

My linear algebra exercises

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Preface

Exercises are from "Linear algebra done right" by Sheldon Axler, 3rd ed. I've already read this book before and completed some exercises from it. Right now I want to brush up the material once again, put all the proofs on a more durable material than paper and to prepare myself to what's gonna happen afterwards.

Chapter 1

1.A R^n and C^n

1

Suppose a and b are real numbers, not both 0. Find real number c and d such that

$$1/(a + bi) = c + di$$

$$\frac{1}{a + bi} = c + di$$

$$\frac{1}{a + bi} - c - di = 0$$

$$\frac{a - bi}{(a + bi)(a - bi)} = c + di$$

$$\frac{a - bi}{(a^2 + b^2)} = c + di$$

$$\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i = c + di$$

Thus $c = \frac{a}{a^2 + b^2}$ and $d = -\frac{b}{a^2 + b^2}$

2

Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1)

$$\left(\frac{-1 + \sqrt{3}i}{2}\right)^3 = \frac{(-1 + \sqrt{3}i)^3}{8} = \frac{(-1 + \sqrt{3}i)(-1 + \sqrt{3}i)^2}{8} = \frac{(-1 + \sqrt{3}i)(1 - 2\sqrt{3}i - 3)}{8} =$$

$$= \frac{(-1 + \sqrt{3}i)(-2 - 2\sqrt{3}i)}{8} = \frac{2 + 2\sqrt{3}i - 2\sqrt{3}i + 6}{8} = \frac{8}{8} = 1$$

as desired.

3

Find two distinct square roots of i

Square root of i , I assume, is a number, whose square is equal to i . Suppose that $(a + bi)^2 = i$. It follows that

$$(a + bi)^2 = a^2 + 2abi - b^2$$

So if we set

$$a = b = 1/\sqrt{2}$$

this equation holds. Also it holds for

$$a = b = -1/\sqrt{2}$$

maxima seems to agree with me on this one

4

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$

Let $\alpha = a_1 + b_1i$ and $\beta = a_2 + b_2i$. It follows

$$\alpha + \beta = a_1 + b_1i + a_2 + b_2i = a_2 + b_2i + a_1 + b_1i = \beta + \alpha$$

as desired.

5

Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$

Let $\alpha = a_1 + b_1i$, $\beta = a_2 + b_2i$, $\lambda = a_3 + b_3i$. It follows that

$$\alpha + (\beta + \lambda) = a_1 + b_1i + (a_2 + b_2i + a_3 + b_3i) = (a_1 + b_1i + a_2 + b_2i) + a_3 + b_3i = (\alpha + \beta) + \lambda$$

6

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

$$\alpha + (\beta + \lambda) = (a_1 + b_1i)((a_2 + b_2i) + (a_3 + b_3i)) = ((a_1 + b_1i)(a_2 + b_2i)) + (a_3 + b_3i) = (\alpha\beta)\lambda$$

7

Show that for every $\alpha \in \mathcal{C}$ there exists a unique $\beta \in \mathcal{C}$ such that $\alpha + \beta = 0$

Suppose that there exist two different $\beta_1 \neq \beta_2$ such that $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$. It follows that

$$\beta_1 = \beta_1 + 0 = \beta_1 + \alpha + \beta_2 = \alpha + \beta_1 + \beta_2 = 0 + \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β .

8

Show that for every $\alpha \in \mathcal{C}$ with $\alpha \neq 0$ there exists a unique $\beta \in \mathcal{C}$ such that $\alpha\beta = 1$

Suppose that it is not true and there exist two different $\beta_1 \neq \beta_2$ such that

$$\alpha\beta_1 = 1 \text{ and } \alpha\beta_2 = 1$$

it follows then that

$$\beta_1 = 1 * \beta_1 = \alpha\beta_2\beta_1 = \alpha\beta_1\beta_2 = 1 * \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β .

etc

The rest of the section is the repetition of this kind of stuff. That is a lot of writing, and not a lot of thinking, so I'll skip it. I don't usually like to skip sections, but I have a feeling, that I've completed this thing on paper somewhere, and there is not much reason to rewrite it here.

Chapter 2

1.B Definition of Vector Space

1

Prove that $-(-v) = v$ for every $v \in V$.

For v there exists only one $-v$. For $-v$ there exists only one $-(-v)$.

Thus

$$v = v + 0 = v + (-v) + (-(-v)) = 0 + (-(-v)) = -(-v)$$

as desired (idk if it's true, I'm not good at axioms and stuff)

2

Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Suppose that $a \neq 0, v \neq 0$ but $av = 0$. It follows that there exist $1/a$ - multiplicative inverse of a . It follows that

$$1/a * av = 1/a * 0$$

$$1v = 0$$

$$v = 0$$

which is a contradiction. Thus either $a = 0$ or $v = 0$.

3

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Suppose that there exists $x_1 \neq x_2$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$. Thus

$$3x_1 = w - v = 3x_2$$

$$x_1 = \frac{1}{3}(w - v) = x_2$$

which is a contradiction.

Same can be stated from the fact that x is a unique additive inverse of $\frac{1}{3}(v - w)$.

4

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Additive identity. Empty set does not have zero element in it. BTW $\{0\}$ is a vector space.

5

Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V .

$$\begin{aligned} 0v &= 0 \\ (1 - 1)v &= 0 \\ 1v - 1v &= 0 \\ v - v &= 0 \\ v + (-v) &= 0 \end{aligned}$$

6

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in R . Define an addition and multiplication on $R \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and the product of two real numbers is as usual, and for $t \in R$ define

$$\begin{aligned} t\infty &= \begin{cases} -\infty & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ \infty & \text{if } t > 0 \end{cases} \\ t(-\infty) &= \begin{cases} \infty & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ -\infty & \text{if } t > 0 \end{cases} \end{aligned}$$

$$t + \infty = \infty + t = \infty$$

$$t + (-\infty) = (-\infty) + t = (-\infty)$$

$$\infty + \infty = \infty$$

$$(-\infty) + (-\infty) = (-\infty)$$

$$\infty + (-\infty) = 0$$

Is $R \cup \{\infty\} \cup \{-\infty\}$ a vector space over R ? Explain.
I don't think that it is, TODO.