My Measure, Integration, & Real Analysis exercises

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Contents

Chapter 1

Riemann Integration

1.1 Riemann Integral

1.1.1

Suppose $f : [a, b] \to R$ is a bounded function such that

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

for some partition P of $[a, b]$. Prove that f is a constant function on $[a, b]$.

Suppose that f is not contant. We want to follow that there is a section of the partition P whose elements are not all equal.

Assume that all images of elements of any subinterval of a partition are equal. This means that for a given x_j, x_{j+1} we have that if $x_1, x_2 \in [x_j, x_{j+1}$, then $f(x_1) = f(x_2)$. We then follow that $f(x_1) = f(x_2)$, $f(x_2) = f(x_3)$, and so on, which implies that images of all elemets of the partition are equal. By our assumption we have that all the elements in between elements of the partition are also equal, which implies that f is a constant function.

Thus if f is not a constant function, then there is a subinterval of the partition $[x_j, x_{j+1}]$ such that there are $q_1, q_2 \in [x_j, x_{j+1}]$ for which $f(q_1) \neq f(q_2)$. Now relabel q_1, q_2 so that $f(q_1) > f(q_2)$. We follow that

$$\sup_{[x_j, x_{j+1}]} f \geq f(q_1) > f(q_2) \geq \inf_{[x_j, x_{j+1}]} f$$

and thus

$$\sup_{[x_j, x_{j+1}]} f > \inf_{[x_j, x_{j+1}]} f$$

$$(x_{j+1} - x_j) \sup_{[x_j, x_{j+1}]} f > (x_{j+1} - x_j) \inf_{[x_j, x_{j+1}]} f$$

We then can state that for any x_k, x_{k+1} we have that

$$(x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f \leq (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

which further implies that

$$\sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f \leq \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

by then adding the partition in question to both sides we have that

$$(x_{j+1} - x_j) \inf_{[x_j, x_{j+1}]} f + \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f <$$

$$< (x_{j+1} - x_j) \sup_{[x_j, x_{j+1}]} f + \sum_{k \in P \setminus \{j\}} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

thus

$$\sum_{k \in P} (x_{k+1} - x_k) \inf_{[x_k, x_{k+1}]} f < \sum_{k \in P} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f$$

and when we apply definitions we get that

$$L(f, P, [a, b]) < U(f, P, [a, b])$$

as desired.

1.1.2

Suppose $a \leq s < t \leq b$. Define $f : [a, b] \to R$ by

$$f(x) = \begin{cases} s < x < t \to 1 \\ 0 \ \textit{otherwise} \end{cases}$$

Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f = t - s$

We can follow that every part of the sum in the definition

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

is nonnegative since $f(x) \geq 0$ for all $x \in [a, b]$. Same goes for $U(f, P, [a, b])$. We then follow that there is $n \in N$ such that

$$s < s + 1/n < t - 1/n < t$$

and for all $m > n$ we have

$$s < s + 1/m < t - 1/m < t < b$$

We then can create a set of partitions

$$P_m = a, s, s + 1/m, t - 1/m, t, b$$

for which we have

$$L(f, P_m, [a, b]) =$$
$$= (s - a) * 0 + (s + 1/m - s) * 0 + (t - 1/m - s - 1/m) * 1 + (t - t + 1/m) * 0 + (b - t) * 0 =$$
$$= t - 1/m - s - 1/m = t - s - 2/m$$

and

$$U(f, P_m, [a, b]) =$$
$$= (s - a) * 0 + (s + 1/m - s) * 1 + (t - 1/m - s - 1/m) * 1 + (t - t + 1/m) * 1 + (b - t) * 0 =$$
$$= s + 1/m - s + t - 1/m - s - 1/m + t - t + 1/m =$$
$$= t - s =$$

thus we have

$$U(f, P_m, [a, b]) - L(f, P_m, [a, b]) = 2/m$$

and since for all $\epsilon > 0$ we have $m \in N$ such that $n \geq m \Rightarrow 2/m < \epsilon$ we follow that for all $\epsilon > 0$ there is $m \in N$ such that

$$U(f, [a, b]) - L(f, [a, b]) \leq U(f, P_m, [a, b]) - L(f, P_m, [a, b]) < \epsilon$$

thus proving that f is indeed Riemann integrable on $[a, b]$. We then follow that

$$L(f, P_m, [a, b]) \leq \int_a^b f \leq U(f, P_m, [a, b])$$

for all m, and thus

$$t - s - 2/m \leq \int_a^b f \leq t - s$$

for all m. This in turn implies that

$$\int_a^b f = t - s$$

as desired.