My concrete mathematics exercises

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Preface

Exercises are from Concrete Mathematics, 2nd ed., by Graham, Knuth and Patashnik.

Probably going to cover warmup, basic and homework exercises, while the rest are going to be left for the better times.

Chapter 1

Recurrent Problems

1.1

Too long of a text of exercise

By saying that 2 through n horses have the same color, we imply that for any given set of horses of length $n-1$ we follow that they have the same color. But we've assumed that for a given set (or even better - list) of horses $[1, n-1]$, we've got that if $x, y \in [1, n-1]$, then x and y have the same color.

1.2

Find the sortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never apper above a smaller one.)

By doing some mental gymnastics we get that

$$n = 1 \rightarrow f(n) = 2$$

$$n = 2 \rightarrow f(n) = 8$$

$$n = 3 \rightarrow f(n) = 26$$

which doesn't give me much of a clue.

Then we conclude, that in order to move n disks from A to B, we need to move $n-1$ disks from A to B, then move n'th disk to the middle peg, then move $n-1$ disks back from B to A, then move the n'th disk to its final place at the bottom of B, and to finish it all we need to move $n-1$ disks again from A to B (During this discuttion I realized, that good guess for the initial values would be $3^n - 1$). Thus we can follow that

$$f(1) = 2$$

$$f(n) = f(n-1) + 1 + f(n-1) + 1 + f(n-1) = 3f(n-1) + 2$$

Thus let us prove that our guess is correct. We're going to do it by induction. Base case is covered, therefore we can assume that our guess is true for $n-1$. Thus

$$3f(n-1) + 2 = 3 * (3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1$$

as desired.

1.3

Show that, in the process of transferring a tower under the restrictions of the preceding exercise, we will actually encounter every properly stacked arrangement of n disks on three pegs.

I think that we can even do this one by induction. Base case with 1 disk is clear, we gotta move it firstly to the middle one, then to the last one, making it all the possible arrangements. Because we use a language of "bottom disk" in further proof we can probably kick it up a notch and make the case for 2 disks as base one, just in case that it matters.

Now assume that we get this property for the case of $n-1$ disks. Then it follows that there are 3 possible positions for the bottom disk, and for all of them we move all other disks from first peg to the last peg, or vice versa, making all the possible combinaitons on the way. Thus we can conclude that disks make all the possible arrangements in the case of n disks, providing us with the desired iteration. n

1.4

Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?

I want to say "no" on this one. Maybe some sort of a shift is the counterexample. It seems like it isn't. Maybe we can somehow show that $2^n - 1$ is the maximum. Suppose that we've arranged disks in order and then we proceed with the fact, that it takes $2T_{n-1} + 1$ moves to move n'th largest disk from its original place to some other place. Thus we can follow that $2^n - 1$ is indeed the largest amount of moves, because it moves every disk from one place to another.

1.5

A Venn diagram with three overlapping circles is often used to illustrate the eight possible subsets associated with three given sets. Can the sixteen possibilities that arise with four given sets be illustrated by four overlapping circles?

Nope.

1.6

Some of the regions defined by n lines in the plane are infinite, while other are bounded. What's the maximum possible number of bounded regions?

I think that it is the same as with any regions, but shifted by 3, because we need to bound the first region, and then proceed with its dissection as per normal rules.

1.7

Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$ and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1)-1) - (2J(n)+1) = 2H(n) - 2$ for all $n \geq 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?

$$H(2n) = J(2n+1) - J(2n) = 2J(n) + 1 - 2J(n) + 1 = 2$$

So everything holds for $H(2n)$.

$$H(2n+1) = J(2n+2) - J(2n+1) = J(2(n+1)) - J(2n+1) =$$

$$= 2J(n+1) - 1 - 2J(n) - 1 = 2(J(n+1) - J(n)) - 2 = 2H(n) - 2$$

Thus our math holds for odd numbers, that are greater then 2.

This thing doesn't work on $n = 1$, therefore we need to have our base case to be set at $n = 2$. Thus we're assuming that it's true for $n \geq 2$ and our induction hypothesis states that if $n \geq 2$, then $H(n) = 2$. Thus we can't follow anything for the induciton step, therefore the whole thing is false.

1.8

Solve the recurrence

$$Q_0 = a$$

$$Q_1 = b$$

$$Q_n = (1 + Q_{n-1})/Q_{n-2}$$

Assume that $Q_n \neq 0$ for all $n \geq 0$

Let's try to extend this thing a bit

$$Q_2 = \frac{1+b}{a}$$

$$Q_3 = \frac{1 + \frac{1+b}{a}}{b} = \frac{\frac{a+1+b}{a}}{b} = \frac{a+1+b}{ab}$$

$$Q_4 = \frac{1+\frac{a+1+b}{ab}}{\frac{1+b}{a}} = \frac{ab+a+1+b}{ab}\frac{1}{\frac{1+b}{a}} = \frac{ab+a+1+b}{\frac{ab(1+b)}{a}} = \frac{ab+a+1+b}{b(1+b)} = \frac{a(b+1)+1(b+1)}{b(1+b)} =$$

$$= \frac{(a+1)(b+1)}{b(1+b)} = \frac{a+1}{b}$$

$$Q_5 = \frac{1+\frac{1+a}{b}}{\frac{a+1+b}{ab}} = \frac{\frac{(b+1+a)}{b}}{\frac{a+1+b}{ab}} = \frac{1}{b}ab = a$$

$$Q_6 = \frac{1+a}{\frac{a+1}{b}} = b$$

Thus we come a full circle and conclude that

$$Q(n) = Q(n \mod 5)$$

and all of the others are computed. Good ol' brute force, nothing beats that.

1.9

Sometimes it's possible to use induction backwards, proving things from n to $n-1$ instead of vice versa! For example, consider the statement

$$P(n) : x_1...x_n \leq \left(\frac{x_1+...+x_n}{n}\right)^n \text{ if } x_1,...,x_n \geq 0$$

This is true when $n=2$, since $(x_1+x_2)^2 - 4x_1x_2 = (x_1-x_2)^2 \geq 0$.

I'm assuming that $P(n) : N \to bool$, which is a little unorthodox, but will do.

(a) By setting $x_n = (x_1+...+x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

Suppose that $P(n)$ is true. Then we follow that for any $x_1,...,x_{n-1} \geq 0$ and $x_n = (x_1+...+x_{n-1})/(n-1)$ we've got that

$$x_1...(x_1+...+x_{n-1})/(n-1) \leq \left(\frac{x_1+...+x_n}{n}\right)^n$$

$$\frac{x_1...(x_1+...+x_{n-1})}{n-1} \leq \left(\frac{x_1+...+x_{n-1}+\frac{x_1+...+x_{n-1}}{n-1}}{n}\right)^n$$

$$\frac{x_1...(x_1+...+x_{n-1})}{n-1} \leq \left(\frac{\frac{(n-1)(x_1+...+x_{n-1})+x_1+...+x_{n-1}}{n-1}}{n}\right)^n$$

$$\frac{x_1...(x_1+...+x_{n-1})}{n-1} \leq \left(\frac{\frac{(n-1)(x_1+...+x_{n-1})+(x_1+...+x_{n-1})}{n-1}}{n}\right)^n$$

$$\frac{x_1 ... (x_1 + ... + x_{n-1})}{n-1} \le \left(\frac{\frac{n(x_1+...+x_{n-1})}{n-1}}{n} \right)^n$$

$$\frac{x_1 ... (x_1 + ... + x_{n-1})}{n-1} \le \left(\frac{x_1 + ... + x_{n-1}}{n-1} \right)^n$$

$$x_1 ... x_{n-1} \frac{(x_1 + ... + x_{n-1})}{n-1} \le \left(\frac{x_1 + ... + x_{n-1}}{n-1} \right)^n$$

$$x_1 ... x_{n-1} \le \left(\frac{x_1 + ... + x_{n-1}}{n-1} \right)^{n-1}$$

which is precisely $P(n-1)$. Thus we follow that $P(n) \to P(n-1)$.

(b) Show that $P(n)$ and $P(2)$ imply $P(2n)$

Suppose that $P(n)$ is true and $P(2)$ is also true (we've proven $P(2)$ beforehand). Let $x_1, ..., x_n \ge 0$. Then we can follow that

$$x_1 ... x_n \le \left(\frac{x_1 + ... + x_n}{n} \right)^n$$

By $P(2)$ we've got that

$$y_1 y_2 \le \left(\frac{y_1 + y_2}{2} \right)^2$$

Suppose that we take a set of numbers $x_1, ..., x_{2n}$. Then we can divide this set into two equaly sized halves and get that

$$(x_1 ... x_n)(x_{n+1} ... x_{2n}) \le \left(\frac{(x_1 ... x_n) + (x_{n+1} ... x_{2n})}{2} \right)^2$$

Since we've got $P(n)$ we can follow that

$$(x_1 ... x_n) \le \left(\frac{x_1 + ... + x_n}{n} \right)^n$$

and

$$(x_{n+1} ... x_{2n}) \le \left(\frac{x_{n+1} + ... + x_{2n}}{n} \right)^n$$

Thus we follow that

$$(x_1 ... x_n)(x_{n+1} ... x_{2n}) \le \left(\frac{x_1 + ... + x_n}{n} \right)^n \left(\frac{x_{n+1} + ... + x_{2n}}{n} \right)^n$$

$$x_1 ... x_{2n} \le \frac{(x_1 + ... + x_n)^n}{n^n} \frac{(x_{n+1} + ... + x_{2n})^n}{n^n}$$

$$x_1 ... x_{2n} \leq \frac{(x_1 + ... + x_n)^n (x_{n+1} + ... + x_{2n})^n}{n^{2n}}$$
$$x_1 ... x_{2n} \leq \left(\frac{(x_1 + ... + x_n)(x_{n+1} + ... + x_{2n})}{n^2} \right)^n$$

and by applying $P(2)$ on the numerator of the inside fraction we get that

$$x_1 ... x_{2n} \leq \left(\frac{\left(\frac{x_1 + ... + x_{2n}}{2}\right)^2}{n^2} \right)^n$$

$$x_1 ... x_{2n} \leq \left(\frac{\left(\frac{x_1 + ... + x_{2n}}{2}\right)}{n} \right)^{2n}$$

$$x_1 ... x_{2n} \leq \left(\frac{x_1 + ... + x_{2n}}{2n} \right)^{2n}$$

as desired.

(c) Explain why this implies the truth of $P(n)$ for all n

$P(1)$ and (b) implies that $P(n)$ is true for $n = 2^k$ for any $k \in N$ by continous application of (b). By (a) we get that if $n < k$ and $P(k)$ is true, then $P(n)$ is true as well. Thus we can follow that for any $n \in N$ there exists $k \in N$ such that $n < 2^k$, and therefore $P(n)$ is true, as desired.

1.10

Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise – that is, from A to B, or from B to the other peg, or from the other peg to A. Also, let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that

$$Q_n = \begin{cases} 0 \text{ if } n = 0 \\ 2R_{n-1} + 1 \text{ if } n > 0 \end{cases}$$

and

$$R_n = \begin{cases} 0 \text{ if } n = 0 \\ Q_n + Q_{n-1} + 1 \text{ if } n > 0 \end{cases}$$

For the remainder of the exercise let us denote the other peg as C

If we've got no disks, then we don't need to momve nothing, therefore we've got the case for the zero.

Let's try firstly to handle some of the cases with low number of disks. If we want to move just 1 disk, then we have got to

$$A_1 \to B_1$$

and that's the case. To move it back to the original peg we've got to

$$B_1 \to C_1 \to A_1$$

Thus $Q_1 = 1 = 2R_0 + 1$ and $R_1 = 2 = Q_0 + Q_1 + 1$. Therefore our assumption seems to be working for 1 disk.

Now suppose that we've gotta move n disks. Firstly, we have got to follow that if we re-label the pegs as such

$$A \to B, B \to C, C \to A$$

then our rules about moving the disks clockwise still makes sence. Thus we can follow that it takes R_k moves to move k disks counter-clockwise (i.e. $A \to C$, $C \to B$, $B \to A$.) and it takes Q_k moves to move k disks clockwise ($A \to B$, $B \to C$, $C \to A$). Thus we can follow that if formulas for R_n and Q_n hold for $n-1$, then we can move $n-1$ disks from A to C in R_{n-1} moves, then move the last disk from A to B in one move, and then move $n-1$ disks from B to A in R_{n-1} moves.

$$\begin{cases} n-1, A \to C : R_{n-1} \\ 1, A \to B : 1 \\ n, C \to B : R_{n-1} \end{cases}$$

Thus we follow that it takes

$$Q_n = 2R_{n-1} + 1$$

moves to move n disks from A to B. Thus we follow that if Q_n and R_n hold for $n-1$, then Q_n holds for n.

For R let's try to expand formula a bit:

$$R_n = Q_n + Q_{n-1} + 1 = 2R_{n-1} + 1 + Q_{n-1} + 1$$

therefore suppose that R_n and Q_n holds for $n-1$, and we've got n disks on B. Then we can use those moves

$$\begin{cases} n-1, B \to A : R_{n-1} \\ 1, B \to C : 1 \\ n-1, A \to B : Q_{n-1} \\ 1, C \to A : 1 \\ n-1, B \to A : R_{n-1} \end{cases}$$

that sum up to our desired formula.

Thus we can conclude, that if our formulas hold for $n-1$, then Q_n holds for n and R_n also holds for n, therefore by induction we get that for $n \in N$ our formulas hold, as desired.

1.11

A Double Tower of Hanoi contains 2n disks of n different sizes, two of each size. As usual, we're required to move only one disk at a line, without putting a larger one over a smaller one.

(a) How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?

Naive guess tells me that it takes a double of moves, that we had in the original problem. The reasoning is simple: it's the same problem, but we have got to move each "disk" two times to effectively put it in the other place.

(b) is a bonus problem, therefore I'll skip it for a while

1.12

Let's generalze exercise 11a even further, by assuming that there are n different sizes of disks and exactly m_k disks of size k. Determine $A(m_1, ..., m_k)$, the minimum number of moves needed to transfer a tower when equal-size disks are considered to be indistinguishable.

The number of disks doesn't affect the overall strategy, therefore the number is determined recursively.