

My real analysis exercises

Evgeny Markin

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Chapter 1

Differentiation

1.1 Standard differentiation

1.1.1

$$g(x) = 4x + 7$$

$$g'(x) = 4$$

1.1.2

$$g(x) = 5x + 4x^2$$

$$g'(x) = 5 + 8x$$

1.1.3

$$f(x) = x^{75} - x + 3$$

$$f'(x) = 75x^{74} - 1$$

1.1.4

$$g(x) = 7/4x^2 - 3x + 12$$

$$g'(x) = 7/2x - 3$$

1.1.5

$$f(t) = -2e^t$$

$$f'(t) = -2te^t$$

1.1.6

$$F(t) = t^2 + e^3$$

$$F(t) = 2t + 3e^3$$

1.1.7

$$y = (4x^2 + 3)(2x + 5)$$

$$y' = 8x(2x + 5) + 2(4x^2 + 3)$$

1.1.8

$$y = x^3 e^x$$

$$y = 3x^2 e^x + x^3 e^x$$

1.1.9

$$f(x) = (3x^2 - 5x)e^x$$

$$f'(x) = (6x - 5)e^x + (3x^2 - 5x)e^x$$

1.1.10

$$f(x) = x \cos x \sin x$$

$$f'(x) = \cos x \sin x - x \sin^2 x + x \cos^2 x$$

1.2 Inverse trigonometric and logarithmic integration**1.2.1**

$$f(x) = \ln(3 + x^2)$$

$$f'(x) = \frac{2x}{3 + x^2}$$

1.2.2

$$f(x) = \ln(x^2 + 3x + 5)$$

$$f'(x) = \frac{2x + 3}{x^2 + 3x + 5}$$

1.2.3

$$\begin{aligned}f(x) &= x \ln x - x \\f'(x) &= \ln x + 1 - 1 = \ln x\end{aligned}$$

1.2.4

$$\begin{aligned}f(x) &= \sin(\ln x) \\f'(x) &= \cos(\ln x) \frac{1}{x}\end{aligned}$$

1.2.5

$$\begin{aligned}f(x) &= \ln(\sin^2 x) \\f'(x) &= \frac{2 \sin x \cos x}{\sin^2 x} = \frac{2 \cos x}{\sin x}\end{aligned}$$

1.2.6

$$\begin{aligned}f(x) &= \ln \frac{1}{x} \\f'(x) &= x(-x^{-2}) = -\frac{1}{x}\end{aligned}$$

1.2.7

$$\begin{aligned}f(x) &= \frac{1}{\ln x} \\f'(x) &= -\frac{1}{(\ln x)^2 x}\end{aligned}$$

1.2.8

$$\begin{aligned}g(x) &= \ln(xe^{-2x}) \\g'(x) &= \frac{e^{-2x} - 2xe^{-2x}}{xe^{-2x}}\end{aligned}$$

1.2.9

$$\begin{aligned}g(t) &= \sqrt{1 + \ln t} \\g'(t) &= \frac{1}{2t\sqrt{1 + \ln t}}\end{aligned}$$

1.2.10

$$f(t) = (\ln t)^2 \sin t$$
$$f'(t) = 2 \ln t \frac{1}{t} \sin t + \cos t (\ln t)^2$$

1.2.11

$$f(t) = \ln(\sqrt{t^2 + 1})$$
$$f'(t) = \frac{2t}{2(t^2 + 1)} = \frac{t}{t^2 + 1}$$

1.2.12

$$f(x) = \log_8(x^2 + 3x)$$
$$f'(x) = \frac{2x + 3}{\ln(8)(x^2 + 3x)}$$

1.2.13

$$f(x) = \sin^{-1}(5x)$$
$$f'(x) = \frac{5}{\sqrt{1 - (5x)^2}}$$

1.2.14

$$f(x) = \sec^{-1}(e^x)$$
$$f'(x) = \frac{2e^{2x}}{e^x \sqrt{e^{2x} - 1}}$$

1.2.15

$$f(x) = \tan^{-1}(\sqrt{x - 1})$$
$$f'(x) = \frac{1}{2\sqrt{x - 1}(1 + |x - 1|)}$$

1.2.16

$$f(x) = \tan^{-1}(x^2)$$
$$f'(x) = \frac{2x}{1 + x^4}$$

1.2.17

$$f(x) = (\tan^{-1} x)^2$$
$$f'(x) = \frac{2(\tan^{-1} x)}{1 + x^2}$$

1.2.18

$$f(x) = \arccos \sqrt{x}$$
$$f'(x) = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

1.2.19

$$f(x) = (\arcsin x) \ln x$$
$$f'(x) = \frac{\ln x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x}$$

1.2.20

$$f(x) = \ln(\arctan(x^4))$$
$$f'(x) = \frac{4x^3}{(1+x^8)\arctan x^4}$$

1.2.21

$$f(x) = e^{\arcsin(x^2)}$$
$$f'(x) = e^{\arcsin(x^2)} \frac{2x}{\sqrt{1-x^4}}$$

Chapter 2

Basic integration

2.1 Basic integrals

2.1.1

$$\int_1^3 (x^2 + 2x - 4)dx = \left[\frac{x^3}{3} + x^2 - 4x\right]_1^3 = (9 + 9 - 12) - (1/3 + 1 - 4) = 6 + \frac{8}{3} = \frac{26}{3}$$

2.1.2

The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

(a) *Show that*

$$\int_a^b e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}[\operatorname{erf}(b) - \operatorname{erf}(a)]$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) = \int_0^x e^{-t^2} dt$$

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(a) = \int_0^a e^{-t^2} dt$$

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(b) - \frac{\sqrt{\pi}}{2} \operatorname{erf}(a) = \int_0^b e^{-t^2} dt - \int_0^a e^{-t^2} dt$$

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(b) - \frac{\sqrt{\pi}}{2} \operatorname{erf}(a) = \int_0^b e^{-t^2} dt + \int_a^0 e^{-t^2} dt$$

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(b) - \frac{\sqrt{\pi}}{2} \operatorname{erf}(a) = \int_a^b e^{-t^2} dt$$

$$\frac{1}{2} \sqrt{\pi} [\operatorname{erf}(b) - \operatorname{erf}(a)] = \int_a^b e^{-t^2} dt$$

(b) Show that the function $y = e^{x^2} \operatorname{erf}(x)$ satisfies the differential equation

$$y' = 2xy + 2/\sqrt{\pi}$$

$$y' = 2xe^{x^2} \operatorname{erf}(x) + e^{x^2} \frac{2}{\pi} e^{-x^2} = 2xe^{x^2} \operatorname{erf}(x) + \frac{2}{\pi}$$

$$2xy + 2/\sqrt{\pi} = 2xe^{x^2} \operatorname{erf}(x) + 2/\sqrt{\pi}$$

as desired.

2.2 Substitution

2.2.1

$$\int \cos 2x dx$$

$$u = 2x$$

$$\int \cos 2x dx = \int \frac{1}{2} 2 \cos 2x dx = \int \frac{1}{2} \cos u du = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u = \frac{1}{2} \sin 2x$$

2.2.2

$$\int x e^{-x^2} dx = \int \left(-\frac{1}{2}\right) - 2x e^{-x^2} dx = -\frac{1}{2} \int e^u dx = -\frac{1}{2} e^u = -\frac{1}{2} e^{-x^2}$$

2.2.3

$$\int x^2 \sqrt{x^3 + 1} dx = \int \frac{1}{3} 3x^2 \sqrt{x^3 + 1} dx = \frac{1}{3} \int \sqrt{u} dx = \frac{1}{3} u^{3/2} \frac{2}{3} = \frac{2}{9} (x^3 + 1)^{3/2}$$

2.2.4

$$\int \sin^2 x \cos x dx = \int u^2 = \frac{u^3}{3} = \frac{\sin^3 x}{3}$$

2.3 Integration by parts

2.3.1

$$\int x e^{2x} dx = x \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x}$$