My real analysis exercises

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4.4.1

Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

a

In order to show, that f is continous we need to show, that $\forall \epsilon \in \mathbf{R}\ \exists \delta$ s.t.

$$|x - c| < \delta \rightarrow |f(x) - f(c)| < \epsilon$$

Let's rewrite the first formula

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + cx + c^2)| = |x - c||x^2 + cx + c^2|$$

We can put $|x - c|$ can be as small as we want it to be. Therefore we need an upper bound for $|x^2 + cx + c^2|$.

$$|x^2 + cx + c^2| \leqslant |x^2| + |cx| + |c^2| \leqslant (|c| + 1)^2 + |c|(|c| + 1) + |c|^2$$

Therefore if we take $\delta = min\{1, \epsilon/((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)\}$ then

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2| \leqslant \epsilon \frac{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)}{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2}) = \epsilon$$

Therefore $f(x) = x^3$ is continous on $\mathbf{R}$.

(b)

Argue, using Theorem 4.4.6, that f is not uniformly continuous on R

Theorem 4.4.6 (Sequential Criterion for Nonuniform Continuity). A function $f : A \rightarrow \mathbf{R}$ fails to be uniformly continuous on A if $\exists \epsilon > 0$ and two sequences (x_n) and (y_n) in A satisfying

$|x_n - y_n| \rightarrow 0$ *but* $|f(x_n) - f(y_n) \leqslant \epsilon_0$

In order to show that $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$ let us use sequences

$$x_n = n$$

$$y_n = (n + 1/n)$$

Firstly

$$|x_n - y_n| = |n - n - 1/n| = | - 1/n| = 1/n \rightarrow 0$$

on the other hand

$$|f(x_n) - f(y_n)| = |n^3 - (n + 1/n)^3| = |n^3 - (n^3 + 3\frac{n^2}{n} + 3\frac{n}{n^2} + \frac{1}{n^3})| =$$

$$= |-3n - \frac{3}{n} - \frac{1}{n^3}| \leqslant |3n| \to \infty$$

rmaxima seems to eraborate this statement, therefore $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n) \to \infty$

Therefore $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$.

(c)

Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Suppose that $A \subset \mathbf{R}$ and $\exists M \in \mathbf{R}$ s.t. $\forall x \in A\ x \leqslant M$ (i.e. A is bounded M)

Then, $\forall c \in A$ and $\forall \epsilon \in \mathbf{R}$

$$\frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2} \leqslant \frac{\epsilon}{((|c|+1)^2 + |c|(|c|+1) + |c|^2}$$

Therefore if we take

$$\delta = min\{1, \frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2}\}$$

then $|x - c| < \delta$ implies, that $|f(x) - f(c)| < \epsilon$, therefore making $f(x)$ uniformly continous by definition

4.4.2

Show that $f(x) = 1/x^3$ is uniformly continous on the set $[1, \infty)$, but is not on the set $(0, 1]$