My algorithms exercises

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Contents

Preface

Exercises for Introduction to Algorithms by Cormen et al., 4th ed. It has some exercises, that should be written down, moslty in math and whatnot.

Part I

Appendix: Mathematical Background

Chapter 1

Summations

1-1

Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$

Short answer:

$$\sum cg(x) = c \sum g(x)$$

Long answer:

Suppose that $g \in O(f_k(i))$. It follows that there exists n_i and c_i such that $0 \leq g(n) \leq cf_i(n)$. Thus we can pick $n = \max\{n_0, n_1, ...\}$ and $c = \max\{c_0, c_1, ...\}$. We know that both n and c will work all of functions f_k. Therefore by linearity of summations

$$\sum_{k=1}^{n} O(f_k(i)) = \sum_{k=1}^{n} cf_k(i) == c \sum_{k=1}^{n} f_k(i) == O(\sum_{k=1}^{n} f_k(i))$$

(notation is a little abused and there is nothing is rigorously proven, but it'll do).

1-2

Find a simple formula for $\sum_{k=1}^{n} (2k-1)$.

$$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} (2k) - \sum_{k=1}^{n} (1) = 2 \sum_{k=1}^{n} (k) - n = 2\frac{n(n+1)}{2} - n = n(n+1) - n = n^2$$

1-3

Interpret the decimal number $111,111,111$ *in light of equation A.6*

$$111,111,111 = \sum_{k=0}^{9} 10^k = \frac{10^{10}-1}{10-1}$$

1-4

Evaluate the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ...$

The series converges absolutely to 2, so we are free to do anything with it.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ... = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \sum_{k=0}^{\infty} \frac{1}{2}^{1+2k} = \sum_{k=0}^{\infty} \frac{1}{2}^{2k} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2}^{2k} = \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{2}^{2k} =$$

$$= \left(1 - \frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{1}{4}^{k} = \left(1 - \frac{1}{2}\right)\frac{1}{1-\frac{1}{4}} = \frac{1}{2} * \frac{4}{3} = \frac{2}{3}$$

1-5

Let $c \geq 0$ *be a constant. Show that* $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$

$$\sum_{k=1}^{n} k^c = \sum_{k=1}^{n-1} k^c + n^c = n^c \sum_{k=1}^{n} \frac{k^c}{n^c} =$$

Let $f(n) = n^c$. It can be seen from the graph that

$$\int_0^n f(x)dx \leq \sum_{k=1}^{n} k^c \leq \int_0^n f(x+1)dx$$

Thus

$$\int_0^n f(x)dx = \int_0^n x^c = \frac{n^{c+1}}{c+1} \in$$

$$\int_0^n f(x+1)dx = \int_0^n (x+1)^c = \frac{(n+1)^{c+1}}{c+1}$$

Thus we can state that $\sum_{k=1}^{n} k^c = \Theta(n^{c+1})$ (I'm not good enough yet to show that $\frac{(n+1)^{c+1}}{c+1} \in \Theta(n^{c+1})$, but I'm pretty sure that it's true TODO).

1-6

Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ *for* $|x| < 1$

We know that for $|x| < 1$

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

thus if we differentiate both sides we get

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

and then if we multiply all of it by x we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

thus if we factor all of this jazz we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = -\frac{x(x+1)}{(x-1)^3}$$

and if we tuck this minus into denominator we'll get (which we can do because the power is odd)

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(x+1)}{(1-x)^3}$$

as desired.

1-7

Prove that $\sum_{k=1}^{n} \sqrt{k \lg k} = \Theta(n^{3/2} \lg^{1/2} n)$

Chapter 2

Sets, Etc.

1-1

Draw Venn diagrams that illustrate the first of the distributive laws (B.1)

TODO, add picture here

1-2

Prove the generalization of DeMorgan's laws to any finite collection of sets

Copy from real analysis exercises

Suppose that $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. It follows, that x is not in the union of given sets. Therefore there is no set E_n such that $x \in E_n$ (because if there would be such a set, then x wouldn't be in $(\cup_{\lambda\in\Lambda} E_\lambda)^c$). Therefore $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$. Therefore

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cap_{\lambda\in\Lambda} E_\lambda^c$$

The proof of reverse inclusion is the same as with the forward, but in reverse order.

$x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c$ implies that x is not in every E_n. Therefore there exists $x \in E_n^c$ for some E_n. therefore it is in $\cup_{\lambda\in\Lambda} E_\lambda^c$. The proof of reverse inclusion uses the same argument, but in other direction.

1-3

TODO

1-4

Show that the set of odd natural numbers is countable.

Let us set a function $f : A \to N$, where A denotes the set of odd natural numbers

$$f(n) = (n+1)/2$$

for this function we've got

$$f^{-1}(n) = 2n - 1$$

Both funcitons are injective and therefore f is bijective. Therefore we've got a bijective function betweeen A and N, therefore $A \sim N$, therefore it is conuntable, as desired.

1-5

Show that for any finite set S, the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

Another copy from real analysis

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary system: 0 for excluded, 1 for included. Therefore there exist 2^n possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of $\emptyset$ are $\emptyset$ itseft ($2^0 = 1$ in total). Subsets of set with one element are $\emptyset$ and set itself ($2^1 = 1$ in total).

Proposition is that set with n elements has 2^n subsets.

Inductive step is that for set with $n+1$ elements can either have or hot have the $n+1$'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

1-6

Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.

The tuple is actually just a re-writing of particular set

$$(a_1, a_2, ..., a_n) = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}...\{a_1, a_2, a_3, ..., a_n\}\}$$