# Part I

# Appendix: Mathematical Background

# Chapter 1

# Summations

## A.1 Summation formulas and properties

### A.1-1

*Prove that* $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$

Short answer:

$$\sum cg(x) = c \sum g(x)$$

Long answer:

Suppose that $g \in O(f_k(i))$. It follows that there exists $n_i$ and $c_i$ such that $0 \leq g(n) \leq cf_i(n)$. Thus we can pick $n = \max\{n_0, n_1, ...\}$ and $c = \max\{c_0, c_1, ...\}$. We know that both $n$ and $c$ will work all of functions $f_k$. Therefore by linearity of summations

$$\sum_{k=1}^{n} O(f_k(i)) = \sum_{k=1}^{n} cf_k(i) == c \sum_{k=1}^{n} f_k(i) == O(\sum_{k=1}^{n} f_k(i))$$

(notation is a little abused and there is nothing is rigorously proven, but it'll do).

### A.1-2

*Find a simple formula for* $\sum_{k=1}^{n} (2k-1)$.

$$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} (2k) - \sum_{k=1}^{n} (1) = 2\sum_{k=1}^{n} (k) - n = 2\frac{n(n+1)}{2} - n = n(n+1) - n = n^2$$

### A.1-3

*Interpret the decimal number* $111,111,111$ *in light of equation A.6*

$$111,111,111 = \sum_{k=0}^{9} 10^k = \frac{10^{10}-1}{10-1}$$

## A.1-4

*Evaluate the infinite series* $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ...$

The series converges absolutely to 2, so we are free to do anything with it.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ... = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \sum_{k=0}^{\infty} \frac{1}{2}^{1+2k} = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2}^{2k} = \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{2}^{2k} =$$

$$= \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{4}^{k} = \left(1 - \frac{1}{2}\right)\frac{1}{1-\frac{1}{4}} = \frac{1}{2} * \frac{4}{3} = \frac{2}{3}$$

## A.1-5

*Let* $c \geq 0$ *be a constant. Show that* $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$

$$\sum_{k=1}^{n} k^c = \sum_{k=1}^{n-1} k^c + n^c = n^c \sum_{k=1}^{n} \frac{k^c}{n^c} =$$

Let $f(n) = n^c$. It can be seen from the graph that

$$\int_0^n f(x)dx \leq \sum_{k=1}^{n} k^c \leq \int_0^n f(x+1)dx$$

Thus

$$\int_0^n f(x)dx = \int_0^n x^c = \frac{n^{c+1}}{c+1} \in$$

$$\int_0^n f(x+1)dx = \int_0^n (x+1)^c = \frac{(n+1)^{c+1}}{c+1}$$

Thus we can state that $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$ (I'm not good enough yet to show that $\frac{(n+1)^{c+1}}{c+1} \in \Theta(n^{c+1})$, but I'm pretty sure that it's true TODO).

### A.1-6

*Show that* $\sum_{k=0}^{\infty} k^2x^k = x(1+x)/(1-x)^3$ *for* $|x| < 1$

We know that for $|x| < 1$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

thus if we differentiate both sides we get

$$\sum_{k=0}^{\infty} k^2x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

and then if we multiply all of it by $x$ we'll get

$$\sum_{k=0}^{\infty} k^2x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

thus if we factor all of this jazz we'll get

$$\sum_{k=0}^{\infty} k^2x^k = -\frac{x(x+1)}{(x-1)^3}$$

and if we tuck this minus into denominator we'll get (which we can do because the power is odd)

$$\sum_{k=0}^{\infty} k^2x^k = \frac{x(x+1)}{(1-x)^3}$$

as desired.

### A.1-7

*Prove that* $\sum_{k=1}^{n} \sqrt{k \lg k} = \Theta(n^{3/2} \lg^{1/2} n)$

$$\int \sqrt{k \lg k} =$$

TODO

### A.1-8

*Show that*

$$\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$$

*by manipulating the harmonic series*

In the book we're reassured that

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

We want to find the sum of reciprocals of odd numebers. Since $n \in Z_+$ is either odd or even, but not both, we follow that

$$\sum_{k=1}^{n} 1/(2k-1) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2}\frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}$$

and since

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

we follow that

$$\sum_{k=1}^{n} 1/(2k-1) = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2}(\ln(n) + O(1)) = \ln(n^{1/2}) + 1/2O(1) = \ln(\sqrt{n}) + O(1)$$

as desired (justification that $1/2O(1) = O(1)$ follows directly from the definition of $O$).

### A.1-9

*Show that*

$$\sum_{k=0}^{\infty} (k-1)/2^k = 0$$

We can use standard series manipulations to get

$$\sum_{k=0}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} (k-1)/2^k = -1 + \sum_{k=2}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} k/2^{k+1} =$$

$$= -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k$$

We can also manipulate it differently to get

$$\sum_{k=0}^{\infty} (k-1)/2^k = \sum_{k=0}^{\infty} k/2^k - 1/2^k = \sum_{k=0}^{\infty} k/2^k - \sum_{k=0}^{\infty} 1/2^k = \sum_{k=0}^{\infty} k/2^k - 2 = \sum_{k=1}^{\infty} k/2^k - 2$$

Now assuming that the original sum converges we get an equation

$$\sum_{k=1}^{\infty} k/2^k - 2 = -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k$$

$$\sum_{k=1}^{\infty} k/2^k - \frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\sum_{k=1}^{\infty} k/2^k = 2$$

and by substituting the result into any of the previous results (I'll take the first) we get that

$$\sum_{k=0}^{\infty} (k-1)/2^k = -1 + \frac{1}{2}\sum_{k=1}^{\infty} k/2^k = -1 + 1 = 0$$

as desired.

### A.1-11

*Evaluate the product*

$$\prod_{k=2}^{n} 1 - \frac{1}{k^2}$$

$$\prod_{k=2}^{n} 1 - \frac{1}{k^2} = \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \prod_{k=2}^{n} \frac{(k+1)(k-1)}{k^2} = \frac{\prod_{k=2}^{n}(k+1)\prod_{k=2}^{n}(k-1)}{\prod_{k=2}^{n}(k^2)} =$$

$$= \frac{\prod_{k=3}^{n+1} k \prod_{k=1}^{n-1} k}{(\prod_{k=2}^{n} k)^2} = \frac{\frac{1}{2} * 1 * 2 * \prod_{k=3}^{n} k * (n+1) * \frac{1}{n} * n * \prod_{k=1}^{n-1} k}{(1 * \prod_{k=2}^{n} k)^2} = \frac{\frac{1}{2} * \prod_{k=1}^{n} k * (n+1) * \frac{1}{n} * \prod_{k=1}^{n} k}{(1 * \prod_{k=2}^{n} k)^2} =$$

$$= \frac{\frac{1}{2} * (n+1) * \frac{1}{n} * (\prod_{k=1}^{n} k)^2}{(\prod_{k=1}^{n} k)^2} = \frac{1}{2} * (n+1) * \frac{1}{n} = \frac{1}{2n} + \frac{1}{2}$$

as desired.

## A.2 Bounding summations

# Chapter 2

# Sets, Etc.

## 1-1

*Draw Venn diagrams that illustrate the first of the distributive laws (B.1)*

TODO, add picture here

## 1-2

*Prove the generalization of DeMorgan's laws to any finite collection of sets*

*Copy from real analysis exercises*

Suppose that $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. It follows, that $x$ is not in the union of given sets. Therefore there is no set $E_n$ such that $x \in E_n$ (because if there would be such a set, then $x$ wouldn't be in $(\cup_{\lambda\in\Lambda} E_\lambda)^c$). Therefore $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$. Therefore

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cap_{\lambda\in\Lambda} E_\lambda^c$$

The proof of reverse inclusion is the same as with the forward, but in reverse order.

$x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c$ implies that $x$ is not in every $E_n$. Therefore there exists $x \in E_n^c$ for some $E_n$. therefore it is in $\cup_{\lambda\in\Lambda} E_\lambda^c$. The proof of reverse inclusion uses the same argument, but in other direction.

## 1-3

TODO

## 1-4

*Show that the set of odd natural numbers is countable.*

Let us set a function $f : A \to N$, where $A$ denotes the set of odd natural numbers

$$f(n) = (n+1)/2$$

for this function we've got

$$f^{-1}(n) = 2n - 1$$

Both funcitons are injective and therefore $f$ is bijective. Therefore we've got a bijective function betweeen $A$ and $N$, therefore $A \sim N$, therefore it is conuntable, as desired.

## 1-5

*Show that for any finite set $S$, the power set $2^S$ has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of $S$).*

*Another copy from real analysis*

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary system: 0 for excluded, 1 for included. Therefore there exist $2^n$ possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of $\emptyset$ are $\emptyset$ itseft ($2^0 = 1$ in total). Subsets of set with one element are $\emptyset$ and set itself ($2^1 = 1$ in total).

Proposition is that set with n elements has $2^n$ subsets.

Inductive step is that for set with $n+1$ elements can either have or hot have the $n+1$'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

## 1-6

*Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.*

The tuple is actually just a re-writing of particular set

$$(a_1, a_2, ..., a_n) = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}...\{a_1, a_2, a_3, ..., a_n\}\}$$