My real analysis exercises

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2022

4.4.1

a

Show that $f(x) = x^3$ is continuous on all of $\boldsymbol{R}$.

In order to show, that f is continous we need to show, that $\forall \epsilon \in \mathbf{R}\ \exists \delta$ s.t.

$$|x - c| < \delta \to |f(x) - f(c)| < \epsilon$$

Let's rewrite the first formula

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + cx + c^2)| = |x - c||x^2 + cx + c^2|$$

We can put $|x - c|$ can be as small as we want it to be. Therefore we need an upper bound for $|x^2 + cx + c^2|$.

$$|x^2 + cx + c^2| \leq |x^2| + |cx| + |c^2| \leq (|c| + 1)^2 + |c|(|c| + 1) + |c|^2$$

Therefore if we take $\delta = min\{1, \epsilon/((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)\}$ then

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2| \leq \epsilon \frac{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)}{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2}) = \epsilon$$

Therefore $f(x) = x^3$ is continous on $\mathbf{R}$.

(b)

Argue, using Theorem 4.4.6, that f is not uniformly continuous on $\boldsymbol{R}$

Theorem 4.4.6 (Sequential Criterion for Nonuniform Continuity). A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if $\exists \epsilon > 0$ and two sequences (x_n) and (y_n) in A satisfying

$|x_n - y_n| \to 0$ *but* $|f(x_n) - f(y_n) \leq \epsilon_0$

In order to show that $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$ let us use sequences

$$x_n = n$$

$$y_n = (n + 1/n)$$

Firstly

$$|x_n - y_n| = |n - n - 1/n| = | - 1/n| = 1/n \to 0$$

on the other hand

$$|f(x_n) - f(y_n)| = |n^3 - (n + 1/n)^3| = |n^3 - (n^3 + 3\frac{n^2}{n} + 3\frac{n}{n^2} + \frac{1}{n^3})| =$$

$$= | - 3n - \frac{3}{n} - \frac{1}{n^3}| \leq |3n| \to \infty$$

rmaxima seems to eraborate this statement, therefore $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n) \to \infty$

Therefore $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$.

(c)

Show that f is uniformly continuous on any bounded subset of $\boldsymbol{R}$.

Suppose that $A \subset \mathbf{R}$ and $\exists M \in \mathbf{R}$ s.t. $\forall x \in A\ x \leq M$ (i.e. A is bounded M)

Then, $\forall c \in A$ and $\forall \epsilon \in \mathbf{R}$

$$\frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2} \leq \frac{\epsilon}{((|c|+1)^2 + |c|(|c|+1) + |c|^2}$$

Therefore if we take

$$\delta = min\{1, \frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2}\}$$

then $|x - c| < \delta$ implies, that $|f(x) - f(c)| < \epsilon$, therefore making $f(x)$ uniformly continous by definition

4.4.2

Show that $f(x) = 1/x^3$ is uniformly continous on the set $[1, \infty)$, but is not on the set $(0, 1]$

In order to show, that $f(x)$ is continous on the set $[1, \infty)$ let us first prove that it is just continous, with the hope that δ is not depentant on x

$$|\frac{1}{x^3} - \frac{1}{c^3}| = |\frac{c^3 - x^3}{x^3c^3}| = |\frac{(c-x)(x^2+cx+c^2)}{x^3c^3}| = |(c-x)\frac{x^2+cx+c^2}{x^3c^3}| = |c-x||\frac{x^2+cx+c^2}{x^3c^3}| =$$

$$= |x - c||\frac{x^2+cx+c^2}{x^3c^3}|$$

Therefore we need to show that if δ is bounded above at 1, then $|\frac{x^2+cx+c^2}{x^3c^3}|$ is bounded above at $[1, \infty)$ by some constant, but $(0, 1]$ isn't.

$$|\frac{x^2+cx+c^2}{x^3c^3}| = |\frac{1}{c^3x} + \frac{1}{c^2x^2} + \frac{1}{cx^3}| \leq |\frac{1}{c^3x}| + |\frac{1}{c^2x^2}| + |\frac{1}{cx^3}|$$

for $x \in [1, \infty)$ each of those fractions are bounded above by 1, therefore for $x \in [1, \infty)$

$$|\frac{x^2+cx+c^2}{x^3c^3}| \leq 3$$

therefore if we pick $\delta < \epsilon/3$ then it follows, that $|f(x) - f(c)| < \epsilon$ for $x \in [1, \infty)$
on the other hand,

$$\lim_{x \to 0}(|\frac{x^2 + cx + c^2}{x^3 c^3}|) \to \infty$$

Therefore we will need smaller deltas as we approach 0; to put it more concretely let's use the theorem for **Sequential Criterion for Nonuniform Continuity**.

Let us pick

$$x_n = 1/n$$

$$y_n = 1/(n+1)$$

then

$$|x_n - y_n| = |\frac{1}{n} - \frac{1}{n+1}| = |\frac{n+1-n}{n(n+1)}| = |\frac{1}{n^2+1}| \to 0$$

but

$$|f(x_n) - f(y_n)| = |1/(\frac{1}{n})^3 - 1/(\frac{1}{n+1})^3| = |1/(\frac{1}{n^3}) - 1/(\frac{1}{(n+1)^3})| = |n^3 - (n+1)^3| =$$

$$= |n^3 - (n^3 + 3n^2 + 3n + 1)| = |3n^2 + 3n + 1| \to \infty$$

therefore by **4.4.6** $f(x)$ is not uniformly continous on $(0, 1]$, as desired

4.4.3

Furnish the details (including an argument for Exercise 3.3.1 if it is not already done) for the proof of the Extreme Value Theorem (Theorem 4.4.3).

Let us first complete 3.3.1

Exercise 3.3.1. Show that if K is compact, then sup K and inf K both exist and are elements of K.

Because K is compact, it is both closed and bound; therefore, because it is bounded,

$$\exists M \in \mathbf{R} > 0 : \forall x \in K$$

$$|x| \leq M$$

Therefore there exist lower and upper bound of K. Therefore, by axiom of completenss, there exist both $sup(K)$ and $inf(K)$ (i.e. both least upper bound and greatest lower bound)

Now let's prove that there exists a sequence that converges to either $sup(k)$ or $inf(k)$.