# My abstract algebra exercises

Evgeny Markin

2023

# Contents

# Part I

# Preliminaries

# Chapter 1

# Relations and Functions

# Chapter 2

# The Integers and Modular Arithmetic

# Part II

# Groups

# Chapter 3

# Introduction to Groups

## 3.1  An Important Example

### 3.1.1

In $S_4$, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. Calculate $\sigma\tau$, $\tau\sigma$ and $\sigma^{-1}$.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

### 3.1.2

In $S_5$, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$ calculate $\sigma\tau\sigma$, $\sigma\sigma\tau$, $\sigma^{-1}$.

$$\sigma\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

$$\sigma\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

### 3.1.3

*How mamy permutations are there in $S_n$? In $S_5$, how many permutations $\alpha$ satisfy $\alpha(2) = 2$?*

We can follow that there are $n!$ permutations total, and if we've got a restriction $\alpha(2) = 2$, then we've got $(n-1)!$ permutation. For the case $S_5$ it means that there are $4! = 24$ such permutations.

### 3.1.4

*Let $H$ be the set of all permutations $\alpha \in S_5$ satisfying $\alpha(2) = 2$. Which of the properties of closure, associativity, identit, inverses does $H$ enjoy under composition?*

All of them

### 3.1.5

*Consider the set of all functions from 6 to 6. Which of the ...*

Everything other then inverse

### 3.1.6

*Let $G$ be the set of all ...*

All of them

## 3.2 Groups

### 3.2.1

*Give group tables for following additive grops: $Z_3$, $Z_3 \times Z_2$*

|   | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

last one is ommited

### 3.2.2

*Give group tables for the following groups: $U(12), S_3$*

We follow that $U(12) = \{1, 5, 7, 11\}$. THus

| | 1 | 5 | 7 | 11 |
|----|----|----|----|----|
| 1 | 1 | 5 | 7 | 11 |
| 5 | 5 | 1 | 11 | 7 |
| 7 | 7 | 11 | 1 | 5 |
| 11 | 11 | 7 | 5 | 1 |

One of the programs in progs folder produces desired table for $S_3$ (and can produce one for any $S_n$ for that matter).

### 3.2.3

*Show that $G \times H$ is abelian iff $G$ and $H$ are both abelian*

Was proven in dummit and foote, check 1.1.29

*Rest of the exercises in this section were either already proven in D&F, are trivial, or could be solved at a later time if I encounter some gaps in the theory.*

## 3.3

## 3.4

## 3.5

## 3.6 Cyclic Groups

### 3.6.1

*Let $G = \langle a \rangle$ be a cyclic group of order 12. List every subgroup of $G$. List every group of $Z_{12}$*

12's divisors are $\{1, 2, 3, 4, 6, 12\}$, therefore subgroups of $G$ are $\langle a^i \rangle$ for $i \in \{0, 1, 2, 3, 4, 6\}$

Since $Z_{12}$ is cyclic, we follow that $\langle [0, 1, 2, 3, 4, 6] \rangle$ are the subgroups of $Z_{12}$.

### 3.6.2

*Let $G = \langle a \rangle$ be a cyclic group of order 120. List all of the groups of order 120. List all of the elements of order 12 in $G$.*

Divisors of 120 are $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 24, 60, 120\}$, thus we can state that subgroups of a cyclic group are $a$ to powers of those numbers

According to the theorems, there should be $\phi(12) = 4$ elements of order 12. All of them lie in a subgroup $\langle a^{120/12} \rangle = \langle a^{10} \rangle$ and are in form $(a^{10})^k$ where $k \in 1, 5, 7, 11$.

*How many element of order 12 are there in a cyclic group of order 1200?*

Also 4.

### 3.6.3

*Let $p$ be a prime and $n$ a positive integer. Show that $\phi(p^n) = p^n - p^{n-1}$*

If $j \in Z_+$ is such that $j = pi$ for some $i \in Z_+$, then we follow that $(p^n, j) = p$, therefore they are not relatively prime. Suppose that $(p^n, j) = 1$ for some $j \in Z_+$. Let $S$ be a multiset of prime divisors of $p^n N$ and $T$ be a multiset of divisors of $j$. Then we follow that $S \cap T = \emptyset$, since otherwise we would've had that $j$ is a multiple of $p$, which is not relatively prime to $p^n$. Thus we follow that the set of not relatively prime numbers to $p^n$ is equal to the set of multiples of $p$.

We can follow that there are pricicely $p^{n-1}$ of multiples of $p$ that are less or equal to $p^n$ (don't think that we need to prove that), therefore the total amount of numbers that are less or equal to $p^n$, which are relatively prime to $p^n$ is $p^n - p^{n-1}$, as desired.

### 3.6.4

*Find all positive integers $n$ such that $|U(n)| = 24$.*

We can follow that $\phi(n)$ is an function that tends to infinity (i.e. for every $n \in Z_+$ there exists $j \in Z_+$ such that $m > n$ implies that $\phi(m) > j$) since $\phi(n)$ is larger than the number of prime numbers that is in the set $Z_+ \cap [1, n)$. Therefore we conclude that there is an upper bound for a number of numbers $n$ such that $\phi(n) = 24$.

Brute-force shows that those numbers are

$$35, 39, 45, 52, 56, 70, 72, 78, 84, 90$$

Can't come up with a better answer than that, but I'm sure that it's there.

### 3.6.5

*Let $G$ be a nonabelian group. If $H$ and $K$ are cyclic subgroups of $G$, does it follow that $H \cap K$ is also a cyclyc subgroup? Prove that it does, or provide a counterexample.*

We follow that every subgroup has an identity in it, thus $e \in H \cap K$. Suppose that $j \in H \cap K$. We follow that $j \in H \wedge j \in K$. Since $H$ and $K$ are both subgroups, we follow that $j^{-1} \in H \wedge j^{-1} \in K$. Thus $j^{-1} \in H \cap K$. Therefore $H \cap K$ is closed under inverses. We can follow also by the same logic that $j, l \in H \cap K$ implies that $jl \in H \cap K$. Therefore we can conclude that $H \cap K$ is a subgroup.

We can follow that if $H \cap K = \{e\}$, then it's cyclic. We can follow that $H \cap K$ can be not only a trivial subgroup by setting $H = K$. Suppose that $H \cap K \neq \{e\}$. By the fact that both $H$ and $K$ are cyclic we follow that $H \cap K = \{a^i : i \in$ some subset of $Z_+\}$. Since $H \cap K \neq \{e\}$, we follow that there exists an element $a \in G$ and two sets $H', K' \in \mathcal{P}(Z_+)$ such that $H = \{a^i : i \in H'\}$ and $K = \{a^i : i \in K'\}$. Since both $H$ and $K$ are cyclic we follow that

both $H'$ and $K'$ are the sets of multiples of some number. Thus $H' \cap K'$ is a set of multiples of some number as well (proof ommited). Thus we follow that $H \cap K = \{a^i : i \in H' \cap K'\}$ is a cyclic group as well.

### 3.6.6

*Let $G = \langle a \rangle$ be an infinite cyclic. If $m$ and $n$ are positive integers, find a generator for $\langle a^m \rangle \cap \langle a^n \rangle$.*

We can follow pretty easily that $\langle a^m \rangle \cap \langle a^n \rangle = \langle a^{lcm(m,n)} \rangle$

### 3.6.7

*Let $n$ be a positive integer and let $T$ be the set of positive integers that divide $n$. Show that $\sum_{k \in T} \phi(k) = n$.*

For 12 we've got

$$T = \{1, 2, 3, 4, 6, 12\}$$
$$\phi(1) = 1$$
$$\phi(2) = 1$$
$$\phi(3) = 2$$
$$\phi(4) = 2$$
$$\phi(6) = 2$$
$$\phi(12) = 4$$

and we follow that result works.

$$\sum_{k=0}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} (k-1)/2^k = -1 + \sum_{k=2}^{\infty} (k-1)/2^k = -1 + \sum_{k=1}^{\infty} k/2^{k+1} =$$

$$= -1 + \frac{1}{2} \sum_{k=1}^{\infty} k/2^k$$

$$\sum_{k=0}^{\infty} (k-1)/2^k = \sum_{k=0}^{\infty} k/2^k - 1/2^k = \sum_{k=0}^{\infty} k/2^k - \sum_{k=0}^{\infty} 1/2^k = \sum_{k=0}^{\infty} k/2^k - 2 = \sum_{k=1}^{\infty} k/2^k - 2$$

$$\sum_{k=1}^{\infty} k/2^k - 2 = -1 + \frac{1}{2} \sum_{k=1}^{\infty} k/2^k$$

$$\sum_{k=1}^{\infty} k/2^k - \frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\frac{1}{2}\sum_{k=1}^{\infty} k/2^k = 1$$

$$\sum_{k=1}^{\infty} k/2^k = 2$$