My "Calculus on Manifolds" exercises

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Contents

Chapter 1

Functions on Euclidean Space

1.1 Norm and Inner Product

1-1

Prove that

$$|x| \leq \sum_{i=1}^{n} |x^i|$$

We follow that both rhs and lhs are nonnegative, thus we follow that

$$|x| \leq \sum_{i=1}^{n} |x^i| \Leftrightarrow |x|^2 \leq \left(\sum_{i=1}^{n} |x^i| \right)^2 \Leftrightarrow \sum |x^i|^2 \leq \left(\sum_{i=1}^{n} |x^i| \right)^2$$

We can follow that terms in rhs contain lhs, and since all the terms are nonnegative, we deduce the desired result. More rigorous result can be obtained through some basic induction.

1-2

Whn does equality hold in Theorem 1-1(3)?

When one vector is a scalar product of the other.

Other exercises were already handled in earlier courses. 1-3, 1-4, 1-5 at readl analysis and linear algebra, 1-6 is undetermined, 1-7 is just isometry thing, and everythng else in the mix of those two.

1.2 Subsets of Euclidean Space

1-14 and 1-15 were handled in a topology course.

1-16

First is closed, second is also closed, third one is R^n.

1-17

Diagonal except for $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$ will do

1-18

We follow that A is open, and thus equal to its interior. We can also follow that $\overline{A} = [0, 1]$, and thus we conclude the desired result.

1-19

There are sequences of rationals that converge to any irrational number, thus irrationals are limit points, which produces this result.

1-20

TBD in topology course pretty soom

1-21

(a) If there's no such nnomber, then there's a sequence in A that converges to x, thus x is a limit point of A and thus it's contained in A.

(b) and (c) skip

1-22

Each $c \in C$ has got a basis neighborhood inside U. Each one of those basis neighborhoods have smaller basis neighborhoods inside of them. Thus we can create a function $f : C \to \mathcal{P}(U)$ to those small neighborhoods, then take a closure of the union of the range of f, and this will produce the desired set.

1.3 Functions and Continuity

1-23

Follows from the definition of product topology. Also was probably handled with a case of 2 abstract spaces in procut topology and can be extended to this case by induction

1-24

handled in topology course.

the rest was taken care of in linear algebra course or somewhere else

Chapter 2

Differentiation

2.1 Basic Definitions

2-1

Prove that if f is differentiable at $a \in R^n$, then it is continous at a.

We can screw around with original definition of continuity to get

$$\lim \frac{|f(a+h) - f(a) - (\lambda(a+h) - \lambda(a))|}{|h|} = \lim \frac{|(f-\lambda)(a+h) - (f-\lambda)(a)|}{|h|} =$$

$$= \lim_{x \to a} \frac{|(f-\lambda)(x) - (f-\lambda)(a)|}{|x-a|} = 0$$

Let B be a basis element around $f(a)$. Let $x \in f^{-1}[B]$. We follow that there's a ball B' around x such that $y \in B' \Rightarrow \lambda(y) \in \lambda(B')$. Using metrics we get

$$|y-a| < \delta \Rightarrow |\lambda(y) - \lambda(a)| < \epsilon$$

Since function

2-2

We follow that we ca define

$$g(x) = f(x, 0)$$

and we follow that if $q = \langle a, b \rangle \in R^2$, then

$$f(q) = f(a, b) = f(a, 0) = g(a)$$

If there's g such that $f(x, y) = g(x)$ and f is not independent of second variable, then we follow that there exist $y_1, y_2, x \in R$ such that

$$f(x, y_1) \neq f(x, y_2)$$

and thus $g(x) \neq f(x, y_1)$ or $g(x) \neq f(x, y_2)$, which is a contradiction.

We follow that $f'(a, b) = \langle g'(x), 0 \rangle$.

2-3

Close to the previous one.

2-4

(a) We follow that if $x \in R^2$ and $x = 0$, then $h(t) = f(tx) = f(0) = 0$, thus h is constant and therefore continous. If $x \neq 0$, then we follow that

$$h(t) = f(tx) = |tx| \cdot g(\frac{tx}{|tx|}) = |tx| \cdot g(\frac{tx}{|t||x|})$$

if $t >= 0$, then we follow that

$$h(t) = |tx| \cdot g(\frac{tx}{t|x|}) = t|x| \cdot g(x/|x|)$$

and if $t < 0$, then

$$h(t) = -t|x| \cdot g(\frac{tx}{-t|x|}) = t|x| \cdot g(x/|x|)$$

since x is fixed, we follow that $|x| \cdot g(x/|x|)$ is a constant, and thus we conclude that h is a linear function, which is differentiable, as desired.

(b) If $g = 0$, then we follow that $f(x) = 0$, and thus it's differentiable at every point.