My topology exercises

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Contents

Preface

Those are my solutions for the James Munkres' "Topology", 2nd edition.

Majority of the notation that is used here migrated from my course on the set theory. In my very personal opinion, notation that is used there is far superior that whatever is happening in Munkres' book. Sometimes I use some abusive notation when it is painfully clear what's going on.

If you decide to persue the study of topology yourself, then I highly recommend firstly to go through a course on axiomatic set theory and logic, because first chapter of this book is highly insufficient in this regard. My personal recommendations are the combo by Cunningham, which includes "Set theory: A first course" and "A Logical Introduction to Proof", or "A first course in Mathematical Logic and Set Theory" by Michael L. O'Leary for both subjects.

Notation

Sometimes I use specific notation, that migrated from my previous endeavours in pure maths. This notation includes:

$$V_\epsilon(x) = (x - \epsilon, x + \epsilon)$$

Part I

General Topology

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

1.1.1

Check distributive and DML laws
GOTO set theory book

1.1.2

Determine which of the following are true.
(a) - impl
(b) - impl
(c) - true
(d) - rimpl
(e) - $\subseteq$, true if $B \subseteq A$.
(f) - $\supseteq$; $A - (B - A) = A$.
(g) - true
(h) - $\supseteq$
(i) - true
(j) - true
(k) - false
(l) - true
(m) - $\subseteq$
(n) - true
(o) - true
(p) - true
(q) - $\supseteq$

1.1.3

(a) Write a contrapositive and converse of the following statement: "If $x < 0$, then $x^2 - x > 0$" and determine which ones are true

Contrapositive:

$$x^2 - x \leq 0 \Rightarrow x \geq 0$$

Converse

$$x^2 - x > 0 \Rightarrow x < 0$$

Contrapositive is correct, converse is incorrect $(2^2 - 2 > 0)$

(b) Do the same for the statement $x > 0 \Rightarrow x^2 - x > 0$

Contrapositive:

$$x^2 - x \leq 0 \Rightarrow x \leq 0$$

Converse

$$x^2 - x > 0 \Rightarrow x > 0$$

Contrapositive is false $(1^2 - 1 = 0)$; Converse is also false $((-2)^2 - (-2) = 6)$.

1.1.4

Let A and B be the sets of real numbers. Write the negation of each of the following statements:

(a)

$$(\exists a \in A)(a^2 \notin B)$$

(b)

$$(\forall a \in A)(a^2 \notin B)$$

(c)

$$(\exists a \in A)(a^2 \in B)$$

(d)

$$(\forall a)(a \notin A \Rightarrow a^2 \notin B)$$

1.1.5

Let A be a nonempty collection of sets. Determine the truths of each of the following and their converses

(a)

$$x \in \bigcup A \Leftrightarrow (\exists B \in A)(x \in B)$$

(b)

$$x \in \bigcup A \Leftarrow (\forall B \in A)(x \in B)$$

(c)

$$x \in \bigcap A \Rightarrow (\exists B \in A)(x \in B)$$

(d)

$$x \in \bigcap A \Leftrightarrow (\forall B \in A)(x \in B)$$

1.1.6

Skip

1.1.7

skip

1.1.8

GOTO set theory book

1.1.9

Formulate DML for arbitrary unions and intersections

$$A \setminus \bigcap(B) = \bigcup(A \setminus B)$$
$$A \setminus \bigcup(B) = \bigcap(A \setminus B)$$

For the proof goto set theory or real analisys book

1.1.10

(a, b, d) are true

1.2 Functions

1.2.1

Let $f : A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$.

(a) Show that $A_0 \subseteq f^{-1}[f[A_0]]$ and that equality holds if f is injective.

Suppose that $x \in A_0$. We follow that there exists $\langle x, y\rangle \in f$ for some $y \in f[A_0]$. Therefore there exists $\langle y, x\rangle \in f^{-1}$. Because $y \in f[A_0]$, we follow that $x \in f^{-1}[f[A_0]]$. Therefore $A_0 \subseteq f^{-1}[f[A_0]]$.

Suppose that f is injective. Suppose that there exists $x_0 \in f^{-1}[f[A_0]]$ such that $x_0 \notin A_0$. We follow that $\langle y, x_0\rangle, \langle y, x\rangle, \in f^{-1}$, therefore $\langle x_0, y\rangle, \langle x, y\rangle \in f$, and because $x_0 \neq x$ we follow that we've got a contradiction.

((b)

pretty simular to (a)

This chapter practicly mirrors the content of my set theory course . Gonna skip it for now, and will come back if the need arises.

Chapter 2

Topological Spaces and Continous Functions

2.1 Topological Spaces

I want to state here that if $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfies properties

$$\{X, \emptyset\} \subseteq \mathcal{T}$$

$$(\forall Y \in \mathcal{P}(\mathcal{T}))(\bigcup U \in \mathcal{T})$$

$$(\forall Y \in \mathcal{P}(\mathcal{T}))(Y \neq \emptyset \wedge |Y| <_c |\omega| \to \bigcap U \in \mathcal{T})$$

then $\mathcal{T}$ is a topology on X.

2.2 Basis for a Topology

Let $Y \subseteq \mathcal{P}(X)$. If

$$(\forall x \in X)(\exists y \in Y)(x \in y)$$

and

$$(\forall x \in X)(\exists y_1, y_2, y_3 \in Y)(x \in y_1 \cap y_3 \to x \in y_3 \wedge y_3 \subseteq y_1 \cap y_2)$$

then Y is a basis for a topology on X.

2.2.1

Let X be a topological space; Let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Let $U : A \to \mathcal{P}(A)$ be an indexed function such that

$$x \in U(x) \wedge U(x) \subseteq A \wedge U(x) \in \mathcal{T}(X)$$

We want to show that $A = \bigcup \text{ran}(U)$. Suppose that $x \in A$. We follow that $x \in U(x)$. Thus $x \in \bigcup \text{ran}(U)$. Therefore $A \subseteq \bigcup \text{ran}(U)$.

Suppose that $z \in \bigcup \text{ran}(U)$. We follow that

$$(\exists Y \in \text{ran}(U))(z \in Y) \Rightarrow (\exists x \in A)(z \in U(x))$$

Since $(\forall x \in A)(U(x) \subseteq A)$, we follow that $z \in A$. Thus $\bigcup \text{ran}(U) = A$.

Because $(\forall x \in A)(U(x) \in \mathcal{T}(X))$, we follow that

$$\text{ran}(U) \subseteq \mathcal{T}(A)$$

, therefore by definition of topology we follow that

$$\bigcup \text{ran}(U) \in \mathcal{T}(X)$$

as desired.

2.2.2

Too tedious, skip

2.2.3

Show that the collection $\mathcal{T}_c$ given in Example 4 of p. 12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \in \mathcal{P}(X) : |X \setminus U| \geq_c |\omega| \vee X \setminus U = \emptyset \vee X \setminus U = X\}$$

a topology on X?

We firstly state that

$$\mathcal{T}_c = \{U \in \mathcal{P}(X) : |X \setminus U| \leq_c |\omega| \vee X \setminus U = X\}$$

We can follow that $X \setminus X = \emptyset$, which is countable, thus $X \in \mathcal{T}_c$. $X \setminus \emptyset = X$, therefore $\emptyset \in \mathcal{T}_c$.

Suppose that $U' \subseteq \mathcal{T}_c$. If $U' = \{\emptyset\}$, then $X \setminus \bigcap U' = X$ and $X \setminus \bigcup U' = X$. Thus assume that $U' \neq \{\emptyset\}$.

We follow that

$$(\forall u \in U')(|X \setminus u| \leq_c |\omega| \vee X \setminus u = X)$$

We follow that if $\emptyset \in U'$, then $\bigcup U' = \bigcup(U' \setminus \{\emptyset\})$. Then we follow by DML that

$$X \setminus \bigcup \{U'\} = X \setminus \bigcup \{U' \setminus \{\emptyset\}\} = \bigcap_{U' \setminus \{\emptyset\}} X \setminus u$$

we know that $(\forall u \in U')(|X \setminus u| \leq_c |\omega|)$. For any $u \in U'$ we follow that

$$\bigcap_{u \in U' \setminus \{\emptyset\}} X \setminus u \subseteq X \setminus u'$$

and given that $X \setminus u'$ is countable, we follow that $\bigcap_{u \in U'} X \setminus u$ is countable as well, thus $\bigcup U' \in \mathcal{T}_c$.

Now let $U' \subseteq \mathcal{T}_c$ and $|U'| < |\omega|$ and $U' \neq \{\emptyset\}$. We follow that if $\emptyset \in U'$, then $\bigcap U' = \emptyset$, and therefore $X \setminus \bigcap U' = X$. Therefore assume that $\emptyset \notin U'$.

Then we can follow that

$$X \setminus \bigcap U' = \bigcup_{u \in U'} X \setminus u$$

Given that U' is countable and $X \setminus u$ is countable we follow that $\bigcup_{u \in U'} X \setminus u$ is countable, thus $X \setminus \bigcap U'$ is countable.

Therefore we conclude that $\mathcal{T}_c$ is a topology on X.

Now let us consider T_∞. We can state that $X \in T_\infty$ because $X \setminus X = \emptyset$. Because $X \setminus \emptyset = X$, we follow that $\emptyset \in T_\infty$.

Suppose that X is not infinite and $T_\infty \neq \{\emptyset, X\}$. Then there exists $u \in T_\infty$ such that $u \neq \emptyset$ and $u \neq X$. Therefore $X - u$ is nonempty finite set, therefore $u \notin T_\infty$, which is a contradiction. Therefore we conclude that if X is finite, then T_∞ is a trivial topology.

If X is infinite, then we follow that we can have an injection $f : \omega \to X$. Let O be the set of odd naturals and E be the set of evens. Then we follow that

$$|X \setminus f[O]| = |f[E]| \geq_c |\omega|$$

and

$$|X \setminus f[E]| =_c |f[O]| \geq_c |\omega|$$

which tells us that $f[O]$ and $f[E]$ are both in X. We can also follow that

$$|X \setminus f[O \cup \{0\}]| \geq |\omega|$$

thus $f[O \cup \{0\}] \in \mathcal{T}_\infty$. This gives us that

$$f[E] \cap f[O \cup \{0\}] = \{f(0)\} \in \mathcal{T}_\infty$$

but $\{f(0)\}$ is a finite nonempty set for which none of the conditions of $\mathcal{T}_\infty$ hold. Therefore we conclude that if X is infinite, then $\mathcal{T}_\infty$ is not a topology.

Therefore we conclude that if X is a finite set, then T_∞ is equal to a trivial topology; if X is infinite, then T_∞ is not a topology at all, since it is not closed under finite intersections.

2.2.4

(a) if $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

Since every topology on X has X and $\emptyset$ as elements, we follow that

$$\{X, \emptyset\} \subseteq \bigcap \mathcal{T}_\alpha$$

If $Y \subseteq \bigcap \mathcal{T}_\alpha$, then we follow that

$$(\forall Z \in \{\mathcal{T}_\alpha\})(\bigcap \mathcal{T}_\alpha \subseteq Z)$$

$$(\forall Z \in \{\mathcal{T}_\alpha\})(Y \subseteq Z)$$

since every Z is a topology, we follow that

$$(\forall Z \in \{\mathcal{T}_\alpha\})(\bigcup Y \in Z)$$

$$\bigcup Y \in \bigcap \mathcal{T}_\alpha$$

If Y is finite and nonempty, we can also follow that

$$(\forall Z \in \{\mathcal{T}_\alpha\})(Y \in Z) \Rightarrow (\forall Z \in \{\mathcal{T}_\alpha\})(\bigcap Y \in Z) \Rightarrow \bigcap Y \in \bigcap \mathcal{T}_\alpha$$

thus we conclude that $\bigcap \mathcal{T}_\alpha$ is a topology.

$\bigcup \mathcal{T}_\alpha$ is not necessarily a topology. Although $\{X, \emptyset\} \in \bigcup \mathcal{T}_\alpha$, we cannot follow that the topology is closed under unions. Case in point: Let $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}\}, \mathcal{T}_1 = \{\emptyset, X, \{b\}\}$$

then $Y = \mathcal{T}_1 \cup \mathcal{T}_2$ does not contain $\{a, b\}$, which would be necessary for this case. Thus we conclude that in general we can't have implications for $\bigcup \mathcal{T}_\alpha$.

(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$ and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Let us take $\bigcup\{\mathcal{T}_\alpha\}$. We cannot follow that presented set is a topology on X, nor can we state that it is a basis of a topology. Former is followed from the discussion in the previous section of this exercise, and the latter cannot be followed because we don't necessarily satisfy the second point of the definition of the basis. Namely, we don't have that

$$(\forall x \in X)(\exists y_1, y_2, y_3 \in \bigcup\{\mathcal{T}_\alpha\})(x \in y_1 \cap y_3 \to x \in y_3 \wedge y_3 \subseteq y_1 \cap y_2)$$

Let Q be a set of all of the intersections of finite nonempty subsets of $\bigcup\{\mathcal{T}_\alpha\}$. We follow that $(\forall x \in \bigcup\{\mathcal{T}_\alpha\})(x = \bigcap\{x\})$, therefore $\bigcup\{\mathcal{T}_\alpha\} \subseteq Q$. Thus we follow that Q satisfies the first requirement for the basis of X. Now let $x \in X$ be such that there exist $y_1, y_2 \in Q$ such that $x \in y_1 \cap y_2$. We follow that there exist finite subsets $Y_1, Y_2 \subseteq \bigcup\{\mathcal{T}_\alpha\}$ such that

$$y_1 = \bigcap Y_1 \wedge y_2 = \bigcap Y_2$$

therefore

$$y_1 \cap y_2 = \bigcap Y_1 \cap \bigcap Y_2$$

which is an intersection of a finite subset of $\bigcup \mathcal{T}_\alpha$. Thus we follow that there exists $y_3 \in Q$ such that $x \in y_3 \wedge y_3 \subseteq y_1 \cap y_2$. Therefore we can follow that the set Q is indeed a basis for a topology on X. Let us name the topology generated by this set as $\mathcal{T}_q$.

Suppose that there is a topology, which contains all of the topologies $\{\mathcal{T}_\alpha\}$. Then we follow that it contains $\bigcup\{\mathcal{T}_\alpha\}$, therefore we follow that it contains all of the unions of $\bigcup\{\mathcal{T}_\alpha\}$, and finite intersections of subsets of $\bigcup\{\mathcal{T}_\alpha\}$, and thus it contains $\mathcal{T}_q$. Therefore we follow that $\mathcal{T}_q$ is the smallest topology, which contains all the topologies of $\{\mathcal{T}_\alpha\}$.

Suppose that $\mathcal{T}_p$ is a topology, which is contained in all of the $\{\mathcal{T}_\alpha\}$. Then we follow that $\mathcal{T}_p \subseteq \bigcap \mathcal{T}_\alpha$. Because $\bigcap \mathcal{T}_\alpha$ is a topology itself, we follow that it is the largest topology, which is contained in all of the $\{\mathcal{T}_\alpha\}$.

(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1, \mathcal{T}_2$.

We can follow from previous discussions that largest contained topology is

$$\{\emptyset, X, \{a\}\}$$

and the smallest containing topology is

$$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$$

2.2.5

Show that if A is a basis for a topology on X, then the topology generated by A equals the intersection of all topologies on X that contains A. Prove the same if A is a subbasis.

Let A be a subbasis. Let $\{\mathcal{T}_\alpha\}$ be a set of topologies, that contain A and $\mathcal{T}_A$ is a topology generated by A. We can follow that $\mathcal{T}_A \in \{\mathcal{T}_\alpha\}$, therefore $\bigcap\{\mathcal{T}_\alpha\} \subseteq \mathcal{T}_A$. If $x \in \mathcal{T}_A$, then we follow that there exists a subset $B \subseteq A$ such that x is equal to some union of some finite intersections of B. Since $B \subseteq A$, we follow that $(\forall y \in \mathcal{T}_\alpha)(B \subseteq y)$. Therefore all of the finite intersections of B are in any topology of $\mathcal{T}_\alpha$. Therefore all of the unions of those intersections are in any $\mathcal{T}_\alpha$. Therefore we conclude that $(\forall y \in \mathcal{T}_\alpha)(x \in y)$.

and thus $x \in \bigcap \mathcal{T}_\alpha$. Therefore we conclude that $\mathcal{T}_A \subseteq \bigcap \mathcal{T}_\alpha$, and by double inclusion we get that $\mathcal{T}_A = \bigcap \mathcal{T}_\alpha$, as desired.

Since every basis of a topology is a subbasis by first clause of the definition, we follow that the desired result holds for bases as well.

2.2.6

Show that the topologies of R_l and R_k are not comparable.

Let $[0, 1)$ be an element of a basis of topology R_l. Then we follow that there are no elements of basis of standart topology on R that contains 0 and lies inside $[0, 1)$. We can follow this by contradiction

Suppose that $0 \in (x, y)$ and $(x, y) \subseteq [0, 1)$. Since $0 \in (x, y)$, we follow that $x < 0$. Thus we conclude that there exists $n \in Z_+$ such that $1/n < |x|$. Therefore $-1/n \in (x, y)$ and $-1/n \notin [0, 1)$ which gives us that $(x, y) \not\subseteq [0, 1)$, which is a contradiction. The same logic applies to any element of basis of R_k.

Now let us look at the basis element $(-1, 1) \setminus K$ and the point 0. We can follow that $0 \in (-1, 1) \setminus K$ and suppose that there exists basis element of R_l $[a, b)$ that has point 0 and is contained within $(-1, 1) \setminus K$. Since $0 \in [a, b)$, we follow that $a \leq 0 < b$. Thus we conclude that there exists $n \in Z_+$ such that $0 < 1/n < b$. Thus we conclude that $1/n \in [a, b)$ and $1/n \notin (-1, 1) \setminus K$, since $1/n \in K$ for all $n \in Z_+$. Thus we conclude that R_k and R_l are not comparable, as desired.

2.2.7

Consider the following topologies on R:

$$\mathcal{T}_1 = \textit{the standart topology on } R$$

$$\mathcal{T}_2 = \textit{the topology of } R_k$$

$$\mathcal{T}_3 = \textit{the finite complement topology}$$

$$\mathcal{T}_4 = \textit{the upper limit topology, having all sets } (a, b] \textit{ as basis}$$

$$\mathcal{T}_5 = \textit{the topology having all sets } (-\infty, a) = \{x : x < a\} \textit{ as a basis}$$

Determine, for each of these topologies, which of the others it contains

We can follow that T_2 contains T_1, since it's finer, as proven in the chapter. The reverse is not true, as proven in the chapter.

We can follow that T_3 does not contain T_1, because if it is, then we follow that $(-\infty, a] \cup [b, \infty)$ has finite number of points. The revese is true, since we can divide each element of a finite complement into a union of open intervals. For example, if $x \in T_3$ is such that $x = R \setminus \{x_1, x_2, x_3\}$ and $x_1 < x_2 < x_3$, then we can state that $x = (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup (x_3, \infty)$. We can follow that middle 2 intervals are in the basis of standart topology, and two infinite intervals are unions of infinite set of intervals of basis. Thus $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_3$.

We can follow that the same logic, that worked with lower limit, works with upper limit as well. thus we conclude that T_4 is strictly finer than T_1.

We can follow that for $(-\infty, a) \in T_5$ we can get a sequence $(x_n) = a - n$, then get a set of intervals $\{(a, a-1), (x_{n+1}, x_n)\}$, all of which are in the basis of standart topology, get another set$\{V_{0.1}(x_n)\}$ to path the holes in this set, and take union of unions of both sets to get that $(-\infty, a) \in T_1$.

For (a, b) - a set in the basis of standard topology we follow that every set in the basis of T_5 contains $a-1$, thus we conclude that $(a, b) \notin T_5$. Thus we conclude that T_1 is strictly finer than T_5.

Topology $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$, therefore we follow that topologies that are finer than $\mathcal{T}_1$ are a subset of $\mathcal{T}_2$. This includes $\mathcal{T}_3$ and $\mathcal{T}_5$. (Almost) the same reasoning that worked with R_k and R_l can be applied to show that $\mathcal{T}_2$ is not finer than $\mathcal{T}_4$. On the other hand, suppose that $x \in X$ and $y \in \mathcal{T}_2$ is such that $x \in y$. We follow that if $y \in \mathcal{T}_1$, then there exists an element of $\mathcal{T}_4$ that is finer than y. Thus assume taht $y \notin \mathcal{T}_1$ and therefore is in the form $y = (a, b) \setminus K$ for some $a, b \in R$. If $x \leq 0$, then we can have set $(a, x] \subseteq y$ that will satisfy. Thus assume that $x > 0$. We follow that there exists $n \in Z_+$ such that $1/n < x$. By well-ordering properties of Z_+ we follow that there exists lowest $n \in Z_+$ such that $1/nx$. Therefore we follow that there are no elements $z \in K$ such that $1/n < z < x$. Since $x \in (a, b) \setminus K$, we follow that $x \notin K$, therefore $(\forall y \in (1/n, x])(y \in x \in (a, b) \setminus K)$. Therefore we conclude that $\mathcal{T}_4$ is strictly finer than $\mathcal{T}_2$, which is neat.

$\mathcal{T}_3$ is strictly coarser than $\mathcal{T}_1$, $\mathcal{T}_2$. Since $\mathcal{T}_4$ is strictly finer than $\mathcal{T}_2$, we follow that $\mathcal{T}_3$ is coarser than $\mathcal{T}_4$. Suppose that $a < x < b$ and let $y = R \setminus \{a, b\}$ be an element of $\mathcal{T}_3$. Then we follow that no element of basis of $\mathcal{T}_5$ has x and does not have a. If $(-\infty, a)$ is an element of $\mathcal{T}_5$, then we follow that every element of topology $\mathcal{T}_3$ has numbers greater than a in it (since there are infinitly many of them). Thus we conclude that no element of $\mathcal{T}_3$ is a subset of $(-\infty, a)$. Thus we conclude that $\mathcal{T}_3$ and $\mathcal{T}_5$ are not comparable.

And after all of the discussion, we can conclude that

$$[\mathcal{T}_3 \,|\, \mathcal{T}_5] \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_4$$

is the desired conclusion.

2.2.8

(a) Apply Lemma 13.2 to show tha the countable collection

$$B = \{(a, b) : a < b \wedge a, b \in Q\}$$

is a basis that generates the standard topology on R.

Denote $\mathcal{T}$ as a standard topology on R. Let $x \in \mathcal{T}$. We follow that there exists an interval (a, b) in basis of standard topology such that $x \in (a, b)$. We can follow that there exist $a', b' \in Q$ such that $a < a' < x < b' < b$ (otherwise we run into some problem with density of rationals in reals). Therefore we follow that $x \in (a', b')$. Lemma 13.2 tells us that the presented result implies that B is a basis for standard topology, as desired.

(b) Show that the collection

$$C = \{[a, b) : a < b \wedge a, b \in Q\}$$

is a basis that genenrates a topology different from the lower limit topology on R.

Proof that C is a basis is trivial. Let us look at $[\sqrt{2}, 2)$ - an element of R_l. Suppose that $c = [a, b) \in C$ is such that $\sqrt{2} \in c$. Because $\sqrt{2} \notin Q$, we follow that $a \neq \sqrt{2}$, therefore $a < \sqrt{2} < b$. Therefore we can conclude that C is not finer than R_l. Proving that C is a subset of R_l is trivial, thus we conclude that R_l is strictly finer than C, and thus C generates a topology different than R_l, as desired.

2.3 The Order Topology

2.4 The Product Topology on $X \times Y$

2.5 The Subspace Topology

2.5.1

Show that if Y is a subspace of X, and A is a subspace of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Suppose that Q is an open set in A with respect to topology, inherited from X. We follow that there exists an open set in X $Q_x \subseteq X$ such that $Q = Q_x \cap A$ by definition of a subspace topology. We follow that there exists open in Y set $Q_y \subseteq Y$ such that $Q_y = Q_x \cap Y$. With respect to Q_y there exists an open in A set $Q' = Q_y \cap A$. Thus

$$Q' = Q_y \cap A$$

$$Q' = Q_x \cap Y \cap A$$

Since $A \subseteq Y$, we follow that $Y \cap A = A$. THus

$$Q' = Q_x \cap (Y \cap A)$$

$$Q' = Q_x \cap A$$

$$Q' = Q$$

Therefore we conclude that if Q is in topology of A inherited from X, then Q is also in a topology of A inherited from Y. Proof of the converse is pretty much the same proof

Here's another, more logical and rigorous proof. Denote topology of A inherited from Y by $\mathcal{T}_A$ and topology of A inherited from X by $\mathcal{T}'_A$. Also denote topology of X by $\mathcal{T}_X$ and topology of Y inherited from X by $\mathcal{T}_Y$. Then we can state that

$$Q \in \mathcal{T}_A \Leftrightarrow (\exists Q_y \in \mathcal{T}_Y)(Q = Q_y \cap A) \Leftrightarrow (\exists Q_X \in \mathcal{T}_X)(Q_y = Q_x \cap Y \wedge Q = Q_y \cap A) \Leftrightarrow$$

$$\Leftrightarrow (\exists Q_X \in \mathcal{T}_X)(Q = Q_x \cap Y \cap A) \Leftrightarrow (\exists Q_X \in \mathcal{T}_X)(Q = Q_x \cap (Y \cap A)) \Leftrightarrow$$

$$\Leftrightarrow (\exists Q_X \in \mathcal{T}_X)(Q = Q_x \cap A) \Leftrightarrow Q \in \mathcal{T}'_A$$

thus $\mathcal{T}'_A = \mathcal{T}_A$ by extensionality axiom.

2.5.2

if $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you cay about the corresponding topologies on the subset Y of X.

Denote corresponding topologies by $\mathcal{T}'_Y$ and $\mathcal{T}_Y$. There're three plausible cases:

1 - we can't say nothing

2 - $\mathcal{T}'_Y \supset \mathcal{T}_Y$

3 - $\mathcal{T}'_Y \supseteq \mathcal{T}_Y$

I'm betting on the second case, so let us try to prove that. In order to do that, let us firstly prove the third case, which is a "subcase" of the second.

Suppose that $Q \in \mathcal{T}_Y$. We follow that there exists $Q_X \in \mathcal{T}$ such that $Q = Q_X \cap Y$. Since $Q_X \in \mathcal{T}$, we follow by $\mathcal{T} \subset \mathcal{T}'$ that $Q_x \in \mathcal{T}'$. Thus $Q = Q_X \cap Y$ implies that $Q \in \mathcal{T}'_Y$. Therefore we follow that $\mathcal{T}'_Y \supseteq \mathcal{T}_Y$.

Although I'm betting on the second case, it seems that I'm not getting my money back. We can follow that second case is not always true, if we substitute $\emptyset$ for Y. Then $\mathcal{T}_Y = \mathcal{T}'_Y = \emptyset$. If we look into topologies of some almost-trivial set, such as $X = \{a, b, c\}$, then I think that we can come up with a more persuasive case as well. Therefore we conclude that presented conditions imply that $\mathcal{T}_Y \subseteq \mathcal{T}'_Y$.

2.5.3

Considet the set $Y = [1, 1]$ as a subspace of R. Which of the following sets are open in Y? Which are open in R?

$$A = \{x : \frac{1}{2} < |x| < 1\}$$

$$B = \{x : \frac{1}{2} < |x| \leq 1\}$$

$$C = \{x : \frac{1}{2} \leq |x| < 1\}$$

$$D = \{x : \frac{1}{2} \leq |x| \leq 1\}$$

$$E = \{x : 0 < |x| < 1 \wedge 1/x \notin Z_+\}$$

We can follow that $A = (-1, -1/2) \cup (1/2, 1)$ is open in both Y and R.

$B = [-1, -1/2) \cup (1/2, 1]$ is a union of two rays in Y, therefore we follow that it is open in Y. For R we've got that there is no open interval, that contains a point 1 and does not contain anything larger then 1. Therefore we conclude that given set is not a union of open intervals, and therefore it is not open in R.

We can follow pretty easily that C and D are not open in both Y and R since there is no open interval/ray that contains $1/2$ and does not contain anything in the interval $(-1/2, 1/2)$.

We can represent E as

$$E = (-1, 0) \cup ((0, 1) \setminus K)$$

We follow that $(-1, 0)$ is an element of a basis of both Y and R. Suppose that $x \in (0, 1)\setminus K$. Then we follow that there exist lowest $n_1 \in Z_+$ such that $1/n_1 < x < 1/(n_1+1)$. Therefore we can conclude that if $x \in E$, then there exist a basis element Q of both Y and R such that $x \in Q \subseteq Y, R$. Therefore we follow that E is an open set in both Y and R.

2.5.4

A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Suppose that $Q \in X \times Y$ is an open set. Therefore we follow that it is a union of some element of a basis of $X \times Y$, therefore there exist a subset R of a basis of $X \times Y$ such that $Q = \bigcap R$. From a set theory course we know that

$$U[\bigcup G] = \bigcup\{R[C] : C \in G\}$$

for any relation U. Therefore we can follow that the same result holds for functions π_1, π_2. We can follow that for any $r \in R$ we've got that both $\pi_1(r)$ and $\pi_2(r)$ are open by the definition of a basis for the product topology. Therefore we conclude that $\pi_1[Q] = \pi_1[\bigcup R] = \bigcup\{\pi_1[\bigcup r] : r \in R\}$. Therefore we conclude that $\pi_1[Q]$ is a union of open sets of X, therefore we conclude that it is in topology of X. We can follow the simular result for π_2 using simular logic.

2.5.5

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively; let Y and Y' denote a single set in the topologies U and U', respectively. Asuume that these sets are nonempty.

There're a couple of ways to deconstruct the text of this exercise: we can assume that $X = X'$, $Y = Y'$, $X \in \mathcal{T}$, $X' \in \mathcal{T}'$, $Y \in U$ and $Y' \in U'$, or we can assume that $X \in \mathcal{T}$, $X' \in \mathcal{T}'$, $Y \in U$ and $Y' \in U'$ without $X = X'$ and $Y = Y'$. The latter case will obviously present some problems in the proofs, therefore we will assume that the author intended to use the former case.

(a) Show that if $\mathcal{T}' \supseteq \mathcal{T}$ and $U' \supseteq U$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$

Let $\mathcal{B}$ denote the basis for $\mathcal{T}_{X\times Y}$. Let $q \in X \times Y$. Because $\mathcal{B}$ is a basis for $\mathcal{T}_{X\times Y}$ we follow that there exists $b \in \mathcal{B}$ such that $q \in b$. Since $b \in \mathcal{B}$, we follow that there exist $x \in \mathcal{T}$ and $y \in U$ such thaht $b = x \times y$. Since $\mathcal{T} \subseteq \mathcal{T}'$ and $U \subseteq U'$, we follow that $x \in \mathcal{T}'$ and $y \in U'$. Therefore $x \times y \in \mathcal{B}'$ where $\mathcal{B}'$ denotes the basis for $\mathcal{T}_{X'\times Y'}$. Therefore we conclude that for every $x \in X \times Y$ and every basis element $q \in \mathcal{B}$ there exists $q' \in \mathcal{B}'$ such that $q' \subseteq q$ and $x \in q'$. Therefore we conclude that $\mathcal{T}_{X\times Y} \subseteq \mathcal{T}_{X'\times Y'}$, as desired.

(b) Does the converse of (a) hold? Justify your answer.

Let $\mathcal{T}$ and $\mathcal{T}'$ be defined on a set $Q = \{a, b\}$ and U and U' be defined on $W = \{c, d\}$. Let $X = X' = \{a\}$, $Y = Y' = \{c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, Q\}$, $\mathcal{T}' = \{\emptyset, \{a\}, Q\}$, and $U = U'$. Then we follow that topology defined on $X \times Y$ is finer than the topology defined $X' \times Y'$ (and vice versa), but $\mathcal{T}'$ is not finer than $\mathcal{T}$.

2.5.6

Show that the countable collection

$$\{(a, b) \times (c, d) : a < b \wedge c < d \wedge a, b, c, d \in Q\}$$

is a basis for R^2.

Let us denote this set by L. Suppose that $x \in R^2$. We follow that there exist $x_1, x_2, y_1, y_2 \in Q$ such that $x_1 < x < x_2$ and $y_1 < y < y_2$, therefore $(\exists l \in L)(x \in l)$. Thus we follow that the first condition of a definition of a basis is sastisfied. The last condition can be satisfied by through the argument about the density of rationals in reals.

We can follow that topology, that is presented by given basis is a subset of the standard topology on R^2, and we can follow though pretty much the same argument that given topology is finer than the standard topology. Therefore I'm pretty sure that we can state that given basis generates the standard topology (I'll not provide any proof of that, just stating what I think).

2.5.7

Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

Don't think so. I think that the author tries to give us a hint to what's going to come afterwards (probably something about the completeness and whatnot).

Pretty sure, that we don't need to prove that Q is a totally ordered set, so we're going to take it as a given. Let

$$M = \{x \in Q : x^2 < 2 \wedge x \geq 0\}$$

(I've added the latter condition in order not to be bogged down by several cases, depending on the sign). Let $x < y \in M$. Suppose that $z \in Q$ is such that $x < z < y$. Then we follow that $z > x \geq 0$, thus $z > 0$. Since all of the numbers are positive, we're justified to square them and get that $x^2 < z^2 < y^2$. Given that $y^2 < 2$, we conclude that $z^2 < 2$ as well. Therefore we follow that $z \in M$. Thus we can follow that $z \in (x, y) \Rightarrow z \in M$. Therefore we can state that presented set is convex.

Given that M is bounded above and below, we follow that it is not a ray. Suppose that it is an interval. Then we follow that there exists $k \in Q$ such that $M = [0, k)$. Therefore we follow that k is a least lower bound of M, which is not the case, as proven in numerous real analysis books. Thus we conclude that M is not an interval.

2.5.8

If L is a straight line in the plane, describe the topology L inherits as a subspace of $R_l \times R$ and as a subspace of $R_l \times R_l$. In each case it is a familiar topology.

Let $\mathcal{B}$ be the basis for $R_l \times R$ and $\mathcal{B}'$ be the basis for $R_l \times R$. Let $q \in \mathcal{B}$ and suppoe that $b = L \cap q \neq \emptyset$. From plotting elements of the basis and the line itself on the graph, we can conclude that b is some sort of an interval on the plane (either closed or open), and it might as well be a ray (once again, open or closed). In case with $R_l \times R_l$ we conclude that the topology here is once again open or closed intervals on the plane.

2.5.9

Show that the dictionary order topology on the set $R \times R$ is the same as the product topology $R_d \times R$, where R_d denotes r in the discrete topology. Compare this topology with the standard topology on R^2.

Let $\langle x, y\rangle \in R^2$. We follow that there exists q - element of basis of $R_d \times R$ such that $\langle x, y\rangle \in q$. Because q is an element of a basis, we follow that $q = w \times r$, where $w \in \mathcal{T}_{R_d}$ and $r \in \mathcal{T}_R$. Because $r \in \mathcal{T}_R$, we follow that there exists (a, b) such that $(a, b) \subseteq r$. Therefore we follow that element $\{x\} \times (a, b)$ is an element of a basis of dictionary order such that $\{x\} \times (a, b) \subseteq q$. Therefore we follow that $R_d \times R$ is coarser than dictionary order topology.

Suppose that $\langle x, y\rangle$ is in R^2 and q is in basis of dictionary topology of R^2 such that $\langle x, y\rangle \in q$. By definition (and immediate implications of thereof) we follow that the set $q \cap \{x\} \times R$ is nonempty. Since $\langle x, y\rangle$ is in q, and q is a basis element, we follow that there exist $a, b \in R$ such that $\{x\} \times (a, b) \subseteq q$ (follows from definitions and maybe some trivial manipulations of definition of dictionary order). Since $\{x\} \times (a, b)$ is an element of a basis of $R_d \times R$, we conclude that dictionary order topology is coarser than $R_d \times R$, and thus by double inclusion we conclude that topology over $R_d \times R$ and dictionary order topologies are equal, as desired.

We can follow that topology in R_d is strictly finer than standard topology of R since $R_d = \mathcal{P}(R)$, and thus it is the largest possible topology. Strictness follows from the fact that $\{0\} \in \mathcal{T}_{R_d}$ and $\{0\} \notin \mathcal{T}_R$.

Thus we can be pretty sure that there is no element of basis of standard topology on $R \times R$ that contains $\{0\} \times R$. Since every element of basis of $R \times R$ is also contained in $R_d \times R$, we conclude that standard topology on R^2 is coarser than $R_d \times R$.

2.5.10

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $R \times R$ in the dictionary order topology.

Wanted to skip this one, since I've solved it incorrecttly the first time, but instead of skipping I'll just present the half-assed proof here for completeness' sake.

Important thing to remember: sets such as $\{x\} \times [0, 0.1)$ are not elements of basis of dictionary topology on $I \times I$

Let us look at the point $\langle 0.5, 1\rangle$ and basis element of standard topology $[0, 1] \times (0.5, 1]$. We can follow that since the elements of dictionary bases cannot just stop at the corners and must wrap around, we follow that there is no element of basis of dictionary order topology, that is contained in presented element of basis and contains the desired point.

We can follow also that $\{0.5\} \times (0, 1)$ cannot be presented in standard topology as well. Thus the first two are not comparable

Suppose that $\langle x, y\rangle \in I \times I$ and q is the basis element with respect to the dictionary order topology. We follow that we can take a "strand" from dictionary (i.e. take a set $\{x\} \times R \cap q$, where q is an element of the basis, that contains point $\langle x, y\rangle$) and get that dictionary order over $I \times I$ is coarser than the topology $I \times I$ inherits as a subspace of $R \times R$ in the dictionary order topology, since the strand is the element of the basis of the latter. We can pull the same trick that we've used in the previous paragraph to show that the inherited topology is strictly coarser than the dictionary topology. Using the "strand" method (i.e. taking basis elements in form $\{x\} \times (a, b)$ or its closed analogs) we can prove that the last topology is strictly finer than the standard topology on $I \times I$.

2.6 Closed Sets and Limit Points

2.6.1

let C be a collection of subsets of the set X. Suppose that $\emptyset$ and X are in C, and that finite unions and arbitrary intersections of elements of C are in C. Show that ther collectionn

$$T = \{X \setminus C : C \in C\}$$

is a topology on X.

We're gonna use the definition of topology on this one. We follow that since X and $\emptyset$ are in C that

$$X \setminus X = \emptyset \in T$$

$$X \setminus \emptyset = X \in T$$

Assume that J is an arbitrary subset of T. We follow that for every $j \in J$ there exists a unique $k \in C$ such that $j = X \setminus k$. Thus we follow that there exists $C' \subseteq C$ such that

$$\{j : j \in J\} = \{X \setminus k : k \in C'\}$$

thus

$$\bigcup J = \bigcup \{j : j \in J\} = \bigcup \{X \setminus k : k \in C'\} = X \setminus \bigcap \{k : k \in C'\} = X \setminus \bigcap C'$$

where we've used DML to justify one of the equations. Since C' is an arbitrary subset of C we follow that $\bigcap C' \in C$. Thus we follow that $X \setminus \bigcap C' \in T$. Thus we follow that $J \subseteq T \Rightarrow \bigcup J \in T$. Therefore we've got second property of topology.

If J is a finite subset of T, then we follow that we can define C' by the same definition and that C' is finite as well. Thus

$$\bigcap J = \bigcap \{j : j \in J\} = \bigcap \{X \setminus k : k \in C'\} = X \setminus \bigcup \{k : k \in C'\} = X \setminus \bigcup C'$$

thus we follow that if J is a finite subset of T , then $\bigcap J \in T$, therefore we've got the third and final condition of topology. Thus we follow that T is a topology, as desired.

2.6.2

Show that if A is closed in Y and Y is closed in X, then A is closed in X

Since Y is closed in X we follow that $Y \subseteq X$. Assuming that the topology on Y is a subset topology, we follow that if A is a closed set in Y, then there exists $A' \subseteq X$ such that A' is closed and $A = A' \cap Y$. Since both A' and Y are closed in X we follow that $A = A' \cap Y$ is closed in X as well by definition of topology, as desired.

2.6.3

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

If A is closed in X, then we follow that there exists A' such that $A = X \setminus A'$, where A' is an open set in X. Same goes for $B = Y \setminus B'$. Thus we follow that $A' \times B'$ is an open set in $X \times Y$. one of the exercises in chapter 1 gives us that

$$(A \times B) = (X \setminus A') \times (Y \setminus B') = (X \times Y \setminus A' \times Y) \setminus X \times B'$$

We follow that $A' \times Y$ is an open set, then $(X \times Y \setminus A' \times Y)$ is a closed set. We follow also that $X \times B'$ is an open set, thus $X \times Y \setminus X \times B'$ is a closed set. Thus

$$(X \times Y \setminus A' \times Y) \setminus X \times B' = (X \times Y \setminus A' \times Y) \cap (X \times Y \setminus X \times B')$$

is an intersection of closed sets and therefore is closed itself. Thus we conclude that $A \times B$ is a closed set, as desired. (we can also follow the same thing by the following exercise)

2.6.4

Show that if U is open in X and A is closed in X, then $U \setminus A$ is open in X and $A \setminus U$ is closed in X.

Firstly I want to prove that if $A, B \subseteq X$, then

$$A \setminus B = A \cap (X \setminus B)$$

We follow that by

$$x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B \Leftrightarrow x \in A \wedge x \in X \wedge x \notin B \Leftrightarrow x \in A \wedge (x \in X \setminus B) \Leftrightarrow x \in A \cap (X \setminus B)$$

We can follow by the fact that $U, A \subseteq X$ that

$$U \setminus A = U \cap (X \setminus A)$$

and

$$A \setminus U = A \cap (X \setminus U)$$

In the former case we've got finite intersection of two open sets, and in the latter we've got finite intersection of two closed sets, thus proving that $U \setminus A$ is open and $A \setminus U$ is closed, as desired.

2.6.5

Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subseteq [a,b]$. Under what conditions does equality hold?

Let us firstly state that $a, b \in X$. We follow that $(a,b) \subseteq [a,b]$, thus $[a,b]$ is a closed set that contains (a,b), therefore by definition of closure we follow that $\overline{(a,b)} \subseteq [a,b]$.

We follow that $\overline{(a,b)} = [a,b]$ if and only if a, b are limit points of (a,b).

2.6.6

Let A, B and A_α denote subsets of a space X. Prove the following

(a) If $A \subset B$, then $\overline{A} \subseteq \overline{B}$

Assume that $A \subseteq B$. Let $x \in \overline{A}$. We follow that every neighborhood of x intersects A. Thus every heighborhood of x intersects B by the fact that $A \subseteq B$. Therefore $x \in \overline{B}$. Therefore $\overline{A} \subseteq \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Let $x \in \overline{A \cup B}$. We follow that every heighborhood of x intersects $A \cup B$. Thus every neighborhood of x intersects A or B. Assume that $x \notin \overline{A}$ and $x \notin \overline{B}$. Then we follow that there exists a neighborhood U of x such that $U \cap A = \emptyset$. There also exists neighborhood U' of x such that $U' \cap B$. Thus we follow that $U \cap U'$ is a neighborhood of x such that it does not intersect A nor B. Thus $x \notin \overline{A \cup B}$, which is a contradiction. Thus we conclude that $x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cup \overline{B}$.

If $x \in \overline{A} \cup \overline{B}$, then $x \in \overline{A}$ or $x \in \overline{B}$. Assume that the former is true. Then we follow that $x \in \overline{A}$. Thus we follow that every neighborhood of x intersects A. Since $A \subseteq A \cup B$, we follow that every neighborhood of x intersects $A \cup B$. Thus $x \in \overline{A \cup B}$, as desired.

(c) $\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}$.

I think that we need to assume here that $A_\alpha \subseteq \mathcal{P}(X)$ and what we actually need to prove is that

$$\overline{\bigcup A_\alpha} \supseteq \bigcup\{\overline{a} : a \in A_\alpha\}$$

if that's the case, then we follow that if $x \in \bigcup\{\overline{a} : a \in A_\alpha\}$, then there exists $x \in a \in A_\alpha$ such that $x \in \overline{a}$. This means that every neighborhood of x intersects a at some point. Since $a \subseteq \bigcup A_\alpha$, we follow that every neighborhood of x intersects $\bigcup A_\alpha$ at some point and thus $x \in \overline{\bigcup A_\alpha}$.

We can follow that if $A_\alpha = \{\{1/n\} : n \in Z_+\}$ and we've got standard topology on reals, then $0 \in \overline{\bigcup A_\alpha}$, but $0 \notin \bigcup\{\overline{a} : a \in A_\alpha\}$, since there is no $a \in A_\alpha$ such that $0 \in \overline{a}$.

2.6.7

Critsize the following "proof" that $\overline{\bigcup A_\alpha} \subseteq \bigcup \overline{A_\alpha}$*: if* $\{A_\alpha\}$ *is a collection of sets in* X *and if* $x \in \overline{\bigcup A_\alpha}$*, then every neighborhood of* U *intersects* $\bigcup \overline{A_\alpha}$*. Thus* U *must intersets some* A_α*, so that* x *must belong to the closure of some* A_α*. Therefore* $x \in \bigcup \overline{A_\alpha}$*.*

We don't have implication "Thus U must intersects some A_α, so that x must belong to the closure of some A_α", as it was just made up. Althought every neighborhood of x indeed intersects some A_α, there's no implication that there exists A_α such that every neighborhood of x intersects A_α.

2.6.8

Let A, B *and* A_α *denote subsets of a space* X*. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions holds.*

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

If $x \in \overline{A \cap B}$, then we follow that every neighborhood of x intersects $A \cap B$ at some point. Thus every neighborhood of x intersects A and B. Thus $x \in \overline{A}$ and $x \in \overline{B}$. Thus we've got forward inclusion.

If $x \in \overline{A} \cap \overline{B}$, then we follow that every neighborhood of x intersects A and every neighborhood of x intersects B. Thus every neighborhood of x intersects both A and B. This does not mean that every neighborhood of x intersects $A \cup B$ since points of intersection can be different. We can come up with some counterexample for this claim: for example we can set $A = \{1/2n : n \in Z_+\}$ and $B = \{1/(2n + 1) : n \in Z_+\}$. We follow that $A \cap B = \emptyset$ and thus $\overline{A \cap B} = \emptyset$, but $\overline{A} \cap \overline{B} = \{0\}$.

Therefore we follow only the forward inclusion.

(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$

We're once again struck with this awful notation, so let's change that

$$\overline{\bigcap A_\alpha} = \bigcap_{a \in A_\alpha} \overline{a}$$

we follow that reverse inclusion is not true, since that would imply the correctness of counterexample in previous point.

We follow the forward inclusion by pretty much the same logic as in previous one

If $x \in \overline{\bigcap A_\alpha}$, then we follow that every neighborhood of x intersects $\bigcap A_\alpha$ at some point. Thus every neighborhood of x intersects every $a \in A_\alpha$. Thus $(\forall a \in A_\alpha)(x \in \overline{a})$. Therefore we follow that $a \in \bigcap_{a \in A_\alpha} \overline{a}$. Thus we've got forward inclusion.

(c) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$

We've got a case against $\overline{A \setminus B} \subseteq \overline{A} \setminus \overline{B}$ by setting once again $A = \{1/2n : n \in Z_+\}$, $B = \{1/(2n+1) : n \in Z_+\}$. We follow that $A \setminus B = A$ since they are disjoint. Therefore we follow that $0 \in \overline{A \setminus B}$ but $0 \notin \overline{A} \setminus \overline{B}$, since $0 \in \overline{B}$.

If $x \in \overline{A} \setminus \overline{B}$, then we follow that every neighborhood of x intersects A, but x is not in $\overline{B}$. Assume that some neighborhood of x intersects A only at a points $A \cap B$. Then we follow that $x \in \overline{A \cap B}$ and thus $x \in \overline{A} \cap \overline{B}$. Therefore $x \in \overline{B}$, which is a contradiction. Therefore we follow that if $x \in \overline{A} \setminus \overline{B}$ then every neighborhood of x intersects $A \setminus B$. Therefore $x \in \overline{A \setminus B}$, which gives us reverse inclusion.

2.6.9

Let $A \subseteq X$ and $B \subseteq Y$. Show that in the space $X \times Y$

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

.

Let $x \in X$ and $y \in Y$ be such that $\langle x, y \rangle \in \overline{A \times B}$. Let U be an arbitrary neighborhood of X and V be an arbitrary neighborhood for y. We follow that $U \times V$ is an evement of basis of $X \times Y$ that contains $\langle x, y \rangle$, and thus it intersects $A \times B$. Therefore there exists $\langle q, w \rangle \in A \times B \cap U \times V$. Thus we follow that $\langle q, w \rangle \in (A \cap U) \times (B \cap V)$. Therefore we follow that U intersects A and V intersects B. Since U and V are arbitrary, we follow that every neighborhood of x intersects A and every neighborhood of y intersects B. Thus $x \in \overline{A}$ and $y \in \overline{B}$ and thus $\langle x, y \rangle \in \overline{A} \times \overline{B}$.

Suppose that $x \in X, y \in Y$ are such that $\langle x, y \rangle \in \overline{A} \times \overline{B}$. We follow that $x \in \overline{A}$ and $y \in \overline{B}$. Assume that $U \times V$ is a basis element of $X \times Y$ that contains $\langle x, y \rangle$. We follow that U intersects A and V intersects B. Thus there exist $u \in U \cap A$ and $v \in V \cap B$. Therefore $\langle u, v \rangle \in A \times B \cap U \times V$. Therefore $U \times V$ intersects $A \times B$. Thus we follow that every basis element that contains $\langle x, y \rangle$ intersects $A \times B$, and thus $\langle x, y \rangle \in \overline{A \times B}$.

Double inclusion produces the desired equality, as desired.

2.6.10

Show that every order topology is Hausdorff.

Let X be a toset, and let $\mathcal{T}$ be respective order topology. I think that definition, that is presented in the book implies that if there is only one element in X, then the topology is vacuously Hausdorff (same goes for empty set). Thus assume that X contains at least two elements, and let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Since $x_1, x_2 \in X$ and $x_1 \neq x_2$, we follow that $x_1 \prec x_2$ or $x_2 \prec x_1$. Assume the former.

Essentially we want to produce two open sets, that will prove that the space is Hausdorff, and those two sets will be just plain old intervals. But we've got two cases: there could exist x_3 such that $x_1 \prec x_3 \prec x_2$, or there might not. If such an element exists, then set $b_1 = a_2 = x_3$. If there is no such element, then set $b_1 = x_2$ and $a_2 = x_1$. Let a_1 be either the lowest element of X if such exists, or some element that is less than x_1 in case that it does not exist. Let b_2 be either the largest element of X if such exists, or some element that is larger than x_2 in case that it does not exist.

Then we can follow that $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$ and $(a_1, b_1) \cap (a_2, b_2) = \emptyset$ by their respective definitions. Thus we follow that the space is Hausdorff, as desired.

2.6.11

Show that the product of two Hausdorff spaces is Hausdorff.

Let X be the first spacce and let Y be the second. Let $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in X \times Y$. We follow that $x_1 \neq x_2$ or $y_2 \neq y_2$. If we've got $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$ such that $A_1 \cap A_2 = \emptyset$ or $B_1 \cap B_2 = \emptyset$, then we follow that

$$A_1 \times B_1 \cap A_2 \times B_2 = (A_1 \cap A_2) \times (B_1 \cap B_2) = \emptyset$$

the desired result follows easily from that.

2.6.12

Show that a subspace of a Hausdorff space is Hausdorff.

Assume that X is a Hausdorff space and let $Y \subseteq X$. Let $y_1, y_2 \in Y$. We follow that there exist open sets $U, V \subseteq X$ such that $y_1 \in U, y_2 \in V, U \cap V = \emptyset$ by the fact that X is Hausdorff. We follow that $U \cap Y$ and $V \cap Y$ are open sets in Y, and since $U \cap V = \emptyset$, we follow that $(U \cap Y) \cap (V \cap Y) = \emptyset$ by commutatitvity and assocoativity of $\cap$. Therefore we conclude that supspace generated by Y is a Hausdorff. Since Y is arbitrary, we follow that any subspace of X is Hausdorff, as desired.

2.6.13

Show that X is Hausdorff iff diagonal $\Delta = \{x \times x : x \in X\}$ is closed in $X \times X$.

Since X is Hausdorff, we follow that for all $x \in X$ we've got that $\{\langle x, x\rangle\}$ is closed.

Assume that $\Delta \neq \overline{\Delta}$ and thus there exists $\langle y_1, y_2 \rangle \in \overline{\Delta} \setminus \Delta$ We follow that since $\langle y_1, y_2 \rangle \notin \Delta$ that $y_1 \neq y_2$. We also follow that $y_1, y_2 \in X$. Since X is Hausdorff, we follow that $y_1, y_2 \in X$ implies that there exist neighborhoods $U, V \subseteq X$ of y_1 and y_2 respestively, such taht $U \cap V = \emptyset$. Thus we follow that $U \times V$ is an open set in $X \times X$. Then we follow that since $\langle y_1, y_2 \rangle \in \overline{\Delta}$ that there exists $d \in \Delta$ such that $d \in U \cap V$. Thus we follow that there exists $d_1 \in X$ such that $\langle d_1, d_1 \rangle = d$. Therefore $\langle d_1, d_1 \rangle \in U \times V$ and therefore $d_1 \in U \wedge d_1 \in V$, which implies that $d_1 \in U \cap V$, which is a contradiction, since $U \cap V = \emptyset$ by the fact that X is Hausdorff, as desired.

Therefore we conclude that $\overline{\Delta} \setminus \Delta = \emptyset$, therefore $\Delta = \overline{\Delta}$ and Δ is a closed set, thus giving us forward implication.

Now assume that Δ is a closed set and X is not Hausdorff. Since X is not Hausdorff, we follow taht there exists $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and for all $U \in \mathcal{T}$ we've got that $x_1 \in U \Rightarrow x_2 \in U$.

Let B be an arbitrary basis element that contains $\langle x_1, x_2 \rangle$. We follow that since it is a basis element, that $B = U \times V$, where U, V are open subsets of X. We follow that $x_1 \in U$ and $x_2 \in V$, that implies that $x_1, x_2 \in U$ and $x_1, x_2 \in V$. Thus we follow that $\langle x_1, x_2 \rangle \in B$. Therefore we follow that if B is a basis element, that contains $\langle x_1, x_2 \rangle$, then B intersects Δ by the fact that $\langle x_1, x_1 \rangle$ and $\langle x_2, x_2 \rangle$ are both in B. Thus we follow that $\langle x_1, x_2 \rangle \in \overline{\Delta}$. Since $x_1 \neq x_2$ we follwo that $\langle x_1, x_2 \rangle \notin \Delta$, which implies that $\Delta \neq \overline{\Delta}$, which implies that Δ is not closed, which is a contradiction. Thus we follow that if Δ is a closed set, then X is Hausdorff, which gives us reverse implication, as desired.

2.6.14

In the finite complement topology on R, to what point or points does the sequence $x_n = 1/n$ converge?

We can follow pretty easily that finite topology on R is not Hausdorff by the fact that if U, V are nonempty open sets in finite complement topology, then $U = R \setminus S_1$, $V = R \setminus S_1$ for some finite subsets S_1, S_2 of R. Thus we follow that $U \cap V = R \setminus S_1 \cap R \setminus S_2 = R \setminus (S_1 \cup S_2)$, where we follow that $S_1 \cup S_2$ is finite and therefore $R \setminus (S_1 \cup S_2)$ is infinite and therefore nonempty.

Assume that U is a neighborhood of 0. We follow that $U = R \setminus S_1$ such that $0 \notin S_1$. and S_1 is finite. Since x_n is an infinite sequence, we follow that there must exist n such that $n_0 > n \Rightarrow x_n \notin S_1 \Rightarrow x_n \in U$. Thus we conclude that x_n converges to 0.

By the same logic we follow that if U is a neighborhood of any other number, then the same logic applies. Therefore we follow that x_n converges to any number in finite complement topology.

2.6.15

Show that T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

Let $x_1, x_2 \in X$ are such that $x_1 \neq x_2$. T_1 axiom implies that $\{x_1\}$ and $\{x_2\}$ are closed and thus $X \setminus \{x_1\}$ and $X \setminus x_2$ are open. Since $x_1 \neq x_2$, we follow that $x_2 \in X \setminus \{x_1\}$ and $x_1 \in X \setminus \{x_2\}$, thus implying that each point has a neighborhood, that does not contain the other point.

2.6.16

Consider the five topologies on R given in Exercise 7 of paragraph 13.

(a) Determine the closure of the set $K = \{1/n : n \in Z_+\}$ under each of these topologies.

Under standard topology we follow that $\overline{K} = K \cap \{0\}$ from real analysis course.

Under K topology we follow that R is open by default, and thus $R \setminus K$ is a basis element, therefore it is open and thus $K = R \setminus (R \setminus K)$ is closed.

Under finite complement topology we follow that all closed sets that are not R are finite, and thus only R contains K. Therefore $\overline{K} = R$.

Since the upper limit topology is finer than the K-topology, we follow that $R \setminus K$ is an open set in upper limit topology, and thus $\overline{K} = K$.

For the topology that has sets $(-\infty, a)$ as a basis, we follow that closed sets there are either R or $[a, \infty)$ for some $a \in R$, and thus $\overline{K} = [0, \infty)$.

(b) Which of these topologies satisfy the Hausdorff axiom? The T_1 axiom?

We follow that standard topology satisfies the Hausdorff axiom, as proven in the chapter or in the real analysis course. Thus we follow that since standard topology is coarser than K-topology and upper limit topology, that both of them satisfy the axiom as well. In exercise 14 of this section we've shown that finite complement topology is definetly not Hausdorff, and for the last topology we follow that if $0 \in U$, then $-1 \in U$, therefore it is not Hausdorff as well.

In the chapter we've discussed that T_1 axiom is weaker than Hausdorff, thus we follow that standard topology, K-topology and upper limit topology all satisfy the T_1 axiom. We can follow that finite complement topology satisfies T_1 by some trivial implications and the last topology is definetly does not satisfy T_1.

2.6.17

Consider teh lower limit topology on R and the topology given by the basis C of exercise 8 of paragraph 13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

Let's talk about lower limit topology first.

Suppose that $0 \in [a, b)$. We follow that $a \leq 0 < b$, and by density of reals we follow that there exists $d \in R$ such that $0 < d < b$ and thus $d \in [a, b)$. Thus we follow that every basis element that contains 0 also intersects A, and thus $0 \in \overline{A}$. We follow that $[\sqrt{2}, 2)$ is a neighborhood of $\sqrt{2}$ such that it does not intersect A at any point. Thus we follow that $\sqrt{2} \notin \overline{A}$. There're plenty of methods to state that if $x < 0$ or $x > \sqrt{2}$, then it is not in closure of A, therefore we conclude that $[0, \sqrt{2})$ is clousre of A under lower limit topology.

For B we follow that pretty much the same logic holds with respect to different numbers, and thus $\overline{B} = [\sqrt{2}, 3)$.

Now let's talk about the weird topology.

We can follow that pretty much the same argument holds for $0 \in \overline{A}$. Now assume that $a \leq \sqrt{2} < b$. We follow that since $a \notin I$, that $a \neq \sqrt{2}$, and thus we follow that $a < \sqrt{2} < b$. Thus we follow that $\sqrt{2} \in \overline{A}$. If $\sqrt{2} < c$, then we follow that there exist rational a, b such that $\sqrt{2} < a < c < b$, and thus $c \notin \overline{A}$. If $c < 0$, then we have pretty much the same result. Therefore we follow that $\overline{A} = [0, \sqrt{2}]$.

For the case of $B = (\sqrt{2}, 3)$, we follow that $[3, 5) \cap B = 0$, thus we follow that $3 \notin \overline{B}$. We can easily follow that $\sqrt{2} \in \overline{B}$, and the rest is followed by pretty much the same logic as in the previous paragraph. Therefore we conclude that $\overline{B} = [\sqrt{2}, 3)$.

2.6.18

Determine the closures of the following subsets of the ordered square:

We're talking about the lexicographical order on the set $I \times I$ for $I = [0, 1]$. We're also not gonna use the dumb notation.

$A = \{\langle 1/n, 0\rangle : n \in Z_+\}$

We follow that $A \subseteq \overline{A}$. Let $\langle x_1, x_2\rangle \in I \times I$.

Assume that $x_1 \neq 0$. If $x_1 = 1/n$ for some $n \in Z_+$ and $x_2 = 0$, then we follow that $\langle x_1, x_2\rangle \in A$, thus assume that $x_2 \neq 0$. We follow that if $x_2 > 0$ and $x_2 < 1$, then there exist $a, b \in R$ such that $0 < a < x_2 < b < 1$ and thus $\langle x_1, x_2\rangle \in (\langle x_1, a\rangle, \langle x_1, b\rangle)$, where we follow that $(\langle x_1, a\rangle, \langle x_1, b\rangle)$ does not intersect A at any point.

If $x_1 = 1/n$ and $x_2 = 1$ then we follow that there exists some space between x_1 and the previous point in the sequence, therefore we can have interval $[x_1, y)$ such that $A \cap [x_1, y) = \{x_1\}$. Thus if we take open inerval $(\langle x_1, 0.5\rangle, \langle y_1, 0.5\rangle)$, then $(\langle x_1, 0.5\rangle, \langle y_1, 0.5\rangle) \cap A = \emptyset$. Therefore we follow that $\{1/n\} \times I \cap A \subseteq A$. We can follow pretty much the same thing for $x_1 \neq 0$ in a more general case.

Now assume that $x_1 = 0$. For the point $\langle 0, 0\rangle$ we've got that $(\langle 0, 0\rangle, \langle 0, 0.5\rangle) \cap A = 0$, therefore we conclude that $\langle 0, 0\rangle \notin \overline{A}$. Simple logic implies that for $x_2 \neq 1$ we've got pretty much the same result.

The case with $\langle 0, 1\rangle$ is an interesting one. Suppose that we've got an element of the basis $(\langle j_1, j_2\rangle, \langle k_1, k_2\rangle)$ such that

$$\langle j_1, j_2\rangle < \langle 0, 1\rangle < \langle k_1, k_2\rangle$$

We can follow that $j_1 = 0$ by the definition of order. We also follow that $0 < k_1$. Thus

$$\langle 0, j_2\rangle < \langle 0, 1\rangle < \langle k_1, k_2\rangle$$

we can follow that for any $k_1 > 0$ there exists $n \in Z_+$ such that $0 < 1/n < k_1$, and therefore

$$\langle 0, j_2 \rangle < \langle 1/n, 0 \rangle < \langle k_1, k_2 \rangle$$

Therefore we follow that if B is an element of the basis such that $\langle 0, 1 \rangle \in B$, then it intersects A, and thus $\langle 0, 1 \rangle$ is the only point outside of A that is in the closure of A. Thus we follow that

$$\overline{A} = A \cap \{\langle 0, 1 \rangle\}$$

$B = \{1 - 1/n) \times 1/2 : n \in Z_+\}$

$$\overline{B} = B \cup \{\langle 1, 1 \rangle\}$$

$C = \{x \times 0 : 0 < x < 1\}$

$$\overline{C} = C \cup \{\langle x, 1 \rangle : 0 \leq x \leq < 1\}$$

$D = \{x \times 1/2 : 0 < x < 1\}$

$$\overline{D} = D \cup \{\langle x, 1 \rangle : 0 \leq x \leq 1\} \cup \{\langle x, 0 \rangle : 0 < x < 1\}$$

$E = \{1/2 \times x : 0 < x < 1\}$

$$\overline{E} = E \cup \{\langle 1/2, 0 \rangle, \langle 1/2, 1 \rangle\}$$

last answers are probably wrong, but I just want to move on.

2.6.19

If $A \subseteq X$, we define the boundary of A by the equation

$$BdA = \overline{A} \cap \overline{(X - A)}$$

(a) Show that $IntA$ and BdA are disjoint and $\overline{A} = IntA \cup BdA$.

Suppose that $x \in IntA$. We follow that there exists open set U such that $x \in U$ and $U \subseteq A$. We follow that U and $X \setminus A$ are disjoint. Suppose that $x \in \overline{X \setminus A}$. Then we follow that every neighborhood of x intersects $X \setminus A$. Since U is a neighborhood of x we follow that $U \cap X \setminus A \neq \emptyset$. Therefore there exists $j \in U$ such that $j \in X \setminus A$. Therefore $j \in U$ and $j \notin A$. This contradicts the fact that $U \subseteq A$. Thus we follow that if $x \in IntA$ then $x \notin \overline{X \setminus A}$. Thus we follow that $IntA \cap \overline{X \setminus A} = \emptyset$. Now we can follow that

$$BdA \cap IntA = \overline{A} \cap \overline{X \setminus A} \cap IntA = \overline{A} \cap \emptyset = \emptyset$$

We can also follow that $BdA \subseteq \overline{A}$ by definition and $IntA \subseteq \overline{A}$ by the fact that $IntA \subseteq A \subseteq \overline{A}$. Therefore we follow that $IntA \cup BdA \subseteq \overline{A}$.

Now assume that $x \in \overline{A}$ and $x \notin IntA$. Since $x \notin IntA$ we follow that there is no neighborhood U of x such that $U \subseteq A$. We follow that every neighborhood U of x has an element $y \in U$ such that $y \notin A$. And since $U \subseteq X$, we follow that $U \cap (X \setminus A) \neq \emptyset$ for every neighborhood of x. Thus we follow that $x \in \overline{(X \setminus A)}$. And since $x \in \overline{A}$, we follow that $x \in BdA$ and thus

$$(\forall x \in \overline{A})(x \notin IntA \Rightarrow x \in BdA)$$

Thus we follow that $\overline{A} \setminus BdA = IntA$ and since both of BdA and $IntA$ are subsets of A we conclude that $\overline{A} = BdA \cup IntA$, as desired.

(b) Show that $BdA = \emptyset \Leftrightarrow A$ is both open and closed.

If $BdA = \emptyset$ we follow that $A = IntA$ and thus it is open. We also follow that $\overline{A} = BdA \cup IntA = IntA = A$ since $IntA \subseteq A \subseteq \overline{A}$, and thus A is closed as well, as desired.

If A is both opened and closed we follow that $IntA = A$ and $\overline{A} = A$, thus $BdA = \emptyset$.

(c) Show that U is open $\Leftrightarrow BdU = \overline{U} \setminus U$.

If U is open, then we follow that $X \setminus U$ is closed and thus $\overline{X \setminus U} = X \setminus U$. This implies that $BdU = \overline{U} \cap (X \setminus U)$ and since $\overline{U} \subseteq X$ and $U \subseteq \overline{U}$ we follow by some identity with a $\setminus$ that $BdU = \overline{U} \setminus U$, as desired.

Suppose that $BdU = \overline{U} \setminus U$. Since $\overline{U} = BdU \cup IntU$ and $IntU \cap BdU = \emptyset$, we follow that $IntU = \overline{U} \setminus BdU$. Thus $IntU = \overline{U} \setminus (\overline{U} \setminus U)$ and since $U \subseteq \overline{U}$ we follow that $IntU = U$, thus proving that U is open, as desired.

(d) If U is open, is it true that $U = Int\overline{U}$? Justify your answer.

Suppose that $x \in U$. We follow that there exists open $V \subseteq U$ such that $x \in V \subseteq U$. Therefore $V \subseteq \overline{U}$. Thus we follow that $V \subseteq Int\overline{U}$ and thus $x \in Int\overline{U}$. Therefore we follow that $U \subseteq Int\overline{U}$.

If $x \in Int(\overline{U})$. We follow that there exists V such that $x \in V \subseteq \overline{U}$. If $x \notin U$, then we follow nothing.

Let $U = R \setminus \{0\}$ and assume stanadrd topology. We follow that $\overline{U} = R = Int\overline{U} \neq U$, which gives us a solid contradiction of the reverse inclusion.

2.6.20

Skip

2.6.21

(Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \overline{A}$ and complementation $A \to X \setminus A$ are functions from this collection to itself.

(a) Show that starting with a given set A, ine can form no more than 14 distinct sets by applying therse two operations successively.

We follow that if $X \neq \emptyset$, then $A \neq X \setminus A$. We also know that $A = X \setminus (X \setminus A)$

Let $A = [0, 2] \setminus \{1\}$ so that it is neither open nor closed.

$$\overline{A} = [0, 2]$$
$$X \setminus \overline{A} = (-\infty, 0) \cup (2, \infty)$$
$$\overline{X \setminus \overline{A}} = (-\infty, 0] \cup [2, \infty)$$
$$X \setminus \overline{X \setminus \overline{A}} = (0, 2)$$
$$\overline{X \setminus \overline{X \setminus \overline{A}}} = [0, 2]$$

$$A = [0, 1) \cup \{2\} \cup (3, 4]$$
$$\overline{A} = [0, 1] \cup \{2\} \cup [3, 4]$$
$$X \setminus \overline{A} = (-\infty, 0) \cup (1, 2) \cup (2, 3) \cup (4, \infty)$$
$$\overline{X \setminus \overline{A}} = (-\infty, 0] \cup [1, 3] \cup [4, \infty)$$
$$X \setminus \overline{X \setminus \overline{A}} = (0, 1) \cup (3, 4)$$
$$\overline{X \setminus \overline{X \setminus \overline{A}}} = [0, 1] \cup [3, 4]$$
$$X \setminus \overline{X \setminus \overline{X \setminus \overline{A}}} = (-\infty, 0) \cup (1, 3) \cup (4, \infty)$$
$$\overline{X \setminus \overline{X \setminus \overline{X \setminus \overline{A}}}} = (-\infty, 0] \cup [1, 3] \cup [4, \infty)$$

SKIP for now, probably will come back later

2.7 Continous Functions

2.7.1

Prove that for functions $f : R \to R$, the $\epsilon - \delta$ definition of continuity implies the open set definition.

Let f be continous with respect to the $\epsilon - \delta$ definition of continuity. Let (a, b) be an element of basis of R. If $(a, b) \cap \operatorname{ran}(f) = \emptyset$, then we follow that $f^{-1}[(a, b)] = \emptyset$, which is an open set.

Suppose that $(a, b) \cap \operatorname{ran}(f) \neq \emptyset$. Let $y \in (a, b) \cap \operatorname{ran}(f)$. Since $y \in (a, b)$, we follow that there exists ϵ such that $V_\epsilon(y) \subseteq (a, b)$.

Let $x \in f^{-1}[\{y\}]$. For that $V_\epsilon(y)$ there exists $\delta > 0 \in R$ with corresponding $V_\delta(x)$ such that $z \in V_\delta(x) \Rightarrow f(z) \in V_\epsilon(y)$. By AC (not sure that we actually need AC at this point, but we're doing topology, so why not) we follow that for each y we can pick exclusive δ such that everything holds. Define $K : (a, b) \to \mathcal{P}(R)$ by $K(y) = \bigcup_{x \in f^{-1}[\{y\}]} V_\delta(x)$ in case that $y \in \text{ran}(f)$ and empty set otherwise. Since $V_\delta(x)$ are all open intervals and empty sets are also open, we follow that for all $y \in (a,b)$ $K(y)$ is an open set. Moreover, we follow that $\bigcup \text{ran}(K)$ is an open set as well. By definition of K and $\epsilon - \delta$ continuity of f we follow that $x \in \bigcup \text{ran}(K) \Rightarrow f(x) \in (a,b)$

Now let $x \in R$ be such that $x \in f^{-1}[(a,b)]$. We follow that $x \in \bigcup \text{ran}(K)$ by definition. Therefore we conclude that $f^{-1}[(a,b)] = \bigcup \text{ran}(K)$. As proven earler, $\bigcup \text{ran}(K)$ is an open set, and therefore we conclude that if f is $\epsilon - \delta$-continous, then for arbitrary interval (a,b) we've got that $f^{-1}[(a,b)]$ is open, and thus f is continous according to our definition. Taking into account stuff that we were given in the chapter, we follow that f is $\epsilon - \delta$-continous if and only if f is continous, as desired.

2.7.2

Suppose that $f : X \to Y$ is continous. If x is a limit point of the subset A of X, is it necessary true that $f(x)$ is a limmit point of $f(A)$?

Short answer: no.

Let $U \subseteq Y$ be a neighborhood of $f(x)$. We follow that $f^{-1}[U]$ is open and $x \in f^{-1}[U]$. Thus we follow that there exists a point $u \in f^{-1}[U]$ such that $u \neq x$ and $u \in A$. Thus $f(u) \in f[A]$. This means that we can't follow crap on the account that f might not be injective.

Let $f : R \to R$ be defined by $f(x) = 5$ and let us assume standard topology. For $A = (0,1)$ we follow that $x = 0$ is a limit point of A, but $\text{ran}(f) = \{5\}$ and it doesn't have no limit points. Thus we've got a contradiction of our conjecture.

2.7.3

Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Let $X' \to X$ be the identity function.

Just to be clear: we've assumed that we've got two topological spaces $\langle X, \mathcal{T} \rangle$ and $\langle X', \mathcal{T}' \rangle$ such that $X = X'$, but $\mathcal{T}$ might be different to $\mathcal{T}'$.

(a) Show that i is continous $\Leftrightarrow \mathcal{T} \subseteq \mathcal{T}'$

Suppose that i is continous. Let $U \in \mathcal{T}$. Thus U is an open set in X. We follow by continuity of i that $i^{-1}[U] = U$ is an open set in $\mathcal{T}'$. Thus we follow that $U \in \mathcal{T}'$. Thus we follow that $\mathcal{T} \subseteq \mathcal{T}'$, as desired.

Same logic in reverse gets us the reverse implication.

(b) Show that i is a homeomorphism $\Leftrightarrow \mathcal{T}' = \mathcal{T}$.

If i is a homeomorphism, then we follow that $i^{-1} : X \to X'$ is continous. Previous point implies that $\mathcal{T}' = \mathcal{T}$.

If $\mathcal{T}' = \mathcal{T}$, then previous point implies that both i and i^{-1} are continous and thus i is an homeomorphism.

2.7.4

Given $x_0 \in X$ *and* $y_0 \in Y$, *show that the maps* $f : X \to X \times Y$ *and* $g : Y \to X \times Y$ *defined by*

$$f(x) = x \times y_0$$

$$g(x) = x_0 \times y$$

are imbeddings