My probability and statistics exercises

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Contents

Chapter 1

Introduction to Probability

1.1 The History of Probability

1.2 Interpretations of Probability

1.3 Experiments and Events

1.4 Set Theory

Exercises in this section (or exercises similar to them) are handled in the set theory course

1.5 The Definition of Probability

1	2/5
2	0.7
3a	1/2
3b	1/6
3c	3/8
4	0.6
5	0.4
6	0.5
8	30
11a	1 - $\pi/4$
11b	0.75
11c	2/3
11d	0
14a	0.38, 0.16
14b	0.04

A little notation, related to 6:

$$Pr(A) = 0.5$$

$$Pr(B) = 0.2$$

$$Pr(A \cap B) = 0.1$$

$$Pr(A \cup B) = 0.6$$

$$Pr((A \cup B) \cap (A \cap B)^c) = P(A \cup B) - P((A \cup B) \cap (A \cap B)) = P(A \cup B) - P(A \cap B) = 0.5$$

1.5.7

If $Pr(A) = 0.4$ and $Pr(B) = 0.7$, then we follow that the maxium $Pr(A \cap B)$ is attained if $A \subset B$, in which case $Pr(A \cap B) = Pr(A) = 0.4$. The minimum is obtained if $A \cup B = S$, in which case $Pr(A \cap B) = 0.1$

1.5.9

The event that exaclty one of the events occurs can be expressed as

$$(A \cap B^c) \cup (A^c \cap B)$$

which comes from either the definition of xor, common sense or something else, depending on your preferences. Thus we follow that

$$Pr((A \cap B^c) \cup (A^c \cap B)) = Pr(A \cap B^c) + Pr(A^c \cap B) - Pr((A \cap B^c) \cap (A^c \cap B)) =$$

$$= Pr(A \cap B^c) + Pr(A^c \cap B) - Pr((A \cap A^c) \cap (B^c \cap B)) =$$

$$= Pr(A \cap B^c) + Pr(A^c \cap B) = Pr(A) - Pr(A \cap B) + Pr(B) - Pr(B \cap A) =$$

$$= Pr(A) - Pr(A \cap B) + Pr(B) - Pr(A \cap B) = Pr(A) + Pr(B) - 2Pr(A \cap B)$$

as desired (rules used in this derivitation: association of unions, $A \cap A^c = \emptyset$ and other trivial stuff)

1.5.10

$$Pr(A \cap B^c) = Pr(A) - Pr(A \cap B)$$

$$Pr(A \cap B^c) + Pr(A \cap B) = Pr(A)$$

as desired.

1.5.12

Suppose that $n > m \in N$. Then we follow that by definition

$$B_m \subseteq A_m$$

and

$$B_n \subseteq A_m^c$$

thus we follow that

$$B_m \cap B_n \subseteq A_m \cap A_m^c = \emptyset$$

thus

$$B_m \cap B_n = \emptyset$$

therefore we conclude that $B_1, B_2...$ are disjoint sets. Thus we follow that

$$Pr(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} Pr(B_i)$$

For $n = 2$ we've got that

$$B_1 \cup B_2 = A_1 \cup (A_1^c \cap A_2) = (A_1 \cup A_1^c) \cap (A_1 \cup A_2) = A_1 \cup A_2$$

and by induction we can follow that

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

thus

$$Pr(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} Pr(B_i)$$

implies that

$$Pr(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} Pr(B_i)$$

for $n \in N$. Given that n is arbirtary, we can follow that

$$Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(B_i)$$

as desired.

1.5.13

First equation follow from induction on the result that

$$Pr(A \cup B) \leq Pr(A) + Pr(B)$$

the second equation follows from the first equation, DeMorgan laws and induction on the form

$$Pr(A \cap B) = Pr((A^c \cup B^c)^c) = 1 - Pr(A^c \cup B^c) \geq 1 - (Pr(A^c) + Pr(B^c))$$

1.5.14

$$Pr(A) = 0.34$$
$$Pr(B) = 0.12$$
$$Pr(O) = 0.5$$
$$Pr(AB) = 1 - 0.34 - 0.12 - 0.5 = 0.04$$
$$Pr(a - A) = 0.34 + 0.04 = 0.38$$
$$Pr(a - B) = 0.12 + 0.04 = 0.16$$

1.6 Finite Sample Spaces

1	1/2
2	1/2
3	2/3
4	1/7
5	4/7
6	1/4
8b	1/4

1.6.7

The possinble genotypes are Aa and aa with probabilities 1/2 and 1/2 respectively

1.6.8a

The sample space of the experiment is $\{heads, tails\} \times \{1, 2, 3, 4, 5, 6\}$,

1.7 Counting Methods

1	14
2	9000
3	120
4	24
5	5/18
6	5/324
7	0.014731
8	360 / 2401
9	1 / 20
10a	r/100
10b	r/100
10c	r/100

1.7.11

$$s(n) = \frac{1}{2}\log(2\pi) + (n+\frac{1}{2})\log n - n \approx \log n!$$

$$\log n! - \log{(n-m)!} = \log\frac{n!}{(n-m)!}$$

$$s(n)-s(n-m) = \frac{1}{2}\log(2\pi)+(n+\frac{1}{2})\log n - n - (\frac{1}{2}\log(2\pi)+((n-m)+\frac{1}{2})\log n - m-(n-m)) =$$

$$= (n+\frac{1}{2})\log n - n - ((n-m)+\frac{1}{2})\log{(n-m)} + (n-m) =$$

$$= (n+\frac{1}{2})\log n - ((n-m)+\frac{1}{2})\log{(n-m)} - m \approx \log\frac{n!}{(n-m)!}$$

$P(n,m) = \frac{n!}{(n-m)!} = \exp(s(n)-s(n-m))$

1.8 Combinatorial Methods

1	184756
2	latter
3	equal
4	1 / 10626
5	-
6	2/n
7	(n - k - 1)/C(n, k)
8	(n - k)/C(n, k)
9	(n + 1)/C(2n, n)
10	C(10, 2)/ C(24, 2)

1.8.5

Prove that

$$\frac{\prod_{4155\leq i\leq 4251} i}{\prod_{2\leq i\leq 97} i}$$

is an integer

$$\frac{\prod_{4155\leq i\leq 4251} i}{\prod_{2\leq i\leq 97} i} = \frac{\prod_{4155\leq i\leq 4251} i}{\prod_{1\leq i\leq 97} i} =$$

$$= \frac{\prod_{4155\leq i\leq 4251} i}{97!} = \frac{4251!}{4154!97!} = \frac{4251!}{4154!(4251-4174)!} = C(4251, 4154)$$

and binomial coefficients are integers (pretty sure that we can follow that by induction in some more advanced course).

1.8.14

Prove that for all positive integers n, k such that $n \geq k$

$$C(n,k) + C(n,k-1) = C(n+1,k)$$

$$C(n,k) + C(n,k-1) = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} =$$

$$= \frac{n!}{k(n-k)!(k-1)!} + \frac{n!}{(n-k+1)(n-k)!(k-1)!} =$$

$$= \frac{(n-k+1)n!}{k(n-k+1)(n-k)!(k-1)!} + \frac{kn!}{k(n-k+1)(n-k)!(k-1)!} =$$

$$= \frac{(n-k+1)n! + kn!}{k(n-k+1)(n-k)!(k-1)!} = \frac{n!((n-k+1)+k)}{k(n-k+1)(n-k)!(k-1)!} =$$

$$= \frac{n!(n+1)}{k(n-k+1)(n-k)!(k-1)!} = \frac{(n+1)!}{((n+1)-k)!k!} = C(n+1,k)$$

as desired.