My abstract algebra exercises

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Introduction to Groups

3.1 An Important Example

3.1.1

In S_4, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. Calculate $\sigma\tau$, $\tau\sigma$ and σ^{-1}.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

3.1.2

In S_5, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$ calculate $\sigma\tau\sigma$, $\sigma\sigma\tau$, σ^{-1}.

$$\sigma\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

$$\sigma\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

3.1.3

How mamy permutations are there in S_n? In S_5, how many permutations α satisfy $\alpha(2) = 2$?

We can follow that there are $n!$ permutations total, and if we've got a restriction $\alpha(2) = 2$, then we've got $(n-1)!$ permutation. For the case S_5 it means that there are $4! = 24$ such permutations.

3.1.4

Let H be the set of all permutations $\alpha \in S_5$ satisfying $\alpha(2) = 2$. Which of the properties of closure, associativity, identit, inverses does H enjoy under composition?

All of them

3.1.5

Consider the set of all functions from 6 to 6. Which of the ...

Everything other then inverse

3.1.6

Let G be the set of all ...

All of them

3.2 Groups

3.2.1

Give group tables for following additive grops: Z_3, $Z_3 \times Z_2$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

last one is ommited

3.2.2

Give group tables for the following groups: $U(12), S_3$

We follow that $U(12) = \{1, 5, 7, 11\}$. THus

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

One of the programs in progs folder produces desired table for S_3 (and can produce one for any S_n for that matter).

3.2.3

Show that $G \times H$ is abelian iff G and H are both abelian

Was proven in dummit and foote, check 1.1.29

Rest of the exercises in this section were either already proven in D&F, are trivial, or could be solved at a later time if I encounter some gaps in the theory.

3.3

3.4

3.5

3.6 Cyclic Groups

3.6.1

Let $G = \langle a \rangle$ be a cyclic group of order 12. List every subgroup of G. List every group of Z_{12}

12's divisors are $\{1, 2, 3, 4, 6\}$, therefore subgroups of G are $\langle a^i \rangle$ for $i \in \{1, 2, 3, 4, 6\}$.

Since Z_{12} is cyclic, we follow that $\langle [1, 2, 3, 4, 6] \rangle$ are the subgroups of Z_{12}.

3.6.2

Let $G = \langle a \rangle$ be a cyclic group of order 120. List all of the groups of order 120. List all of the elements of order 12 in G.