}(R)$ , but we want to prove specifically that we can restrict codomains of those functions to  $R/I$ . Material in the book and the exercises certainly implies that ranges of functions over underlying sets are subsets of a single coset (i.e. given  $a, b \in R/I$  there's  $c \in R/I$  such that  $a + b \subseteq c$  and the same goes for  $*$ ).

UPDATE: after doing some exercises, I became convinced that this thing won't work after all. Defining product as a sum of products (as with ideals) might solve the problem, but I won't investigate this thing further.

In order to try to prove that our definitions are indeed equivalent to the ones in the book, we might want to prove some other things about cosets. Notably we need to show

that if  $b' \in R/I$ ,  $w \in I$ , and  $b \in b'$ , then  $b + w \in b'$ . We can do that by a straightforward application of the definition of coset.

Let  $a', b', w \in R/I$  be such that  $a' + b' \subseteq w$ . Let  $q \in w$ . We follow that since  $q \in w$  that  $w = q + I$ . Let  $a \in a'$ ,  $b \in b'$ . We follow that  $a + b \in w$ , and thus  $a + b \in q + I$ . Therefore there is an element  $e \in I$  such that  $a + b = q + e$ . Therefore  $a + b - e = q$ . Since  $e \in I$  we follow that  $-e \in I$ , thus  $b - e \in b'$ , and therefore we conclude that for  $q \in w$  there is a pair of element  $\langle a, b - e \rangle \in a' \times b'$  such that  $q = a + b - e$ , and thus  $w \subseteq a' + b'$ , which implies that  $a' + b' = w$ .

With our definitions we can try to produce formulas, that were given in the book: Let  $a', b' \in R/I$ . Let  $a \in a'$ ,  $b \in b'$ . We follow that

$$a' = a + I$$

and

$$b' = b + I$$

as discussed previously. Now let us prove the identity in the book. Let

$$q \in (a + b) + I$$

We follow that it is the case if and only if there's  $w \in I$  such that

$$q = (a + b) + w \Leftrightarrow q = (a + 0) + (b + w)$$

which implies that

$$q \in (a + I) + (b + I)$$

Now assume that  $q \in (a + I) + (b + I)$ . We follow that there are  $w, e \in I$  such that

$$q = a + w + b + e$$

thus

$$q = (a + b) + (w + e)$$

given that  $w, e \in I$  we follow that  $w + e \in I$  and thus

$$q \in (a + b) + I$$

thus giving us an identity

$$(a + I) + (b + I) = (a + b) + I$$

which gives us the formula from the book.

Let  $a, b \in R$ . We follow that

$$q \in (a + I) * (b + I)$$

implies that there are  $c, d \in I$  such that

$$q = (a + c) * (b + d)$$

$$q = ab + cb + ad + cd$$

we follow that  $cb, ad, cd \in I$  by definition of ideal, and thus

$$cb + ad + cd \in I$$

therefore

$$q \in ab + I$$

which gives us a statement

$$(a + I) * (b + I) \subseteq a * b + I$$

### 3.5.1

Let  $R$  be a commutative ring.

(a) Let  $c \in R$ . Prove that

$$\{cR : r \in R\}$$

is an ideal of  $R$ .

Let's name the unnamed set  $Q$  and assume that  $q, w \in Q$ . We follow that there are  $a, b \in R$  such that  $q = ca$  and  $w = cb$ . We then follow that

$$q + w = ca + cb = c(a + b) \in Q$$

where we derive the last relation by definition of  $Q$ .

If  $r \in R$ , then we conclude that

$$rq = rca = cra \in Q$$

where deriviations come from the fact that  $R$  is commutative. Hence we conclude that  $Q$  is an ideal by definition.

(b) ... straightforward applications of definitions and commutative/distributive properties of  $R$  will give us this conclusion

### 3.5.2

Let  $R$  be a commutative ring. Prove that  $R$  is a field if and only if its only ideals are the zero ideal and the entire ring  $R$ .

Assume that  $R$  is a field and let  $c \in R$  be nonzero. We follow that there's  $c^{-1} \in R$ , and thus  $c^{-1}R$  is a subset of  $R$  by closure. Multiplying  $c$  by this subset we get that  $R \subseteq cR$ , and thus  $c \neq 0 \Rightarrow cR = R$ . If  $c = 0$  we can trivially conclude that  $cR = \{0\}$ .



Now assume that  $R$  is a commutative ring whose only ideals are  $\{0\}$  and  $R$ . Let  $q \in R$  be nonzero. We follow that  $qR = R$ , and thus there's  $w \in R$  such that  $qw = 1$ . This implies that  $w = q^{-1}$ , and thus we conclude that the fact that  $q$  is arbitrary together with this implication constitutes a definition of a field, as desired.

### 3.5.3

Let  $R$  be a commutative ring, and let  $I$  be an ideal of  $R$ . The radical of  $I$  is defined to be the set

$$\text{Rad}(I) = \{a \in R : (\exists n \in \omega)(n \geq 1 \wedge a^n \in I)\}$$

Prove that  $\text{Rad}(I)$  is an ideal of  $R$ .

Let  $a \in \text{Rad}(I)$  and  $b \in R$ . We follow that there's  $n \in \omega$  such that  $a^n \in I$  by definition of  $\text{Rad}(I)$ . We also follow that since  $b \in R$  that  $b^n \in R$  and thus we follow that  $(ab)^n = a^n b^n$  is a product of an element of  $I$  and an element of  $R$ , and thus is in  $I$  by the fact that  $I$  is an ideal.

Let  $a \in \text{Rad}(I)$ . We follow that there's a  $n \in \omega$  such that  $a^n \in I$ . We then follow that for all  $m \geq n$  that  $a^m = a^n a^{m-n}$  and thus  $a^m$  is a multiple of an element of  $I$  ( $a^n$ ) and an element of  $R$ , and thus it is itself an element of  $I$ .

Now assume that  $a, b \in \text{Rad}(I)$  and  $m, n \in \omega$  are such that  $a^m, b^n \in I$ . We follow that

$$(a + b)^k = \sum_{i \in \{0..k\}} bc(i, k) a^i b^{k-i}$$

for any  $k \in \omega$ . Thus if  $k \geq 2 \max(m, n)$ , then for any element of the sum

$$bc(i, k) a^i b^{k-i}$$

either  $i$  or  $k - i$  is greater than  $\max(m, n)$ , and thus  $a^i$  or  $b^{k-i}$  is in  $I$ , thus the whole product is in  $I$ , and thus then the sum is a sum of elements of  $I$ , thus making the whole sum an element of  $I$ . Thus we follow that there's an element  $n$  of  $\omega$  such that  $(a + b)^n \in I$ , and therefore we conclude that  $a + b \in \text{Rad}(I)$ .

Those two properties constitute a definition of the ideal, as desired.

### 3.5.4

The goal of this exercise is to prove that every ideal in  $Z$  is principal ideal.

(a) Let  $I$  be a non-zero ideal in  $Z$ . Prove that  $I$  contains a positive integer.

If  $I$  is a non-zero ideal, then it's got  $a \in I$  such that  $a \neq 0$ . If  $a > 0$ , then we're done, and if not, then we follow that  $-1 * a > 0 \in I$ .

(b) Let  $I$  be a non-zero ideal in  $Z$ . Let  $c$  be the smallest positive integer in  $I$ . Prove that every element of  $I$  is a multiple of  $c$ .

We follow that all the multiples of  $c$  are in  $I$ . If there's some element  $q \in I$  that is not a multiple of  $c$ , then we follow that  $\gcd(q, c) < c$ , and thus there are integers  $a, v \in \mathbb{Z}$  such that  $aq + vc = \gcd(q, c)$ . Since  $q, c \in I$  we follow that  $\gcd(q, c) \in I$ , and thus we conclude that there's a positive element of  $I$  that is less than  $c$ , which is a contradiction.

(c) *Prove that every ideal in  $\mathbb{Z}$  is principal.*

Directly follows from the previous point.

### 3.5.5

*Prove the remaining parts of Proposition 3.32. Let  $R$  be a commutative ring, and let  $I$  be an ideal of  $R$ .*

(a) *Let  $a + I$  and  $a' + I$  be two cosets. Prove that  $a + I = a' + I$  if and only if  $a - a' \in I$ .*

Suppose that  $a + I = a' + I$ . We follow that there are elements  $i, i' \in I$  such that  $a + i = a' + i'$ . Then we follow that  $a - a' = i' - i$ . Since  $i', i \in I$  we follow that  $-i \in I$  and  $i' - i \in I$ , and thus  $a - a' \in I$ .

If  $a - a' \in I$ , then we follow that there's an element  $i \in I$  such that  $a - a' = i$ , thus  $a = a' + i$ . We then follow that for every element  $q \in a + I$  there's an element  $i' \in I$  such that  $q = a + i'$ , and thus  $q = a' + i + i'$ . By the additive closure of  $I$  we follow that  $i + i' \in I$ , and thus we conclude that  $q \in a' + I$ , which gives us the  $\subseteq$  relation between those sets. Reverse direction is kinda similar, and thus we conclude the desired reverse implication.

(b) *Prove that addition of cosets is well-defined.*

Was taken care of in the notes.

(c) *Prove that addition and multiplication of cosets turns  $R/I$  into a commutative ring.*

Definition of the sum and the axioms of underlying ring imply all the necessary conditions in a straightforward fashion.

### 3.5.6

*Let  $R$  be a commutative ring.*

(a) *Let  $I$  be an ideal of  $R$ . Prove that the map  $f$*

$$R \rightarrow R/I, a \mapsto a + I$$

*sending an element to its coset is a surjective ring homomorphism whose kernel is  $I$ . This is proposition 3.34(a)*

$R/I$  is a partition of  $R$ , and thus if  $q \in R/I$ , then there's  $w \in q \subseteq R$ , thus  $f(w) = q$ , which implies that  $f$  is surjective. Identities are mapped properly by definition, and the fact that the thing is a homomorphism is handled by definitions of  $R/I$ . If  $q \in I$ , then  $f(q) = q + I = 0 + I$ , and otherwise it isn't, which implies that the whole  $f(q) = 0$  if and only if  $q \in I$ , thus making  $I$  into the kernel of  $f$ .

(b) Let  $I$  and  $J$  be ideals of  $R$ . Prove that the map  $g$

$$R \rightarrow R/I \times R/J, a \rightarrow (a + I, a + J)$$

is homomorphism. What is its kernel? Give an example where it is surjective and an example for which it is not surjective.

Although this thing looks unconventional, it is nothing but a concatenation of homomorphisms and thus a homomorphism itself. Kernel of this thing is the intersection of underlying rings ( $I$  and  $J$  respectively in this case).

We can set  $I = 2Z$ ,  $J = 3Z$ , and thus if  $(a, b) \in Z/2Z \times Z/3Z$ , then we follow that we have set  $b + 3Z$  has both even and odd elements, and thus we're able to produce proper element depending on  $a$ , thus making the map surjective. We can also set  $I = J = 2Z$ , from which we can't produce element  $(1, 0)$ , which gives us a non-surjective map.

### 3.5.7

Let  $I$  be the principal ideal of  $\mathbb{R}[x]$  generated by polynomial  $x^2 + 1$ . Prove that the map

$$\phi : \mathbb{R}[x]/I \rightarrow C, \phi(f(x) + I) = f(i)$$

is a well-defined isomorphism, where  $i = \sqrt{-1}$ , as usual.

If  $q$  is an element in  $R/I$ , then we follow that for all  $f \in q$  we get

$$q = f + I$$

Thus if for any  $q \in R/I$  we follow that for all  $f_1, f_2 \in q$

$$f_1(i) = f_2(i)$$

then can conclude that any element of  $q$  maps  $i$  to the same value, and thus there's a function from  $R/I$  to  $C$  that maps the things exactly in a way that the exercise describes. Now let us try to prove that and let  $q \in R/I$  and  $f_1, f_2 \in q$ . We follow that there's  $w \in C$  such that

$$f_1(i) = w$$

Since  $f_1, f_2 \in q$  we follow that  $f_1 - f_2 \in I$ , and by definition of  $I$  we follow that there's  $c \in \mathbb{R}[x]$  such that

$$f_1 - f_2 = c(x^2 + 1)$$

therefore

$$f_2 = f_1 - c(x^2 + 1)$$

and thus

$$f_2(i) = f_1(i) - c(x^2 + 1) = w - c(-1 + 1) = w - c0 = w$$

which all implies the fact that the whole shebang is "well-defined".

Now we need to prove that this thing is an isomorphism. Identities are easy, functions are as well, and hence this thing is a homomorphism. Bijectivity is kinda interesting though.

If  $r \in I$ , then  $r(i) = c(-1 + 1) = 0$ . From our previous exercises know that for any polynomials in commutative rings

$$r(k) = 0 \iff r(x) = (x - k)f(x)$$

for some  $f \in R$ , and thus  $r(i) \neq 0$  implies that there's no polynomial  $g \in R$  such that  $r = (x - i)g$ . In polynomials in real coefficients it's also not too complicated to prove that  $\overline{r(x)} = \overline{r(i)}$ , which means that  $-i$  is also not a root for this polynomial, which gives us the fact that  $r \neq (x - i)(x + i)g = (x^2 + 1)$ . This implies that  $r \notin I$ , and thus kernel of  $\phi$  is  $I$ . Theorem in the book proves then that  $\phi$  is injective.

If  $k \in C$ , then we follow that there are  $a, b \in R$  such that  $k = a + bi$ , and thus

$$f(x) = a + bx$$

is an element of  $R$  such that  $f(i) = k$ , which implies that  $\phi$  is surjective, which gives us the fact that  $\phi$  is an isomorphism, as desired.

### 3.5.8

Let  $R$  be a commutative ring and let  $I$  be an ideal of  $R$ . For  $a \in R$ , we will denote the coset  $a + I$  by  $\bar{a}$ , i.e.

$$R \rightarrow R/I : a \rightarrow \bar{a}$$

is the reduction modulo  $I$  homomorphism. We define a map of polynnomial rings

$$\phi : R[x] \rightarrow (R/I)[x], \phi(\sum a_i x^i) = \sum \bar{a}_i x^i$$

by reducing the coefficieents modulo  $I$ . Prove that  $\phi$  is a ring homomorphism.

We can easily follow that identities are mapped as expected. The rest is basically taken care of by the definitions and previous proves:

$$\phi(a + b) = \phi(\sum a_i x^i + \sum b_i x^i) = \phi(\sum (a_i + b_i) x^i) = \sum \overline{(a_i + b_i)} x^i$$

$$\phi(a) + \phi(b) = \sum \bar{a}_i x^i + \sum \bar{b}_i x^i = \sum \overline{(a_i + b_i)} x^i$$

$$\phi(a * b) = \phi(\sum a_i x^i * \sum b_j x^j) = \phi(\sum \sum a_i b_j x^{i+j}) = \sum \sum \overline{a_i b_j} x^{i+j}$$

$$\phi(a) * \phi(b) = \sum \bar{a}_i x^i * \sum \bar{b}_j x^j = \sum \sum \overline{a_i b_j} x^{i+j}$$

as desired.

## 3.5.9

Let  $R$  be a commutative ring and let  $I$  and  $J$  be ideals of  $R$

(a) Prove that the intersection  $I \cap J$  is an ideal of  $R$ .

We follow that if  $w, e \in I \cap J$ , then  $w + e \in I$  by axioms of  $I$  and  $w + e \in J$  by axioms of  $J$ , and thus  $w + e \in I \cap J$ . If  $q \in R$  is arbitrary, then  $wq \in I$  by axioms of  $I$  and  $wq \in J$  by axioms of  $J$ , which implies that  $wq \in I \cap J$ . This constitutes a definition of an ideal, as desired.

(b) Prove that the ideal sum (that is defined in an obvious way) is an ideal of  $R$

If  $q, w \in I + J$ , then there are  $a, b \in I$  and  $c, d \in J$  such that  $q = a + c, w = b + d$ , and we can follow that then

$$q + w = a + b + c + d = (a + c) + (b + d) \in I + J$$

If  $e \in R$  is arbitrary, then

$$e + q = e + a + b = (e + a) + b \in I + J$$

which implies that  $I + J$  is indeed an ideal, as desired.

(c) The ideal product of two ideals is defined to be

$$IJ = \left\{ \sum a_i b_i : n \geq 1 \wedge a_n \in I \wedge b_n \in J \right\}$$

Prove that  $IJ$  is an ideal of  $R$

Pretty much the same idea as before: sums do not even need to be rearranged, and

$$c \sum a_i b_i = \sum c a_i b_i = \sum (c a_i) b_i$$

which by power of the fact that  $I$  is an ideal gives us the desired result.

(d) One might ask why the product  $IJ$  of ideals isn't simply defined as the set of products. The answer is that the set of product need not be an ideal. Here is an example. Let  $R = \mathbb{Z}[x]$ , and let  $I$  and  $J$  be the ideals

$$I = 2\mathbb{Z}[x] + x\mathbb{Z}[x], J = 3\mathbb{Z}[x] + x\mathbb{Z}[x]$$

Prove that the set of products is not an ideal.

We follow that

$$a_1 = x, a_2 = -2 \in I$$

$$b_1 = x, b_2 = 3 \in J$$

We then follow that  $a_1 b_2 = 3x, a_2 b_1 = -2x$  are in the set of products. We then follow that  $3x - 2x = x$  is a sum of elements of the set of products. Let  $i \in I$  and  $j \in J$  be arbitrary. We follow that if  $i * j = x$ , then degrees of  $i$  and  $j$  got to be 1 and 0 in order for the product to be of order 1. Since 1 is a multiple of neither 2 or 3 we can follow that

there are no polynomials, which implies that the set is not closed under addition, and thus is not an ideal.

(e) On the other hand, prove in general that if either  $I$  or  $J$  is a principal ideal, then the set of products is an ideal.

Assume that  $I$  is a principal ideal. We follow that if  $a', c' \in I$  and  $b, d \in J$  are elements, then their pairwise products are in the set of products, and thus for arbitrary  $e \in R$  we've got that

$$ca'b = (ea') * b$$

which satisfies the multiplication part of the definition of an ideal. Since  $I$  is a principal ideal, we follow that there's  $q \in R$  that is a defining element of  $I$ , and thus there are  $a, c \in R$  such that  $a' = qa, c' = qc$ . Thus

$$a'b + c'd = qab + qcd = q(ab + cd)$$

$b, d \in J$ , and thus  $ab \in J$  and  $cd \in J$ . Together with the fact that  $q \in I$  we follow that the abovementioned element is indeed an element of the set of products, which gives us the closure of the set under addition. ] Together with the previous part it constitutes a definition of an ideal, as desired.

### 3.5.10

Let  $R$  be a ring, let  $I$  be an ideal of  $R$ , and for any other ideal  $J$  of  $R$ , let  $\bar{J}$  be the following subset of the quotient ring  $R/I$ :

$$\bar{J} = \{a + I : a \in J\}$$

(a) Prove that  $\bar{J}$  is an ideal of  $R/I$ .

If  $q', w' \in \bar{J}$ , we follow that

$$q' + w' = q + I + w + I = (q + w) + I$$

$q + w \in J$  by the powers of  $J$ , and thus the whole thing is  $\bar{J}$ . By similar logic product of any element of  $R/I$  and  $\bar{J}$  is in  $\bar{J}$ , which concludes the proof.

(b) Let  $\bar{K}$  be an ideal of  $R/I$ . Prove that the set

$$Q = \bigcup_{a+I \in \bar{K}} (a + I)$$

is an ideal of  $R$  that contains  $I$ .

By definition of  $\bar{K}$  and underlying  $K$  we follow that  $0 \in K$ , and thus  $0 + I = I \in \bar{K}$ , which implies that  $I \subseteq Q$ . If  $a, b \in Q$ , then there are appropriate  $a', b' \in K$ , and thus

$$ca = c(a + k) = ca + ck$$

$ck \in K$ , and thus the abovementioned thing is an element of  $ca + K$ .

$$a + b = a_1 + k_1 + b_1 + k_2 = (a_1 + b_1) + (k_1 + k_2)$$

and thus blah-blah-blah, it is all covered.

(c) Conclude that there is a bijective map

$$\{\text{ideals of } R \text{ that contain } I\} \rightarrow \{\text{ideals of } R/I\}, J \rightarrow J/I$$

By point (a) for every ideal of  $R$  (whether or not it contains  $I$ ) there's an appropriate ideal of  $R/I$ . If  $R_1 \neq R_2$  are ideals that contain  $I$ , then we follow that there is  $r_1 \in R_1 \setminus I$  such that  $r_1 \notin R_2$  (if not, then swap'em around and assume so). Thus  $r_1 + I$  is in the  $\overline{R_1}$  but not in  $\overline{R_2}$  by their respective definitions. Thus the whole thing is injective. If  $Q$  is an ideal of  $R/I$ , then by point (b) we've got an ideal of  $R$  that contains  $I$ . If  $I \subseteq J$  and  $J$  is an ideal, then we can follow that point (b) is essentially an inverse function of construction of  $\overline{J}$ , which produced the desired result.

### 3.5.11

*Prove that the set of nilpotent elements is an ideal of  $R$ .*

It's the radical of ideal  $\{0\}$ .

## 3.6 Prime Ideals and Maximal Ideals

### Notes

There's gotta be an easier way to do this, but we gonna try to prove that for every non-invertible element of an  $R$  there's a maximum ideal, that contains it. Let us assume that  $q \in R$  is such that  $q \notin R^*$ . We follow that all of the  $\mathcal{P}(R)$  is a set, and thus there's a set of all ideals of  $R$ . Therefore there's a set of all ideals of  $R$  that contain  $q$ , and if we remove  $R$  from this set, then we get a set of all ideals of  $R$ , that are not  $R$ , but contain  $q$  (this set is not empty due to restriction on  $q$ ; this can be derived, but it's not that complicated, key is to assume that since  $q \notin R^*$  that there's no nonzero power of  $q$  that is equal to 1). We then follow that this thing is a poset under  $\subseteq$ , and let  $C$  be a chain in it. We then conclude that

$$1 \in \cup C \Rightarrow \exists Q' \in C : 1 \in Q' \Rightarrow Q' = R$$

which implies that  $1 \notin \cup C$ . Now if  $r, w \in \cup C$ , then we follow that  $r \in Q_1, w \in Q_2$ , and by the fact that  $C$  is a chain we follow that  $r \in Q_2$  or  $w \in Q_1$ , which in turn implies that  $r + w \in Q_1$  or  $r + w \in Q_2$ , and thus  $r + w \in \cup C$ . By similar process we conclude that for arbitrary  $r \in R$  and  $c \in \cup C$  we've got that  $rc \in \cup C$ , which implies that  $\cup C$  is an ideal that contains  $q$ , and thus  $\cup C \in C$ .

By Zorn's lemma then we follow that there's a maximal element  $M$  in the set of ideals of  $R$  that aren't  $R$  and contain  $q$  under  $\subseteq$ . Let now  $r \in R \setminus M$ . We follow that if  $J$  is an ideal such that  $M \subseteq J$  and  $r \in J$ , then by definition of  $M$  and  $r$ ,  $J$  is not an ideal of  $R$  that isn't  $R$ , and thus  $J = R$ , which implies that  $M$  is a maximal ideal. Thus we conclude the following:

**Let  $R$  be a commutative ring. If  $q \notin R^*$ , then there's a maximal ideal  $M$  that contains  $q$ .**

### 3.6.1

Let  $I$  be the following subset of the ring  $Z[x]$  of polynomials having integer coefficient.

$$I = \{2a(x) + xb(x) : a(x), b(x) \in Z[x]\}$$

(a) Prove that  $I$  is an ideal of  $Z[x]$ .

We follow that both  $a(x), b(x)$  are ideals, both  $2a(x)$  and  $xb(x)$  are principal ideals, and thus  $I$  is a set of sums of two ideals, and thus itself an ideal (as proven in the exercises).

(b) Prove that  $I \neq Z[x]$

1 is not present in  $I$

(c) Prove that  $I$  is not a principal ideal

Assume that  $q(x)$  is an element, that generates  $I$ . We follow that it generates both 2 and  $x$ , for which the only common multiple is 1, which implies that  $q = 1$ , which implies that  $I = Z[x]$ , which is not true.

(d) Prove that  $I$  is a maximal ideal of  $Z[x]$

Let  $J$  be an ideal of  $Z[x]$  that properly contains  $I$ . Assume that  $q \notin I$ . We follow that since  $q \notin I$  that it cannot be a multiple of  $x$ , and thus it has to contain an element  $b$  of sum of degree 0. Since  $q \notin I$  we follow that this element is not even (and thus odd). Thus we then follow that  $q - b$  is a multiple of  $x$  that does not have an element of the sum of degree 0, and thus is a multiple of  $x$ . Therefore we can state that  $q - b \in I$ , which implies that  $q - (q - b) = b \in J$ . Since  $b \in J$  is an odd number, we follow that  $b - 2n \in J$  for all  $n \in Z$ , which implies that  $1 \in J$ , which in turn implies that  $J = Z[x]$ , as desired.

### 3.6.2

(a) Let  $m \neq 0$  be an integer. Prove that the ideal  $mZ$  is a maximal ideal if and only if  $|m|$  is a prime number

Assume that  $mZ$  is a maximal ideal. If  $m$  is not a prime, then there are  $a \in Z \setminus \{0, 1\}, b \geq 2$  such that  $m = ab$ . By the restriction on  $b$  we follow that  $a, b$  aren't multiples of  $m$ , and thus  $mZ$  is not a prime ideal. Thus contrapositive of this statement implies that  $m$  is a prime, as desired.

Assume now that  $m$  is a prime and let  $b \in Z \setminus mZ$ . We follow that  $\gcd(m, b) = 1$ , thus there are integers  $a, v$  such that  $am + bv = 1$ , which implies that if  $J$  is an ideal



that contains both  $mZ$  and  $b$ , then  $J = Z$ , which implies that  $mZ$  is a maximal ideal, as desired.

(b) Let  $F$  be a field, and let  $a, b \in F$  with  $a \neq 0$ . Prove that the principal ideal  $(ax + b)F[x]$  is a maximal ideal of the polynomial ring  $F[x]$ .

Since  $a \in F$  and  $a \neq 0$  we follow that  $a^{-1} \in F$ . Thus by pretty much the same logic as in previous exercise (part c) we conclude the same result.

(c) Skip

### 3.6.3

Let  $R$  be a ring, let  $M \subseteq R$  be a maximal ideal, and let  $R^*$  be its group of units. Prove that the following is equivalent

(a)  $M$  is the only maximal ideal of  $R$

(b)  $R = M \cup R^*$ .

A ring having a unique maximal ideal is called a local ring.

What is interesting is that if  $r \in R^*$ , then there is  $r^{-1} \in R^*$ , and thus if  $r \in M$  then  $rr^{-1} = 1 \in M$  by properties of an ideal, and thus  $M = R$ , which is not true. This implies that  $M \cap R^* = \emptyset$  for any maximal ideal  $M$ . Thus the whole thing is a partition.

Assume (a) and contradict (b). Let  $q \in R \setminus (M \cup R^*)$ . Theorem in the notes implies that there's a maximal ideal, that contains  $q$ , and thus definition of  $q$  implies that  $M$  is not the only maximal ideal, which is a contradiction (holy heck, that was an overkill)

Let us assume (b). Prerequisites of the exercise imply that  $M$  has got to be a maximal ideal, so we are only required to derive the uniqueness of it. Assume that  $M'$  is a maximal ideal of  $R$ , that is different from  $M$ . We follow that  $M'$  cannot be a subset of  $M$ , since  $M'$  is a maximal ideal, and thus there's  $q \in M' \setminus M$ . (b) now implies that  $q \in R^*$ , and thus there's  $q^{-1} \in R$ , which means that  $qq^{-1} = 1 \in M'$ , which implies that  $M' = R$ , which is false, and thus we conclude the desired result, which was pretty easy.

### 3.6.4

Let  $R$  be a ring, let  $b, c \in R$ , and let  $E_{b,c} : R[x, y] \rightarrow R$  be the evaluation homomorphism as described in 3.13

(a) If  $R$  is an integral domain, prove that  $\ker(E_{b,c})$  is a prime ideal of  $R[x, y]$ .

We firstly follow that since  $E_{b,c}$  is a ring homomorphism, that  $\ker(E_{b,c})$  is an ideal. Now let  $q, w, e \in R[x, y]$  are such that  $e \in \ker(E_{b,c})$  and

$$q * w = e$$

we follow that

$$E_{b,c}(q * w) = E_{b,c}(e) = 0$$

$$E_{b,c}(q) * E_{b,c}(w) = 0$$

since  $R$  is an integral domain, we follow that  $E_{b,c}(w) = 0$  or  $E_{b,c}(q) = 0$ , which implies that  $E_{b,c}(w) \in \ker(E_{b,c})$  or  $E_{b,c}(q) \in \ker(E_{b,c})$ , which implies that  $\ker(E_{b,c})$  is a prime domain, as desired.

(b) *If  $R$  is a field, prove that  $\ker(E_{b,c})$  is a maximal ideal of  $R[x, y]$ .*

We can try to use our fancy theorem from the cahpter this time. Presume that  $R$  is a field. We can follow that we can define an injective

$$\phi : R[x, y] / \ker(E_{b,c}) \rightarrow R$$

we then follow that  $\phi(x) = x$  for all  $x \in R$  which implies that  $\phi$  is surjective, thus making  $\phi$  an isomorphism, and thus  $R[x, y] / \ker(E_{b,c})$  is a field, therefore  $\ker(E_{b,c})$  is a maximal ideal, as desired.

*The rest of the exercises are skipped for now*

## Chapter 4

# Vector Spaces - Part 1

*Practically all the material here was covered in my course on linear algebra*

# Chapter 5

## Fields - Part 1

### 5.1 Introduction to Fields

### 5.2 Abstract Fields and Homomorphisms

#### 5.2.1

Let  $F$  be a field and let  $f(x) \in F[x]$  be a non-zero polynomial.

(a) Suppose that  $\alpha \in F$  is a root of  $f(x)$ . Prove that there is a polynomial  $g(x) \in F[x]$  such that  $f(x) = (x - \alpha)g(x)$ .

This one is a repeat of an exercise two chapters before (3.8(b))

(b) More generally, suppose that  $\alpha_1, \dots, \alpha_n \in F$  are distinct roots of  $f(x)$ . Prove that there is a polynomial  $g(x)$  such that

$$f(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)g(x)$$

We're gonna use the induction on this one. Part (a) will suffice as a base case. Let  $f(x) = (x - \alpha_1)g(x)$  and  $\alpha_2 \neq \alpha_1$  is a root of  $f(x)$ . We follow then that

$$(\alpha_2 - \alpha_1)g(\alpha_2) = 0$$

Since  $F$  is a field, it's an integral domain, and thus it's got cancellation property.  $(\alpha_2 - \alpha_1) \neq 0$  by our assumption, and thus we follow that  $g(\alpha_2) = 0$ . Part(a) implies all the things that it needs to imply, which gives us the desired result.

(c) Use (b) to deduce the following important result:

**Theorem** Let  $F$  be a field, and let  $f(x) \in F[x]$  be a nonzero polynomial. Then  $f(x)$  has at most  $\deg(f)$  distinct roots in  $F$ .

If there are more roots, than the degree, then part (b) implies that it's got such a decomposition, which gives it greater degree than it's got, which gives us a contradiction.

**5.2.2**

Done this one.

**5.3 Interesting Examples of Fields****5.3.1**

*Prove that each of the following subsets of  $R$  is a field*

(a)  $Q(\sqrt{3})$

The same idea as with  $\sqrt{2}$  works here as well: inverses are found in a same way as inverses of complex numbers, closure is kinda trivial, and the rest is inherited from  $R$ .

(b)  $Q(\sqrt{4})$

Since  $\sqrt{4} = 2$ , it's just  $Q$

(c) *Skip*

**5.3.2**

*Prove that each of the following rings is not a field:*

(a)  $Z[i]$

We can follow that there's no inverse of 2 (or any other non-1 element), simplest proof of that comes from the fact that by multiplying any given element by that thing gives us a tuple, that consists of multiples of that number, and 1 is not a multiple of any non-1 number

(b)  $Z[\sqrt{2}]$

Same idea as in (a)

(c)  $Z/p^2Z$ , where  $p$  is a prime

We can follow that  $p \neq 0$ , but  $p^2 = 0$ , which implies that the whole thing is not an integral domain, and thus not a field.

(d)  $Z/mnZ$ , where  $m, n \geq 2$

Same idea as in (c)

**5.3.3**

*The fields  $Q(i)$  and  $Q(\sqrt{2})$  are both contained in the field  $C$ .*

(a) *Prove that  $Q(i)$  is not contained in  $Q(\sqrt{2})$*

$i$  is not present in  $Q(\sqrt{2})$

(b) *Prove that  $Q(\sqrt{2})$  is not contained in  $Q(i)$*

$\sqrt{2}$  is not present in  $Q(i)$ .

**5.3.4**

(a) Prove that

$$\sqrt{6} \notin \{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in \mathbb{Q}\}$$

and conclude that this set of real numbers is not a ring

Suppose that there are  $a, b, c \in \mathbb{Q}$  such that

$$\sqrt{6} = a + b\sqrt{2} + c\sqrt{3}$$

we follow then that

$$\begin{aligned}\sqrt{6} - a &= b\sqrt{2} + c\sqrt{3} \\ (\sqrt{6} - a)^2 &= (b\sqrt{2} + c\sqrt{3})^2 \\ 6 - 2a\sqrt{6} + a^2 &= 2b^2 + 2bc\sqrt{6} + 3c^2 \\ 6 + a^2 - 3c^2 - 2b^2 &= 2bc\sqrt{6} + 2a\sqrt{6} \\ 6 + a^2 - 3c^2 - 2b^2 &= \sqrt{6}(2bc + 2a)\end{aligned}$$

lhs here is rational, and thus if  $(2bc + 2a) \neq 0$  we've got a contradiction. Thus we conclude that  $2bc + 2a = 0 \Rightarrow bc + a = 0 \Rightarrow a = -bc$ .

$$\begin{aligned}\sqrt{6} - c\sqrt{3} &= a + b\sqrt{2} \\ 6 + c^2 \cdot 3 - 6c\sqrt{2} &= a^2 + 2b^2 + 2ab\sqrt{2} \\ 6 + c^2 \cdot 3 - a^2 - 2b^2 &= 2ab\sqrt{2} + 6c\sqrt{2} \\ 6 + c^2 \cdot 3 - a^2 - 2b^2 &= \sqrt{2}(2ab + 6c)\end{aligned}$$

by the same logic  $2ab + 6c = 0$

$$\begin{aligned}\sqrt{6} - b\sqrt{2} &= a + c\sqrt{3} \\ 6 + 2b^2 - 4b\sqrt{3} &= a^2 + 3c^2 + ac\sqrt{3} \\ 6 + 2b^2 - (a^2 + 3c^2) &= ac\sqrt{3} + 4b\sqrt{3} \\ 6 + 2b^2 - (a^2 + 3c^2) &= \sqrt{3}(ac + 4b)\end{aligned}$$

by the same logic  $ac + 4b = 0$  thus

$$\begin{cases} a = -bc \\ 2ab + 6c = 0 \\ ac + 4b = 0 \end{cases}$$

$$\begin{cases} 2(-bc)b + 6c = 0 \\ (-bc)c + 4b = 0 \end{cases}$$

$$\begin{cases} -2b^2c + 6c = 0 \\ -bc^2 + 4b = 0 \end{cases}$$

$$\begin{cases} c(-b^2 + 3) = 0 \\ b(-c^2 + 4) = 0 \end{cases}$$

the fact that  $b \in Q$  implies that  $b^2 \neq 3$ , and thus  $c = 0$ . Therefore  $-c^2 + 4 \neq 0$ , which implies that  $b = 0$ , and thus  $a = 0$ . Therefore we conclude that  $a = b = c = 0$ , which implies

$$a + b\sqrt{2} + c\sqrt{3} = 0$$

which is a contradiction (I might've messed up some of the calculations here, but it has got to get the same result w/ the right ones). More general case of this thing is somewhat whacky, maybe I'll try to get a hold of it sometime in the future.

(b) *Prove that the set*

$$Q(\sqrt{2}, \sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}, a, b, c, d \in Q\}$$

*is a subring of  $R$ .*

It's closed under all the respective properties and it has zeroes.

(c) *Prove that  $Q(\sqrt{2}, \sqrt{3})$  is a subfield of  $R$*

Assume that

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \in Q(\sqrt{2}, \sqrt{3})$$

we want to find  $a' + b'\sqrt{2} + c'\sqrt{3} + d'\sqrt{6} \in Q(\sqrt{2}, \sqrt{3})$  such that

$$(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) * (a' + b'\sqrt{2} + c'\sqrt{3} + d'\sqrt{6}) = 1$$

moreover, we can relax this thing, to get any given number

$$(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) * (a' + b'\sqrt{2} + c'\sqrt{3} + d'\sqrt{6}) = j$$

so that  $j \in Q$  and  $j \neq 0$  we follow that

$$(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) * (a' + b'\sqrt{2} + c'\sqrt{3} + d'\sqrt{6}) =$$

$$\begin{aligned} & (aa' + ab'\sqrt{2} + ac'\sqrt{3} + ad'\sqrt{6}) + \\ & (b\sqrt{2}a' + 2bb' + b\sqrt{2}c'\sqrt{3} + b\sqrt{2}d'\sqrt{6}) + \\ & (c\sqrt{3}a' + c\sqrt{3}b'\sqrt{2} + 3cc' + c\sqrt{3}d'\sqrt{6}) + \\ & (d\sqrt{6}a' + d\sqrt{6}b'\sqrt{2} + d\sqrt{6}c'\sqrt{3} + 6dd') = \end{aligned}$$

$$(aa' + ab'\sqrt{2} + ac'\sqrt{3} + ad'\sqrt{6}) +$$

$$\begin{aligned}
& (2bb' + a'b\sqrt{2} + bc'\sqrt{6} + 3b\sqrt{3}d') + \\
& (3cc' + c\sqrt{3}a' + cb'\sqrt{6} + 3c\sqrt{2}d') + \\
& (6dd' + d\sqrt{6}a' + d2\sqrt{3}b' + 3d\sqrt{2}c') =
\end{aligned}$$

$$\begin{aligned}
& (aa' + ab'\sqrt{2} + ac'\sqrt{3} + ad'\sqrt{6}) + \\
& (2bb' + a'b\sqrt{2} + 3bd'\sqrt{3} + bc'\sqrt{6}) + \\
& (3cc' + 3cd'\sqrt{2} + ca'\sqrt{3} + cb'\sqrt{6}) + \\
& (6dd' + 3c'd\sqrt{2} + 2db'\sqrt{3} + da'\sqrt{6})
\end{aligned}$$

thus given (presumed) linear independence of roots of integers under  $Q$ , we follow that we need to find such  $a', b', c', d' \in Q$  so that

$$\begin{cases} ab' + a'b + 3cd' + 3c'd = 0 \\ ac' + 3bd' + ca' + 2db' = 0 \\ ad' + bc' + cb' + da' = 0 \end{cases}$$

now let us move some variables around (the ones without the dash are our given constants in this case)

$$\begin{cases} ba' + ab' + 3dc' + 3cd' = 0 \\ da' + cb' + bc' + ad' = 0 \\ -ca' + 2db' + ac' + 3bd' = 0 \end{cases}$$

and probably this one leads to the satisfying answer TODO

### 5.3.5

(a) Let  $F$  be a finite field. Prove that

$$\prod_{\alpha \in F^*} \alpha = -1$$

Given  $\alpha \in F^*$  we can follow that there is  $\alpha^{-1} \in F^*$  as well, but they do not need to be distinct (i.e. there's a case when  $\alpha = \alpha^{-1}$ ). Thus we can partition the whole set into subsets, that contain one or two elements, where each subset that contains two elements is a set of an element and its inverse. Product of those sets is then 1, and thus we follow that the entire product is the product of all elements, that are equal to its inverses.

For our not-so general case of  $F_{13}$  we have a case that the whole product can be partitioned into

$$\{2, 7\}, \{3, 9\}, \{4, 10\}, \{5, 8\}, \{6, 11\}, \{12\}$$



(needed a hint on this one) Suppose that there is an element  $q \in F$  such that  $q^2 = 1$  and such that  $q$  is distinct from 1 and  $-1$ . We follow that

$$(q - 1)(q + 1) = q^2 - 1 = 1 - 1 = 0$$

thus

$$(q - 1)(q + 1) = 0$$

$F$  is an integral domain, this we have that  $q - 1 = 0$  or  $q + 1 = 0$ , which implies that  $q = \pm 1$ , which is a contradiction. Thus we conclude several things, chief among them is the fact that any element that is not equal to  $\pm 1$  is distinct from its inverse. Thus we conclude by previous point (1st paragraph) the desired result.

(b) As a follow-up to (a), let  $m \geq 2$  be an integer, that need not be prime. Prove that

$$\prod_{a \in (Z/mZ)^*} a = \pm 1$$

Suppose that  $a \in (Z/mZ)^*$ . We can follow that since  $(Z/mZ)^*$  is a group under multiplication, that there is  $a^{-1} \in (Z/mZ)^*$ . Thus we can once again partition the whole set into pairs/singletons, and if

$$a = a^{-1}$$

then

$$a^2 = 1$$

the difference between this exercise and the previous point is that we can't state that

$$a^2 - 1 = 0 \Leftrightarrow a = \pm 1$$

since  $(Z/mZ)^*$  is not an integral domain and thus

$$(a - 1)(a + 1) = 0$$

does not give us that  $a - 1 = 0$  or  $a + 1 = 0$ . And thus we have nothing.

We can try to generalize this case and ask the question of what exactly is the product of all of elements in the finite group. We then have the problem that we can't adequately generalize what exactly  $-1$  is in this case is, since it's just an additive inverse of the multiplicative identity, and there is nothing special about it in the context of the multiplicative group.

We can follow though that the square of the lhs is 1 by the fact that then we can partition the elements into pairs of elements and its inverses (where the case with the  $a = a^{-1}$  resolves into giving us a pair with identical elements). Thus we have that

$$\left( \prod_{a \in (Z/mZ)^*} a \right)^2 = 1$$

and therefore

$$\prod_{a \in (Z/mZ)^*} (a^2) = 1$$

Let  $j \in (Z/mZ)^*$  be such that

$$\prod_{a \in (Z/mZ)^*} a = j$$

and assume that  $j \neq \pm 1$ . We follow that since  $j \in (Z/mZ)^*$  that

$$\prod_{a \in (Z/mZ)^* \setminus \{j\}} a = 1$$

we can also follow that  $j \in (Z/mZ)^*$  implies that  $j^{-1} \in (Z/mZ)^*$ , and thus

$$j^{-1} \prod_{a \in (Z/mZ)} a = 1$$

by taking away the elements of the lhs of the former equation from lhs on the latter equation we get that

$$j^{-1} j = 1$$

which gives us absolutely nothing.

Maybe induction then? We can follow that this thing is true for 2, thus if

$$\prod_{a \in (Z/nZ)^*} a = \pm 1$$

we follow that in  $Z$  there is  $k \in Z$  such that

$$\prod_{a \in (Z/nZ)^*} a = nk \pm 1$$

and thus  $(Z/nZ)^*$  and  $(Z/(n+1)Z)^*$  don't have much in common, thus giving us absolutely nothing.

We can follow that  $m-1$  is always in  $(Z/mZ)^*$  by following that

$$(m-1)^2 = m^2 - 2m + 1 \equiv 1$$

Maybe we should go back a bit. If

$$(a-1)(a+1) = 0$$

we follow that in  $Z$  we have that there is  $k \in Z$  such that

$$(a-1)(a+1) = mk$$

and thus

$$a^2 - mk - 1 = 0$$

we then follow that

$$\begin{aligned} a^2 &= mk + 1 \\ a &= \pm\sqrt{mk + 1} \end{aligned}$$

which does not appear to give us much.

For small cases (i.e.  $m = 2, 3, \dots$ ) we have that

$$\begin{aligned} 1 &= 1 \\ 1 * 2 &= 2 \equiv -1 \pmod{3} \\ 1 * 3 &= 3 \equiv -1 \pmod{4} \\ 1 * 2 * 3 * 4 &= 24 \equiv -1 \pmod{5} \\ 1 * 5 &= 5 \equiv -1 \pmod{6} \\ 1 * \dots * 7 &\equiv -1 \pmod{7} \\ 1 * 3 * 5 * 7 &= 105 \equiv 1 \pmod{8} \end{aligned}$$

We know that the answer should be  $\pm 1$ , which kinda means that we might be having squares lying around in this thing. Suppose that  $j \in (Z/mZ)$ . Assume also that  $j^2 = 1$ . Can we follow that  $j = \pm 1$ ? Well, we can't since for example in  $(Z/8Z)$  we have that  $3^2 = 9 \equiv 1$ , and  $3 \neq \pm 1$ .

After some careful search for hints on the internet, I got an idea to pair up elements that procude  $-1$ . Let  $a \in (Z/mZ)^*$ . We can follow that if  $a * b = -1$ , then there is  $a^{-1} \in (Z/mZ)^*$ , and thus

$$b = -a^{-1}$$

can  $a = -a^{-1}$  then? If  $a = -a^{-1}$ , then  $a^2 = -1$ , thus  $a^2 + 1 \equiv 0 \pmod{m}$ . We follow that the order of  $a$  is 4 in this case.

Suppose that  $a \in (Z/mZ)^*$ . We follow that  $-a * -(a^{-1}) = 1$ , and thus  $-a \in (Z/mZ)^*$ . We can also follow that  $a = -a \Rightarrow 2a = 0 \Rightarrow a \notin (Z/mZ)^*$ . Thus we can pair up  $(a, -a)$  with  $a \neq -a$ .

Therefore let us look at  $(Z/mZ)^*$ . We can divide this set into two subsets: elements that are equal to its inverse, and the ones that aren't. If an element  $a$  is equal to its inverse, then  $(-a)^2 = 1$ , and thus  $-a \neq a$  and  $-a$  is also present and not equal to its inverse, and thus we can pair up  $(a, -a)$ , whose product is  $-1$ . If an element is not equal to its inverse, then we can pair this element with its inverse. Thus we have a set of pairs, whose individual product is either 1 or  $-1$ , which implies that the product of the entire set is  $\pm 1$ , as desired.

**5.3.6**

(a) is pretty much just checking, and the answer to (b) is the fact that  $F_4$  is a field, and  $(\mathbb{Z}/4\mathbb{Z})$  is not an integral domain, let alone a field.

**5.4 Subfields and Extension Fields****5.4.1**

(a) Let  $K/F$  be an extension of fields. Prove that

$$[K : F] = 1 \Leftrightarrow K = F$$

Since  $K/F$  is an extension, by definition we have that  $F \subseteq K$ .

Suppose that  $[K : F] = 1$ . This means that  $\dim_F(K) = 1$ , and therefore  $\{1\}$  spans  $K$ . Thus if  $k \in K$ , then there is  $f \in F$  such that  $f \cdot 1 = k$ , and thus  $f = k$ . Therefore we have that  $k \in F$ , and thus  $K \subseteq F$ , which implies the right side. Reverse is trivial.

(b) Let  $L/F$  be a finite extension of fields, and suppose that  $[L : F]$  is a prime. Suppose further that  $K$  is a field such that  $F \subseteq K \subseteq L$ . Prove that either  $K = F$  or  $K = L$ .

We follow that  $[L : K]$  is finite (if it's infinite, then  $[L : F]$  is infinite, which is not the case), and thus

$$[L : F] = [L : K] * [K : F]$$

the fact that  $[L : F]$  is prime implies that  $[L : K]$  or  $[K : F]$  is equal to one, which by previous point implies the desired result

**5.4.2**

Let  $K/F$  be a finite extension of fields. Prove that there is a finite set of elements  $\alpha_1, \dots, \alpha_n \in K$  such that

$$K = F(\alpha_1, \dots, \alpha_n)$$

The fact that  $K/F$  is finite extension by definition implies that there is  $n \in \omega$  such that

$$\dim_F(K) = n$$

thus there is a basis (don't know whether or not we need AoC in this case)  $A$  of elements of  $K$ . Linear combination under  $F$  of elements of  $A$  will span  $K$  by the fact that it is the basis, which gives us the desired conclusion.

## 5.4.3

let  $L/K/F$  be extensions of fields.

(a) Assume that  $L/K$  and  $K/F$  are finite extensions. During the proof of Theorem 5.18 we defined a set

$$C = \{a_i b_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

claimed that  $C$  is a basis for  $L$  as an  $F$ -vector space, and proved that  $C$  is linearly independent. Prove that  $C$  is a spanning set

Let  $l \in L$ . We follow that there are  $c_1, \dots, c_n \in K$  such that

$$l = \sum c_i b_i$$

each  $c_i$  is an element of  $K$  and thus there are  $q_1, \dots, q_m \in F$  such that

$$c_i = \sum q_{i,j} a_j$$

and thus

$$l = \sum \sum q_{i,j} a_j b_i$$

by unfolding this sum we get the desired result.

(b) Prove that  $|C| = mn$ , i.e. prove that the  $mn$  products of  $a_i b_j$  are distinct.

Assume that there are  $a_i b_j$  and  $a_q b_w$  such that  $a_i \neq a_q$  and  $b_j \neq b_w$ , but

$$a_i b_j = a_q b_w$$

One thing that we can follow is that  $a_i, a_q, b_j, b_w \neq 0$  since they are part of a basis for  $L$  and  $K$  respectively. We then follow that

$$a_i a_q^{-1} b_j = b_w$$

which implies that  $b_j$  is a nonzero multiple of  $b_w$  in  $L/K$ , which implies that the list is then not linearly independent, thus making  $B$  not a basis, which gives us a contradiction

(c) Prove that  $L/F$  is an infinite extension if and only if at least one of  $L/K$  or  $K/F$  is an infinite extension

Part (a) of this theorem is fully cooked at this point, which gives us an easy forward implication. Thus assume that one of the  $L/K$  or  $K/F$  is infinite. We can then see that the proof for finite case does not depend on finality of  $A$  nor  $B$ , and thus we conclude that  $C$  in that case is indeed a basis for  $L/F$ . We then see that

$$[L : F] = \dim_F(L) = |C|$$

which is still the case for infinite sets, which gives us the desired conclusion

*The rest of this section was previously attempted in the notes, not foint to repeat it here*

## 5.5 Polynomial Rings

### Notes

Let us reiterate a couple of things here: given a ring  $R$  there is a set of finite lists of elements of  $R$ , whose first element is not zero. We call those lists polynomials, and denote the set of them by  $R[x]$ . Degree of a polynomial is the length of a given list minus 1 for nonzero polynomials, and  $-\infty$  for the zero polynomial. Thus  $\deg$  is a function

$$\deg : R[x] \rightarrow \omega \cup \{-\infty\}$$

for any given  $R$ . Purpose of including  $-\infty$  to the mix is to ensure that we've got an identity

$$\deg(f_1 f_2) = \deg(f_1) + \deg(f_2)$$

where we define addition between a random element of omega (or  $-\infty$ ) with  $-\infty$  in an obvious manner.

Another thing that is clear here is the fact that  $F[x]$  is an integral domain. We can also follow that if  $R$  is an integral domain, then by the same idea  $R[x]$  is an integral domain as well.

### 5.5.1

*Repeat of a previous exercise*

### 5.5.2

Let  $F$  be a field, let  $f(x), g(x) \in F[x]$  be polynomials with  $g(x) \neq 0$ , and suppose that there are polynomials  $q_1(x), q_2(x), r_1(x), r_2(x) \in F[x]$  satisfying

$$f(x) = g(x)q_1(x) + r_1(x) \text{ with } \deg(r_1) < \deg(g)$$

$$f(x) = g(x)q_2(x) + r_2(x) \text{ with } \deg(r_2) < \deg(g)$$

prove that  $q_1(x) = q_2(x)$  and  $r_1(x) = r_2(x)$ .

If  $\deg(f) < \deg(g)$ , then we follow that if  $q_1 \neq q_2$ , then one of  $q_1, q_2$  aren't zero. Assume that  $q_1 \neq 0$ , and then

$$\deg(g(x)q_1(x)) = \deg(g(x)) + \deg(q_1(x))$$

since  $\deg(r(x)) < \deg(g(x))$ , we follow that

$$\deg(g(x)q_1(x) + r(x)) = \deg(g(x)) + \deg(q_1(x)) \geq \deg(g(x)) > \deg(f(x))$$

and thus

$$g(x)q_1(x) + r(x) \neq f(x)$$

which gives us a contradiction.

Since

$$f(x) = g(x)q_1(x) + r_1(x)$$

$$f(x) = g(x)q_2(x) + r_2(x)$$

we follow that

$$g(x)q_1(x) + r_1(x) - (g(x)q_2(x) + r_2(x)) = f(x) - f(x) = 0$$

$$g(x)q_1(x) + r_1(x) - (g(x)q_2(x) + r_2(x)) = 0$$

$$g(x)(q_1(x) - q_2(x)) + (r_1(x) - r_2(x)) = 0$$

$$g(x)(q_1(x) - q_2(x)) = r_2(x) - r_1(x)$$

thus

$$\deg(g(x)(q_1(x) - q_2(x))) = \deg(r_2(x) - r_1(x))$$

If  $q_1 \neq q_2$ , then we follow that  $q_1 - q_2 \neq 0$ , and thus  $\deg(g(x)(q_1(x) - q_2(x))) \geq \deg(g(x))$ . Degree of rhs here is less than  $\deg(g(x))$ , which gives us a contradiction. Thus we conclude that  $q_1 = q_2$ , which then implies that

$$r_2(x) - r_1(x) = g(x) * 0 = 0$$

which implies the desired result.

## 5.6 Building Extension Fields

### Notes

One thing that we can say is that if a polynomial has roots, then an exercise from chapter 3 implies that it is reducible. Equivalently irreducible polynomials don't have roots. Reverse is not necessarily true, we can have rootless reducible polynomials.

We also follow that if  $f \in F[x]$  is of degree 1, then we can follow that  $f(x) = a_1x + a_0$  and thus

$$a_1x + a_0 = 0$$

$$a_1x = -a_0$$

$$x = -a_0/a_1$$

and thus  $f$  has roots.

## 5.6.1

Let  $F$  be a field

(a) Prove that every polynomial of degree 1 in  $F[x]$  is irreducible

Let  $g \in F[x]$  be a polynomial of degree 1. We follow that if there are  $h, k \in F[x]$  such that

$$g = h * k$$

then

$$\deg(g) = \deg(h * k) = \deg(h) + \deg(k)$$

thus

$$\deg(h) + \deg(k) = 1$$

Thus we conclude that  $\deg(h) = 0$  or  $\deg(k) = 0$ , which implies the desired result.

(b) Let  $f \in F[x]$  be a polynomial of degree 2. Prove that  $f(x)$  is irreducible if and only if it has no roots in  $F$ .

Irreducibility implies lack of roots in general, as indicated in the notes. Now assume that  $f$  has no roots in  $F$ . We follow that if  $f$  is reducible, then there are nonconstant  $g, h \in F[x]$  such that

$$f(x) = g(x) * h(x)$$

we then follow that

$$\deg(f(x)) = \deg(g(x) * h(x)) = \deg(g(x)) + \deg(h(x)) = 2$$

Since  $g, h$  are nonconstant, we follow that  $\deg(g(x)) = \deg(h(x)) = 1$ , which implies that they have roots, and thus  $f$  has roots as well, which is a contradiction.

(c) Let  $f \in F[x]$  be a polynomial of degree 3. Prove that  $f(x)$  is irreducible if and only if it has no roots in  $F$ .

Same kind of idea as in point (b), but we follow that either  $g$  or  $h$  has roots.

(d) Let  $f(x) = x^4 + 2$ . Prove that  $f(x)$  is irreducible in  $Q[x]$ .

We can follow that if  $f$  has roots, then

$$x^4 + 2 = 0$$

$$x^4 = -2$$

which is a contradiction, and thus  $f$  has no roots. If  $f$  is reducible, then there are  $g, h \in Q[x]$  such that

$$f(x) = g(x)h(x)$$

this implies that either  $g(x)$  or  $h(x)$  has degree 1 or 2 (other needs to have degree 3 or 2).

If  $g(x)$  has degree 1, then it's got roots, and thus  $f$  has roots, which is not the case.



Suppose that there is  $g(x)$  such that there is  $h(x)$  so that

$$x^4 + 2 = g(x)h(x)$$

we follow that both  $g$  and  $h$  have degree 2. Assume that  $g, h$  are both monic polynomials. Then we can give indicate indices of  $g$  as  $a$  and indices of  $h$  as  $b$ , which gives us

$$(x^2 + a_1x + a_2) * (x^2 + b_1x + b_2) = x^4 + 2$$

$$x^4 + b_1x^3 + b_2x^2 + a_1x^3 + a_1b_1x^2 + a_1b_2x + a_2x^2 + a_2b_1x + a_2b_2 = x^4 + 2$$

$$x^4 + (b_1 + a_1)x^3 + (b_2 + a_1b_1 + a_2)x^2 + (a_1b_2 + a_2b_1)x + a_2b_2 = x^4 + 2$$

$$(b_1 + a_1)x^3 + (b_2 + a_1b_1 + a_2)x^2 + (a_1b_2 + a_2b_1)x + a_2b_2 = 2$$

therefore

$$\begin{cases} b_1 + a_1 = 0 \\ b_2 + a_1b_1 + a_2 = 0 \\ a_1b_2 + a_2b_1 = 0 \\ a_2b_2 = 2 \end{cases}$$

$$\begin{cases} a_1 = -b_1 \\ b_2 - b_1^2 + b_2^{-1} 2 = 0 \\ -b_1b_2 + b_2^{-1} 2b_1 = 0 \\ a_2 = b_2^{-1} 2 \end{cases}$$

$$\begin{cases} a_1 = -b_1 \\ b_2 - b_1^2 + b_2^{-1} 2 = 0 \\ b_1(-b_2 + b_2^{-1} 2) = 0 \\ a_2 = b_2^{-1} 2 \end{cases}$$

Let us concentrate on the third equation, which by the fact that  $Q$  is a field, and thus an integral domain, gives us two cases:  $b_1 = 0$  or  $-b_2 + b_2^{-1} 2 = 0$ . If we assume the former, then second equation becomes

$$b_2 + b_2^{-1} 2 = 0$$

$$b_2^2 + 2 = 0$$

and no such elements of  $Q$  exist. Thus we conclude the latter case, which gives us that

$$-b_2 + b_2^{-1} 2 = 0$$

$$-b_2^2 + 2 = 0$$

$$b_2^2 = 2$$

and once again, no such element of  $Q$  exists, which implies that there are no monic polynomials of degree 2, that multiply to  $f$ . Now if  $g(x)$  and  $h(x)$  are arbitrary polynomials of degree 2, that multiply to  $f$ , then we conclude that coefficients for  $x^2$  in both  $g$  and  $h$  has got to multiply to 1. Thus we conclude that multiplying both polynomials by inverses of their coefficients for  $x^2$  gives us a pair of monic polynomials, which gives us a hasty contradiction, and thus implies the desired result.

(e) Let  $f = x^4 + 4$ . Prove tht  $f$  is reducible in  $Q[x]$ , despite the fact that it has no roots in  $Q$ .

$$x^4 + 4 = (x^2 + 2x + 2) * (x^2 - 2x + 2)$$

### 5.6.2

Let  $F$  be a field, and suppose that the polynomial  $X^2 + X + 1$  is irreducible in  $F[X]$ . Let

$$K = F[X]/(X^2 + X + 1)F[X]$$

be the quotient ring.

(a) Find a polynomial  $p(X) \in F[X]$  of degree at most 1 such that

$$\overline{p(X)} = \overline{(X + 3)} * \overline{(2X + 1)}$$

We follow that

$$(X + 3) * (2X + 1) = 2X^2 + 7X + 3$$

we then want to get an element of  $(X^2 + X + 1)F[X]$ , whose sum with given element is of power of 1 or less. This is easy, we can get

$$2(X^2 + X + 1) = 2X^2 + 2X + 2 * 1$$

and then subtract one from another to get

$$2X^2 + 7X + 3 - 2X^2 - 2X - 2 = 5X + 1$$

(b) Find a polynomial  $q(X) \in F[X]$  of degree at most 1 satisfying

$$\overline{q(X)} * \overline{X + 1} = \overline{1}$$

Exercise says that we can get an element of degree 1 in here, so let's try that

$$x * (x + 1) = x^2 + x$$

initial guess wont do, so let us try dividing the initial polynomial with the remainder

$$(x^2 + x + 1) = x(x + 1) + 1$$

thus we might get a guess of plugging  $-x$ , which gives us

$$(-x) * (x + 1) = -x^2 - x \equiv 1$$

as desired. Here we might try to develop a proper way of finding multiplicative inverses, but I'm too eager to move forward to actually do that

(c) Find a polynomial  $r \in F[x]$  of degree at most 1 satisfying

$$\overline{r^2} = \overline{-3}$$

We can follow that

$$\overline{-3} = \overline{x^2 + x + 1 - 3} = \overline{x^2 + x - 2} = \overline{(x - 1) * (x + 2)}$$

we then try to plug in some coefficients into  $x^2 + x + 1$  so that its sum with  $-3$  is a square. We follow that 4 is such a number:

$$4x^2 + 4x + 1 = (2x + 1)^2$$

which in turn implies that

$$\overline{(2x + 1)^2} = \overline{4x^2 + 4x + 1} = \overline{-3}$$

as desired.

### 5.6.3

Let

$$f(x) = \sum_{0 \leq i \leq d} c_i x^i \in Z[x]$$

be a polynomial of degree  $d \geq 1$

(a) Prove that if  $a/b \in \mathbb{Q}$  is a root of  $f(x)$ , written in lowest terms, then  $a|c_0$  and  $b|c_d$

We follow that

$$\sum_{0 \leq i \leq d} c_i x^i = c_0 + \sum_{1 \leq i \leq d} c_i x^i$$

and thus

$$f(a/b) = c_0 + \sum_{1 \leq i \leq d} c_i (a/b)^i = c_0 + \sum_{1 \leq i \leq d} c_i a^i b^{-i} = c_0 + a \sum_{1 \leq i \leq d} c_i a^{i-1} b^{-i}$$

since  $a/b$  is a root of  $f(x)$  we follow that

$$f(a/b) = c_0 + a \sum_{1 \leq i \leq d} c_i a^{i-1} b^{-i}$$

$$\begin{aligned}
0 &= c_0 + a \sum_{1 \leq i \leq d} c_i a^{i-1} b^{-i} \\
-c_0 &= a \sum_{1 \leq i \leq d} c_i a^{i-1} b^{-i} \\
-c_0 b^d &= a b^d \sum_{1 \leq i \leq d} c_i a^{i-1} b^{-i} \\
-c_0 b^d &= a \sum_{1 \leq i \leq d} c_i a^{i-1} b^{d-i}
\end{aligned}$$

by our multiplication by  $b^d$  we got rid of elements of  $Q$ , and now this whole shebang is in  $Z$ . We then follow that since  $a/b$  are in their lowest terms that they don't have any common primes as multiples, which implies that

$$a | c_0$$

as desired.

We can then do similar trick with  $c_d$  to get

$$\begin{aligned}
0 &= f(a/b) = \sum_{0 \leq i \leq d} c_i a^i b^{-i} = c_d b^{-d} + \sum_{0 \leq i \leq d} c_i a^i b^{-i} \\
0 &= c_d a^d b^{-d} + \sum_{0 \leq i \leq d-1} c_i a^i b^{-i} \\
0 &= c_d a^d + b^d \sum_{0 \leq i \leq d-1} c_i a^i b^{-i} \\
-c_d a^d &= b^d \sum_{0 \leq i \leq d-1} c_i a^i b^{-i} \\
-c_d a^d &= b \sum_{0 \leq i \leq d} c_i a^i b^{d-1-i}
\end{aligned}$$

which also implies the desired result.

(b) Use (a) to find the small finite set of integers  $A$  and  $B$  so that all of the rational roots of the polynomial

$$g(x) = 45x^4 + 62x^3 + 56x^2 + 11x - 6$$

are in the set

$$\left\{ \pm \frac{a}{b} : a \in A, b \in B \right\}$$

Part (a) here gives us the idea that we can get a total set of divisors of the first and the last coefficients of any given polynomials, then get a set of fractions out of those two sets, and this set in the end will contain all of the rational roots of the given polynomial. I've written the desired program (polyq.py in progs/, bruteforced a couple of things), and the result is pretty fun

## 5.6.4

Let  $F$  be a field, and let  $f(x, y) \in F[x, y]$  be a polynomial in two variables. For example, you could take  $f$  to be irreducible

Definitions of irreducibility (and so on) are transferring nicely to two variables. On top of that, polynomials of one variable can be viewed as a polynomial in two variables, which allows us to conclude that for example  $(x + 1)$  is an irreducible polynomial of two variables.

Definitions of degrees is kinda non-applicable, but the fact that we include only non-constant polynomials are applicable in this case

(a) Suppose that there are values  $a, b \in F$  such that  $f(a, b) = 0$ . Let  $I$  be the ideal

$$I = \{(x - a)g(x, y) + (y - b)h(x, y) : g, h \in F[x, y]\}$$

Prove that

$$f(x, y)F[x, y] \subset I \subset R$$

(proper subsets) and conclude that the principal ideal  $f(x, y)F[x, y]$  generated by  $f(x, y)$  is not maximal

Firstly, let us check that  $I$  is indeed an ideal. For addition we have

$$\begin{aligned} (x - a)g(x, y) + (y - b)h(x, y) + (x - a)g'(x, y) + (y - b)h'(x, y) = \\ = (x - a)g(x, y)g'(x, y) + (y - b)h(x, y)h'(x, y) \in I \end{aligned}$$

and for multiplication we follow that

$$q(x, y)((x - a)g(x, y) + (y - b)h(x, y)) = (x - a)q(x, y)g(x, y) + (y - b)q(x, y)h(x, y)$$

thus we have that  $I$  is indeed an ideal.

Since  $f(a, b) = 0$  we follow that

$$f(a, b) = \sum q_c a^i b^j = 0$$

and thus

$$f(x, y) = f(x, y) - f(a, b) = \sum q_c x^i y^j - \sum q_c a^i b^j = \sum q_c x^i y^j - q_c a^i b^j = \sum q_c (x^i y^j - a^i b^j) =$$

We can follow that if  $q \in I$ , then  $q = (x - a)g(x, y) + (y - b)h(x, y)$  for some  $g, h \in F[x, y]$ , and for it we have

$$q(a, b) = 0 * g(a, b) + 0 * h(a, b) = 0$$

thus proving that  $I \subseteq \ker(E_{a,b})$ .

We can fudge around a bit in order to get ourselves closer to the right idea:

$$f(x, y) = x + y - a - b = (x - a) + (y - b)$$

$$\begin{aligned}
f(x, y) &= x^2 + y^2 - (a^2 + b^2) = (x^2 - a^2) + (y^2 - b^2) = (x - a)(x + a) + (y - b)(y + b) \\
f(x, y) &= x^2 + y^2 + xy - (a^2 + b^2 + ab) = (x^2 - a^2) + (y^2 - b^2) + (xy - ab) = \\
&= (x - a)(x + a) + (y - b)(y + b) + (xy - ab) = \\
&= (x - a)(x + a) + (y - b)(y + b) + (xy - ab - ay + bx + ay - bx) = \\
&= (x - a)(x + a) + (y - b)(y + b) + (xy - ab - ay + bx) + (ay - bx) = \\
&= (x - a)(x + a) + (y - b)(y + b) + (x - a)(y + b) + (ay - bx) = \dots
\end{aligned}$$

Which seems to be boiling down to the question of how does  $f(x, y) = xy - ab$  factor in the desired way? Let's then try to fudge around with this one:

$$(x - a) * y + (y - b) * x = xy - ay + xy - bx$$

this one is a non-starter

$$\begin{aligned}
(x - a)(y + b) + (y - b) * (x + a) &= xy - ay + xb - ab + xy - bx + ay - ab = \\
&= xy - ay + xb - ab + xy - bx + ay - ab = 2xy - 2ab
\end{aligned}$$

which is hell of a lot closer. Thus we have that

$$1/2(x - a)(y + b) + 1/2(y - b) * (x + a) = xy - ab$$

which is dope. This lets us conclude that  $xy - ab \in I$ . Next task is to push it into the higher degree:

$$f(x, y) = x^2y^2 - a^2b^2 = (xy)^2 - (ab)^2 = (xy - ab) * (xy + ab)$$

We then know that the lhs of the latter product is in  $I$ , which implies that the whole thing is in  $I$ . In general we have that

$$x^n y^n - a^n b^n = (xy)^n - (ab)^n = (xy - ab) * \sum \dots$$

just as with the single variable polynomial (look up an exercise in 3rd chapter with polynomial evaluation to get details), which proves that

$$x^n y^n - a^n b^n \in I$$

which is cool as heck. This case tackles only polynomials where we have equal degrees for  $x$  and  $y$ , but polynomials of two variables have unequal ones as well. For example there is a polynomial

$$f(x, y) = x^2y - a^2b$$

that exemplifies this case. Let's try to solve this one then:

$$x^2y - a^2b$$

after some fudging around with maxima I got that

$$x^2y - a^2b = (x + a)(xy - ab) - ax(y - b)$$

we know that  $(xy - ab)$  is an element of  $I$ , latter part of the sum (i.e.  $ax(y - b)$ ) is in  $I$  by definition, thus the whole thing is in  $I$ . Generalizing this thing is not going to be trivial. We can follow that  $x^i y^j$  is a term of a sum in  $(x - y)^{i+j}$ . Fudging around a bit more we get that

$$x^{n+1}y^n - a^{n+1}b^n = (x - a) * (x^n y^n - a^n b^n) + a * x * (x^{n-1} * y^n - a^{n-1} b^n)$$

which by induction gives us that

$$x^{n+1}y^n - a^{n+1}b^n \in I$$

We then follow that

$$(x^2y + a^2b) * (x - a) + ax(xy - ab) = x^3y - a^3b$$

and in general

$$x^{n+1}y - a^{n+1}b = (x - a)(x^n * y + a^n * b) + ax(x^{n-1}y - a^{n-1}b)$$

which by induction gives us that for arbitrary  $n$  we follow that  $x^n y - a^n b \in I$

In way more general sense we get

$$x^n y^m - a^n b^m = (x - a) * (x^{n-1} y^m + a^{n-1} b^m) + ax(x^{n-2} y^m - a^{n-2} b^m)$$

thus for an arbitrary  $i > j$  we can get up to

$$x^j y^j - a^j b^j \in I$$

with a point that was made a couple of paragraphs ago, and then by using this formula by induction we get that

$$x^i y^j - a^i b^j \in I$$

with some notation we can get the same case for  $i < j$  as well.

Thus we can get back to our main point. If  $f(a, b) = 0$  for some  $a, b \in F$ , then

$$f(x, y) = f(x, y) - f(a, b) = \sum q_c x^i y^j - \sum q_c a^i b^j = \sum q_c (x^i y^j - a^i b^j)$$

each  $x^i y^j - a^i b^j$  is in  $I$ , thus  $q_c(x^i y^j - a^i b^j) \in I$ , therefore the whole sum is in  $I$ , and thus  $f(x, y) \in I$ , which proves that  $I = \ker E_{a,b}$ , and since our  $f(x, y) \in \ker E_{a,b}$  by definition, we get the first inclusion (not proper inclusion though).

We then follow that  $I \subset R$  (where  $R$  is presumed to be  $F[x, y]$ ) with proper inclusion, since  $R$  can have arbitrary polynomials, whose root is not  $a, b$  (for example constant polynomials), that aren't present in  $I$ .

Thus the last thing that is left to prove is proper inclusion of  $f(x, y)F[x, y]$  in  $I$ . If  $f$  is a zero polynomial, then we follow that  $f(x, y)F[x, y] = \{0\}$ , and since  $I$  has nonzero polynomials in it, we follow proper inclusion.  $f$  cannot be a constant nonzero polynomial (i.e. a number) since it's got roots. Thus let  $m$  and  $n$  be the highest powers of  $x$  and  $y$  respectively in  $f(x, y)$ . We follow that at least one of those is greater than 0 (assume with no loss in generality that it is  $m$ ), and thus every polynomial in  $f(x, y)F[x, y]$  is either 0 or its power of  $y$  is greater or equal than  $m$ . Polynomials in  $I$  can have no powers in  $y$  (simple  $(x - a)$  will do as an example), and thus we conclude that this polynomial is not a multiple of  $f$ , thus giving us proper inclusion, as desired.

*Skipping the rest of this exercise*

### 5.6.5

*The mathematical term for the property that at non-constant polynomials have at least one roots is 'algebraically closed'.*

*(a) Prove that the only irreducible polynomials in  $C[x]$  are linear polynomials.*

If  $C[x]$  is a polynomial of degree two or more, then it's got a root  $a$ , and thus can factor into  $(x - a)g(x)$  for some polynomial  $g(x)$ .

*(b) Let  $f(x) \in R[x]$  and suppose that  $a + bi \in C$  is a root of  $f(x)$ . Prove that  $a - bi$  is also a root of  $f(x)$*

Follows directly from the fact that  $\overline{a + bi} = a - bi$ .

*(c) Let  $f(x) \in R[x]$  be an irreducible polynomial in  $R[x]$ . Prove that  $\deg(f) \leq 2$*

We can follow that there is  $a \in C$  such that  $f(a) = 0$ . If  $a \in R$ , then we're done, thus assume that  $a \notin R$ . From previous point we get that  $f(\bar{a}) = 0$ . Thus

$$f(x) = (x - a)(x - \bar{a})g(x)$$

and thus

$$f(x) = (x^2 - x\bar{a} - xa + |a|^2)g(x) = (x^2 - x(\bar{a} + a) + |a|^2)g(x)$$

we then follow that  $\bar{a} + a \in R$ , and thus lhs is in  $R$  as well, thus producing the desired result.

### 5.6.6

*Let  $F$  be a field, let  $f(x) \in F[x]$  be a possibly reducible non-constant polynomial, and let  $\deg(f) = d$ .*



(a) *Prove that there exists a field extension  $K/F$  satisfying  $[K : F] \leq d$  such that  $f(x)$  has a root in  $K$*

If  $f$  has roots in  $F$ , then we're done, thus assume that it does not have them.

If  $f(x)$  is irreducible then let  $g(x) = f(x)$ , and otherwise let  $g(x)$  be an irreducible factor of  $f$  (we can follow that those exist since all the linear polynomials are irreducible, and  $\deg(f) \geq 1$ ). We then follow that we have that  $\deg(g) \leq \deg(f)$ , and since  $g$  is irreducible by theorem 5.27 we have that there is an extension field  $K$  of  $F$  such that it's got roots for  $g$ . Since  $g$  is a factor for  $f$  we follow that the same root is also a root for  $f$ , and we also have that

$$[K_f : F] = \deg(g) \leq \deg(f)$$

as desired.

(b) *Prove that there exists a field extension  $L/F$  and elements  $c \in F$  and  $\alpha_1, \dots, \alpha_d \in L$  such that*

$$f(x) = c(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_d)$$

( $\alpha$ 's need not be distinct). *Prove that it is always possible to find such an  $L$  that also satisfies*

$$[L : F] \leq d!$$

*The field  $L$  is called a splitting field for the polynomial  $f(x)$  over the field  $F$ .*

From previous point we follow that there is  $K/F$  such that

$$[K : F] \leq d$$

and such that  $f(x)$  has a root  $\alpha_1$  in  $K$ . We then follow that there is  $g(x)$  such that

$$f(x) = (x - \alpha_1)g(x)$$

Then we can follow that  $\deg(g(x)) = \deg(f(x)) - 1$ , and by applying the same logic we follow by induction that there is a desired field.

## 5.7 Finite Fields

### 5.7.1

*Let  $F$  be a finite field with  $q$  elements*

(a) *Prove that every non-zero element of  $F$  is a root of the polynomial  $x^{q-1} - 1$*

We can follow that  $F^*$  is an abelian group under multiplication, and since it is finite, we follow that for  $a \in F : a \neq 0$  its order  $m$  divides the order of  $F^*$ , which is precisely  $q - 1$ . We then follow that

$$a^{q-1} = (a^m)^{(q-1)/m} = 1^{(q-1)/m} = 1$$

which implies the desired result.

(b) Prove that every element of  $F$  is a root of the polynomial  $x^q - x$

We follow that

$$x^q - x = x(x^{q-1} - 1)$$

and thus its zeroes are all nonzero elements, and zero (i.e. the whole field)

(c) Prove the formula

$$\prod_{a \in F} (x - a) = x^q - x$$

Since every  $a \in F$  is a root of  $x^q - x$  we follow that it can be deconstructed into lhs (exercise 5.1(b) directly implies it)

### 5.7.2

Let  $m$  and  $n$  be positive integers

(a) Prove that  $x^m - 1$  divides  $x^n - 1$  in  $F[x]$  if and only if  $m$  divides  $n$  in  $\mathbb{Z}$ .

Suppose that  $m$  divides  $n$ . Definition states that there exists  $k \in \mathbb{N}_+$  such that  $n = mk$ .

Thus

$$x^n - 1 = x^{mk} - 1$$

and thus

$$\begin{aligned} (x^m - 1) * \sum_{0 \leq i \leq k-1} x^{im} &= \sum_{0 \leq i \leq k-1} x^{im} * (x^m - 1) = \sum_{0 \leq i \leq k-1} x^{(i+1)m} - x^{im} = \\ &= \sum_{0 \leq i \leq k-1} x^{(i+1)m} - \sum_{0 \leq i \leq k-1} x^{im} = \sum_{1 \leq i \leq k} x^{im} - \sum_{1 \leq i \leq k-1} (x^{im}) - 1 = \\ &= x^{km} + \sum_{1 \leq i \leq k-1} x^{im} - \sum_{1 \leq i \leq k-1} (x^{im}) - 1 = x^{km} - 1 \end{aligned}$$

as desired (derivation of the desired polynomial was done by observing polynomial division for lower powers, i.e.  $x^6 - 1$  and  $x^2 - 1$ , and then generalizing the hypothesis).

Now assume that  $x^m - 1$  divides  $x^n - 1$ . Thus there is a polynomial  $h(x)$  such that

$$(x^m - 1) * h(x) = x^n - 1$$

We follow that there is  $g(x)$  such that

$$(x^m - 1) * (x^{n-m} + g(x)) = x^n - 1$$

$$(x^m - 1) * (x^{n-m} + g(x)) = x^n - 1$$

$$x^n + x^m g(x) - x^{n-m} - g(x) = x^n - 1$$

$$x^m g(x) - x^{n-m} - g(x) = -1$$

$$(x^m - 1)g(x) = x^{n-m} - 1$$

and this is the moment where induction should kick somewhere here.

Our hypothesis is that if  $x^m - 1$  divides  $x^n - 1$ , then for all  $k \in N_+$  such that  $mk \leq n$  we have that  $x^m - 1$  divides  $x^{n-mk} - 1$ . Base case is provided, and the inductive step is not much different from the base case, which gives us the desired conclusion. We then can state that if  $m$  does not divide  $n$ , then there is  $k \in N_+$  and  $0 < r < m$  such that

$$n = km + r$$

and thus

$$x^n - 1 = x^{km+r} - 1$$

our little lemma gives us that since  $x^m - 1$  divides  $x^n - 1$  that  $x^m - 1$  divides  $x^r - 1$ , which is impossible by the degrees rule, which gives us the desired contradiction.

(b) Let  $a \geq 2$  be an integer. Prove that  $a^m$  divides  $a^n - 1$  in  $Z$  if and only if  $m$  divides  $n$  in  $Z$ .

Proof is similar to the proof of previous point, where we swap appropriate things.

*The rest of this section is marked as TODO*

## Chapter 6

# Groups - Part 2

### 6.1 Normal Subgroups and Quotient Groups

#### Notes

With definition of  $G/H$  in terms of a set of cosets of  $H$  we note that the Lagrange's Theorem can be rephrased as

$$|G/H| = |G|/|H|$$

#### 6.1.1

Let  $\psi : G \rightarrow G'$  be a homomorphism of groups.

(a) Prove that the image of  $\psi$  is a subgroup of  $G'$

We can follow that if  $h', g' \in \psi[G]$ , then there are by definition  $h, g \in G$  such that

$$\psi(h) = h', \psi(g) = g'$$

thus

$$\psi(h^{-1}g) = \psi(h^{-1})\psi(g) = \psi(h)^{-1}\psi(g) = h'^{-1}g'$$

as desired.

(b) Suppose that  $G$  is a finite group. Prove that

$$|G| = |\ker(\phi)| * |\phi[G]|$$

Let  $a * \ker(\phi)$  be a left coset of  $G$ . We follow that if  $q \in a * \ker(\phi)$ , then there is  $b \in \ker(\phi)$  such that

$$q = a * b$$

and thus

$$\phi(q) = \phi(a * b) = \phi(a) * \phi(b) = \phi(a) * 1 = \phi(a)$$

Thus we can follow that different element of the same cosets are sent to the same element of  $G'$ , which implies that there is a bijection between a set of cosets, and elements in the range of  $\phi[G]$ . Thus there is by definition the same number of elements of  $\phi[G]$  and cosets of  $\ker(\phi)$ . Then Lagrange's theorem implies the desired conclusion

### 6.1.2

Let  $G$  be a group, and let  $H \subseteq G$  be a subgroup.

(a) Suppose that  $g^{-1}Hg \subseteq H$  for all  $g \in G$ . Prove that  $H$  is a normal subgroup of  $G$ .

Fix  $g \in G$  and let us assume that

$$g^{-1}Hg \subseteq H$$

we can also follow that by our assumptions we've got that

$$gHg^{-1} \subseteq H$$

Now let  $h \in H$ . We want to prove that there is  $h' \in H$  such that

$$h = g^{-1}h'g$$

we follow that

$$h' = ghg^{-1}$$

thus  $h' \in gHg^{-1}$ , and since  $gHg^{-1} \subseteq H$  we follow that  $h' \in H$ , as desired.

(b) Let  $g \in G$ . Prove that  $g^{-1}Hg$  is a subgroup of  $G$

Since  $e \in H$ , we follow that

$$e = g^{-1}g = g^{-1}eg \in g^{-1}Hg$$

We can also follow that if  $q, w \in g^{-1}Hg$ , then there are  $h, h' \in H$  such that

$$q = g^{-1}hg, w = g^{-1}h'g$$

then

$$q^{-1}w = g^{-1}hgg^{-1}h'g = g^{-1}hh'g$$

and since  $hh' \in H$  we conclude that  $q^{-1}w \in g^{-1}Hg$ , thus making  $g^{-1}Hg$  a subgroup

(c) Let  $g \in G$ . Prove that the following map is a group isomorphism:

$$H \rightarrow g^{-1}Hg, h \rightarrow g^{-1}hg$$

Multiplication by an element is injective, and definition of  $g^{-1}Hg$  implies that the map is surjective, thus making the map bijective.

We can also see that

$$\psi(ab) = g^{-1}abg = g^{-1}aebg = g^{-1}agg^{-1}bg = \psi(a)\psi(b)$$

as desired

(d) If  $H$  is finite. prove that every conjugate of  $H$  has the same number of elements as  $H$

Bijection in the previous point implies that

### 6.1.3

*Skip*

### 6.1.4

*Prove that every subgroup of the quaternion group is normal*

Quaternion group has 8 elements, which implies that its only subgroups are of order 1, 2, 4, 8. We then follow that 1 and 8 are trivial and the entire group respectively, the only group of order 2 is  $\{\pm 1\}$ , and thus the only order that is left is 4. We then follow that the only groups of order 4 are in the form

$$\{\pm 1, \pm x\}$$

where  $x \in \{i, j, k\}$ . After this we can proceed with a simple enumeration, that we're gonna skip, which implies the desired result

### 6.1.5

*Let  $G$  be a group and let  $H \subseteq G$  be a subgroup of  $G$  of index 2. Prove that  $H$  is a normal subgroup of  $G$ .*

Suppose that there is an element  $g \in G$  such that there exists  $h \in H$  so that  $g^{-1}hg \notin H$ . We then follow that if  $g^{-1}h \in H$ , then  $g^{-1} \in H$ , and thus  $g \in H$  and  $g^{-1}hg \in H$ , which is a contradiction, and thus  $g, g^{-1} \notin H$ . This implies that  $g^{-1}H, gH \neq H$ , and thus we follow that  $g^{-1}hg \in gH$ . Thus there is  $h' \in H$  such that

$$g^{-1}hg = g^{-1}h'$$

$$hg = h'$$

$$g = h^{-1}h'$$

which implies that  $g \in H$ , which is a contradiction. Thus we conclude that there is no element  $g \in G$  such that there is  $h \in H$  so that  $g^{-1}hg \notin H$ . This implies that for all  $g \in G$  and all  $h \in H$  we have that

$$g^{-1}hg \in H$$

which implies that

$$g^{-1}Hg \subseteq H$$

which implies the desired result. (did this one een without a hit, what a rush)

### 6.1.6

Let  $G$  be a group, let  $H \subseteq G$  and  $K \subseteq G$  be subgroups, and assume that  $K$  is a normal subgroup of  $G$

(a) Prove that  $HK = \{hk : h \in H, k \in k\}$  is a subgroup of  $G$

We can follow that  $e \in H$  and  $e \in K$ , and thus  $e = e * e \in HK$ . Now let  $q, w \in HK$ . We folllow that they can be deconstructed to  $q = hk, w = h'k'$ . We then conclude that

$$qw^{-1} = hk(h'k')^{-1} = hkk'^{-1}h'^{-1} = hekk'^{-1}h'^{-1} = hh'^{-1}h'kk'^{-1}h'^{-1} = hh'^{-1}(h'kk'^{-1}h'^{-1})$$

We then can see that  $hh'^{-1} \in H$  and by the fact that  $K$  is normal we follow that  $(h'kk'^{-1}h'^{-1}) \in K$ , which implies that the whole thing is in  $HK$ , as desired.

(b) Prove that  $H \cap K$  is a normal subgroup of  $H$  and that  $K$  is a normal subgroup of  $HK$

We can follow that  $H \cap K$  is a subgroup of  $H$  by properties of subgroups. Let  $h \in H$  and  $q \in H \cap K$  be arbitrary. We follow that  $h^{-1}qh \in H$  by the fact that it's a product of elements of  $H$ , and  $h^{-1}qh \in K$  by the fact that  $K$  is normal subgroup of  $G$ . Thus  $h^{-1}qh \in H \cap K$ , which means that  $H \cap K$  is a normal subgroup of  $H$ , as desired

$K$  is a normal subgroup of  $G$ , and thus it's a normal subgroup of any subgroup that contains it by a pretty straightforward application of definition.

(c) Prove that  $HK/K$  is isomorphic to  $H/(H \cap K)$

We can follow that  $f_1 : H \rightarrow HK$  defined by identity is a homomorphism, and since  $K$  is a normal subgroup of  $HK$  we can follow there is a homomorphism  $f_2 : HK \rightarrow HK/K$ . Thus there is a homomorphism  $f_3 : H \rightarrow HK/K$ . We follow that if  $q \in H \cap K$ , then  $q \in K$ , and thus

$$f_3(q) = qK = K = f_3(0)$$

If  $w \in H \setminus K$ , then

$$f_3(w) = wK \neq K = f_3(0)$$

which implies that  $H \cap K$  is a kernel of  $f_3$ . Thus we can define  $f_4 : H/(H \cap K) \rightarrow HK/K$ , which will be an injective homomorphism. We can also follow that if  $C \in HK/K$ , then there is  $q \in HK$  where  $q = hk$  such that

$$C = qK = hkK = hK = f_4(h(H \cap K))$$

which implies that  $f_4$  is surjective, and thus it is bijective, which implies the desired result.

(d) Skip

## 6.1.7

Let  $G$  be a group, let  $K \subseteq H \subseteq G$  be subgroups, and assume that  $K$  is normal.

(a) Prove that  $H/K$  is naturally a subgroup of  $G/K$

We can follow that if  $C \in H/K$ , then there is  $h \in H$  such that  $C = hK$ , and since  $H \subseteq G$  we follow that  $C \in G/K$ , which implies that  $f : H/K \rightarrow G/K$  defined as an identity with the expanded codomain is an injective homomorphism

(b) Conversely, prove that every subgroup of  $G/K$  looks like  $H/K$  for some subgroup  $H$  satisfying  $K \subseteq H \subseteq G$

Let  $Q \subseteq G/K$  be a subgroup. We follow that each  $q \in Q$  is of the form  $q = gK$  for some  $g \in G$ . We then can unionize  $Q$  in order to get

$$H = \bigcup Q$$

we then follow that if  $w, r \in H$ , then there are cosets  $wK, rK \in Q$ , and since  $Q$  is a group there is  $w^{-1}rK$ . Thus

$$w^{-1}KrK = w^{-1}rK \in Q$$

this implies that  $w^{-1}rK \subseteq H$ , and thus

$$w^{-1}re = w^{-1}r \in H$$

which implies that  $H$  is a subgroup of  $G$ . We also follow that since  $Q$  is a group that  $eK = K \in Q$ , which implies that  $K \subseteq H$ . Then we follow that if  $q \in Q$ , then there for all  $t \in q$  we have that  $q = tK$ . Thus  $t \in H$  and therefore  $tK \in H/K$ , which implies that  $Q \subseteq H/K$ . Similar argument shows the inverse inclusion, which implies the desired result.

(c) Prove that  $H$  is a normal subgroup of  $G$  if and only if  $H/K$  is a normal subgroup of  $G/K$

Assume that  $H$  is normal, and let  $gK \in G/K, hK \in H/K$ . We follow that

$$g^{-1}KhKgK = (g^{-1}hg)K$$

since  $h$  is normal we follow that  $(g^{-1}hg) \in H$  and this implies that  $H/K$  is normal.

Assume that  $H/K$  is normal. We follow that if  $g \in G, h \in H$ , then

$$g^{-1}hg \in g^{-1}hgH = g^{-1}HhHgH = g^{-1}HgH = H$$

which implies that  $H$  is normal

(d) If  $H$  is a normal subgroup of  $G$ , prove that

$$\frac{G/K}{H/K} \cong G/H$$

There's a surjective homomorphism  $f_1 : G \rightarrow G/H$  with kernel  $H$ , and since  $K \subseteq H$  we follow that there is a surjective homomorphism  $f_2 : G/K \rightarrow G/H$ .



Let  $q \in G/K$ . If  $q \in H/K$ , then  $q = hK$  for some  $h \in H$  and thus

$$f_2(q) = f_2(hK) = f_1(h) = e$$

if  $q \notin H/K$ , then there is  $g \in G \setminus H$  such that  $q = gK$ . Thus

$$f_2(q) = f_2(gK) = f_1(g) \neq e$$

which implies that  $H/K$  is the kernel of  $f_2$ . This implies that there is an isomorphism  $f_3 : \frac{G/K}{H/K} \rightarrow G/H$ , which implies the desired result.

### 6.1.8

Let  $G$  be a group, let  $K \subseteq G$  be a normal subgroup of  $G$ , and let  $H \subseteq H' \subseteq G$  be subgroups of  $G$ .

(a) Finished this one before

(b) Prove that  $H/(H \cap K)$  is naturally a subgroup of  $H'/(H' \cap K)$

We can define  $\phi : H/(H \cap K) \rightarrow H'/(H' \cap K)$  by

$$\phi(h(H \cap K)) = h(H' \cap K)$$

we then follow that if  $h, h' \in h(H \cap K)$  then there is  $k \in (H \cap K)$  such that  $h = h'k$ , and thus

$$h(H' \cap K) = h'k(H \cap K) = h'(H \cap K)$$

where the second definition is given by properties of cosets. Thus  $\phi$  is well defined. The proof that  $\phi$  is a homomorphism is somewhat trivial, and we also have that if there are  $h(H \cap K), h'(H \cap K)$  such that

$$\phi(h(H \cap K)) = \phi(h'(H \cap K))$$

then

$$h'(H' \cap K) = h'(H' \cap K) \Leftrightarrow \exists k \in K : h' = hk \Rightarrow h(H \cap K)h(H' \cap K)$$

thus proving that  $\phi$  is injective, as desired.

(c) Suppose further that  $H$  is a normal subgroup of  $H'$ . Prove that  $H/(H \cap K)$  is a normal subgroup of  $H'/(H' \cap K)$ .

There is a somewhat trivially followed with a lot of notation involved

## 6.2 Group Acting on Sets

### Notes

Initial definition of the group action is a tad bit lacking, by action we denote a function  $\cdot : G \times A \rightarrow A$  with all the axioms attached

Then, I want to follow some things. If  $x, y \in A$  and  $g \in G$ , then we follow that

$$gx = gy$$

implies that

$$\begin{aligned} g^{-1}(gx) &= g^{-1}(gy) \\ (g^{-1}g)x &= (g^{-1}g)y \\ ex &= ey \\ x &= y \end{aligned}$$

which implies that with fixed  $g \in G$ , our initial map is injective.

If  $x \in A$  is fixed, then if  $g_1, g_2 \in G$  we have

$$\begin{aligned} g_1x &= g_2x \\ g_1^{-1}g_1x &= g_1^{-1}g_2x \\ ex &= (g_1^{-1}g_2)x \end{aligned}$$

and we might not have cancelation properties here, so we're assuming that injectivity is not implied here.

Orbit is the image of an element of  $X$ , and the stabilizer is the values of  $G$  that leave particular  $x \in X$  unchanged. Thus orbit is a subset of  $X$  and stabilizer is a subset of  $G$ . Both of them are indexed by a particular  $x \in X$ .

Now I want to re-word the proof of 6.19(b):

With fixed  $x \in X$  our initial map becomes a function of  $G$ , whose range is the orbit of  $x$ . A union of the set of all orbits constitutes  $X$  (we have that  $e * x = x$ , which implies this). And on top of that, if there are two elements  $x, y$  such that

$$Gx \cap Gy \neq \emptyset$$

then there are  $g_1, g_2 \in G$  such that

$$\begin{aligned} g_1x &= g_2y \\ x &= g_1^{-1}g_2y \\ x &= (g_1^{-1}g_2)y \end{aligned}$$

which means that  $x \in Gy$  and  $y \in Gx$ .

If  $q \in Gx$ , then there is  $g' \in G$  such that  $q = gx \Leftrightarrow x = g^{-1}q$ , and then we follow that if  $r \in Gq$ , then

$$(\exists g \in G)(r = gq) \Rightarrow (\exists g, g' \in G)(r = gg'x) \Rightarrow r \in Gx$$

thus  $Gq \subseteq Gx$ . By the similar logic if  $r \in Gx$  then

$$(\exists g \in G)(r = gx) \Rightarrow (\exists g, g' \in G)(r = gg'q) \Rightarrow r \in Gq$$

which implies that  $Gq = Gx$ .

Combining our discovery into the previous point, we get that

$$Gy = Gx$$

which means that orbits of elements of  $X$  constitute partition of  $X$ . This gives us a thing, that is similar to cosets: they can be indexed by any on their elements. Moreover, if we have  $x \in X$ , then  $ex = x \in Gx$ , which implies that  $x \in Gy \iff Gy = Gx$ .

Let us also re-word the 6.19(c): *If  $G$  is a group that acts on  $X$ , then for every  $x \in X$  there is a bijection*

$$\alpha : G/G_x \rightarrow Gx$$

such that

$$(\forall C \in G/G_x) : (\forall g \in C)(\alpha(C) = gx)$$

*Proof:* If  $C \in G/G_x$  and  $g, g' \in C$ , then by definition of the coset for all  $q \in C$  we have  $C = qG_x$ . Thus  $C = gG_x = g'G_x$ . This implies that there is  $h \in G_x$  such that

$$g = g'h$$

and therefore

$$gx = g'hx = g'(hx) = g'x$$

where  $hx = x$  is derived from the definition of  $G_x$ . This implies that

$$(\forall C \in G/G_x)(\forall g \in C)(\exists! y \in X)(gx = y)$$

which means that there is a function  $\beta : G/G_x \rightarrow X$  that satisfies

$$(\forall C \in G/G_x)(\forall g \in C)(\beta(C) = gx)$$

Furthermore, since for every  $C \in G/G_x$  we have that  $\beta(C) = gx$  for some  $g \in G$  we can follow that  $\beta[G/G_x] \subseteq Gx$ , and thus we can restrict its codomain to  $\beta : G/G_x \rightarrow Gx$ .

We then follow that if  $q \in G_x$ , then  $qx = x$ , and thus

$$\beta(G_x) = x$$

Moreover, if  $g \in G$ , then  $g \in gG_x$ , and thus by restriction on  $\beta$  we have that

$$\beta(gG_x) = gx$$

thus if  $y \in Gx$ , then there is  $g \in G$  such that  $y = gx$  by definition, and thus

$$\beta(gG_x) = gx = y$$

which implies that  $\beta$  is surjective.

Let  $C_1, C_2 \in G/G_x$  and assume that  $\beta(C_1) = \beta(C_2)$ . We follow then that for all  $g \in C_1$  and  $g' \in C_2$  we have

$$gx = g'x$$

by definition of  $\beta$ . This gives us

$$x = g^{-1}g'x \Rightarrow g^{-1}g' \in G_x$$

Thus there is  $r \in G_x$  such that  $g^{-1}g' = r$ , and therefore  $g' = gr$ . Therefore we conclude that

$$C_1 = g'G_x = grG_x = gG_x = C_2$$

which implies that  $\beta$  is injective, and thus we have the desired bijection.

### 6.2.1

Let  $G = C_n$  be a cyclic group of order  $n$ , and let  $X = \{x_1, \dots, x_n\}$  be a set containing  $n$  elements. For each part, find an action of  $G$  on  $X$  that there is an element  $x \in X$  having the indicated property

Firstly, for convenience's sake we note that every cyclic group of order  $n$  is isomorphic to  $Z/nZ$ , and thus we're gonna take  $G = Z/nZ$ . We're also gonna take  $X = Z/nZ$  for some easy notation.

(a) The stabilizer is  $G_x = \{e\}$

We can set  $\times$  to be normal  $+$ , which gives this thing trivial.

We can also see that  $|G|/|G_x| = |Gx|$ , which knocks out point (d)

(b) The stabilizer is  $G_x = G$

We can set

$$j \times x_i = x_i$$

same idea applies to point (c)

### 6.2.2

Let  $G$  be a group, let  $X$  be a set, and let  $S_X$  be the symmetry group of  $X$ . Let

$$\alpha : G \rightarrow S_X$$

be a function, and for  $g \in G, x \in X$ , let  $g * x = \alpha(g)(x)$ . Prove that this defined a group action if and only if  $\alpha$  is a group homomorphism.

Suppose that this thing defines a group action. This means that we can define a function  $\times : G \times X \rightarrow X$  by

$$g \times x = \times(g, x) = \alpha(g)(x)$$

with properties that

$$e \times x = \times(e, x) = x$$

and

$$(g * g') \times x = \times(g * g', x) = \alpha(g * g')(x) = \times(g, \times(g', x)) = \alpha(g)(\alpha(g')(x))$$

Thus for all  $g, g' \in G, x \in X$

$$\alpha(gg')(x) = (gg') \times x = g(g' \times x) = \alpha(g)(\alpha(g')(x))$$

which gives us

$$\alpha(g * g') = \alpha(g) \cdot \alpha(g')$$

which proves that this thing is a homomorphism.

Now if  $\alpha$  is a homomorphism, then

$$\alpha(e) = Id$$

and thus

$$\alpha(e)(x) = x$$

thus satisfying the first condition. We also have the same derivation, but in reverse for the rest of conditions.

### 6.2.3

(a) Prove that  $G$  acts transitively in  $X$  if and only there is at least one  $x \in X$  such that  $Gx = X$ .

Forward implication is handled by definition.

Suppose that there exists  $x \in X$  such that  $Gx = X$ . This means that for all  $y \in X$  there is  $g \in G$  such that  $gx = y \Leftrightarrow x = g^{-1}y$ . Thus we follow that if  $j \in X$ , then there is  $g' \in G$  such that  $g'x = j$ , and therefore  $g'g^{-1}y = j$ , thus implying that  $j \in Gy$ , thus implying that  $X \subseteq Gy$ , which gives us the desired result.

(b) Prove that  $G$  acts transitively on  $X$  if and only if for every pair of elements  $x, y \in X$  there exists a group element  $g \in G$  such that  $gx = y$ .

If  $G$  acts transitively, then if  $x, y \in X$  we follow that  $y \in X = Gx$ , and thus there is  $g \in G$  such that  $y = gx$ .

We then follow that if for all  $x, y \in X$  there is  $g \in G$  such that  $gx = y$  then by fixing  $x$  and shoving  $y$  into the thing we get a surjective function from  $Gx$  to  $G$ , which implies the desired result.

(c) If  $G$  acts transitively in  $X$ , prove that  $|X|$  divides  $|G|$ .

Probably we're assuming here that everything is finite.

If  $G$  acts transitively, then  $X = Gx$  for all  $x \in X$ , and thus

$$|X| = |Gx| = |G|/|G_x|$$

which implies the desired result.

#### 6.2.4

Let  $G$  be a group that acts on a set  $X$ . We say that the action is doubly transitive if it has the following property:

For all  $x_1, x_2, y_1, y_2 \in X$  with  $x_1 \neq x_2$  and  $y_1 \neq y_2$  there exists an element  $g \in G$  of the group satisfying  $gx_1 = y_1$ ,  $gx_2 = y_2$ .

(a) Let  $Z$  be the following set of ordered pairs:

$$Z = \{(z_1, z_2) \in X \times X : z_1 \neq z_2\}$$

Let  $G$  act on  $Z$  by the rule

$$g(z_1, z_2) = (gz_1, gz_2)$$

Prove that the action of  $G$  on  $X$  is doubly transitive if and only if the action of  $G$  on  $Z$  is transitive.

Suppose that the action on  $X$  is doubly transitive. Fix  $q = (q_1, q_2) \in Z$ . Since  $q \in Z$  we follow that  $q_1 \neq q_2$ . Let  $z \in Z$ . We follow that  $z = (z_1, z_2)$  for some  $z_1, z_2 \in X$  such that  $z_1 \neq z_2$ . We then follow that for  $q_1, q_2 \in X$  there is  $g \in G$  such that  $gq_1 = z_1$  and  $gq_2 = z_2$ . Thus  $g(q_1, q_2) = (z_1, z_2)$  and thus  $z \in Gq$ . Since  $z$  is arbitrary we conclude that  $Z \subseteq Gq$ , which implies the desired result. Running about the same result but in reverse gives us the reverse implication.

(b) skip

### 6.3 The Orbit-Stabilizer Counting Theorem

#### 6.3.1

Let  $G$  be a group, and let  $X$  be a set on which  $G$  acts.

(a) Suppose that  $|G| = 15$  and  $|X| = 7$ . Prove that there is some element of  $X$  that is fixed by every element of  $G$ .

OSCT states that

$$|X| = \sum_{i=1}^k \frac{|G|}{|G_{x_i}|}$$

elements in the sum in rhs are nonzero divisors of 15 (by the fact that  $|G| = 15$ ), of which there are 1, 3, 5. The only way that we can have them sum up to  $|X| = 7$  is

$$3 + 3 + 1$$

or

$$5 + 1 + 1$$

which implies that there's gotta be an element  $x_i$  such that  $|G_{x_i}| = |G|$ , which implies that there is  $x_i$  such that  $G_{x_i} = G$ , which implies the desired result.

(b) What goes wrong with your proof in (a) if  $|G| = 15$  and  $|X| = 6$  or  $|X| = 8$ ?

The fact that then we can sum up the divisors into this number:

$$6 = 3 + 3$$

$$8 = 5 + 3$$

### 6.3.2

*Skip*

### 6.3.3

Let  $G$  act on itself by conjugation as described in 6.25. Let  $x \in G$ . The conjugacy class of  $x$  is the orbit of  $X$  for this action.

(a) Prove that  $G$  is the disjoint union of its conjugacy classes.

We can follow that for every  $g \in G$  we have  $g = ege$ , thus implying that every element of  $G$  is in some conjugacy class, which implies the union part.

We then follow that if  $C$  is a conjugacy class, and  $q \in C$ , then there is  $g, g' \in G$  such that  $g$  generates this class, and  $q = g'gg'^{-1}$ . Thus  $g = g'^{-1}qg'$ , and if  $w \in C$ , then there is  $r \in G$  such that

$$w = rgr' = rg'^{-1}qg'r^{-1} = rg'^{-1}q(rg'inv)^{-1}$$

which implies that  $w$  is in conjugacy class of  $q$ , and thus  $C$  is a subset of the conjugacy class of  $q$ . Reverse inclusion is similar to this one, which implies that

$$C_1 \cap C_2 \neq \emptyset \Rightarrow C_1 = C_2$$

which implies the desired result.

(b) Prove that  $G$  is abelian if and only if each conjugacy class of  $G$  contains a single element.

If  $G$  is abelian, then  $gqg^{-1} = gg^{-1}q = q$  for all  $g \in G$ , which implies that conjugacy classes are singletons.

If  $G$ 's conjugacy classes are singletons, then for every pair  $g, q \in G$  we have

$$gqg^{-1} = q$$

(the only element in conjugacy class is  $q$  since  $e \in G$ ). This implies that

$$gq = qg$$

which implies the desired result.

*Skip the rest*

**6.3.4**

Let  $p$  be a prime. We proved in 6.26 that a group with  $p^2$  elements must be abelian. Let  $G$  be a group with  $p^3$  elements

(a) Mimic the proof of corollary 6.26 to try to prove that  $G$  is abelian. Where does the proof do wrong?

We firstly can show that  $p$  divides  $|Z(G)|$ , since nothing else in that part of the proof is dependent on the square. We then follow that

$$G/Z$$

does not have to be cyclic, which diverts from our initial proof.

(b) Give two examples of non-abelian groups with  $2^3$  elements

$Q$  and  $D_4$ .

(c) What sort of information about  $G$  can you deduce from the proof in (a) that failed?

That  $G/Z(G)$  is an abelian group (since its order is  $p^2$ )

**6.3.5**

Let  $G$  be a group, and let  $a, b \in G$ . We say that  $a$  commutes with  $b$  if  $ab = ba$ . Is "commutes with" an equivalence relation

By direct implication of various definitions, "commutes with" is reflexive and symmetric, but I think that transitivity is not exactly covered here. For example  $a$  commutes with  $e$  and  $e$  commutes with  $b$  does not imply that  $a$  commutes with  $b$  (e.g.  $i * 1, 1 * k, ik \neq ki$  in  $Q$ )

**6.3.6**

Let  $G$  be a group.

(a) Prove that the center  $Z(G)$  of  $G$  is a normal subgroup of  $G$ .

We follow that  $e$  commutes with everything, and thus  $e \in Z(G)$ . Let  $g, h \in Z(G)$ , then let  $q$  be arbitrary. We follow that

$$qg^{-1} = (g^{-1}g)qg^{-1} = g^{-1}(gq)g^{-1} = g^{-1}qgg^{-1} = g^{-1}q$$

which gives us that  $g^{-1} \in Z(G)$ . We also have that

$$ghq = gqh = qgh$$

thus implying that  $Z(G)$  is a subgroup, as desired.

Now assume that  $g \in G, z \in Z(G)$ . We follow that

$$g^{-1}zg = g^{-1}gz = z \in Z(G)$$



which implies that  $Z(G) \subseteq g^{-1} Z(G)g$ , thus making  $Z(G)$  normal, as desired

(b) Let  $H \subseteq G$  be a subgroup of  $G$ . Prove that the centralizer  $Z_G(H)$  is a subgroup of  $G$ .

We follow that  $e \in Z_G(H)$  since  $eHe = H$ . If  $g \in Z_G(H)$ , then for all  $q \in H$

$$qq^{-1} = (g^{-1}g)qq^{-1} = g^{-1}(gq)g^{-1} = g^{-1}qgg^{-1} = g^{-1}q$$

and if  $g, g' \in Z_G(H)$  then

$$gg'h = ghg' = hgg'$$

thus implying the desired result

(c) Let  $H \subseteq G$  be a subgroup of  $G$ . Prove that the normalizer  $N_G(H)$  is a subgroup of  $G$ .

We once again follow that  $e \in N_G(H)$ . If  $g, g' \in N_G(H)$  then

$$gHg^{-1} = gg^{-1}Hgg^{-1} = H$$

thus  $g^{-1} \in N_G(H)$  and

$$gg'H(gg')^{-1} = gg'Hg'g = gHg = H$$

as desired.

### 6.3.7

*Skip*

## 6.4 Sylow's Theorem

### 6.4.1

Fix a prime  $p$  and an integer  $n \geq 1$ . For  $m \geq 1$ , we consider the product

$$A_m = \prod_{i=0}^{p^n-1} p^n m - i$$

Suppose that we factor  $A_m$  as  $A_m = p^k B_m$  with  $p \nmid B_m$

(a) Assume that  $p \nmid m$ . Find a simple closed formula for  $k$  that depends only on  $p$  and  $n$ .

Let us look at  $p^n m - i$ . Let us factor  $p$  out of  $i$  and get  $i = p^j s$ , where  $p$  does not divide  $s$ . We follow then that

$$p^n m - i = p^j (p^{n-j} m - s)$$

For every  $i \in \omega$  such that  $i \neq 0$  there is  $j$  such that  $i = p^j s$  and  $p \nmid s$ , and thus we conclude that there is a function  $f : \omega_+ \rightarrow \omega$  such that  $f(i) = j$ . We then follow that if  $a, b \in \omega_+$ , then

$$f(a * b) = f(a) + f(b)$$

(as can be proven quite trivially) and we also note that  $k = f(A_m)$  in our conditions. Thus

$$A_m = \prod_{i=0}^{p^n-1} p^n m - i$$

$$k = f(A_m) = f\left(\prod_{i=0}^{p^n-1} p^n m - i\right) = \sum_{i=0}^{p^n-1} f(p^n m - i)$$

(none of the elements in the product are zero, which gives us this result) we then can follow that if  $p|a$ , and  $b \neq 0$  then

$$f(a + b) = \min(f(a), f(b))$$

and in our case where  $0 \leq i \leq p^n - 1$  we have that

$$f(p^n m - i) = f(i)$$

for  $i \neq 0$  and

$$f(p^n m - 0) = n$$

If  $S$  is a finite sequence in  $\omega$ , then we can follow an interesting thing. Namely that if  $J$  is a finite sequence of elements of  $\omega$ , then we can define  $Q_i : J \rightarrow \omega$  as

$$Q_i(S) = \text{the number of the elements of } S \text{ that are greater or equal to } i$$

or more formally

$$Q_i(S) = |\{j \in S[\omega] : j \geq i\}|$$

and then

$$\sum S = \sum_{i=1}^{\omega} Q_i(S)$$

and also if  $j$  is the highest element of  $S$ , then

$$\sum S = \sum_{i=1}^j Q_i(S)$$

thus we have that

$$k = \sum_{i=0}^{p^n-1} f(p^n m - i) = f(p^n m) + \sum_{i=1}^{p^n-1} f(p^n m - i) =$$

$$= f(p^n m) + \sum_{i=1}^{\omega} Q_i(f[\{p^n m - j : 1 \leq j \leq p^n - 1\}])$$

we then follow that since  $j \neq 0$  and  $j \leq p^n - 1 < p^n m$  we have

$$k = f(p^n m) + \sum_{i=1}^{\omega} Q_i(f[\{j : 1 \leq j \leq p^n - 1\}])$$

then we follow that the maximum possible value of  $f[\{j : 1 \leq j \leq p^n - 1\}]$  is  $n - 1$ , which gives us

$$k = f(p^n m) + \sum_{i=1}^{n-1} Q_i(f[\{j : 1 \leq j \leq p^n - 1\}])$$

and then we can also follow that for  $i \in N_+$

$$Q_i(f[\{j : 1 \leq j \leq p^n - 1\}])$$

is the number of divisors of  $p^i$  in the set  $1 \leq j \leq p^n - 1$ . We then can follow that for  $j \in N_+$  we have  $p^{n-j} - 1$  (from  $p^i$  to  $p^n - p^i = p^{n-i}p^i - p^i = p^i(p^{n-i} - 1)$ ) multiples of  $p^i$ . Thus

$$\begin{aligned} k &= f(p^n m) + \sum_{i=1}^{n-1} Q_i(f[\{j : 1 \leq j \leq p^n - 1\}]) = \\ &= f(p^n m) + \sum_{i=1}^{n-1} (p^{n-i} - 1) = f(p^n m) - n + 1 + \sum_{i=1}^{n-1} (p^{n-i}) \end{aligned}$$

then we can refactor the sum to get

$$\sum_{i=1}^{n-1} (p^{n-i}) = p^{n-1-1} + \dots + p^1 = \sum_{i=1}^{n-2} p^i = \sum_{i=0}^{n-2} p^i - 1 = \frac{p^n - 1}{p - 1} - 1$$

and thus we have

$$k = f(p^n m) - n + 1 + \frac{p^n - 1}{p - 1} - 1 = f(p^n m) - n + \frac{p^n - 1}{p - 1}$$

Since  $p$  does not divide  $m$  we conclude that  $f(p^n m) = n$  and thus

$$k = \frac{p^n - 1}{p - 1}$$

as desired

(b) Use (a) to give a quick proof of Lemma 6.30

Proof is essentially the same, but we note that the maximal power of  $p$  in both numerator and denominator are the same, and thus the whole shebang is not divisible by  $p$

(c) What happens if we allow  $p$  to divide  $m$ ?

We get that if  $j$  is the highest power of  $p$  that divides  $m$  then the whole product's highest power of  $p$  is also  $j$  (as seen in the proof of (a)).

## 6.4.2

Let  $p$  be a prime, and let  $G$  be a group of order  $p^n$ . Prove that for every  $0 \leq r \leq n$  there is a subgroup  $H$  of order  $p^r$ .

Sylow's theorem implies the existence for  $p^n$  (i.e. the whole group), and we can state also that  $\{e\}$  is also a subgroup of order  $p^0$ .

Assume that  $|G| \geq 2$ , and let  $q \in G$  be non-identity. We follow that its order  $j$  divides  $|G|$ , and thus since it's a non-identity, its order is  $p^k$  for some  $1 \leq k \leq n$ . Then we follow that order of  $j^k$  is  $p$ , which implies that there is a subgroup of  $G$ , whose order is  $p$ .

Assume that for some  $n \in \omega$  we have the desired property. Let  $G$  be an arbitrary group of order  $p^{n+1}$ . We follow that its center is non-trivial, and then follow that its center has a subgroup  $H$  of order  $p$ . Since  $H$  is a subgroup of the center, we can quickly follow that it's normal, and thus  $G/H$  constitutes a group. Order of  $G/H$  is going to be  $p^n$  by Lagrange's theorem. We then follow that by our assumption, there are subgroups of  $G/H$  of order  $p^0, p^1, \dots, p^n$ . Let us label them by  $C_0, C_1, \dots, C_n$ . Exercise 6.7(b) (one, that is also solved in this work), implies that for each  $C_j$  there is a subgroup  $Q_j \subseteq G$  such that  $C_j = Q_j/H$ . In the proof of that exercise we essentially state that the union of all the elements of  $C_j$  is itself a subgroup of  $G$ , and now we're adding that by the fact that all the elements of  $C_j$  are disjoint (by properties of cosets) we conclude that  $|Q_j| = |C_j| * |H| = p^j * p = p^{j+1}$ . Thus we conclude that for all  $0 \leq j \leq n$  there are subgroups of  $G$  of order  $p^{j+1}$ . In combination with the fact that the trivial subgroup  $\{e\}$  is a subgroup of  $G$ , we conclude that  $G$  has subgroups of  $p^j$  for all  $0 \leq j \leq n+1$ . By the fact that  $G$  is arbitrary, we conclude that it is true for all groups of order  $p^{n+1}$ .

If we now take our previous proof, and combine it with one of the derivations at the start, we get a proof by induction of the fact all groups  $G$  of order  $p^n$  with  $n \in \omega$  have the desired property. We also note, that we did not restrict  $p$ , which also implies that the result is true for any prime  $p$ .

## 6.4.3

*This exercise asks you to give two different proofs of the following stronger version of the first part of Sylow's Theorem*

**Theorem:** Let  $G$  be a finite group. let  $p$  be a prime, and suppose that  $|G|$  is divisible by  $p^r$  for  $r \in \omega$ . There is a subgroup of  $G$  of order  $p^r$ .

(a) Give a proof that directly mimics the proof in the book.

Let  $n$  be the highest power of  $p$  that divides  $|G|$ . We follow then that

$$|G| = p^n m$$

where  $p \nmid m$ , and thus for any given  $1 \leq j \leq n-1$  we have

$$|G| = p^{n-j} p^j m$$

then we state that there are  $C(p^{n-j}p^jm, p^j)$  subsets of  $|G|$  that are of size  $p^j$ , and thus

$$C(p^{n-j}p^jm, p^j) = \sum_{i=1}^r \frac{p^{n-j}p^jm}{|G_{A_r}|}$$

we then follow that by first exercise in this section, the maximal power of  $p$  in  $C(p^{n-j}p^jm, p^j)$  is  $p^j$ , and thus there is  $B$  such that  $p \nmid B$  and

$$C(p^{n-j}p^jm, p^j) = p^j B = \sum_{i=1}^r \frac{p^{n-j}p^jm}{|G_{A_r}|}$$

thus

$$\begin{aligned} p^j B &= \sum_{i=1}^r \frac{p^{n-j}p^jm}{|G_{A_r}|} \\ p^j B_m &= p^j \sum_{i=1}^r \frac{p^{n-j}m}{|G_{A_r}|} \\ B_m &= \sum_{i=1}^r \frac{p^{n-j}m}{|G_{A_r}|} = \end{aligned}$$

which implies that there is  $r$  such that  $|G_{A_r}| \mid p^{n-j}$ . After this point, the proof is practically the same as well, thus giving us implication that for all  $1 \leq j \leq n-1$  we have a subgroup of  $G$  of order  $p^{n-j}$ , which is equivalent to the desired conclusion.

(b) *Combine the version of Sylow's Theorem that we did prove with the previous exercise.*

Let  $G$  be a finite group. We follow that if  $n$  is the highest order of a prime  $p$  such that  $p^n \mid |G|$ , then there is a subgroup of  $G$  of order  $p^n$ . Previous exercise implies that the resulting subgroup has subgroups of order  $p^0, p^1, \dots, p^n$  of its own, and since subgroup  $J$  of a subgroup  $K$  of a group  $L$  is a subgroup of  $L$  we get the desired result.

#### 6.4.4

*In example 6.36 we showed that there are exactly two groups of order 10. Do a similar calculation to find all groups of order 15*

15 is divided by 5 and 3, and thus we follow that there are  $H_5$  and  $H_3$ . We then once again follow that the only  $k$  such that  $k \mid |G|$  and  $k \equiv 1 \pmod{5}$  is 1, thus implying that  $H_5$  is the only subgroup of order 5 and thus normal

Then let us uncombine those groups to get

$$H_3 = \{e, a, a^2\}, H_5 = \{e, b, b^2, b^3, b^4\}$$

we then note that since  $\gcd(3, 5)$  that  $H_3 \cap H_5 = \{e\}$ , and thus  $a, a^2 \notin H_5$ . This in turn gives us that

$$eH_5 = H_5, aH_5, a^2H_5$$

are cosets of  $H_5$  that decompose the group, and thus any given  $g \in G$  is of the form  $g = a^i b^j$  for some  $i, j \in \omega$ .

We then note that  $a^{-1}ba \in H_5$  and thus there is  $0 \leq j \leq 4$  such that

$$aba^{-1} = b^j$$

which is equivalent to

$$b = a^{-1}b^ja$$

thus we can follow that

$$\begin{aligned} b &= a^{-1}b^ja = (a^{-1}ba)^j = (a^{-1}(a^{-1}b^ja)a)^j = \\ &= (a^{-1}(a^{-1}ba)^ja)^j = (a^{-1}(a^{-1}(a^{-1}b^ja)a)^ja)^j = \\ &= (a^{-1}(a^{-1}b^ja)a)^{j^2} = (a^{-1}b^ja)^{j^2} = b^{j^3} \end{aligned}$$

which is practically the same derivation as in the book, but with an extra application of substitution of  $b$  for  $a^{-1}b^ja$ . Thus we have that

$$b = b^{j^3} \Leftrightarrow e = b^{j^3-1}$$

thus we have that

$$j^3 \equiv 1 \pmod{5}$$

we then follow that the only such  $j$  is  $j = 1$ , which implies that  $G$  is abelian, and then by the same derivation we conclude that  $G$  is cyclic group of order 15.

### 6.4.5

Let  $G$  be a finite group, and let  $H_1, H_2$  be normal subgroups having the property that  $\gcd(|H_1|, |H_2|) = 1$ . Prove that elements of  $H_1$  and  $H_2$  commute.

Let us look at

$$aba^{-1}b^{-1} = (aba^{-1})b^{-1}$$

since  $H_2$  is normal we follow that  $aba^{-1} \in H_2$  and thus

$$aba^{-1}b^{-1} \in H_2b^{-1}$$

by the fact that  $b^{-1} \in H_2$  we conclude thta

$$H_2b^{-1} = H_2$$

and thus

$$aba^{-1}b^{-1} \in H_2$$

by similar logic we have

$$aba^{-1}b^{-1} = a(ba^{-1}b) \in aH_1 = H_1$$

and thus

$$aba^{-1}b^{-1} \in H_1 \cap H_2$$

If  $H_1 \cap H_2$  is non-trivial, then we follow that there is  $h \in H_1 \cap H_2$  such that  $h \neq e$ . Thus the order of  $h$  is some  $j > 1$  and therefore  $h$  generates a cyclic subgroup of order  $j$ . Therefore we follow that  $j$  divides both  $|H_1|$  and  $|H_2|$  by Lagrange's theorem, which implies that  $\gcd(|H_1|, |H_2|) > 1$ , which is a contradiction. Thus we conclude that  $H_1 \cap H_2$  is trivial, and therefore

$$aba^{-1}b^{-1} = e \Rightarrow ab = ba$$

which is the desired result.

#### 6.4.6

Let  $G$  be a finite group of order  $|G| = pq$  where  $p, q$  are primes satisfying  $p > q$ . Assume further that  $p \not\equiv 1 \pmod{q}$ .

(a) Prove that  $G$  is an abelian group.

We follow that there are  $H_p, H_q$  - respective  $p$ -Sylow and  $q$ -Sylow subgroups of  $G$  of order  $p$  and  $q$  respectively. Their order implies that  $H_p$  and  $H_q$  are abelian. Example 6.37 implies that  $H_p$  is normal (while we know nothing of the  $H_q$ ). We then follow that our assumption that  $p \not\equiv 1 \pmod{q}$  gives us an ability to use example 6.37 to prove by same idea mutatis mutandis that  $H_q$  is also normal. Thus previous exercise implies that elements of  $H_p$  and  $H_q$  commute.

We then follow that every element  $g \in G$  can be expressed as

$$g = ab$$

for some  $a \in H_p$  and  $b \in H_q$  (by the same derivation as in example 6.36 or two exercises ago). Thus let  $g_1, g_2 \in G$ . We follow that there are  $a, a' \in H_p$  and  $b, b' \in H_q$  such that

$$g_1 = ab, g_2 = a'b'$$

and thus

$$g_1g_2 = aba'b' = a(ba')b' = aa'bb' = (aa')(bb') = a'(ab')b = a'b'ab = g_2g_1$$

which implies the desired result.

(b) Prove that  $G$  is a cyclic group

We can use the same derivation as in example 6.36

## 6.4.7

Let  $G$  be a group. An isomorphism from  $G$  to itself is called an automorphism of  $G$ . The set of automorphisms is denoted

$$\text{Aut}(G) = \{ \text{group isomorphisms } G \rightarrow G \}$$

We define a composition law on  $\text{Aut}(G)$  by composition of functions

(a) Prove that this composition law makes  $\text{Aut}(G)$  into a group

We can follow that the general identity is the identity for the group. If  $f \in \text{Aut}(G)$  is an element, then  $f^{-1}$  is a function by the fact that everything in  $\text{Aut}(G)$  is a bijection. Inverse of an isomorphism is an isomorphism itself, and thus we have inverses in the  $\text{Aut}(G)$ . Association is taken care of by the general association of the composition of functions, which gives us the desired result

(b) Let  $a \in G$ . Define a map  $\phi_a : G \rightarrow G$  by the formula

$$\phi_a(g) = aga^{-1}$$

Prove that  $\phi_a \in \text{Aut}(G)$  and that the map

$$q : G \rightarrow \text{Aut}(G), q(a) = \phi_a$$

is a group homomorphism.

We follow that  $\phi_a$  is a composition of products of elements of a group, and thus is injective. If  $h \in G$ , then

$$\phi_a(a^{-1}ha) = h$$

thus making  $h$  surjective, which implies that  $\phi_a \in \text{Aut}(G)$ . Then we follow that

$$q(ab) = \phi_{ab}$$

$$q(a)q(b) = \phi_a\phi_b$$

now let  $g \in G$ . We follow that

$$\phi_{ab}(g) = abg(ab)^{-1} = abgb^{-1}a$$

$$\phi_a\phi_b(g) = \phi_a(bgb^{-1}) = abgb^{-1}a^{-1}$$

which implies that  $q$  is indeed a homomorphism, as desired.

(c) Prove that the kernel of homomorphism 6.23 is the center  $Z(G)$  of  $G$ .

Let  $c \in Z(G)$ . We follow that for  $g \in G$

$$\phi_c(g) = cgc^{-1} = cc^{-1}g = eg = g$$



and if  $v \notin Z(G)$ , then if  $q(v) = id$  then for all  $g \in G$

$$\phi_v(g) = g$$

thus

$$vgv^{-1} = g$$

$$vg = gv$$

which implies that  $v \in Z(G)$ , which is a contradiction. Thus we have the desired result.

(d) Elements of  $Aut(G)$  that are equal to  $\phi_a$  for some  $a \in G$  are called inner automorphisms, and all other elements of  $Aut(G)$  are called outer automorphisms. Prove that  $G$  is abelian if and only if its only inner automorphism is the identity map

Inner automorphisms are essentially images of  $q$ . If  $G$  is abelian, then  $Z(G) = G$ , and thus  $q(g) = id$  as per the previous exercise. If the range of  $q$  is  $\{id\}$ , then for all  $v, g \in G$

$$\phi_v(g) = g$$

$$vgv^{-1} = g$$

$$vg = gv$$

which implies the desired result.

(e) More generally, if  $H$  is a normal subgroup of  $G$ , prove that there is a well-defined group homomorphism

$$w : G \rightarrow Aut(H), q(a)(h) = aha^{-1}$$

and that the kernel of this homomorphism is the centralizer of  $H$  in  $G$

We can follow that if  $f$  is an inner automorphism of  $G$ , then for  $h \in H$  we have

$$f(h) = aha^{-1}$$

and more generally

$$f[H] = aHa^{-1} = H$$

thus we can make  $f$  into an automorphism on  $H$  by restricting its domain. We call the function that does such a thing  $j$ . Thus by composing homomorphism from (b) with  $j$  we get the desired function. We then follow that if  $k \in Z_G(H)$ , then

$$w(k)(h) = khk^{-1} = kk^{-1}h = h$$

for all  $h \in H$ , thus implying that  $w(k) = id$ . Now assume that  $k \notin Z_G(H)$  and we have that  $w(k) \neq id$ . Then for all  $h \in H$

$$h = w(k)(h) = khk^{-1} \Rightarrow kh = hk \Rightarrow k \in Z_G(H)$$

which is a desired contradiction.

**6.4.8**

Let  $C_n$  be a cyclic subgroup of order  $n$ , and let  $\text{Aut}(C_n)$  be the automorphism group of  $C_n$ ; Prove that  $\text{Aut}(C_n)$  is isomorphism to  $(\mathbb{Z}/n\mathbb{Z})^*$ .

We notice that  $C_n$  by definition is generated by some element  $g \in C_n$ . That is,

$$C_n = \{g^i : 0 \leq i < n\}$$

Let  $q \in [0, n) \cap \omega$  be such that it's relatively prime to  $n$ . We follow that  $g^q$  also generates  $C_n$ , as was proven somewhere in the book. Thus we can follow that there is a function  $h : C_n \rightarrow C_n$

$$h(g^j) = (g^q)^j$$

that is bijective. We then can trivially follow that this thing is a homomorphism, and thus isomorphism, and thus an element of  $\text{Aut}(C_n)$ .

We then follow that if  $k : C_n \rightarrow C_n$  is an isomorphism, then there is  $0 < j < n$  such that

$$k(g) = g^j$$

which gives us the fact that for every element of  $\text{Aut}(C_n)$  there is a unique  $0 < j < n$  and vice versa. We then notice that  $j$  can only be a number that is relatively prime to  $n$ , which in combination with the previous paragraph gives us the idea that  $\phi : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow \text{Aut}(C_n)$  defined by

$$\phi(j)(g) = g^j$$

is the desired isomorphism

**6.4.9**

Let  $G$  be a group of order  $|G| = 75$ .

(a) Prove that  $G$  has a subgroup  $H$  with all three of the following properties:  $|H| = 25$ ,  $H$  is a normal subgroup of  $G$ ,  $H$  is abelian

Let  $H$  be a 5-Sylow subgroup of  $G$ . We follow that its order is 25. Since its order is a square of a prime, it's abelian. Numbers that divide 75 are

$$1, 3, 5, 15, 25, 75$$

none of which give 1 mod 5, except for 1, which implies that this is the only 5-Sylow subgroup, which implies that the thing is normal.

(b) Suppose that the subgroup  $H$  in (a) is cyclic of order 25. Prove that  $G$  is abelian.

We follow that there is also a 3-Sylow subgroup. Since its order is 3 we follow that it's cyclic. By the example's derivation we have

$$aba^{-1} = b^j$$

$$\begin{aligned}
b &= a^{-1} b^j a \\
b &= b^{j^3} \\
j^3 &\equiv 1 \pmod{25}
\end{aligned}$$

the only such  $j$  is 1, which implies that 3-Sylow and  $H$  commute. Let  $a$  now generate  $H$  and  $b$  generate the 3-Sylow subgroup. We follow that  $ab$ 's order is finite and let this order be  $j$ . We follow that

$$\begin{aligned}
e &= (ab)^j \Rightarrow e = a^j b^j \Rightarrow a^{-j} = b^j \in H \cap H_3 = \{e\} \Rightarrow \\
&\Rightarrow a^k = b^k = e \Rightarrow 3|k \wedge 25|k \Rightarrow k = 75
\end{aligned}$$

which is pretty much the same derivation as in one of the examples in the chapter

## 6.5 Double Cosets and Sylow's Theorem

### 6.5.1

Let  $G$  be a finite group of order  $|G| = pm$ , where  $p$  is prime and  $p \nmid m$ . Assume further that  $m$  has the following property: The only divisor  $d$  of  $m$ , that satisfies  $d \equiv m \pmod{p}$  is  $d = m$ .

(a) Prove the  $p$ -Sylow subgroup of  $G$  is a normal subgroup of  $G$ .

Let us name the  $p$ -Sylow subgroup of  $G$  as  $H$ .

We firstly follow that if  $p$ -Sylow subgroup is normal, then its conjugates are equal to the group, thus making this group the one and only  $p$ -Sylow subgroup of  $G$ . Same goes in the other direction, thus  $p$ -Sylow is normal if and only if  $k = 1$  (variables definition here is the same as in the text in the book).

Now assume that  $H$  is not normal. We follow that  $k > 1$ . We also have that  $k|pm$ . If  $p|k$ , then  $k \equiv 0 \pmod{p}$ , which is a contradiction, which implies that  $p \nmid k$ . Since  $p \nmid m$  we follow then that

$$k|m$$

which implies that there is  $n \in N_+$  such that  $m = kn$ . Moreover we have that  $n < m$  since  $k > 1$ . Now we can follow that

$$\begin{aligned}
k &\equiv 1 \pmod{p} \\
nk &\equiv n \pmod{p} \\
m &\equiv n \pmod{p} \\
n &\equiv m \pmod{p}
\end{aligned}$$

which directly contradicts the premise (this proof is somewhat awkward, and can be easily converted to the direct one).

*Skipping the rest of those exercises, maybe will come back for them later*

## Chapter 7

# Rings - Part 2

### 7.1 Irreducible Elements and Unique Factorization Domains

#### 7.1.1

Let  $R$  be an integral domain, and let  $a, b \in R$ . Prove that the following are equivalent:

- (a)  $a|b$  and  $b|a$
- (b) There is a unit  $u \in R^*$  satisfying  $a = bu$ .
- (c) There is an equality of principal ideals  $aR = bR$ .

There's probably a cyclical implication here, so assume that  $a|b$  and  $b|a$ . Definition implies that  $a, b \neq 0$  and also that there are  $k_1, k_2 \in R$  such that  $a = bk_1$  and  $b = ak_2$ . We follow then that

$$a = bk_1 = ak_2k_1$$

thus

$$a * 1 = ak_2k_1$$

which by the fact that  $a \neq 0$  and  $R$  being integral domain implies that

$$k_2k_1 = 1$$

thus implying that  $k_1, k_2 \in R^*$ , which implies (b) from derivation that  $a = bk_1$ .

Now if  $a = bu$  for some  $u \in R^*$ , then we follow that

$$h \in aR \Rightarrow (\exists r \in R)(h = ar) \Rightarrow (\exists r \in R)(h = bur) \Rightarrow (\exists ur \in R)(h = b(ur)) \Rightarrow h \in bR$$

thus implying that  $aR \subseteq bR$ . Reverse inclusion is kinda similar here, which implies (c)

If  $aR = bR$ , then  $a \in bR$ , which implies that there is  $r \in R$  such that  $a = br$ , which implies that  $b|a$ . Similar implication holds for the other condition.

## 7.1.2

Let  $R$  be a ring, and let  $a, b \in R$ .

(a) Suppose that  $b = au$  for some unit  $u \in R^*$ . Prove that the principal ideals  $aR$  and  $bR$  are equal.

In the previous exercise we had the assumption that  $R$  is an integral domain, which we don't have here. However our proof holds the same since the derivation from (b) to (c) does not involve any properties of an integral domain.

(b) Give an example of a ring  $R$  and elements  $a, b \in R$  having the property that  $aR = bR$ , but there does not exist a unit  $u \in R^*$  satisfying  $b = au$ .

Hint gives us that  $R$  cannot be an integral domain (which is kinda obvious from the previous point). The fact that it is a challenging problem perhaps suggests that we're going to have to deal with the polynomials.

Simplest ring that is not an integral domain that I can think of is the  $Z/4Z$ . We can follow that the only units are 1 and  $-1$  that give us the only nonzero element that exists apart from that is 2, which does not give us much room to move around.

With  $Z/6Z$  we have that 1, 5 are units, and 2, 3, 4 are non-units. We follow that their respective principal ideals are

$$2 * Z/6Z = \{0, 2, 4\} = 4 * Z/6Z$$

$$3 * Z/6Z = \{0, 3\}$$

we follow that  $2 * 5 = 4 \pmod{6}$  which does not give us the desired result.

After some fudging around with python we can find out that none of the  $Z/nZ$  for  $n$  up to 100 are satisfying those ideas.

Looking into the proof of 7.1 we can follow that  $(b) \Rightarrow (c), (c) \Rightarrow (a)$  without the usage of the fact that  $R$  is an ID. The only requirement of ID comes from the b to c implication.

TODO

## 7.2 Euclidian Domains and Principal Ideal Domains

## 7.2.1

Let  $R$  be a PID, let  $a, b \in R$  be elements that are not both 0, and let  $d \in R$  be an element that generates the (principal) ideal generated by  $a$  and  $b$ :

$$dR = aR + bR$$

(a) Prove that  $d$  divides both  $a$  and  $b$

We can follow that since  $dR = aR + bR$  we have that  $0 \in bR$  and thus  $aR = aR + 0 \subseteq dR$ . Therefore we have that  $a \in dR$  and thus there is  $r \in R$  such that  $a = dr$ , which implies that  $d|a$ , as desired. Similar thing holds for  $b$ .

(b) Let  $c \in R$  be a non-zero element that divides both  $a$  and  $b$ . Prove that  $c$  divides  $d$ .

Since  $c|a$  and  $c|b$  we follow that there are  $n, k \in R$  such that  $nc = a$  and  $kc = b$ . Since  $dR = aR + bR$  and  $d \in dR$  we follow that there are  $r, r' \in R$  such that

$$d = ar + br' = ncr + kcr' = c(nr + kr')$$

which implies the desired result

### 7.2.2

Let  $R$  be a Euclidian domain with size function  $\sigma$  and let  $a \in R$  be a nonzero element of  $R$ . Prove that

$$a \in R^* \iff \sigma(a) = \sigma(1)$$

$a \in R^*$  implies that for all  $b \in R \setminus \{0\}$  we have

$$\sigma(ba) = \sigma(b)$$

thus since  $1 \neq 0$  we have

$$\sigma(a) = \sigma(1 * a) = \sigma(1)$$

which gives us the forward implication.

Since  $\sigma(1) = \sigma(a) = \sigma(1 * a)$  we follow the desired result by 7.15

### 7.2.3

Let  $R$  be an integral domain, and let

$$\sigma : R \setminus \{0\} \rightarrow \omega$$

be a function that has property (1) in the definition of a Euclidian domain. Define a new function

$$\tau : R \setminus \{0\} \rightarrow \omega, \tau(r) = \min\{\sigma(rc) : c \in R \setminus \{0\}\}$$

Prove that  $\tau$  has both properties (1) and (2)

Suppose that  $b, c \in R$  are arbitrary nonzero elements. We follow that there are  $a, a' \in R \setminus \{0\}$  such that  $\tau(b) = \sigma(ab)$ ,  $\tau(cb) = \sigma(acb)$ . Since  $R$  is an integral domain and  $a, c$  are nonzero we follow that  $ac \neq 0$ . By definition of  $\tau$  we follow that for all  $q \in R \setminus \{0\}$  we have that  $\tau(b) \leq \sigma(qb)$ , and thus  $\tau(b) \leq \sigma((ac)b) = \tau(cb)$  which implies the part (2) of the definition.

Assume that  $b, r \in R \setminus 0$  are such that  $\sigma(b) > \sigma(r)$ . We firstly can follow that  $\sigma(r) \geq \tau(r)$  by definition of  $\tau$ , and  $\tau(b) = \sigma(cb) \geq \sigma(b)$  by previous point, which implies that

$$\tau(b) \geq \sigma(b) > \sigma(r) \geq \tau(r)$$

which gives us point (1) and the desired result.

**7.2.4**

*this one was taken care of in the linear algebra course*

**7.2.5**

*Explain why proof of the fact that  $Z[i]$  is a Euclidean domain won't work on  $Z[\sqrt{3}]$   
here i've copied the exercise wrongly*

We firstly follow that  $Z[\sqrt{3}]$  is a subring of  $R$ , and thus it's got to be an integral domain. We can also state that there is a bijection between  $Z[\sqrt{3}]$  and  $Z^2$  which can also give us a plane, just as in  $Z[i]$  case. We can also define  $\sigma : Z[\sqrt{3}] \rightarrow R$  by setting

$$\sigma(a + b\sqrt{3}) = a^2 + b^2$$

we can follow that  $\sigma(\zeta) = 0 \Leftrightarrow \zeta = 0$ , but can we do the second part? We can see that

$$\sigma(\sqrt{3} * \sqrt{3}) = \sigma(3) = 3^2 + 0^2 = 9$$

$$\sigma(\sqrt{3}) * \sigma(\sqrt{3}) = 1 * 1 = 1 \neq 9$$

which gives us problems.

**7.2.6**

*For each of the following rings  $R$  and elements  $\alpha$ , determine whether  $\alpha$  is irreducible in  $R$ . Justify your answer by either factoring  $\alpha$  or proving that it is irreducible*

(a)  $R = Z[i]$  and  $\alpha = 2 + 3i$

We can follow that we have a function  $\sigma(x + yi) = x^2 + y^2$  is having a property  $\sigma(ab) = \sigma(a)\sigma(b)$ . We then follow that

$$\sigma(\alpha) = \sigma(2 + 3i) = 4 + 9 = 13 = 3 * 5$$

thus we can try to make up elements with sizes 3 and 5. The problem is that 3 is not a sum of squares, and thus we can follow that there is no element  $a$  of  $Z[i]$  such that  $\sigma(a) = 3$ , which implies that the element is irreducible.

(b)  $R = Z[i]$  and  $\alpha = 4 + 3i$

We follow that

$$\sigma(\alpha) = \sigma(4 + 3i) = 16 + 9 = 25 = 5^2$$

we then can heck around with  $Z[i]$  to get that

$$(2 - i) * (1 + 2i) = 2 + 4i - i + 2 = 4 + 3i$$

thus giving us the desired result.

(c)  $R = F_2[x]$  and  $\alpha = x^5 + x + 1$

Here we firstly note that the size function is  $\deg$ , which has property  $\deg(a * b) = \deg(a) + \deg(b)$ , thus stating that our resulting elements are of order 1, 4 or 2, 3. We also note that both elements should have a non-zero constant value in order to result in a non-zero constant value in the result. There are only a finite amount of polynomials of both degrees, and thus we can try to go through all of them

$$(x + 1)(x^4 + 1) = x^5 + x^4 + x + 1$$

$$(x + 1)(x^4 + x + 1) = \dots$$

we then note an interesting thing: polynomial multiplication in  $F_2$  looks kinda similar to multiplication in a base 2. We follow that in base 2

$$\alpha = 2^5 + 2 + 1 = 35 = 5 * 7$$

and  $5 = x^2 + 1$ ,  $7 = x^2 + x + 1$ . We then follow that

$$(x^2 + 1) * (x^2 + x + 1) = x^4 + x^3 + x^2 + x^2 + x + 1 = x^4 + x^3 + x + 1$$

thus concluding that we were wrong on this one (but it's still pretty interesting how exactly this thing translates to binary). Thus let us go back to just calculating the things

$$(x + 1)(x^4 + 1) = x^5 + x^4 + x + 1$$

$$(x + 1)(x^4 + x + 1) = x^5 + x^2 + x + x^4 + x + 1 = x^5 + x^4 + x^2 + x + 1$$

$$(x + 1)(x^4 + x^2 + x + 1) = x^5 + x^3 + x^2 + x + x^4 + x^2 + x + 1 = x^5 + x^4 + x^3 + 1$$

$$(x + 1)(x^4 + x^3 + x^2 + x + 1) = x^5 + x^4 + x^3 + x^2 + x + x^4 + x^3 + x^2 + x + 1 = x^5 + 1$$

after doing this thing for a while, I thought that maybe a better idea is to think about the problem a bit. If we're multiplying  $(x + 1)$  by any polynomial with 4th power, then the fourth power will be in the resulting polynomial, which does not give us the desired result. Thus we can skip this thing to get to the powers 3, 2:

$$(x^2 + 1)(x^3 + 1) = x^5 + x^2 + x^3 + 1$$

$$(x^2 + 1)(x^3 + x + 1) = x^5 + x^3 + x^2 + x^3 + x + 1 = x^5 + x^2 + x + 1$$

$$(x^2 + 1)(x^3 + x^2 + 1) = x^5 + x^4 + x^2 + x^3 + x^2 + 1 = x^5 + x^4 + x^3 + 1$$

$$(x^2 + 1)(x^3 + x^2 + x + 1) = x^5 + x^4 + x^3 + x^2 + x^3 + x^2 + x + 1 = x^5 + x^4 + x + 1$$

$$(x^2 + x + 1)(x^3 + 1) = x^5 + x^4 + x^3 + x^2 + x + 1$$

$$(x^2 + x + 1)(x^3 + x + 1) = x^5 + x^3 + x^2 + x^4 + x^2 + x + x^3 + x + 1 = x^5 + x^4 + 1$$

$$(x^2 + x + 1)(x^3 + x^2 + 1) = x^5 + x^4 + x^2 + x^4 + x^3 + x + x^3 + x^2 + 1 = x^5 + x + 1$$



which gives us  $\alpha$ , and thus the desired result.

(d)  $R = F_2[x]$  and  $\alpha = x^5 + x^2 + 1$  we've got only one polynomial to check, thus let us do this:

$$(x^2 + x + 1)(x^3 + x^2 + x + 1) = x^5 + x^4 + x^3 + x^2 + x^4 + x^3 + x^2 + x + x^3 + x^2 + x + 1x^5 + x^3 + x^2 + 1$$

thus we've gone through all the applicable polynomials, which implies that there  $\alpha$  is irreducible.

### 7.2.7

*This one mirrors the proof of 3.51(c) since we haven't used degrees in that proof*

## 7.3 Factorization in PID

### 7.3.1

*Let  $R$  be a principal ideal domain, and let  $a, bc \in R$ . Prove that if  $a$  is irreducible, does not divide  $b$ , and  $a|c$ ,  $b|c$ , then  $ab|c$ .*

We can follow that there are nonzero  $n, k \in R$  such that

$$an = c, bk = c$$

thus

$$an = bk \Rightarrow a|bk$$

since  $a$  does not divide  $b$  we follow that  $a|k$ . Thus there is  $j \in R$  such that  $aj = k$ , and therefore

$$bk = baj = c$$

which implies that  $ba = ab$  divides  $c$ , as desired.

### 7.3.2

*Let  $R$  be a PID. Prove the 7.19 without the assumption that  $R$  is a Euclidean domain by assuming that  $a \in R$  is non-zero non-unit, not a product of irreducibles, and following the following steps.*

(a) Show that you can write  $a = a_1b_1$  as a product of non-units such that  $b_1$  is not a product of irreducibles and then that you can write  $a = a_1a_2b_2$  as a product of non-units such that  $b_2$  is not a product of irreducibles etc. In this way, for every  $n$  you get a product of non-units  $a = a_1a_2\dots a_nb_n$  such that  $b_n$  is not a product of irreducibles.

Since  $a$  is not a non-unit, and not irreducible, we follow that there are  $a_1, b_1$  such that

$$a = a_1b_1$$

We follow that if both of  $a_1, b_1$  are irreducibles, then  $a$  is a product of irreducibles, which is a contradiction. Therefore we conclude that at least one of  $a_1, b_1$  is not irreducible. We then follow that at least one of  $a_1, b_1$  also has to be a product of non-irreducibles, otherwise we have that  $a$  is a product of irreducibles. By relabeling if necessary and proceeding with induction we get the desired result.

(b) Prove that the ideals  $I_n = b_n R$  satisfy  $I_1 \subseteq I_2 \subseteq I_3 \dots$

We can follow that by construction of  $b_n$  we have that  $b_{n+1} | b_n$ , and thus  $b_{n+1} R \subseteq b_n R$ , which implies this result.

(c) Let  $J = I_1 \cup I_2 \dots$  be the union of all of the  $I_n$ . Prove that  $J$  is an ideal of  $R$ . Then since we have assumed that  $R$  is a PID, the ideal  $J$  is principal, so we can write  $J = cR$  for some  $c \in R$ .

We can follow that any given  $a_1, a_2 \in J$  are both contained in some  $I_n$ , and thus their sum is also in  $I_n$ . We can also follow that  $a_1 c \in I_n$  by the fact that  $I_n$  is an ideal, thus making  $J$  an ideal.

(d) Since  $J$  is the union of the  $I_n$ , there is an index  $k$  so that  $c \in I_k$ . Prove that  $I_k = I_{k+1} = I_{k+2} = \dots$

We know that  $cR = J$ ,  $cR \subseteq I_k$  and  $I_k \subseteq J$ , which implies that  $I_k = J$ . By similar reasoning we have the rest.

(e) Use  $I_k = I_{k+1}$  to deduce that  $a_{k+1} b_k + 1R = b_{k+1} R$ , and from this, deduce that  $a_{k+1} \in R^*$ .

We can follow that  $I_{k+1} = b_{k+1} R$  by definition, and  $b_k = a_{k+1} b_{k+1}$  by its definition, thus implying that

$$b_k R = a_{k+1} b_{k+1} R = I_k = I_{k+1} = b_{k+1} R$$

thus we have that there is  $c \in R$  such that

$$a_{k+1} c b_{k+1} = b_{k+1}$$

we then follow that  $a_{k+1} c = 1$  by cancellation property, that implies in turn that  $a_{k+1} \in R^*$ .

### 7.3.3

Let  $R = \mathbb{Z}[\sqrt{-3}]$  be the ring that we studies in Example 7.21.

(a) Prove that  $\pm 1$  are the only units in  $R$ .

$\pm 1$  are obviously units here, as can be proven trivially.

If we take a number  $q$  that is in  $\mathbb{Z}$  and not equal to  $\pm 1$ , then we follow that if we multiply it by another member of  $\mathbb{Z}$ , then we're not getting 1 (due to basic properties of  $\mathbb{Z}$ ). If we multiply them by any  $a + b\sqrt{-3}$  with non-zero  $b$ , then we're getting non-zero  $b$  in the result as well, thus proving that no member of  $\mathbb{Z}$  is a unit.

If  $q \notin \mathbb{Z}$ , then we follow by previous point that its supposed multiplicative inverse cannot be in  $\mathbb{Z}$  by previous point. If  $q = b\sqrt{-3}$ , then we follow that by multiplying this thing by some  $b'\sqrt{-3}$  will give us  $-bb'$ , which cannot be equal to 1 with  $b, b' \in \mathbb{Z}$ .

Multiplication by  $a' + b'\sqrt{-3}$  with nonzero  $a'$  will give us nonzero  $\sqrt{-3}$  in the result, so it won't do. Thus the only other case happens if  $q = a + b\sqrt{-3}$  with  $a, b \neq 0$ . Multiplicating this  $q$  by any  $a' + b'\sqrt{-3}$  with zero  $a$  or  $b$  will give us previous point, thus we need to assume that  $a', b' \neq 0$ . We follow that

$$(a + b\sqrt{-3})(a' + b'\sqrt{-3}) = 1$$

we then follow that

$$(a - b\sqrt{-3})(a' - b'\sqrt{-3}) = 1$$

and multiplying them we get

$$(a^2 + 3b^2)(a'^2 + 3b'^2) = 1$$

we then follow that  $(a^2 + 3b^2)$  for nonzero  $a, b$  is greater or equal to 4, thus implying that there is no way to pick such numbers, thus implying the desired result

(b) *Prove that  $1 + \sqrt{-3}$  and  $1 - \sqrt{-3}$  are irreducible elements of  $R$ .*

Suppose that there are non-units  $q, w \in R$  such that

$$qw = 1 + \sqrt{-3}$$

we follow that

$$(a + b\sqrt{-3})(a' + b'\sqrt{-3}) = 1 + \sqrt{-3}$$

and therefore

$$(a - b\sqrt{-3})(a' - b'\sqrt{-3}) = 1 - \sqrt{-3}$$

and thus

$$(a^2 + 3b^2)(a'^2 + 3b'^2) = 1^2 + 3 = 4$$

$$(a^2 + 3b^2)(a'^2 + 3b'^2) = 4$$

and then we're going with pretty much the same derivation as in the example.

(c) *Prove that 2 is irreducible in  $R$*

Skip

(d) *Use previous point to deduce that  $R$  is not a UFD*

Directly follows from definitions and trivial deriviations

## 7.4 The Chinese Remainder Theorem

### 7.4.1

*Solve the following simultaneous congruences:*

(a)  $x \equiv 3 \pmod{7}$  and  $x \equiv 5 \pmod{11}$

$38 + 77n$  for  $n \in \mathbb{Z}$ .

(b)  $x \equiv 31 \pmod{117}$  and  $x \equiv 41 \pmod{119}$

$13726 + 117 * 119 * n$  for  $n \in \mathbb{Z}$ .

Next one is 423, and the rest is skipped

**7.4.2**

Let  $R$  be a commutative ring

(a) Suppose that  $a, b \in R$  have the property that  $aR + bR = R$ . Prove that for all  $m, n \geq 1$  we have

$$a^m R + b^n R = R$$

We follow that there are  $v, w \in R$  such that

$$a^2 v + baw = a$$

and thus we have that

$$1 = av + bw = (a^2 v + baw)v + bw = a^2 v^2 + baww + bw = a^2 v^2 + b(aw^2 + w)$$

and therefore we follow that  $a^2 R + bR = R$ . By induction we can prove that  $a^m R + bR = R$ , which would imply the desired result

*Next one is also proven by induction*

**7.4.3**

Let  $R$  be a PID

(a) Let  $a, b \in R$ . Prove that

$$aR + bR = R \Leftrightarrow aR \cap bR = abR$$

Assume the former. We follow that if  $x \in abR$  then there is  $q \in R$  such that  $x = abq$ , and thus  $x \in aR, x \in bR$ , which implies that

$$abR \subseteq aR \cap bR$$

If  $x \in aR \cap bR$ , then we follow that there are  $u, w \in R$  such that  $au = bw = x$ . Since  $aR + bR = R$  we follow also that there are  $q, e \in R$  such that

$$aq + be = 1$$

and thus

$$u = u * 1 = u(aq + be) = auq + bue = bwq + bue = b(wq + ue)$$

and thus we have that

$$x = au = ab(wq + ue) \in abR$$

which gives us the forward implication.

Since  $R$  is a PID we follow that there is  $d \in R$  such that

$$aR + bR = dR$$

thus there are  $u, w \in R$  such that

$$au + bw = d$$

We then follow that by one of the previous exercises  $d|a$  and  $d|b$ . Therefore there are  $n, j \in R$  such that

$$dn = a$$

$$dk = b$$

and thus we follow that

$$dnk = ak = bn \in aR \cap bR = abR$$

therefore we have that there is  $q \in R$  such that

$$dnk = abq$$

and thus

$$dnk = dndkq$$

since  $R$  is a PID we follow that it's an integral domain, and thus we can cancel out the terms to obtain

$$dnk = dndkq$$

$$1 * (dnk) = (dnk) * dq$$

$$1 = dq$$

which implies that  $d \in R^*$ , and thus  $dR = R$ , which gives us the desired result.

(b) Let  $c_1, \dots, c_n \in R$ , and suppose that  $c_i R + c_j R = R$  for all  $i \neq j$ . Prove that

$$c_1 R \cap c_2 R \dots \cap c_n R = c_1 c_2 \dots c_n R$$

As before, we can follow that  $c_1 c_2 \dots c_n R \subseteq c_1 R \cap c_2 R \dots \cap c_n R$  by a straightforward application of the definition.

Now the proof of 7.25 (particularly the IH part) gives us that

$$c_1 c_2 \dots c_n R + c_{n+1} R = R$$

namely through the derivation that

$$c_1 R + c_3 R = R$$

$$c_2 R + c_3 R = R$$

$$c_1 u_1 + c_3 w_1 = 1$$

$$c_2 u_2 + c_3 w_2 = 1$$

$$(c_1 u_1 + c_3 w_1)(c_2 u_2 + c_3 w_2) = 1$$

$$c_1c_2u_1u_2 + c_3c_1u_1w_2 + c_3\dots = c_1c_2u_1u_2 + c_3(\dots) = 1$$

thus we follow that

$$c_1R \cap c_2R \cap c_3R = (c_1R \cap c_2R) \cap c_3R = c_1c_2R \cap c_3R = c_1c_2c_3R$$

which by induction gives us the desired result.

(c) Prove that (a) may be false if  $R$  is not a PID by giving a counterexample for the ring  $F[x, y]$  where  $F$  is a field.

Let us look at  $xF[x, y]$  and  $yF[x, y]$ . We follow that if  $q \in xF[x, y] \cap yF[x, y]$ , then every sum element of  $q$  is a multiple of  $x$  and  $y$ , and thus  $xy$ , thus implying that  $q \in xyR$ , and the reverse inclusion is also trivial. However, we have that

$$1 \notin xF[x, y] + yF[x, y]$$

which implies the desired result.

#### 7.4.4

*Prove the generalized version of the CRT*

We're going to try to replicate the proof of the version 2 here, but with additional points. Thus we firstly need to prove the version for 2 ideals.

Let us denote  $I = I_1 \cap I_2$ . We follow that since  $I_1 + I_2 = R$ , that there are  $i \in I_1, j \in I_2$  such that

$$i + j = 1$$

there is an obvious map

$$\psi : R \rightarrow R/I_1 \times R/I_2, \psi(r) = (r \bmod I_1, r \bmod I_2)$$

we then follow that if  $x \in I = I_1 \cap I_2$ , then  $\psi(x) = (x \bmod I_1, x \bmod I_2) = (0, 0)$ , thus implying that  $I \subseteq \ker(\psi)$ . (by  $\bmod$  here we mean standard addition of the cosets, the same as in the chapter).

Now assume that  $r \in \ker(\psi)$ . We follow that  $r + I_1 = I_1$  and  $r + I_2 = I_2$ , thus we have that  $r \in I_1$  and  $r \in I_2$ , and thus  $r \in I$ . Therefore we conclude that  $I \subseteq \ker(\psi)$ , and thus  $\ker(\psi) = I$ .

Thus we have an injective homomorphism

$$\phi : R/I \rightarrow R/I_1 \times R/I_2$$

we've assumed in the beginning that  $i + j = 1$ , and thus we have that  $i = 1 - j$ , and  $j = 1 - i$ . We follow that

$$\phi(1 - j) = \phi(i) = (i + I_1, 1 - j + I_2) = (0, 1)$$

and similar for the other case, which implies that  $\psi$  is surjective, as desired.

Now for the inductive step. We're going to state it only for the case of 3, which can be generalized further. We follow that there are  $i_1 \in I_1, i_2 \in I_2, i_{3,1}, i_{3,2} \in I_3$  such that

$$i_1 + i_{3,1} = 1$$

$$i_2 + i_{3,2} = 1$$

thus giving us that

$$(i_1 + i_{3,1})(i_2 + i_{3,2}) = 1$$

$$i_1 i_2 + i_1 i_{3,2} + i_{3,1} i_2 + i_{3,1} i_{3,2} = 1$$

we follow that

$$i_1 i_{3,2} + i_{3,1} i_2 + i_{3,1} i_{3,2} \in I_3$$

and

$$i_1 i_2 \in I_1 \cap I_2$$

which implies that there are  $q \in I_1 \cap I_2, r \in I_3$  such that

$$q + r = 1$$

thus givin us the fact that

$$I_1 \cap I_2 + I_3 = R$$

this is the point where our previous proof will kick in, and by the same deriviation as in the original proof, we get the desired result.

*Skipping the rest of the exercises*

## 7.5 Field of Fractions

### 7.5.1

*Prove the following results, which are needed to complete the proof of Theorem 7.30*

(a) *Prove that the relation  $(a, b) \equiv (a', b')$  defined by  $ab' = a'b$  is an equivalence relation.*

Firstly it's obvious that

$$ab = ab$$

thus giving us the reflexivity. Symmetry is also kinda trivial. If

$$(a_1, b_1) \equiv (a_2, b_2), (a_2, b_2) \equiv (a_3, b_3)$$

then we follow that

$$a_1 b_2 = a_2 b_1, a_2 b_3 = a_3 b_2$$

we thus can multiply both equations to get

$$a_1 b_2 a_2 b_3 = a_2 b_1 a_3 b_2$$

since  $R$  is an indegrel domain, we follow that we can cancel and thus

$$a_1 a_2 b_3 = a_2 b_1 a_3$$

$$a_1 b_3 = b_1 a_3$$

thus implying that

$$(a_1, b_1) \equiv (a_3, b_3)$$

which is the desired conclusion.

(b) If  $(a'_1, b'_1) \equiv (a_1, b_1)$  and  $(a'_2, b'_2) \equiv (a_2, b_2)$ , prove that their multiplication is well-defined

We can follow that we need to prove that

$$(a'_1 a'_2, b'_1 b'_2) \equiv (a_1 a_2, b_1 b_2)$$

by definition of the multiplication. Thus we need to prove that

$$a'_1 a'_2 b_1 b_2 = a_1 a_2 b'_1 b'_2$$

by definition of  $\equiv$ . By our assumptions we have that

$$a'_1 b_1 = a_1 b'_1$$

$$a'_2 b_2 = a_2 b'_2$$

Thus we have

$$a'_1 a'_2 b_1 b_2 = a_1 a_2 b'_1 b'_2$$

$$a'_1 b_1 a'_2 b_2 = a_1 b'_1 a_2 b'_2$$

$$(a'_1 b_1)(a'_2 b_2) = a_1 b'_1 a_2 b'_2$$

$$(a_1 b'_1)(a_2 b'_2) = a_1 b'_1 a_2 b'_2$$

$$a_1 b'_1 a_2 b'_2 = a_1 b'_1 a_2 b'_2$$

(which you can chain, or use the same thing as in the proof for the addition to get a solid proof).

(c) Prove that the map  $\lambda$  defined by 7.23 is a field homomorphism from  $F$  to  $K$ .

We can follow that

$$\lambda(1, 1) = \phi(1) * \phi(1)^{-1} = \phi(1) * \phi(1) = 1 * 1 = 1$$



where we follow that  $\phi(1) = 1$  by the fact that it's a homomorphism (with 1's in their respective rings/fields).

We also have

$$\begin{aligned}
 \lambda((a, b) + (c, d)) &= \lambda((ad + bc, bd)) = \\
 &= \phi(ad + bc) * \phi(bd)^{-1} \\
 \lambda(a, b) + \lambda(c, d) &= \phi(a) * \phi(b)^{-1} + \phi(c) * \phi(d)^{-1} = \\
 &= \phi(a) * \phi(b)^{-1} * 1 + \phi(c) * \phi(d)^{-1} * 1 = \phi(a) * \phi(b)^{-1} * 1 + \phi(c) * \phi(d)^{-1} * 1 = \\
 &= \phi(a) * \phi(b)^{-1} * \phi(d) * \phi(d)^{-1} + \phi(c) * \phi(d)^{-1} * \phi(b) * \phi(b)^{-1} = \\
 &= \phi(ad) * \phi(b)^{-1} * \phi(d)^{-1} + \phi(cb) * \phi(d)^{-1} * \phi(b)^{-1} = \\
 &= (\phi(ad) + \phi(cb)) * \phi(b)^{-1} * \phi(d)^{-1} = (\phi(ad + cb)) * \phi(bd)^{-1}
 \end{aligned}$$

which states that addition is fine, and we also have that

$$\begin{aligned}
 \lambda((a, b) * (c, d)) &= \lambda((ac, bd)) = \phi(ac) * \phi(bd)^{-1} \\
 \lambda((a, b)) * \lambda((c, d)) &= \phi(a) * \phi(b)^{-1} * \phi(c) * \phi(d)^{-1} = \phi(ac) * \phi(bd)^{-1}
 \end{aligned}$$

which proves that multiplication is also fine, thus proving that  $\lambda$  is indeed a homomorphism, as desired

### 7.5.2

Let  $F$  be a field. An informal definition of the projective line is that it is the union

$$P^1(F) = F \cup \{\infty\}$$

of the field  $F$  and one extra point "at infinity". Then for  $\alpha \in F$ , we define an evaluation map

$$E_\alpha : F(X) \rightarrow P^1(F)$$

as follows. For rational functions  $\phi \in F(X)$ , we write  $\phi(X) = f(X)/g(X)$  as a ratio of polynomials having no common factors in  $F[X]$ , and then we set

$$E_\alpha(\phi) = \begin{cases} f(\alpha)/g(\alpha) & \text{if } g(\alpha) \neq 0 \\ \infty & \text{if } g(\alpha) = 0 \end{cases}$$

(a) Prove that  $E_\alpha(\phi)$  is well-defined; i.e. that it does not depend on the choice of  $f$  and  $g$ .

Let  $\phi \in F(X)$ . We know that  $F(X)$  is a field of fractions, and thus assume that  $(f_1, g_1), (f_2, g_2)$  are both representations of  $\phi$ . We follow that by definition

$$f_1 g_2 = f_2 g_1$$

We know also that  $F[X]$  is PID, and thus a UFD. Therefore we follow that we can deconstruct  $f_1, g_1$  into irreducible polynomials, and then remove common factors, (or the factors that differ by multiplication by a unit). Thus we can get  $f, g \in F[X]$  with such property. Since we've removed common factors, we follow that there is a product of those factors  $q$ , and thus we have that

$$f_1 = fq, g_1 = gq$$

and thus we have that

$$f_1 g q = f q g_1$$

and since  $F[X]$  is an integral domain we follow that

$$f_1 g = f g_1$$

thus concluding that

$$(f_1, g_1) \sim (f, g)$$

transitivity of  $\sim$  implies that

$$(f_2, g_2) \sim (f, g)$$

Now we've proven that two representations of a rational "function" can be mapped into one single pair of polynomials, that have no common factors, but we don't have that outputting pair is unique. And it does not have to be, since we can differ for example factors of polynomials in  $R$  by a unit, and get different pair. Thus we still need to prove that if  $(f, g)$  and  $(f', g')$  are both polynomials, that have no common factors that are equivalent to each other, then their quotient at  $\alpha$  is the same.

Since  $(f, g) \sim (f', g')$  we have that

$$fg' = f'g$$

and therefore we have

$$(fg')(\alpha) = (f'g)(\alpha)$$

or in other words

$$E_\alpha(fg') = E_\alpha(f'g)$$

evaluation of  $F[X]$  is a ring homomorphism, thus giving us that

$$E_\alpha(fg') = E_\alpha(f)E_\alpha(g') = E_\alpha(fg') = E_\alpha(f')E_\alpha(g)$$

or in other words

$$f(\alpha)g'(\alpha) = f'(\alpha)g(\alpha)$$

we then can follow that if  $g'(\alpha), g(\alpha) \neq 0$ , then

$$f(\alpha)g(\alpha)^{-1} = f'(\alpha)g'(\alpha)^{-1}$$

which gets us the desired result. If  $g(\alpha) = 0$  and  $g'(\alpha) \neq 0$ , then we follow that there is  $q \in F[X]$  such that

$$g = q(x - \alpha)$$

by exercise 5.1. Since  $g'(\alpha) \neq 0$  we follow that  $(x - \alpha)$  is not a factor of  $g'$ . But we have that

$$fg'(\alpha) = f'g(\alpha) = 0$$

$$f(\alpha)g'(\alpha) = 0$$

$$f(\alpha) = 0$$

thus implying that  $f$  is also factored by  $(x - \alpha)$ , thus implying that  $f$  and  $g$  have common factors, which is false. And in conclusion we have that if  $g(\alpha) = g'(\alpha) = 0$ , then  $E_\alpha(\phi) = \infty$ , which implies the desired result.

*Skip this one*

### 7.5.3

*This exercise describes a generalization of the construction of the field of fractions. Let  $R$  be a ring. A subset  $S \subseteq R$  is said to be multiplicatively closed if it has the following properties:*

$$1 \in S \text{ and } 0 \notin S$$

*and*

$$\text{If } a, b \in S \text{ then } ab \in S$$

*We define an equivalence relation  $\sim_S$  on the set of pairs  $(a, b) \in R \times S$  by*

$$(a, b) \sim_S (a', b') \iff cab' = ca'b \text{ for some } c \in S$$

*and we write  $R_S$  for the set of equivalence classes. We define addition and multiplication on  $R_S$  exactly as we did for the field of fractions.*

Little note: for this chapter we assume that rings are all associative, and also we are not assuming that  $R_S$  is a field, only indicating that it is a ring.

(a) *Prove that  $\sim_S$  is an equivalence relation*

We follow that for  $(a, b), (a', b'), (a'', b'') \in R \times S$  we have

$$ab = ab \Rightarrow (a, b) \sim_S (a, b)$$

thus giving us reflexivity. We also follow that if

$$(a, b) \sim_S (a', b')$$

then there is  $c \in S$  such that

$$cab' = ca'b \Rightarrow (a', b') \sim_S (a, b)$$

thus giving us symmetry. We also have that if

$$(a, b) \sim_S (a', b') \wedge (a', b') \sim_S (a'', b'')$$

then there are  $c, c' \in S$  such that

$$cab' = ca'b \wedge c'a'b'' = c'a''b'$$

and therefore we can just as before multiply both sides to get

$$cab'c'a'b'' = ca'bc'a''b'$$

$$cb'c'a'(ab'') = ca'c'b'(a''b)$$

setting  $q = cb'c'a'$  we get

$$q(ab'') = q(a''b)$$

thus proving that

$$(a, b) \sim_S (a'', b'')$$

thus giving us transitivity and the desired result.

(b) *Prove that addition and multiplication on  $R_S$  are well-defined.*

Let  $(a_1, b_1), (a'_1, b'_1), (a_2, b_2), (a'_2, b'_2) \in R \times S$  be such that

$$(a_1, b_1) \sim_S (a'_1, b'_1), (a_2, b_2) \sim_S (a'_2, b'_2)$$

We follow that

$$(a_1, b_1) + (a_2, b_2) = (a_1b_2 + a_2b_1, b_1b_2)$$

$$(a'_1, b'_1) + (a'_2, b'_2) = (a'_1b'_2 + a'_2b'_1, b'_1b'_2)$$

thus we need to show that there is  $c \in S$  such that

$$c(a_1b_2 + a_2b_1)(b'_1b'_2) = c(a'_1b'_2 + a'_2b'_1)(b_1b_2)$$

we then pull the same trick as in the book

$$0 = 0$$

$$(c_1a_1b'_1 - ca'_1b_1) = (c_2a'_2b_2 - c'a_2b'_2)$$

$$c_1(a_1b'_1 - a'_1b_1) = c_2(a'_2b_2 - a_2b'_2)$$

we then multiply lhs by  $c_2$  and rhs by  $c_1$  (which we can do since both are still zero)

$$c_1c_2(a_1b'_1 - a'_1b_1) = c_1c_2(a'_2b_2 - a_2b'_2)$$

we then set  $c = c_1c_2$  and get that

$$\begin{aligned}
 cb_2b'_2(a_1b'_1 - a'_1b_1) &= cb_1b'_1(a'_2b_2 - a_2b'_2) \\
 ca_1b_2b'_1b'_2 - ca'_1b'_2b_1b_2 &= ca'_2b'_1b_1b_2 - ca_2b_1b'_1b'_2 \\
 ca_1b_2b'_1b'_2 + ca_2b_1b'_1b'_2 &= ca'_1b'_2b_1b_2 + ca'_2b'_1b_1b_2 \\
 (ca_1b_2 + ca_2b_1)(b'_1b'_2) &= (ca'_1b'_2 + ca'_2b'_1)(b_1b_2) \\
 c(a_1b_2 + a_2b_1)(b'_1b'_2) &= c(a'_1b'_2 + a'_2b'_1)(b_1b_2)
 \end{aligned}$$

which is precisely the desired result.

We could've also simplified this proof by firstly stating that if

$$(a_1, b_1) \sim_S (a'_1, b'_1), (a_2, b_2) \sim_S (a'_2, b'_2)$$

then there are  $c_1, c_2 \in S$  such that

$$c_1a_1b'_1 = c_1a'_1b_1, c_2a_2b'_2 = c_2a'_2b_2$$

and therefore we follow that

$$c_1c_2a_1b'_1 = c_1c_2a'_1b_1, c_1c_2a_2b'_2 = c_1c_2a'_2b_2$$

and thus there is a single  $c = c_1c_2 \in S$  (product is in  $S$  since it's multiplicatively closed) such that

$$ca_1b'_1 = ca'_1b_1, ca_2b'_2 = ca'_2b_2$$

For multiplication we gotta prove that

$$(a_1, b_1) * (a_2, b_2) = (a_1a_2, b_1b_2)$$

$$(a'_1, b'_1) * (a'_2, b'_2) = (a'_1a'_2, b'_1b'_2)$$

By our assumptions and previous paragraph we have that there is  $c \in S$  such that

$$ca'_1b_1 = ca_1b'_1$$

$$ca'_2b_2 = ca_2b'_2$$

which then workd out in the same manner, as our proof for intergal domains

(c) Prove that  $R_S$  with the indicated addition and multiplication is a ring. The ring  $R_S$  is called the localization of  $R$  in  $S$ .

We firstly follow that the identitiies proof is the same as in the usual field of fractions, same goes for distributivity. For identity we need to prove that  $(0, 1) \not\sim_S (1, 1)$ , thus firstly assume that  $(0, 1) \sim_S (1, 1)$ . Then we have that there is  $c \in S$  such that

$$0 * 1 * c = 1 * 1 * c \Rightarrow 0 = 1$$

which is not true by the assumption that  $R$  is a ring.

The fact that  $R_S$  under plus is commutative follows directly from the definition, we follow that for  $(a, b) \in R_S$  we have that

$$(a, b) + (-a, b) = (ab - ba, bb) = (0, ab) = (0, 1)$$

thus proving inverses for the addition. We also follow that

$$((a_1, b_1) + (a_2, b_2)) + (a_3, b_3) = (a_1b_2 + a_2b_1, b_1b_2) + (a_3, b_3) = (a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2, b_1b_2b_3)$$

$$(a_1, b_1) + ((a_2, b_2) + (a_3, b_3)) = (a_1, b_1) + (a_2b_3 + a_3b_2, b_2b_3) = (a_1b_2b_3 + a_2b_3b_1 + a_3b_2b_1, b_1b_2b_3)$$

thus giving us the associativity and proving that  $R$  is indeed an abelian group under  $+$ .

The only thing that is left for the ring properties is associativity, which is also trivial:

$$((a_1, b_1) * (a_2, b_2)) * (a_3, b_3) = (a_1a_2a_3, b_1b_2b_3) = (a_1, b_1) * ((a_2, b_2) * (a_3, b_3))$$

One note: we see that in all of the given proofs we have denominators being products of elements of  $S$ , which by assumption of  $S$  give us the fact that they aren't 0.

Here we are also note why this thing is a ring, and not a field:  $R_S$  is a cartesian of  $R$  and  $S$ , which implies that if  $q \in R \setminus S$  and  $q \neq 0$ , then  $(q, 1) \neq 0$  for example may not have an inverse in  $R_S$  since  $(1, q) \notin R_S$ . If  $R \setminus S = \{0\}$  however, then

$$(a, b) * (b, a) = (ab, ab)$$

but here we can have a problem that there might be the case that  $ab = 0$  since  $R$  is not an integral domain, which would imply that  $(ab, ab) \notin R_S$ .

(d) Suppose that  $R$  is an integral domain and that  $S$  is multiplicatively closed. Prove that  $R_S$  is a subring of the field of fractions of  $R$ .

If  $R$  is an integral domain, then we follow that there is indeed a field of fractions.  $R_S$  in this case by the way it's defined is indeed a subset of a field of fractions, with the same addition and multiplication functions, and as proven in the previous point it is a ring, thus making it a subring of the field of fractions.

(e) Let  $I$  be an ideal of  $R$  and let

$$S = R \setminus I$$

Prove that  $S$  is multiplicatively closed if and only if  $I$  is a prime ideal.

Here we're gonna assume that  $I$  is given as an ideal, and then multiplicative closure of  $S$  is followed directly, as indicated in the definition in the book.

(f) Let  $b \in R$  be an element, and let

$$\langle b \rangle = \{1, b^1, b^2, \dots\}$$

be the set of powers of  $b$ . Prove that  $\langle b \rangle$  is multiplicatively closed if and only if  $b$  is not nilpotent.

Suppose that  $\langle b \rangle$  is multiplicatively closed. We follow that there are no  $q, w \in \langle b \rangle$  such that  $qw = 0$  since  $qw \in \langle b \rangle$  and  $0 \notin \langle b \rangle$ . Thus we conclude that there are no  $m, n \in \mathbb{Z}$  such that  $b^m b^n = 0$ , and thus  $b$  is not nilpotent, as desired.

If  $b$  is not nilpotent, then we follow that there is no  $m \in \mathbb{Z}$  such that  $b^m = 0$ . We follow that  $1 \in \langle b \rangle$ ,  $0 \notin \langle b \rangle$ , and if  $b^m, b^n \in \langle b \rangle$  for some  $m, n \in \mathbb{Z}$  then  $b^m b^n \in \langle b \rangle$ , as desired.

*Skipping the rest due to triviality*

## 7.6 Multivariate and Symmetric Polynomials

### 7.6.1

Let  $F$  be a field, let  $F(x, y)$  be the 2-variable field of rational functions as described in Definition 7.34. For  $(b, c) \in F^2$ , we might try to define an evaluation-at- $(b, c)$  map

$$E_{b,c} : F(x, y) \rightarrow F, \frac{f(x, y)}{g(x, y)} \rightarrow \frac{f(b, c)}{g(b, c)}$$

but just as in the 1-variable case, things may go wrong if we get 0 in the denominator. In this exercise we look at what might happen for the function

$$\phi(x, y) = \frac{x}{x^2 + y^2} \in F(x, y)$$

(a) If  $F = \mathbb{R}$ , prove that  $E_{b,c}(\phi)$  is well-defined except at the one point  $(0, 0)$

We can easily follow that  $x^2 + y^2 = 0 \Leftrightarrow x = 0 \wedge y = 0$  by various methods.

(b) If  $F = \mathbb{C}$ , for which points  $(b, c) \in \mathbb{C}^\oplus$  is  $E_{b,c}\phi$  is not defined?

We have that

$$x^2 + y^2 = 0$$

$$y^2 = -x^2$$

$$y = \pm \sqrt{-x^2}$$

which is as much as I can define in this case

(c) If  $F = \mathbb{F}_5$ , for which points  $(b, c) \in \mathbb{F}_5^2$  is  $E_{b,c}(\phi)$  is not well-defined. in particular, how many such points are there

There are in total 25 points in  $\mathbb{F}_5$ , and python says that points

$$(0, 0), (1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2), (4, 3)$$

are such points.

(d) Same as in previous

**7.6.2**

We defined the  $n$ -variable polynomial ring  $R[X_1, \dots, X_n]$  in Definition 7.33

(a) Prove that there is a natural ring inclusions

$$R \subseteq R[X_1] \subseteq R[X_1, X_2] \subseteq \dots$$

Follows directly from the definitions of those things

(b) Prove that the union

$$Q = \bigcup_{n=0}^{\infty} R[X_1, \dots, X_n]$$

using the inclusion from (a) is a ring. It is denoted by  $R[X_1, X_2, \dots, X_n]$  and is called the ring of polynomials in infinitely many variables.

Identities are present in the  $R$ . We follow that any given  $a, b \in Q$  are contained in some  $R[X_1, \dots, X_n]$  for some  $n \in \omega$  (we can pick the largest of the proper containers of  $a$  or  $b$ ), in which rules for rings work just fine, and thus we conclude that  $Q$  is also a ring, as desired.

**7.6.3**

This exercise asks you to prove Proposition 7.39, which says that permutations  $\pi \in S_n$  act on the ring of  $R[X_1, \dots, X_n]$  as homomorphisms.

In this exercise we're going to use somewhat liberal notation and proofs. As usual, with enough proofs of somewhat trivial things, we can make this thing way more rigorous.

(a) Prove that  $\pi$  sends sums to sums.

Let  $a, b \in R[X_1, \dots, X_n]$ . Let  $\omega^n$  be a set of vectors of length  $n$  in  $\omega$ . We can follow that this set is countable, and thus there is an injection  $f$  from  $\omega$  to  $\omega^n$ . Thus by definition each element of  $R[X_1, \dots, X_n]$  can be represented by a sequence  $j$  of elements of  $R$  such that

$$a = \sum_{i \in \omega} f(i)j(i)$$

We then follow that  $\pi$  is essentially acting exclusively on  $f(i)$ , i.e.

$$\pi(a) = \pi\left(\sum_{i \in \omega} f(i)j(i)\right) = \sum_{i \in \omega} \pi(f(i))j(i)$$

Thus we have that

$$\begin{aligned} \pi(a+b) &= \pi\left(\sum_{i \in \omega} f(i)j(i) + \sum_{i \in \omega} f(i)k(i)\right) = \pi\left(\sum_{i \in \omega} f(i)(j(i) + k(i))\right) = \sum_{i \in \omega} \pi(f(i))(j(i) + k(i)) = \\ &= \sum_{i \in \omega} \pi(f(i))j(i) + \sum_{i \in \omega} \pi(f(i))k(i) = \pi\left(\sum_{i \in \omega} f(i)j(i)\right) + \pi\left(\sum_{i \in \omega} f(i)k(i)\right) = \pi(a) + \pi(b) \end{aligned}$$



as desired

(b) *Prove that  $\pi$  sends products to products*

We follow that

$$\begin{aligned}\pi(a * b) &= \pi\left(\sum_{i \in \omega} f(i)k(i) * \sum_{i \in \omega} f(i)l(i)\right) = \pi\left(\sum_{i \in \omega} \sum_{j \in \omega} f(i) * f(j) * k(i) * l(j)\right) = \\ &= \sum_{i \in \omega} \sum_{j \in \omega} \pi(f(i) * f(j)) * k(i) * l(j) = \sum_{i \in \omega} \sum_{j \in \omega} \pi(f(i)) * \pi(f(j)) * k(i) * l(j) = \\ &= \pi\left(\sum_{i \in \omega} \pi(f(i))k(i) * \sum_{i \in \omega} \pi(f(i))l(i)\right) = \pi(a * b)\end{aligned}$$

(c) *Similar to the previous one*

#### 7.6.4

Let  $s_k(X_1, \dots, X_n)$  be the  $k$ th elementary symmetrical polynomial in  $R[X_1, \dots, X_n]$

(a) *Let  $\pi \in S_n$ . Prove that*

$$\pi(X_1 \dots X_k) = X_1 \dots X_k \Leftrightarrow \pi \in S_k$$

If  $\pi(X_1 \dots X_k) = X_1 \dots X_k$ , then we follow that for each  $i \in \{1, \dots, k\}$  there is unique  $j \in \{1, \dots, k\}$  such that  $\pi(i) = j$ , which means that  $\pi$  is a bijection, and thus  $\pi \in S_k$ .

Reverse direction can be seen easily as well.

(b) *Prove that*

$$s_k(X_1, \dots, X_n) = \sum_{\pi \in S_n/S_k} \pi(X_1 X_2 \dots X_k)$$

where the sum is over one permutation from each coset of  $S_k$  in  $S_n$

Elements on both sides of the equal sign are the sums of finite sets of elements of  $R[X_1, \dots, X_n]$ . Thus if the underlying sets are equal, then the whole sums are equal as well.

Let  $j$  be an element of the sum on lhs. We can follow that there is a set  $J \subseteq \{1, \dots, n\}$  such that  $|J| = k$  and such that

$$j = \prod_{i \in J} X_i$$

by definition of  $s_k$ . We then also can follow that there is a permutation  $\phi$  on  $\{1, \dots, n\}$  such that

$$\phi(i) = \begin{cases} i \leq k \Rightarrow i\text{'th lowest element of } J \\ i > k \Rightarrow (i - k)\text{'th lowest element of } \{1, \dots, n\} \setminus J \end{cases}$$

Let  $u \in S_k$ . We follow that

$$(\phi \circ u)[\{1, \dots, k\}] = \phi[u[\{1, \dots, k\}]] = \phi[\{1, \dots, k\}] = J$$

Thus  $\phi S_k$  is an element of  $S_n/S_k$  such that  $j = \phi(X_1, \dots, X_k)$ . Therefore we conclude that if  $L$  is a set of elements of the sum on lhs, and  $R$  is similar thing for the rhs, then  $L \subseteq R$ .

I think that this exercise is wrong because number of elements of the sum on the lhs and rhs are wrong. We can follow that number of elements of the sum on the lhs is  $bc(n, k)$  and on the rhs due to Lagrange it is  $n!/k!$ , which aren't equal. For example we know that

$$s_1(X_1, X_2, X_3) = X_1 + X_2 + X_3$$

we know that there is only one permutation in  $S_1$ , and 6 in  $S_3$ , namely

$$S_3 = (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$$

thus we follow that

$$S_3/S_1 = S_3$$

and therefore

$$\sum_{\pi \in S_3/S_1} \pi(X_1) = X_1 + X_1 + X_2 + X_2 + X_3 + X_3 = 2X_1 + 2X_2 + 2X_3$$

*Skiping the rest for now*

## Chapter 8

# Fields - Part 2

### 8.1 Algebraic Numbers and Transcendental Numbers

#### Notes

The proof that I was making in the third chapter in this book (the one about interesting types of rings about the injections into the ID) is becoming relevant in this section (see proof of the 8.5).

We can follow that 8.6(a) that states that

$$\dim_F F[x]/f(x)F[x] = \deg(f)$$

for some  $f(x) \in F[x]$  is independent of the fact that  $f$  might have some roots in some extension fields, and thus is true in general.

One thing that is not proven in the chapter, and I cannot find the proof in the book is the proof of the following:

**Theorem:** If  $f : R \rightarrow R'$  is a homomorphism, and  $H \subseteq \ker(f)$  is an ideal of  $f$  that is also contained in  $\ker(f)$ , then range of  $f$  and range of  $\bar{f} : R/H \rightarrow R'$  defined by

$$\bar{f}(a + H) = f(a)$$

are the same

*Proof:* If  $r' \in \text{ran}(f)$ , then there is  $r \in R$  such that  $f(r) = r'$ , and thus  $\bar{f}(r) = r'$ , thus implying that  $\text{ran}(f) \subseteq \text{ran}(\bar{f})$ . Reverse inclusion is similar.

We can follow that  $f : R \rightarrow R$  defined by  $f(x) = -x$  is a vector field isomorphism, however we follow that  $f(1) = -1 \neq 1$ , thus implying that  $f$  is not a ring homomorphism. Thus we can follow that vector field homomorphisms need not be ring homomorphisms. Reverse implication might not be true as well: if  $f$  is a ring homomorphism, then we can only follow that

$$f(c_1a + c_2b) = f(c_1a) + f(c_2b) = f(c_1)f(a) + f(c_2)f(b)$$

which falls just short of the proof of the desired condition. Thus we can state that in proof of 8.6(b) we only state that a particular function in question is both ring and vector homomorphism. That particular statement about  $E_a$  being both vector and ring homomorphism follows from exercises in previous chapters.

### 8.1.1

Let  $F$  be a field, let  $f(x) \in F[x]$  be a non-zero polynomial, and let  $\alpha$  be a root of  $f(x)$  in some extension field of  $F$ . Prove that

$$[F(\alpha) : F] = \deg(f) \Rightarrow f(x) \text{ is irreducible in } F[x]$$

By definition we have that

$$[F(\alpha) : F] = \dim_F(F(\alpha))$$

and thus

$$\dim_F(F(\alpha)) = \deg(f) = \dim_F F[x]/f(x)F[x]$$

By the same reasoning as in proof of 8.6(b) we have that there is a well-defined surjective ring homomorphism

$$\overline{E_\alpha} : F[x]/f(x)F[x] \rightarrow F(a)$$

which is also a linear map between two  $F$ -spaces. Given that we have that

$$\dim_F(F[x]/f(x)F[x]) = \dim_F(F(a))$$

we conclude that the function  $\overline{E_\alpha}$  is bijective, thus we have that  $\overline{E_\alpha}$  is a ring homomorphism that is bijective, thus implying that by 7.16  $f(x)$  is irreducible, as desired.

### 8.1.2

Prove that each of the following numbers is algebraic over  $Q$  by finding a polynomial in  $Q[x]$  for which it is a root:

(a)  $\sqrt[n]{c}$  where  $c \in Q$  and  $n \geq 1$ .

We can say that

$$f(x) : x^n - c$$

is such a polynomial.

(b)  $\sqrt{2} + \sqrt{3}$

We can follow that

$$(\sqrt{2} + \sqrt{3})^2 = 2\sqrt{6} + 5$$

thus

$$((\sqrt{2} + \sqrt{3})^2 - 5)/2 = \sqrt{6}$$

and therefore

$$\begin{aligned} (((\sqrt{2} + \sqrt{3})^2 - 5)/2)^2 &= 6 \\ (((\sqrt{2} + \sqrt{3})^2 - 5)/2)^2 - 6 &= 0 \end{aligned}$$

then we sub  $\sqrt{2} + \sqrt{3}$  for  $x$  and get

$$((x^2 - 5)/2)^2 - 6 = 0.25(x^2 - 5)^2 - 6 = 0.25(x^4 - 10x^2 + 25) - 6 = \frac{1}{4}x^4 - \frac{10}{4}x^2 + \frac{1}{4}$$

which is the desired polynomial.

(c)  $\sqrt[3]{2} + \sqrt{3}$ .

We can follow that

$$\begin{aligned} (\sqrt[3]{2} + \sqrt{3})^0 &= 1 \\ (\sqrt[3]{2} + \sqrt{3})^1 &= \sqrt[3]{2} + \sqrt{3} \\ (\sqrt[3]{2} + \sqrt{3})^2 &= (\sqrt[3]{2})^2 + 2\sqrt[3]{2}\sqrt{3} + 3 \\ (\sqrt[3]{2} + \sqrt{3})^3 &= 2 + 3\sqrt[3]{2}^2\sqrt{3} + 9\sqrt[3]{2} + 3\sqrt{3} \end{aligned}$$

and so on. In general we can later create a matrix with coefficients in

$$1, \sqrt[3]{2}, (\sqrt[3]{2})^2, \sqrt{3}, \sqrt{3}\sqrt[3]{2}, \sqrt{3}(\sqrt[3]{2})^2$$

from powers to coefficients, to get a surjective matrix with nonzero kernel, which would in turn give us the desired polynomial (we need to go for at least 6th power here, and that's too tedious).

### 8.1.3

*This exercise sketches a proof of the following result, which says that if a number is the root of a polynomial in  $\mathbb{Q}[x]$ , then it cannot be too closely approximated by rational numbers.*

*Let  $f(x) \in \mathbb{Q}[x]$  be a polynomial of degree  $d \geq 1$ . There is a positive constant  $C_f > 0$  such that if  $\alpha \in \mathbb{C} \setminus \mathbb{Q}$  is a non-rational root of  $f(x)$ , then*

$$|p/q - \alpha| \geq C_f/q^d \text{ for all } p/q \in \mathbb{Q}$$

Skipping the rest due to the fact that this exercise is wrong, and fixing it is a mess

## 8.2 Polynomial Roots and Multiplicative Subgroups

### Notes

We can see that during the proof of 8.10 we haven't used the fact that  $F$  is a field, and only used the fact that  $F$  is an integral domain, thus we can relax the prerequisites of the theorem.

It is also important to remember that any finite integral domain is a field (see exercises for section 3.4). Thus we can ask a somewhat interesting question: If  $U$  is a finite subset of some integral domain  $R$ , then is there a finite field  $F$ , that is contained within  $R$  and also contains  $U$ ? The answer to that is no: we can take  $U = \{1\}$ , which is a subset of  $Z$ , and then follow that additive closure of this set is the entirety of  $Z$ , which implies that no such field exists.

### 8.2.1

*Suppose that  $R$  is a commutative ring that is not an integral domain. Prove that there is a polynomial  $f(x) \in R[x]$  that has more distinct roots in  $R$  than its degree.*

Let  $a, b \in R$  be nonzero and such that  $ab = 0$ . We can follow that

$$(x - a)(x - b) = x^2 - bx - ax + ab = x^2 + (-a - b)x + ab = x^2 + (-a - b)x$$

We then follow that roots for this polynomial are  $a, b, 0$ , which gives us the desired result.

### 8.2.2

*Let  $R$  be a ring, let  $a_1, \dots, a_n \in R$ , and define a set of polynomials*

$$K_a = \{f(x) \in R[x] : f(a_1) = \dots = f(a_n) = 0\}$$

*(a) Prove that  $K_a$  is an ideal of  $R[x]$ .*

Let  $f, g \in K_a$ . We follow that

$$(f + g)(a_j) = f(a_j) + g(a_j) = 0 + 0 = 0$$

and if  $q \in R$  then

$$(fq)(a_j) = f(a_j)q(a_j) = 0 * q(a_j) = 0$$

thus proving to us that  $K_a$  is indeed an ideal, as desired.

*(b) Suppose that  $R$  is an integral domain, and that  $a_1, \dots, a_n$  are distinct. Prove that  $K_a$  is the principal ideal of  $R[x]$  generated by the polynomial  $(x - a_1)\dots(x - a_n)$ .*

Firstly we follow that

$$(x - a_1)\dots(x - a_n) \in K_a$$

which is pretty evident. Then we follow that if  $f \in K_a$  then it can be destructed as

$$(x - a_1)\dots(x - a_n)g(x)$$

by theorem 8.8(b), which implies the desired result.

## 8.2.3

Let  $H$  be the ring of quaternions.

(a) It was noted that the quadratic polynomial  $x^2 + 1$  has more than two roots in  $H$ . Explain which step of the proof Theorem 8.8 goes wrong for  $H$ .

Theorem 8.8 is not applicable to  $H$  due to the fact that  $H$  is a non-commutative ring. For the case of (a) we can follow that

$$x^k - a^k = (x - a) \sum_{i=0}^{k-1} x^{k-i-1} a^i$$

is wrong:

$$i^2 - j^2 = (-1) - (-1) = 0$$

$$(i - j)(i + j) = i^2 + ij - ji + j^2 = (-1) + k + k + (-1) = k - 2$$

(b) Prove that  $x^2 + 1$  has infinitely many roots in  $H$ .

We can follow that for some  $h \in H$  we have that

$$h = q + xi + yj + zk$$

for some  $q, x, y, z \in R$ , and thus we have that

$$\begin{aligned} h^2 &= (q + xi + yj + zk)^2 = (q + xi + yj + zk)(q + xi + yj + zk) = \\ &= (q^2 + qxi + qyj + qzk) + (qxi - x^2 + x y i j + x z i k) + \\ &+ (q y j + x y j i - y^2 + y z j k) + (q z k + z x k i + z y k j - z^2) = \\ &= (q^2 + qxi + qyj + qzk) + (qxi - x^2 + x y k - x z j) + \\ &+ (q y j - x y k - y^2 + y z i) + (q z k + z x j - z y i - z^2) = \\ &= (q^2 - x^2 - y^2 - z^2) + i(qx + qx + yz - zy) + j(qy - xz + qy + zx) + k(qz + xy - xy + qz) = \\ &= (q^2 - x^2 - y^2 - z^2) + (2qx)i + (2qy)j + (2qz)k \end{aligned}$$

We then follow that if we set  $q = 0$ , then  $(2qx)i = (2qy)j = (2qz)k = 0$ . Thus we only need to find some  $x, y, z \in R$  so that

$$-x^2 - y^2 - z^2 = -1$$

in order to find appropriate root. For example we can set  $x = 0$  and get

$$-y^2 - z^2 = -1$$

$$-y^2 = -1 + z^2$$

$$y^2 = 1 - z^2$$

$$y = \sqrt{1 - z^2}$$

we then follow that for any  $z \in [-1, 1]$  there is an appropriate  $y$ , which gives us that

$$(\sqrt{1 - z^2}j + zk)^2 = -1$$

which gives us the desired result.

(c) Describe all the roots of  $x^2 + 1$  in  $H$

This can be easily derived from our previous formula for the square.

(d) Skip

### 8.2.4

Let  $F$  be a finite field of order  $q$ , and assume that  $q$  is odd.

(a) Let  $a, b \in F^*$ . If  $a^2 = b^2$ , prove that either  $a = b$  or  $a = -b$ .

Here we only use the assumption that  $F$  is an integral domain. Namely we have that

$$a^2 = b^2 \Leftrightarrow a^2 - b^2 = 0 \Rightarrow (a - b)(a + b) = 0 \Rightarrow a = b \vee a = -b$$

as desired.

(b) Show by the way of example that (a) is not true for the rings  $\mathbb{Z}/8\mathbb{Z}$  and  $\mathbb{Z}/15\mathbb{Z}$

We follow that

$$4^2 = 16 \equiv 0 = 0^2$$

and

$$11^2 = 121 \equiv 1 \equiv 196 = 14^2$$

as desired.

(c) Let

$$R = \{a^2 : a \in F^*\}, N = F^* \setminus R$$

be, respectively, the set of squares and non-squares in  $F^*$ . Prove that  $R$  and  $N$  each contain exactly  $(q - 1)/2$  elements.

Since  $F^*$  is a finite subset of itself, and  $F$  is a finite field, we follow that  $F^*$  is a cyclic group under multiplication. This gives us the fact that there is  $w$  such that for each  $a \in F^*$  there is  $n \in \mathbb{N}$  such that

$$a = w^n$$



thus we then follow that either  $n$  is odd or even, which in conjunction with the fact that  $|F^*| = q - 1$  gives us the desired result.

(d) Let  $f(x)$  be the polynomial

$$f(x) = x^{(q-1)/2} - 1$$

Prove that  $R$  is exactly the set of roots of  $f(x)$  in  $F$ .

We follow that if  $a^2 \in R$  then

$$(a^2)^{(q-1)/2} = a^{q-1}$$

since  $a \in F^*$  we follow that its order  $j$  (which is finite since  $F^*$  is finite) divides the order of  $F^*$ , and thus

$$a^{q-1} = a^{jk} = e^k = 1$$

which gives us that all elements of  $R$  are roots of the given polynomial. Since

$$|R| = (q - 1)/2$$

we follow that  $f$  is fully destructed by elements of  $R$ , which implies that  $R$  is a complete set of roots of  $f$ , as desired.

(e) Let  $c \in F^*$ . Prove that

$$c^{(q-1)/2} \equiv \begin{cases} c \in R \Rightarrow 1 \\ c \in N \Rightarrow -1 \end{cases}$$

Presuming that there is a typo and the author meant  $R$  instead of  $Q$ .

We follow that if  $x \in F^*$  then

$$x^{q-1} = 1$$

we also follow that

$$(x^{(q-1)/2} - 1)(x^{(q-1)/2} + 1) = x^{q-1} - 1$$

The fact that  $(x^{(q-1)/2} - 1)$  destructs by  $R$  is the previous point, which gives us the desired result.

(f) Let  $a_1, a_2 \in R$  and  $b_1, b_2 \in N$ . Prove that

$$a_1 a_2 \in R, b_1 b_2 \in R$$

We can follow that for  $a_1, a_2 \in R$  there are even  $m, n \in N$  such that

$$a_1 = w^m, a_2 = w^n$$

thus

$$a_1 a_2 = w^{mn}$$

since  $mn$  is even, we follow that it's in  $R$ , as desired. Similar reasoning holds for  $b$ , but in it we follow that sum of odd numbers is even.

## 8.2.5

Let  $F$  be a finite field with  $q$  elements, and let  $m|q-1$ .

(a) Prove that  $F^*$  has a unique subgroup of order  $m$ .

$F^*$  is a cyclic group, and exercise 2.27(b) implies the desired result.

(b) Let  $\alpha \in F^*$ . Prove that the following are equivalent:

(i)  $\alpha$  is an  $m$ th power in  $F$  (i.e. there's  $\beta \in F^*$  such that  $\beta^m = \alpha$ ).

(ii)  $\alpha^{(q-1)/m} = 1$

Assuming (i) we follow that

$$\alpha^{(q-1)/m} = \beta^{q-1} = 1$$

by Lagrange or whatever

Assuming (ii) we follow that there is an element  $\beta$  that generates  $F^*$ , and  $n \in \mathbb{Z}$  such that

$$\alpha = \beta^n$$

thus

$$\alpha^{(q-1)/m} = \beta^{n(q-1)/m} = 1$$

thus we follow that

$$(q-1)|n(q-1)/m$$

thus there is  $i \in \mathbb{Z}$  such that

$$i(q-1) = n * (q-1)/m$$

thus  $n/m \in \mathbb{Z}$  (there's gotta be a better way to phrase this) and thus  $m|n$ . Therefore there is  $j \in \mathbb{Z}$  such that  $n = mj$ , and therefore

$$\alpha = \beta^n = \beta^{mj} = (\beta^j)^m$$

as desired.

(c) Suppose that  $q$  is odd. Prove that

$$-1 \text{ is a square in } F^* \Leftrightarrow q \equiv 1 \pmod{4}$$

Suppose that  $-1$  is a square. We follow then that there is  $\beta \in F^*$  such that

$$\beta^2 = -1$$

thus

$$(-1)^{(q-1)/2} = 1$$

and therefore there is  $j \in \mathbb{Z}$  such that

$$(q-1)j = (q-1)/2$$

$$2(q-1)j = (q-1)$$

$$2(q-1)j + 1 = q$$

Since  $q$  is odd we follow that  $q-1$  is even. Since  $F$  is a field, it's a ring, and by our standards it's got to have at least two elements, and therefore  $q \geq 3 \Rightarrow q-1 \geq 2$ . Thus there is  $l \in N$  such that  $q-1 = 2l$ . Thus

$$q = 2(q-1)j + 1 = 2 * 2 * l * j + 1$$

modding both sides by 4 gives the desired result.

### 8.3 Splitting Fields, Separability, and Irreducibility

#### Notes

In regards with theorem 8.24 (that states that  $f \in F[x]$  is separable iff  $\gcd(f, f') = 1$ ): definition of  $\gcd$  for arbitrary PID is found in the first exercise of section 7.2. This also encapsulates the footnote: if  $aR + bR = 1R$  then we conclude that  $\gcd(a, b) = 1$ .

When we talk about separability of a given polynomial, the question arises about the dependency of separability of given polynomial with respect to different splitting fields. I.e. suppose that  $L$  and  $L'$  are both fields such that  $f(x) \in F[x]$  splits completely in them. If  $\alpha \in L$  is such that  $f(x) = (x - \alpha)^n g(x)$  for some  $g(x) \in F[x], n \in \mathbb{Z}_+$ , then does there exist  $\alpha' \in L'$  such that  $f(x) = (x - \alpha')^n g(x)$ ?

Firstly, if we take on faith for now the assertion that all splitting fields of  $f(x)$  are isomorphic, then we follow that separability makes sense and our initial assertion is correct.

Secondly, if  $f(x)$  is separable in one splitting field, but not the other, then we follow that  $\gcd(f(x), f'(x)) = 1$  from the former assumption, but  $\gcd(f(x), f'(x)) \neq 1$  from the latter, which given us the contradiction, and thus implies that if  $f(x)$  is separable in  $L$ , then it is also separable in  $L'$  (mutatis mutandis with inseparable)

#### 8.3.1

*In each case, prove that  $L$  is a splitting field for  $f(x)$  over  $F$ .*

(a)  $F = \mathbb{Q}$ ,  $f(x) = x^2 - 4x + 13$ ,  $L = \mathbb{Q}(i)$

We can follow that  $x = 2 \pm 3i$  are the roots of this polynomial, which are located in  $L$  and not in  $\mathbb{Q}$ . Since  $[L : \mathbb{Q}] = 2$  as per examples in 5th chapter, we follow from exercise 5.4.1 that there are no proper subfields of  $L$  that properly contain  $\mathbb{Q}$ , which gives the desired result.

(b)  $F = \mathbb{Q}$ ,  $f(x) = x^4 - 4$ ,  $L = \mathbb{Q}(i, \sqrt{2})$

We firstly follow that

$$x^4 - 4 = (x^2 - 2)(x^2 + 2) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{2}i)(x + \sqrt{2}i)$$

with respect to  $L$ , and thus we need to show that  $L$  is the smallest field splits  $f$ . We can follow that  $[L : Q] = 4$ , which gives us the idea that there might be extension fields of degree 2 that are located in between  $L$  and  $Q$  (here we assume that the proper subfield of  $L$ , that contains but is not equal to  $Q$  is a should be of degree not 1 or 4 due to some trivial linear algebra derivation). Thus we need to prove that

$$Q(\sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i) = Q(i, \sqrt{2})$$

here we can state firstly that

$$Q(\sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i) = Q(\sqrt{2}, \sqrt{2}i)$$

and thus

$$Q(\sqrt{2}, \sqrt{2}i) = Q(\sqrt{2}, \sqrt{2}^{-1} * \sqrt{2}i) = Q(\sqrt{2}, i)$$

as desired

*Skiping due to triviality*

### 8.3.2

(a) Let  $f(x) = x^4 - 1 \in Q[x]$ . Factor  $f(x)$  into irreducible factors in  $Q[x]$ , and then prove that  $Q(\sqrt{-1})$  is the splitting field of  $f(x)$  over  $Q$ .

We can follow that

$$(x^4 - 1) = (x^2 + 1)(x^2 - 1) = (x - i)(x + i)(x - 1)(x + 1)$$

the fact that  $Q(i) = Q(1, -1, i, -i)$  is evident.

(b) Let  $f(x) = x^6 - 1 \in Q[x]$ . Factor  $f(x)$  into irreducible factors in  $Q[x]$ , and then prove that  $Q(\sqrt{-3})$  is the splitting field of  $f(x)$  over  $Q$ .

We can follow that

$$\begin{aligned} (x^6 - 1) &= (x - 1)(x + 1)(x^4 + x^2 + 1) = (x - 1)(x + 1)(x^4 + 2x^2 + 1 - x^2) = \\ &= (x - 1)(x + 1)((x^2 + 1) - x^2) = (x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1) = \\ &= (x - 1)(x + 1)((x^2 + 1) - x^2) = \\ &= (x - 1)(x + 1)\left(x + \frac{\sqrt{3}i - 1}{2}\right)\left(x - \frac{\sqrt{3}i + 1}{2}\right)\left(x + \frac{\sqrt{3}i + 1}{2}\right)\left(x - \frac{\sqrt{3}i - 1}{2}\right) \end{aligned}$$

as desired.

**8.3.3**

*Which of the following rings are fields, and which are not? Be sure to give a reason for your answer*

We can follow that the whole thing is going to be a field if and only if the polynomial is irreducible (per theorem 7.16)

$$(a) R[x]/(x^2 + 1)R[x]$$

We can follow that  $x^2 + 1$  is irreducible, and thus the whole thing is a field. Also, as proven in some earlier exercise, this thing is isomorphic to  $C$ .

$$(b) F_3[x]/(x^2 + 1)F_3[x]$$

We can follow that

$$(0^2 + 1) = 1$$

$$(1^2 + 1) = 2$$

$$(2^2 + 1) = 5 \equiv 2$$

thus this polynomial does not have roots, therefore it's irreducible (if it would be, then a factor would be linear, and thus have roots), therefore the whole thing is a field

$$(c) F_5[x]/(x^2 + 1)F_5[x] \text{ We can follow that}$$

$$(0^2 + 1) = 1$$

$$(1^2 + 1) = 2$$

$$(2^2 + 1) = 5 \equiv 0$$

thus we have root of 2, and thus

$$x^2 + 1 = (x - 2)(x + 2)$$

thus making the polynomial reducible, and the whole thing not a field (once again, per 7.16)

**8.3.4**

*Was taken care of in the real analysis course mutatis mutandis*

**8.3.5**

*Let  $F$  be a field, and let  $f(x) \in F[x]$  be a polynomial.*

*(a) Prove that there is a unique polynomial  $\phi(x) \in F[x]$  so that in the 2-variable polynomial ring  $F[x, y]$  we have*

$$f(x + y) = f(x) + \phi(x)y + (\text{element in the ideal } y^2F[x, y])$$

We can start our reasoning with some direct computation first:

$$\begin{aligned}
\sum a_i(x+y)^i &= \sum \left( a_i \sum_{0 \leq j \leq i} bc(j, i) x^{i-j} y^j \right) = \sum \left( a_i + (x^i + \sum_{1 \leq j \leq i} bc(j, i) x^{i-j} y^j) \right) = \\
&= \sum \left( a_i x^i + a_i \sum_{1 \leq j \leq i} bc(j, i) x^{i-j} y^j \right) = f(x) + \sum \left( a_i \sum_{1 \leq j \leq i} bc(j, i) x^{i-j} y^j \right) = \\
&= f(x) + \sum \left( a_i (bc(1, i) x^{i-1} y + \sum_{2 \leq j \leq i} bc(j, i) x^{i-j} y^j) \right) = \\
&= f(x) + \sum \left( a_i (i x^{i-1} y + \sum_{2 \leq j \leq i} bc(j, i) x^{i-j} y^j) \right) = \\
&= f(x) + \sum \left( a_i i x^{i-1} y + a_i \sum_{2 \leq j \leq i} bc(j, i) x^{i-j} y^j \right) = \\
&= f(x) + y \sum a_i i x^{i-1} + \sum \left( a_i \sum_{2 \leq j \leq i} bc(j, i) x^{i-j} y^j \right) =
\end{aligned}$$

and since  $j \geq 2$  we note that the latter part is indeed an element of the ideal  $y^2 F[x, y]$

(b) Prove that the polynomial  $\phi(x)$  in (a) is the derivative  $f'(x)$

Follows directly from definition.

*Skipping the next one*

## 8.4 Finite Fields Revisited

### 8.4.1

Let  $q$  be a power of a prime, let  $d \geq 1$ , let  $F$  be a field with  $q$  elements, and let  $K/F$  be an extension of  $F$  having  $q^d$  elements. Prove that  $K$  is a splitting field of  $x^{q^d} - x$  over  $F$ .

We firstly follow by the same reasoning as in the proof in the chapter that  $x^{q^d} - x$  is separable, and thus has  $q^d$  distinct roots. We then follow that due to theorem 8.11 we have that  $K^*$  is cyclic, and thus for any elements  $k \in K$  we have that order of  $k$  divides  $|K^*| = q^d - 1$ . Thus we have that  $k^{q^d} = k^{q^d-1} k = 1 * k = k$ , and thus

$$k^{q^d} - k = 0$$

therefore implying that  $k$  is the root of  $x^{q^d} - x$ . Thus every element of  $K$  is a root, and all the roots are accounted for, which implies the desired result.

**8.4.2**

Let  $K$  be a field with  $p^d$  elements, so in particular  $K$  contains a copy of  $F_p$ .

(a) Prove that there exists an element  $\gamma \in K$  such that the evaluation map

$$E_\gamma : F_p[x] \rightarrow K$$

is surjective

We can follow that  $K^*$  is cyclic, and thus there is an element  $\gamma$  such that

$$k \in K^* \Rightarrow \exists n \in \omega : k = \gamma^n$$

thus if  $k \in K$  then there is  $f(x) = x^n \in F[x]$ , and thus

$$E_\gamma(f) = k$$

thus implying the desired result.

(b) Prove that  $F_p[x]$  contains an irreducible polynomial of degree  $d$ .

We firstly have  $\ker(E_\gamma)$ . We know that  $F_p[x]$  is a PID, and thus there is  $h(x) \in \ker(E_\gamma)$  that generated  $\ker(E_\gamma)$ . We firstly follow that we have that

$$F_p[x]/h(x)F_p[x] \rightarrow K$$

is isomorphic, and thus  $F_p[x]/h(x)F_p[x]$  is a field, and thus  $h(x)$  is irreducible. Moreover, we have that

$$\dim_F F_p[x]/h(x)F_p[x] = \dim_F K$$

On top of that we have that

$$\dim_F(F_p[x]/h(x)F_p[x]) = \deg(h)$$

thus implying that

$$\deg(h) = \dim_F(K) = d$$

thus implying the desired result.

**8.5 Gauss's Lemma and Eisenstein's Irreducibility Criterion****8.5.1**

Let  $u, v \in \mathbb{Z}$ , and let  $f(x)$  be the polynomial

$$f(x) = x^3 + ux^2 + vx - 1$$

Prove that  $f(x)$  is irreducible in  $\mathbb{Q}[x]$  if and only if  $(u+v)(u-v-2) \neq 0$

Suppose that  $f(x)$  is reducible. We follow that there are  $g, h \in Z(x)$  such that

$$f(x) = g(x) * h(x)$$

With no loss in generality assume that  $\deg(g) = 1, \deg(h) = 2$ . We follow that

$$g(x) = (b_g x + c_g)$$

$$h(x) = (a_g x^2 + b_h x + c_h)$$

for some  $b_g, c_g, a_h, b_h, c_h \in Z$ . We follow then that

$$a_h b_g = 1$$

and thus either  $a_h = b_g = 1$  or  $a_h = b_g = -1$ . If latter is the case, then we can find other appropriate coefficients so that

$$g(x) = (x + c_g)$$

$$h(x) = (x^2 + b_h x + c_h)$$

we then also follow that

$$c_g c_h = -1$$

thus implying that  $c_g = -1, c_h = 1$  or  $c_g = 1, c_h = -1$ .

We follow that

$$u = a_g c_g + b_h b_g$$

$$v = b_g c_h + b_h c_g$$

thus we have that

$$u = c_g + b_h$$

$$v = c_h + b_h c_g$$

if  $c_g = 1$  then we have that

$$u = 1 + b_h \wedge v = b_h - 1 \Rightarrow u - v - 2 = 1 + b_h - b_h + 1 - 2 = 0$$

and if  $c_g = -1$  then

$$u = -1 + b_h \wedge v = 1 - b_h \Rightarrow u - v = 0$$

thus implying that if  $f$  is reducible, then

$$u + v = 0 \vee u - v - 2 = 0 \Leftrightarrow (u + v)(u - v - 2) = 0$$

which gives us the reverse implication.

Now assume that

$$(u + v)(u - v - 2) = 0$$



We follow that

$$u + v = 0 \vee (u - v - 2) = 0$$

by assuming former we have that

$$\begin{aligned} x^3 + ux^2 - ux - 1 &= x^3 + ux(x-1) - 1 = x^3 - 1 + ux(x-1) = \\ &= (x-1)(x^2 + x + 1) + ux(x-1) = (x-1)(x^2 + x + 1 + ux) \end{aligned}$$

and if latter holds, then

$$x^3 + (v+2)x^2 + vx - 1 = x^3 + vx^2 + 2x^2 + vx - 1 = vx(x+1) + x^3 + 2x^2 - 1 = vx(x+1) + (x+1)(x^2 + x - 1) = (x+1)(x^2 + x - 1 + vx)$$

which gives us the forward implication.

### 8.5.2

(a) Let

$$f(x) = 3x^6 - x^5 - 4x^4 + x^3 - x^2 + 3$$

Use Corollary 8.32 to prove that  $f(x)$  is irreducible in  $\mathbb{Q}$

The most obvious prime of choice in this case has to be 2. We follow that

$$\bar{f}(x) = x^6 + x^5 + x^3 + x^2 + 1$$

By trivial computation we follow that this thing has no roots. Thus we follow that if this thing is reducible, then one of its factors has got to be irreducible of order 2 or 3.

We then follow that

$$x^6 + x^5 + x^3 + x^2 + 1 = (x^2 + x + 1)(x^4 + x^2) + 1$$

and similarly

$$x^6 + x^5 + x^3 + x^2 + 1 = (x^3 + x + 1)(x^3 + x^2 + x + 1) + x^2 + x$$

$$x^6 + x^5 + x^3 + x^2 + 1 = (x^3 + x^2 + 1)(x^3) + x^2 + 1$$

thus giving us the desired conclusion that  $\bar{f}$  is indeed irreducible, and thus by the straightforward implication of the corollary in question we get the desired result.

*The rest of this one is tedious; skip*

## 8.5.3

Let  $\Phi_1(x) = x - 1$ , and for  $n \geq 1$ , define the  $n$ th cyclotomic polynomial  $\Phi(x)$  inductively by the formula

$$x^n - 1 = \prod_{m|n} \Phi_m(x)$$

as described in Definition 8.37

(a) Compute  $\Phi_n(x)$  for all  $2 \leq n \leq 10$

We can firstly get the more expressive form for the recursion:

$$\begin{aligned} \Phi_1(x) &= x - 1 \\ \Phi_n(x) &= \frac{x^n - 1}{\prod_{m|n, m \neq n} \Phi_m(x)} \end{aligned}$$

and thus we have

$$\begin{aligned} \Phi_2 &= \frac{x^2 - 1}{x - 1} = x + 1 \\ \Phi_3 &= \frac{x^3 - 1}{x - 1} = x^2 + x + 1 \\ \Phi_4 &= \frac{x^4 - 1}{(x - 1)(x + 1)} = x^2 + 1 \\ \Phi_5 &= \frac{x^5 - 1}{(x - 1)} = x^4 + x^3 + x^2 + x + 1 \\ \Phi_6 &= \frac{x^6 - 1}{(x - 1)(x + 1)(x^2 + x + 1)} = x^2 - x + 1 \\ \Phi_7 &= \frac{x^7 - 1}{(x - 1)} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ \Phi_8 &= \frac{x^8 - 1}{(x - 1)(x + 1)(x^2 + 1)} = x^4 + 1 \\ \Phi_9 &= \frac{x^9 - 1}{(x - 1)(x^2 + x + 1)} = x^6 + x^3 + 1 \\ \Phi_{10} &= \frac{x^{10} - 1}{(x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)} = x^6 + x^3 + 1 = x^4 - x^3 + x^2 - x + 1 \end{aligned}$$

(b) Let  $\zeta_n = e^{2\pi i/n} \in C$  be a primitive  $n$ -th root of unity. Prove that  $\Phi_n(x)$  factors in  $C[x]$  as

$$\Phi_n(x) = \prod_{1 \leq d \leq n \wedge \gcd(d, n) = 1} (x - \zeta_n^d)$$

Thus  $\Phi_n(x)$  is a monic polynomial in  $C[x]$

I don't know enough about complex analysis to crack this one open; skip

## 8.5.4

Let  $R$  be a PID with fraction field  $K$ . For a non-zero polynomial

$$f(x) = c_0 + c_1x + \dots + c_dx^d \in R[x]$$

define content of  $f$  to be the ideal generated by the coefficients of  $f$ , and fix a generator  $C_f \in R$  for this principal ideal. Thus

$$\text{Content}(f) = Rc_0 + Rc_1 + \dots + Rc_d = C_f R$$

(a) Let  $f(x), g(x) \in R[x]$  be non-zero polynomials. Prove that

$$C_{fg}R = C_f C_g R$$

One of the ways to prove that is to go through the proof of 8.30 and substitute where appropriate divisibility by inclusion in the ideal.

We can start by assuming that  $C_f R = C_g R = R$  and  $C_{fg} R \neq R$ . Thus there is an irreducible  $d$  such that  $d|C_{fg}$ . We follow that there are coefficients of  $f, g$ , namely  $a_j, b_k$  respectively such that they aren't divided by  $d$  for lowest possible  $j, k$ . We follow that coefficient of  $x^{j+k}$  is given us by

$$c_{j+k} = \sum_{0 \leq i \leq j+k} a_{j+k-i} b_i$$

which implies that  $a_j b_k$  is the elements of this sum, while the rest of those elements are products of elements that are divided by  $d$ . Since  $d|c_{j+k}$  we conclude that

$$\sum_{0 \leq i \leq j+k} a_{j+k-i} b_i \in dR$$

and since every other element of the sum is the multiple of  $d$  we follow that

$$a_j b_k \in dR$$

which implies that either  $a_j$  or  $b_k$  is divided by  $d$  (as per 7.17), thus giving us the desired contradiction.

Therefore we have that  $C_f R = C_g R = R$  implies that  $C_{fg} = R$ , which by similar reasoning as in 8.30 gives us the desired result.

*Skipping the rest as it is practically rewrite of the theorem proofs mutatis mutandis*

*Skipping this chapter for now, probably will come back at a later date*

## Chapter 9

# Galois Theory: Fields + Groups

### 9.1 A Quick Review of Polynomials and Field Extensions

*Fill in the following table with an I if the polynomial is irreducible over the indicated field and with R if it is reducible*

We can state firstly that over  $Q$  the only reducible is  $x^2 - 1$ . With  $Q(\sqrt{2})$  we add  $x^2 - 2$ . With  $Q(i)$  we have  $x^2 - 1$  and  $x^2 + 1$ . With  $R$  we have  $x^2 - 1$ ,  $x^2 - 2$  and  $x^3 - 3$ . With  $C$  all of them are reducible. With finite fields we could check all of the fields for irreducibility with direct computation and existence of roots

### 9.2 Fields of Algebraic Numbers

#### Notes

Given an algebraic  $\alpha \in L$  for  $L/F$  we are lead to believe that there is a unique minimal polynomial for  $\alpha$ . Let's prove its uniqueness. We can follow that since  $\alpha$  is algebraic, there is  $f \in F[x]$  such that

$$f(\alpha) = 0$$

we then follow that  $f \in \ker E_\alpha$ , and  $\ker E_\alpha$  is an ideal, as proven somewhere in the book. Thus since  $F[x]$  is a PID we follow that there is  $g \in \ker E_\alpha$  that is a seed of  $\ker E_\alpha$ . Dividing  $f$  by its leading coefficient we obtain monic  $h$ , that also generates  $\ker E_\alpha$ . Let now  $k$  be a monic seed for  $\ker E_\alpha$ . We follow that there is  $l \in F[x]$  such that

$$k = lh$$

we thus follow that  $\deg k \geq \deg h$ . Since  $k$  also generates  $\ker E_\alpha$  we follow that there is  $l'$  such that

$$h = l'k$$

thus implying that  $\deg h \geq \deg k$ , therefore giving us the fact that

$$\deg h = \deg k$$

therefore we conclude that  $l$  in  $k = lh$  is constant. Thus the equality of leading coefficients implies that  $l = 1$ , and thus

$$h = k$$

as desired.

One of the interesting consequences of this thing is the fact that if we take  $l \in Z$ , then  $\sqrt{l}$  is not a root for

$$x^2 + \beta x + \gamma$$

for non-zero  $\beta$  (i.e. all the roots of those kinds of polynomials are whacky). We also follow that

### 9.2.1

*Let  $L/F$  be an extension of fields, let  $\alpha \in L$ , and suppose that  $\alpha \neq 0$  and that  $\alpha$  is algebraic over  $F$ . Prove directly from definition that  $\alpha^{-1}$  is algebraic over  $F$ .*

Since  $\alpha \in L$  is algebraic over  $F$  directly from definition we follow that there is  $f \in F[x]$  such that

$$f(\alpha) = 0$$

thus there are  $a_0, a_1, \dots, a_n$  such that

$$f(x) = \sum a_i x^i$$

and thus

$$f(\alpha) = \sum a_i \alpha^i$$

Since  $\alpha$  is the root we follow that

$$\sum a_i \alpha^i = 0$$

and thus

$$\alpha^{-n} \sum a_i \alpha^i = 0$$

thus implying that (we're gonna expand the list under sum here for clarity and relabel order in the last equalities)

$$\sum_{0 \leq i \leq n} a_i \alpha^{i-n} = \sum_{0 \leq i \leq n} a_i (\alpha^{-1})^{n-i} = \sum_{0 \leq i \leq n} a_{n-i} (\alpha^{-1})^i = 0$$

Therefore we are able to conclude that  $\alpha^{-1}$  is the root of

$$g(x) = \sum_{0 \leq i \leq n} a_{n-i} x^i$$

(long story short: if we reverse the order of the coefficients, then roots invert;)

## 9.2.2

Let  $L/F$  be an extension of fields, and let  $\alpha_1, \dots, \alpha_r$  be algebraic over  $F$ .

(a) Prove that  $F[a_1, \dots, a_r] = F(a_1, \dots, a_r)$

From definition we follow that  $F[a_1, \dots, a_n][a_{n+1}] = F[a_1, \dots, a_{n+1}]$ , and similarly  $F(a_1, \dots, a_n)(a_{n+1}) = F(a_1, \dots, a_{n+1})$ . Thus the fact that every single  $a_1, \dots, a_r$  is algebraic implies the desired result.

(b) Prove that

$$[F(a_1, \dots, a_r) : F] \leq \prod [F(a_i) : F]$$

We can follow that since we have a tower of fields

$$F \subseteq F(a_1) \subseteq F(a_1, a_2) \subseteq \dots \subseteq F(a_1, \dots, a_r)$$

we've got (due to 5.18)

$$[F(a_1, \dots, a_r) : F] = \prod [F(a_1, \dots, a_i) : F(a_1, \dots, a_{i-1})]$$

Now let us look at

$$[F(a_2) : F]$$

We can follow that since  $\alpha_2$  is algebraic over  $F$  that there is a minimal polynomial  $\Phi_{F, \alpha_2}(x)$  with degree  $d$ , that is also a (not necessarily minimal) polynomial in both  $F(\alpha_1, \alpha_2)$  and  $F(\alpha_1)$  (since  $F$  is a subfield of both of them) whose roots include  $\alpha_2$ . Therefore if  $\Phi_{F(\alpha_1), \alpha_2}(x)$  is a minimal polynomial with respect to  $F(\alpha_1)$ , then its degree by definition is less or equal to  $\Phi_{F, \alpha_2}(x)$ , thus giving us

$$\deg(\Phi_{F(\alpha_1), \alpha_2}(x)) \leq \deg(\Phi_{F, \alpha_2}(x))$$

which by 9.10(c) implies that

$$[F(\alpha_1, \alpha_2) : F(\alpha_1)] = [F(\alpha_1)(\alpha_2) : F(\alpha_1)] = \deg(\Phi_{F(\alpha_1), \alpha_2}(x)) \leq \deg(\Phi_{F, \alpha_2}(x)) = [F(\alpha_2) : F]$$

We can generalize this result to give us the inequality

$$[F(a_1, \dots, a_i) : F(a_1, \dots, a_{i-1})] \leq [F(\alpha_i) : F]$$

which implies desired result.

(c) Suppose that the degrees  $[F(\alpha_i) : F]$  are pairwise relatively prime. Prove that the inequality in (b) is an equality

We follow that

$$[F(\alpha_i, \alpha_j) : F] = [F(\alpha_i, \alpha_j) : F(\alpha_i)][F(\alpha_i) : F]$$

$$[F(\alpha_i, \alpha_j) : F] = [F(\alpha_i, \alpha_j) : F(\alpha_j)][F(\alpha_j) : F]$$

thus  $[F(\alpha_i, \alpha_j) : F]$  is divided by both  $[F(\alpha_i) : F]$  and  $[F(\alpha_j) : F]$ . Since  $[F(\alpha_i) : F]$  and  $[F(\alpha_j) : F]$  are pairwise prime we follow that their minimal positive common multiple is equal to their product, thus implying the fact that

$$[F(\alpha_i, \alpha_j) : F] = [F(\alpha_i) : F][F(\alpha_j) : F]$$

We can generalize this fact to get the desired result.

(d) Suppose that

$$F(\alpha_i) \cap F(\alpha_j) = F$$

for all  $i \neq j$ . Does this imply the inequality in (b)? Prove or give a counterexample

Let  $F(\alpha_i) \cap F(\alpha_j) \neq F$ . We follow that  $K = F(\alpha_i) \cap F(\alpha_j)$  is itself a field, and thus

$$[F(\alpha_i, \alpha_j) : F] = [F(\alpha_i, \alpha_j) : K][K : F]$$

TODO

### 9.2.3

Let  $L/K/F$  be a tower of fields. Prove that if  $L$  is algebraic over  $K$ , and  $K$  is algebraic over  $F$ , then  $L$  is algebraic over  $F$ .

Suppose that  $\alpha \in L$ . We follow that there is  $g \in K[x]$  such that

$$g(\alpha) = 0$$

We also follow that

$$g = \sum a_i x^i$$

We then follow that  $F(a_1, a_2, \dots, a_n)$  is finite extension of  $F$ . We also have that  $g \in F(a_1, a_2, \dots, a_n)[X]$ , thus implying that  $\alpha$  is algebraic over  $F(a_1, \dots, a_n)$ . Thus we have that  $F(a_1, \dots, a_n)(\alpha) = F(a_1, \dots, a_n, \alpha)$  is finite extension of  $F$ .  $F(\alpha)$  is a subfield of  $F(a_1, \dots, a_n, \alpha)$ , which implies that it is a finite extension of  $F$ , thus implying that  $\alpha$  is algebraic over  $F$ . And since  $\alpha$  is arbitrary we follow the desired result.

### 9.2.4

Compute the minimal polynomials of the indicated numbers over the indicated fields

$$x^2 - 3$$

$$x^2 - 3$$

$$x - \sqrt{3}$$

$$x^2 - 1$$

$$x - i$$

$$x - i$$

TODO the rest

### 9.2.5

Let  $K/F$  be an algebraic extension of fields, and let  $\alpha, \beta \in K$ . Suppose that

$$\gcd(\deg \Phi_{F,\alpha}(X), \deg \Phi_{F,\beta}(X)) = 1$$

Prove that  $\Phi_{F,\beta}(X)$  is irreducible in  $F(\alpha)[X]$

Suppose that

$$\gcd(\deg \Phi_{F,\alpha}(X), \deg \Phi_{F,\beta}(X)) = 1$$

We can follow that

$$\gcd([F(\alpha), F], [F(\beta), F]) = 1$$

Thus one of the previous exercises dictates that

$$[F(\alpha, \beta) : F] = [F(\alpha) : F][F(\beta) : F]$$

Now if  $\Phi_{F,\beta}$  is reducible within  $F(\alpha)[X]$ , then we follow that

$$\deg \Phi_{F(\alpha),\beta} < \deg \Phi_{F,\beta}$$

thus giving us that

$$[F(\alpha, \beta), F(\alpha)] < [F(\beta), F]$$

and therefore

$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\alpha)] * [F(\alpha) : F] < [F(\beta), F] * F[(\alpha), F]$$

which gives us contradiction.

### 9.2.6

For every  $n \geq 1$ , let

$$K_n = \mathbb{Q}(\sqrt[n]{2})$$

(a) Prove that  $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ . Let

$$K = \bigcup_{n \geq 1} K_n$$

Prove that  $K$  is a field.

Let  $n > 1$ . We follow that

$$\sqrt[n-1]{2} = \sqrt[n]{2} \sqrt[n]{2}$$

Thus implying that  $K_{n-1} \subseteq K_n$ , which implies the desired tower of  $\subseteq$ . The fact that each of them is a field follows directly from definition, and the fact that a union of a tower of fields is a field follows trivially (see 5th chapter for clarity).



(b) *Prove that every  $\alpha \in K$  is algebraic over  $Q$ .*

We can follow that by definition, or trivial implication thereof we have that each  $K_n$  is algebraic. For  $\alpha \in K$  we have  $n \in \omega$  such that  $\alpha \in K_n$ , which implies the desired result.

(c) *Prove that  $[K : Q] = \infty$ ; i.e. prove that  $K$  is an infinite-dimensional vector space.*

We can follow that each  $K_n$  is an  $2^n$  dimensional subspace of  $K$  with respect to  $Q$ , and since  $K_n \subseteq K$  for each  $n$  we conclude that there is no  $n \in \omega$  such that  $[K : Q] = n$ , as desired.

### 9.3 Algebraically Closed Fields

#### 9.3.1

*Prove that a field  $K$  is an algebraically closed field iff every non-zero polynomial in  $K[x]$  splits completely over  $K$ .*

We can follow that if  $K$  is algebraically closed, then nonconstant polynomial  $f(x)$  of degree  $n$  splits into

$$f(x) = (x - a)g(x)$$

where  $a$  is a root of  $f(x)$  (due to some theorem in 8th chapter) where  $\deg g = \deg f - 1$ . Thus by induction we have the forward implication.

If every polynomial splits completely in  $K[x]$ , then for nonconstant polynomial  $f$  we have some factor  $(x - a)$ , thus implying that  $a$  is a root for  $f$ , which gives us the reverse.

#### 9.3.2

*Let  $F$  be a field, let  $K/F$  be an extension such that  $K$  is algebraically closed. Let*

$$L = \{a \in K : a \text{ is algebraic over } F\}$$

*Prove that  $L$  is an algebraically closed field.*

Let  $f \in L[x]$  be nonconstant. We follow that there is  $a \in K$  such that  $f(a) = 0$ . Thus  $a$  is algebraic over  $L$ . By definition we follow that  $L$  is an algebraic extension of  $F$ . Thus by an exercise in previous section that asserts that algebraic extensions are transitive over towers of fields we have that  $a$  is algebraic over  $F$ . Therefore by definition of  $L$  we have that  $a \in L$ , which implies that  $L$  is algebraically closed, as desired.

#### 9.3.3

*Let  $F$  be a field, let  $K/F$  be an extension field, and assume that  $K$  is algebraically closed. Prove that the following are equivalent:*

- (1) *The only algebraically closed field  $L$  satisfying  $F \subseteq L \subseteq K$  is  $L = K$*
- (2)  *$K$  is algebraic over  $F$ .*

Assume 1. We follow that the field defined as in the previous exercise is equal to  $K$  and is an algebraic extension of  $F$ , which implies 2.

Assume 2. Suppose that  $F/L/K$  is a tower of fields such that  $L$  is algebraically closed. Thus since  $L \subseteq K$  we have that  $L$  is an algebraic extension of  $F$ . Let  $k \in K$ . We follow that  $k$  is algebraic over  $F$ , and thus a root of a minimal polynomial  $\Phi_{F,k}(x) \in F[x]$ . This polynomial splits completely in  $L$  due to the fact that  $L$  is algebraically closed, which implies that  $k \in L$ , which further implies that  $K \subseteq L$ , which gives us 1.

## 9.4 Automorphisms of Fields

### 9.4.1

Let  $K$  be a field of characteristic 0, so  $K$  contains a (unique) copy of  $\mathbb{Q}$ . Prove that every field automorphism  $\sigma : K \rightarrow K$  fixes every element of  $\mathbb{Q}$ .

We can follow that  $\sigma(1) = 1$  by definition, and thus we have that

$$\sigma(n+1) = \sigma(n) + \sigma(1) = n+1$$

therefore by induction we have that  $\sigma$  fixes  $\mathbb{Z}$ . Then we have that  $\sigma(n^{-1}) = \sigma(n)^{-1} = n^{-1}$  by properties of field homomorphism (see 5.4), and thus

$$\sigma(n/m) = \sigma(n * (1/m)) = \sigma(n) * \sigma(1/m) = n * (1/m) = n/m$$

thus proving that  $\sigma$  indeed fixes  $\mathbb{Q}$ , as desired.

### 9.4.2

Prove that  $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$  is the trivial group.

Let  $\sigma$  be an isomorphism of  $\mathbb{Q}(\sqrt[3]{2})$ . We know that the identity is present within the group, thus let us assume that  $\sigma$  is not an identity. This means that there is  $q \in \mathbb{Q}(\sqrt[3]{2})$  such that

$$\sigma(q) \neq q$$

From the previous point we know that  $\sigma$  fixes  $\mathbb{Q}$ , and thus we follow that  $q \notin \mathbb{Q}$ . Therefore we follow that

$$q = a + b\sqrt[3]{2} + c\sqrt[3]{2}^2$$

for some  $a, b, c \in \mathbb{Q}$  such that  $b$  or  $c$  is nonzero. Therefore we have that

$$\sigma(q) = \sigma(a) + \sigma(b\sqrt[3]{2}) + \sigma(c\sqrt[3]{2}^2) = \sigma(a) + \sigma(b)\sigma(\sqrt[3]{2}) + \sigma(c)\sigma(\sqrt[3]{2}^2) = a + b\sigma(\sqrt[3]{2}) + c\sigma(\sqrt[3]{2}^2)$$

where we have the latter equality due to the previous exercise.

We follow that  $\sqrt[3]{2}$  is the root of

$$f(x) = x^3 - 2$$

Thus  $\sigma(\sqrt[3]{2})$  is also the root of

$$x^3 - 2$$

Since this polynomial's only root is  $\sqrt[3]{2}$  we follow that  $\sigma$  fixes  $\sqrt[3]{2}$ . Thus

$$a + b\sigma(\sqrt[3]{2}) + c\sigma(\sqrt[3]{2}^2) = a + b\sigma(\sqrt[3]{2}) + c\sigma(\sqrt[3]{2})^2 = a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 = q$$

which means that  $q = \sigma(q)$ , which is a contradiction, that gives us the desired result.

### 9.4.3

Let  $K/F$  be a finite extension of fields, and let  $\phi : K \rightarrow K$  be a field homomorphism, that fixes the elements of  $F$ . Prove that  $\phi$  is an isomorphism.

Every field homomorphism is an injection (due to 5.4). Since  $K$  is finite we follow that  $\phi$  is also surjective, which gives us the desired result.

### 9.4.4

Let  $F$  be a field, and let  $F(T)$  be the field of rational functions. Define

$$\sigma, \tau : F(T) \rightarrow F(T) : \sigma(p(T)) = p(T^{-1}); \tau(p(T)) = p(T^2)$$

(a) Prove that  $\sigma, \tau$  are field homomorphisms that fix  $F$ . Prove that  $\sigma$  is a field automorphism of  $F(T)$ , but that  $\tau$  is not

Since neither of them include elements of  $F$  in their definition, it is trivial to follow that both functions fix  $F$ .

By expansion we can prove that the given functions are homomorphisms.

There is no  $x$  in the image of  $\tau$ , which implies that it is not an automorphism.

(b) Prove that  $\sigma^2 = e$ , but that no iterate of  $\tau$  is the identity element.

Former derives directly from the definition, and the latter is proven by induction on the application on  $x$  (i.e. every application of  $\tau$  increases the degree of  $x$ )

(c) Find an element  $u \in F(T)$  such that

$$\{p(T) \in F(T) : \sigma(p(T)) = p(T)\} = F(u)$$

Skip due to confusion about the requirements of the exercise.

(d) What are elements of  $F(T)$  that are fixed by  $\tau$ ?

Those elements are exclusively  $F$  since if  $T$  is somewhere within the definition, then its power will increase (both in numerator and denominator)

## 9.5 Splitting Fields - Part 1

### 9.5.1

*Show that Lemma 9.23 is false for*

$$F_1 = F_2 = Q, f_1(x) = f_2(x) = x^4 - 5x^2 + 6, a_1 = \sqrt{2}, a_2 = \sqrt{3}$$

*Why does this not provide a counterexample to Lemma 9.23*

We can follow that  $x^2 - 3$  is irreducible in the former field, and not in the other. If an isomorphism between given field exists, then we would expect it to be either both reducible or both irreducible (as per the footnote under the proof of the main theorem of the section). This particular case does not fit the requirements of the lemma, since the given polynomial is reducible

$$x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$$