My linear algebra exercises

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Preface

Exercises are from "Linear algebra done right" by Sheldon Axler, 3rd ed. I've already read this book before and completed some exercises from it. Right now I want to brush up the material once again, put all the proofs on a more durable material than paper and to prepare myself to what's gonna happen afterwards.

Glossary

FTLM - Fundamental Theorem of Linear Maps
lhs - left hand side
rhs - right hand side
GSP - Gram-Schmidt Procedure
RRT - Riesz Representation Theorem
PD - polar decomposition
SVD - singular value decomposition
IH - inductive hypothesis
N is defined to be $Z \cap [1, \infty)$.

Chapter 1

Vector Spaces

1.1 R^n and C^n

1.1.1

Suppose a and b are real numbers, not both 0. Find real nuber c and d such that

$$1/(a+bi) = c+di$$

$$\frac{1}{a+bi} = c+di$$

$$\frac{1}{a+bi} - c - di = 0$$

$$\frac{a-bi}{(a+bi)(a-bi)} = c+di$$

$$\frac{a-bi}{(a^2+b^2)} = c+di$$

$$\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = c+di$$

Thus $c = \frac{a}{a^2+b^2}$ and $d = -\frac{b}{a^2+b^2}$

1.1.2

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 *(meaning that its cube equals 1)*

$$(\frac{-1+\sqrt{3}i}{2})^3 = \frac{(-1+\sqrt{3}i)^3}{8} = \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)^2}{8} = \frac{(-1+\sqrt{3}i)(1-2\sqrt{3}i-3)}{8} =$$

$$= \frac{(-1+\sqrt{3}i)(-2-2\sqrt{3}i)}{8} = \frac{2+2\sqrt{3}i-2\sqrt{3}i+6}{8} = \frac{8}{8} = 1$$

as desired.

1.1.3

Find two distinct square roots of i

Square root of i, I assume, is a number, whose square is equal to i. Suppose that $(a+bi)^2 = i$. It follows that

$$(a+bi)^2 = a^2 + 2abi - b^2$$

So if we set

$$a = b = 1/\sqrt{2}$$

this equation holds. Also it holds for

$$a = b = -1/\sqrt{2}$$

maxima seems to agree with me on this one

1.1.4

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$ and $\beta = a_2 + b_2 i$. It follows

$$\alpha + \beta = a_1 + b_1 i + a_2 + b_2 i = a_2 + b_2 i + a_1 + b_1 i = \beta + \alpha$$

as desired.

1.1.5

Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$, $\beta = a_2 + b_2 i$, $\lambda = a_3 + b_3 i$. It follows that

$$\alpha + (\beta + \lambda) = a_1 + b_1 i + (a_2 + b_2 i + a_3 + b_3 i) = (a_1 + b_1 i + a_2 + b_2 i) + a_3 + b_3 i = (\alpha + \beta) + \lambda$$

1.1.6

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

$$\alpha + (\beta + \lambda) = (a_1 + b_1 i)((a_2 + b_2 i) + (a_3 + b_3 i)) = ((a_1 + b_1 i)(a_2 + b_2 i)) + (a_3 + b_3 i) = (\alpha\beta)\lambda$$

1.1.7

Show that for every $\alpha \in \boldsymbol{C}$ there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha + \beta = 0$

Suppose that there exist two different $\beta_1 \neq \beta_2$ such that $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$. It follows that

$$\beta_1 = \beta_1 + 0 = \beta_1 + \alpha + \beta_2 = \alpha + \beta_1 + \beta_2 = 0 + \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

1.1.8

Show that for every $\alpha \in \boldsymbol{C}$ with $\alpha \neq 0$ there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha\beta = 1$

Suppose that it is not true and there exist two different $\beta_1 \neq \beta_2$ such that

$$\alpha\beta_1 = 1 \textit{ and } \alpha\beta_2 = 1$$

it follows then that

$$\beta_1 = 1 * \beta_1 = \alpha\beta_2\beta_1 = \alpha\beta_1\beta_2 = 1 * \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

1.1.9

The rest of the section is the repetition of this kind of stuff. That is a lot of writing, and not a lot of thinking, so I'll skip it. I don't ususally like to skip sections, but I have aa feeling, that I've completed this thing on paper somewhere, and there is not much reason to rewrite it here.

1.2 Definition of Vector Space

1.2.1

Prove that $-(-v) = v$ for every $v \in V$.

For v there exists only one $-v$. For $-v$ there exists only one $-(-v))$.

Thus

$$v = v + 0 = v + (-v) + (-(-v)) = 0 + (-(-v)) = -(-v)$$

as desired (idk if it's true, I'm not good at axioms and stuff)

1.2.2

Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Suppose that $a \neq 0$, $v \neq 0$ but $av = 0$. It follows that there exist $1/a$ - multiplicative inverse of a. It follows that

$$1/a * av = 1/a * 0$$

$$1v = 0$$

$$v = 0$$

which is a contradiction. Thus either $a = 0$ or $v = 0$.

1.2.3

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Suppose that there exists $x_1 \neq x_2$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$. Thus

$$3x_1 = w - v = 3x_2$$

$$x_1 = \frac{1}{3}(w - v) = x_2$$

which is a contradiction.

Same can be stated from the fact that x is a unique additive inverse of $\frac{1}{3}(v - w)$.

1.2.4

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Additive indentity. Empty set does not have zero element in it. BTW $\{0\}$ is a vector space.

1.2.5

Show that n the defintition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \textit{ for all } v \in V$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

$$0v = 0$$

$$(1 - 1)v = 0$$

$$1v - 1v = 0$$
$$v - v = 0$$
$$v + (-v) = 0$$

1.2.6

Let ∞ and $-\infty$ denote two distinct object, neither of which is in R. Define an addition and multiplication on $R \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and the product of two real numbers is as usual, and for $t \in R$ define

$$t\infty = \begin{cases} -\infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ \infty \text{ if } t > 0 \end{cases}$$

$$t(-\infty) = \begin{cases} \infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ -\infty \text{ if } t > 0 \end{cases}$$

$$t + \infty = \infty + t = \infty$$
$$t + (-\infty) = (-\infty) + t = (-\infty)$$
$$\infty + \infty = \infty$$
$$(-\infty) + (-\infty) = (-\infty)$$
$$\infty + (-\infty) = 0$$

Is $R \cup \{\infty\} \cup \{-\infty\}$ a vector space over R? Explain.

I don't think that it is.

$$(t + \infty) - \infty = \infty - \infty = 0$$
$$t + (\infty - \infty) = t + 0 = t$$

thus

$$t + (\infty - \infty) \neq (t + \infty) - \infty$$

thus $R \cup \{\infty\} \cup \{-\infty\}$ is not associative, therefore it is not a vector space.

1.3 Subspaces

1.3.1

For each of the following subsets of F^3, determine whether it is a subspace of F^3:

(a) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$

Yes, it is. 0 is contained within it.

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

therefore

$$x_1 + y_1 + 2(x_2 + y_2) + 3(x_3 + y_3) = x_1 + 2x_2 + 3x_3 + y_1 + 2y_2 + 3y_3 = 0 + 0 = 0$$

therefore it is closed under addition

$$n(x_1, x_2, x_3) = (nx_1, nx_2, nx_3)$$

$$nx_1 + 2nx_2 + 3nx_3 = n(x_1 + 2x_2 + 3x_3) = 0n = 0$$

therefore it is closed under multiplication.

(b) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$

It's not a subspace, because it does not contain zero.

(c) $\{(x_1, x_2, x_3) \in F^3 : x_1 x_2 x_3 = 0\}$

It's not a subspace, because

$$(0, 1, 1) + (1, 0, 0) = (1, 1, 1)$$

therefore it's not closed under addition.

(d) $\{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}$

It's a subspace, proof is the same as in (a), can be seen more clearly when we rewrite constraint as

$$x_1 = 5x_3 \rightarrow x_1 + 0x_2 - 5x_3 = 0$$

1.3.2

Verify all the assertions in Example 1.35

(a) if $b \in F$, then

$$\{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

is a subspace of F^4 if and only if $b = 0$

If $b \neq 0$, then 0 is not an element of this set.

Proving that it's a subspace when $b = 0$ is trivial

(b) The set of continous real-valued functions on the interval $[0, 1]$ is a subspace of $R^{[0,1]}$.

$(kf) = kf$ by algebraic properties of continous functions. If f and g are continous, then $(f+g)$ is continous as well by the same property. $f(x) = 0$ is continous because it's a constant functions.

By the way, same (probably) applies to a set of uniformly continous functions.

(c) The set of differentiable real-valued functions on R is a subspace of R^R.

Same deal, algebraic proerties imply linearity, adn zero is included.

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $R^{(0,3)}$ if and only if $b = 0$.

Same deal as in previous one, $f'(2)$ needs to be equal to zero in order to include zero. Previous part does not include it, because it does not have specific restrictions on derivatives being particular values at particular places.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of C^∞.

Here we can take zero to be $(x_n) = 0$. Linearity is implied by aldebraic properties of limits of sequences.

1.3.3

Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(1) = 3f(2)$ is a subspace of $R^{[-4,4]}$.

Zero is included here. Suppose that f and g are functions in given set. It follows that

$$f'(1) + g'(1) = 3f(2) + 3g(2)$$

$$f'(1) + g'(1) = 3(f(2) + g(2))$$

$$(f+g)'(1) = 3(f+g)(2)$$

thus it's closed under addition.

$$(kf)'(1) = 3(kf)(2)$$

implies

$$kf'(1) = 3kf(2)$$

therefore it's closed under multiplication by scalar. Therefore we can state that given subset is a vector subspace.

1.3.4

analogous to previous

1.3.5

Is R^2 a subspace of the complex vector space C^2?

No, it's not closed under scalar multiplication.

1.3.6

(a) Is

$$\{(a,b,c) \in R^3 : a^3 = b^3\}$$

a subspace if R^3?

Yes. it is. $a^3 = b^3 \to a = b \to a - b = 0$, the rest of proof is trivial.

(b) Is

$$\{(a,b,c) \in C^3 : a^3 = b^3\}$$

a subspace if C^3?

I want to say no to this one, example is

$$(1/2 + i\frac{\sqrt{3}}{2}, -1, 0) + (1/2 - i\frac{\sqrt{3}}{2}, -1, 0) = (1, -1, 0)$$

thus it's not closed under additon.

1.3.7

Give an example of a nonemplty subset U of R^2 such that U is closed under addition and under additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of R^2

Q^2. On the other thouhgh, Z will do as well.

1.3.8

Give an example of a nonempty subset U of R^2 such that U is closed under scalar multiplication, but U is not a subspace of R^2.

Two lines through origin.

1.3.9

A function is called periodic if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in R$. Is the set of periodic functions from R to R a subspace of R^R? Explain.

Zero is a periodic function. Set is certainly closed under scalar multiplication.

Suppose that f and g are both periodic and f has a period of $p1$ and g has a period of $p2$. Thus if $p2/p1 \in I$, then functions will be constantly out of phase, therefore the set is not closed under addition. Thus this subset is not a subspace.

1.3.10

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Zero is included in any subspace, therefore zero is included.

Suppose that $u_1, u_2 \in U_1 \cap U_2$. It follows that for $z \in F$ $zu_1 \in U_1$ and $zu_1 \in U_2$ by closure of those two subspaces. Therefore $zu_1 \in U_1 \cap U_2$ for any scalar, thus the set is closed under scalar multiplication.

$u_1 + u_2 \in U_1$ and $u_1 + u_2 \in U_2$ by closure under addition for both subspaces. Thus $u_1 + u_2 \in U_1 \cap U_2$ for any such vectors. Therefore the set is closed under addition.

Thus the set satisfies all requirements to be a subspace. Therefore it is a subspace.

1.3.11

Prove that the interseection of every collection of subspace of V is a subspace of V

Intersection of two subspaces is a subspace. Therefore by induction intersection of any finite collection of subspaces is a subspace.

Suppose that Λ is an arbitrary collection of subspaces. Every subspace contains a zero element, therefore

$$0 \in \cap\Lambda$$

Any vector in $\cap\Lambda$ will be closed under scalar multiplication for every $U \in \Lambda$. Thus, it will be contained in every $U \in \Lambda$. Therefore it is contained in $\cap\Lambda$.

Any two vectors in $\cap\Lambda$ will be closed under addition, for every $U \in \Lambda$. Thus, their sum will be contained in every $U \in \Lambda$. Therefore it is contained in $\cap\Lambda$.

Thus $\cap\Lambda$ is a vector space.

1.3.12

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Suppose that a union of two subspaces $U_1 \cup U_2$ is a subspace of V.

Zero is included in eveery subspace, so in case of the union we don't worry about it. Scalar multiplication is also trivial, as we are working only with one vector.

Now for the interesting part: addition. Let $u_1, u_2 \in U_1 \cup U_2$. In case when u_1, u_2 are contained only in one subspace we've got a trivial case. Interesting part comes when $u_1 \in U_1$ and $u_2 \in U_2$.

What we want to prove is that it is impossible to have $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$ and we're going to use contradiction. Suppose that $u_1 \in U_1 \setminus U_2$, $u_2 \in U_2 \setminus U_1$ and $u_1 + u_2 \in U_1 \cup U_2$. Thus it must be the case that $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$. Suppose that the former is true; then it follows that $u_1 + u_2 - u_1 = u_2 \in U_1$, which is a contradiction (same thing happens if we assume the latter). Thus given case is impossible. Therefore there cannot exist $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. Thus $U_1 = U_1 \cup U_2$ or $U_2 = U_1 \cup U_2$.

The reverse case is trivial: if we have two subspaces and one of it is a subset of another, then larger subspace is is subspace.

1.3.13

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Same thing applies as in previous exercise: zero and multiplication are trivial.

We are going to proceed with a proof by contradiction, but firstly we want to state precisely what we want to prove in a first place. We want to state, that if a union of three subspaces is a subspace, then this union is equal to one of the subpsaces. So let us start: suppose that the union of three subspaces is not equal to one of the subspaces.

Firstly, we can eliminate the case, when one of the subspaces is a subset of another subspace, but third isn't, because it will mean that union of first two subspaces constitues a subspace, and thus we'll default to result in the previous exercise.

Thus let us assume that none of the subspaces is a subset of another subspace. Now we've got two cases to sort out: suppose that if we take $u_2 \in U_2$ and $u_3 \in U_3$ we get that

$$u_2 + u_3 \in U_1$$

for every $u_2 \in U_2$ and $u_3 \in U_3$. Then we can follow, by setting $u_2 = 0$ to the case that

$$\forall u_3 \in U_3 \to u_3 + u_2 \in U_1 \to u_3 + 0 \in U_1 \to u_3 \in U_1$$

thus U_3 is a subset of U_1, which raises a contradiction (in our assumptions that U_3 is not a subset of U_1 and by extension for the default 2-subspace case).

The case when $u_2 \in U_2$, $u_3 \in U_3$ and $u_2 + u_3 \notin U_1 \cup U_2 \cup U_3$ implies that $U_1 \cup U_2 \cup U_3$ is not a vector space, thus it cannot happen.

The case when $u_2 \in U_2$, $u_3 \in U_3$ and $u_2 + u_3 \notin U_1$ implies that $u_2 + u_3$ is in $U_2 \cup U_3$. This raises the case that U_2 is a subspace of U_3, which is a contradiction.

Thus we can follow that there exists $u_1 \in U_1$ such that it cannot be represented in terms of vectors from U_2 and U_3. Thus we can follow that analogous vectors $u_2 \in U_2$ and $u_3 \in U_3$ also exist.

Because we are still assuming that $U_1 \cup U_2 \cup U_3$ we can follow that

$$u_1 + u_2 + u_3 \in U_1 \cup U_2 \cup U_3$$

Thus this sum is bound to be located in one of the U_1, U_2 or U_3. Let us assume for simplicity of notation that it is located in U_1. Then we can follow that

$$u_1 + u_2 + u_3 - u_1 = u_2 + u_3 \in U_1$$

Suppose that we take $u_2 \in U_2 \setminus (U_3 \cup U_1)$ and $u_3 \in U_3 \setminus (U_1 \cup U_2)$. It follows that $u_2 + u_3$ cannot be in either U_2 nor in U_3 because in this case we have that

$$u_2 + u_3 - u_2 = u_3 \in U_2$$

which is a contradiction. Thus

$$u_2 + u_3 \in U_1 \setminus (U_2 \cup U_3)$$

let us call it u_1'. In the same fashion we can define u_2' and u_3'.

Thus $u_1' + u_2' + u_3' \in U_1 \cup U_2 \cup U_3$. Thus it needs to be in one of U_1, U_2 or U_3. Suppose that it is included in U_1. Then we can follow that

$$u_1' + u_2' + u_3' \in U_1$$

$$u_2' + u_3' \in U_1$$

$$u_1 + u_3 + u_1 + u_2 \in U_1$$

$$2u_1 + u_3 + u_2 \in U_1$$

$$u_3 + u_2 \in U_1$$

TODO

1.3.14

Verify the assertion in Example 1.38

1.38 states that

Suppose that $U = \{(x, x, y, y) \in F^4 : x, y \in F\}$ *and* $W = \{(x, x, x, y) \in F^4 : x, y \in F\}$. *Then*

$$U + W = \{(x, x, y, z) \in F^4 : x, y, z \in F\}$$

as you should verify

Let $u \in U$ and $w \in W$. It follows that

$$u = (x_1, x_1, x_1, y_1)$$

$$w = (x_2, x_2, y_2, y_2)$$

Suppose that $q \in U + W$. It follows that

$$q = (x_1 + x_2, x_1 + x_2, x_1 + y_2, y_1 + y_2)$$

thus we can set $x = x_1 + x_2$, $y = x_1 + y_2$ and $z = y_1 + y_2$ and call it a day.

1.3.15

Suppose U is a subspace of V. What is $U + U$.

By properties of vector space, if we take $u_1, u_2 \in U$ then

$$u_1 + u_2 \in U$$

for every $u_1, u_2 \in U$. Thus we can follow that

$$U + U = U$$

1.3.16

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

If $q \in U + W$ it follows that there exists $u \in U$ and $w\imath W$ such that

$$q = v + w = w + v = q'$$

where $q' \in W + U$. Thus we can follow that $W + U = U + W$.

1.3.17

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Yes it is. We can apply the same logic as in the previous exercise and it'll do the job.

1.3.18

Does the operation of addition on the subspaces of V have an additive identity? Which subspace have additive inverces?

Every subspace contains zero, therefore

$$U + 0 = U$$

thus we've got additive identity.

By adding two subspaces together we get a larger subspace, thus we can follow that the only way to get 0 vector space as the result of addition of two subspaces is to add

$$0 + 0 = 0$$

thus the only subspace that contains additive inverse is 0.

Prove or give counterexample: if U_1, U_2, W are subspaces of V, such that

$$U_1 + W = U_2 + W$$

then $U_1 = U_2$

This is wrong: suppose that U_2 is a nonzero subspace of W and $U_1 = 0$. Then it follows that

$$U_1 + W = 0 + W = W = W + U_2$$

and

$$U_1 \neq U_2$$

as desired.

1.3.19

Suppose

$$U = \{(x, x, y, y) \in F^4 : x, y \in F\}$$

Find a subspace W *of* F^4 *such that* $F^4 = U \bigoplus W$

$$W = \{(0, x, y, 0) \in F^4 : x, y \in F\}$$

1.3.20

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in F^5 : x, y \in F\}$$

Find a subspace W *of* F^5 *such that* $F^5 = U \oplus W$

$$W = \{(0, 0, x, y, z) \in F^5 : x, y, z \in F\}$$

1.3.21

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in F^5 : x, y \in F\}$$

Find a thee subspaces W_1*,* W_2*,* W_3 *of* F^5 *such that* $F^5 = U \oplus W_1 \oplus W_2 \oplus W_3$

$$W_1 = \{(0, 0, x, 0, 0) \in F^5 : x \in F\}$$
$$W_2 = \{(0, 0, 0, y, 0) \in F^5 : y \in F\}$$
$$W_3 = \{(0, 0, 0, 0, z) \in F^5 : z \in F\}$$

1.3.22

Prove or give a counterexample: if U_1*,* U_2*,* W *are subspaces of* V *such that*

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

then $U_1 = U_2$

This one is false;

$$U_1 = \{(x, x) \in F^2 : x \in F\}$$
$$U_2 = \{(x, 0) \in F^2 : x \in F\}$$
$$W = \{(0, y) \in F^2 : y \in F\}$$

1.3.23

A function $f : R \to R$ is called even if

$$f(-x) = f(x)$$

for all $x \in R$. A function $f : R \to R$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in R$. Let U_e denote the set of real-valued even functions on R and let U_o denote the set of real-valued odd functions on R. Show that

$$R^R = U_e \oplus U_o$$

Let $f : R \to R$ be arbitrary. It follows that

$$f_e(x) = \begin{cases} 2f(x) - f(-x) \text{ if } x \geq 0 \\ f(x) \text{ if } x = 0 \\ 2f(-x) - f(x) \text{ if } x < 0 \end{cases}$$

Every odd function satisfies $f(0) = 0$. Therefore for even function we've got to have $f_e(0) = f(0)$

$$f_e(x) = \begin{cases} a_1 f(x) + b_1 f(-x) \text{ if } x > 0 \\ a_1 f(-x) + b_1 f(x) \text{ if } x < 0 \end{cases}$$

$$f_o(x) = \begin{cases} a_2 f(x) + b_2 f(-x) \text{ if } x > 0 \\ -a_2 f(-x) - b_2 f(x) \text{ if } x < 0 \end{cases}$$

$$\begin{cases} a_1 + a_2 = 1 \\ b_1 + b_2 = 0 \\ a_1 - a_2 = 0 \\ b_1 - b_2 = 1 \end{cases}$$

$$\begin{cases} a_1 = 0.5 \\ b_1 = 0.5 \end{cases}$$

$$f_e(x) = \begin{cases} 1/2f(x) + 1/2f(-x) \text{ if } x > 0 \\ f(x) \text{ if } x = 0 \\ 1/2f(x) + 1/2f(-x) \text{ if } x < 0 \end{cases}$$

$$f_o(x) = \begin{cases} 1/2f(x) - 1/2f(-x) \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1/2f(-x) + 1/2f(x) \text{ if } x < 0 \end{cases}$$

Thus

$$f_e(x) = f_e(-x)$$

$$f_o(-x) = -f_o(x)$$

and

$$f_e(x) + f_o(x) = f(x)$$

as desired.

Also, the only function that is odd and even at the same time is 0, therefore we've got a direct sum, as desired.

Chapter 2

Finite-Dimentional Vector Spaces

2.1 Span and Linear Independence

2.1.1

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v4$$

also spans V.

Let $v \in V$ be represented as

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$$

then we can follow that

$$v = a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4$$

therefore any $v \in V$ can be represented using given list, therefore given list spans V, as desired.

2.1.2

Verify the assertion in Example 2.18

Suppose that $v \in V$. Then it follows from some exercise in previous chapter that $a_1v = 0$ iff $a_1 = 0$ or $v = 0$. Thus if $v \neq 0$ we can follow that the only way to represent zero is to set a_1 to 0. Thus list is linearly independent.

Suppose that we've got linearly independent list of two vectors. We therefore can follow that the only way to represent 0 is to set $a_1 = 0$ and $a_2 = 0$. Thus vectors are not a scalar multiples of each other. In other directon we've got a trivial case.

For the list

$$v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 01, 0)$$

we've got that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 = (a_1, a_2, a_3, 0)$$

therefore the only way to represent zero is to set all of a's into 0.

Same case applies for the last one.

2.1.3

Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly inependent in R^3

The only way that this list is not linearly independent is if we can represent last vector as a linear combiination of the other two. Thus

$$\begin{cases} 3a_1 + 2a_2 = 51a_1 - 3a_2 = 9 \end{cases}$$

$$\begin{cases} 3a_1 + 2a_2 = 5a_1 = 9 + 3a_2 \end{cases}$$

$$3(9 + 3a_3) + 2a_2 = 5$$

$$27 + 9a_2 + 2a_2 = 5$$

$$11a_2 = -22$$

$$a_2 = -2$$

thus

$$a_1 = 3$$

therefore

$$3 * 4 - 5 * 2 = t$$

$$t = 2$$

2.1.4

Verify the assertion in the second bullet point in Example 2.20

$c = 8$ is the only solution such that third vector is a scalar multipe of first vector plus scalal multiple of second. Thus we can follow that the last vector is not in the span of first two, therefore the list is linearly independent.

2.1.5

(a) Show that if we think of C as a vector space over R, then the list $(1+i, 1-i)$ is linearly independent.

$$(1+i+1-i)/2 = 1$$
$$(1+i-1+i)/2 = i$$

thus the only way to represent 0 is to set all of a's to zero

(b) Show that if we think of C as a vector space over C, then the list $(1+i, 1-i)$ is linearly dependent

List (1) spans C, and its length is less that the length of given set. THus given set is linearly dependent.

2.1.6

Suppose v_1, v_2, v_3, v_4 is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v4$$

is also linearly independent.

As we've shown before, spans of two sets are equal, therefore the only way to represent 0 is to put all a's to 0.

2.1.7

Prove or give counterexample: If $v_1, v_2, ...v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, ...v_m$$

is linearly independent

Both sets span the same space and have the same length, therefore they are both linearly independent.

2.1.8

Trivial, equivalent to previous

2.1.9

Prove or give counterexample: If $v_1, ...v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

False: set $w_1 = -v_1$ and get the desired result.

2.1.10

Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, v_2 + w, ...v_m + w$ is linearly dependent, then $w \in span(v_1, v_2, ...v_m)$.

Suppose that resulting list is linearly dependent. It follows that there exists a way to represent

$$\sum_{n=1}^{m} a_n(v_n + w) = 0$$

such that not all a's are zeroes. Thus

$$\sum_{n=1}^{m} a_n(v_1 + w) = \sum_{n=1}^{m} (a_n w + a_n v_n) = \sum_{n=1}^{m} a_n w + \sum_{n=1}^{m} a_n v_n = w \sum_{n=1}^{m} a_n + \sum_{n=1}^{m} a_n v_n = 0$$

$$-w \sum_{n=1}^{m} a_n = \sum_{n=1}^{m} a_n v_n$$

$\sum_{n=1}^{m} a_n \neq 0$, because otherwise left side is zero and therefore right side is zero, which is not assumed.

$$w = \sum_{n=1}^{m} -\frac{a_n}{\sum_{j=1}^{m} a_j} v_n$$

thus $w \in span(v_1, v_2, ...v_m)$, as desired.

2.1.11

Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin span(v_1, ..., v_m)$$

Because otherwise we've got a bigger linearly independent list, that spans V.

2.1.12

Explain why there does not exist a list of six polinomials that is linearly independent of $\mathcal{P}_\triangle(F)$.

Because the list of length 5 spans this space.

2.1.13

Explain why no list of four polynomials spans $\mathcal{P}_\triangle(F)$.

Because the list of length 5 spans this space.

2.1.14

Prove that V is infinite-dimentional if and only if there is a sequence $v_1, v_2, ...$ of vectors in V such that $v_1, ...v_m$ is linearly independent for every possible integer m.

Forward is coming from the fact that we can always add new vectors to a given linearly independent list of vectors, that are outside of span of given list.

Because there always exists list that is bigger than given list and is linearly independent in V we can follow that no final list of vectors spans V, therefore it is infinite-dimentional.

2.1.15

Prove that F^∞ is infinite-dimentional.

Infinite list

$$(1, 0, ...), (0, 1, 0, ...), ...$$

is all linearly indepenent, therefore no finite set spans the space.

2.1.16

PRove that the real vector space of all continous real-valued functions on the interval $[0, 1]$ is infinite-dimentional.

We can create a countable sequence $(r_1, r_2, ...)$ of rationals in this space, and correspod each one of them with some number, thus creating a infinite linearly inedependent list.

2.1.17

Suppose $p_0, p_1, ...p_n$ are polynomials in $\mathcal{P}_{\Updownarrow}(F)$ such that $p_j(2) = 0$ for each j. Prove tat $p_0, p_1, ...p_m$ is not linearly independent in $\mathcal{P}_{\Updownarrow}(F)$.

Because it has the same length as $1, x, x^2...$, but doesn't span the same space.

2.2 Bases

There are no challenging exercises in this section, just a recap of the material. Looked them over, brushed up the material, not gonna waste my time writing them down.

2.3 Dimention

2.3.1

Suppose V is finite-dimentional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$

They have the same length of basis, thus basis of U is a basis of V.

2.3.2

Show that the subspaces of R^2 are precisely $\{0\}, R^2$ and all lines through the origin

For 0 dimention we've got null

For dimention 1 we've got scalar multiple of any vector, which are lines through the origin

For dimention 2 we've got the space itself

2.3.3

Show that the subspaces of R^3 are precisely $\{0\}, R^3$, all lines through the origin, and all planes through the origin

Same idea as in previos exericise, but list of length 2 defines a plane through the origin and 3 defined space itself

2.3.4

(a) Let $U = \{p \in P_4(F) : p(6) = 0$. Find a basis of U.

$$(x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, (x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

Find a subspace W of $P_4(F)$ such that $P_4(F) = U \oplus W$

$$\{c : c \in F\}$$

2.3.5

(a) Let $U = \{p \in P_4(F) : p''(6) = 0$. Find a basis of U.

$$1, (x-6), (x-6)^3, (x-6)^4$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, (x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

Find a subspace W of $P_4(F)$ such that $P_4(F) = U \oplus W$

$$(x-6)^2$$

2.3.6

(a) Let $U = \{p \in P_4(F) : p(2) = p(5)$. *Find a basis of* U.

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$x$$

2.3.7

(a) Let $U = \{p \in P_4(F) : p(2) = p(5) = p(6)$. *Find a basis of* U.

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, x^2, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$x, x^2$$

2.3.8

(a) Let $U = \{p \in P_4(F) : \int\_ 1^1 p = 0\}$. *Find a basis of* U.

$$x, x^3$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, x^2, x^3, x^4$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$1, x^2, x^4$$

2.3.9

Suppose $v_1, ...v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim span(v_1 + w, ..., v_m + w) \geq m - 1$$

Because $v_1, ...v_m$ is linearly independent we can follow that w is either in $span(v_1, ..., v_m)$ or not. In the latter case we've got that the case that we increase the span. In the former we've got by linear independence of $v_1, ...v_m$ that the maximum decline of degree is 1. Thus

$$\dim span(v_1 + w, ..., v_m + w) \geq m - 1$$

as desired.

2.3.10

Suppose $p_0, p_1, ..., p_m \in P(F)$ are such taht each p_j has degree j. Prove that $p_0, ...p_m$ is a basis of $P_m(F)$.

Suppose that $p \in P_m(f)$. Because each p_n has a degree of n we can follow that there exists only 1 $a_m \in F$ such that of p_m such that

$$p - a_m p_m \in P_{m-1}(F)$$

.

Bu applying the same procedure again repeatedly we get unique $a_m, ..., a_0$ such that

$$\sum a_n p_m = p$$

for every $p \in P_m(f)$. Thus we can follow that given list spans $P_m(F)$ and by unique representation we get that this list is linearly independent. Thus we can follow that given list is a basis of $P_m(F)$, as desired.

2.3.11

Suppose that U and W are subspaces of R^8 such that $\dim U = 3$, $\dim W = 5$, and $U + W = R^8$. Prove that $R^8 = U \oplus W$.

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Thus we can follow that in this particular case

$$\dim(R^8) = \dim U + \dim W - \dim(U \cap W)$$

$$8 = 3 + 5 - \dim(U \cap W)$$

$$\dim(U \cap W) = 0$$

thus we can follow that $U \cap W = \{0\}$. Therefore

$$U + W = U \oplus W = R^8$$

as desired.

2.3.12

Suppose that U and W are both five-dimentional subspaces of R^9. Prove that $U \cap W \neq \{0\}$

Once again we get that

$$\dim R^9 = \dim U + \dim W - \dim(U \cap W)$$

$$9 = 5 + 5 - \dim(U \cap W)$$

$$\dim(U \cap W) = 1$$

thus

$$U \cap W \neq 0$$

as desired.

2.3.13

Suppose U and W are both 4-dimentional subspaces of C^6. Prove that there exists two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other

Goto previous exercise for concretee explanation if needed, but we can conclude that

$$\dim U \cap W = 2$$

thus there exists a linearly independent list of length 2 in $U \cap W$ (basis) so that neither of them is a scalar multiple of another by some exercise in 2.A

2.3.14

Suppose $U_1, ...U_m$ are finite-dimentional subspaces of V. Prove that $U_1 + ... + U_m$ is finite-dimentional and

$$\dim(\sum U_n) \leq \sum \dim U_n$$

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

given that $\dim W \geq 0$ for any vector space W we follow that

$$\dim(U_1 + U_2) \leq \dim U_1 + \dim U_2$$

Thus by induction

$$\dim(\sum U_n) \leq \sum \dim U_n$$

which in presented case get us desired result.

2.3.15

Suppose V is finite-dimentional, with $\dim V = n \geq 1$. Prove that there exist 1-dimentional subspaces $U_1, ...U_n$ of V such that

$$V = U_1 \oplus ... \oplus U_n$$

For V there exists a basis of length n. Thus by setting

$$U_j = \{cv_j : c \in F\}$$

we get desired result.

2.3.16

Suppose $U_1, ..., U_m$ are finite-dimentional subspaces of V such that $U_1 + .. + U_m$ is a direct sum. Prove that $U_1 + ... + U_m$ is finite dimentional and that

$$\dim \sum U_n = \sum \dim U_n$$

We can just go by induction on the case that

$$\dim(U \oplus W) = \dim U + \dim W + \dim(U \cap W) = \dim U + \dim W + 0$$

Or we can use the fact, that we can combine all bases of subspaces together in one mega-basis for their sum. Both will suffice.

2.3.17

You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of finite-dimentional vector space, then

$$\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) -$$

$$- \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$$

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

and

$$U_1 + U_2 + U_3 = (U_1 + U_2) + U_3$$

thus

$$\dim(U_1+U_2+U_3) = \dim((U_1+U_2)+U_3) = \dim(U_1+U_2)+\dim U_3-\dim((U_1+U_2)\cap U_3) =$$

$$= \dim U_1 + \dim U_2 - \dim U_1 \cap U_2 + \dim U_3 - \dim((U_1 + U_2) \cap U_3) =$$

here we get a little problem because we don't know how to reduce $(U_1 + U_2) \cap U_3$ to some managable pieces. After this discovery one might even glance over the equation once again in order to try to disprove the theorem. And indeed we've found a counterexample: suppose that U_1, U_2, U_3 are lines through the origin in R^3 such that they are located on the same plane. Then it follows that left-hand side becomes 2, and the right side is equal to 3. Thus we've got a contradiction (which is a shame, because the formula looks nice :( ).

Chapter 3

Linear maps

3.1 The Vector Space of Linear Maps

3.1.1

Suppose $b, c \in R$. Define $T : R^3 \to R^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that T is linear if and only of $b = c = 0$.

Suppose that T is linear. Then it follows that

$$T(0) = 0 = (0 + b, 0)$$

thus we can follow that $b = 0$.

Also,

$$T((1, 1, 1) + (2, 2, 2)) = (6 - 12 + 9, 18 + 27c) = (3, 18 + 27c) =$$

$$= T((1, 1, 1)) + T(2, 2, 2) = (2-4+3, 6+c) + (4-8+6, 12+8c) = (1, 6+c) + (2, 12+8c) = (3, 18+9c)$$

Thus

$$27c = 9c$$

$$3c = c$$

$$c = 0$$

as desired.

Reverse implication is trivial, thus we get the desired result.

3.1.2

Suppose $b, c \in R$. Define $T : (P)(R) \to R^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c\sin p(0)\right)$$

Show that T is linear if and only if $b = c = 0$.

Suppose that T is linear. Then it follows that if $p(0) = \pi/2$, then latter part of resulting vector has additive property only when $c = 0$. For the former we've got result that

$$\lambda^2 b = b$$

for all $\lambda \in R$, which happens only if $b = 0$. Thus $b = c = 0$.

Reverse implication is trivial, thus we have the desired result.

3.1.3

Suppose $T \in \mathcal{L}(F^n, F^m)$. Show that there exists scalars $A_{j,k} \in F$ for $j = 1, ..., m$ and $K = 1, ..., n$ such that

$$T(x_1, ..., x_n) = (A_{1,1}x_1 + ... + A_{1,n}x_n, ..., A_{m,1}x_1 + ... + A_{m,n}x_n)$$

for every $(x_1, ..., x_n) \in F^n$.

Because $(1, 0, ...), (0, 1, ...), ...$ is a basis of F^n we can follow that there vector in F^m, such that $T(v) \in F^m$. Thus let us denote

$$T(1, 0, ...) = (A_{1,1}, A_{2,1}, ..., A_{m,1})$$

$$T(0, 1, ...) = (A_{1,2}, A_{2,2}, ..., A_{m,2})$$

$$...$$

Thus given given arbitrary vector $v = (x_1, x_2, ..., x_n) \in T^n$ we get that

$$T(v) = T(x_1, x_2, ...) = T(x_1, 0, 0, ...) + T(0, x_2, 0, ...) + ... = x_1 T(1, 0, 0, ...) + x_2 T(0, 1, 0, ...) + ... =$$

$$= (x_1 A_{1_1}, x_1 A_{2,1}, ...) + (x_2 A_{1_2}, x_2 A_{2,2}, ...) = (x_1 A_{1,1} + x_2 A_{1,2} + ..., x_1 A_{2,1} + x_2 A_{2,2} + ...)$$

as desired.

3.1.4

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, v_2, ...v_m$ is a list of vectors in V such that $Tv_1, ..., , Tv_m$ is a linearly inndependent list in W. Prove that $v_1, v_2, ..., v_m$ is linearly independent.

Suppose that it isn't. Then we can follow that there exist $w_1 \in W$ such that

$$w_1 = \sum a_j v_j = 0$$

and not all of a_j's are zeroes. Thus we can follow that

$$T(w) = T(\sum a_j v_j) = \sum T(a_j v_j) = \sum a_j T(v_j) = 0$$

But $T(v_j)$ is a list of linearly independent vectors, and therefore their sum is equal to zero iff all a_j's are zeroes, which is false. Thus we've got a contradiction.

3.1.5

Prove the assertion in 3.7

Let $T_1 = T, T_2 = S, T_3 \in L(V, W)$. Then it follows that

(1)

$$(T_1 + T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2 + T_1)(v)$$

(2)

$$(T_1 + (T_2 + T_3))(v) = T_1(v) + (T_2 + T_3)(v) = T_1(v) + T_2(v) + T_3(v) =$$

$$= (T_1 + T_2)(v) + T_3(v) = ((T_1 + T_2) + T_3)(v)$$

(3)

$$\lambda((S + T)(v)) = \lambda(S(v) + T(v)) = \lambda S(v) + \lambda T(v) = (\lambda S + \lambda T)(v)$$

(4)

$$T + 0 = T$$

(5)

$$1T = T$$

(6)

$$T + -1T = (1 - 1)T = 0T = 0$$

Thus $L(V, W)$ satisfies all reqirements of a vector space, as desired.

3.1.6

Prove the assertion in 3.9

Let $v \in V$.

(1) Then it follows that

$$((T_1T_2)T_3)(v) = (T_1T_2)(T_3(v)) = T_1(T_2(T_3(v))) = T_1((T_2T_3)(v)) = (T_1(T_2T_3))(v)$$

directly from definition. (I wonder if it's true in general for all functions; it probably is).

(2)

$$TIv = T(I(v)) = T(v) = I(T(v))$$

(3)

$$(S_1 + S_2)T(v) = (S_1 + S_2)(T(v)) = S_1(T(v)) + S_2(T(v)) = S_1Tv + S_2Tv$$

$$S(T_1 + T_2)(v) = S((T_1 + T_2)(v)) = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v)) = ST_1v + ST_2v$$

as desired.

3.1.7

Show that every linear map from a 1-dimentional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ *and* $T \in L(V, V)$*, then there exists* $\lambda \in F$ *such that* $Tv = \lambda v$ *for all* $v \in V$*.*

Because we've got a 1-dimentional space, it follows that there exists a basis of V - v_1. For this vector we've got that

$$Tv_1 = v_2 = \lambda v_1$$

Thus we can follow that if $u \in V$ then

$$Tu = T\sigma v_1 = \sigma T v_1 = \sigma\lambda v_1 = \lambda\sigma v_1 = \lambda u$$

as desired.

3.1.8

Give an example of a function $\phi : R^2 \to R$ *such that*

$$\phi(av) = a\phi(v)$$

for all $a \in R$ *and all* $v \in R^2$ *but* ϕ *is not linear.*

$$\phi(x, y) = \begin{cases} x \text{ if } x \neq y \\ 0 \text{ otherwise} \end{cases}$$

3.1.9

Give an example of a function $\phi : C \to C$ such that

$$\phi(w+z) = \phi(w) + \phi(z)$$

for all $w, z \in C$ but ϕ is not linear.

Let us define

$$\phi(a+bi) = b + ai$$

Thus

$$\phi(a+bi+c+di) = ai + ci + b + d = \phi(a+bi) + \phi(c+di)$$

but

$$i\phi(a+bi) = -a + bi$$

$$\phi(i(a+bi)) = \phi(ai-b) = -bi + a \neq i\phi(a+bi)$$

3.1.10

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in L(V,W)$ and $S \neq 0$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv \text{ if } v \in U \\ 0 \text{ if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that T is not a linear map on V.

Let $u \neq 0 \in U$ such that $Su \neq 0$ and $v \in V \setminus U$. Then it follows that

$$v + u \notin U$$

(because otherwise $-(v+u)$ is in U, therefore $u-(v+u) = -v \in U$ and thus $v \in U$, which is a contradiction) Thus we can follow that

$$T(v+u) = 0$$

but

$$T(v) + T(u) = 0 + Su = Su \neq 0 = T(v+u)$$

therefore the function is not linear, as desired.

3.1.11

Suppose V is finite-dimentional. Prove that every linear map on a subspace of V can be extended to a lineaer map on V. In other words, show that if U is a subspace of V and S is a subspace of V and $S = L(V, W)$, then there exists $T \in L(V, W)$ such that $Tu = Su$ for all $u \in U$.

Because V is finite-dimentional and U is a subspace of V, we can follow that U is finite-dimentional as well. Thus we can follow that there exists $u_1, ..., u_m$ - basis of U. As we know, we can extend this basis to a basis of V - $u_1, ..., u_m, v_1, ...v_n$. Therefore we can define a map $P \in L(V, U)$ by

$$P(x_1, x_2, ...) = (x_1, x_2, ...x_m, 0, 0, ...)$$

(basically trim every element of basis that is not in U). Thus we can follow that $P(u) = u$ if $u \in U$. Proof that P is linear is trivial. Thus if $S \in L(U, W)$, then $T = SP \in (V, W)$ with the desired properties.

3.1.12

Suppose V is finite-dimentional with $\dim V > 0$, and suppose W is infinite-dimentional. Prove that $L(V, W)$ is infinite-dimentional.

Let $v_1, ..., v_m$ be a basis of V and let $w_1, w_2, ...$ be a list of linearly independent vectors in W. Now let us look at $T_n : V \to W$

$$T_n((x_1, x_2, ...)) = x_1 w_n$$

Then it follows that by linear independence of w_n there does not exist a linear combination of T_m such that

$$\sum_{m \neq n} a_m T_m \neq T_n$$

Thus we can follow that list T_n is linearly independent. Because list is not finite we can follow that the space $L(V, W)$ is infinite-dimentional, as desired.

3.1.13

Suppose $v_1, ..., v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, ...w_m \in W$ such that no $T \in L(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, ..., m$.

Because $v_1, ..., v_m$ is linearly dependent we can reduce it to a linearly independent list $v'_1, ..., v'_n$. Thus resulting list will span some subspace of V and will be its basis.

Thus we can take vector v_j from the original list, that does not appear in basis. Then take some vectors $w_1, ...w_n$ in W. We know that there exists a unique map

$$Tv'_n = w_n$$

thus by adding to list $w_1, ...w_n$ any vectors from W, apart from $T(v_j)$ we create desired list.

3.1.14

Suppose V is finite-dimentional with $\dim V \geq 2$. Prove that there exists $S, T, \in L(V, V)$ such that $ST \neq TS$

Let v_1, v_2 be a basis of V and let

$$S(x, y) = (y, x)$$
$$T(x, y) = (x, 0)$$

Then

$$ST = (0, x)$$

and

$$TS = (y, 0)$$

as desired.

3.2 Null Spaces and Ranges

3.2.1

Give an example of a linear map T such that $\dim nullT = 3$ and $\dim rangeT = 2$.

$T(x, y, z) = (x, y)$

3.2.2

Suppose V is a vector space and $S, T \in L(V, V)$ are such that

$$rangeS \subset nullT$$

Prove that $(ST)^2 = 0$.

Let $v \in V$. Then it follows that $S(T(v)) \in rangeS$. Thus $ST(v) \in nullT$. Therefore $TST(v) = 0$. And thus $STST = (ST)^2 = 0$, as desired.

3.2.3

Suppose $v_1, ..., v_m$ is a list of vectors in V. Define $T \in L(F^m, V)$ by

$$T(z_1, ..., z_m) = z_1v_1 + ... + z_mv_m$$

(a) What property of T corresponds to $v_1, ..., v_m$ spanning V?

Surjectivity

(b) What property of T corresponds to $v_1, ..., v_m$ being linearly independent?

Injectivity

3.2.4

Show that

$$\{T \in L(R^5, R^4) : \dim nullT > 2\}$$

is not a subspace of $L(R^5, R^4)$.

We can set

$$T_1(x, y, z, w, q) = (x, 0, 0, 0)$$
$$T_2(x, y, z, w, q) = (0, y, 0, 0)$$
$$T_3(x, y, z, w, q) = (0, 0, z, 0)$$
$$T_4(x, y, z, w, q) = (0, 0, 0, w)$$

all of which are in the desired subset, but their sum is

$$T(x, y, z, w, q) = (x, y, z, w, 0)$$

which has $\dim null = 1$. Thus this subset is not closed under addition and therefore it is not a subspace.

3.2.5

Give an example of a linear map $T : R^4 \to R^4$ *such that*

$$rangeT = nullT$$

$$T(x, y, z, w) = (z, w, 0, 0)$$

.

3.2.6

Prove that there does not exist a linear map $TR^5 \to R^5$ *such that*

$$rangeT = nullT$$

dim is always an integer, therefore for $\dim rangeT = \dim nullT = n$ and

$$\dim T = 2n = 5$$

which is impossible.

3.2.7

Suppose V and W are finite-dimentional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in L(V, W) : T$ is not injective$\}$ is not a subspace of $L(V, W)$.

Suppose that $v_1, ..., v_m$ is a basis for V and $w_1, ..., w_n$ is a basis of W. We can follow that there exist, which maps v_1 to w_1 and so on. By adding all of those maps together we get an injective map. Thus we can follow that given set is not closed under addition and therefore is not a subspace.

3.2.8

Suppose V and W are finite-dimentional with $2 \leq \dim W \leq \dim V$. Show that $\{T \in L(V, W) : T$ is not surjective$\}$ is not a subspace of $L(V, W)$.

By following the simular logic as in previous exercise, we get a desired result.

3.2.9

Suppose $T \in L(V, W)$ is injective and $v_1, ..., v_n$ is linearly independent in V. Prove that $Tv_1, ..., Tv_n$ is linearly independent in W.

Suppose that it is not the case. Then it follows that there exists $a_1, ...a_n \in F$ such that not all of them are equal to zero and

$$\sum a_n T v_n = 0$$

Thus we can follow that

$$T \sum a_n v_n = 0$$

Thus $\sum a_n v_n \in nullT$. Because T is injective we can follow that

$$\sum a_n v_n = 0$$

and some of a_n's are not equal to zero. But $v_1, ..., v_n$ is linearly independent, thus we get a contradiction.

3.2.10

Suppose $v_1, ..., v_n$ spans V and $T \in L(V, W)$. Prove that the list $Tv_1, ...Tv_n$ spans range T.

Suppose $w \in rangeT$. Thus we can follow that there exists $v \in V$ such that

$$Tv = w$$

Given that $v_1, ..., v_n$ spans V we can follow that there exists $a_1, ...a_n$ such that

$$v = \sum a_n v_n$$

and thus

$$w = T\sum a_n v_n$$

$$w = \sum T a_n v_n$$

thus we can follow that $v_1, ..., v_n$ spans the range of T, as desired.

3.2.11

Suppose $S_1, ..., S_n$ are injective linear maps such that $S_1 S_2 ... S_n$ makes sense. Prove that $S_1 S_2 ... S_n$ is injective.

Suppose that T and S are injective such that ST makes sence. Suppose that

$$STv = 0$$

Then by injectivity of S we get that $Tv \in nullS$ and thus $Tv = 0$. Thus, by injectivity of T we get that $v = 0$. Therefore $nullST = 0$. Therefore ST is injective.

The case in the exercise is derived from induction on presented argument.

3.2.12

Suppose that V is finite-dimentional and that $T \in L(V, W)$. Prove that there exists a subspace U of V such that $U \cap nullT = 0$ and $rangeT = \{Tu : u \in U\}$.

Let N be a nullspace of T. It follows that it is a subspace of V. Now let $n_1, ..., n_m$ be a basis of N and extend it to a basis of V: $n_1, ..., n_m, v_1, ..., v_n$. Then if follows that $span(v_1, ...v_n) \cap N = 0$ (because otherwise the vector is in nullspace) and if $w \in rangeT$, then there exists $u \in span(v_1, ...v_n)$ such that $Tu = w$. Thus $span(v_1, ...v_n)$ is the desired subspace.

3.2.13

Suppose T is a linear map from F^4 to F^2 such that

$$nullT = \{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 5x_2, x_3 = 7x_4\}$$

Prove that T is surjective.

$\dim nullT = 2$, thus $\dim rangeT = 2$, therefore T is surjective, as desired.

3.2.14

Suppose U is a 3-dimentional subspace of R^8 and that T is a linear map from R^8 to R^5 such that $nullT = U$. Prove that T is surjcetive.

We can follow that $\dim rangeT = 5$, and therefore T is surjective, as desired.

3.2.15

Very similar to previous one

3.2.16

Same

3.2.17

Same

3.2.18

Same

3.2.19

Same

3.2.20

Suppose W is finite-dimentional and $T \in L(V, W)$. Prove that T is injective if and only if there exists $S \in L(W, V)$ such that ST is the identity map on V.

I don't know why it isn't stated explicitly, but by existence of injective T we can follow that $\dim V \leq \dim W$, and thus V is finite-dimentional.

In forward direction:

Suppose that T is injective. Now let $v_1, ..., v_m$ be a basis of V. Then we can follow that $Tv_1, ..., Tv_n$ is a basis of range T. Thus, extend this basis to a basis of W: $Tv_1, ..., Tv_n, w_1, ..., w_m$. Now let us define $S \in L(W, V)$ such that

$$STv_n = v_n$$

$$Sw_n = 0$$

Which will exist, and by the way, will be unique because we're pairing basis of W with a list of vectors in V. Thus we can follow that if $v \in V$ then

$$STv = ST \sum a_n v_n = S \sum T a_n v_n = S \sum a_n T v_n = \sum a_n v_n = v$$

thus $ST = I$, as desired.

In reverse dierction:

Suppose that there exists $S \in L(W, V)$ such that ST is an identity map on V. Suppose that T is not injective. Then we follow that $nullT \neq 0$. Then let $v_1 \in nullT \neq 0$. Then we can follow that

$$STv_1 = S(Tv_1) = S(0) = 0 \neq Iv_1$$

which is a contradiction. Thus we can conclude that T is injective, as desired.

3.2.21

Suppose W is finite-dimentional and $T \in L(V, W)$. Prove that T is surjective if and only if there exists $S \in L(W, V)$ such that TS is the identity map on W.

In forward direction:

Suppose that T is surjective and let $w_1, ..., w_n$ be a basis of W. Then we can follow that there exists $v_1, ..., v_m$ such that $Tv_1 = w_1, ...Tv_m = w_m$. Thus we can follow that there exists a map in $L(W, V)$ such that it maps

$$Sw_1 = v_1$$

$$Sw_n = v_n$$

Thus if $w \in W$, then we can follow that

$$TSw = TW(\sum a_n w_n) = T(\sum a_n W w_n) = T(\sum a_n v_n) = \sum a_n T v_n = \sum a_n w_n = w$$

for every $w \in W$. Thus we can follow that $TS = I$, as desired.

In reverse direction:

Suppose that there exists a map $S \in L(W, V)$ such that TS is an identity map on W.

Suppose now that T is not surjective. Then we can follow that there exists $w \in W$ such that there is no $v \in V$ such that $Tv = w$. But we've got that

$$TSw = T(Sw) = w$$

thus we've got a contradiction.

3.2.22

Suppose U and V are finite-dimentional vector spaces and $S \in L(V, W)$ andd $T \in L(U, V)$. Prove that

$$\dim nullST \leq \dim nullS + \dim nullT.$$

We know that if T maps a vector to zero, then $STv = S(Tv) = S0 = 0$. Thus we can follow that

$$\text{null } T \subseteq \text{null } ST$$

Suppose that $STv = 0$. Then we can follow that $Tv \in nullS$. Thus $nullST$ exhaustively decomposes into two sets: $nullT$ and $\{u \in U : Tu \in rangeT \cap nullS\}$. We know that

$$\dim(rangeT \cap nullS) \leq \dim nullS$$

. thus we can follow that

$$\dim nullST = \dim nullT + \dim(rangeT \cap nullS) \leq \dim nullS + \dim nullT$$

as desired.

3.2.23

Suppsoe U and V are finite-dimentional vector spaces and $S \in L(V, W)$ and $T \in L(U, V)$. Prove that

$$\dim rangeST \leq \min\{\dim rangeS, \dim rangeT\}$$

Given that $rangeST \subseteq rangeS$ we can follow that

$$\dim rangeST \leq \dim rangeS$$

Suppose that U' is a preimage of range of ST. Then we can follow that if $u' \in U'$, then u' is also in preimage of range of T. Thus we can follow that preimage of ST is a subset of preimage of T, and thus

$$\dim rangeST \leq \dim rangeT$$

Because both equations must hold, in follows that we get out desired inequality.

3.2.24

Suppose W is finite-dimentional and $T_1, T_2 \in L(V, W)$. Prove that $nullT_1 \subset nullT_2$ if and only if there exists $S \in L(W, W)$ such that $T_2 = ST_1$.

Firstly I should state that proposition in the exercise holds if we state that $\subset$ does not donote a proper subset, but a regular subset.

In forward direction: Suppose $nullT_1 \subset nullT_2$. This implies that $\dim rangeT_1 \geq \dim rangeT_2$. Let $v_1, ..., v_n, u_1, ...u_n, r_1, ..., r_n$ be a basis of V such that $r_1, ...r_n$ is a basis of $nullT_1$, $u_1, ..., u_n, r_1, ...r_n$ is a basis of T_2. Then we can follow that $T_2v_1, ..., T_2v_n$ is a basis of range of T_2 and $T_1v_1, ...T_1v_n$ is a basis of a subspace of range of T_1. Thus we can create a map $S : W \to W$ such that $ST_1v_n = T_2v_n$ Suppose that $v \in V$. Then it follows that

$$ST_1v = ST_1 \sum a_nv_n = \sum a_nST_1v_n = \sum a_nT_2v_n = T_2v$$

Thus we get our desired result.

In reverse direction: Suppose that there exists $S \in L(W, W)$ such that $T_2 = ST_1$. Suppose that $v \in nullT_1$. Thus $T_1v = 0 = ST_1v = T_2v$. Thus $v \in nullT_2$. Therefore $nullT_1 \subset T_2$, as desired.

3.2.25

Suppose W is finite-dimentional and $T_1, T_2 \in L(V, W)$. Prove that $rangeT_1 \subset rangeT_2$ if and only if there exists $S \in L(V, V)$ such that $T_2 = T_1 S$.

In forward direction:

Suppose that $rangeT_1 \subset rangeT_2$. Then let $q_1, ...q_n$ be a basis of range of T_1. Thus we can extend it to be a basis of range of T_2 be adding $w_1, ..., w_m, q_1, ..., q_n$. Thus we can follow that there exist $v_1, ...v_k \in V$ such that

$$T_1 v_n = w_n$$

and $v'_1, ..., v'_k \in V$ such that

$$T_2 v'_n = w_n$$

Thus we can create a map $S \in L(V, V)$ such that

$$S v_n = v'_n$$

and thus

$$T_2 S v = T_2 S \sum a_n v_n = T_2 \sum a_n S v_n = T_2 \sum a_n v'_n = \sum a_n T_1 v_n = T_1 \sum a_n v_n = T_1 v$$

as desired.

In reverse direction:

Suppose that there exists S such that $T_1 = T_2 S$. Then it follows that if $u \in rangeT_1$, then $u \in T_2$ as well. Thus $rangeT_1 \subset T_2$, as desired.

3.2.26

Suppose $D \in L(P(R), P(R))$ is such that $\deg Dp = (degp) - 1$ for every nonconstant polynomial $p \in P(R)$. Prove that D is surjective.

Let us define a list of polynomials p_n such that $\deg(p_n) = n$. Then it follows that the list $D(p_n)$ is a list of polynomials such that $\deg(D(p_n)) = n - 1$, thus it spans the space of polynomials. Thus we can follow that D is surjective.

3.2.27

Suppose $p \in P(R)$. Prove that there exists a polynomial $q \in P(R)$ such that $5q'' + 3q' = p$.

By the exercise above we can state that differentiation is surjective. Thus double differentiation is also surjective. Thus there exists $k \in P(R)$ such that $q'' = k'$, therefore $5q'' = 5k'$. Thus by surjectivity of differentiation we've got the desired result.

3.2.28

Suppose $T \in L(V,W)$. and $w_1, ..., w_m$ is a basis of range T. Prove that there exist $\phi_1, ...\phi_m \in L(V,F)$ such that

$$T(v) = \phi_1(v)w_1 + ...\phi_n(v)w_n$$

for every $v \in V$

Suppose that $v_1, ..., v_n$ is a basis of V. It follows that we can get coefficients

$$Tv_j = A_{j,1}w_1 + ... + A_{j,n}w_n$$

thus if we set

$$\phi_j(v) = \phi_j(\sum a_n v_n) = \sum a_j A_{j,n}$$

then we get that

$$Tv = T\sum a_n v_n = \sum a_n T v_n = \sum a_n \sum A_{n,j} w_j = \sum\sum a_n A_{n,j} w_j = \sum \phi_n(v) w_n$$

as desired.

3.2.29

Suppose $\phi \in L(V,F)$. Suppose $u \in B$ is not in null ϕ. Prove that

$$V = null\phi \oplus \{au : a \in F\}$$

ϕ maps into a space of dimention one. Thus we can follow that its range is either 1 or 0. In this case there exists $u \in V$, such that it is not in null space of ϕ, therefore we can follow that $\dim range\phi = 1$. Thus the space, that is not in $nullT$ has dimention 1. Thus we can follow that this space is scalar multiples of u. Therefore

$$null\phi + \{au : a \in F\} = V$$

because $u \notin null\phi$ we follow that

$$null\phi \cap \{au : a \in F\} = 0$$

and thus we can state that

$$null\phi \oplus \{au : a \in F\} = V$$

as desired.

3.2.30

Suppose ϕ_1 and ϕ_2 are linear maps from V to F that have the same null space. Show that there exists a constant $c \in F$ such that $\phi_1 = c\phi_2$.

If $\dim range\phi = 0$, then the case is trivial. Thus let us assume that $\dim range\phi = 1$. Because they have the same null space we can follow that they have the same preimage of the range. Thus we follow that if $v_1, ..., v_n$ is a basis of nullspace, then $v_1, ..., v_n, w$ is a basis of V, therefore w is a basis of a preimage. Thus

$$\phi_1 v = a_{n+1}\phi_1 w = a_{n+1}c_1$$

$$\phi_2 v = a_{n+1}\phi_2 w = a_{n+1}c_2$$

thus

$$\phi_1 = c_2/c_1\phi_2$$

as desired.

3.2.31

Give an example of two linear maps T_1 and T_2 from R^5 to R^2 that have the same null space but are such that T_1 is not a scalar multiple of T_2

$$T_1(x, y, z, w, q) = (x, y)$$
$$T_2(x, y, z, w, q) = (y, x)$$

3.3 Matrices

3.3.1

Suppose V and W are finite-dimentional and $T \in L(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim rangeT$ *nonzero entries.*

Suppose that we've there exists a choice of bases of V and W, such that matrix of this linear map has less nonzero entries, then $\dim rangeT$. Then it follows, that $\dim rangeT$ is spanned by list of vectors, that has length less than $\dim rangeT$, which is impossible.

3.3.2

Suppose $D \in L(P_3(R), P_2(R))$ is the differentiation map defined by $Dp = p'$. Find a basis of $P_3(R)$ and a basis of $P_2(R)$ such that the matrix of D with resprect to these bases is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

I think that we can use standart basis for $P_3(R)$, and for $P_2(R)$ we gotta use basis $1, 2x, 3x^2$.

3.3.3

Suppose V and W are finite-dimentional and $T \in L(V, W)$. Prove that there exist a basis of V and a basis of W. such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim rangeT$.

We can create a basis out of preimage of range of T. Thus if we set $v_1, ..., v_n$ to be a basis of preimage and $Tv_1, ..., Tv_n$ to be the basis of range. Thus if we extend those lists to be a bases of V and W respectively, we get the desired result.

3.3.4

Suppose $v_1, ..., v_m$ is a basis of V and W is finite-dimentional. Suppose $T \in L(V, W)$. Prove that there exists a basis $w_1, ..., w_n$ of W such that all the entries in the first column of $M(T)$ (with respect to the bases $v_1, ..., v_m$ and $w_1, ..., w_m$) are 0 except for possibly a 1 in the first row, first column.

We can plug in v_1 into T to get Tv_1. If $Tv_1 = 0$, then v_1 is in the nullspace and any basis will do. Otherwise we can extend Tv_1 to a basis of W and get the desired result.

3.3.5

Suppose $w_1, ..., w_n$ is a basis of W and V is finite-dimentional. Suppose $T \in L(V, W)$. Prove that there exists a basis $v_1, ..., v_m$ of V such that all the entries in the first row of $M(T)$ (with respect to the bases $v_1, ..., v_m$ and $w_1, ..., w_n$) are 0 except for possibly a 1 in the first row, first column.

Suppose that we've got a random basis $v_1, ..., v_n$ of V and then map it through T. Then pick a vector v_j such that $Tv_j = a_1w_1 + ... + a_mw_m$ such that $a_1 \neq 0$. If there is no such vector, then we're set. Otherwise go through all the other vecrors v_k and look at the representation

$$v_k = a'_1w_1 + ... + a'_mw_m$$

and set

$$v'_k = v_k - b_nv_n$$

where b_n satisfies a_1/a'_1 (or vice versa) such that v'_k represented as

$$Tv'_k = 0w_1 + ... + a_mw_m$$

Thus by linear independence of $v_1, ..., v_n$ we've got that $v_j, ..., v'_1, ..., v'_n$ is also linearly independent. Then, by plugging this vector into a matrix in this order, we get the desired result.

3.3.6

Suppose V and W are finite-dimentional and $T = L(V, W)$. Prove that $\dim rangeT = 1$ *if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ equal 1.*

In forward direction: Suppose that we've got T and $\dim rangeT = 1$. Suppose that $v_1, ..., v_n$ is the resulting basis of V. Thus we can follow that $Tv_1 = Tv_2 = ... = Tv_n = w \neq 0$. Thus we can follow that $w_1 + ... + w_m = w$ is the basis of range of T.

Thus we can create a vector v_1 such that $Tv_1 = w$, then expand it to a basis of V $v_1, ..., v_n$ and then it'll follow that $v_2, ..., v_n$ is a basis of a nullspace. Thus we can make a list $v_1, v_2 + v_1, v_3 + v_1, ..., v_n + v_1$, that will also a basis of V and for it it'll follow that

$$Tv_j' = T(v_j + v_1) = 0 + w = w$$

Thus the only thing that is left is to create a basis of W such that

$$w_1 + ... + w_n = w$$

we can actually do it by expanding w to a basis of W, and getting $w, w_1, ..., w_n$. Then we can set the first vector to be $w - w_1 - w_2 - ... - w_n$ and we'll get the desired property. Thus we can construct the bases such that we have the desired property.

In reverse direction: Suppose that there exist a basis of V and basis of W such that all entries of $M(T)$ are equal to 1. Then we can follow that if we plug any vector into T, then we'll get the constant multiple of the vector $w_1 + ... + w_n$. Thus we can follow that $\dim rangeT = 1$, as desired.

3.3.7

Verify 3.36

3.36 states that if $S, T \in L(V, W)$, then $M(S + T) = M(S) + M(T)$.

Suppose that $S, T \in L(V, W)$, $v_1, ..., v_n$ is a basis of V and $w_1, ..., w_m$ is a basis of W. Then we can follow that values at j'th column of $M(S + T)$ are obtained through

$$(S + T)(v_j) = S(v_j) + T(v_j) = (a_1 + a_1')w_1 + ... + (a_m + a_m')w_m$$

where $a_1, ...a_m$ will be numbers in j'th row of $M(S)$ and $a_1', ..., a_m'$ will be numbers in j'th row of $M(T)$. Thus we can follow that $M(S + T) = M(S) + M(T)$.

3.3.8

Verify 3.38

3.38 states that if $\lambda \in F$ and $T \in L(V, W)$, then $M(\lambda T) = \lambda M(T)$

$$(\lambda T)v_j = \lambda a_1 w_1 + ... + \lambda a_m w_m = \lambda(a_1 w_1 + ... + a_m w_m) = \lambda(Tv_j)$$

thus by the same reasoning as in previous exercise we've got that $M(\lambda T) = \lambda M(T)$, as desired.

3.3.9

Verify 3.52

Follows directrly from the definition.

3.3.10

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

$$(AC)_{j,k} = A_{j,\cdot}C_{k,\cdot}$$

thus

$$(AC)_{j,\cdot} = (A_{j,\cdot}C_{1,\cdot}, A_{j,\cdot}C_{2,\cdot}, ..., A_{j,\cdot}C_{k,\cdot}) = A_{j,\cdot}C$$

3.3.11

Suppose $a = (a_1, ..., a_n)$ is a 1-by-n matrix and C is an n-by-p matrix. Prove that

$$aC = a_1C_{1,\cdot} + ... + a_nC_{n,\cdot}$$

In other words, show that aC is a linear combination of the rows of C, with the scalars that multiply the rows coming from a.

This follows directly from a definition of matrix multiplication.

3.3.12

Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices A and C such that $AC \neq CA$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$CA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The rest of the exercises are just basic applications of definitions, and equating them rigorously. Nothing interesting in there

3.4 Invertibility and Isomorphic Vector Spaces

3.4.1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$

Let $u \in U$. Then

$$u = Iu = (T^{-1}T)u = T^{-1}(Tu) = T^{-1}I(Tu) = T^{-1}(S^{-1}S)(Tu) = (T^{-1}S^{-1})(ST)u$$

thus we can follow that (ST) is invertible and $(T^{-1}S^{-1}) = (ST)^{-1}$, as desired.

3.4.2

Suppose V is finite-dimentional and $\dim V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

We can have

$$S_1(a_1v_1 + ...a_n + v_n) = a_1v_1 + ...a_j + v_j$$

$$S_2(a_1v_1 + ...a_n + v_n) = a_{j+1}v_{j+1} + ...a_n + v_n$$

for some $1 \leq j < n$. (you can interpret it as an upper part of the identity and a lower part). Both of them are non-invertible (by non-surjectivity), but their sum is the identity, which is invertible. Thus the subset is not closed under addition and therefore it is not a subspace.

3.4.3

Suppose V is finite-dimentional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T(u) = S(u)$ for every $u \in U$ if and only if S is injective.

In forward direction: Suppose that there exists such an operator and S is not injective. Thus there exists u_1 such that $u_1 \neq 0$ and $Su_1 = 0$. Thus $Tu_1 = Su_1 = 0$, therefore T is not injective, therefore it is not invertible, which is a contradiction.

In reverse direction: Suppose that S is injective. Let $u_1, ..., u_n, v_1, ..., v_m$ be a basis of V such that $u_1, ..., u_n$ is a basis of U. Thus, by FTLM we've got that

$$\dim U = \dim \operatorname{range} S + \dim \operatorname{null} S$$

Given that S is injective, we can follow that $\dim \operatorname{null} S = 0$. Thus

$$\dim U = \dim \operatorname{range} S$$

Thus, because $\operatorname{range} S$ is a subspace, we can create a basis of it $u'_1, ..., u'_n$, which will have the same length as the basis of U. By expanding this basis to a basis of V we can create

$u'_1, ..., u'_n, v'_1, ..., v'_m$. Then if we map $u_1, ..., u_n, v_1, ..., v_m$ to $u'_1, ..., u'_n, v'_1, ..., v'_m$, we'll get $T \in \mathcal{L}(V)$, which by uniqueness of representation that will have

$$Tu = Su$$

and because $u'_1, ..., u'_n, v'_1, ..., v'_m$ is a basis of V we'll get that T is surjective, and thus invertible, as desired.

3.4.4

Suppose W is finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

In forward direction: Suppose that $\operatorname{null} T_1 = \operatorname{null} T_2$. Then we can follow that $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2$. We can also follow that there exists $v_1, ...v_n$ - basis of preimage of T_1 and T_2. Thus, $T_1 v_1, ...T_1 v_n$ is a basis of $\operatorname{range} T_1$ and $T_2 v_1, ..., T_2 v_n$ is a basis of $\operatorname{range} T_2$. Thus we can create a unique map $S \in \mathcal{L}(\operatorname{range} T_2, \operatorname{range} T_1)$

$$S'(a_1 T_2 v_1 + ... + a_n T_2 v_n) = a_1 T_1 v_1 + ... + a_n T_1 v_n$$

Thus, if $v \in V$, then

$$ST_2 v = S(a_1 T_2 v_1 + ... + a_n T_2 v_n) = a_1 T_1 v_1 + ... + a_n T_1 v_n = T_1(a_1 v_1 + ... + a_n v_n) = T_1 v$$

Given that S' is injective and $S' \in (\operatorname{range} T_2, \operatorname{range} T_1) \to S' \in (\operatorname{range} T_2, W)$ (because $\operatorname{range} T_2 \subseteq W$) , we can follow by results of our previous exercise, that there exists invertible $S \in \mathcal{L}(W)$ such that

$$S(w) = S'(w)$$

and therefore

$$T_1 = ST_2$$

as desired.

In reverse direction: Suppose that there exists invertible $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. Thus if $v \in \operatorname{null} T_1$, then we can follow that $ST_2 v = 0$. By invertability of S we've got that there exists S^{-1} and therefore

$$ST_2 v = 0$$

$$S^{-1} ST_2 v = S^{-1} 0$$

$$T_2 v = 0$$

Thus, $v \in \operatorname{null} T_2$. Therefore we can follow that $\operatorname{null} T_1 \subseteq \operatorname{null} T_2$. By the same logic, but in other direction we've got that $\operatorname{null} T_2 \subseteq \operatorname{null} T_1$, thus

$$\operatorname{null} T_1 = \operatorname{null} T_2$$

as desired.

3.4.5

Suppose that V is finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ *if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = T_2 S$.*

In forward direction: Let $w_1, ...w_n$ be a basis of $\operatorname{range} T_1 = \operatorname{range} T_2$. Thus we can follow that there exists basis $v_1, ...v_n \in V$ of preimage of T_1 and $v'_1, ...v'_n \in V$ - basis of preimage of T_2 such that $T_1 v_j = w_j = T_2 v'_j$. Because those lists are linearly independent and have the same length, we can follow that there exists an isomorphy $S \in \mathcal{L}(V)$ such that

$$S(a_1 v_1 + ...a_n v_n) = a_1 v'_1 + ...a_n v'_n$$

Thus we can follow that for $v \in V$

$$T_2 S v = T_2 S(a_1 v_1 + ...a_n v_n) = a_1 w_1 + ... + a_n w_n = T_1 v$$

as desired.

In reverse direction: Suppose that there exists $S \in \mathcal{L}(W)$ such that $T_1 = T_2 S$. Then it is obvious that $\operatorname{range} T_1 = \operatorname{range} T_2$.

3.4.6

Suppose that V and W are finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exists invertible operators $R \in \mathcal{L}\, L(V)$ and $S \in L(W)$ such that $T_1 = S T_2 R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$*.*

In forward direction: By resullts of previous exercise we can follow that $\operatorname{null} T_1 = \operatorname{null} T_2 R$. Thus by injectivity of R we've got that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

In reverse direction: Suppose that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$. We can follow that there exists isomorphism $S \in \mathcal{L}(W)$ such that $\operatorname{null} T_1 = \operatorname{null} T_2 R$. Thus, by results of previous exercises we can folllow that there exists $S \in \mathcal{L}(V)$ such that $T_1 = S T_2 R$, as desired.

3.4.7

Suppose V and W are finite-dimentional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

Suppose that $S, T \in E$. Then for $v \in V$ we've got that

$$(S + T)v = S(v) + T(v) = 0$$

thus $S + T \in E$ and E is closed under addition.

Let $\lambda \in F$. Then

$$(\lambda T)v = \lambda(Tv) = \lambda 0 = 0$$

thus $(\lambda T) \in E$, therefore E is closed under scalar multiplication. Given that $\mathcal{L}(V, W)$ is a vector space we can follow that E is a subspace, as desired.

(b) Suppose $v \neq 0$. What is $\dim E$*?*

Extend v to a basis $v, v_1 ... v_{n-1}$ of V and let $w_1, ..., w_n$ be arbitrary basis of W. Because $Tv = 0$, we require that first column of $M(T)$ will be zeroes. Then we can follow that $\mathcal{M}$ is an isomorphism between E and $F^{m,(n-1)}$. Thus

$$\dim E = (\dim V - 1)(\dim W)$$

3.4.8

Suppose V is finite-dimentional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of W such that $T|_U$ is an isomorphism of U onto W.

Let $v_1, ..., v_n$ be a basis of $\operatorname{null} V$. Then we can extend it to $v_1, ..., v_n, u_1, ..., u_m$, which will be a basis of V. Let $U = \operatorname{span}(u_1, ..., u_m)$. We can follow by FTLM that $\dim U = \dim \operatorname{range} T$. Also, because U is surjective, we can follow that it is invertible and therefore is isomorphism, as desired.

3.4.9

Suppose V is finite-dimentional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

In forward direction:

Suppose that ST is invertible. Suppose that T is not invertible. Then it is not injective. Therefore theree exists $v \in V \neq 0$ such that $Tv = 0$. Therefore $STv = S0 = 0$, which is a contradiction. Thus T is injective and therefore invertible.

Suppose that S is not invertible. Then we can follow that there exists $v \in V \neq 0$ such that $Sv = 0$. Given that T must be invertible we can follow that there exists $w \in V \neq 0$ such that $Tw = v$. Thus $STw = Sv = 0$. Therefore ST is not injective, which is a contradiction.

Thus we can follow that in order for ST to be invertible, both S and T must be invertible as well, as desired.

In reverse direction: Suppose that S and T are invertible. Then we can follow that both of them are injective. Thus by some exercise in this chapter (looked it up, it's 3.2.11) ST is injective as well. Thus ST is invertible, as desired.

3.4.10

Suppose V is finite0dimentional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

From previous exercise we can follow that both S and T, as well as ST and TS are invertible.

All of the following are equivalences and not implications, therefore we can prove everything in one go.

$$ST = I$$

$$STS = IS$$

$$STS = SI$$

$$TS = I$$

3.4.11

Suppose V is finite-dimentional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Firstly I should state that we assume here that S and U are invertible, otherwise we default to one of the previous exrecises.

Suppose that T is not invertible. Then it isn't injective, and therefore there exists $v \in V \neq 0$ such that $Tv = 0$. Because U is invertible we can follow that there exists $w \in V \neq 0$ such that $Uw = v$. Thus

$$STUw = STv = S0 = 0 \neq w$$

Therefore $STU \neq I$, which is a contradiction.

Now we can use some algebra in here

$$STU = I$$

$$TU = S^{-1} I$$

$$TU = S^{-1} U^{-1}$$

$$T = S^{-1} U^{-1}$$

and by first exercise in this chapter

$$T^{-1} = (S^{-1} U^{-1})^{-1} = US$$

as desired.

3.4.12

Show that the result in the previous exercise can fail withoud the hypothesis that V is finite-dimentional

We can set U to be I,

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...)$$

$$S(x_1, x_2, ...) = (x_2, x_3, x_4, ...)$$

and so on. Then T will not be surjective, and therefore will not be invertible (which doesn't follow from our usual equivalence, but by the fact that there does not exist a map such that $TS = I$, because there is no way to represent (1, 1, ...))

3.4.13

Suppose V is a finite-dimentional vector space and $R, S, T \in \mathcal{L}(V)$. are such that RST is surjective. Prove that S is injective.

Suppose that it isn't. We can follow that RST is invertible. Then we can follow that there exits $v \in V \neq 0$ such that $Sv = 0$. By invertability of T we follow that there exists $w \in V \neq 0$ such that $Tw = v$. thus

$$RSTw = RSv = R0 = 0$$

Thus RST is not invertible and isn't surjective, which is a contradiction.

3.4.14

Suppose $v_1, ..., v_n$ is a basis of V. Prove that the map $T : V \to F^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $F^{n,1}$.

Only way to represent 0 in $M(v)$ is that if $v = 0$, therefore the map is injective. Also, by common sense, map is surjective. Thus it is invertible and therefore it is an isomorphism.

3.4.15

Trivial, seen simular in previous chapter

3.4.16

Suppose V is finite-dimentional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Forward direction is trivial.

Suppose that $ST = TS$ for every $S \in \mathcal{L}(V)$. Suppose that $Tv \neq \lambda v$ for some $v \neq 0$ and $\lambda \in F$. Then we can follow that there exsits S such that

$$S(Tv) = v$$

and

$$S(v) = 0$$

Thus

$$S(Tv) = v \neq 0 = S(v) = T(Sv)$$

Thus we can follow that for every $v \in V$ there exists $a \in F$ such that $Tv = av$.

Now suppose that $v, w \in V$. If $v \neq \lambda w$, then

$$T(v + w) = a_{v+w}(v + w) = a_{v+w}v + a_{v+w}w = T(v) + T(w) = a_v v + a_w w$$

thus

$$a_v = a_w = a_{v+w}$$

by the unique representation of zero.

If $v = \lambda w$, then

$$Tv = a_v v = T(\lambda w) = a_w \lambda w$$

$$a_v v = a_w \lambda w$$

$$a_v v = a_w v$$

thus $a_v = a_w$. Therefore for any given v, w we follow that $a_v = a_w$. Thus, T is a scalar multiple of I, as desired. (Proof aquired after reading another proof in supplimentary material).

3.4.17

Suppose V is finite-dimentional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

It's trivial to proof that $\{0\}$ and $\mathcal{L}(V)$ are such supspaces. We also should mention that in order for it not to be a trivial case we assume that $\dim V > 1$ (otherwise we don't have any other subspaces other than $\mathcal{L}(V)$ and $\{0\}$).

Our strategy with this proof is to show that there exists an invertible map in $\mathcal{E}$. If we've got that, then the rest of the proof is trivial.

Thus suppose that $E \neq \{0\}$ and $E \neq \mathcal{L}(V)$. Then we can follow that there exists a map $T \in E$ such that $T \neq 0$. Because we want to prove that there exists an invertible map in E, suppose that T is not invertible (otherwise $TT^{-1} = I \in \mathcal{E}$, and we can skip to the later part). Thus there exists a vector $v \in V \neq 0$ such that $Tv \neq 0$ and $w \in V \neq 0$ such that $Tw = 0$. Therefore extend v to a basis $v, v_1, ..., v_n$ of V and make a map

$$S(v) = v$$

.

$$S(v_k) = 0$$

. Then it follows that there exist maps $TS \in E_j$ such that

$$TS(a_0v + a_1v_1 + ...a_nv_n) = Ta_0v = a_0Tv$$

Thus there exists a map

$$Q_j(a_0v + ... + a_nv_n) = a_jv_j$$

also in $\mathcal{E}$. Thus we can follow that $I \in \mathcal{E}$. Thus we can follow that for every $S \in \mathcal{L}(V)$ we've got that

$$SI = S$$

is also in $\mathcal{E}$. Thus $\mathcal{L}(V) \subseteq \mathcal{E}$. Thus we can follow that $\mathcal{L}(V) = \mathcal{E}$, which is a contradiction.

Thus we can follow that $\mathcal{E}$ is either $\mathcal{L}(V)$ and $\{0\}$, as desired.

3.4.18

Show that V and $\mathcal{L}(F, V)$ are isomorphic vector spaces.

For finite-dimentional spaces we've got that

$$\dim \mathcal{L}(F, V) = (\dim F)(\dim V) = 1(\dim V) = \dim V$$

Thus they are isomorphic.

Otherwise we suppose that $v_1, ...$ is a basis of V. Given that

$$v = \sum a_jv_j$$

we can follow that we've got bijectivity between $\mathcal{L}(F, V)$ and V, as desired.

3.4.19

Suppose $T \in \mathcal{L}(P(R))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in P(R)$.

(a) Prove that T is surjective.

Because T is injective, we can follow that $\operatorname{null} T = \{0\}$ (looked it up, this one applies to any vector space).

Now let's try to prove that $\deg Tp = \deg p$. We probably don't need the induction here, but we'll use it anyways.

For $p = 0$ we've got that $Tp = 0$. Let us prove that this is the case also for $\deg p = 0$, just in case. Suppose that $\deg p = 0$ and $\deg Tp < 0$, then $\operatorname{null} T$ is not equal to zero, which is a contradiction.

For inductive step let us assume that $\deg Tp = \deg p$ for p such that $\deg p = n - 1$. Now suppose that $\deg p = n$ and $\deg Tp < n$. By our assumption, it follows that there exists a basis $p_0, ..., p_{n-1}$ such that $\deg Tp_j = j$. Thus we can follow that $Tp_0, ...Tp_n$ spans $P_{n-1}(R)$. Thus we follow that there exists $a_0, ...a_{n-1}$ such that

$$a_0 Tp_0 + ...a_{n-1} Tp_{n-1} = Tp$$

then it follows that T is not injective, which is a contradiction.

This concludes the proof that $\deg p = \deg Tp$. By this we can follow that given $p \in P(R)$ with $\deg p = n$ there exists $p_0, ...p_n$ and by extension $Tp_0, ...Tp_n$ such that $p \in \text{range}(Tp_0, ...Tp_n)$. Thus there exists $p' \in P(R)$ such that $Tp' = p$. Thus T is surjective, as desired (this dragged on for waaay too long).

3.4.20

not gonna repeat the text of the exercise, but it basically reduces to our usual eqivalence of surjectivity/injectivity on finite-dimentional operators.

3.5 Products and Quotients of Vector Spaces

3.5.1

Suppose T is a function from V to W. The graph of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Firstly, $G(T)$ denotes the graph of T.

In forward direction: Suppose that T is a linear map. Suppose that $v, w \in G(T)$. Then it follows that

$$v + w = (v + w, T(v + w))$$

Thus $G(T)$ is closed under addition.

$$\lambda v = (\lambda v, \lambda Tv)$$

thus it is also closed under scalar multiplication. Given that $G(T)$ is a subset of a product of vector spaces, which is a vector space, we follow that $G(T)$ is a subspace, as desired.

In reverse direction:

Suppose that $G(T)$ is a subspace of $V \times W$. Suppose that $v, w \in V$. Then

$$T(v + w) = T(v) + T(w)$$

and

$$\lambda T(v) = T(\lambda v)$$

by properties of a product of vector spaces. Thus we can follows that T is linear, as desired.

3.5.2

Suppose $V_1, ..., V_m$ are vector spaces such that $V_1 \times ... \times V_m$ is finite-dimentional. Prove that V_j is finite-dimentional for each $j = 1, ..., m$.

We can follow that the combined list of bases of V_n's spans $V_1 \times ... \times V_m$ and is linearly independent. Given that this list is finite, we can follow that $V_1 \times ... \times V_m$ is finite-dimentional, as desired.

3.5.3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$, but $U_1 + U_2$ is not a direct sum.

$$U_1 = (0, x_1, 0, x_2, 0, x_3, ...)$$
$$U_2 = (x_1, x_2, x_3, 0, x_4 ...)$$

3.5.4

Suppose $V_1, ...V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times ... \times V_m, W)$ and $\mathcal{L}(V_1, W) \times ... \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

For finite-dimentional spaces we can just follow the standard equations for calculating the dimentions of given spaces and produce the desired result.

Conversely, if any of the spaces is infinite-dimentional, then we gotta produce the bijective map between the two spaces.

Our strategy here will be to prove that there exists a bijective map between the two. Let $T : \mathcal{L}(V_1 \times ... \times V_m, W) \to \mathcal{L}(V_1, W) \times ... \times \mathcal{L}(V_m, W)$ be defined by

$$T(S) = S_1 \times ... \times S_n$$

where

$$S_j(v_j) = S(0, ..., v_j, .., 0) = w_j$$

thus

$$T(\lambda S) = \lambda S_1 \times ... \times \lambda S_n$$

and

$$T(S + M) = (S + M)_1 \times ... \times (S + M)_n = S_1 \times ... \times S_n + M_1 \times ... \times M_n$$

thus it is linear.

Injectivity and surjectivity are given with this one. Thus we follow that there exists a linear bijectivity between the two, thus they are isomorphic.

We can actually follow here that if the space is infinite-dimentional, then they are isomorhic by default.

3.5.5

Suppose $W_1, ...W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times ...W_n)$ and $\mathcal{L}(V, W_1) \times ... \times \mathcal{L}(V, W_n)$ are isomorphic

Same logic as in previous ones works on this oen too.

3.5.6

For a posisitivee integer n, define V^n by

$$V^n = V \times ... \text{ n times } ...V$$

Prove that V^n and $\mathcal{L}(F^n, V)$ are isomorphic vector spaces.

$$\dim \mathcal{L}(F^n, V) = (\dim F^n)(\dim V) = n \dim V = \dim V^n$$

as desired.

3.5.7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v+U = x+W$. Prove that $U = W$.

Let $v' = x - v$. It follows that $x = v + v'$. Given that $v + U = x + W$ we follow that there exists $-w \in W$ such that $x - w = v + 0$. Thus

$$x - w = v + 0$$

$$v + v' - w = v$$

$$v' - w = 0$$

$$v' = w$$

Thus we follow that $v' \in W$. Therefore we've got that

$$x + W = v + v' + W = v + W = v + U$$

thus we follow that

$$W = U$$

as desired.

3.5.8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v+(1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in F$.

In forward direction:

Suppose that A is an affine subset of V. Then we follow that there exists vector $x \in V$ and subspace $U \subseteq V$ such that

$$A = x + U$$

Now let $v, w \in A$. Then we can follow that there exist $v', w' \in U$ such that

$$v = x + v'$$

$$w = x + w'$$

Thus we follow that

$$\lambda v + (1-\lambda)w = \lambda x + \lambda v' + x + w' - \lambda x - \lambda w' = \lambda v' + x + w' - \lambda w'$$

Because $v', w' \in U$ we follow that

$$\lambda v' + w' - \lambda w' \in U$$

thus

$$\lambda v' + x + w' - \lambda w' \in A$$

as desired.

In reverse direction:

Suppose that for every $\lambda \in F$ and $v, w \in A$ we've got that

$$\lambda v + (1-\lambda)w \in A$$

Fix $x \in A$ and suppose that $v, w \in A$. Then it follows that

$$\lambda(v-x) = \lambda v - \lambda x = \lambda v - \lambda x + x - x = (\lambda v - (1-\lambda)x) - x$$

Thus space $A - x$ is closed under scalar multiplication. And

$$(v-x) + (w-x) = 2(v/2 + w/2 - x) = 2(\frac{1}{2}v + (1-\frac{1}{2})w - x)$$

by the fact that $v, w \in A \to \frac{1}{2}v + (1-\frac{1}{2})w \in A$ we follow that $(\frac{1}{2}v + (1-\frac{1}{2})w - x) \in A - x$. Thus, by the fact that $A - x$ is close under scalar multiplication we follow that

$$2(\frac{1}{2}v + (1-\frac{1}{2})w - x) \in A - x$$

thus we can conclude that $A - x$ is a subspace of V. Thus, $A - x + x = A$ is an affine subset of V, as desired.

3.5.9

Suppose A_1 and A_2 aare affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Fisrtly, let us denote that

$$A_1 = x_1 + U_1$$

$$A_2 = x_2 + U_2$$

Firstly, if A_1 and A_2 are parallel, then they are either equal to each other (in which case their intersection is an affine subset), or empty. Thus we can follow that intersection of two affine subsets can be empty.

Suppose that their intersection is nonempty and let $x \in A_1 \cap A_2$. Then it follows that $U_1 = A_1 - x$ and $U_2 = A_2 + x$. Intersection of two vector spaces is a vector space, thus we follow that

$$x + U_1 \cap U_2 = A_1 \cap A_2$$

is an affine space, as desired.

3.5.10

Prove that the intersection of every collection of affine subsets of V is either an affine subset of V or the empty set

It follows by induction from the previous exercise.

3.5.11

Suppose $v_1, ..., v_m \in V$. Let

$$A = \{\lambda_1 v_1 + ... + \lambda_m v_m : \lambda_{[1,m]} \in F \text{ and } \sum \lambda_j = 1\}$$

(a) Prove that A is an affine subset of V.

Let $v, m \in A$. Then it follows that

$$\kappa v + (1-\kappa)w = \kappa(\lambda_1 v_1 + ... + \lambda_m v_m) + (1-\kappa)(\lambda'_1 v_1 + ... + \lambda'_m v_m) =$$

$$= \kappa(\lambda_1 v_1 + ... + \lambda_m v_m) + (1-\kappa)(\lambda'_1 v_1 + ... + \lambda'_m v_m) = (\kappa\lambda_1 + (1-\kappa)\lambda'_1)v_1 + ...(\kappa\lambda_n + (1-\kappa)\lambda'_n)v_n$$

Thus the sum of the coefficients is equal to

$$\sum(\kappa\lambda_n + (1-\kappa)\lambda'_n) = \sum(\kappa\lambda_n) + \sum[(1-\kappa)\lambda'_n] = \kappa\sum(\lambda_n) + (1-\kappa)\sum[\lambda'_n] = \kappa + (1-\kappa) =$$

$$= 1$$

Thus we follow that A is an affine subset by equivalence, that was proven in a couple of exercises above.

(b) Prove that every affine subspae of V that contains $v_1, ..., v_m$ also contains A

Suppose that some affine subset B contains $v_1, ..., v_m$. Let $a \in A$.

We know that there exists $U \subseteq V$ such that $v_1 + U = A$ and $W \in V$ such that $B = v_1 + W$. Thus we follow that $v_2 - v_1 \in B - v_1$, ..., $v_n - v_1 \in B - v_1$. Thus

$$\operatorname{span}(v_2 - v_1, ...v_n - v_1) \subseteq B - v_1$$

Suppose that $a \in A$. Then

$$a - v_1 \in \operatorname{span}(v_2 - v_1, ...v_n - v_1)$$

Thus

$$a - v_1 \in B$$

as desired.

(c) Prove that $A = v+U$ for some $v \in V$ and some subspace U of V with $\dim U \leq m-1$

We know, that the list $(v_2 - v_1, ...v_m - v_1)$ spans $A - v_1$. Therefore we can follow that $\dim U \leq m - 1$.

3.5.12

Suppose U is a subspace of V such that V/U is finite-dimentional. Prove that V is isomorphic to $U \times (V/U)$.

For finite-dimentional case is trivial, for infinite-dimentional we have isomorphism by default.

3.5.13

Suppose U is a subspace of V and $v_1 + U, ..., v_m + U$ is a basis of V/U and $u_1, ..., u_n$ is a basis of U. Prove that $v_1, ..., v_m, u_1, ..., u_m$ is a basis of V.

Because $v_1 + U, ..., v_m + U$ is a basis of V/U, we can follow that $v_1, ..., v_m$ is linearly independent in V.

If $v_j \in U$, then $v_j + U = U = 0 + U$, therefore we follow that $v_j \notin U$.

Thus we can follows that $v_1, ..., v_n, u_1, ..., u_m$ is a linearly independent list of vectors.

Given that $v_1, ..., v_n, u_1, ..., u_m$ is a list of vectors in V, we can follow that $\operatorname{span}(v_1, ..., v_n, u_1, ..., u_m)$ is a subspace of V.

Now suppose that $v \in V$ and $v \notin \operatorname{span}(v_1, ..., v_n, u_1, ..., u_m)$. We can follow that $v \notin U$. Thus we can follow that $v + U$ is an element of V/U. In this case we follow that $v \in \operatorname{span}(v_1, ..., v_n)$, which is a contradiction. Thus we conclude that such a vector does not exist and therefore $V = \operatorname{span}(v_1, ..., v_n, u_1, ..., u_m)$. Given that $v_1, ..., v_n, u_1, ..., u_m$ is linearly independent, we follow that it is a basis of V, as desired.

3.5.14

Suppose $U = \{(x_1, x_2, ...) \in F^\infty : x_j \neq 0$ *for only finitely many* $j\}$

(a) Show that U *is a subspace of* F^∞

Let $v, u \in U$. We can follow, that sinse finitely many x's are not zero, then we follow that for $v + u$

$$\{j_1, ..., j_n\} \cup \{j'_1, ..., j'_n\}$$

is a set of positions, in which the $v + u$ might not be zero. Since union of finite sets is finite, we follow that U is closed undeer addition. The same reasoning, but applied to intersection, rather then union, can be applied to get closure for multiplication. Thus we follow that U is a subspace of F^∞.

(b) Prove that F^∞/U *is infinite-dimentional*

We can follow that there exist

$$x_j = (0, 0, 0, ... \text{ j times } ...0, 1, 1, 1...)$$

such that each x_j is linearly independent from one another. Since $x_j \notin U$, we can follow that there is no basis of F^∞/U, therefore it is infinite-dimentional, as desired.

3.5.15

Suppose $\phi \in \mathcal{L}(V, F)$ *and* $\phi \neq 0$. *Prove that* $\dim V/(\operatorname{null}\phi) = 1$

Because $\phi \neq 0$, we can follow that there exists $v \in V$ such that

$$\phi v \neq 0$$

thus we follow that $\dim \operatorname{range} \phi = 1$. Given that $V/\operatorname{null}\phi$ is isomorphic to $\operatorname{range}\phi$, we follow that its dimention is also one, as desired.

3.5.16

Suppose U *is a subspace of* V *such that* $\dim V/U = 1$. *Prove that there exists* $\phi \in \mathcal{L}(V, F)$ *such that* $\phi = U$.

Given that $\dim V/U = 1$ we can follow that there exists $v \in V$ such that $v \notin U$. Thus v is a basis for V/U. Thus we can define $g : V/U \to F$

$$g(kv + U) = k$$

Then, by plugging $\pi : V \to V/U$ such that $\pi(v) = v + U$ and making

$$\phi(v) = g(\pi(v))$$

we get $\phi : V \to F$ such that $u \in U \to \phi(u) = 0$. Thus $U \subseteq \operatorname{null}\phi$.

Then suppose that $v \neq 0$ and $v \notin U$. Then it follows that $\phi(v) \neq 0$. Thus we can conclude that $\operatorname{null}\phi = U$, as desired.

3.5.17

Suppose U is a subspace of V such that V/U is finite-dimentional. Prove that there exists a subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Because V/U is finite-dimentional, we can follow that there exists a basis of V.

$$v_1 + U, ..., v_n + U$$

We can follow that $v_1, ..., v_n$ is linearly independent. It is also will be helpful later to mention that none of v_j's are in U. Then, define

$$W = \operatorname{span}(v_1, ..., v_n)$$

Suppose that $v \in V$. Then we can follow that $v \in v' + U$ for some v. Thus we follow that $v' = a_1v_1 + ...a_nv_n$ and therefore there exists $u \in U$ and $v' \in W$ such that $v = u + v'$. Thus $V = W + U$.

Suppose that $w \in W$ and $w \in U$. Then it follows that

$$w = a_1v_1 + ...a_nv_n$$

given that none of $v_1, ...v_n$ are in U, we follow that the only way that it is possible is when $a_1, ..., a_n = 0$. Thus we follow that

$$W \cap U = \{0\}$$

. Thus we can conclude that

$$V = W \oplus V$$

as desired.

3.5.18

Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in (V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \operatorname{null} T$.

In forward direction:

Suppose that there exists S such that $T = S \circ \pi$. Let $u \in U$. Then it follows that

$$\pi(u) = u + U = 0 + U$$

Because S is a linear function, we follow that $S(0) = S(U) = 0$. Thus we follow that $u \in \operatorname{null} T$. Thus we can conclude that $U \subseteq T$

In reverse direction:

Suppose that $U \subseteq \operatorname{null} T$. Then we follow that $\pi(U) = 0$. Create a map $S' : V/U \to V/U$ such that $\operatorname{null}(S' \circ \pi) = \operatorname{null} T$. Then there exists an invertible operator R such that $RS' \circ \pi = T$. Thus we follow that there exists a map $RS' = S$ such that $S \circ \pi = T$, as desired.

3.5.19

Find a correct statement analogous to 3.78 that is applicable to finite sets, with unions analogous to sums of subspaaces and disjoint unions analagous to direct sums

Suppose A is a finite set and $U_1, ..., U_m$ are subsets of A. Then

$$U_1 \cap ... \cap U_n = A$$

is a disjoint union if and only if

$$\sum |U_j| = |A|$$

3.5.20

Suppose U is a subspace of V. Define $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$$\Gamma(S) = S \circ \pi$$

(a) Show that Γ is a linear map

Let $S, T \in \mathcal{L}(V/U, W)$. Then

$$\Gamma(S + T) = (S + T) \circ \pi = S \circ \pi + T \circ \pi = \Gamma(S) + \Gamma(T)$$

If $\lambda \in F$, them

$$\Gamma(\lambda S) = (\lambda S) \circ \pi = \lambda S \circ \pi = \lambda \Gamma(S)$$

Therefore we can follow that Γ is linear

(b) Show that Γ is injective

Let $\Gamma(T) = 0$. It follows that

$$T \circ \pi = 0$$

Then we can follow that for $v \in V$

$$T \circ \pi(v) = 0 = T(v + U)$$

Thus $T = 0$. Therefore we follow that $\operatorname{null} \Gamma = \{0\}$, therefore it is injective.

(c) Show that $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0$ *for every* $u \in U\}$

Suppose that $T \in \operatorname{range} \Gamma$. Then it follows that there exists $S \in \mathcal{L}(V/U, W)$ such that

$$S \circ \pi = T$$

thus if $u \in U$ then

$$S \circ \pi(u) = S(u + U) = S(U) = 0$$

Suppose that $S \in \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$. Then it follows by the results in exercise 18, that there exists $T \in \mathcal{L}(V/U, W)$ such that

$$T = S \circ \pi$$

therefore by double inclusion we've got that $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$ as desired.

3.6 Duality

3.6.1

Explain why every linear functional is either surjective or the zero map

Dimention of its codomain is 1, therefore we've got that range is either 1 or 0. In former case it is surjective, in latter it's a null map.

3.6.2

Give three distinct examples of linear functionals on $R^{[0,1]}$

$$f(1) - f(0)$$
$$f(0.5)$$
$$f(0.2) + f(0.3)$$

3.6.3

Suppose V *is finite-dimentional and* $v \in V$ *with* $v \neq 0$. *Prove that there exists* $\phi \in V'$ *such that* $\phi(v) = 1$

We can extend v to a basis $v, v_1, ..., v_n$ of V, then define the dual basis $\phi, \phi_1, ..., \phi_n$ on this basis, and then get the desired function.

3.6.4

Suppose V *is finite-dimentional and* U *is a subspace of* V *such that* $U \neq V$. *Prove that there exists* $\phi \in V'$ *such that* $\phi = 0$ *for every* $u \in U$ *but* $\phi \neq 0$.

Simular to the previous one, we define $u_1, ..., u_n$ to be the basis of U, then expand it to a basis of V - $v_1, ..., v_m, u_1, ..., u_n$, and define dual basis on this basis.

The property that $U \neq V$ guaranteed that there exists v_1, therefore there exists ϕ_1 such that

$$\phi_1(v_1) = 1$$

thus $\phi_1 \neq 0$. And by definition of dual basis we get that $\phi_1(u_j) = 0$. Thus we follow that

$$u \in U \to u \in \text{span}(u_1, ..., u_n) \to \phi(u) = \phi(a_1 u_1) + ... + \phi(a_n u_n) = 0$$

as desired.

3.6.5

Suppose $V_1, ..., V_m$ *are vector spaces. Prove that* $(V_1 \times ... \; V_m)'$ *and* $V_1' \times ... \times V_m'$ *are isomorphic vector spaces.*

We follow it from the exercise 3.5.4.

3.6.6

Suppose V is finite-dimentional and $v_1, ..., v_m \in V$. Define a linear map $\Gamma : V' \to F^m$ by

$$\Gamma(\phi) = (\phi(v_1), ..., \phi(v_m))$$

(a) Prove that $v_1, ..., v_m$ spans V if and only if Γ is injective.

In forward direction: Suppose that $v_1, ...v_m$ spans V. Let $\phi \in V'$ be such that

$$\Gamma(\phi) = 0$$

Then we follow that

$$\Gamma(\phi) = (0, 0, ..., 0)$$

thus for $1 \leq j \leq m$ we've got that

$$\phi(v_j) = 0$$

Because $v_1, ..., v_m$ spans V we follow that if $v \in V$ then there exist $a_1, ..., a_m$ such that

$$v = \sum a_j v_j$$

thus we follow that

$$\phi v = \sum \phi v_j = 0$$

for any $v \in V$. Thus $\phi = 0$. Therefore we've got that

$$\Gamma \phi = 0 \to \phi = 0$$

thus we can follow that Γ is injective.

In reverse direction: Suppose that Γ is injective. We've going to proceed with a proof by contradiction on this one.

Suppose that $v_1, ..., v_m$ does not span V. Then we follow that we can create linearly independent list $v_1, ..., v_n$ by removing some elements from $v_1, ..., v_m$, if it is not already linearly independent. Then we can expand this list to $v_1, ..., v_n, v'_1, ...$ - basis of V. Then we can create a dual basis for this basis and get our ϕ', which will be a dual basis for v'_1. Given that $\phi'(v_j) = 0$ by definition of the dual basis, we follow that

$$\Gamma(\phi') = 0$$

Given that $\phi'(v'_1) \neq 0$ we follow that $\phi' \neq 0$. Thus $\operatorname{null} \Gamma \neq \{0\}$, therefore it is not injective, which is a contradiction.

(b) Prove that $v_1, ..., v_m$ is linearly independent if and only if Γ is surjective

In forward direction: Suppose that $v_1, ..., v_m$ is linearly independent. Then we follow that there exists a dual basis $\phi_1, ..., \phi_m$ of $v_1, ..., v_m$. Thus we follow that range of Γ has

$$(\phi_1(v_1), ..., \phi_1(v_m)) = (1, 0, ..., 0)$$

Therefore we can conclude that range of Γ contains a list of length m of linearly independent vectors. Given that $\dim(F^m) = m$, we follow that $F^m = \operatorname{range}\Gamma$. Thus Γ is surjective, as desired.

In reverse direction: Suppose that $v_1, ..., v_m$ are linearly independent. Then we follow that there exists a space $\operatorname{span}(v_1, ..., v_m)$, for which $v_1, ..., v_m$ is a basis. Thus we follow that for this basis there exists a dual basis $\phi_1, ..., \phi_m$, where $\phi_j \in V'$. Thus we follow that

$$\Gamma(\phi_1) = (\phi_1(v_1), ..., \phi_1(v_m)) = (1, 0, ...0) \in \operatorname{range}\Gamma$$

$$\Gamma(\phi_j) = (\phi_j(v_1), ..., \phi_j(v_j), ..., \phi_1(v_m)) = (0, ..., 1, ...0) \in \operatorname{range}\Gamma$$

Thus we follow that there exist a linearly independent list of length m in $\operatorname{range}\Gamma$. Thus we can follow that the dimention of $\operatorname{range}\Gamma$ is at least m. Then we can state, that because $\dim F^m = m$, $\operatorname{range}\Gamma$ is a subspace of F^m and the fact that a subspace has a dimention less or equal to the original space, we can follow that $\operatorname{range}\Gamma = F^m$. Thus we follow that Γ is surjective, as desired.

3.6.7

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, x^2, ..., x^m$ of $P_m(R)$ is $\phi_0, ..., \phi_m$ where $\phi_j(p) = \frac{p^{(j)}(0)}{j!}$.

It's straightforward to check that $(x^j)^{(j)} = j!$ (proof by induction will be useful in this case). Thus we follow that

$$\phi_j(x^j) = 1$$

If $k < j$, then we can follow that

$$\phi_j(x^k) = 0$$

and if $k > j$, then

$$\phi_j(x^k) = \frac{k!}{j!}0^{k-j} = 0$$

thus we get that

$$\phi_j(p_k) = \begin{cases} 0 \text{ if } k \neq j \\ 1 \text{ if } k = j \end{cases}$$

thus it is a dual basis by definition.

3.6.8

Suppose m is a positive

(a) Show that $1, x-5, ..., (x-5)^m$ is a basis for $P_m(R)$.

All of them are linearly independent, since all of them have different degrees, and the fact that length of the list is equal to the dimention of the space that it is in, implies that it is a basis for this space.

(b) What is a dual basis of the basis in part (a)

If we try to create this basis by drawing inspiration from the previous exercise, then we'll get

$$\phi_j(p_k) = \frac{p_k^{(j)}(5)}{j!}$$

some basic implication will show that it is indeed the dual basis for given basis.

3.6.9

Suppose $v_1, ..., v_m$ isi a basis of V and $\phi_1, ..., \phi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \sum \psi(v_j)\phi_j$$

We know that beacause $\phi_1, ..., \phi_m$ is a basis of V' that

$$\psi = \sum a_j \phi_j$$

for some $a_1, ..., a_m \in F$. Thus we follow that

$$\psi(v_j) = \sum a_j \phi_j(v_j) = \sum_{k \neq j} [a_k \phi_k(v_j)] + a_j \phi_j(v_j) = 0 + a_j * 1 = a_j$$

thus

$$\psi = \sum a_j \phi_j = \sum \psi(v_j)\phi_j$$

as desired.

3.6.10

Prove the first two bullet points in 3.101

Specifically we want to prove that

$$(S+T)' = S' + T'$$

$$(\lambda T)' = \lambda T'$$

$$(S+T)'(\phi) = \phi(S+T) = \phi S + \phi T = S' + T'$$

where the second equality comes from distributive properties of linear maps.

$$(\lambda T)'(\phi) = \phi(\lambda T) = \lambda \phi T = \lambda T'$$

3.6.11

Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, ..., c_m) \in F^m$ and $(d_1, ..., d_n) \in F^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, ..., m$ and every $k = 1, ..., n$.

In forward direction: Suppose that rank of A is equal to 1. Then we follow that the dimension of span of columns is 1. Thus we follow that every column is a scalar multiple of the first non-zero column. Thus we can set

$$v_j = a_j v_m$$

where v_m is the first non-zero column and a_j is corresponding coefficient. Thus we can create $(1, a_2, ..., a_m)$ - vector of corresponding multiplicities. Thus we can conclude that there exists vectors v_1 and $(1, a_2, ..., a_m)$ with the desired properties.

In reverse direction: Suppose that there exist two vectors, as defined in the exercise. Then we can follow that every column is a scalar multiple of the first non-zero column. Therefore we follow that the column rank of given matrix is 1, as desired.

3.6.12

Show that the dual map of the identity map on V is the identity map on V'

Let I' be the dual map of the identity. Then we can follow that for every $\phi \in V'$

$$I'(\phi) = \phi I = \phi$$

thus I' is the identity on V', as desired.

3.6.13

Define $T : R^3 \to R^2$ by $T(x, y, z) = (4x+5y+6z, 7x+8y+9z)$. Suppose ϕ_1, ϕ_2 denotes the dual basis of the standart basis of R^2 and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of R^3.

(a) Describe the linear functions $T'(\phi_1)$ and $T(\psi_2)$

We can follow that

$$M(T) = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

thus

$$M(T') = (M(T))^t = \begin{pmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{pmatrix}$$

also, ϕ_1 is represented by $(1, 0)$ and ϕ_2 is represented by $(0, 1)$. Therefore

$$M(T'(\phi_1)) = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

and

$$M(T'(\phi_2)) = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$$

thus

$$T'(\phi_1) = 4x + 5y + 6z$$

$$T'(\phi_2) = 7x + 8y + 9z$$

(b) Write $T'(\phi_1)$ and $T'(\phi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.
Given that

$$\psi_1 = (1, 0, 0)$$

$$\psi_2 = (0, 1, 0)$$

$$\psi_3 = (0, 0, 1)$$

we follow that

$$T'(\phi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$$

$$T'(\phi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$$

3.6.14

Define $T : P(R) \to P(R)$ by $(Tp)(x) = x^2p(x) + p''(x)$ for $x \in R$.

(a) Suppose $\phi \in P(R)'$ is defined by $\phi(p) = p'(4)$. Describe the linear functional $T'(\psi)$ on $P(R)$.

$$T'(\psi)(p) = \psi(Tp) = \phi(x^2p + p''(x)) = 2xp + x^2p + p'''(x)$$

(b) Suppose $\phi \in P(R)'$ is defined by $\phi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\phi))(x^3)$

$$T'(\phi)(p) = \phi(Tp) = \phi(x^2p + p''(x)) = \phi(x^2x^3 + 6x) = \phi(x^5 + 6x) =$$

$$= \int_0^1 x^5 + 6x = [x^6/6 + 3x^2]_0^1 = 1/6 + 3 = 3\frac{1}{6}$$

3.6.15

Suppose W is finite-dimentional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

In forward direction: Suppose that $T' = 0$. Then we can follow that $\phi T(v) = 0$ for every $\phi \in W'$ and $v \in v$. Suppose that $T \neq 0$. Then there exists $v \in V$ such that $Tv \neq 0$. Thus we can create a basis $Tv, w_1, ..., w_m$ of W and define the dual basis of this basis. Thus we follow that there exists ϕ_1 such that

$$\phi_1(Tv) = 1$$

Thus

$$\phi_1(Tv) = T'(\phi)(v) \neq 0$$

therefore we follow that $T' \neq 0$, which is a contradiction. Thus we can conclude that $T' = 0$ implies that $T = 0$.

In reverse direction: Suppose that $T = 0$. Then we follow that $\phi T v = T'(\phi)(v) = \phi 0 = 0$. Thus we conclude that for any $\phi \in V$ $T'(\phi) = 0$. Thus $T' = 0$.

3.6.16

Suppose V and W are finite-dimentional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Let us denote this map as $\pi : \mathcal{L}(V, W) \to \mathcal{L}(W', V')$

We know that

$$\pi(S + T) = (S + T)' = S' + T' = \pi S + \pi T$$

and

$$\pi(\lambda T) = (\lambda T)' = \lambda(T') = \lambda(\pi T)$$

thus we can follow that π is a linear map.

We've already proven that $T = 0$ if and only if $T' = 0$, thus we can follow that given map is injective.

Because V and W are finite-dimentional, we can follow that $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$ are finite dimentional. Thus we can make bases of those vector spaces

$$\kappa_1, ..., \kappa_m \in \mathcal{L}(V, W)$$

$$\gamma_1, ..., \gamma_n \in \mathcal{L}(W', V)$$

and create an invertible map $\zeta \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(W', V'))$ such that

$$\zeta(\sum a_j \kappa_j) = \sum a_j \gamma_j$$

therefore we can follow that $\pi\zeta^{-1}$ is an operator on $\mathcal{L}(W', V')$. By invertibility of ζ we follow that ζ^{-1} is invertible as well and therefore it is injective. Thus $\pi\zeta^{-1}$ is a composition of injective linear maps, and therefore it is itself injective. Given that $\pi\zeta^{-1}$ is an operator on a finite-dimentional vector space, we follow that its injectivity implies inversibility. Therefore we can follow that π is also inversible, as desired. (I just wanted to go this road to get surjectivity, but got the whole inversibility from it.)

3.6.17

Suppose $U \subseteq V$. Explain why $U^0 = \{\phi \in V' : U \subseteq \operatorname{null}\phi\}$

Suppose that $\phi \in U^0$. Then we can follow that $\phi(u) = 0$, therefore $u \in \operatorname{null}\phi$ for every $u \in U$. Thus $U \subseteq \operatorname{null}\phi$. Thus $U^0 \subseteq \{\phi \in V' : U \subseteq \operatorname{null}\phi\}$.

Now let ϕ be such that $U \subseteq \operatorname{null}\phi$. Then we can follow that for all $u \in U$

$$\phi(u) = 0$$

thus $\phi \in U^0$. Therefore $U^0 \supseteq \{\phi \in V' : U \subseteq \operatorname{null}\phi\}$ Therefore by double inclusion we've got the desired equality.

3.6.18

Suppose V is finite-dimentional and $U \subseteq V$. Show that $U = \{0\}$ is and only if $U^0 = V'$.

In forward direction: Suppose that $U = \{0\}$. Then we can follow that

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}$$

U^0 is a subspace of V', therefore we've got that $U^0 \subseteq V'$. Suppose that $\phi \in V'$. Then it follows that

$$\phi(0) = 0$$

because it's a linear function. Thus we follow that $V \subseteq U^0$. Thus by double inclusion we've got that $V' = U^0$.

In reverse direction: Suppose that $U^0 = V'$. Now suppose that $u \in U$. Then we can follow that there exist $a_1, ..., a_m$ such that

$$u = \sum a_j v_j$$

for some basis $v_1, ..., v_m$ of V. For this basis there exists a dual basis $\phi_1, ..., \phi_m$. Thus we can follow that

$$0 = \phi_j(u) = \phi(\sum a_k v_k) = \phi_j(\sum_{k \neq j} a_k v_k) + \phi(a_j v_j) = a_j$$

Thus $a_1 = ... = a_m = 0$. Therefore $u = 0$. Thus $U = \{0\}$, as desired.

3.6.19

Suppose V is finite-dimentional and U is a subpace of V. Show that $U = V$ if and only if $U^0 = \{0\}$.

In forward direction: Suppose that $U = V$. Then it follows that $\dim U = \dim V$. Thus

$$\dim V = \dim U + \dim U^0$$

$$\dim V = \dim V + \dim U^0$$
$$\dim U^0 = 0$$

thus $U^0 = \{0\}$.

In reverse direction:

Suppose that $U^0 = 0$. Then

$$\dim V = \dim U + \dim U^0$$

$$\dim V = \dim U$$

Thus $U = V$.

3.6.20

Suppose U and W are subsets of V with $U \subseteq W$. Prove that $W^0 \subseteq U^0$.

Suppose that $\phi \in W^0$. Then we follow that if $u \in U$ then $u \in W$, and therefore $\phi(u) = 0$ for any $u \in U$. Thus $\phi \in U^0$. Therefore $\phi \in W^0 \to \phi \in U^0$. Thus $W^0 \subseteq U^0$, as desired.

3.6.21

Suppose V is finite-dimetionmal and U and W are subspaces of V with $W^0 \subseteq U^0$. Prove that $U \subseteq W$.

Let $w_1, ..., w_m$ be a basis of W and let us extend this basis to basis $w_1, ..., w_m, ..., w_n$ of V. Then we can follow that there exists a dual basis $\phi_1, ..., \phi_m, ..., \phi_n$ of V. Therefore we can follow that if $\psi \in \operatorname{span}(\phi_{m+1}, ..., \phi_n)$, then

$$\psi(w) = 0$$

therefore $\operatorname{span}(\phi_{m+1}, ..., \phi_n) \subseteq W^0$. Conversely if $\psi \in W^0$, then we can follow that

$$\psi = 0\phi_1 + ... + 0\phi_m + a_{m+1}\phi_{m+1} + ...$$

(otherwise $\psi(w_j) \neq 0$.) Therefore we can follow that $W^0 \subseteq \operatorname{span}(\phi_{m+1}, ..., \phi_n)$. Thus we've got that $W^0 = \operatorname{span}(\phi_{m+1}, ..., \phi_n)$. Therefore $\phi_{m+1}, ..., \phi_n$ is a basis for W^0.

Suppose now that there exists $u \in U$ such that $u \notin W$. Then we can follow that

$$u = \sum a_j w_j$$

Because $u \notin W$ we can follow that there exists $k > m$ such that $a_k \neq 0$. Thus $\phi_k(u) = a_k \neq 0$. Given that $k > m$, we follow that $\phi_k \in W^0$. Thus we follow that $\phi_k \in U^0$. Therefore $\phi_k(u) = 0$, which is a contradiction.

Therefore we follow that there does not exist $u \in U$ such that $u \notin W$. Thus we can follow that $u \in U \to u \in W$. Therefore $U \subseteq W$, as desired.

3.6.22

Suppose U, W are subspaces of V. Show that $(U+W)^0 = U^0 \cap W^0$.

Suppose that $\phi \in (U+W)^0$. Then we can follow that for every $u \in U$, $\phi(u) = 0$. therefore $\phi \in U^0$. By the same logic we have that $\phi \in W^0$. Thus we can follow that

$$\phi \in (U+W)^0 \to \phi \in U^0 \wedge \phi \in W^0$$

$$\phi \in (U+W)^0 \to \phi \in U^0 \cap W^0$$
$$(U+W)^0 \subseteq U^0 \cap W^0$$

Conversely, suppose that $\phi \in U^0 \cap W^0$. Then we follow that if $v = u + w \in U + W$, then

$$\phi(v) = \phi(u+w) = \phi(u) + \phi(w) = 0 + 0 = 0$$

Thus we follow that

$$\phi \in U^0 \cap W^0 \to \phi \in (U+W)^0$$
$$U^0 \cap W^0 \subseteq (U+W)^0$$

thus by double inclusion we've got the desired equality.

3.6.23

Suppose V is finite-dimentional and U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$

Because V is finite-dimentional we can follow that U and W are both finite-dimentional, and therefore $U \cap W$ is finite dimentional (the fact that it is a subspace was proven in the exercises before). We can therefore make a basis $u_1, ..., u_n, ..., u_m, ..., u_l, ..., u_k$ where $u_n, ..., u_m$ is a basis of $U \cap W$ (empty in case if the dimention of intersection is zero), $u_1, ..., u_{n-1}$ is the basis of U, $u_{m+1}, ..., u_l$ is the rest of the basis of W and the rest is the basis of V. Then we can create a dual basis on this basis and get that

$$U^0 + W^0 = \text{span}(\phi_1, ..., \phi_{n-1}, ..., \phi_{m+1}, ..., \phi_k) = (U \cap W)^0$$

as desired.

3.6.24

Prove 3.106 using the ideas scketched in the discussion before the statement of 3.106

We've outlined this proof in 3.6.21.

3.6.25

Suppose V is finite-dimentional and U is a subspace of V. Show that

$$U = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

Let $u_1, ..., u_n$ be a basis of U, and extend this basis to $u_1, ..., u_n, ..., u_m$ - a basis of V. Then let us define dual basis on this basis $\phi_1, ..., \phi_m$, and it'll follow that if $v \notin U$, then there will exist ϕ_k for some $k > n$ such that

$$\phi_k(v) \neq 0$$

Thus we can follow that

$$v \notin U \to v \notin \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

and therefore

$$v \in \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\} \to v \in U$$

thus

$$\{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\} \subseteq U$$

Conversely, suppose that $u \in U$. Then it follows that $\phi \in U^0 \to \phi(u) = 0$. Thus we've got that $U \subseteq \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$.

Thus by double inclusion we've got our desired equality.

3.6.26

Suppose V is finite-dimentional and Γ is a subspace of V'. Show that

$$\Gamma = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

Suppose that Γ is a subspace of V'. Then we can follow that there exists a basis of Γ - $\phi_1, ..., \phi_n$, and that we can extend this basis to a basis of V' - $\phi_1, ..., \phi_m$.

For every ϕ_j we've got that $\phi_j \neq 0$ and therefore by FTLM we've got that

$$\dim V' = \dim \operatorname{null} \phi_j + \dim \operatorname{range} \phi_j$$

$$\dim V' = \dim \operatorname{null} \phi_j + 1$$

$$\dim V' - 1 = \dim \operatorname{null} \phi_j$$

thus we can follow that if we take a basis of $\operatorname{null} \phi_j$, extend it to a basis of V by adding one vector v_j', then

$$\phi_j v_j' \neq 0$$

then we can define

$$v_j = v'_j * \frac{1}{\phi_j v_j}$$

so that

$$\phi_j(v_j) = 1$$

By making v_j for each corresponding ϕ_j we can get the list $v_1, ..., v_m$. Because every ϕ_m has a diffenent nullspace (otherwise they are scalar multiple of each other) we can follow that they have different preimage, and therefore $v_1, ..., v_m$ is linearly independent. By length of this linearly independent list we can follow that it is a basis of V.

Then we can follow that there exist vectors $v_1, ..., v_m$ such that

$$\phi_j(v_j) = 1$$

and for $k \neq j$

$$\phi_k(v_j) = 0$$

Therefore we'll have a subspace $U = \operatorname{span}(v_{n+1}, ..., v_m)$. Then it follows that

$$\Gamma = U^0$$

and by results of our previous exercise we'll have that

$$U = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

$$U^0 = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}^0$$

$$\Gamma = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

as desired.

3.6.27

Suppose $T \in \mathcal{L}(P_5(R), P_5(R))$ and $\operatorname{null} T' = \operatorname{span}(\phi)$, where ϕ is the linear functional on $P_5(R)$ defined by $\phi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in P_5(R) : p(8) = 0\}$.

By a theorem in the chapter we've got that

$$\operatorname{null} T' = (\operatorname{range} T)^0$$

and by results of the previous exercise we've got that

$$\operatorname{span}(\phi) = \{p \in P_5(R) : \phi(p) = 0\}^0$$

(we've shortened the right-hand side, since $\psi \in \operatorname{span}(\phi) \wedge \psi(v) = 0 \to \phi(v) = 0$). thus we can follow taht

$$\operatorname{null} T' = \operatorname{span}(\phi)$$

is equivalent to stating that

$$(\operatorname{range} T)^0 = (\{p \in P_5(R) : \phi(p) = 0\})^0$$

For the next implication we'll probably need a little lemma

Lemma: Equivalent annihilators implies equivalent subspaces

Suppose that V is finite-dimentional, U and W are subspaces of V. Then $U^0 = W^0$ implies $U = W$.

By exercise 21 in this chapter we've got that

$$W^0 \subseteq U^0 \to U \subseteq W$$

thus we follow that

$$W^0 = U^0 \to W^0 \subseteq U^0 \wedge U^0 \subseteq W^0 \to U \subseteq W \wedge W \subseteq U \to W = U$$

as desired. (If we think about it, exercise 20 gives us that this statement is actually an equivalence and not implicaton. For infinite-dimentional vector spaces we probably got just the implication)

Thus we can indeed follow that

$$(\operatorname{range} T)^0 = (\{p \in P_5(R) : \phi(p) = 0\})^0$$

implies that

$$\operatorname{range} T = \{p \in P_5(R) : \phi(p)$$

as desired

3.6.28

Suppose V and W are finite-dimentional, $T \in \mathcal{L}(V, W)$, and there exists $\phi \in W'$ such that $\operatorname{null} T' = \operatorname{span} \phi$. *Prove that* $\operatorname{range} T = \operatorname{null} \phi$

This is a generalization of the previous exercise

$$\{v \in V : \psi(v) = 0 \text{ for every } \psi \in \operatorname{span}(\phi)\} = \{v \in V : \lambda\phi(v) = 0\} = \{v \in V : \phi(v) = 0\} = \operatorname{null} \phi$$

Thus

$$\operatorname{null} T = (\operatorname{range} T)^0 = (\operatorname{null} \phi)^0$$

thus

$$\operatorname{range} T = \operatorname{null} \phi$$

as desired.

3.6.29

Suppose V and W are finite-dimentional, $T \in \mathcal{L}(V, W)$, and there exists $\phi \in V'$ such that $\operatorname{range} T' = \operatorname{span} \phi$. *Prove that* $\operatorname{null} T = \operatorname{null} \phi$

$$\operatorname{range} T' = (\operatorname{null} T)^0 = \operatorname{span} \phi = (\operatorname{null} \phi)^0$$

thus

$$(\operatorname{null} T)^0 = (\operatorname{null} \phi)^0$$
$$\operatorname{null} T = \operatorname{null} \phi$$

as desired.

3.6.30

Suppose V is finite-dimentional and $\phi_1, ..., \phi_m$ is a linearly independent list in V'. Prove that

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) = (\dim V) - m$$

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) + \dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = (\dim V)$$

Thus what we probably are intended to prove is that

$$((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = \operatorname{span}(\phi_1, ..., \phi_m)$$

By exercise 26 we can get that

$$\operatorname{span}(\phi_1, ..., \phi_m) = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

Since $\phi \in \operatorname{span}(\phi_1, ..., \phi_m) \to \phi = \sum a_j \phi_j$, we follow that

$$\operatorname{span}(\phi_1, ..., \phi_m) = \{v \in V : \phi_j(v) = 0 \text{ for every } \phi_j\}^0$$

If $\phi_j(v) = 0$ for every ϕ_j, then $v \in \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$ du definition of nullspace. If $v \in \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$, then it is obliously true that $\phi_j(v) = 0$ by the same definition. Thus by double inclusion we get that

$$\{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\} = \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$$

thus

$$\operatorname{span}(\phi_1, ..., \phi_m) = (\operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m)^0$$

and therefore

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) + \dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = (\dim V)$$

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + \dim \operatorname{span}(\phi_1, ..., \phi_m) = (\dim V)$$

since $\phi_1, ..., \phi_m$ are linearly independent, we can follow that

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + m = (\dim V)$$

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + = (\dim V) - m$$

as desired.

3.6.31

Suppose V is finite-dimentional and $\phi_1, ..., \phi_n$ is a basis of V'. Show that there exists a basis of V, whose dual basis is $\phi_1, ..., \phi_n$.

From previous exercice we can follow that there for given basis there exists a space

$$U_k = \cap_{i \neq k} \operatorname{null}\phi_i$$

such that

$$\dim U_k = \dim V - (n-1) = n - (n-1) = 1$$

From the definition we can follow that

$$u \in U_k \to u \in \operatorname{null}\phi_{i \neq k} \to \phi_{i \neq k}(u) = 0$$

Suppose that $u \neq 0$. Then we can follow that we can extend this vector to the basis of V. Thus we can follow that we can create a map $\psi \in \mathcal{L}(V, F) \iff \psi \in V'$ such that

$$\psi(u) = 1$$

Given that $\phi_1, ..., \phi_n$ spans V', we can follow that there exist $a_1, ..., a_n$ such that

$$\psi = \sum a_j \phi_j$$

from this we can follow that

$$\psi(u) = \sum a_j \phi_j(u) = \sum_{j \neq k} a_j \phi_j(u) + a_k \phi_k(u) = 0 + a_k \phi(u) = a_k \phi_k(u)$$

Thus we can follow that

$$a_k \phi(u) = \psi(u) = 1$$

$$a_k \phi(u) = 1$$

thus we follow that $a_k \neq 0$ and $\phi(u) \neq 0$. Thus we follow that $\phi(a_k u) = 1$. Set $v_k = a_k u$.

In this fashion we can create list $v_1, ..., v_n$ with the property that

$$\phi_k(v_k) = 1$$

$$\phi_{i\neq k}(v_k) = 0$$

The only thing that is left is to show that this list is linearly independent. Suppose that it isn't. Then it follows that

$$v_k = \sum_{j\neq k} a_j v_j$$

thus

$$\phi_k(v_k) = \phi_k(\sum_{j\neq k} a_j v_j)$$
$$1 = 0$$

which is false. Thus we follow that $v_1, ..., v_n$ is linearly independent list with desired properties in V. Given that its length is the dimention of the space, we follow that it is a basis, for which $\phi_1, ..., \phi_n$ is dual basis, as desired.

3.6.32

Suppose $T \in \mathcal{L}(V)$, and $u_1, ..., u_n$ and $v_1, ..., v_n$ are bases of V. Prove that the following are equivalent:

(a) T is invertible

(b) The columnns of $M(T)$ are linearly independent in $F^{n,1}$.

(c) The columns of $M(T)$ span $F^{n,1}$

Because T is invertible, we can follow that the dimention of its range is equal to n. Thus we can follow that rank of its matrix is n. Therefore its row rank and column rank is n. Thus (a) implies (b). By the size of the mateix we get that (b) implies (c). And (c) implies that with given bases the function is invertible, therefore (c) implies (a).

Because T is invertible if and only if T' is invertible, we get the (d) and (e) for free.

3.6.33

Suppose m and n are positive integers. Prove that teh function that takes A to A^t is a linear map from $F^{m,n}$ to $F^{n,m}$. Furtermore, prove that this linear maps is invertible.

Linearity follows directly from definitions of the transpose, matrix addition and scalar multiplication. Invertability follows directily from injectifity and surjectivity of this transformation.

3.6.34

The double dual space of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda(v))(\phi) = \phi(v)$$

for $v \in V$ and $\phi \in V'$

Since I haven't fully understood the definition, I'll try to dumb it down a notch. Suppose that $v \in V$. Then we follow that there exists $\kappa \in V''$ such that

$$\Lambda(v) = \kappa$$

Thus

$$\kappa \in V'' \to \kappa \in (V')' \to \kappa \in \mathcal{L}(V', F) \to \kappa \in \mathcal{L}(\mathcal{L}(V, F), F)$$

Thus we can plug in some $\phi \in V'$ into κ, and get some number from it.

What could be an example of such a map? Suppose that $p \in P(R)$. Then we can define $\phi(p) = \int_0^1 p$. And thus we can define Λ to be a function, that inputs a polynomial, and returns a function, that inputs a linear functional on a polynomial, and returns the result of applying inputed polynomial into the linear functional.

That kind of made a dent in the understanding of what the hell is going on, but nothing major had happened.

(a) Show that Λ is a linear map from V to V''.

$$(\Lambda(\lambda v))(\phi) = \phi(\lambda v) = \lambda(\phi(v)) = \lambda((\Lambda(v))(\phi))$$
$$(\Lambda(v + w))(\phi) = \phi(v + w) = \phi(v) + \phi(w) = (\Lambda(w))(\phi) + (\Lambda(w))(\phi)$$

Thus we follow that Λ is linear, as desired.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

Let $v \in V$ and $\phi \in V'$. Then we can follow that

$$(T'' \circ (\Lambda v))(\phi) = (\Lambda v)(T'(f)) = (T'f)(v) = f(T(v)) = \Lambda(Tv)(f)$$

as desired.

(c) Show that if V is finite-dimentional, then Λ is an isomorphism from V onto V''

If V is finite-dimentional, then V'' is finite-dimentional, and their dimentions are the same. Thus injectivity of Λ implies the invertibility. Suppose that $\Lambda v = 0$. Then we follow that $\phi(v) = 0$ for any $\phi \in V'$. By exercise 19 we've got that it happens if and only if $v = 0$. Thus Λ is injective, and therefore invertible, as desired.

3.6.35

Show that $(P(R))'$ and R^∞ are isomorphic.

$P(R)$ and $(P(R))'$ are isomorphic, $P(R)$ and R^∞ are also isomorphic, therefore $(P(R))'$ and R^∞ are isomorphic, as desired.

3.6.36

Suppose U is a subspace of V. Let $i : U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$

$$\operatorname{null} i' = (\operatorname{range} i)^0 = U^0$$

(b) Prove that if V is finite-dimentional, then $\operatorname{range} i' = U'$.

Let $v_1, ..., v_n, ..., v_m$ be a basis of V, where $v_1, ..., v_n$ is a basis of U. Define dual basis of it. Then we follow that $v_{n+1}, ..., v_n$ is a basis of $\operatorname{null} i$, and therefore $\psi_1, ...\psi_n$ is a basis of $(\operatorname{null} i)^0$, which is equal to basis of U'. THus

$$(\operatorname{null} i)^0 = U'$$

and since

$$\operatorname{range} i' = (\operatorname{null} i)^0$$

we follow that

$$\operatorname{range} i' = U'$$

as desired.

(c) Prove that if V is finite-dimentional, then $\overline{i'}$ is an isophormism from V'/U^0 onto U'.

$$\dim V'/U^0 = \dim V' - \dim U^0 = \dim V - (\dim V - \dim U) = \dim U$$

and

$$\dim U' = \dim U$$

thus we follow that the dimentions of those vector spaces are the same.

Since $\operatorname{range} i' = U'$, we can follow that $\operatorname{range} \overline{i'} = \operatorname{range} i' = U'$. Thus we follow that this map is sujective, and therefore by their identical dimention we follow that $\overline{i'}$ is an isomorphism, as desired.

3.6.37

Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the usual quotent map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

Sinse π is surjective (suppose that $a \in (V/U)$, then $a = v + U$, then there exists $v \in V$, then $\pi(v) = v + U$, therefore it is surjective.) we can follow that π' is injective, as desired.

(b) Show that $\operatorname{range} \pi' = U^0$.

$$\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$$

(the fact that null $\pi = U$ is somewhat followed in the proof of 3.89, where it is derived from 3.85)

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

π' is inhective and is surjective, if we restrict its codomain to U^0. Thus it is invertible, therefore it is an isomorphism.

Chapter 4

Polynomials

4.1 Polynomials

4.1.1

Verify all assertions in 4.5 except the last one.

Firstly, let us state that $z, w \in C$ and

$$z = a + bi$$

$$w = c + di$$

Then we follow that

$$z + \overline{z} = a + bi + a - bi = 2a = 2Re(z)$$

$$z - \overline{z} = a + bi - a + bi = 2bi = 2Im(z)i$$

$$z\overline{z} = (a+bi)(a-bi) = a^2+abi-abi-b^2 = a^2+b^2 = (\sqrt{a^2+b^2})^2 = (\sqrt{Re(z)^2+Im(z)^2})^2 = |z|^2$$

$$\overline{w+z} = \bar{(}a+bi+c+di) = \bar{(}a+c+(b+d)i) = a+c-(b+d)i = a-bi+c-di = \overline{z}+\overline{w}$$

$$\overline{\overline{z}} = \overline{\overline{a+bi}} = \overline{a-bi} = a+bi = z$$

$$|a^2| = a^2 \leq a^2+b^2 = |a^2+b^2| \rightarrow |a|^2 \leq a^2+b^2 \rightarrow |a| \leq \sqrt{a^2+b^2} \rightarrow |Re(z)| \leq |z|$$

$$|\overline{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} = |z|$$

$$|wz| = |(a+bi)(c+di)| = |ac+adi+cbi-bd| = |ac-bd+(ad+cb)i| = \sqrt{(ac-bd)^2+(ad+cb)^2} =$$

$$= \sqrt{(ac)^2-2abcd+(bd)^2+(ad)^2+2abcd+(cb)^2} = \sqrt{(ac)^2+(bd)^2+(ad)^2+(cb)^2} =$$

$$= \sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)} = \sqrt{(a^2+b^2)(c^2+d^2))} = |z||w| = |w||z|$$

4.1.2

Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(F) : \deg p = m\}$$

a subspace of $\mathcal{P}(F)$?

Suppose $m \neq 1$ Let $p_1 = x^m + 1$ and $p_2 = -x^m + 1$. Then we follow that

$$\deg(p_1 + p_2) = \deg(2) = 0 \neq m$$

(we use the fact that m must be positive). Thus given set is not closed under addition, therefore it is not a subspace.

4.1.3

Is the set

$$\{0\} \cup \{p \in \mathcal{P}(F) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(F)$?

Let $p_1 = x^2 + x$ and $p_2 = -x^2 + x$. Then we follow that

$$\deg(p_1 + p_2) = \deg(2x) = 1$$

thus the space is not closed under addition, therefore it is not a subspace.

4.1.4

Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, ..., \lambda_m \in F$. Prove that there exists a polynomial $p \in P(F)$ with $\deg(p) = n$ such that $0 = p(\lambda_1) = ... = p(\lambda_n)$ and such that p has no other zeroes.

Let

$$p = (z - \lambda_1)...(z - \lambda_m)^{m-n+1}$$

Then we can follow that the only zeroes of p are precisely zeroes of

$$p_1 = (z - \lambda_1)...(z - \lambda_m)$$

which are $\lambda_1, ..., \lambda_m$ Then we follow that zeroes of

$$p_2 = (z - \lambda_1)...(z - \lambda_m)^2 = p_1(z - \lambda_m)$$

are zeroes of p_1 and λ_m. Given that zeroes of p_1 already have λ_m, we follow that p_1 and p_2 have the same zeroes. Then by induction we follow that zeroes of p are zeroes of p_1, which are $\lambda_1, ..., \lambda_m$, as desired.

4.1.5

Suppose m is nonnegative integer, $z_1, ..., z_{m+1}$ are distinct elements of F, and $w_1, ..., w_{m+1} \in F$. Prove that there exists a unique polynomial $p \in P_m(F)$ such that

$$p(z_j) = w_j$$

for $j = 1, ...m + 1$.

Let $T : P_m(F) \to F^{m+1}$ be defined as

$$T(p) = (p(z_1), p(z_2), ..., p(z_{m+1}))$$

We can follow that if

$$T(p) = 0$$

then

$$p(z_1) = p(z_2) = ... = p(z_{m+1}) = 0$$

But if $p \neq 0$, then it has at most m roots. Thus we follow that $p = 0$. Therefore T is injective. By FTLM we have that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Therefore for this case we have

$$\dim P_m(F) = \dim \operatorname{null} T + \dim \operatorname{range} T$$

$$m + 1 = \dim \operatorname{range} T$$

given that $\dim F^{m+1} = m + 1$ we follow that $\operatorname{range} T = F^{m+1}$. Therefore we follow that it is surjective. Thus for any vector $(w_1, ..., w_{m+1}) \in F^{m+1}$ there exists $p \in P_m(F)$ such that

$$p(z_j) = w_j$$

as desired.

4.1.6

Suppose $p \in \mathcal{P}(C)$ has degree m. Prove that p has m distinct zeroes if and only if p and its derivative p' have no zeroes in common

Suppose that p has m distinct zeroes and suppose that $p(\lambda_j) = 0$ for some $\lambda \in F$. Then we follow that

$$p = a(x - \lambda_1)(x - \lambda_2)...(x - \lambda_n)$$

By the product rule of differentiation we've got that

$$p' = a(\prod_{i \neq j}(x - \lambda_i) + \text{... some terms that have } (x - \lambda_j)\text{...})$$

and therefore

$$p'(\lambda_j) = a(\prod_{i \neq j} \lambda_j - \lambda_i)$$

Given that every λ is distinct, we follow that $(\lambda_j - \lambda_i) \neq 0$, and therefore $p'(\lambda_j) \neq 0$. Thus λ_j is not a zero for p'. for any $1 \leq j \leq n$.

Now supppose that p and p' don't have any zeroes in common. Thus we can follow that

$$p' = a(\sum_{i=1}^{n} \prod_{i \neq j} (x - \lambda_i))$$

thus we follow that if p has less then m distinct values, then $p'(\lambda) = 0$, therefore we've got that the polinomial and its derivative does not have any common zeroes.

4.1.7

Prove that every polynomial of odd degree with real coefficient has a real zero.

Suppose that p is a polynomial of an odd degree m. Then it follows that it has can be factorized as

$$p(z) = c(z - \lambda_1)...(z - \lambda_m)$$

By the fact that $p(\lambda) = 0 \to p(\overline{\lambda}) = 0$ we can follow that there must exsist λ_j such that $\lambda_j = \overline{\lambda_j}$, therefore $\lambda_j \in R$, as desired.

4.1.8

Define $T : P(R) \to R^R$ by

$$Tp = \begin{cases} \frac{p-p(3)}{x-3} \text{ if } x \neq 3 \\ p'(3) \text{ if } x = 3 \end{cases}$$

Show that $Tp \subseteq P(R)$ for every polynomial $p \in P(R)$ and that T is a linear map.

We know that p is a polynomial, therefore $p(3)$ is a constant, which is also a polynomial. Thus $p(x) - p(3)$ is a polynomial. Thus if $x = 3$, then $p(x) - p(3) = 0$. Thus we can follow that $p(x) - p(3)$ has a factorization

$$p(x) - p(3) = (x - 3)q(x)$$

thus

$$\frac{p - p(3)}{x - 3} = \frac{(x - 3)q(x)}{x - 3}$$

given that $x \neq 3$ we follow that

$$\frac{(x-3)q(x)}{x-3} = q(x)$$

When I looked at this thing one more time, I've noticed that it looks suspiciously simular to a definition of the derivative of the polynomial.

Thus we can conclude that by definition of derivative

$$p'(3) = \lim_{x \to 3} \frac{p(x) - p(3)}{x-3} = \lim_{x \to 3} \frac{(x-3)q(x)}{x-3} = \lim_{x \to 3} q(x) = q(3)$$

Thus we can follow that $Tp = q$, where q is a polynomial.

Proof of linearity is trivial, therefore I'll skip it.

4.1.9

Suppose $p \in P(C)$. Define $q : C \to C$ by

$$q(z) = p(z)\overline{p(\overline{z})}$$

Prove that q is a polynomial with real coefficients.

Given that $p(z)$ is a polynomial, we can state that we can factor it as

$$p(z) = c \prod (z - \lambda_j)$$

thus

$$\overline{p(\overline{z})} = \overline{c \prod (\overline{z} - \lambda_j)} = \overline{c} \prod \overline{(\overline{z} - \lambda_j)} = \overline{c} \prod (z - \overline{\lambda}_j)$$

Thus we follow that

$$q(z) = p(z)\overline{p(\overline{z})} = c \prod (z - \lambda_j) \overline{c} \prod (z - \overline{\lambda}_j) = |c|^2 \prod (z - \overline{\lambda}_j)(z - \lambda_j) =$$

$$= |c|^2 \prod (z^2 - z(\overline{\lambda}_j + \lambda_j) + |\lambda_j|^2) = |c|^2 \prod (z^2 - 2\Re(\lambda_j)z + |\lambda_j|^2)$$

Since every expression in the parenthesis is a polynomial in real coefficients, we follow that their product is a polynomial in real coefficients. And given that $|c|^2$ is a polynomial in real coefficients as well, we conclude that $q(z)$ is a polynomial in real coefficients, as desired.

4.1.10

Suppose m is a nonnegative integer and $p \in P_m(C)$ is such that there exist distinct real numbers $x_0, ..., x_m$ such that $p(x_j) \in R$ for $j = 0, 1, ..., m$. Prove that all the coefficients of p are real.

We can follow that from the exercise 5, where we add a restriction that $F = R$.

From the uniqueness clause in the exercise, we can follow that for any set $x_0, ..., x_m \in R$ we'll have unique collection of real coefficients.

4.1.11

Suppose $p \in P(F)$ *with* $p \neq 0$*. Let* $U = \{pq : q \in P(F)\}$*.*

(a) Show that $\dim P(F)/U = \deg p$*.*

We know that

$$p = sq + r$$

thus we can follow that

$$P(F) = U \oplus P_{(\deg p - 1)}(F)$$

thus we conclude that

$$\dim(P(F)/U) = \deg p$$

(b) Find a basis of $\dim P(F)/U$*.*

$$1 + U, x + U, ...x^m + U$$

Chapter 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

5.1 Invariant Subspaces

5.1.1

Suppose $T \in \mathcal{L}(V)$ *and* U *is a subspace of* V*.*

(a) Prove that if $U \subseteq \operatorname{null} t$*, then* U *is invariant under* T

Suppose that $u \in U$. Then $u \in \operatorname{null} T$ and therefore $Tu = 0 \in U$, because U is a subspace. Thus $u \in U \to Tu \in U$, therefore U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$*, then* U *is invariant under* T*.*

Suppose that $u \in U$. Then $Tu \in \operatorname{range} T \to Tu \in U$. Thus $u \in U \to Tu \in U$, therefore U is invariant under T.

5.1.2

Suppose $S, T \in \mathcal{L}(V)$ *are such that* $ST = TS$*. Prove that* $\operatorname{null} S$ *is invariant under* T*.*

Suppose that $v \in \operatorname{null} S$. Then we follow that $Sv = 0$ and therefore $TSv = 0$. Thus by our equality we've got that $STv = 0$. Therefore we follow that $Tv \in \operatorname{null} S$. Thus $v \in \operatorname{null} S \to Tv \in \operatorname{null} S$. Thus we follow that $\operatorname{null} S$ is invariant under T.

5.1.3

Suppose $S, T \in \mathcal{L}(V)$ *are such that* $ST = TS$*. Prove that* $\operatorname{range} S$ *is invariant under* T*.*

Suppose that $w \in \operatorname{range} S$. Then we follow that there exists $v \in V$ such taht $Sv = w$. Thus $TSv = Tw = STv$. Because $Tw = S(Tv)$, we follow that $Tw \in \operatorname{range} S$. Thus we conclude that $w \in \operatorname{range} S \to Tw \in \operatorname{range} S$. Thus $\operatorname{range} S$ is invariant under T.

5.1.4

Suppose $T \in \mathcal{L}(V)$ and $U_1, ..., U_m$ are subspaces of V invariant under T. Prove that $U_1 + ... + U_m$ is invariant under T.

Suppose that $u \in U_1 + ... + U_m$. Then we follow that

$$u = \sum u_j$$

for $u_j \in U_j$ for $1 \leq j \leq m$. Thus we follow that

$$Tu = T \sum u_j = \sum Tu_j$$

Given that U_j is invariant under T, we follow that $Tu_j \in U_j$, and therefore $Tu \in U_1 + ... + U_m$. Thus we follow that $U_1 + ... + U_m$ is invariant under T, as desired.

5.1.5

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Suppose that $U_1, ..., U_m$ are invariant subspaces. Then we follow that $U_1 \cap ... \cap U_m$ is also a subspace. Suppose that $u \in U_1 \cap ... \cap U_m$. Then it follows by invariance of respective subspace that $Tu \in U_1 \wedge ... Tu \in U_m$, thus we can conclude that $Tu \in U_1 \cap ... U_m$. Therefore we conclude that $U_1 \cap ... \cap U_m$ is an invariant subspace under T.

5.1.6

Prove or give a counterexample: if V is finite-dimentional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Suppose that U is a subspace of V such that $U \neq \{0\}$ and $U \neq V$. Then we can follow that $0 \neq \dim U$ and $\dim U \neq \dim V$. Thus we can follow that there exists a non-empty basis $u_1, ..., u_n$ of U, that we can extend with a non-empty list of vectors to $u_1, ..., u_m$ - basis of V. Now define a linear map

$$Tu_j = u_{m-j+1}$$

then we follow that $Tu_1 = u_m \notin U$. Therefore we conclude that U is not invariant under T. Thus we can conclude that the statement is correct.

5.1.7

Suppose $T \in \mathcal{L}(R^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

$$\lambda(x, y) = (-3y, x)$$

$$(\lambda x, \lambda y) = (-3y, x)$$

$$\begin{cases} \lambda x = -3y \\ \lambda y = x \end{cases}$$

$$\lambda^2 y = -3y$$

$$\lambda^2 = -3$$

which cannot be the case in real numbers. In complex numbers though we've got that

$$\lambda = [\sqrt{3}i, -\sqrt{3}i]$$

Suppose that we've got another real eigenvalue of T. Then we can follow that this eigenvalue is also an eigenvalue in complex numbers, therefore we've got more distinct eigenvalues, then dimentions, which is a contradiction. Thus we can conclude that T does not have real eigenvalues.

5.1.8

Define $T = \mathcal{L}(F^2)$ *by*

$$T(w, z) = (z, w)$$

. Find all eivenvalues and eigenvectors of T

$$\lambda(w, z) = (z, w)$$

$$(\lambda w, \lambda z) = (z, w)$$

$$\begin{cases} \lambda w = z \\ \lambda z = w \end{cases}$$

$$\lambda^2 w = w$$

$$\lambda = [1, -1]$$

Thus we follow that for $w, z \in F \setminus \{0\}$ we've got that eigenvectors (w, w) have an eigenvalue of 1, and eigenvectors $(z, -z)$ have eigenvalues of -1.

5.1.9

Define $T \in \mathcal{L}(F^3)$ *by*

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of T.

$$\lambda(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

$$(\lambda z_1, \lambda z_2, \lambda z_3) = (2z_2, 0, 5z_3)$$

$$\begin{cases} \lambda z_1 = 2z_2 \\ \lambda z_2 = 0 \\ \lambda z_3 = 5z_3 \end{cases}$$

the only values that work here are 0 with vectors $(z, 0, 0)$ and 5 with $(0, 0, z)$ with $z \in F \backslash \{0\}$.

5.1.10

Define $T \in \mathcal{L}(F^n)$ *by*

$$T(x_1, x_2, ..., x_n) = (x_1, 2x_2, 3x_3, ..., nx_n)$$

(a) Find all eigenvalues and eigenvectors of T.

We obliously have 1 for $(z, ..., 0)$, 2 for $(0, z, 0, ..., 0)$ and so on. Given that the number of distinct eigenvectors is equal to the number of the dimentions, we follow that those are the only eigenvalues and eigenvectors.

(b) Find all invariant subspaces of T.

We can state that $\lambda(1, 0, ..0)$ and such are the invariant subspaces. By the things that we've concluded earlier we can follow that sums of those subspaces are also invariant.

5.1.11

Define $T : P(R) \to P(R)$ *by* $Tp = p'$. *Find all eigenvalues and eigenvectors of* T.

Let $Tp = p' = \lambda p$. Since differentiation always reduces a degree of a polynomial by 1, we follow that the only eigenvalue that can exist is 0 with eigenvector $c \in F \setminus \{0\}$.

5.1.12

Define $T \in \mathcal{L}(P_4(R))$ *by*

$$(Tp)(x) = xp'(x)$$

for all $x \in R$. *Find all eigenvalues and eigenvectorss of* T.

$$\lambda p = xp'(x)$$

0 is still a viable eigenvalue with eigenvector $c \in F \setminus \{0\}$.

$$\sum \lambda a_j x^j = \sum j a_j x^j$$

since polynomials are uniquely determined by their coefficients, we follow that

$$\lambda a_j = j a_j$$

thus we can follow that we've got eigenvalues $0, 1, 2, 3, 4, 5$ for eigenvectors

$$1, x, x^2, x^3, x^4$$

and their non-zero scalar multiples.

Since the number of distinct eigenvalues is equal to the dimention of the space, we are done.

5.1.13

Suppose V is finite-dimentional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Prove that there exists $\alpha \in F$ such that $|\alpha - \lambda| < 1/1000$ and $T - \alpha I$ is invertible.

Since V is finite-dimentional, we know that there can exist only a finite amount of distinct eigenvalues. Since the amount of elements of F is infinite in the neighborhood around λ (I don't know if neighborhoods apply to the complex numbers, but I think that they do), we follow that there exists a number α such that $|\alpha - \lambda| < 1/1000$ and such that α is not an eigenvalue of T. Thus we follow that $T - \alpha I$ is invertible, as desired.

5.1.14

Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

0 and 1 for eigenvectors u and w respectively.

5.1.15

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

let λ be an eigenvalue of T and v is a corresponding eigenvector. Then we follow that

$$Tv = \lambda v$$

Because S is invertible, we follow that it is surjective, and there exists $w \in V$ such that $Sw = v$ and thus $S^{-1}v = w$. Thus

$$S^{-1}TSw = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v = \lambda w$$

Thus if λ is an eigenvalue of T, then it is an eigenvalue of $S^{-1}TS$.

Suppose that λ is an eigenvalue of $S^{-1}TS$ with corresponding eigenvector w. Then we follow that

$$S^{-1}TSw = \lambda w$$
$$S(S^{-1}TSw) = \lambda Sw$$
$$T(Sw) = \lambda(Sw)$$

thus we follow that λ is an eigenvalue for T. Therefore we follow that the set of eigenvalues of T and $S^{-1}TS$ is equal by double inclusion.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

If v is an eigenvector of T, then $S^{-1}v$ is an eigenvalue of $S^{-1}TS$.

5.1.16

Suppose V is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V contains only real entries. Show that if λ is an eigenvalue of T, then so is $\overline{\lambda}$.

Suppose that λ is an eigenvalue of T with corresponding eigenvector w. Let $v_1, ..., v_n$ be a basis of V, with respect to which the matrix of T contains only real entries. Then we follow tha t

$$w = \sum a_j v_j$$

$$Tw = T\sum a_j v_j = v_1 \sum a_j A_{1,j} + ... + v_n \sum a_j A_{n,j} = \lambda w = \lambda a_1 v_1 + ... + \lambda a_n v_n$$

thus we follow that

$$\sum a_j A_{1,j} = \lambda a_1$$

from this we follow that

$$\overline{\sum a_j A_{1,j}} = \overline{\lambda a_1}$$
$$\sum \overline{a_j} A_{1,j} = \overline{\lambda}\overline{a_1}$$

thus we follow that for

$$w' = \sum \overline{a_j} v_j$$

it is true that

$$Tw' = T\sum \overline{a_j} v_j = v_1 \sum \overline{a_j} A_{1,j} + ... + v_n \sum \overline{a_j} A_{n,j} = \overline{\lambda}\overline{a_1} v_1 + ... \overline{\lambda}\overline{a_n} v_n = \overline{\lambda} \sum \overline{a_j} v_j = \overline{\lambda} w'$$

thus w' is an eigenvector of T with eigenvalue $\overline{\lambda}$, as desired.

5.1.17

Give an example of an operator $T \in \mathcal{L}(R^4)$ such that T has no (real) eigenvalues.

Since the operator in exercise 7 didn't have any real eigenvalues, the idea is to double it, and see what happens. If we create a map

$$T(x, y, z, w) = (-3y, x, 3w, z)$$

then after plugging this thing into some sophisticated software we get that it doesn't have any real eigenvalues.

5.1.18

Show that the operator $T \in \mathcal{L}(C^\infty)$ defined by

$$T(z_1, z_2, ...) = (0, z_1, ...)$$

has no eigenvalues.

Suppose that it has one. Then we follow that

$$\lambda z_1 = 0$$

$$\lambda z_2 = z_1$$

$$\lambda z_3 = z_2$$

and so on. If $z_1 = 0$, then we follow that the other values in the supposed eigenvector are also equal to zero, thus this won't do. If $z_1 \neq 0$, then we follow that

$$\lambda z_2 = z_1 \neq 0$$

but

$$\lambda z_1 = 0$$

thus we can follow that $\lambda = 0$, and by extension $\lambda z_2 = z_1 = 0$, which is a contradiction. Thus there is no suitable value for z_1, therefore there does not exist an eigenvector for this map.

5.1.19

Suppose n is a positive integer and $T \in \mathcal{L}(F^n)$ is defined by

$$T(x_1, ..., x_n) = (x_1 + ... + x_n, ..., x_1 + ...x_n)$$

In other words, T is the operator whose matrix (with respect to the standart basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

Since $\dim \operatorname{range} T = 1$, we follow that $\dim \operatorname{null} T = n - 1$, and as long as $n \neq 1$, we follow that T is not injective. Thus

$$T - 0I$$

is not injective, and 0 is an eigenvalue of T. Eigenvectors of T consist of any vector in $\operatorname{null} T$ with the exception of 0.

The other eigenvalue is n with the eigenvector $(x, x, ..., x)$

5.1.20

Find all eigenvectors fo the backward shift operator $T \in \mathcal{L}(F^\infty)$ defined by

$$T(z_1, ...) = (z_2, z_3, ...)$$

Obliously $(1, 1, ...)$ and its nonzero scalar are eigenvectors of T with the eigenvalue of 1.

$$\lambda z_1 = z_2$$
$$\lambda z_2 = z_3$$

Suppose that $z_j \neq z_k$. Then we follow that all the values before z_j are equal to z_j and all the values after z_k are equal to z_k. There is no such λ that can do the trick, thus we follow that 1 is the only eigenvalue of T.

5.1.21

Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in F$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}

Suppose that $\lambda \in F$ with $\lambda \neq 0$ is an eigenvalue of T. Then we follow that

$$Tv = \lambda v$$

thus

$$T^{-1}Tv = Iv = T^{-1}\lambda v = \lambda T^{-1}v$$
$$v = \lambda T^{-1}v$$
$$1/\lambda v = T^{-1}v$$

thus $1/\lambda v$ is the eigenvalue of T^{-1}.

Given that $T = (T^{-1})^{-1}$ and $1/(1/\lambda) = \lambda$ we follow the converse result from forward implication (how nice).

(b) Prove that T and T^{-1} have the same eigenvectors

If we look at the deriviation of the previous result more closely, we see that we've shown it for the forward case, and by the fact that $T^{-1-1} = T$ we get the converse as well.

5.1.22

Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors v and w in V such that

$$Tv = 3w \text{ and } Tw = 3v$$

Prove that 3 or -3 is an eigenvalue of T

$$Tv + Tw = T(v + w) = 3w + 3v = 3(v + w)$$

and

$$Tv - Tw = T(v - w) = 3w - 3v = -3(v - w)$$

given that v and w are nonzero, we follow that $v + w$ or $v - w$ bound to be nonzero. Thus either 3 or -3 (or both) are eigenvalues of T.

5.1.23

Suppose V is finite-dimentional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Let λ be ean eigenvalue of ST with corresponding eigenvector v. Then we follow that

$$STv = \lambda v$$

$$TSTv = T\lambda v$$

$$TS(Tv) = \lambda(Tv)$$

thus λ is an eigenvalue of TS.

Since we can apply this by setting $S = T$ and $T = S$ and getting the same result, we can follow that they have the same eigenvalues.

5.1.24

Suppose A is an $n - by - n$ matrix with entries in F. Define $T \in \mathcal{L}(F^n)$ by $Tx = Ax$, where elements of F^n are thought of as $n - by - 1$ column vectors.

Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T

By plugging vector $(1, ..., 1)$ into the matrix we get the desired result.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

Let V be finite-dimentional vector space and let $T \in \mathcal{L}(V)$. Suppose that λ is an eigenvalue of T. Then we follow that

$$T - \lambda I$$

is not invertible and therefore not surjective. Thus we can also follow that

$$(T - \lambda I)'$$

is not injective and therefore also not invertible. Thus we conclude that

$$T' - \lambda I'$$

is not invertible and thus λ is an eigenvalue of T'.

Since every implication in previous paragraph is also an equivalence, we follow that T and T' have the same eigenvalues. Thus we follow that since T' for which $(T') = A^t$ has 1 as an eigenvalue by previous paragraph, then we conclude that T also has an eigenvalue 1, as desired.

5.1.25

Suppose $T \in \mathcal{L}(V)$ and u, v are eigenvectors of T such that $u + v$ is also an eigenvector of T. Prove that u and v are eigenvectors of T corresponding to the same eigenvalue.

Suppose that they aren't. Then we follow that

$$T(u + v) = \lambda_3(u + v) = \lambda_3 u + \lambda_3 v = T(u) + T(v) = \lambda_1 u + \lambda_2 v$$

$$\lambda_3 u + \lambda_3 v = \lambda_1 u + \lambda_2 v$$
$$(\lambda_3 - \lambda_1)u = -(\lambda_3 - \lambda_2)v$$

Since $\lambda_1 \neq \lambda_1$ we follow that either of (or both) $\lambda_3 - \lambda_1$ or $\lambda_3 - \lambda_2$ is not equal to zero. If $\lambda_3 - \lambda_1$ is equal to zero, then we follow that

$$0 = (\lambda_3 - \lambda_2)v$$

since $(\lambda_3 - \lambda_2)$ is not equal to zero we follow that $v = 0$, which is a contradiction. Thus we follow that both $\lambda_3 - \lambda_2$ and $\lambda_3 - \lambda_2$ are not equal to zero. Thus we conclude that

$$\frac{-(\lambda_3 - \lambda_2)}{(\lambda_3 - \lambda_1)} u = v$$

therefore u and v are not linearly independent, which is a contradiction. Thus we conclude that the case when $\lambda_1 \neq \lambda_2$ is impossible, and therefore we've got our desired conclusion.

5.1.26

Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Let $v, w \in V \setminus \{0\}$. Then we follow that $v + w$ is also an eigenvector, and therefore by the results of the previous exercise we've got that they correspond to the same eigenvalue.

Thus we follow that all of the vectors in $V \setminus \{0\}$ correspond to the same eigenvalue. Let λ be this eigenvalue. Then we follow that for $v \in V$

$$Tv = \lambda v$$

if $v \neq 0$ and if $v = 0$, then

$$Tv = T0 = 0 = \lambda * 0 = \lambda v$$

thus we conclude that

$$Tv = \lambda v = (\lambda I)v$$

for every $v \in V$. Thus we follow that $T = \lambda I$, as desired.

5.1.27

Suppose V is finite-dimentional and $T \in \mathcal{L}(V)$ is such that every subspace of V with dimention $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

If $T = 0$, then we've got that $T = 0I$, thus assume that $T \neq 0$. Let v_1 be such that $Tv_1 \neq 0$. Expand v_1 to the basis of V - $v_1, ..., v_n$. Then we follow that $\operatorname{span}(v_2, ..., v_n)$ is invariant under T. Let $Tv = \sum a_j v_j$. We can follow that since $\operatorname{span}(a_1, ..., a_{i-1}, a_{i+1} ..., a_n)$ is invariant under T, then $a_i = 0$ for $i \neq 1$. Thus we conclude that

$$Tv = \sum a_j v_j = a_1 v_1 + \sum_{j \neq 1} a_j v_j = a_1 v_1$$

Thus we can follow that v is an eigenvalue of T. Thus we've got that every nonzero vector of V is an eigenvalue of T, therefore by previous exercise we follow that T is a scalar multiple of identity.

5.1.28

Suppose V is finite-dimentional with $\dim V \geq 3$ and $T \in \mathcal{L}(V)$ is such that every 2-dimentional subspace of V is invariant under T. Prove that T is a scalar multiple of the identity operator.

Let U be a subspace of V such that $\dim U = 3$ and let u_1, u_2, u_3 be a basis of this subspace. Let $w \in U$. Then we follow that

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3$$

thus

$$Tw = T(a_1 v_1 + a_2 v_2 + a_3 v_3) = T(a_1 v_1 + a_2 v_2) + T(a_3 v_3)$$

since every 2-dimentional subspace is invariant in V, we follow that $T(a_1 v_1 + a_2 v_2) + T(a_3 v_3) =$ is in $\operatorname{span}(u_1, u_2) + \operatorname{span}(u_2 + u_3) = \operatorname{span}(u_1, u_2, u_3)$ Thus we follow that U

is invariant under T. Therefore we've got that every 3-dimentional subspace is invariant under T, and by induction we can follow that this is true for any subspace of V. Thus we follow that every subspace with dimention $n-1$ is invariant under T, therefore by our previous exercise we've got the desired result.

5.1.29

Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k+1$ distinct eigenvalues.

Because eigenvectors for distinct eigenvalues have got to be linearly independent, we can follow that we can we can have at most k nonzero distinct eigenvalues, since $Tv = \lambda v \in \operatorname{range} T \neq 0$.

If T is not injective, we can have an additional distinct eigenvalue of 0, thus increasing total number of disctinct eigenvalues to $k+1$.

5.1.30

Suppose $T \in \mathcal{L}(R^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in R^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Since 9 is not an eigenvalue of T, we can follow that $T - 9I$ is injective and therefore invertible and surjective. Thus by surjectivity of $T - 9I$ we follow that there exists a desired vector.

5.1.31

Suppose V is finite-dimentional and $v_1, ..., v_m$ is a list of vectors in V. Prove that $v_1, ..., v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, ..., v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

In forward direction:

We can make a map

$$T(a_1v_1 + ... + a_mv_m) = (a_1v_1 + 2a_2v_2 + ... + ma_mv_m)$$

that satisfies required parameters.

In reverse direction:

Trivial, follows directly from the fact that eigenvectors for distinct eigenvalues are linearly independent.

5.1.32

Suppose $\lambda_1, ..., \lambda_n$ is a list of distinct real numbers. Prove that the list $\exp(\lambda_1 x), ..., \exp(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on R.

Let $V = \operatorname{span}(\exp(\lambda_1 x), ..., \exp(\lambda_n x)$ and define an operator $T \in \mathcal{L}(R^R)$ by $Tf = f'$ (linearity of this thing was proven somewhere in the 3rd chapter). Then we follow that

$$T(\exp(\lambda_j x)) = \lambda_j \exp(\lambda_j x)$$

which comes from calculus. Thus we follow that $\lambda_1, ..., \lambda_m$ are distinct eigenvalues of T. Therefore we can conclude that $\exp(\lambda_1 x), ..., \exp(\lambda_n x)$ is linearly independent.

5.1.33

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$

Let $v \in V$. Then we follow that

$$Tv \in \operatorname{range} T$$

and

$$0 \in \operatorname{range} T$$

Thus we can follow that

$$T(v + (\operatorname{range} T)) = 0 + (\operatorname{range} T)$$

Thus $T/(\operatorname{range} T) = 0$, as desired.

5.1.34

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is injective if and only if $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

Suppose that $T/(\operatorname{null} T)$ is injective. Then we can follow that

$$T/(\operatorname{null} T)(v + \operatorname{null} T) = (Tv + \operatorname{null} T) = 0 + \operatorname{null} T$$

if and only if $Tv + \operatorname{null} T = 0 + \operatorname{null} T$. $Tv + \operatorname{null} T = 0 + \operatorname{null} T$ if and only if

$$Tv - 0 \in \operatorname{null} T$$

$$Tv \in \operatorname{null} T$$

thus we follow that $\operatorname{null} T \cap \operatorname{range} T = \{0\}$. Since we've only used equivalences here, we've got both cases at the same time.

5.1.35

Suppose V is finite-dimentional, $T \in \mathcal{L}(V)$, and U is invariant under T. Prove that each eigenvalue of T/U is an eigenvalue of T.

Let λ be an eigenvalue of T/U. Then we follow that there exists $v \in V$ such that

$$(T/U)(v+U) = Tv + U = \lambda v + U$$

If $v \in U$, then $v + U = 0$, which is impossible, because eigenvectors are not zero. Let $u_1, ..., u_m$ be a basis of U and extend this basis to $u_1, ..., u_m, v_1, ..., v_n$ - basis of V. Then we follow that

$$Tv + U = \lambda v + U$$

$$Tv - \lambda v + U = 0$$

$$(Tv - \lambda v) \in U$$

$$(T - \lambda I)(v) \in U$$

$$(T - \lambda I)(\sum a_j v_j + \sum a_i u_i) = \sum b_j u_j$$

$$(T - \lambda I)(\sum a_j v_j) + (T - \lambda I)(\sum a_i u_i) = \sum b_j u_j$$

Since U is invariant under U, we can follow that $(T - \lambda I)(\sum a_i u_i) \in U$. Thus

$$(T - \lambda I)(\sum a_j v_j) = \sum b_j u_j - (T - \lambda I)(\sum a_i u_i)$$

$$(T - \lambda I)(\sum a_j v_j) \in U$$

Since $v \notin U$, we follow that $\sum a_j v_j \neq 0$ and therefore $\sum a_j v_j$ and $u_1, ..., u_m$ are linearly independent. Therefore we follow that $(T - \lambda I)$ maps $\dim U + 1$ linearly independent vectors into a space with $\dim U$. Thus we follow that it is not injective, and therefore λ is an eigenvalue of T, as desired.

5.1.36

Give an example of a vector space V, an operator $T \in \mathcal{L}(V)$, and a subspace U of V that is invariant under T such that T/U has an eigenvalue that is not an eigenvalue of T.

$$T(z_1, z_2, ...) = (0, z_2 + z_1, z_3 + z_1, ...)$$

If we set $U = (0, z_1, z_2, ...)$, then T/U has an eigenvalue of 0 and T doesn't.

5.2 Eigenvectors and Upper-Triangular Matrices

5.2.1

Suppose $T \in \mathcal{L}(V)$ and there exists a positivee integer n such that $T^n = 0$.

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$

Suppose that $I - T$ is not invertible. Then we follow that 1 is an eigenvalue of T. Thus we follow that there exists an eigenvector $v \neq 0$, corresponding to this value, for which it is true that

$$v = Tv = T(Tv) = T^2v = T^3v = ... = T^nv$$

Since $v \neq 0$, we follow that $T^nv \neq 0$, and therefore $T^n \neq 0$ which is a contradiction. Therefore we can follow that $T - I$ is invertible.

$$(I - T)(I + T + ... + T^{n-1}) = (I - T)I + (I - T)T + ... + (I - T)T^{n-1} =$$

$$= I - T + T - T^2 + ... + T^{n-1} - T^n = I - T^n = I - 0 = I$$

$$(I - T)^{\text{-}1} = I + T + ... + T^{n-1}$$

(b) Explain how you would guess the formula above.

From its deriviation

5.2.2

Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$, $\lambda = 3$ or $\lambda = 4$.

Suppose that $\lambda \neq 2$, $\lambda \neq 3$ and $\lambda \neq 4$. Then we follow that $(T - 2I)$, $(T - 3I)$, and $(T - 4I)$ are all invertible. Since the product of invertible matrices is invertible, we follow that it is also injective and therefore for $v \neq 0$

$$(T - 2I)(T - 3I)(T - 4I)v \neq 0$$

thus

$$(T - 2I)(T - 3I)(T - 4I) \neq 0$$

which is a contradiction.

5.2.3

Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

$$T^2 = I$$
$$T^2 - I = 0$$
$$(T + I)(T - I) = 0$$

from the previous exercise we follow that if λ is an eigenvalue of T, then it's either 1 or -1. Because -1 is not an eigenvalue of T, we follow that $(T + I)$ is invertible and therefore

$$(T + I)(T - I) = 0$$
$$(T + 1)^{-1}(T + I)(T - I) = (T + 1)^{-1}\,0$$
$$I(T - I) = 0$$
$$T - I = 0$$
$$T = I$$

as desired.

5.2.4

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$

$V = \operatorname{null} P \oplus \operatorname{range} P$ means that $V = \operatorname{null} P + \operatorname{range} P$ and $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

To prove the latter part we suppose that $v \in \operatorname{null} P$ and $v \in \operatorname{range} P$. If $v \neq 0$, then we follow that there exists $w \in V \neq 0$ such that $Pw = v$. Because $v \in \operatorname{null} P$ we follow that

$$Pv = 0$$
$$P(v) = 0$$
$$P(Pw) = 0$$
$$P^2 w = 0$$
$$Pw = 0$$
$$v = 0$$

which is a contradiction. Thus we follow that there does not exist $v \neq 0$ such that $v \in \operatorname{null} P$ and $v \in \operatorname{range} P$ at the same time. Therefore we can conclude that

$$\operatorname{null} P \cap \operatorname{range} P = \{0\}$$

We know that for every $v \in V$ we've got that

$$v = v$$

$$v = v - Pv + Pv$$

Since

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = P(v - v) = P(0) = 0$$

we follow that $v - Pv \in \operatorname{null} P$. $Pv \in \operatorname{range} P$. Thus we follow that

$$V = \operatorname{null} P + \operatorname{range} P$$

Thus we can conclude that $V = \operatorname{null} P \oplus \operatorname{range} P$, as desired.

5.2.5

Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in P(F)$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}$$

$$(STS^{-1})^m = (STS^{-1})(STS^{-1})...(STS^{-1}) = ST(S^{-1}S)T(S^{-1}...S)TS^{-1}) = STITI...ITS^{-1} = ST^mS^{-1}$$

thus we follow that

$$p(STS^{-1}) = \sum a_j(STS^{-1})^j = \sum a_j ST^j S^{-1} = S \sum [a_j T^j] S^{-1} = Sp(T)S^{-1}$$

as desired.

5.2.6

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in P(F)$

Suppose that $u \in U$ is invariant under T. Then we follow that

$$Tu \in U$$

and by induction

$$T^m u = T(T^{m-1}u) \in U$$

Thus we follow that $a_j T^j u \in U$ by closure under scalar multiplication of subspaces and

$$\sum a_j T^j u = p(T) \in U$$

by additive closure of subspaces for any $p \in P(F)$.

5.2.7

Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Suppose that 9 is an eigenvalue of T^2. Then we follow that $(T^2 - 9I)$ is not injective. Thus

$$(T^2 - 9I) = (T - 3I)(T + 3I)$$

is not injective. Thus we follow that either one of $T - 3I$ of $T + 3I$ is not injective. Thus we follow that -3 or 3 is an eigenvalue of T.

Suppose that either 3 or -3 is an eigenvalue of T. Then we follow that

$$T^2 v = 3(3v) = 9v$$

or

$$T^2 v = -3(-3v) = 9v$$

thus 9 is an eigenvalue of T, as desired.

5.2.8

Give an example of $T \in \mathcal{L}(R^2)$ such that $T^4 = -1$

After some though I came up with the

$$\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Which seems to be working.

5.2.9

Suppose V is finite-dimentional, $T \in \mathcal{L}(V)$ and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v \neq 0$. Prove that every zero of p is an eigenvalue of T.

We've got that

$$p(T)v = 0$$

$$\prod (T - \lambda_j I)v = 0$$

Suppose that λ_i is not an eigenvalue of T. Then we follow that $(T - \lambda_i)$ is invertible and therefore we've got that for

$$(T - \lambda_i)\left(\prod_{j \neq i} (T - \lambda_j I)v\right) = 0$$

$$\prod_{j\neq i}(T-\lambda_j I)v = 0 = (T-\lambda_i)\left(\prod_{j\neq i}(T-\lambda_j I)v\right)$$

thus we can follow that p is not in its lowest degree, which is a contradiction.

5.2.10

Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppoe $p \in P(F)$. Prove that $p(T)v = p(\lambda)v$

Suppose that λ is an eigenvalue for an eigenvector v. Then we follow that

$$T^m(v) = \lambda^m v$$

thus for our problem we've got that

$$p(T)v = \sum(a_j T^j)v = \sum(a_j T^j v) = \sum(a_j \lambda^j v) = \sum(a_j \lambda^j)v = p(\lambda)v$$

as desired.

5.2.11

Suppose $F = C$, $T \in \mathcal{L}(V)$, $p \in P(C)$ is a polynomial, and $a \in C$. Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

In forward direction:

Suppose that a is an eigenvalue of $p(T)$. Then we can follow that there exists eigenvector $v \neq 0$ such that

$$p(T)v = av$$

$$p(T)v - av = 0$$

$$p(T)v - aIv = 0$$

$$(p(T) - aI)v = 0$$

Since $p \in P(C)$, we follow taht $p - a \in P(C)$. Thus there exists factorization $c\prod(z-\lambda_i) = p - a$. Thus

$$c\prod(T-\lambda_i I)v = 0$$

If $c = 0$, then we follow that $p(z) - a = 0$, and therefore $p(z) = a$ and $p(\lambda) = a$ for any eigenvalue of T. Thus suppose that $c \neq 0$. Then we've got that one of $(T - \lambda_i I)$ is not injective and therefore λ_i is an eigenvelue of T, for which

$$c(z - \lambda_i) = 0$$

thus

$$p(\lambda_i) - a = 0$$

$$p(\lambda_i) = a$$

as desired.

In reverse direction:

Suppose that $a = p(\lambda)$ for some eigenvalue of p. Then we follow that for eigenvector v , that correspond to this eigenvalue

$$Tv = \lambda v$$

thus

$$p(T)v = p(\lambda)v = av$$

thus a is an eigenvalue of $p(T)$, as desired.

5.2.12

Show taht the result in the previous exericse does not hold if C is replaced with R.

With $F = R$ we're not guaranteed that T has an eigenvalue in finite dimnetions. Thus we can follow that for our map from previous chapteer

$$T(x, y) = (-3y, x)$$

and polynomial

$$p(z) = z^2$$

we've got that $p(T)$ has an eigenvalue 3, but no eigenvalues exist for T.

5.2.13

Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimentional.

Suppose that U is an invariant finite-dimentional nonzero subspace of W. Then we follow that it has an eigenvalue λ under T with a corresponding eigenvector $v \neq 0$ (as does any finite-dimentional nonzero complex space). Thus we follow that $Tv = \lambda v$, therefore T has an eigenvalue, which is a contradition. Thus we follow that there does not exist an invariant finite-dimentional nonzero subspace of W, that is invariant under T. Therefore we follow that if a subspace is invariant under T, then it's either not finite-dimentional (i.e. infinite dimentional), or zero, as desired.

5.2.14

Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operation is invertible

$$T \in \mathcal{L}(C^2) : T(x, y) = (-3y, x)$$

5.2.15

Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

5.2.16

Rewrite the proof of 5.21 using the linear map that sends $p \in P_n(C)$ to $(p(T))v \in V$ (and use 3.23)

We follow that there exists a linear map S from $P_n(C)$ to V defined by

$$S(p) = p(T)v$$

thus we follow that $\dim P_n(C) = n + 1 > n = \dim V$, therefore S is not injective (by 3.23), and thus there exists a polynomial $p \in P_n(C) \neq 0$ such that

$$S(p) = \sum (a_j T^j)v = 0$$

after this we've got the same proof.

5.2.17

Rewrite the proof of 5.21 using the linear map that sends $p \in P_{n^2}(C)$ to $p(T) \in \mathcal{L}(V)$ (and use 3.23)

We follow that $\dim \mathcal{L}(V) = n^2 < n^2 + 1 = \dim P_{n^2}(C)$. Thus we follow that the map that sends

$$S(p) = p(T)$$

is not injective. Thus there exists $p \in P_{n^2}(C)$ such that

$$S(p) = p(T) = 0$$

the rest is the same.

5.2.18

Suppose V is a finite-dimentional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : C \to R$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I)$$

Prove that f is not a continous function

Let $V_\delta(\lambda)$ be a neighborhood for λ for arbitrary δ. Since any neighborhood has infinite amount of distinct numbers, we follow that there exists $x \in V_\delta(\lambda)$ that is not an eigenvalue of T. Thus we follow that there is no neighborhood of λ such that $x \in V_\delta(\lambda) \to \dim \operatorname{range}(T - xI) = \dim \operatorname{range}(T - \lambda I)$. Thus we follow that the function is not continous, as desired.

5.2.19

Suppose V is finite-dimentional with $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that

$$\{p(T) : p \in P(F)\} \neq \mathcal{L}(V)$$

For now, let us concentrate on $F = C$. If T has only 1 eigenvalue, then we follow that by the virtue of the fact that there exists an upper-triangular matrix for it, that there exist at least two vectors such that

$$p(T)v_1 = p(\lambda)v_1 = p(\lambda)v_2 = p(T)v_2$$

therefore we follow that $\{p(T) : p \in P(F)\}$ does not have maps S such that $Sv_1 \neq Sv_2$. If T has more then 1 eigenvalues, then the case is reversed.

Case of $F = R$ is a somewhat specific case of $F = C$, which is resolved with the same logic and maybe minor modifications

5.2.20

Suppose V is finite-dimentional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant supspace of dimention k for $1 \leq k \leq \dim V$.

T has an upper-triangular map for some basis $v_1, ..., v_n$. Thus it has a subspace

$$\operatorname{span}(v_1, ..., v_k)$$

with desired properties

5.3 Eigenspaces and Diagonal Matrices

5.3.1

Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

Although it wasn't mentioned specifically, I think that we can follow that if T is diagonalizable in some space V, then this space if finite-dimentional. Thus we can follow that V has a basis $v_1, ..., v_n$ of eigenvectors of T with corresponding eigenvalues $\lambda_1, ..., \lambda_n$, where λ's can repeat. Thus for $v \in V$

$$v = \sum a_j v_j$$

If $\lambda_j \neq 0$, then we follow that there exists $\frac{a_j}{\lambda_j} v_j$ such that

$$T\frac{a_j}{\lambda_j} v_j = a_j v_j$$

If $\lambda_j = 0$, then we follow that $v_j \in \operatorname{null} T$. Thus

$$v = \sum a_j v_j = \sum_{\lambda \neq 0} a_j v_j + \sum_{\lambda = 0} a_j v_j$$

where $\sum_{\lambda \neq 0} a_j v_j \in \operatorname{range} T$ and $\sum_{\lambda = 0} a_j v_j \in \operatorname{null} T$. Thus we follow that $V = \operatorname{null} T + \operatorname{range} T$.

Since $v_1, ..., v_n$ is a basis, we follow that it is linearly independent, and therefore $V = \operatorname{null} T \oplus \operatorname{range} T$, as desired.

5.3.2

Prove the converse of the statement in the exercise above or give a couneterexample to the converse.

Suppose that $V = \operatorname{null} T \oplus \operatorname{range} T$. I think that we're assumnming once again that V is finite-dimentional.

We know that there exist bases of both $\operatorname{null} T$ and $\operatorname{range} T$ that add up to a basis of V. Every member of basis of $\operatorname{null} T$ is an eigenvector for an eigenvalue of 0.

If $F = R$, then we can follow that there exists a map such that it has no eigenvalues, therefore making the whole thing not diagonalizable. However if $F = C$, then we follow that $T|_{\operatorname{range} T}$ has an eigenvalue (which will be nonzero, since all zeroes are in the basis of $\operatorname{null} T$) with a corresponding eigenvector. If we take one of those eigenvectors away, we can follow that the span of rest of the vectors is invariant, because if it isn't, then for vector $u \in \operatorname{span}(v_k)$ and $w \in \operatorname{span}(v_{k+1}, ..., v_n)$

$$T(u) = T(u)$$

$$T(u - w) = 0$$

for which $u \neq w$, $u - w \in \operatorname{range} T$ and $u - w \in \operatorname{null} T$, and therefore it violates the fact that $V = \operatorname{null} T \oplus \operatorname{range} T$. Thus the process can be repeated and T is diagonalizable.

Therefore we conclude that if in addition to the fact that $V = \operatorname{null} T \oplus \operatorname{range} T$ we've got that V is a finite-dimentional vector space, we can follow that T is diagonalizable.

5.3.3

Suppose V is finite-dimentional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$

(b) $V = \operatorname{null} T + \operatorname{range} T$

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

(a) implies (b) and (c) from the definition of $\oplus$.

If $\operatorname{null} T + \operatorname{range} T = V$, then combined bases of $\operatorname{null} T$ and $\operatorname{range} T$ span V. Since

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

we follow that list that consists of combined bases of $\operatorname{null} T$ and $\operatorname{range} T$ is a list of length $\dim V$, that spans V, making it a basis of V. Thus we can follow that this list is linearly independent, and therefore

$$\operatorname{null} T \cap \operatorname{range} T = \{0\}$$

Thus (b) implies (c).

If

$$\operatorname{null} T \cap \operatorname{range} T = \{0\}$$

then we follow that list, that consists of combined bases of $\operatorname{null} T$ and $\operatorname{range} T$ is linearly independent. Therefore by FTLM we've got linearly independent list of length $\dim V$, thus making it a basis of V. Therefore we can follow that (c) implies (b)

(b) and (c) implies (a), and since (b) and (c) are equivalent, we follow that both (b) and (c) are eqivalent to (a).

Give an example to show that the exercise above is false without the hypothesis that V is finite-dimentional.

$$U = (0, 0, 0, x_1, ...)$$
$$W = (0, 0, x_1, 0, ...)$$
$$U \cap W = \{0\}$$

which doesn't imply anything.

5.3.4

Suppose V is finite-dimentional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$$

for every $\lambda \in C$.

Suppose that T is diagonalizable. Then we follow that V has a basis of eigenvectors of T. For any of this vectors it is true that

$$Tv = \kappa v$$

for some $\kappa \in F$. Thus we follow that

$$(T - \lambda I)v = \kappa v - \lambda v = (\kappa - \lambda)v$$

and therefore we follow that v is also an eigenvector for $(T - \lambda I)$. Thus we conclude that V has a basis, that consists of eigenvectors of $(T - \lambda I)$. Therefore $(T - \lambda I)$ is also diagonalizable.

By equivalence that we've proven in exercises 1 and 2, and presented proof we have the desired result.

5.3.5

Suppose V is finite-dimentional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$

Because T has $\dim V$ distinct eigenvalues we follow that it is diagonalizable. Thus V has a basis, that consists of eigenvectors of T. Since S has the same eigenvectors, we follow that V has a basis, consisting of eigenvectors of S, thus making S diagonalizable. Let $v_1, ..., v_n$ be this basis, $\lambda_1, ..., lambda_n$ be the corresponding eigenvalues of T and $\kappa_1, ..., \kappa_n$ be corresponding eigenvalues of S. Let $v \in V$. Then we follow that

$$v = \sum a_j v_j$$

$$STv = ST \sum a_j v_j = \sum a_j STv_j = \sum a_j \kappa_j \lambda_j v_j = \sum a_j \lambda_j \kappa_j v_j = \sum a_j TSv_j = TS \sum a_j v_j = TSv$$

thus $ST = TS$, as desired.

5.3.6

Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in F$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

This diagonal basis will consist of eigenvectors of T. Thus we follow that since

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

and for $j \neq m$,

$$v \in E(\lambda_j, T) \to v \notin E(\lambda_m, T)$$

there will exist $\dim E(\lambda_j, T)$ vectors, that correspond to λ_j.

5.3.7

Suppose $T \in \mathcal{L}(F^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

In order for $T - 6I$ to be non-injertible, we've got to have

$$\dim E(6, T) \geq 1$$

Since

$$\dim V = \sum \dim E(\lambda_j T) = 5$$

$$\sum \dim E(\lambda_j T) + \dim E(8, T) = 5$$
$$\sum \dim E(\lambda_j T) = 1$$

thus we follow that there exists at most one more non-invertible $T - \lambda I$. Thus one of the provided maps is invertible, as desired.

5.3.8

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda T) = E(\frac{1}{\lambda} T^{-1})$ for every $\lambda \in F$ with $\lambda \neq 0$.

From exercise 5.1.21 we follow that if T is invertible and λ is an eigenvalue for T with corresponding eigenvector v, then v is an eigenvector of T^{-1} for value $1/\lambda$ ($\lambda \neq 0$ follows from invertability of T). Thus we can follow that presented sets are equal by definition of eigenspace.

5.3.9

Suppose that V is finite-dimentional and $T \in \mathcal{L}(V)$. Let $\lambda_1, ..., \lambda_n$ denote distinct nonzero eigenvalus of T. Prove that

$$\sum \dim E(\lambda_j, T) \leq \dim \operatorname{range} T$$

$$\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T$$

since $\operatorname{null} T = E(0, T)$ (follows from definitions and a moments' thought) we follow that

$$\dim V = \dim \operatorname{range} T + \dim E(0, T)$$

thererfore since

$$\sum \dim E(\lambda_j, T) \leq V$$

we follow that

$$\sum \dim E(\lambda_j, T) \leq \dim \operatorname{range} T + \dim E(0, T)$$
$$\dim E(0, T) + \sum_{\lambda_j \neq 0} \dim E(\lambda_j, T) \leq \dim \operatorname{range} T + \dim E(0, T)$$
$$\sum_{\lambda_j \neq 0} \dim E(\lambda_j, T) \leq \dim \operatorname{range} T$$

as desired.

5.3.10

Verify the assertion in Example 5.40

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 69 \\ 276 \end{pmatrix} = 69 \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 322 \\ 230 \end{pmatrix} = 46 \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

thus we get that for this basis everything holds.

5.3.11

Suppose $R, T \in \mathcal{L}(F^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(F^3)$ such that $R = S^{-1}TS$

Since R and T have $\dim F^3 = 3$ distinct eigenvalues, we follow that they are diagonalizable. Thus we can follow that there exist a basis v_1, v_2, v_3 of V, that consists of eigenvalues of R, and basis u_1, u_2, u_3 of T that also consists of eigenvalues of T, both of which correspond to eigenvalues 2, 3 and 7 respectively.

Define map

$$S(\sum a_j v_j) = \sum a_j u_j$$

which is invertible, because $\dim \operatorname{range} T = 3$. Thus we follow that for $v \in V$

$$v = a_1v_1 + a_2v_2 + a_3v_3$$

$$Rv = 2a_1v_1 + 6a_2v_2 + 7a_3v_3$$

$$S^{-1}TSv = S^{-1}TS(\sum a_jv_j) = S^{-1}T(\sum a_ju_j) = S^{-1}(2a_1u_1 + 6a_2u_2 + 7a_3u_3) =$$

$$= (2a_1S^{-1}u_1 + 6a_2S^{-1}u_2 + 7a_3S^{-1}u_3) = 2a_1v_1 + 6a_2v_2 + 7a_3v_3 = R(v)$$

as desired.

5.3.12

Find $T \in \mathcal{L}(C^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of C^3

In order for it now to be diagonalizable, we need for T not to have 3 distinct eigenvalues. Thus let us frick around the matrix of the standard diagonale matrix.

After some deloberation I came up with

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 7 \\ 0 & 0 & 7 \end{pmatrix}$$

which seems to satisfy the desired properties.

5.3.13

Suppose $T \in \mathcal{L}(C^3)$ is such that 6 and 7 are eigenvalueus of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of C^3. Prove that there exists $(x, y, z) \in F^3$ such that

$$T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$$

Since T is not diagonalizable, we follow that 6 and 7 are the only eigenvalues of T. Thus we follow that 8 is not an eigenvalue of T. Therefore $T - 8I$ is invertibe and surjective. Threrefore we follow that by surjectivity of $T - 8I$, that there exists $(x', y', z') \in C^3$ such that

$$(T - 8I)(x', y', z') = (17, \sqrt{5}, 2\pi)$$

thus

$$(T - 8I)(x', y', z') = (17, \sqrt{5}, 2\pi)$$
$$T(x', y', z') - 8(x', y', z') = (17, \sqrt{5}, 2\pi)$$
$$T(x', y', z') = (17, \sqrt{5}, 2\pi) + 8(x', y', z')$$

$$T(x', y', z') = (17 + 8x', \sqrt{5} + 8y', 2\pi + 8z')$$

as desired.

5.3.14

The Fibonacci sequence $F_1, F_2, ...$ is defined by

$$F_0 = 0$$

$$F_1 = 1$$

$$F_2 = 1$$

$$F_n = F_{n-2} + F_{n-1}$$

Define $T \in \mathcal{L}(R^3)$ by $T(x, y) = (y, x + y)$

I've extended the sequence back a bit becasuse I like zeroes.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each positive integer n.

This kinda follows from the definition, but we're going to use induction just in case. For the case $n = 1$ we've got

$$T^1(0, 1) = T(0, 1) = (1, 1) = (F_1, F_2)$$

For our hypothesis we've got that

$$T^{n-1}(0, 1) = (F_{n-1}, F_n)$$

and for our step we've got

$$T^n(0,1) = T(T^{n-1}(0,1)) = T(F_{n-1}, F_n) = (F_n, F_{n-1} + F_n) = (F_n, F_{n+1})$$

thus we follow that our inductive hypothesis is true for $n \in N$.

(b) Find the eigenvalues of T.

Let λ be an eigenvalue of T. Then we follow that

$$\begin{cases} \lambda x = y \\ \lambda y = x + y \end{cases}$$

$$\lambda(\lambda x) = x + \lambda x$$
$$\lambda^2 x = (1+\lambda)x$$
$$\lambda^2 = (1+\lambda)$$
$$\lambda^2 - \lambda - 1 = 0$$
$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

(c) Find a basis of R^2 consisting of eigenvalues of T

Let $c_1 = \frac{1+\sqrt{5}}{2}$ and $c_2 = \frac{1-\sqrt{5}}{2}$

Suppose that v_1, v_2 are such vectors. Then we follow that

$$Tv_1 = T(x,y) = c_1(x,y) = (y, x+y)$$

thus if we fix $x = 1$, then we'll get that

$$c_1(1,y) = (y, x+y)$$
$$c_1(1,y) = (c, 1+c)$$
$$c_1(1,c) = (c, 1+c)$$

thus one of our vectors is $v_1 = (1, c_1)$ and the other is $v_2 = (1, c_2)$. Octave seems to concur.

(d) Use the solution to part (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

We follow that

$$v_1 - v_2 = (0, c_1 - c_2)$$

thus

$$\frac{1}{c_1 - c_2}(v_1 - v_2) = (0,1)$$

therefore

$$T^n(0,1) = T^n(\frac{1}{c_1 - c_2}(v_1 - v_2)) = \frac{1}{c_1 - c_2}T^n(v_1 - v_2) = \frac{1}{c_1 - c_2}(c_1^n v_1 - c_2^n v_2)$$

thus

$$F_n = \frac{1}{c_1 - c_2}c_1^n - c_2^n$$

since

$$c_1 - c_2 = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

thus by expanding c_1 and c_2 we've got

$$F_n = \frac{1}{c_1 - c_2}(c_1^n - c_2^n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

as desired.

(e) Use part (d) to conclude that for each positive integer n, the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n =$$

Since $|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)| < 0.5$ and $|\left(\frac{1-\sqrt{5}}{2}\right)| < 1$ we follow that

$$|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n| < 0.5$$

therefore we've got our desired conclusion.

Chapter 6

Inner Product Spaces

6.1 Inner Products and Norms

6.1.1

Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in R^2 \times R^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on R^2.

Suppose that $\lambda < 0$ and $|x_1y_1| + |x_2y_2| \neq 0$. Then we follow that

$$|\lambda x_1y_1| + |\lambda x_2y_2| = |\lambda||x_1y_1| + |\lambda||x_2y_2| = |\lambda|(|x_1y_1| + |x_2y_2|) \neq \lambda(|x_1y_1| + |x_2y_2|)$$

thus we can follow that this function is not homogenous in the first slot, therefore it isn't linear.

6.1.2

Show that the function that takes $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in R^3 \times R^3$ to $x_1y_1 + x_3y_3$ is not an inner product on R^3.

Let $v = (0, 1, 0)$. Then we follow that

$$x_1x_1 + x_3x_3 = 0$$

but $v \neq 0$. Thus this function does not have definiteness property, and therefore it is not an inner product.

6.1.3

Suppose $F = R$ and $V \neq 0$. Replace the positivity condition (which states that $\langle v, v\rangle \geq 0$ for all $v \in V$) in the definition of an inner product with the condition that $\langle v, v\rangle > 0$ for some $v \in V$. Show that this change in the definition does not change the set of functions from $V \times V$ to R that are inner product on V.

We need to show that two sets of functions are equal. Forward inclusion is obvious

$$\forall v \in V : f(v) \geq 0 \wedge (f(v) = 0 \iff v = 0) \wedge V \neq 0 \to \exists v \neq 0 \in V : f(v) > 0$$

Thus let us work on a reverse inclusion. Suppose that for some function $f : V \times V \to R$ we've got that there exists $v \in V$ such that $\langle v, v\rangle > 0$ and all of the conditions for inner product hold, except for positivity, which is unknown. Suppose that $u \in V$. If $u = 0$, then by definiteness we've got that $\langle u, u\rangle = 0$, thus $\langle u, u\rangle \geq 0$. Therefore suppose that $u \neq 0$. Suppose that $\langle u, u\rangle < 0$. Then we can follow that $u \neq v$ and that $u \neq \lambda v$ because $F = R$ implies that $\lambda \in F \to \lambda^2 \geq 0$ and therefore

$$\langle \lambda v, \lambda v\rangle = \lambda^2 \langle v, v\rangle \geq 0$$

Because they are not a scalar multiple of each other, we can follow that there exists a decomposition

$$u = cv + w$$

where w is an orthogonal vector to v (proof of existence of such a vector and orthogonal decomposition does not depend on truthfullness of positivity). Thus we follow that

$$\langle u, u\rangle = \langle cv + w, cv + w\rangle = \langle cv, cv\rangle + \langle w, w\rangle$$

thus

$$\langle w, w\rangle = \langle u, u\rangle - \langle cv, cv\rangle = \langle u, u\rangle - c^2\langle v, v\rangle < 0$$

thus we follow that there exists w, which is orthogonal to v and $\langle w, w\rangle < 0$

Now let

$$\kappa = -\frac{\langle v, v\rangle}{\langle w, w\rangle}$$

we follow that

$$\kappa > 0$$

and thus

$$\langle v, v\rangle = -\kappa\langle w, w\rangle$$

$$\langle v, v\rangle + \kappa\langle w, w\rangle = 0$$

$$\langle v, v\rangle + \langle \sqrt{\kappa}w, \sqrt{\kappa}w\rangle = 0$$

and by pythagorean theorem we've got that

$$\langle v + \sqrt{\kappa}w, v + \sqrt{\kappa}w\rangle = 0$$

Since u is not a scalar multiple of v, we follow that w is not a scalar multiple of v and therefore $v + \sqrt{\kappa}w \neq 0$. Thus we follow that there exists a vector $v + \sqrt{\kappa}w \neq 0$, for which it is true that

$$\langle v + \sqrt{\kappa}w, v + \sqrt{\kappa}w\rangle = 0$$

which is a contradition of the definiteness clause. Thus we follow that there is no vector u such that $\langle u, u\rangle < 0$. Thus we follow that

$$u \in U \to \langle u, u\rangle \geq 0$$

thus our clause implies positiveness, and therefore by double inclustion we follow that two sets are equal, as desired.

6.1.4

Suppose V is a real inner product space.

(a) Show that $\langle u+v, u-v\rangle = ||u||^2 - ||v||^2$ for every $u, v \in V$.

$$\langle u+v, u-v\rangle = \langle u, u-v\rangle + \langle v, u-v\rangle = \langle u-v, u\rangle + \langle u-v, v\rangle = \langle u, u\rangle + \langle -v, u\rangle + \langle u, v\rangle + \langle -v, v\rangle =$$

$$= \langle u, u\rangle - \langle v, u\rangle + \langle v, u\rangle - \langle v, v\rangle = \langle u, u\rangle - \langle v, v\rangle = ||u||^2 + ||v||^2$$

(b) Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$

$$\langle u+v, u-v\rangle = ||u||^2 - ||v||^2 = ||u||^2 - ||u||^2 = 0$$

where second equality comes from previous point.

(c) Use part (b) to show that the diagonals of a rhombus are perpendicular to each other

It would've been nice to draw a picture here, but we follow that one of the diagonals is equal to sum of two vectors, while the other is equal to their difference. Thus we follow that by (b), they are orthogonal and therefore perperdicular.

6.1.5

Suppose $T \in \mathcal{L}(V)$ is such that $||Tv|| \leq ||v||$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Suppose that it isn't. Then we follow that $\sqrt{2}$ is an eigenvalue of T, and therefore there exists a vector v such that

$$Tv = \sqrt{2}v$$

thus

$$||Tv|| = ||\sqrt{2}v|| = |\sqrt{2}|||v|| > ||v||$$

where the last inequality comes from the fact that $v \neq 0$ because it is an eigenvector, and therefore $||v|| > 0$ and $\sqrt{2} = |\sqrt{2}| > 1$. Thus we've got a contradition.

6.1.6

Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$$||u|| \leq ||u + av||$$

for all $a \in F$.

In forward direction:

Suppose that $\langle u, v \rangle = 0$. Then we follow that

$$\langle u, av \rangle = \overline{a} \langle u, v \rangle = \overline{a} 0 = 0$$

thus we can conclude that u and av are orthogonal for every $a \in F$. Thus by pythagorean theorem we've got that

$$||u + av||^2 = ||u||^2 + ||av||^2$$
$$||u||^2 \leq ||u + av||^2$$
$$||u|| \leq ||u + av||$$

and we're justified to make those implication because we're working with non-negative reals.

In reverse direction:

Suppose that

$$||u|| \leq ||u + av||$$

for all $a \in F$. Then we follow that

$$||u||^2 \leq ||u + av||^2$$

$$\langle u, u \rangle \leq \langle u + av, u + av \rangle$$

$$\langle u, u \rangle \leq \langle u, u \rangle + \langle av, av \rangle + \langle u, av \rangle + \langle av, u \rangle$$

$$0 \leq \langle av, av \rangle + \langle u, av \rangle + \langle av, u \rangle$$

$$\langle av, av \rangle \geq -\langle u, av \rangle - \langle av, u \rangle$$

$$\langle av, av \rangle \geq \langle -u, av \rangle + \langle av, -u \rangle$$

$$|a^2| \langle v, v \rangle \geq \langle -u, av \rangle + \overline{\langle -u, av \rangle}$$

$$|a^2| \langle v, v \rangle \geq \overline{a} \langle -u, v \rangle + a \overline{\langle -u, v \rangle}$$

$$\langle v, v \rangle \geq \frac{\overline{a}}{|a|^2} \langle -u, v \rangle + \frac{a}{|a|^2} \overline{\langle -u, v \rangle}$$

$$\langle v, v \rangle \geq \frac{\overline{a}}{|a|^2} \langle -u, v \rangle + \overline{\frac{\overline{a}}{|a|^2} \langle -u, v \rangle}$$

$$\langle v, v\rangle \geq 2\Re\left(\frac{\overline{a}}{|a|^2}\langle -u, v\rangle\right)$$

Suppose that $\langle u, v\rangle \neq 0$. Then we follow that $v \neq 0$ and therefore $\langle v, v\rangle \neq 0$. Thus we can follow that we can make right hand side as large as we want, since $\frac{\overline{a}}{|a|^2}$ is unbounded. Thus we follow that there exists a such that above-mentioned inequality does not hold, which is a contradiction. Therefore we conclude that $\langle u, v\rangle = 0$, as desired.

6.1.7

Suppose $u, v \in V$. Prove that $||au + bv|| = ||bu + av||$ for all $a, b \in R$ if and only if $||u|| = ||v||$.

Suppose that $||au + bv|| = ||bu + av||$ for $a, b \in R$. Then we follow that it is true for $a = 1$ and $b = 0$. Thus

$$||au + bv|| = ||1u + 0v|| = ||u|| = ||bu + av|| = ||0u + 1v|| = ||v||$$

Conversely, suppose that $||u|| = ||v||$. Then we follow that there exists decomposition

$$u = cv + w$$

thus

$$||au + bv||^2 = \langle au + bv, au + bv\rangle = \langle au, au\rangle + \langle bv, bv\rangle + \langle au, bv\rangle + \langle bv, au\rangle =$$

$$= |a|^2||u||^2 + |b|^2||v||^2 + ab\langle u, v\rangle + ab\langle v, u\rangle$$

$$||bu + av||^2 = \langle bu + av, bu + av, \rangle = \langle bu, bu\rangle + \langle av, av\rangle + \langle bu, av\rangle + \langle av, bu\rangle =$$

$$= |a|^2||v||^2 + |b|^2||u||^2 + ab\langle u, v\rangle + ab\langle v, u\rangle = |a|^2||u||^2 + |b|^2||v||^2 + ab\langle u, v\rangle + ab\langle v, u\rangle =$$

thus

$$||au + bv||^2 = ||bu + av||^2$$

$$||au + bv|| = ||bu + av||$$

as desired.

6.1.8

Suppose $u, v \in V$ and $||u|| = ||v|| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Let us use the orthogonal decomposition on that one and costruct u in terms of v.

$$c = \frac{\langle u, v\rangle = 1}{||v||^2} = \frac{1}{1} = 1$$

$$w = u - 1v = u - v$$

then we follow that

$$u = cv + w = v + u - v = u$$

which is not helpful. But if we try to construct v in terms of u, then we'll get also not a helpful result.

We can try to use a Cauchy-Schwartz inequality to get that

$$\langle u, v\rangle = ||u||||v||$$

and thus we can follow that u is a scalar multiple of V (and vice versa, since they are not equal to zero, because they are not orthogonal.)

$$\langle u-v, u-v\rangle = ||u||^2+||v||^2-\langle u, v\rangle-\langle v, u\rangle = ||u||^2+||v||^2-\langle u, v\rangle-\overline{\langle u, v\rangle} = 1+1-1-1 = 0$$

thus we follow that $u - v = 0$ by definiteness property of inner product. Therefore $u = v$, as desired.

6.1.9

Suppose $u, v \in V$ and $||u|| \leq 1$ and $||v|| \leq 1$. Prove that

$$\sqrt{1 - ||u||^2}\sqrt{1 - ||v||^2} \leq 1 - |\langle u, v\rangle|$$

$$\sqrt{1 - ||u||^2}\sqrt{1 - ||v||^2} \leq 1 - |\langle u, v\rangle|$$

By Cauchy-Schwartz

$$|\langle u, v\rangle| \leq ||u||||v||$$

$$1 - |\langle u, v\rangle| \geq 1 - ||u||||v||$$

$$\sqrt{1 - ||u||^2}\sqrt{1 - ||v||^2} \leq 1 - ||u||||v||$$

$$(1 - ||u||^2)(1 - ||v||^2) \leq (1 - ||u||||v||)^2$$

$$1 - ||u||^2 - ||v||^2 + (||u||||v||)^2 \leq 1 - 2||u||||v|| + (||u||||v||)^2$$

$$1 - ||u||^2 - ||v||^2 \leq 1 - 2||u||||v||$$
$$||u||^2 + ||v||^2 \geq 2||u||||v||$$
$$||u||^2 - 2||u||||v|| + ||v||^2 \geq 0$$
$$(||u|| - ||v||)^2 \geq 0$$

which is given.

6.1.10

Find vectors $u, v \in R^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$ and $(1, 2) = u + v$

Let $u' = (1, 3)$. We follow that $v' = (-3, 1)$ will suffice as a vector that is orthogonal to u. Thus we need to represent $(1, 2)$ in terms of basis of u' and v', which we can do by reducing some linear equation

$$\begin{cases} 1x - 3y = 1 \\ 3x + 1y = 2 \end{cases}$$

$$\begin{cases} 1x - 3y = 1 \\ 10y = -1 \end{cases}$$

$$\begin{cases} 1x - 3y = 1 \\ y = -0.1 \end{cases}$$

$$\begin{cases} x + 0.3 = 1 \\ y = -0.1 \end{cases}$$

$$\begin{cases} x = 0.7 \\ y = -0.1 \end{cases}$$

Thus we follow that if we set $v = (0.3, -0.1)$ and $u = (0.7, 2.1) = 0.7(1, 3)$ then we'll get the desired result.

6.1.11

Prove that

$$16 \leq (a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d})$$

for all positive numbers a, b, c, d.

Let

$$v = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$$

and

$$u = \left(\sqrt{\frac{1}{a}}, \sqrt{\frac{1}{b}}, \sqrt{\frac{1}{c}}, \sqrt{\frac{1}{d}}\right)$$

Then we follow that

$$v \cdot u = \sum_{i \in [a,b,c,d]} \sqrt{i}\sqrt{\frac{1}{i}} = \sum_{i \in [a,b,c,d]} \sqrt{i\frac{1}{i}} = \sum_{i \in [a,b,c,d]} 1 = 4 = |4| = |\langle v, u \rangle|$$

thus

$$|\langle v, u \rangle|^2 = 16$$

and

$$||v||^2 = \sum_{i \in [a,b,c,d]} \sqrt{i}\sqrt{i} = \sum_{i \in [a,b,c,d]} |i| = \sum_{i \in [a,b,c,d]} i = (a + b + c + d)$$

$$||u||^2 = \sum_{i \in [a,b,c,d]} \sqrt{\frac{1}{i}}\sqrt{\frac{1}{i}} = \sum_{i \in [a,b,c,d]} \frac{1}{i} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

By Cauchy-Schwartz we've got that

$$|\langle u, v \rangle| \leq ||u||||v||$$

$$|\langle u, v \rangle|^2 \leq ||u||^2||v||^2$$

$$16 \leq (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

and in general we've got that

$$2^n \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right)$$

for $a_i > 0$, which is neat.

6.1.12

Prove that

$$(x_1 + ... + x_n)^2 \leq n(x_1^2 + ... + x_n^2)$$

for all positive integers and all real numbers $x_1, ..., x_n$.

maybe something along the lines of

$$u = (..x_j...)$$

$$v = (1, 1, ..., 1)$$

$$||u||^2 = \sum x_j^2$$
$$||v||^2 = \sum 1^2 = n$$

and

$$|\langle u, v\rangle| = |\sum x_j|$$

and thus

$$|\langle u, v\rangle|^2 = |\sum x_j|^2 = (\sum x_j)^2$$

thus by using Cauchy-Schwartz inequality and presented vectors, we'll get the desired conclusion.

6.1.13

Suppose u, v are nonzero vectors in R^2. Prove that

$$\langle u, v\rangle = ||u||||v|| \cos\theta$$

By using the picture, which I will not provide, and the humble law of cosines, we follow that

$$||u - v||^2 = ||u||^2 + ||v||^2 - 2||u||||v|| \cos\theta$$
$$||u||^2 + ||v||^2 - \langle u, v\rangle - \langle v, u\rangle = ||u||^2 + ||v||^2 - 2||u||||v|| \cos\theta$$
$$-2\langle v, u\rangle = -2||u||||v|| \cos\theta$$
$$\langle v, u\rangle = ||u||||v|| \cos\theta$$

as desired.

6.1.14

Not gonna copy that text

arccos is a function, which is defined on domain $[-1, 1]$, therefore in order for to restrict domain we gotta have the Cauchy-Schwatrz

6.1.15

Prove that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right)$$

for $a_j, b_j \in R$.

Let

$$u = (..., \sqrt{j}a_j, ...)$$

$$v = (..., \frac{b_j}{\sqrt{j}}, ...)$$

then we follow that

$$||u||^2 = \sum j a_j^2$$

$$||v||^2 = \sum \frac{b_j^2}{j}$$

$$\langle u, v\rangle = \sum \frac{b_j}{\sqrt{j}} \sqrt{j} a_j = \sum a_j b_j$$

$$|\langle u, v\rangle|^2 = \langle u, v\rangle^2 = (\sum a_j b_j)^2$$

thus Cauchy-Schwartz inequality implies the desired result.

6.1.16

Suppose $u, v \in V$ are such that

$$||u|| = 3, ||u + v|| = 4, ||u - v|| = 6$$

What number does $||v||$ equal?

$$\langle u + v, u + v\rangle = \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle = 16$$
$$\langle u - v, u - v\rangle = \langle u, u\rangle + \langle v, v\rangle - \langle u, v\rangle - \langle v, u\rangle = 36$$
$$\langle u + v, u + v\rangle + \langle u - v, u - v\rangle = 2\langle u, u\rangle + 2\langle v, v\rangle = 52$$
$$\langle u, u\rangle + \langle v, v\rangle = 26$$
$$9 + \langle v, v\rangle = 26$$
$$\langle v, v\rangle = 17$$
$$||v|| = \sqrt{\langle v, v\rangle} = \sqrt{17}$$

6.1.17

Prove or disprove: there is an inner product on R^2 such that the associated norm is given by

$$||(x, y)|| = \max\{x.y\}$$

for all $(x, y) \in R^2$

for $x = (-1, -1)$ we've got that

$$||x|| = \sqrt{\langle x, x\rangle} = -1$$

and since we're working with real numbers, we follow that there is no such number.

We also don't have such a product in complex numbers, since such norm would violate positivity clause of inner product definition.

6.1.18

Suppose $p > 0$. Prove that there is an inner product on R^2 such that the associated norm is given by

$$||(x,y)|| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in R^2$ if and only if $p = 2$.

Since

$$||(x,y)|| = (x^p + y^p)^{1/p} = \sqrt{\langle (x,y),(x,y)\rangle}$$

thus

$$(x^p + y^p)^{1/p} = \sqrt{\langle (x,y),(x,y)\rangle}$$
$$(x^p + y^p)^{2/p} = \langle (x,y),(x,y)\rangle$$

We follow

$$((\lambda x)^p+(\lambda y)^p)^{2/p} = (\lambda^p x^p+\lambda^p y^p)^{2/p} = (\lambda^p)^{2/p}(x^p+y^p)^{2/p} = (\lambda^p)^{2/p}(x^p+y^p)^{2/p} = (\lambda)^2(x^p+y^p)^{2/p}$$

therefore we don't lose homogeneity.

$$((x+x')^2+(y+y')^2) = x^2+2xx'+x'^2+y^2+2yy'+y'^2 = \langle (x,y),(x,y)\rangle+\langle (x',y'),(x',y')\rangle+2\langle (x,y),(x',y')\rangle$$

$$\langle (x+x', y+y'),(x+x', y+y')\rangle =$$
$$\langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle + 2\langle u,v\rangle$$

we've got this expantion only for the power of 2, therefore we follow that in order to have valid additivity in the first slot we've got to have $p = 2$.

Reverse case is trivial, if $p = 2$, then we've got a Eucledian inner product, that will suffice.

6.1.19

Suppose V is a real innner product space. Prove that

$$\langle u,v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for all $u, v \in V$.

$$\frac{||u+v||^2 - ||u-v||^2}{4} = \frac{\langle u+v\rangle - \langle u-v\rangle}{4} = \frac{2\langle u,v\rangle + 2\langle u,v\rangle}{4} = \langle u,v\rangle$$

as desired. 6.1.16 has proper expansions, that will demonstrate the case more clearly.

6.1.20

Suppose V is a complex inner product space. Prove that

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}$$

for all $u, v \in V$.

$$\frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4} = \frac{\langle u+v\rangle - \langle u-v\rangle + \langle u-iv\rangle i - \langle u-iv\rangle i}{4} =$$

$$= \frac{2\langle u,v\rangle + 2\langle v,u\rangle + 2\langle u,iv\rangle i + 2\langle iv,u\rangle i}{4} = \frac{2\langle u,v\rangle + 2\langle v,u\rangle + 2\langle u,v\rangle \bar{i} i + 2\langle v,u\rangle i^2}{4} =$$

$$= \frac{2\langle u,v\rangle + 2\langle v,u\rangle + 2\langle u,v\rangle - 2\langle v,u\rangle}{4} = \frac{4\langle u,v\rangle +}{4} = \langle u,v\rangle$$

how nice

6.1.21

Show that if $||||$ is a notm on U satisfying the parallelogram equality, then there is an inner product on u such that $||u|| = \langle u, u\rangle^{1/2}$ for all $u \in U$.

TODO

6.1.22

Show that the square of an average is less than or equal to the average of squares. More precisely, show that if $a_1, ..., a_n \in R$, then the square of the average is less than or equal to the average of $a_1^2, ..., a_n^2$

From exercise 6.1.12 we've got that for reals $a_1, ..., a_n$ we've got that

$$(\sum a_j)^2 \leq n\left(\sum a_j^2\right)$$

thus we follow that

$$\frac{1}{n^2}(\sum a_j)^2 \leq \frac{1}{n}\left(\sum a_j^2\right)$$

$$\left(\frac{\sum a_j}{n}\right)^2 \leq \frac{\sum a_j^2}{n}$$

as desired.

6.1.23

Suppose $V_1, ..., V_n$ are inner product spaces. Show that the equation

$$\langle (u_1, ..., u_n), (v_1, ..., v_n) \rangle = \langle u_1, v_1 \rangle + ... + \langle u_n, v_n \rangle$$

defined an inner product on $V_1 \times ... \times V_n$

Given function satisfies positivity and definiteness by positiveness and definiteness of respective inner product spaces.

Let $u = (u_1, ..., u_n), v = (v_1, ..., v_n), w = (w_1, ..., w_n) \in V_1 \times ... \times V_n$ and $\lambda \in F$. Then we follow that

$$\langle \lambda u, v \rangle = \sum \langle \lambda u_j v_j \rangle = \sum \lambda \langle u_j v_j \rangle = \lambda \sum \langle u_j v_j \rangle = \lambda \langle u, v \rangle$$

$$\langle u+w, v \rangle = \sum \langle u_j + w_j, v_j \rangle = \sum (\langle u_j, v_j \rangle + \langle w_j, v_j \rangle) = \sum \langle u_j, v_j \rangle + \sum \langle w_j, v_j \rangle = \langle u, v \rangle + \langle w, v \rangle$$

thus we follow that we've got a linearity in the first slot.

$$\langle u, v \rangle = \sum \langle u_j, v_j \rangle = \sum \overline{\langle v_j, u_j \rangle} = \overline{\sum \langle v_j, u_j \rangle} = \overline{\langle v, u \rangle}$$

therefore we've got conjugate symmetry as well. Thus we follow that given function is an inner product by definition, as desired.

6.1.24

Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define $\langle \cdot, \cdot \rangle_1$ by $\langle u, v \rangle_1 = \langle Su, Sv \rangle$ for $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V.

We've got positivity and conjugate summetry for free from the previous inner product.

$$\langle \lambda u + w, v \rangle_1 = \langle S(\lambda u + w), S(v) \rangle = \langle \lambda Su + Sw, S(v) \rangle = \lambda \langle Su, S(v) \rangle + \langle Sw, Sv \rangle$$

thus we follow that we have linearity in the first slot. From injectivity of S we follow that

$$v = 0 \iff Sv = 0 \iff \langle Sv, Sv \rangle = 0 \iff \langle v, v \rangle_1 = 0$$

thus injectivity of S is equivalent to definiteness of our function. Thus we follow that given function is an inner product, as desired.

6.1.25

Suppose $S \in \mathcal{L}(V)$ is nont injective. Define $\langle \cdot, \cdot \rangle_1$ as in the exercise above. Explain why $\langle \cdot, \cdot \rangle_1$ is not an inner product on V.

Because S is not injective we follow that there exists $v \neq 0$ such that

$$Sv = 0$$

thus

$$v \neq 0 \wedge \langle v, v \rangle_1 = \langle Sv, Sv \rangle = 0$$

which is a violation of definiteness clause.

6.1.26

Suppose f, g are differentiable functions from R to R^n.

(a) Show that

$$\langle f(t), g(t)\rangle' = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$$

Let us define a function

$$q(x) = \langle f(t), g(t)\rangle$$

then I think that we need to find $q'(x)$.

This identity is clearly simular to the identity for the differentiation of the product of functions. Thus we probably want to show that $\langle f(t), g(t)\rangle$ is a product of some functions.

If we want to investigate the Eucledian product, then we're getting that

$$\langle f(t), g(t)\rangle = \sum f_j(t)g_j(t)$$

thus

$$\langle f(t), g(t)\rangle' = \sum f_j(t)'g_j(t) + f_j(t)g_j'(t) = \sum f_j(t)'g_j(t) + \sum f_j(t)g_j'(t) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$$

I can't follow any further for the general case of the innerp product, therefore I'm going to assume that we're only need to cover the Eucledean product for this case.

(b) Suppose $c > 0$ and $||f(t)|| = c$ for every $t \in R$. Show that $\langle f'(t), f(t)\rangle = 0$ for every $t \in R$.

The intuitive notion is simple: if the function does not change its norm, then it's on the circle and therefore its derivative is tangent to the function.

Suppose that $||f(t)|| = c$. Then we follow that

$$||f(t)||^2 = c^2$$
$$\langle f(t), f(t)\rangle = c^2$$
$$\langle f(t), f(t)\rangle' = 0$$

by standard rules of differentiation. By previous point we follow that

$$\langle f(t), f(t)\rangle' = \langle f(t), f'(t)\rangle + \langle f(t), f'(t)\rangle$$

since we're working with real space we follow that $\langle f(t), f'(t)\rangle = \langle f(t), f'(t)\rangle$ and therefore

$$\langle f(t), f(t)\rangle' = 2\langle f(t), f'(t)\rangle$$

by the point made earlier

$$2\langle f(t), f'(t)\rangle = 0$$
$$\langle f(t), f'(t)\rangle = 0$$

as desired.

(c) Interpret the result in part (b) geometrically in terms of the tangent vector to a curve lying on a sphere in R^n contered at the origin

Alrerady did it in the beggining of part (b).

6.1.27

Suppose $u, v, w \in V$. Prove that

$$||w - \frac{1}{2}(u+v)||^2 = \frac{||w-u||^2 + ||w-v||^2}{2} - \frac{||u-v||^2}{4}$$

Firstly, let's reshuffle lhs a bit

$$||w - \frac{1}{2}(u+v)||^2 = ||\frac{1}{2}(w-u) + \frac{1}{2}(w-v)||^2 = \frac{1}{4}||(w-u)+(w-v)||^2 =$$

$$= \frac{1}{4}(||w-u||^2 + ||w-v||^2 + \langle w-u, w-v\rangle + \langle w-v, w-u\rangle)$$

then let's factor rhs onto one denominator

$$\frac{||w-u||^2 + ||w-v||^2}{2} - \frac{||u-v||^2}{4} = \frac{2||w-u||^2 + 2||w-v||^2 - ||u-v||^2}{4}$$

then combine the two

$$\frac{1}{4}(||w-u||^2 + ||w-v||^2 + \langle w-u, w-v\rangle + \langle w-v, w-u\rangle) = \frac{2||w-u||^2 + 2||w-v||^2 - ||u-v||^2}{4}$$

reduce'em a bit

$$||w-u||^2 + ||w-v||^2 + \langle w-u, w-v\rangle + \langle w-v, w-u\rangle = 2||w-u||^2 + 2||w-v||^2 - ||u-v||^2$$

and expand the rest

$$\langle w-u, w-v\rangle + \langle w-v, w-u\rangle = ||w-u||^2 + ||w-v||^2 - ||u-v||^2$$

$$(\langle w\rangle + \langle u,v\rangle - \langle u,w\rangle - \langle w,v\rangle) + (\langle w\rangle + \langle v,u\rangle - \langle w,u\rangle - \langle v,w\rangle) = ||w-u||^2 + ||w-v||^2 - ||u-v||^2$$

$$(2\langle w\rangle + \langle u,v\rangle - \langle u,w\rangle - \langle w,v\rangle) + (\langle v,u\rangle - \langle w,u\rangle - \langle v,w\rangle) =$$

$$= (\langle w\rangle + \langle u\rangle - \langle w,u\rangle - \langle u,w\rangle) + (\langle w\rangle + \langle v\rangle - \langle w,v\rangle - \langle v,w\rangle) - \langle u\rangle - \langle v\rangle + \langle u,v\rangle + \langle v,u\rangle$$

and then pray to anyone who listens that this thing reduces to

$$0 = 0$$

thus we've got the desired identity.

6.1.28

Suppose C is a subset of V with the property that $u, v \in C$ implies $\frac{1}{2}(u+v) \in C$. Let $w \in V$. Show that there is at most one point in C that is closest to w. In other words, show that there is at most one $u \in C$ such that

$$||w-u|| \leq ||w-v|| \text{ for all } v \in C$$

If C is empty or if it contains only one element, then the case is trivial. Thus suppose that C has more than one element.

Suppose that it isn't the case and there exist $u' \neq u$ with desired property. Then we follow that

$$||w-u|| = ||w-u'||$$

We also follow that since $u \neq u'$

$$||u-u'|| \neq 0$$

Since $u, u' \in C$ we follow that $w = \frac{1}{2}(u+u') \in C$. Now let us use previous identity

$$||w - \frac{1}{2}(u+u')||^2 = \frac{||w-u||^2 + ||w-u'||^2}{2} - \frac{||u-u'||^2}{4}$$

since

$$||u-u'|| \neq 0$$

we follow that

$$||u-u'||^2 \neq 0$$

$$\frac{||u-u'||^2}{4} \neq 0$$

thus we can follow that

$$||w - \frac{1}{2}(u+u')||^2 < \frac{||w-u||^2 + ||w-u'||^2}{2}$$

by using our identity we follow that

$$||w - \frac{1}{2}(u+u')||^2 < \frac{2||w-u||^2}{2}$$

$$||w - \frac{1}{2}(u+u')||^2 < ||w-u||^2$$

$$||w - \frac{1}{2}(u+u')|| < ||w-u||$$

thus we follow that point $\frac{1}{2}(u+u')$ is closer than u, which is a contradiction.

6.1.29

For $u, v \in V$ define $d(u, v) = ||u - v||$.

(a) Show that d is a metric on V.

$$d(x, x) = ||x - x|| = ||0|| = 0$$

$$d(x, y) = ||x - y|| = ||(-1)(y - x)|| = |-1|||y - x|| = ||y - x|| = d(y, x)$$

and by definiteness and positivity we follow that

$$x \neq y \rightarrow \langle x - y, x - y\rangle > 0 \rightarrow ||x - y|| > 0 \rightarrow d(x - y) > 0$$

which I think suffices.

(b) Show that if V is finite-dimentional, then d is a complete metric on V (meaning that every Cauchy sequence converges to a point in V).

V is isomorphic to F^n, which is equal to R^n or C^n, both of which are complete (not completely sure about the latter, but it's safe to assume that it is the case). Let $v_1, ..., v_n$ be a basis of V. Then we follow that

$$d(x, y) = d(\sum a_j v_j, \sum a'_j v_j)$$

proof that d with relation to F^n is a metric is trivial. Thus we follow that V is also complete, as desired.

(c) Show that every finite-dimentional subspace of V is a closed subset of V (with respect to the metric d).

Let U be a finite subspace of V.

Suppose that we've got a convergent sequence in U. Then we follow that it is a Cauchy sequence and therefore it has its limit in U by previous point. Thus we follow that U has all of its limit points, and therefore it is a closed set.

6.1.30

A polynomial on R^n is a linear combination of functions of the form $x_1^{m_1}, ..., x_n^{m_n}$, where $m_1, ..., m_n$ are nonnegative integers.

Suppose q is a polynomial R^n. Prove that there exists a harmoniic polynomial p on R^n such that $p(x) = q(x)$ for every $x \in R^n$ with $||x|| = 1$

Taking advice from the hint we're setting

$$p = q + (1 - ||x||^2)r$$

for some polynomial r. Then we follow that

TODO

6.1.31

Use inner product to prove Appolonius's Identity: In a triangle with sides of length a, b and c let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$$

From the picture we've got that

$$a = -\frac{1}{2}c - d$$

$$b = \frac{1}{2}c - d$$

thus

$$a^2 = \langle -\frac{1}{2}c - d\rangle$$

$$b^2 = \langle \frac{1}{2}c - d\rangle$$

$$a^2+b^2 = \langle -\frac{1}{2}c-d\rangle+\langle \frac{1}{2}c-d\rangle = \langle \frac{1}{2}c+d\rangle+\langle \frac{1}{2}c-d\rangle = \langle \frac{1}{2}c\rangle+\langle d\rangle+2\langle \frac{1}{2}c, d\rangle+\langle \frac{1}{2}c\rangle+\langle d\rangle-2\langle \frac{1}{2}c, d\rangle =$$

$$= 2|(\frac{1}{2})^2|\langle c\rangle + 2\langle d\rangle = \frac{1}{2}\langle c\rangle + 2\langle d\rangle = \frac{1}{2}c^2 + 2d^2$$

6.2 Orthonormal Bases

6.2.1

(a) Suppose $\theta \in R$. Show that $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$ *and* $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$ *are orthonnormal bases of R^2.*

$$\cos\theta = 0 \implies \sin\theta \neq 0$$

thus we follow that none of those vectors are zero.

Assuming that we've got Eucledian inner product (otherwise this thing doesn't hold), we follow by using some important trigonometric property that states that

$$\sin^2\theta + \cos^2\theta = 1$$

we follow that norms of all of the vectors is 1.

Now we follow that

$$\langle(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\rangle = -\sin\cos + \sin\cos = 0$$

$$\langle(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\rangle = -\sin\cos + \sin\cos = 0$$

thus those vectors are orthogonal. Given that we've got two nonzero orhogonal vectors in both of the lists, we follow that they are not scalar multiples of each other and therefore linearly independent. Taking into account the dimention of the space and length of lists of vectors, we follow that we've got orthonormal basis, as desired.

(b) Show that each orthonormal basis of R^2 is of the form given by one of the two possibilities of part (a)

Suppose that $(x_1, x_2), (y_1, y_2)$ is an orthonormal basis of R^2. Then we follow that

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 0$$

From the fact that

$$||(x_1, x_2)|| = ||(x_1, x_2)||^2 = x_1^2 + x_2^2 = 1$$

we follow that $|x_1| \leq 1$. Thus we follow that there exists θ such that

$$x_1 = \cos\theta$$

and we can follow from above-mentioned trigonometric identity that

$$x_2 = \pm\sqrt{1 - \cos\theta} = \sin\theta$$

From the fact that bases are orthogonal we follow that two vectors are perpendicular. Thus

$$(y_1, y_2) = \begin{cases} (\cos(\theta + \pi/2), \sin(\theta + \pi/2)) = (-\sin\theta, \cos\theta) \\ (\cos(\theta - \pi/2), \sin(\theta - \pi/2)) = (\sin\theta, -\cos\theta) \end{cases}$$

thus we follow that any orthonormal basis in R^3 has one of the desired forms.

6.2.2

Suppose $e_1, ..., e_m$ is an orthogonal list of vectors in V. Let $v \in V$. Prove that

$$||v||^2 = \sum |\langle v, e_j \rangle|^2$$

if and only if $v \in \text{span}(e_1, ..., e_m)$.

Suppose that

$$||v||^2 = \sum |\langle v, e_j \rangle|^2$$

and $v \notin \text{span}(e_1, ..., e_m)$. Then we follow that $e_1, ..., e_m, v$ is linearly independent list in V. By applying Gram-Schmidt procedure we can aquire vector e_v, such that $v \in \text{span}(e_1, ..., e_m, e_v)$ and $e_1, ..., e_m, e_v$ is orthonormal list. Let

$$v = \sum_{i \in \{1,...,m,v\}} a_i e_i = \sum_{i \in \{1,...,m,v\}} |\langle v, e_i \rangle| e_i$$

then we follow that since $v \notin \text{span}(e_1, ..., e_m)$ that $|\langle v, e_i\rangle| \neq 0$. thus

$$||v||^2 = \sum_{i \in \{1,...,m,v\}} |\langle v, e_i\rangle|^2 \neq \sum_{i \in \{1,...,m\}} |\langle v, e_j\rangle|^2$$

thus we've got the contradiction.

Reverse implication is followed directly from the theorem for orthonormal bases.

6.2.3

Suppose $T \in \mathcal{L}(R^2)$ has an upper-triangular matrix with respect to the basis $(1,0,0), (1,1,1), (1,1,2)$. Find an orthonormal basis of R^3 (use the usual inner product on R^2) with respect to which T has an upper-triangular matrix.

Essentially we're asked to apply GSP to given basis (follows from either proof of Schur's Theorem, clause in GSP itself or common sense), so let's do this.

Let given vectors be also denoted by v_j's

$$||v_1|| = 1$$

thus

$$e_1 = (1,0,0)$$
$$e_2' = v_2 - \langle v_2, e_1\rangle e_1 = (0,1,1)$$
$$||e_2'|| = \sqrt{2}$$
$$e_2 = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$
$$e_3' = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2 = (0,1,2) - \frac{3\sqrt{2}}{2}(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = (0,1,2) - (0, \frac{3}{2}, \frac{3}{2}) = (0, -\frac{1}{2}, \frac{1}{2})$$
$$e_3 = (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

not sure that this is right, but this seems to make sense.

6.2.4

Suppose n is a positive integer. Prove that

$$\frac{1}{2\sqrt{\pi}}, \frac{\cos^1 x}{2\sqrt{\pi}}, ..., \frac{\cos^n x}{2\sqrt{\pi}}, ... \frac{\sin^1 x}{2\sqrt{\pi}}, ..., \frac{\sin^n x}{2\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

From my previous endeavour in real analysis I've got that

For all $n \in N$

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \pi \text{ and } \int_{-\pi}^{\pi} \sin^2(nx)dx = \pi$$

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}dx = \frac{1}{2}\int_{-\pi}^{\pi} 1+\cos 2nxdx = \frac{1}{2}\left(\left[\int_{-\pi}^{\pi} 1dx\right] + \left[\int_{-\pi}^{\pi} \cos 2nxdx\right]\right) =$$

$$= \frac{1}{2}(2\pi + 0) = \pi$$

$$\int_{-\pi}^{\pi} \sin^2(nx)dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}dx = \frac{1}{2}\int_{-\pi}^{\pi} 1-\cos 2nxdx = \frac{1}{2}\left(\left[\int_{-\pi}^{\pi} 1dx\right] - \left[\int_{-\pi}^{\pi} \cos 2nxdx\right]\right) =$$

$$= \frac{1}{2}(2\pi - 0) = \pi$$

thus from linearity of the integral we follow that all of the given functions have norm 1

For all $m, n \in N$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$$

For $m \neq n$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(mx+nx) + \sin(mx-nx)\right]dx =$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} \sin((m+n)x) + \int_{-\pi}^{\pi} \sin((m-n)x)dx\right) = \frac{1}{2}(0+0) = 0$$

thus the functions are orthogonal. Thus we follow that we've got an orthonormal list, as desired.

6.2.5

On $P_2(R)$*, consider the inner product given by*

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

Apply GSP to the basis $1, x, x^2$ *to produce an orthonormal basis of* $P^2(R)$*.*

$$\langle 1, 1\rangle \int_0^1 1^2 dx = x|_0^1 = 1$$

thus we follow that $e_1 = 1$.

$$e_2' = x - \int_0^1 x * e_1 dx = x - \frac{1}{2}$$

$$||e_2'|| = \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 x^2 - x + \frac{1}{4} dx = [\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}]_0^1 = \frac{1}{12} = (\frac{1}{2\sqrt{3}})^2$$

$$e_2 = 2\sqrt{3}(x - \frac{1}{2})$$

$$\langle x^2, 1 \rangle = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\langle x^2, 2\sqrt{3}(x-\frac{1}{2})\rangle = 2\sqrt{3}\int_0^1 x^2(x - \frac{1}{2})dx = 2\sqrt{3}\int_0^1 x^3 - \frac{1}{2}x^2 dx = 2\sqrt{3}[\frac{x^4}{4} - \frac{x^3}{6}]_0^1 = 2\sqrt{3}(\frac{1}{4} - \frac{1}{6}) = \frac{\sqrt{3}}{6}$$

$$e_3' = x^2 - \frac{\sqrt{3}}{6}e_2 - \frac{1}{3}e_1$$

$$e_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6})$$

checked with maxima, seems to be correct.

6.2.6

Find an orthonormal basis of $P_2(R)$ (with inner product as in Exercise 5) such that the differentiation operator (the operator that takes p to p') on $P_2(R)$ has an upper-triangular matrix with respect to this basis.

Given that differentiation operator is upper-triandular with respect to basis $1, x, x^2$, we can follow that the basis from previous exercise will suffice

6.2.7

Find a polynomial $q \in P_2(R)$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)dx$$

for every $p \in P_2(R)$.

We can use the procedure from proof of RRT to get

$$u = -15x^2 + 15x - \frac{3}{2}$$

which seems to be working from what I can test in Maxima

6.2.8

Find a polynomial $q \in P_2(R)$ such that

$$\int_0^1 p(x)(\cos \pi x)dx = \int_0^1 p(x)q(x)dx$$

for every polynomial $p \in P_2(R)$

From the same procedure in proof of RRT we've got

$$u = -\frac{24(x - \frac{1}{2})}{\pi^2}$$

which seems to be working as well.

6.2.9

What happens if the GSP is applied to a list of vectors that is not linearly independet

Then we'll have numerator in first linearly dependent vector in the list become 0, thus its norm will be zero, and then we'll have a division by zero problem.

6.2.10

Suppose V is a real inner product space and $v_1, ..., v_m$ is linearly independent list of vectors in V. Prove that there exists exactly 2^m orthonormal lists $e^1, ..., e^m$ of vectors in V such that

$$\operatorname{span}(v_1, ..., v_j) = \operatorname{span}(e_1, ..., e_j)$$

for all $j \in \{1, ..., m\}$.

2^m is cardinality of a set with m elements, which will probably help us somehow.

If we start with the small cases, then we follow that there are two such lists in one dimention. Precisely we've got normalisation of the first vector and this normalized vector multiplied by -1. For the case with two dimentions we've got result of GSP, where we either multiply e_j with 1 or with -1. Now we can see resemblence of generalized argument.

We now have that we can apply GSP to a given list of vectors, and then we've got the desired result for this list of vectors, or its scalar multiples of -1. There are m vectors which we can multiply by -1, thus we follow that there are 2^m distinct lists of vectors with desired property at least.

Now we need to prove that there are no other lists with the desired property. We're going to proceed with induction for this one. Base case is that there are only two normalized vectors for v_1. Our hypothesis is that there are only 2^{n-1} lists for the case with $n-1$ vectors. Then we follow that by adding another vector to the list we increase dimention of the span by 1, therefore there are only two normalized vectors, that are orthogonal to previous ones and such that span of orthonormal lists is the same as with the original span. Therefore we follow that there are $2 * 2^{n-1} = 2^n$ lists with the desired property, which gives us the desired result.

6.2.11

Suppose $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are inner products on V such that $\langle v, w \rangle_1 = 0$ if and only if $\langle v, w \rangle_2 = 0$. Prove that there is a positive number c such that $\langle v, w \rangle_1 = c \langle v, w \rangle_2$ for every $v, w \in V$.

For now, let $v \neq 0$.

By orthogonal decomposition we've got that

$$c_1 = \frac{\langle v, w \rangle_1}{\langle v, v \rangle_1}$$

$$c_2 = \frac{\langle v, w \rangle_2}{\langle v, v \rangle_2}$$

$$u_1 = w - c_1 v$$

$$u_2 = w - c_2 v$$

such that

$$\langle u_1, v \rangle_1 = 0$$

$$\langle u_2, v \rangle_2 = 0$$

$$w = c_1 v + u_1$$

$$w = c_2 v + u_2$$

thus we follow that

$$\langle u_2, v \rangle_2 = 0 = \langle u_1, v \rangle_2$$

$$\langle u_2, v \rangle_1 = 0 = \langle u_1, v \rangle_1$$

thus

$$\langle v, w \rangle_1 = \langle v, c_1 v + u_1 \rangle_1 = \langle v, c_1 v \rangle_1 + \langle v, u_1 \rangle_1 = \overline{c_1} \langle v, v \rangle_1 + \langle v, u_1 \rangle_1 = \overline{c_1} \langle v, v \rangle_1$$

$$\langle v, w \rangle_2 = \langle v, c_1 v + u_1 \rangle_2 = \langle v, c_1 v \rangle_2 + \langle v, u_1 \rangle_2 = \overline{c_1} \langle v, v \rangle_2 + \langle v, u_1 \rangle_1 = \overline{c_1} \langle v, v \rangle_2$$

Let $c = \frac{\langle v, v \rangle_1}{\langle v, v \rangle_2}$. Then we follow that $c > 0$ and

$$\langle v, w \rangle_1 = \overline{c_1} \langle v, v \rangle_1 = c \overline{c_1} \langle v, v \rangle_2 = c \langle v, w \rangle_2$$

Since c is dependent exclusively on v, we can follow that

$$\langle v, u \rangle_1 = c \langle v, u \rangle_2$$

for every $u \in V$.

Now fix $q \neq 0 \in V$ and we can perform the same manipulations once again to get c'. Then we follow that

$$\begin{aligned}
\langle v, q\rangle_1 &= c\langle v, q\rangle_2 \\
\overline{\langle q, v\rangle_1} &= c\overline{\langle q, v\rangle_2} \\
\langle q, v\rangle_1 &= c'\langle q, v\rangle_2 \\
\overline{\langle q, v\rangle_1} &= \overline{c'\langle q, v\rangle_2} \\
c\overline{\langle q, v\rangle_2} &= \overline{c'\langle q, v\rangle_2} \\
c\overline{\langle q, v\rangle_2} &= c'\overline{\langle q, v\rangle_2} \\
c &= c'
\end{aligned}$$

thus we can follow that c is the same for every nonzero vector, and for zero vector

$$\langle 0, w\rangle_1 = \langle 0, w\rangle_2 = 0 = c0$$

thus we follow that there exists positive $c > 0$ such that

$$\langle v, w\rangle_1 = c\langle v, w\rangle_2$$

for every $v, w \in V$, as desired.

6.2.12

Suppose V is finite-dimentional and $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V with corresponding norms $||\cdot||_1$ and $||\cdot||_2$. Prove that there exists a positive number c such that

$$||v||_1 \leq c||v||_2$$

for every $v \in V$.

Because V is finite-dimentional, we follow that there exists an orthonormal basis $e_1, ..., e_n$ with respect to $\langle\cdot,\cdot\rangle_1$. There also exists orthonormal basis $e'_1, ..., e'_n$ with respect to $\langle\cdot,\cdot\rangle_2$. Then we can follow that

$$e_i = \sum b_{i,j}e'_j$$

Let $c' = \max\{|b_{1,1}|, ...|b_{n,n}|\}$.

Suppose $v \in V$. Then we follow that

$$v = \sum a_j e_j$$

For the first norm we follow that

$$||v||_1 = ||\sum a_j e_j||_1 = \sqrt{\sum |a_j|^2}$$

and for the second we've got that

$$||v||_2^2 = ||\sum a_j e_j||_2^2 = ||\sum a_j \sum [b_i e_i']||_2^2 = ||a_1(b_{1,1}e_1' + ... + b_{1,n}e_n') + ... + a_n(b_{n,1}e_1' + ... + b_{n,n}e_n')||_2^2 =$$

$$= ||(a_1 b_{1,1} + ... + a_n b_{n,1})e_1' + ... + (a_1 b_{1,1} + ... + a_n b_{n,n})e_n'||_2^2 =$$

$$= |(a_1 b_{1,1} + ... + a_n b_{n,1})|^2 + ... + |(a_1 b_{1,1} + ... + a_n b_{n,n})|^2 \leq$$

$$\leq (|a_1 b_{1,1}| + ... + |a_n b_{n,1}|)^2 + ... + (|a_1 b_{1,1}| + ... + |a_n b_{n,n}|)^2 =$$

$$= (|a_1||b_{1,1}| + ... + |a_n||b_{n,1}|)^2 + ... + (|a_1||b_{1,1}| + ... + |a_n||b_{n,n}|)^2 \leq$$

$$\leq (c'|a_1| + ... + c'|a_n|)^2 + ... + (c'|a_1| + ... + c'|a_n|)^2 = c'^2 n (\sum |a_n|)^2 =$$

$$= c'^2 n (\sum |a_n|)^2 \leq c'^2 n (\sum |a_n|^2) = c'^2 n ||v||_1^2$$

where inequalities come from either triangular inequality or the fact that

$$(\sum a_j)^2 \leq (\sum a_j^2)$$

for $a_j > 0$.

Since $c' > 0$ and $n \in N$, we follow that $c = \sqrt{c'^2 n} > 0$. Thus we can follow that

$$||v||_2 \leq c||v||_1$$

for some $c \in R^+$, as desired. (This one might be false, but I'll continue anyways)

6.2.13

Suppose $v_1, ..., v_m$ is linearly independent list in V. Show that there exists exists $w \in V$ such that $\langle w, v_j \rangle > 0$ for all $j \in \{1, ..., m\}$

We're going to proceed with induction on this one.

Base case is trivial: for the space with dimention 1 pick $w_1 = v_1$, and we're golden.

Suppose that we've got vector space, defined by span$(v_1, ..., v_{n-1})$, and this vector space has a vector w_{n-1} such that

$$\langle v_j, w_{n-1} \rangle > 0$$

for every $1 \leq j \leq n-1$. Then add a vector v_n to the list and define a space span$(v_1, ..., v_n)$. We follow that we haven't lost the fact that

$$\langle v_j, w_{n-1} \rangle > 0$$

for $1 \leq j \leq n-1$, but now we need to get a vector w_n that has the desired properties for all of the previous vectors, and the vector v_n as well. Using GSP we can define a vector

u_n to be a vector, that is orthogonal to every $v_1, ..., v_{n-1}$ and for it we'll have that for $1 \leq j \leq n-1$

$$\langle v_j, \kappa u_n + w_{n-1}\rangle = \kappa\langle v_j, u_n\rangle + \langle v_j, w_{n-1}\rangle = \langle v_j, \lambda w_{n-1}\rangle > 0$$

for every $\kappa \in F$. Since u_n was aquired through the GSP, we can follow that

$$\langle v_n, u_n\rangle \neq 0$$

and thereefore we're justified to define

$$\kappa = \frac{-\langle v_n, w_{n-1}\rangle + 1}{\langle v_n, u_n\rangle}$$

for which we have that

$$\kappa\langle v_n, u_n\rangle = -\langle v_n, w_{n-1}\rangle + 1$$
$$\kappa\langle v_n, u_n\rangle + \langle v_n, w_{n-1}\rangle = 1$$
$$\langle v_n, \overline{\kappa} u_n + w_{n-1}\rangle = 1 > 0$$

thus we've got a vector $w_n = \overline{\kappa} u_n + w_{n-1}$ such that

$$\langle v_j, w_n\rangle > 0$$

for every $1 \leq j \leq n$, as desired.

6.2.14

Suppose $e_1, ..., e_n$ is an orthogonal basis of V and $v_1, ..., v_n$ are vectors in V such that

$$||e_j - v_j|| < \frac{1}{\sqrt{n}}$$

for each j. Prove that $v_1, ..., v_n$ is a basis of V.

Suppose that $v_1, ..., v_n$ is linearly dependent. Then we can follow that there exist $a_1, ..., a_n \in F$ such that

$$\sum a_j v_j \neq 0$$

therefore there exists v_j such that

$$v_j = \sum a_i v_i$$

thus

$$||e_j - v_j|| < \frac{1}{\sqrt{n}}$$
$$||e_j - \sum a_i v_i|| < \frac{1}{\sqrt{n}}$$

TODO

6.2.15

6.2.16

Suppose $F = C$, V is finite-dimentional, $T \in \mathcal{L}(V)$, all the eigenvalues of T have absolute values less than 1, and $\epsilon > 0$. Prove that there exists a positive integer m such that $||T^m v|| \leq \epsilon ||v||$

Because V is finite-dimentional and complex, we follow that there exists an orthonormal basis of V, such that T has upper-triangular matrix. Let this basis be denoted by $e_1, ..., e_n$. Let $v \in V$. Then we follow that

$$v = \sum a_n e_n$$

for some $a_1, ..., a_n \in C$.

For e_1 we've got that

$$Te_1 = \lambda_1 e_1$$

thus

$$T^m e_1 = \lambda_1^m e_1$$

for any $m \in N$. Therefore we follow that

$$||T^m e_1|| = ||\lambda_1^m e_1|| = |\lambda_1^m|||e_1|| = |\lambda_1|^m$$

and

$$||T^m a_1 e_1|| = ||a_1 \lambda_1^m e_1|| = |\lambda_1^m|||e_1|| = |a_1||\lambda_1|^m$$

Since $|\lambda_1| < 1$, we follow that

$$\lim_{m \to \infty} ||T^m a_1 e_1|| = \lim_{m \to \infty} (|a_1||\lambda_1|^m) = 0$$

for every $a_1 \in C$

For e_2 we've got that

$$Te_2 = b_1 e_1 + \lambda_2 e_2$$

$$T^2 e_2 = b_1 T e_1 + \lambda_2 T e_2 = b_1 \lambda_1 e_1 + \lambda_2 (b_1 e_1 + \lambda_2 e_2) = b_1 \lambda_1 e_1 + \lambda_2 b_1 e_1 + \lambda_2^2 e_2 =$$

TODO

The rest of the chapter is marked as TODO for the better days

6.3 Orthogonal Complements and Minimization Problems

6.3.1

Suppose $v_1, ..., v_m \in V$. Prove that

$$\{v_1, ..., v_m\}^\perp = (\text{span}(v_1, ..., v_m))^\perp$$

Let

$$v \in \{v_1, ..., v_m\}^\perp$$

which means that for every $v_j \in \{v_1, ..., v_m\}$

$$\langle v_j, v\rangle = 0$$

which is eqivalent to (assuming that $\lambda_j \in F$ is arbitrary)

$$\langle \lambda_j v_j, v\rangle = \lambda_j 0$$

$$\langle \lambda_j v_j, v\rangle = 0$$

$$\sum_{1\leq j\leq m} \langle \lambda_j v_j, v\rangle = 0$$

$$\langle \sum_{1\leq j\leq m} [\lambda_j v_j], v\rangle = 0$$

which is equivalent to

$$v \in (\text{span}(v_1, ..., v_m))^\perp$$

thus we follow that

$$\{v_1, ..., v_m\}^\perp = (\text{span}(v_1, ..., v_m))^\perp$$

as desired.

6.3.2

Suppose U is a finite-dimentional subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$.

Suppose that it is not the case. Then we follow that there exists $v \in V \neq 0$ such that $v \notin U$. Let $u_1, ..., u_n$ be a basis of U. Then we follow that

$$v \notin \text{span}(u_1, ..., u_n)$$

therefore we follow that there exists a subspace $U' = \text{span}(u_1, ..., u_n, v)$. If we apply GSP to a list $u_1, ..., u_n, v$, we get an orthonormal basis $e_1, ..., e_k$ of U' and therefore

$$\langle e_k, e_j\rangle = 0$$

for any $1 \leq j \leq n$. Since $\text{span}(e_1, ..., e_n) = \text{span}(u_1, ..., u_n) = U$, we follow that for every $u \in U$

$$\langle e_k, u\rangle = 0$$

therefore $e_k \in U^\perp$. Thus $U^\perp \neq \{0\}$, which is a contradiction.

6.3.3

Suppose U is a subspace of V with basis $u_1, ..., u_m$ and

$$u_1, ..., u_m, w_1, ..., w_n$$

is a basis of V. Prove that if the GSP is applies to the basis of V above, producing list $e_1, ..., e_m, f_1, ..., f_n$, then $e_1, ..., e_m$ is an orthonormal basis of U and $f_1, ..., f_n$ is an orthonormal basis of $U^\perp$.

We can follow that

$$U = \operatorname{span}(u_1, ..., u_m) = \operatorname{span}(e_1, ..., e_m)$$

. Thus if $w \in \operatorname{span}(f_1, ..., f_n)$ then $w \in U^\perp$. Now suppose that $w \in U^\perp$. Then we follow that

$$w = \sum [a_j e_j] + \sum [b_j f_j]$$

Let $u \in U$ be defined as

$$u = \sum [e_j]$$

thus

$$\langle u, w\rangle = \langle \sum [e_j], \left(\sum [a_j e_j] + \sum [b_j f_j]\right)\rangle = \langle \sum [e_j], \sum [a_j e_j]\rangle + \langle \sum [e_j], \sum [b_j f_j]\rangle =$$

$$= \langle \sum [e_j], \sum [a_j e_j]\rangle = \sum |a_j|^2 = 0$$

thus we follow that $a_1 = ... = a_m = 0$. Therefore

$$w \in \operatorname{span}(f_1, ..., f_n)$$

thus be double inclusion we follow that $U^\perp = \operatorname{span}(f_1, ..., f_n)$, as desired.

6.3.4

Suppose U is a subspace of R^4 defined by

$$U = \operatorname{span}((1, 2, 3, -4), (-5, 4, 3, 2))$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

The idea is to append this list to a basis of R^4 (random vectors will do, the chance of them being linearly dependent is pretty small) and apply GSP to it. By using maxima I got the answer, which is pretty awful and included in maxima files.

6.3.5

Suppose V is finite-dimentional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity on V.

Let $e_1, ..., e_n$ be an orhonormal basis of U and expand it to orhonormal basis of V $e_1, ..., e_n, f_1, ..., f_m$. Let $v \in V$. Because $e_1, ..., e_n, f_1, ..., f_m$ is a basis of V we follow that there exist $a_1, ..., a_n, b_1, ..., b_m$ such that

$$v = \sum a_j e_j + \sum b_j f_j$$

Then we follow that

$$P_U(v) = P_U(\sum a_j e_j + \sum b_j f_j) = \sum a_j e_j$$

thus

$$(I - P_U)v = \sum a_j e_j + \sum b_j f_j - \sum a_j e_j = \sum b_j f_j = P_{U^\perp} v$$

as desired.

6.3.6

Suppose U and W are finite-dimentional subspaces of V. Prove that

$$P_U P_W = 0$$

if and only if $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$

In forward direction: Suppose that $P_U P_W = 0$. Let $u_1, ..., u_n$ be an orthonormal basis of U and $w_1, ..., w_m$ be an orthonormal basis of W. Then we follow that

$$P_U P_W w_j = P_U w_j = \sum \langle u_i, w_j \rangle u_i = 0$$

Since $u_1, ..., u_n$ is a basis of U, we follow that $\langle u_i, w_j \rangle = 0$ for every $1 \leq i \leq n$. Thus for $1 \leq i \leq n$ and $1 \leq j \leq m$ we've got that $\langle u_i, w_j \rangle = 0$. Therefore we follow that for every $u \in U$ and $w \in W$

$$\langle u, w \rangle = \langle \sum a_j u_j, \sum b_i w_i \rangle = 0$$

In reverse direction: If $\langle u, w \rangle = 0$ for every $u \in U$ and $w \in W$, then we can follow that for $v \in V$

$$P_U P_W(v) = \sum \langle u_j, P_W(v) \rangle u_j$$

and since $P_W(v) \in W$ for every $v \in V$ we follow that

$$P_U P_W(v) = \sum \langle u_j, P_W(v) \rangle u_j = \sum 0 u_j = 0$$

6.3.7

Suppose V is finite-dimentional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ *is orthogonal to every vector in* $\operatorname{range} P$. *Prove that there exsits a subspace U of V such that $P = P_U$.*

From exercise 4 in section 5.2 we know that $P^2 = P$ implies that

$$V = \operatorname{null} P \oplus \operatorname{range} P$$

thus we can follow that we can have an orthonormal basis $e_1, ..., e_n, f_1, ..., f_m$ of V with $e_1, ..., e_n$ being basis of $\operatorname{null} P$ and $f_1, ..., f_m$ being basis of $\operatorname{range} P$. Suppose that $v \in V$. Then we follow that

$$v = \sum a_j e_j + \sum b_j f_j$$

thus

$$Pv = P(\sum a_j e_j) + P(\sum b_j f_j)$$

since $e_1, ...e_n$ is a basis of $\operatorname{null} P$, we follow that

$$Pv = P(\sum a_j e_j) + P(\sum b_j f_j) = P(\sum b_j f_j)$$

thus we follow that P is an orthogonal projection for $\operatorname{range} P$. (Don't know why do we need clause about $\operatorname{null} P$ being orthogonal to $\operatorname{range} P$).

6.3.8

Suppose V is finite-dimentional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$||Pv|| \leq ||v||$$

for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.

Same as previous.

6.3.9

Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimentional subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$

If U is invariant under T, then we follow that for every $v \in V$ we've got that

$$P_U v \in U$$

$$T P_U v \in U$$

thus

$$T P_U v = P_U(T P_U v)$$

Suppose that $P_U T P_U = T P_U$ and suppose that T is not invariant under U. Then we follow that there exists $u \in U$ such that $Tu \notin U$. Thus

$$TP_U u = Tu \notin U$$

$$P_U T P_U u = P_U T u \in U$$

thus

$$P_U T P_U \neq P_U T$$

which is a contradiction.

6.3.10

Suppose V is finite-dimentional, $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$

In forward direction: Suppose that U and $U^\perp$ are invariant under T. Let $e_1, ..., e_n, f_1, ..., f_m$ denote the basis of V with $e_1, ..., e_n$ being the basis of U and $f_1, ..., f_m$ being the basis of $U^\perp$. Let $v \in V$. Then we follow that

$$v = u + w = \sum a_j e_j + \sum b_j f_j$$

where $u \in U$ and $w \in U^\perp$. Then we follow that

$$Tu = \sum a'_j e_j$$

$$Tw = \sum b'_j f_j$$

$$Tv = T(\sum a_j e_j + \sum b_j f_j) = \sum a'_j e_j + \sum b'_j f_j$$

for some $a_1, ..., b_n \in F$. Thus

$$P_U T v = P_U T(\sum a_j e_j + \sum b_j f_j) = P_U(\sum a'_j e_j + \sum b'_j f_j) = \sum a'_j e_j$$

and

$$T P_U v = T P_U(\sum a_j e_j + \sum b_j f_j) = T \sum a_j e_j = \sum a'_j e_j = P_U T v$$

In reverse direction: Suppose that $P_U T = T P_U$. Suppose that T is not invariant under U. Then we follow that there exists $u \in U$ such that $Tu \notin U$. Thus

$$Tu = TPu \notin U$$

and

$$PTu = P(Tu) \in U$$

therefore

$$TPu \neq PTu$$

which is a contradiction.

If T is not invariant under $U^\perp$, then we follow that there exists $w \in U^\perp$ such that $Tw \notin U^\perp$. Therefore

$$Tw = \sum a_j e_j + \sum b_j f_j$$

where some of a_j are not equal to 0. Thus we've got that

$$TP_U w = T0 = 0$$

and

$$P_U T w = \sum a_j e_j \neq 0 = TP_U w$$

which is a contradiction.

Last several exercises (with the exception of the 14) are great practice at maxima programming, that I'll do at better times (approximation of the sine is done though)

Chapter 7

Operators on Inner Product Spaces

7.1 Self-Adjoint and Normal Operators

From this chapter on we've got that every space that we're talking about is finite-dimentional, as denoted in the notation notes in the beginning of the chapter

7.1.1

Suppose n is a positive inveger. Define $T \in \mathcal{L}(F^n)$ by

$$T(z_1, ..., z_n) = (0, z_1, ..., z_{n-1})$$

Find a formula for $T^\star(z_1, ..., z_n)$

Let

$$v = (z_1, ..., z_n)$$
$$w = (z'_1, ..., z'_n)$$

Assuming that we've got Eucledean inner product we follow that

$$\langle v, T^\star w\rangle = \langle Tv, w\rangle = \langle (0, z_1, ..., z_{n-1}), (z'_1, z'_2, ..., z'_n)\rangle = \sum_{2 \leq j \leq n} z_{j-1} z'_j = \sum_{1 \leq j \leq n-1} z_j z'_{j+1} =$$

$$= \langle (z_1, ..., z_n), (z'_2, z'_3, ..., z_n, 0)\rangle$$

thus we follow that

$$T^\star(z_1, ..., z_n) = (z_2, ..., z_n, 0)$$

7.1.2

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of $T^\star$

Suppose that λ is an eigenvalue for T with corresponding eigenvector v and $\overline{\lambda}$ is not an eigenvalue for $T^\star$.

We follow that $T^\star - \lambda I$ is surjective. Therefore there exists $w \in W$ such that

$$v = (T^\star - \lambda I)w$$

since $v \neq 0$ we follow that

$$\langle v, v\rangle = \langle v, (T^\star - \lambda I)w\rangle \neq 0$$

but

$$\langle v, v\rangle = \langle v, (T^\star - \lambda I)w\rangle = \langle (T - \lambda I)v, w\rangle = \langle 0, w\rangle = 0$$

which is a contradiction.

We follow converse from the initial implication since $(T^\star)^\star = T$ and $\overline{\overline{\lambda}} = \lambda$.

7.1.3

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under $T^\star$.

We've got that $u \in U \to Tu \in U$. Suppose that $u \in U$ and $u' \in U^\perp$. Then we follow that

$$\langle Tu, u'\rangle = 0$$

$$\langle u, T^\star u'\rangle = 0$$

thus we follow that $u' \in U' \to T^\star u'$, as desired. Proof of is followed from this one, because we've got that $(T^\star)^\star = T$ and $(U^\perp)^\perp = U$

7.1.4

Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if $T^\star$ is surjective

Suppose that T is injective. Then we follow that

$$\operatorname{null} T = \{0\}$$

$$(\operatorname{null} T)^\perp = \operatorname{range} T^\star = (\{0\})^\perp = V$$

thus we follow that T is surjective. Since every implication in this proof is equivalence, we've got a proof of converse for free.

(b) T is surjective if and only if $T^\star$ is injective

Follows from part a and the fact that $(T^\star)^\star = T$.

7.1.5

Prove that

$$\dim \operatorname{null} T^\star = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^\star = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$

Since

$$\operatorname{null} T^\star = (\operatorname{range} T)^\perp$$

we follow that

$$\dim \operatorname{null} T^\star = \dim(\operatorname{range} T)^\perp = \dim V - (\dim \operatorname{range} T) = \dim V - \dim W + \dim \operatorname{null} T$$

where the second equality comes from the fact that $\dim U^\perp = \dim V - \dim U$ and the third equality comes from FTLM.

From the fact that

$$\operatorname{range} T = (\operatorname{null} T^\star)^\perp$$

and

$$\operatorname{null} T^\star = (\operatorname{range} T)^\perp$$

we follow that

$$\dim \operatorname{range} T = \dim(\operatorname{null} T^\star)^\perp = \dim W - \dim \operatorname{null} T^\star = \dim W - (\dim W - \dim \operatorname{range} T^\star) = \dim \operatorname{range} T^\star$$

as desired.

7.1.6

Make $P_2(R)$ *into an inner product space by defining*

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

Define $T \in \mathcal{L}(P_2(R))$ *by* $T(a_0 + a_1x + a_2x^2) = a_1x$

(a) Show that T *is not self-ajoint.*

Let

$$p(x) = x$$

$$q(x) = 1$$

then we follow that

$$\langle Tp, q \rangle = \int_0^1 Tp(x)q(x)dx = \int_0^1 xdx = 1$$

$$\langle p, Tq\rangle = \int_0^1 p(x)Tq(x)dx = \int_0^1 0dx = 0$$

thus

$$\langle Tp, q\rangle \neq \langle p, Tq\rangle$$

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is equal to its conjugate transpose, even though T is not self-ajoint. Explain why this is not a contradiction

Because basis $(1, x, x^2)$ is not orthonormal.

7.1.7

Suppose $S, T \in \mathcal{L}(V)$ are self-ajoint Prove that ST is self-adjoint if and only if $ST = TS$.

$$ST = (ST)^\star = T^\star S^\star = (T^\star)(S^\star) = TS$$

7.1.8

Suppose V is a real inner product space. Show that the set of self-adjoint operators in V is a subspace of $\mathcal{L}(V)$.

Suppose that T, S are self-adjoint. Then we follow that for every $v, w \in V$ and $\lambda \in R$

$$\langle (S+T)v, w\rangle = \langle v, (S+T)^\star w\rangle = \langle v, (S^\star + T^\star)w\rangle = \langle v, (S+T)w\rangle$$

$$\langle (\lambda T)v, w\rangle = \langle v, (\lambda T)^\star w\rangle = \langle v, \overline{\lambda} T^\star w\rangle = \langle v, \lambda Tw\rangle$$

thus we follow that S, T being self-adjoint implies that $S + T$ and λS are self-adjoint, thus the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

7.1.9

Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Stemming from the previous exercise we've got the additivity for this set, but in the case with complex numbers we've got that there exists $\lambda \in C$ such that

$$\lambda T \neq \overline{\lambda} T$$

As the example we can look at vector space C, map $T(x) = x$ and $\lambda = i$.

7.1.10

Suppose $\dim V \geq 2$. *Show that the set of normal operators on* V *is not a subspace of* $\mathcal{L}(V)$.

For the set of normal operators we've got that

$$(\lambda T)^\star = \overline{\lambda} T^\star$$

and therefore

$$||\lambda T v|| = |\lambda| ||Tv|| = |\overline{\lambda}| ||T^\star v|| = ||(\lambda T)^\star v||$$

therefore we've got closure under multiplication for all normal operators.

Thus we need to look at problems with the closure under addition. Firstly we going to look at the small cases, and after some time well spent in octave I found out that for

$$S = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

where S is just normal and T is self-adjoint we've got that

$$S + T = \begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}$$

which is not normal. Thus we've got a case for the real vector space with the dimention 2.

Then we can generalize this case for the finite-dimentional real spaces where we define S and T to be represented by the given matrix in upper-left corner and zeroes in the rest of the matrix. Then we can follow that $S + T$ is still not normal.

Given that this reasoning is still valid if we assume that $F = C$, we follow that the space of normal linear maps is not a subspace of $\mathcal{L}(V)$ for any finite-dimentional space V, with $\dim V \geq 2$, as desired.

7.1.11

Suppose $P \in \mathcal{L}(V)$ *is such that* $P^2 = P$. *Prove that there is a subspace* U *of* V *such that* $P = P_U$ *if and only if* P *is self-adjoint.*

From previous exercises we know that if $P^2 = P$, then we can follow that

$$V = \operatorname{null} P \oplus \operatorname{range} P$$

thus we can follow that $P = P_U$, where $U = \operatorname{range} P$.

Suppose that $e_1, ..., e_n$ is an orthonormal basis of U and $e_1, ..., e_n, f_1, ..., f_m$ is an orthonormal basis of V. Then we follow that

$$P_U(\sum a_j e_j + \sum b_i f_i) = \sum a_j e_j$$

Thus we follow that with resprect to this orthonormal basis the matrix is equal to its conjugate transpose and therefore it is self-ajoint.

If P is self-adjoint, we follow that $P = P_U$ because of the other implication. (as far as I get this, in this case we've got that $P^2 = P$ implies that $P = P_U$ for some U and that P is self adjoint, so it is technically a biconditional, but it is not a biconditional in the standart sense).

7.1.12

Suppose that T is a normal operator in V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $||v|| = \sqrt{2}$ and $||Tv|| = 5$

Given that T is a normal operator, we follow that the eigenvectors with respect to different eigenvalues are orthogonal. Suppose taht u' is an eigenvector for the eigenvalue 3 and w' is the eigenvector for the eigenvalue 4. Then set

$$u = u' * \frac{1}{||u'||}$$

$$w = w' * \frac{1}{||w'||}$$

because of this we get that

$$||u|| = ||w|| = 1$$

set

$$v = u + w$$

then we follow that since u and w are orthogonal, we can use Pythagorean theorem to follow that

$$||v|| = \sqrt{||u||^2 + ||w||^2} = \sqrt{1+1} = \sqrt{2}$$

and we get that

$$||Tv|| = ||T(u+w)|| = ||3u + 4w|| = \sqrt{|3|^2||u||^2 + |4|^2||w||^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

as desired.

7.1.13

Give an example of an operator $T \in \mathcal{L}(C^4)$ such that T is normal but not self-ajoint

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

will do (with respect to standart basis).

7.1.14

Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$||v|| = ||w|| = 2, Tv = 3v, Tw = 4w$$

Show that $||T(v+w)|| = 10$

From the equations we follow that v is an eigenvector for 3 and w is an eigenvector for 4. Thus we can follow that since T is normal that v and w are orthogonal. Thus Tv and Tw are orthonormal. Therefore

$$||T(v+w)|| = \sqrt{||Tv||^2 + ||Tw||^2} = \sqrt{4*9+4*16} = 10$$

as desired.

7.1.15

Fix $u, x \in V$. Define $T = \mathcal{L}(V)$ by

$$Tv = \langle v, u \rangle x$$

for every $v \in V$.

(a) Suppose $F = R$. Prove that T is self-adjoint if and only if u, x is linearly dependent.

If $x = 0$ or $u = 0$ we follow that $T = 0$, therefore x, u are linearly dependent for any other u and x and T is self-adjoint.

Now suppose that $x \neq 0$ and $u \neq 0$. As we've seen in the example in the chapter, for given T

$$\langle v, T^\star w \rangle = \langle Tv, w \rangle = \langle \langle v, u \rangle x, w \rangle = \langle v, u \rangle \langle x, w \rangle = \langle v, \langle w, x \rangle u \rangle$$

therefore

$$T^\star w = \langle w, x \rangle u$$

Since T is self-adjoint, we follow that for fixed $w, v \in V$

$$\langle w, x \rangle u = \langle v, u \rangle x$$

therefore u is a scalar multiple of x and therefore they are linearly dependent.

Conversely, if x and u are linearly dependent, then we follow that

$$Tv = \langle v, u \rangle x = \langle v, \lambda x \rangle x = \lambda \langle v, x \rangle x$$

therefore

$$\langle Tv, w \rangle = \langle \langle v, u \rangle x, w \rangle = \langle \langle v, \lambda x \rangle x, w \rangle = \langle v, \lambda x \rangle \langle x, w \rangle = \langle v, \lambda x \rangle \langle w, x \rangle =$$

$$= \langle v, \langle w, x \rangle \lambda x \rangle = \langle v, \langle w, \lambda x \rangle x \rangle = \langle v, \langle w, u \rangle x \rangle = \langle v, Tu \rangle$$

thus $T = T^\star$ and therefore it's self-adjoint.

(b) Prove that T is normal if and only if u, x is linearly dependent

From the previous argument we can derive that u is a scalar multiple of x.

Now suppose that u, x are linearly dependent and suppose that $w, v \in V$. Then we follow that

$$||Tv||^2 = \langle Tv, Tv\rangle = \langle\langle v, u\rangle x, \langle v, u\rangle x\rangle = |\langle v, u\rangle|^2\langle x, x\rangle = |\langle v, \lambda x\rangle|^2\langle x, x\rangle =$$

$$= |\overline{\lambda}|^2|\langle v, x\rangle|^2\langle x, x\rangle = |\lambda|^2|\langle v, x\rangle|^2\langle x, x\rangle =$$

$$||T^\star v||^2 = \langle T^\star v, T^\star v\rangle = \langle\langle v, x\rangle u, \langle v, x\rangle u\rangle = \langle\langle v, x\rangle\lambda x, \langle v, x\rangle\lambda x\rangle = |\langle v, x\rangle|^2|\lambda|^2\langle x, x\rangle$$

thus we follow that $||Tv|| = ||T^\star v||$, therefore T is normal, as desired.

7.1.16

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{range} T = \operatorname{range} T^*$$

Suppose that T is normal. Then we follow that

$$||Tv|| = ||T^*v||$$

thus we can follow that $\operatorname{null} T = \operatorname{null} T^*$. Therefore

$$\operatorname{null} T = \operatorname{null} T^*$$

$$(\operatorname{range} T^*)^\perp = (\operatorname{range} T)^\perp$$

$$\operatorname{range} T^* = \operatorname{range} T$$

as desired.

7.1.17

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \ \textit{and} \ \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k

We want to prove that if T is normal, then

$$V = \operatorname{null} T \oplus \operatorname{range} T$$

Suppose that $v \in \operatorname{null} T \cap \operatorname{range} T$. Then by the fact that

$$\operatorname{null} T = (\operatorname{range} T^*)^\perp = (\operatorname{range} T)^\perp$$

we follow that v is orthogonal to v. Thus

$$\langle v, v\rangle = 0$$

therefore $v = 0$. Therefore we follow that

$$V = \operatorname{null} T \oplus \operatorname{range} T$$

If $v \in \operatorname{null} T$, then we follow that $T^2 v = T(Tv) = T0 = 0$, thus $v \in \operatorname{null} T^2$ or in other words

$$\operatorname{null} T \subseteq \operatorname{null} T^2$$

Suppose that $v \in \operatorname{range} T$. Then we follow that there exists $w \in V$ such that $Tw = v$. Since $w \in V$ we follow that there exsit $u \in \operatorname{null} T$ and $r \in \operatorname{range} T$ such that

$$w = u + r$$

Since

$$Tw = v$$

we follow that

$$T(u + r) = T(u) + T(r) = 0 + T(r) = T(r) = w$$

thus we follow that there exists $r \in \operatorname{range} T$ such that $Tr = w$ and therefore

$$T^2 r = Tw = v$$

therefore $v \in \operatorname{range} T^2$. Thus we can follow that

$$\operatorname{range} T \subseteq \operatorname{range} T^2$$

Because $V = \operatorname{null} T \oplus \operatorname{range} T$ and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$ we can follow that

$$\operatorname{null} T = \operatorname{null} T^2$$

$$\operatorname{range} T = \operatorname{range} T^2$$

thus by induction we can follow that

$$\operatorname{null} T = \operatorname{null} T^k$$

$$\operatorname{range} T = \operatorname{range} T^k$$

as desired (we skip here the whole basis-building stuff, because it is kind of obvious and/or redundant, and maybe was proven in earlier chapters).

7.1.18

Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, ..., e_n$ of V such that $||Te_j|| = ||T^*e_j||$ for each j, then T is normal.

I want to say that this is not the case: define $T \in \mathcal{L}(C^2)$ by

$$T = \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix}$$

which is not normal, but the constraints hold for standart basis. Also the same thing happens with reals too

$$S = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

by putting those matrices in upper corners of matrices with respect to standard basis for the maps in higher dimentions we get that given proposition doesn't hold for any finite-dimentional F^n.

7.1.19

Suppose $T \in \mathcal{L}(C^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$

Suppose that $(z_1, z_2, z_3) \in \operatorname{null} T$. Because T is normal, we follow that if $u \in \operatorname{range} T$, then

$$\langle (z_1, z_2, z_3), u \rangle = 0$$

Since $(1,1,1) = T(1/2, 1/2, 1/2) \in \operatorname{range} T$, we follow that

$$\langle (z_1, z_2, z_3), (1,1,1) \rangle = z_1 \overline{1} + z_2 \overline{1} + z_3 \overline{1} = z_1 + z_2 + z_3 = 0$$

as desired.

Last two exercises are left for the end of this chapter

7.2 The Spectral Theorem

7.2.1

True of false: There exsits $T \in \mathcal{L}(R^3)$ such that T is not self-adjoint and such that there is a basis of R^3 consisting of eigenvectors of T.

I think that this one is true, the only thing that we need is to get such a map.

From the examples in the 5th chapter we can follow that there exists a map in R^3

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

with respect to the standart basis. Since it is not equal to its conjugate transpose (which in this case is just a normal transpose) in an orthonormal basis (such as the standart one) we follow that this map is not self-adjoint. But for it we've got that

$$(1,0,0),(1,3,0),(1,6,6)$$

are the eigenvalues of T. It's easy to see that the given list is linearly independent (by the position of zeroes). Given that the length of the list is 3, we follow that it is a basis of R^3. Thus we've got a non-self-adjoint map whose eigenvectors are the basis of R^3, as desired.

7.2.2

Suppose that T is a self-adjoint operator on a finite-dimentional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$

Since T is self-adjoint, we follow that it has an orthonormal basis of the eigenvectors of T, with respect to which there is a diagonal matrix of T. Suppose that a_i is an entry on the diagonal of such a matrix. Then we follow that the same entry on the matrix of the map

$$T^2 - 5T + 6I$$

is

$$a_i^2 - 5a_i + 6$$

given that 2 and 3 are the only eigenvalues of T, we follow that a_i is equal to either 2 or 3, for both of which

$$a_i^2 - 5a_i + 6 = 0$$

thus we follow that every entry in the matrix of $T^2-5T+6I$ is zero, thus $T^2-5T+6I = 0$, as desired.

7.2.3

Give an example of an operator $T \in \mathcal{L}(C^3)$ such thaht 2 and 3 are tehe only eigenvalues of T and $T^2 - 5T + 6I \neq 0$

The idea is to fill the entries of the matrix with respect to the standart basis above the main diagonal with some junk, and put $2, 3$ on the diagonal.

$$T = \begin{pmatrix} 2 & 4 & 9 \\ 0 & 3 & 7 \\ 0 & 0 & 2 \end{pmatrix}$$

will do.

Octave shows that

$$T^2 - 5T + 6I = \begin{pmatrix} 0 & 0 & 19 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as desired.

7.2.4

Suppose $F = C$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

where $\lambda_1, ..., \lambda_m$ denote the distinct eigenvalues of T.

In forward direction: It's pretty straightforward. If T is normal, we follow that it has a diagonalizable matrix with respect to the basis of orthonormal eigenvectors. Thus we follow that for different values the eigenvectors are orthogonal and

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

by the fact that the matrix is diagonal, as desired.

In reverse direction: Suppose that

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

Because of this we can follow that the T is diagonalizable. We can also follow that each $E(\lambda_j, T)$ is invariant under T. By getting the orthonormal basis of each of the $E(\lambda_j, T)$ we get a list of vector of length $\dim V$. By the fact that each pair of eigenvectors corresponding to different eigenvalues are orthogonal, we follow that produced list is an orthonormal list of length $\dim V$ in V, thus making it an orthonormal basis of V. Thus we can follow that T is normal, as desired.

7.2.5

Suppose $F = R$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to dictinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

where $\lambda_1, ..., \lambda_m$ denote the distinct eigenvalues of T.

The argument is identical to the one, that is presented in the previous exercise (with a minor adjustment of notation in the forward direction), since none of the implications depend on F.

7.2.6

Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real

Forward implication was taken care of in the previous chapter. With the information, that was presented in this chapter we can follow that if T is normal, and all of its eigenvaleus are real we can follow that the diagonal matrix of this operator with respect to the orthonormal basis of V contains only real numbers, therefore it is equal to its complex conjugate and thus we can follow that T is self-adjoint.

7.2.7

Suppose V s a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

I want to prove that Given that T is a normal operator, we can follow that it has a diagonal matrix with respect to the orthonormal basis of eigenvectors of T. Thus we can follow that if λ is an eigenvalue of T with corresponding eigenvector v, then

$$T^9 v = T^8 v$$

$$\lambda^9 v = \lambda^8 v$$

thus

$$\lambda^9 = \lambda^8$$

thus we can follow that $\lambda = 0$ or $\lambda = 1$ (I'm pretty sure that is the case for the complex numbers as well), in which case the eigenvalues of T are both real numbers, and therefore T is self-adjoint.

Given that the only eigenvalues of T are 0 and 1 we can follow that

$$T^n = T^m$$

for any $n, m \in N$. Thus $T^2 = T$, as desired.

7.2.8

Give an example of an operator T on complex vector space such that $T^9 = T^8$, but $T^2 \neq T$

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

will do.

7.2.9

Suppose V is a complex inner product space. Prove that every normal operator on V has a square root (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$)

Given that T is complex, we follow that it is diagonalizable with respect to some basis $e_1, ..., e_n$. In other words,

$$T(\sum a_j e_j) = \sum d_j a_j e_j$$

where d_i is the i'th entry on the diagonal. Define S as

$$S(\sum a_j e_j) = \sum \sqrt{d_j} a_j e_j$$

where $\sqrt{d_j}$ is well-defined since $F = C$. Then we follow that

$$S^2(\sum a_j e_j) = S(\sum \sqrt{d_j} a_j e_j) = \sum d_j a_j e_j = T \sum a_j e_j$$

thus $S^2 = T$, and therefore S is a square root of T

7.2.10

Give an example of a real inner product space V and $T \in \mathcal{L}(V)$ and real numbers b, c with $b^2 < 4c$ such that $T^2 + bT + cI$ is not invertible

We can set $V = R^2$,

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in which case

$$T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus we can set $b = 0$ and $c = 1$, and

$$T^2 + bT + cI = 0$$

which will do as an example.

7.2.11

Prove or give a counterexample: every self-adjoint operator on V has a cube root.

I want to say that this is true. Suppose that T is self adjoint. Then we can follow that with respect to some basis $e_1, ..., e_n$, T is represented as

$$T(\sum a_j e_j) = \sum d_j a_j e_j$$

where $d_1, ..., d_n$ are reals. Thus we can define a map that satisfies condition for being the cube root

$$S(\sum a_j e_j) = \sum \sqrt[3]{d_j} a_j e_j$$

which is well-defined, since every real number has a cube root

7.2.12

Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in F$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $||v|| = 1$ and

$$||Tv - \lambda v|| < \epsilon$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$

Because T is self-adjoint, we follow that there exists an orthonormal basis of V, which consists of eigenvectors of T. Let us denote this basis by $e_1, ..., e_n$.

Then we follow that

$$v = \sum a_j e_j$$

therefore

$$Tv = \sum a_j d_i e_j$$

where each d_i is an eigenvalue of T (d_i's don't have to be distinct) and

$$\sum a_j^2 = 1$$

thus

$$||Tv - \lambda v||^2 = ||\sum a_j d_j e_j - \lambda \sum a_j e_j||^2 = ||\sum a_j (d_j - \lambda) e_j||^2 = \sum (a_j (d_j - \lambda))^2 < \epsilon^2$$

If every eigenvalue of T is under condition that

$$|\lambda - \lambda'| \geq \epsilon$$

then we follow that there exists $\kappa \in R^+$ such that

$$\kappa = \min\{|\lambda_1 - \lambda|, ...|\lambda_n - \lambda|\}$$

thus we follow that

$$\sum (a_j (d_j - \lambda))^2 = \sum a_j^2 |d_j - \lambda|^2 \leq \sum a_j^2 \kappa^2 = \kappa^2 \sum a_j^2 = \kappa^2 ||v|| = \kappa^2 < \epsilon^2$$

which contradicts the assumption that

$$|\lambda - \lambda'| \geq \kappa \geq \epsilon$$

thus we follow that there exists an eigenvalue λ' of T such that

$$|\lambda - \lambda'| < \epsilon$$

as desired.

7.2.13

Give an alternative proof of the Complex Spectral Theorem that avoids Schur's Theorem and instead follows the pattern of the proof of the Real Spectral Theorem

The proof of equvalence of (b) and (c) does not depend on the Schur's theorem and is instead based on the common sense, therefore we won't prove it. TODO

7.2.14

TODO

7.2.15

Find the matrix entry below that is covered up

It's 1.

7.3 Positive Operators and Isometries

7.3.1

Prove or give a counterexample: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis $e_1, ..., e_n$ of V such that $\langle Te_j, e_j \rangle \geq 0$ for each j, then T is a positive operator.

This one looks like it's true, but it isn't.

Suppose that $T \in \mathcal{L}(R^2)$ is defined by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the standard basis. Let

$$e_1 = \frac{\sqrt{2}}{2}(1, 1)$$

$$e_2 = \frac{\sqrt{2}}{2}(1, -1)$$

then we follow that for Eucledian inner product

$$||e_1|| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 = ||e_2||$$

and

$$\langle e_1, e_2 \rangle = 0$$

thus e_1, e_2 are orthonormal vectors. After some computing we can follow that

$$Te_1 = Te_2 = 0$$

thus $\langle Te_1, e_1 \rangle = \langle Te_2, e_2 \rangle \geq 0$ and we can follow that for vector $v = (0, 1)$

$$\langle Tv, v \rangle = (0, -1) \cdot (0, 1) = -1$$

thus we have the desired counterexample.

7.3.2

Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w$$

and

$$Tw = v$$

Prove that $v = w$

We can follow from the Characterization of positive operators, that if v is an eigenvector of positive operator T, then v is also an eigenvector for its square root.

$$T^2 v = T(Tv) = Tw = v$$

thus we follow that 1 is an eigenvalue for T^2 with an eigenvector v. Since every positive operator has a unique square root, we follow that 1 is an eigenvalue of T. with corresponding eigenvector v. Thus

$$Tv = v = w$$

as desired.

7.3.3

Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U.

For every $u, v \in U$ we've got that

$$\langle T|_U(u), v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, T|_U v \rangle$$

thus T is self-adjoint.

Suppose that $u \in U$. Then we follow that

$$\langle T|_U(u), u \rangle = \langle T(u), u \rangle \geq 0$$

thus $T|_U$ is positive, as desired.

7.3.4

Suppose $T \in \mathcal{L}(V, W)$. Prove that T^*T is a positive operator on V and TT^* is a positive operator on W.

Suppose that $v, u \in V$. Then we follow that

$$\langle T^*Tv, u\rangle = \langle Tv, Tu\rangle = \langle v, T^*Tu\rangle$$

thus T^*T is self-adjoint and

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \geq 0$$

thus T^*T is positive. Similar argument holds for TT^* and W.

7.3.5

Prove that the sum of two positive operators on V is positive.

Suppose that S and T are positive. Then we follow that

$$(S+T)^* = S^* + T^* = S + T$$

thus $S+T$ is self-adjoint. Suppose that $v \in V$. Then

$$\langle (S+T)v, v\rangle = \langle Sv + Tv, v\rangle = \langle Sv, v\rangle + \langle Tv, v\rangle \geq 0$$

thus we follow that $S+T$ is positive, as desired.

7.3.6

Suppose $T \in \mathcal{L}(V)$ is positive. Prove that T^k is positive for every positive integer k.

Because T is positive, we follow that there exists a basis $e_1, ..., e_n$ of V, that consists of eigenvectors of T. Thus we can follow by induction that for every $k \in N$

$$(T^k)* = (TT^{k-1})^* = (T^{k-1})^*T^* = T^{k-1}T = T^k$$

thus T^k is self-adjoint. And

$$T^k e_j = \lambda_j^k e_j$$

thus e_j is an eigenvector of T^k with corresponding eigenvalue λ_j^k. Since every eigenvalue of T^k is a power of a nonnegative real number, we follow that every eigenvalue of T^k is nonnegative as well. Thus we can follow that T^k is a positive operator, as desired.

7.3.7

Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for every $v \in V$ with $v \neq 0$

In forward direction: Suppose that T is invertible. We follow by exercise 3.4.9 that T is invertible if and only if its square root is invertible. Thus we follow that square root of T is invertible. Thus for $v \neq 0$ we've got that

$$Rv \neq 0$$

and therefore

$$\langle Tv, v\rangle = \langle RRv, v\rangle = \langle Rv, Rv\rangle > 0$$

as desired.

In reverse direction: Suppose that for every $v \neq 0$ we've got that

$$\langle Tv, v\rangle > 0$$

and suppose that T is not invertible. Then we follow that T is not injective and thus there exists $v \neq 0$ such that

$$Tv = 0$$

Thus

$$\langle Tv, v\rangle = \langle 0, v\rangle = 0$$

which is a contradiction.

7.3.8

Suppose $T \in map(V)$. For $u, v \in V$ define $\langle u, v\rangle_T$ by

$$\langle u, v\rangle_T = \langle Tu, v\rangle$$

Prove that $\langle \cdot, \cdot\rangle_T$ is an inner product of V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot\rangle$)

If $\langle \cdot, \cdot\rangle_T$ is an inner product, then we follow that for every $v \in V \neq 0$

$$\langle v, v\rangle_T = \langle Tv, v\rangle > 0$$

by definiteness property, thus by previous exercise we follow the forward direction.

We can follow that T is invertible if and only if its square root is invertible. Thus by exercise 6.1.24 and the fact that

$$\langle Tv, u\rangle = \langle RRv, u\rangle = \langle Rv, Ru\rangle$$

we follow the reverse direction.

7.3.9

Prove or disprove: the identity operator on F^2 has infinitely many self-adjoint square roots.

From the top of my head I can come up with only 4 roots for reals.

We can think of it as the rotation matrix, or something of sorts. After some time well-spent in Octave I came up with

$$\begin{pmatrix} \sin k & \cos k \\ \cos k & -\sin k \end{pmatrix}$$

which seems to work for arbitrary $k \in R$. Thus we can follow that there are indeed infinite amount of self-adjoint square roots for F^2 (this case works with both C and R.)

7.3.10

Suppose $S \in \mathcal{L}(V)$. PRove that the following are equivalent:

S is an isometry if and only if S^* is an isometry. The rest follows from the standart isometry equivalence.

7.3.11

Suppose T_1, T_2 are normal operators on $\mathcal{L}(F^3)$ and both operators have 2, 5, 7 as eigenvalues. Prove that there exists an isometry $S \in \mathcal{L}(F^3)$ such that $T_1 = S * T_2 S$

Since both T_1 and T_2 have $\dim V$ distinct eigenvalues, we follow that both maps are diagonalizable. Thus we can follow that there exists a basis of V, that consists of eigenvectors of T_1 – v_1, v_2, v_3, and the same thing exsits for T_2 with vectors u_1, u_2, u_3. Since the eigenvectors, corresponding to distinct eigenvalues in normal maps are orthogonal, we follow that both lists are orthogonal as well. Thus we can follow that we can normalize the vectors in the list and get orthonormal bases, that consist of eigenvectors of their respective maps. Let us denote those bases by e_1, e_2, e_3 and e'_1, e'_2, e'_3. Define $S \in \mathcal{L}(V)$ by

$$S(e_j) = e'_j$$

thus we can follow that for $v \in V$, which is defined by

$$v = \sum a_j e_j$$

we've got that

$$||Sv||^2 = ||S(\sum a_j e_j)||^2 = \sum ||a_j||^2 = ||v||^2$$

thus we follow that S is an isometry. Since S is an isometry we follow that $S^{-1} = S^*$. Now suppose that

$$v = \sum a_j e_j$$

for which we've got that

$$T_1 v = \sum a_j \lambda_j e_j$$

and

$$S^* T_2 S v = S^* T_2 S(\sum a_j e_j) = S^* T_2(\sum a_j e_j') = S^*(a_j \lambda_j e_j') = \sum a_j \lambda_j e_j = T_1 v$$

as desired.

7.3.12

Give an example of two self-adjoint operators $T_1, T_2 \in \mathcal{L}(F^4)$ such that the eigenvalues of both operators are 2, 5, 7, but there does not exist an isometry $S \in F^4$ such that $T_1 = S^* T_2 S$. Be sure to explain why there is no isometry with the required property

We can have

$$T_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Then we can follow that the $\dim E(2, T_2) = 2$, and $\dim E(2, T_1) = 1$. We can prove that multiplying a map by an isometry doesn't change the dimentions of eigenspaces, which needs to happen in this case. TODO: expand this idea a bit

7.3.13

Prove of give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, ..., e_n$ of V such that $||Se_j|| = 1$ for each e_j, then S is an isometry.

Let $V = R^2$ and define

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Then we follow that

$$Se_1 = e_1$$
$$Se_2 = e_1$$

therefore our condition holds, but

$$S(e_1 - e_2) = 0$$
$$||e_1 - e_2|| = \sqrt{2}/2$$

which is a contradiction.

7.3.14

Let T be the second derivative operator in Exercise 21 in Section 7A. Show that $-T$ is a positive operator.

TODO; left till the end of the chapter

7.4 Polar Decomposition and Singular Value Decomposition

7.4.1

Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u\rangle x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{||x||}{||u||}\langle v, u\rangle u$$

for every $v \in V$.

For

$$Tv = \langle v, u\rangle x$$

we've got that

$$T^*v = \langle v, x\rangle u$$

thus

$$T^*Tv = \langle\langle v, u\rangle x, x\rangle u$$

$$T^*Tv = \langle v, u\rangle\langle x, x\rangle u$$

$$T^*Tv = ||x||^2\langle v, u\rangle u$$

For our supposed answer we've got that

$$\sqrt{T^*T}v = \frac{||x||}{||u||}\langle v, u\rangle u$$

thus

$$\sqrt{T^*T}\sqrt{T^*T}v = \frac{||x||}{||u||}\langle\frac{||x||}{||u||}\langle v, u\rangle u, u\rangle u$$

$$\sqrt{T^*T}\sqrt{T^*T}v = \frac{||x||}{||u||}\frac{||x||}{||u||}\langle v, u\rangle\langle u, u\rangle u$$

$$\sqrt{T^*T}\sqrt{T^*T}v = \frac{||x||^2}{||u||^2}\langle v, u\rangle||u||^2 u$$

$$\sqrt{T^*T}\sqrt{T^*T}v = ||x||^2\langle v, u\rangle u$$

$$T^*Tv = ||x||^2\langle v, u\rangle u$$

thus given result is indeed a square root of T^*T.

Now suppose that $v \in V$. Then we follow that

$$\langle\sqrt{T^*T}v, v\rangle = \langle\frac{||x||}{||u||}\langle v, u\rangle u, v\rangle = \frac{||x||}{||u||}\langle\langle v, u\rangle u, v\rangle = \frac{||x||}{||u||}\langle v, u\rangle\langle u, v\rangle =$$

$$= \frac{||x||}{||u||}\overline{\langle u, v\rangle}\langle u, v\rangle = \frac{||x||}{||u||}|\langle u, v\rangle|^2 \geq 0$$

Suppose that $v, w \in V$. Then we follow that

$$\langle\sqrt{T^*T}v, w\rangle = \langle\frac{||x||}{||u||}\langle v, u\rangle u, w\rangle = \frac{||x||}{||u||}\langle v, u\rangle\langle u, w\rangle = \overline{\frac{||x||}{||u||}}\langle v, u\rangle\overline{\langle w, u\rangle} = \langle v, \frac{||x||}{||u||}\langle w, u\rangle u\rangle =$$

$$= \langle v, \sqrt{T^*T}w\rangle$$

thus we follow that given answer for $\sqrt{T^*T}$ is positive, as desired.

7.4.2

Give an example of $T \in \mathcal{L}(C^2)$ such that 0 is the only eigenvalue of T and the singular values of T are 5 and 0

Suppose that

$$T = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$$

with respect to the standard basis. Then we follow that it's upper diagonal and therefore 0 is the only eigenvalue of T. We also have that

$$T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}$$

which is derived from the multiplication of conjugate transpose of the original matrix by the matrix itself. We can follow from it that

$$\sqrt{T^*T} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

is the square root of the resulting product (verification that given resuls is positive is trivial). Thus 0 and 5 are the singular values of this matrix.

7.4.3

Suppose $T \in \mathcal{L}(V)$. Prove that there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = \sqrt{TT^*}S$$

By applying PD to T^* we get that there exists S such that

$$T^* = S\sqrt{(T^*)^*T^*} = S\sqrt{TT^*}$$

thus

$$(T^*)^* = (S\sqrt{TT^*})^*$$
$$T = (\sqrt{TT^*})^*S^*$$

and since the square root is positive and therefore self-adjoint, we follow that

$$T = \sqrt{TT^*}S^*$$

Since the adjoint of the isometry is an isometry, we get the desired result.

7.4.4

Suppose $T \in \mathcal{L}(V)$ and s is a singular value of T. Prove that there exists a vector $v \in V$ such that $||v|| = 1$ and $||Tv|| = s$

Because s is a singular value of V, we follow that it is an eigenvalue of $\sqrt{T^*T}$ and $s \geq 0$. Thus there exists an eigenvector v', that corresponds to this eigenvalue. We can normalize this vector and get

$$v = \frac{1}{||v'||}v'$$

such that

$$||v|| = 1$$

By PD, T can be represented as

$$T = S\sqrt{T^*T}$$

and thus

$$||Tv|| = ||S\sqrt{T^*T}v|| = ||Ssv|| = ||sv|| = |s| = s$$

as desired.

7.4.5

Suppose $T \in \mathcal{L}(C^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of T.

We follow that with respect to the standard basis

$$T = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

thus

$$T^* = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

$$T^*T = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}$$

thus $4, 1$ are the singular values of T.

7.4.6

Find the singular values of the differentiation operator $D \in P(R^2)$ defined by $Dp = p'$, where the inner product on $P(R^2)$ is as in Example 6.33

From this example we've got that

$$e_1 = \sqrt{\frac{1}{2}}$$

$$e_2 = \sqrt{\frac{3}{2}}x$$

$$e_3 = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$$

for which we've got

$$e_1' = 0$$

$$e_2' = \sqrt{\frac{3}{2}} = \sqrt{3}e_1$$

$$e_3' = \frac{3\sqrt{5}}{\sqrt{2}}x = \sqrt{15}e2$$

thus we follow that with respect to this orthonormal basis, matrix of D is

$$D = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$$

therefore

$$D^* = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{pmatrix}$$

thus

$$D^*D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

which means that $\sqrt{3}, \sqrt{15}, 0$ are singular values of D, as desired.

7.4.7

Define $T \in \mathcal{L}(F^3)$ by

$$T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2)$$

Find (explicitly) an isometry $S \in \mathcal{L}(F^3)$ such that $T = S\sqrt{T^*T}$.

We follow that with respect to standard basis

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

and since the standard basis is orthonormal, we follow that

$$T^* = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

thus

$$T^*T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

thus

$$\sqrt{T^*T} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which seems to be working.

7.4.8

Suppose $T \in \mathcal{L}(V)$, $S \in \mathcal{L}(V)$ is an isometry, and $R \in \mathcal{L}(V)$ is a positive operator such that $T = SR$. Prove that $R = \sqrt{T^*T}$.

Suppose that we've got a positive operator such that

$$T = SR$$

by taking adjoint of both sides we get that

$$T^* = (SR)^* = R^*S^*$$

and since R is a positive operator, we follow that it is self-adjoint and thus

$$T^* = R^*S^* = RS^*$$

therefore

$$T^*T = RS^*SR$$

since S is an isometry, we follow that $S^*S = I$. Thus

$$T^*T = RS^*SR = RIR = RR = R^2$$

Since T^*T is a positive operator for any T (which was proven somewhere in the previous section), we follow that it has a unique positive square root, and since R is positive, we follow that $R = \sqrt{T^*T}$, as desired.

7.4.9

Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if there exists a unique isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

In forward direction:

Suppose that T is invertible. Then we follow that $\operatorname{range} T = V$. By PD we've got that $T = S\sqrt{T^*T}$, and therefore $\operatorname{range}\sqrt{T^*T} = V$ as well. Thus we've got that for a basis $e_1, ..., e_n$, the list $\sqrt{T^*T}e_1, ..., \sqrt{T^*T}e_n$ is also a basis of V. Thus there exists a unique map S such that

$$S(\sqrt{T^*T}e_j) = Te_j$$

Thus we can follow the uniqueness of the isometry derived from PD by the fact that this map is unique in general.

In reverse direction:

Suppose that there exists a unique map S such that $T = S\sqrt{T^*T}$ and T is not invertible. Then we follow that T is not injective, and $\dim \operatorname{null} T \neq 0$. Since S is an isometry (and therefore injective) we follow that

$$S^{-1}T = \sqrt{T^*T}$$

thus $\sqrt{T^*T}$ is not injective as well and moreover we can follow that $\operatorname{null} T = \operatorname{null} \sqrt{T^*T}$. Let $e_1, ..., e_j$ be an orthonormal basis of $\operatorname{null} T$ and expand it to an orthonormal basis of V – $e_1, ..., e_n$. Since S is an isometry, we follow that $Se_1, ..., Se_n$ is also an orthonormal basis. Now define a map $S' \in \mathcal{L}(V)$ by

$$S'e_i = -Se_i$$

for $1 \leq i \leq j$ (i.e. all the vectors that form nullspace of T) and

$$S'e_i = Se_i$$

for the rest. From this definition we follow the simple fact that $S \neq S'$. We also follow that for any $v \in V$ such that

$$v = \sum a_j e_j$$

we've got that

$$||S'v||^2 = ||\sum \pm a_j e_j|| = \sum |a_j|^2 = ||v||^2$$

thus S' is an isometry. Now we can follow that for every $v \in V$ such that

$$v = \sum a_j e_j$$

we've got that

$$S\sqrt{T^*T}v = S\sqrt{T^*T}(\sum a_i e_i) = S(\sum a_i \sqrt{T^*T}e_i) = S(\sum_{i \geq j} a_i \sqrt{T^*T}e_i) =$$

$$= S'(\sum_{i \geq j} a_i \sqrt{T^*T}e_i) = S'(\sum a_i \sqrt{T^*T}e_i) = S'\sqrt{T^*T}(\sum a_i e_i) = S'\sqrt{T^*T}v$$

thus $S'\sqrt{T^*T} = S\sqrt{T^*T}$, which is a contradiction of our assumption that S is a unique isometry.

7.4.10

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of eigenvalues of T, repeated appropriately.

Because T is self-adjoint (and therefore normal), we follow by (both) spectral theorems that there exists an orthonormal basis of V with respect to which T has a diagonal map. Then we can follow that if T is in the form $Te_j = \lambda_j e_j$ thus T^*T is in the form

$$T^*Te_j = |\lambda_j|^2 e_j$$

and therefore

$$\sqrt{T^*T}e_j = \sqrt{|\lambda_j|^2}e_j = |\lambda_j|e_j$$

thus we follow the desired result.

7.4.11

Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.

Suppose that e_1 is an eigenvector of $\sqrt{T^*T}$, then we follow that

$$\sqrt{T^*T}e_1 = s_1 e_1$$

Also from PD and one of the previous exercises we've got that

$$T = S\sqrt{T^*T}$$

thus

$$TT^* = S\sqrt{T^*T}\sqrt{T^*T}S^* = ST^*TS^*$$

thus suppose that e_j is an eigenvector of TT^* with an eigenvalue s_j. Then we follow that

$$TT^*e_j = s_j e_j$$

and thus

$$ST^*TS^* = s_j e_j$$
$$T^*TS^* = S^* s_j e_j$$
$$T^*TS^* = s_j S^* e_j$$

thus S^*e_j is an eigenvector of T^*T with an eigenvalue of s_j. Thus we follow that T^*T has the same eigenvalues (with the same multiplicities) as TT^*. Thus we follow that same goes for their square roots. Thus we follow T^* and T have the same singular values, as desired.

7.4.12

Prove of give a counterexample: if $T \in \mathcal{L}(V)$, then the singular values of T^2 equal the squares of the singular values of T.

After some time in the octave I found out that this is not the case with random matrices, therefore we've got to find a good example to show off. Suppose that $T \in \mathcal{L}(V)$ is represented by

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then it can be easily shown that

$$\sqrt{T^*T} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $S = I$. Thus we follow that the singular values of T are 0 and 1. Square of this operator is zero, therefore its singular values are also zeroes. Thus we've got the desired counterexample.

7.4.13

Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T

We know that every matrix can be represented by

$$T = S\sqrt{T^*T}$$

thus T is invertible if and only if both S and $\sqrt{T^*T}$ are invertible. Since S is an isometry (which is always invertible) we follow that T is invertible if and only if $\sqrt{T^*T}$ is invertible. Then we follow that $\sqrt{T^*T}$ is invertible if and only if 0 is not one of its eigenvalues. Since every eigenvalue of $\sqrt{T^*T}$ is a singular value of T we follow that T is invertible if and only if 0 is not a singular value of T, as desired.

7.4.14

Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.

Because T is a linear map, we follow that there exists a decomposition

$$T = S\sqrt{T^*T}$$

Because $\sqrt{T^*T}$ is a positive square root, we follow that it is self-adjoint, and therefore there exists an orthonormal basis of V such that $\sqrt{T^*T}$ is diagonalizable. Let us denote this basis by $e_1, ..., e_n$. Then we follow that for every e_j, if corresponding singular value s_j is not zero, then $Te_j = Ss_je_j$ is in the range of T. Since S is an isometry, we follow that list Ss_je_j, Ss_ke_k is an orthogonal list, therefore we can follow that $\dim \operatorname{range} T$ is equal to the number of nonzero singular values of T.

7.4.15

Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1

S is an isometry if and only if $SS^* = I$. Since the positive square root of the operator is unique and $\sqrt{SS^*} = I$, we follow that S is an isometry if and only if $\sqrt{SS^*} = I$. If T is self-adjoint then we can follow that $T = I$ if and only if all of its eigenvalues are 1, therefore we follow that S is an isometry if and only if all of its singular values are equal to 1, as desired.

7.4.16

Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values if and only if there exist isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1T_2S_2$

If T_1 and T_2 have the same singular values, then we follow that with respect to (perhaps different) ortonormal bases, there exist diagonal matrices of $\sqrt{T_1^*T_1}$ and $\sqrt{T_2^*T_2}$ such that they have the same numbers on the diagonal. Thus we can follow that there exists two isometries S_1' and S_2' (change of basis operators) such that

$$\sqrt{T_1^*T_1} = S_1'\sqrt{T_2^*T_2}S_2'$$

By PD we've got that there exsits S such that

$$T_1 = S\sqrt{T_1^*T_1}$$

thus

$$T_1 = SS_1'\sqrt{T_2^*T_2}S_2'$$
$$(SS_1')^{-1}T_1 = \sqrt{T_2^*T_2}S_2'$$

and since for some isometry U we've got

$$T_2 = U\sqrt{T_2^*T_2}$$

we follow that

$$U(SS_1')^{-1}T_1 = U\sqrt{T_2^*T_2}S_2'$$
$$U(SS_1')^{-1}T_1 = T_2S_2'$$
$$T_1 = SS_1'U^{-1}T_2S_2'$$

thus by setting

$$S_1 = SS_1'U^{-1}$$
$$S_2 = S_2'$$

we get the desired result.

7.4.17

Suppose $T \in \mathcal{L}(V)$ has SVD given by

$$Tv = s_1\langle v, e_1\rangle f_1 + ... + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, ..., s_n$ are the singular values of T and $e_1, ..., e_n$ and $f_1, ..., f_n$ are orthonormal bases of V.

(a) Prove that if $v \in V$, then

$$T^*v = \sum s_j\langle v, f_j\rangle e_j$$

From the proof of SVD we follow that $S \in \mathcal{L}(V)$ is an isometry such that

$$Se_j = f_j$$

For any $w \in V$ we follow that

$$Tw = \sum s_j \langle w, e_j \rangle f_j$$

by decomposing T and looking in the proof of SVD we follow that

$$S\sqrt{T^*T}w = \sum s_j \langle w, e_j \rangle S e_j$$

$$\sqrt{T^*T}w = \sum s_j \langle w, e_j \rangle e_j$$

thus for a given $v \in V$

$$\sqrt{T^*T}(S^*v) = \sum s_j \langle S^*v, e_j \rangle e_j = \sum s_j \langle v, S e_j \rangle e_j = \sum s_j \langle v, f_j \rangle e_j$$

By PD we follow that

$$T = S\sqrt{T^*T}$$

and thus

$$T^* = \sqrt{T^*T}S^*$$

thus

$$T^* = \sum s_j \langle v, f_j \rangle e_j$$

as desired.

(b) Prove that if $v \in V$, then

$$T^*Tv = \sum s_j^2 \langle v, e_j \rangle e_j$$

We need to remember that since

$$S e_j = f_j$$

then

$$e_j = S^* f_j$$

and that e_j is an eigenvector of $\sqrt{T^*T}$ such that

$$\sqrt{T^*T}e_j = s_j e_j$$

also keep in mind that

$$T^* = \sqrt{T^*T}S^*$$

Now we can follow that for

$$Tv = \sum s_j \langle v, e_j \rangle f_j$$

we can apply T^* to both sides to derive that

$$T^*Tv = \sum s_j \langle v, e_j \rangle T^* f_j = \sum s_j \langle v, e_j \rangle \sqrt{T^*T} S^* f_j = \sum s_j \langle v, e_j \rangle \sqrt{T^*T} e_j = \sum s_j \langle v, e_j \rangle s_j e_j =$$

$$= \sum s_j^2 \langle v, e_j \rangle e_j$$

as desired.

(c) Prove that if $v \in V$, then

$$\sqrt{T^*T} v = \sum s_j \langle v, e_j \rangle e_j$$

If there's a trick, then I don't see it. For the proof of this equality go to part (a) or straight to the proof of SVD itself.

(d) Suppose T is invertible. Prove that if $v \in V$, then

$$T^{-1} = \sum \frac{\langle v, f_1 \rangle e_1}{s_1}$$

This one's a bit tricky.

Suppose that T is invertible. By PD we've got that

$$T = S\sqrt{T^*T}$$

thus

$$T^{-1} = (S\sqrt{T^*T})^{-1} = \sqrt{T^*T}^{-1} S^{-1} = \sqrt{T^*T}^{-1} S^*$$

Let's look at the $\sqrt{T^*T}^{-1}$. If we define U by

$$U(e_j) = \frac{1}{s_j} e_j$$

then we follow that for $v \in V$ represented by

$$v = \sum a_j e_j$$

we've got that

$$\sqrt{T^*T} U v = \sum a_j \sqrt{T^*T} U e_j = \sum a_j \sqrt{T^*T} (\frac{1}{s_j} e_j) = \sum a_j s_j (\frac{1}{s_j} e_j) = \sum a_j e_j = v$$

thus $\sqrt{T^*T} U = I$ and therefore $U = \sqrt{T^*T}^{-1}$.

For arbitrary w and given orthonormal basis $e_1, ..., e_n$ we follow that

$$w = \sum \langle v, e_j \rangle e_j$$

thus S^*v with respect to the same basis is represented by

$$S^*v = \sum \langle S^*v, e_j\rangle e_j = \sum \langle v, Se_j\rangle e_j = \sum \langle v, f_j\rangle e_j$$

thus

$$T^{-1} v = \sqrt{T^*T}^{-1} S^*v = \sum \langle v, f_j\rangle \sqrt{T^*T}^{-1} e_j = \sum \langle v, f_j\rangle \frac{1}{s_j} e_j =$$

$$= \sum \frac{\langle v, f_j\rangle e_j}{s_j}$$

as desired.

7.4.18

Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denothe the smallest value of T, and let s denote the largest singular value of T.

(a) Prove that

$$\hat{s}||v|| \leq ||Tv|| \leq s||v||$$

for every $v \in V$.

Suppose that $v \in V$. Then we follow that with respect to orthonormal basis of eigenvectors of $\sqrt{T^*T}$

$$v = \sum \langle v, e_j\rangle e_j$$

thus

$$||v||^2 = \sum |\langle v, e_j\rangle|^2$$

Since

$$Tv = \sum s_j \langle v, e_j\rangle f_j$$

we follow that

$$||Tv||^2 = \sum |s_j \langle v, e_j\rangle|^2 = \sum |s_j^2 \langle v, e_j\rangle^2|$$

since $s_j \geq 0$ we follow that

$$||Tv||^2 = \sum s_j^2 |\langle v, e_j\rangle^2|$$

from which we can follow that

$$\hat{s}^2 \sum |\langle v, e_j\rangle^2| \leq \sum s_j^2 |\langle v, e_j\rangle^2| \leq s^2 \sum |\langle v, e_j\rangle^2|$$

$$\hat{s}^2 ||v||^2 \leq \sum s_j^2 |\langle v, e_j\rangle^2| \leq s^2 ||v||^2$$

thus

$$\hat{s}^2 ||v||^2 \leq ||Tv||^2 \leq s^2 ||v||^2$$

$$\hat{s}||v|| \leq ||Tv|| \leq s||v||$$

as desired.

Suppose λ is an eigenvalue of T. Prove that $\hat{s} \leq |\lambda| \leq s$

For given λ there exists an eigenvector v. Define

$$e = \frac{\lambda}{||v||}$$

thus

$$||e|| = 1$$

then we follow from the previous part that

$$\hat{s}||e|| \leq ||Tv|| \leq s||e||$$

$$\hat{s} \leq ||\lambda e|| \leq s$$

$$\hat{s} \leq |\lambda| \leq s$$

as desired.

Exercise 19 is a bit too complicated for now (although I got a feeling that it follows from 18); gonna leave it untill I complete the required prerequisites in a topology course. Exercise 20 is left for desert.

Chapter 8

Operators on Complex Vector Spaces

8.1 Generalized Eigenvectors and Nilpotent Operators

8.1.1

Define $T \in \mathcal{L}(C^2)$ *by*

$$T(w, z) = (z, 0)$$

Find all generalized eigenvectors of T

Since $T^2 = 0$, we follow that $(T - 0I)^2 v = 0$ for any $v \in V$. Thus we follow that $V \setminus 0$ are generalized eigenvectors of T.

8.1.2

Define $T \in \mathcal{L}(C^2)$ *by*

$$T(w, z) = (-z, w)$$

Because T commutes with its adjoint (proven by matrix multiplication), we follow that V has an orthonornal basis of eigenvectors. Suppose that $v = (v_1, v_2)$ is such a vector. Then we follow that

$$T(v_1, v_2) = \lambda(-v_2, v_1)$$

thus

$$v_1 = \lambda - v_2$$

$$v_2 = \lambda v_1$$

$$v_2 = \lambda^2(-v_2)$$

$$1 + \lambda^2 = 0$$

$$\lambda = \pm i$$

Thus we follow that $(1, i)$ and $(1, -i)$ and their scalar multiples represent all of the eigenvectors of T, which are also generalized eigenvectors.

8.1.3

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in F$ with $\lambda \neq 0$.

Define

$$M = \{j \in N : \operatorname{null}(T - \lambda I)^j \subseteq \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)^j\}$$

We follow that for the case $j = 1$ (usual eigenspaces) we've got that if $v = 0$, then v is is both nullspaces. If $v \in \operatorname{null}(T - \lambda I) \setminus 0$, then

$$(T - \lambda I)v = 0$$

$$Tv = \lambda v$$

thus

$$T^{-1}(Tv) = v$$

$$T^{-1}(\lambda v) = v$$

$$\lambda T^{-1}(v) = v$$

$$T^{-1}(v) = \frac{1}{\lambda} v$$

Thus we follow that $v \in \operatorname{null}(T - \lambda I) \to v \in \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)$. Therefore $1 \in M$.

Suppose that $j - 1 \in M$ and let $v \in \operatorname{null}(T - \lambda I)^j$. Case of $v = 0$ is trivial, thus assume that $v \neq 0$.

If $(T - \lambda I)v = 0$, Then we follow that $v \in \operatorname{null}(T - \lambda I) = \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)$ by induction hypothesis, and by the fact that $\operatorname{null} T \subseteq \operatorname{null} T^j$ we follow that $v \in (T^{-1} - \frac{1}{\lambda} I)^j$ for any $j \geq 1$.

If $(T - \lambda I)v \neq 0$ then let k denote the maximum of the set of natural numbers (which exists, because it's bounded by j because of our assumtions), for which $(T - \lambda I)^k v \neq 0$. If $k < j - 1$, then we follow that the case is handled by the IH.

Now the only case that is left is when

$$(T - \lambda I)^{j-1} v \neq 0$$

thus we follow that

$$(T - \lambda I)^{j-1}(T - \lambda I)v = 0$$

Thus $(T-\lambda I)v \in \operatorname{null}(T-\lambda I)^{j-1}$. By IH we follow that

$$(T-\lambda I)v \in \operatorname{null}(T^{\text{-}1} - \frac{1}{\lambda}I)^{j-1}$$

thus

$$(T^{\text{-}1} - \frac{1}{\lambda}I)^{j-1}(T-\lambda I)v = 0$$

It can be proven that since T and $T^{\text{-}1}$ commute, then the polynomials of those maps also commute (looked in up in the answers after too much time spent on this exercise; proof of the assumption is ommited). Thus

$$(T-\lambda I)(T^{\text{-}1} - \frac{1}{\lambda}I)^{j-1}v = 0$$

Denote $w = (T^{\text{-}1} - \frac{1}{\lambda}I)^{j-1}v$. We follow that

$$(T-\lambda I)w = 0$$

$$Tw = \lambda w$$

$$w = \lambda T^{\text{-}1} w$$

$$\frac{1}{\lambda}w = T^{\text{-}1} w$$

$$(T^{\text{-}1} - \frac{1}{\lambda}I)w = 0$$

$$(T^{\text{-}1} - \frac{1}{\lambda}I)(T^{\text{-}1} - \frac{1}{\lambda}I)^{j-1}v = 0$$

$$(T^{\text{-}1} - \frac{1}{\lambda}I)^{j}v = 0$$

Thus we can follow that $j-1 \in M \to j \in M$. Because of this and because $1 \in M$ we follow that$M = \mathbf{N}$, and therefore we've got that

$$\operatorname{null}(T-\lambda I)^j \subseteq \operatorname{null}(T^{\text{-}1} - \frac{1}{\lambda}I)^j$$

for any $j \in N$. Since $G(\lambda, T) = \operatorname{null}(T-\lambda)^{\dim V}$, we follow that

$$G(\lambda, T) \subseteq G(\frac{1}{\lambda}, T^{\text{-}1})$$

by the fact that the $(T^{\text{-}1})^{\text{-}1} = T$ and $(\lambda^{\text{-}1})^{\text{-}1}$ we follow that we've got double inclusion, and thus

$$G(\lambda, T) = G(\frac{1}{\lambda}, T^{\text{-}1})$$

as desired.

8.1.4

Suppose taht $T \in \mathcal{L}(V)$ and $\alpha, \beta \in F$ with $\alpha \neq \beta$. Prove that

$$G(\alpha, T) \cap G(\beta, T) = \{0\}$$

Suppose that

$$v \in G(\alpha, T) \cap G(\beta, T)$$

Let $u_1, ..., u_n$ be a basis of $G(\alpha, T)$ and $v_1, ..., v_m$ be a basis of $G(\beta, T)$. We follow that

$$v = \sum a_j u_j$$

and

$$v = \sum b_j v_j$$

Since u_j and v_j are in $G(\alpha, T)$ and $G(\beta, T)$ respectively, we follow that they are linearly independent. Thus we conclude that $a_1 = a_2 = ... = b_m = 0$. Thus $v = 0$, as desired.

8.1.5

Suppose $T \in \mathcal{L}(V)$, $m \in N$ and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that

$$v, Tv, ..., T^{m-1}v$$

is linearly independent.

We firstly state some that $v, Tv, ..., T_{m-2} \neq 0$, because if they are, then $T^{m-1}v = 0$, which is a contradiction.

Let

$$a_0 v + a_1 Tv + ... + a_{m-1} T^{m-1} v = 0$$

for some $a_0, ..., a_{m-1} \in F$.

Suppose if $a_0, a_1, ...$ except for a_j are equal to zero, then we follow that

$$a_j T^j v = 0$$

$$T^j = 0$$

$$T^{m-j-1} T^j = T^{m-j-1} 0$$

$$T^{m-1} v = 0$$

which is a contradiction.

Now suppose that there exist two or more coefficients, that are not equal to zero. Let then $j < k \in N$ be two lowest integers such that a_j and a_k are not zero. Then we follow that

$$a_0 v + a_1 Tv + ... + a_{m-1} T^{m-1} v = 0$$

$$a_jT^jv + a_kT^kv + ... + a_{m-1}T^{m-1}v = 0$$

we follow that

$$T^{m-1-k}(a_jT^jv + a_kT^kv + ... + a_{m-1}T^{m-1}v) = T^{m-1-k}0$$

$$a_jT^{m-1-k+j}v + a_kT^{m-1-k+k}v + ... = 0$$

$$a_jT^{m-1-k+j}v + a_kT^{m-1}v + ...) = 0$$

Since n and k are lowest integers such that a_j, a_k are not zero, we follow that terms

$$T^{m-1-k}(a_lT^lv) = a_lT^{m-1-k+l}v = 0$$

since $-k + l \geq 1$ and thus the power of T is greater then m, for which $T^mv = 0$. Thus we follow that

$$a_jT^{m-1-k+j}v + a_kT^{m-1}v + ... = a_jT^{m-1-k+j}v + a_kT^{m-1}v = 0$$

$$a_jT^{m-1-k+j}v + a_kT^{m-1}v = 0$$

$$a_jT^{m-1-k+j}v = -a_kT^{m-1}v$$

$$-\frac{a_j}{a_k}T^{m-1-k+j}v = T^{m-1}v$$

where we're justified to divide by a_k since it is not equal to zero. Thus

$$-\frac{a_j}{a_k}(T^{m-1-k+j}v) = T^{k-j}(T^{m-1-k+j}v)$$

Thus we follow that T^{k-j} has an eigenvalue $-\frac{a_j}{a_k} \neq 0$, which corresponds to the vector $T^{m-1-k+j}v$. Thus we follow that

$$(T^{k-j})^m(T^{m-1-k+j}v) = (-\frac{a_j}{a_k})^mv$$

$$T^{(k-j)*m+m-1-k+j}v = (-\frac{a_j}{a_k})^mv$$

Since $k > j$ we follow that $k - j \geq 1$. Thus $(k - j) * m \geq m$. Since $m - 1 - k + j \geq 0$, we follow that $(k - j) * m + m - 1 - k + j \geq m$. Thus we've got that left-hand side of the equation is supposed to be equal to zero, which is not the case, which gives us a contradiction.

Thus we follow that $a_0 = a_1 = ... = a_{m-1} = 0$. Therefore presented list is linearly independent, as desired.

8.1.6

Suppose $T \in \mathcal{L}(C^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precicely, prove that there does not exist $S \in \mathcal{L}(C^3)$ such that $S^2 = T$.

From definition we can follow that $T^3 = 0$. Suppose that there exists $S \in \mathcal{L}(V)$ such that $S^2 = T$. Then we follow that $T^3 = S^6 = 0$, therefore S is nilpotent. Thus $S^{\dim V} = S^3 = 0$. Thus $T^4 = SS^3 = 0$, which is a contradiction.

8.1.7

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that 0 is the only eigenvalue of N.

Suppose that $\lambda \neq 0$ is an eigenvalue of the nilpotent operator N with corresponding eigenvector v. Then we follow that $N^j v = \lambda^j v \neq 0$, for every $j \in N$, therefore N is not nilpotent, which is a contradiction.

8.1.8

Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$

Let $v_1, ..., v_n$ be a basis of V and define

$$T_1(v_n) = v_1$$

$$T_1(v_{j \neq n}) = 0$$

$$T_2(v_1) = v_n$$

$$T_2(v_{j \neq 1}) = 0$$

We can follow pretty easily that T_1 and T_2 are nilpotent, but we follow that

$$(T_1 + T_2)v_1 = v_n$$

$$(T_1 + T_2)v_n = v_1$$

Therefore we follow that $(T_1 + T_2)^j v_1$ is either equal to v_1 or v_n for every $j \in N$, which means that $T_1 + T_2$ is not nilpotent. Thus the set of nilpotent operators is not closed under addition, which means that it is not a subspace.

8.1.9

Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. PRove that TS is nilpotent.

Suppose that ST is nilpotent. This implies that there exists $j \in N$ such that

$$ST^j = 0$$

which means that

$$(STSTST...ST) = 0$$

$$(STSTST...ST)S = 0$$
$$T(STSTST...ST)S = 0$$
$$(TS)(TS)(TS)...(TS)(TS) = 0$$
$$(TS)^{j+1} = 0$$

thus TS is nilpotent.

8.1.10

Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $n = \dim V$. Show that

$$V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$$

If $n = 1$, then we follow that $T^{n-1} = T^0 = I$, therefore $\operatorname{null} T = \{0\}$, and thus we've got the desired conclusion. Thus assume that $n > 1$.

Since T is not nilpotent, we follow that

$$\dim \operatorname{null} T^n \neq \dim V$$

and since $\operatorname{null} T^n$ is a subspace of $\dim V$, we follow that

$$\dim \operatorname{null} T^n < \dim V$$

or in other words

$$\dim \operatorname{null} T^n \leq \dim V - 1$$

which in this case means that

$$\dim \operatorname{null} T^n \leq n - 1$$

Thus we can conclude that

$$0 \leq \dim \operatorname{null} T \leq ... \leq \dim \operatorname{null} T^n \leq n - 1$$

we follow that at some point in this chain $\dim \operatorname{null} T^{j-1} = \dim \operatorname{null} T^j$.

Suppose that $v \in \operatorname{null} T^{n-1} \cap \operatorname{range} T^{n-1}$. We follow that there exists $w \in V$ such that $T^{n-1}w = v$. Thus we follow that $T^{n-1}T^{n-1}w = T^{n-1}v$. Since $v \in \operatorname{null} T^{n-1}$, we follow that

$$T^{n-1}T^{n-1}w = 0$$
$$T^{2n-2}w = 0$$

thus $w \in \operatorname{null} T^{2n-1}$. Since $n > 1$, we follow that $2n - 2 \geq n$, thus we follow that $w \in \operatorname{null} T^n$, and therefore $w \in \operatorname{null} T^{n-1}$. Thus we conclude that

$$T^{n-1}w = 0$$

and since $v = T^{n-1}w$, we follow that $v = 0$. Thus $\operatorname{null} T^{n-1} \cap \operatorname{range} T^{n-1} = \{0\}$. Because $\dim \operatorname{null} T^{n-1} + \dim \operatorname{range} T^{n-1} = \dim V$, we follow that

$$\dim(\operatorname{null} T^{n-1} + \operatorname{range} T^{n-1}) = \dim V$$

and therefore $\operatorname{null} T^{n-1} + \operatorname{range} T^{n-1} = V$. Thus

$$V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$$

8.1.11

Prove or give a counterexample: If V is a complex vector space and $\dim V = n$ and $T \in \mathcal{L}(V)$, then T^n is diagonalizable.

Solution of exercise 5.3.2 (and exercise 5.3.5 with particular $\lambda = 0$) gives us that in finite-dimentional complex vector space we can state that

$$V = \operatorname{null} T \oplus \operatorname{range} T$$

if and only if T is diagonalizable. Since

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n$$

for any $T \in \mathcal{L}(V)$, we follow that T^n is indeed diagonalizable.

8.1.12

Suppose $T \in \mathcal{L}(V)$ and there exists a basis of V with respect to which T has an upper-triangular matrix with only 0's on the diagonal. Prove that T is nilpotent.

Let us denote the basis with respect to which N has an upper trianguar mamtrix with zeroes on the diagonal with $v_1, ..., v_n$ and let

$$M = \{j \in N \cap [0, \dim V] : (\forall i \leq j \in N)(T^{\dim V} v_i = 0)\}$$

We follow that if $j = 1$, then matrix gives us that $Tv_j = Tv_1 = T^1 v_1 = 0$. Thus $T^{\dim V} v_1 = T^{\dim V - 1} T v_1 = 0$. Therefore $1 \in M$.

Suppose that $j - 1 \in M$. By the fact that matrix is upper triangular we follow that

$$Tv_j \in \operatorname{span}(v_1, ..., v_j)$$

thus

$$Tv_j = \sum_{1 \leq i \leq j} a_i v_i$$

$$Tv_j = \sum_{1 \leq i < j} a_i v_i + a_j v_j$$

Since we've got zeroes on the diagonal, we follow that $a_j = 0$ and therefore $a_j v_j = 0$. Thus

$$Tv_j = \sum_{1 \le i < j} a_i v_i$$

Applying $T^{\dim V}$ to both sides we get that

$$T^{\dim V + 1} v_j = \sum_{1 \le i < j} a_i T^{\dim V} v_i$$

$$T^{\dim V + 1} v_j = \sum_{1 \le i \le j-1} a_i T^{\dim V} v_i$$

by IH we follow that $T^{\dim V} v_i = 0$ for all $1 \le i \le j - 1$. Thus

$$T^{\dim V + 1} v_j = \sum_{1 \le i \le j-1} a_i T^{\dim V} v_i = \sum_{1 \le i \le j-1} a_i 0 = 0$$

Thus $v_j \in T^{\dim V + 1}$. Thus $v_j \in T^{\dim V}$ and therefore we follow that $j \in M$.

Therefore we follow that if $j \in M$ and $j + 1 \le \dim V$, then $j + 1 \in M$. Since $1 \in M$ we follow that $M = N \cap [0; \dim V]$. Therefore for every $j < \dim V$ we've got that $T^{\dim V} v_j$.

Suppose that $v \in V$. Then we follow that

$$v = \sum a_j v_j$$

thus

$$T^{\dim V} v = \sum a_j T^{\dim V} v_j = 0$$

Thus $T^{\dim V} = 0$, and therefore T is nilpotent, as desired.

8.1.13

Suppose V is an inner product space and $N \in \mathcal{L}(V)$ is normal and nilpotent. Prove that $N = 0$

By exercisee 7.1.17 we've got that for every normal operator

$$\operatorname{null} T^k = \operatorname{null} T$$

for every positive k. Thus we follow that

$$\operatorname{null} N^{\dim V} = \operatorname{null} N$$

$$\operatorname{null} N = V$$

thus $N = 0$, as desired.

8.1.14

Suppose V is an inner product space and $N \in \mathcal{L}(V)$ is nilpotent. Prove that there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix

Because N is a nilpotent operator, we follow that there exist a basis such that it has an upper-diagonal matrix with zeroes on the diagonal. By applying GSP to this basis we get the orthonormal basis, and since after applying GSP we get =

$$Tv_j \in \operatorname{span}(e_1, ..., e_j)$$

we follow that the matrix is upper-diagonal with zeroes on the diagonal, as desired.

8.1.15

Suppose $N \in \mathcal{L}(V)$ is such that $\operatorname{null} N^{\dim V-1} \neq \operatorname{null} N^{\dim V}$. Prove that N is nilpotent and that

$$\dim \operatorname{null} N^j = j$$

for every integer j with $0 \leq j \leq \dim V$

Suppoose that $\operatorname{null} N^{\dim V-1} \neq \operatorname{null} N^{\dim V}$. Then we follow that

$$\operatorname{null} N^{\dim V-1} \subset \operatorname{null} N^{\dim V}$$

therefore

$$\operatorname{null} N \subset \operatorname{null} N^2 \subset \operatorname{null} N^3 ... \subset \operatorname{null} N^{\dim V}$$

thus

$$\dim \operatorname{null} N < \dim \operatorname{null} N^2 ... < \dim \operatorname{null} N^{\dim V}$$

Since $\operatorname{null} N^{\dim V}$ is a subspace of V we follow that $\dim \operatorname{null} N^{\dim V} \leq \dim V$. Also, if $\dim T = 0$ then we can follow that $\dim T^2 = 0$ and therefore

$$\dim \operatorname{null} T = \dim \operatorname{null} T^2 ... = \dim \operatorname{null} T^{\dim V} = 0$$

which is not our case. Thus we follow that $\dim N \geq 1$. Thus we've got that

$$1 \leq \dim \operatorname{null} N < \dim \operatorname{null} N^2 ... < \dim \operatorname{null} N^{\dim V} \leq \dim V$$

the only sequence of integers, that would satisfy the presented restrictions is

$$1, 2, 3, ...$$

(can be proven through the fact that the number of distinct integers in required range is $\dim V$, and by transitivity of $<$ in N we've got that presented sequence is the only one that satisfies the requirements. More rigorous proof is ommited, since I've got enough set theory on my hands) thus we follow that

$$\dim \operatorname{null} N^j = j$$

Therefore $\dim \operatorname{null} N^{\dim V} = \dim V$, therefore $N^{\dim V} = 0$ and therefore N is nilpotent.

8.1.16

Suppose $T \in \mathcal{L}(V)$. Show that

$$V = \operatorname{range} T^0 \supseteq T^1 ... \supseteq \operatorname{range} T^k \supseteq \operatorname{range} T^{k+1} ...$$

Let

$$M = \{j \in I : \operatorname{range} T^j \supseteq \operatorname{range} T^{j+1}\}$$

We follow that if $j = 0$, then $T^j = I$, and thus $\operatorname{range} T^0 = V$. Therefore it is true for any map (inluding T) that

$$V \supseteq \operatorname{range} T$$

thus we follow that $0 \in M$.

Let $k - 1 \in M$ and suppose that $v \in \operatorname{range} T^{k+1}$. Then we follow that there exists w such that

$$T^{k+1} w = v$$

or in other words

$$T^k T w = v$$

thus there exists $Tw \in V$ such that $T^k T w = v$. Thus $v \in \operatorname{range} T^k$. Thus we follow that $k - 1 \in M \to k \in M$. Thus $M = N \cup \{0\}$, as desired.

8.1.17

Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that

$$\operatorname{range} T^m = \operatorname{range} T^{m+1}$$

Prove that $\operatorname{range} T^m = \operatorname{range} T^k$ *for all $m < k$*

Suppose that $\operatorname{range} T^m = \operatorname{range} T^{m+1}$. We follow that $\dim \operatorname{range} T^m = \dim \operatorname{range} T^{m+1}$. This means that $\dim \operatorname{null} T^m = \dim \operatorname{null} T^{m+1}$. Which in turn implies that $\operatorname{null} T^m = \operatorname{null} T^{m+1}$. Thus we follow that $\operatorname{null} T^m = \operatorname{null} T^k$, therefore $\dim \operatorname{null} T^k = \dim \operatorname{null} T^m$. Once again this means that $\dim \operatorname{range} T^k = \dim \operatorname{range} T^m$. Since $\operatorname{range} T^k \subseteq \operatorname{range} T^m$, we follow that they have the same basis, and thus they are equal, as desired.

8.1.18

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that

$$\operatorname{range} T^n = \operatorname{range} T^{n+1} = ...$$

Since $\operatorname{null} T^n = \operatorname{null} T^{n+1} = ...$, we follow that $\dim \operatorname{range} T^n = \dim \operatorname{range} T^{n+1} = ...$, thus by the fact that $\operatorname{range} T^n \supseteq \operatorname{range} T^{n+1} \supseteq ...$ we follow that $\operatorname{range} T^n = \operatorname{range} T^{n+1} = ...$, as desired.

8.1.19

Suppose $\mathcal{L}(V)$ and m is a nonnegative integer. Prove that

$$\operatorname{null} T^m = \operatorname{null} T^{m+1} \iff \operatorname{range} T^m = \operatorname{range} T^{m+1}$$

Follows from the previous two exercises.

8.1.20

Suppose $T \in \mathcal{L}(C^5)$ is such that $\operatorname{range} T^4 \neq \operatorname{range} T^5$. Prove that T is nilpotent.

We follow that since $\operatorname{range} T^4 \neq \operatorname{range} T^5$, then $\dim \operatorname{range} T^4 \neq \dim \operatorname{range} T^5$, thus $\dim \operatorname{null} T^4 \neq \dim \operatorname{null} T^5$, and now we've got the particular case of 8.1.15.

8.1.21

Find a vector space W and $T \in \mathcal{L}(W)$ such that $\operatorname{null} T^k \subset \operatorname{null} T^{k+1}$ and $\operatorname{range} T^k \supset \operatorname{range} T^{k+1}$ for every positive integer k.

Let $W = F^\infty$ and

$$T(a_1, a_2, a_3, ...,) = (0, a_4, a_1, a_6, a_3)$$

in other words, we shift oddss by 1 to the right and shifting evens by 1 to the left.

8.2 Decomposition of an Operator

8.2.1

Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N. Prove that N is nilpotent.

Since N is in a complete vector space, we follow that it has an upper-triangular matrix. Since 0 is the only eigenvalue of N we follow that the only numbers on the diagonal are the zeroes. Thus by exercise 8.1.12 we follow that N is nilpotent, as desired.

8.2.2

Give an example of an operator T on a finite-dimentional real vector space such that 0 is the only eigenvalue of T but T is not nilpotent

In this case we need to have an operator, that in complex vector space has eigenvalues $0, \lambda_1, \lambda_2$, where $\lambda_1, \lambda_2 \notin R$. We can do this by firstly setting operator

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

whose eigenvalues are i and $-i$, and then appending another vector to the basis, that goes straight to zero, thus resulting in

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

We can follow through calculation that $T^2 \neq 0$ and $T^3 = T$, thus proving (rigorous proof follows easily from induction) that T is not nilpotent, as desired.

8.2.3

Suppose $T \in \mathcal{L}(V)$. Suppose that $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same mutltiplicities.

From exercise 5.2.5 we follow that

$$p(STS^{-1}) = Sp(T)S^{-1}$$

which in this case gives us that

$$p(S^{-1}TS) = S^{-1}p(T)S$$

and therefore

$$(S^{-1}TS - \lambda I)^{\dim V} = S^{-1}(T - \lambda I)^{\dim V}S$$

thus we follow that if $v_1, ..., v_n$ is the basis of $G(T, \lambda)$, then

$$(S^{-1}TS - \lambda I)^j S^{-1}v = S^{-1}(T - \lambda I)^j SS^{-1}v = S^{-1}(T - \lambda I)^j v = S^{-1}0 = 0$$

thus $S^{-1}v_1, ..., S^{-1}v_n$ are generalized eigenvectors of $S^{-1}TS$. Since invertible functions preserve the linear independence of the lists, we follow that $S^{-1}v_1, ..., S^{-1}v_n$ is linearly independent, and thus

$$\dim G(T, \lambda) \leq \dim G(S^{-1}TS, \lambda)$$

We can state that

$$T = ITI = S(S^{-1}TS)S^{-1}$$

thus by previous point we get that

$$\dim G(S(S^{-1}TS)S^{-1}, \lambda) \geq \dim G(S^{-1}TS, \lambda)$$

and thus

$$\dim G(T, \lambda) \geq \dim G(S^{-1}TS, \lambda)$$

from which by antisymmetry we get that

$$\dim G(T, \lambda) = \dim G(S^{-1}TS, \lambda)$$

thus $S^{-1}TS$ and T have the same eigenvalues with the same multiplicities, as desired.

8.2.4

Suppose V is an n-dimentional complex vector space and T is an operator on V such that $\operatorname{null} T^{n-2} \neq \operatorname{null} T^{n-1}$/ *Prove thaht T has at most two distinct eivenvalues.*

We've got two cases here: first one is when $\operatorname{null} T^{n-1} \neq \operatorname{null} T^{n}$, in which case N is nilpotent, and therefore it has the one eigenvalue - 0.

If $\operatorname{null} T^{n-1} = \operatorname{null} T^{n}$, then we follow that T is not nilpotent, but 0 is still an eigenvalue of T since $\operatorname{null} T^{n-1} \neq \operatorname{null} T^{n}$. Thus we can also follow that $\dim G(0,T) = n-1$, and thus there must exist another subspace $G(\lambda, T)$ such that $\dim G(\lambda, T) = 1$ and $\lambda \neq 0$. Thus we conclude that T has two eigenvalues.

8.2.5

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

V has a basis consisting of eigenvectors of T if and only if it is diagonalizable. T is diagonalizable if and only if

$$V = E(\lambda_1, T) \oplus ... \oplus ...E(\lambda_m, T)$$

by creating bases and the fact that $E(\lambda_j, T) \subseteq G(\lambda_j, T)$ we can prove that

$$V = E(\lambda_1, T) \oplus ... \oplus ...E(\lambda_m, T)$$

if and only if $E(\lambda_j, T) = G(\lambda_j, T)$, or in other words, if every generalized eigenvector is an eigenvector of T.

8.2.6

Define $N \in \mathcal{L}(F^5)$ by

$$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0)$$

Find a square root of $I + N$.

We can follow pretty easily that N is nilpotent, thus we can use the formula that was hinted at in the proof.

If we take the numbers from the proof and set $a_4 = -25.6$, then we'll get the desired rusult

$$\sqrt{I+N} = I + 1/2N - 1/8N^2 + 1/16N^3 - 5/128N^4$$

8.2.7

Suppose V is a complex vector space. Prove that every invertible operator on V has a cube root

We can look once again at the proof of the theorem that identity plus a nilpotent operators, the only thing that would change are the coefficients. After this the application if the same ideas as in 8.33 with the sole difference of taking the cube root instead of the square root will produce the desired result.

8.2.8

Suppose $T \in \mathcal{L}(V)$ and 3 and 8 are eigenvalues of T. Let $n = \dim V$. Prove that

$$V = (\operatorname{null} T^{n-2}) \oplus (\operatorname{range} T^{n-2})$$

We can follow that in this case T has at least eigenvalues 3 and 8, which means that if T is not invertible and has an eigenvalue of 0, then since

$$V = G(0, T) \oplus G(3, T) \oplus G(8, T) \oplus ...$$

and dimentions of $G(3, T)$ and $G(8, T)$ is at least 1 each, we follow that $\dim G(0, T) \leq n-2$. Thus we follow that

$$\operatorname{null} T^{n-2} = \operatorname{null} T^{n-1} = \operatorname{null} T^{n}$$

thus we follow from exercises in the previous section that

$$\operatorname{range} T^{n-2} = \operatorname{range} T^{n-1} = \operatorname{range} T^{n}$$

and thus

$$V = \operatorname{null} T^{n} \oplus \operatorname{range} T^{n}$$

will imply that

$$V = (\operatorname{null} T^{n-2}) \oplus (\operatorname{range} T^{n-2})$$

as desired.

8.2.9

Follows from the application of matrix multiplication rules.

8.2.10

Suppose $F = C$ and $T \in \mathcal{L}(V)$. Prove that there exist $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.

Because $F = C$, we can follow that any T has an upper-triangular matrix with respect to some basis. Thus we can divide T into the sum of upper-triangular matrix with zeroes on the diagonal, and the diagonal itself. If we do this with any given upper-tirangular matrix, then we won't get that $DN = ND$. But if we do this with a block-diagonal matrix, (which exists when $F = C$ by results of 8.29), then we can dissect D into a block-diagonal matrix, where each block will be a scalar multiple of the identity, which in turn will commute with N by the result of the previous exercise.

exercise 11 is left for later

8.3 Characteristic and Minimal Polynomials

8.3.1

Suppose $T \in \mathcal{L}(C^4)$ is such that the eigenvalues of T are 3, 5, 8. Prove that

$$(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$$

Because we've got 3 distinct eigenvalues and the sum of the multiplicities of those eigenvalues are 4 (dimention of the vector space), we follow that one of the eigenvalues of T has a multiplicity of 2, while others have multiplicity of 1. Thus we can follow that given polynomial is in a form

$$q(T)p(T)$$

where $q(T)$ is a characteristic polynomial, and p is just a polynomial. Since $q(T) = 0$, we follow that $q(T)p(T) = 0p(T) = 0$. Thus we conclude that

$$(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$$

as desired.

8.3.2

Generalization of the previous exercise.

8.3.3

Give an example of an operator on C^4 whose characteristic polynomial equals

$$(z - 7)^2(z - 8)^2$$

$$T = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

8.3.4

Give and example of an operator on C^4 whose characteristic polynomial equals $(z-1)(z-5)^3$ and whose minimal polynomial equals $(z-1)(z-5)^2$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

8.3.5

Give an example of an operator on C^4 whose characteristic and minimal polynomials both equal $z(z-1)^2(z-3)$

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

8.3.6

Give an example of an operator on C^4 whose characteristic polynomial equals $z(z-1)^2(z-3)$ and whose minimal polynomial equals $z(z-1)(z-3)$

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

8.3.7

Suppose V is a complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.

We know from exercises of section 5.3 and some other exercises that P is diagonalizable and the only eigenvalues of T are 0 and 1. Also,

$$V = \operatorname{null} P \oplus \operatorname{range} P$$

thus we've got the desired characteristic polynomial.

8.3.8

Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if the constant term in the minimal polynomial is not zero.

T is not invertible if and only if 0 is not the eigenvalue of T. This happens if and only if 0 is not a root of its minimal polynomial. 0 is not a root of a polynomial if and only if its constant term is not zero.

8.3.9

Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $4+5z-6z^2-7z^3+2z^4+z^5$. Find the minimal polynomial of $T^{\text{-}1}$.

From what I can see on the graph, this polynomial does not have handy zeroes lying around, closest that I've gotten was some value around 0.9315, that doesn't seem to be very helpful. Other roots are probably complex and ugly, which also doesn't help.

Suppose that q is a minimal polynomial for T. Then we follow that it can be represented as

$$q(z) = \prod (z - \lambda_j)^{d_j}$$

then we can conjecture that

$$q'(z) = \prod (z - \frac{1}{\lambda_j})^{d_j}$$

is a minimal polynomial for $T^{\text{-}1}$, which in this case does not give us much to work with.

Denote presented polynomial as q and we follow that

$$q(T) = 4I + 5T - 6T^2 - 7T^3 + 2T^4 + T^5 = 0$$

thus if we multiply this thing by T^{-5} (which is $(T^{\text{-}1})^5$) we get

$$4T^{-5} + 5T^{-4} - 6T^{-3} - 7T^{-2} + 2T^{-1} + I = 0$$

we want a monic polynomial, thus we need to divide this thing by 4, which gets us

$$T^{-5} + \frac{5}{4}T^{-4} - \frac{3}{2}T^{-3} - \frac{7}{4}T^{-2} + \frac{1}{2}T^{-1} + \frac{1}{4}I = 0$$

which seems to be a reasonable candidate for being the minimal polynomial of $T^{\text{-}1}$. Now the only thing left to prove is that this is indeed a minimal polynomial.

The only reasonable approach to it is to show the contradiction: suppose that it isn't a minimal polynomial for T^{-1}. Then we follow that there exists a polynomial with degree less than 5, thas is a minimal polynomial of T^{-1}, and this polynomial right there is just a polynomial multiple of it. Then we follow that we can multiply our new found minimal polynomial q' by $T^{\deg q'}$, which will give us a polynomial with lesser degree then our initial minimal polynomial, and such that it is equal to zero, when we plug T into it. Thus we follow that our initial minimal polynomial is not minimal, which is a contradiciton.

Thus we follow that

$$w(z) = z^5 + \frac{5}{4}z^4 - \frac{3}{2}z^3 - \frac{7}{4}z^2 + \frac{1}{2}z^1 + \frac{1}{4}$$

is a minimal polynomial of T^{-1}, as desired.

8.3.10

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Let p denote the characteristic polynomial of T and let q denote the characteristic polynomial of T^{-1}. Prove that

$$q(z) = \frac{1}{p(0)} z^{\dim V} p(\frac{1}{z})$$

for all nonzero $z \in C$.

Don't know how does the phrase "for all nonzero $z \in C$" fits in this context, but nontheless

Suppose that p is a characteristic polynomial of T. We follow that

$$p(z) = \prod_{\lambda \in eig(T)} (z - \lambda)$$

where $eig(T)$ is a tuple of eigenvalues of T, where each of the eigenvalues of T is repeated according to its multiplicity. We've proven in the previous section (or in the one before it) that $G(\lambda, T) = G(\frac{1}{\lambda}, T)$. Thus we can follow that

$$q(z) = \prod_{\lambda \in eig(T)} (z - \frac{1}{\lambda})$$

is a characteristic polynomial of T^{-1}, provided that T is invertible.

To prove the desired identity we need to use a little algebra. Henseforth let us omit the subscript of the product sign

$$p(z) = \prod (z - \lambda)$$

$$p(\frac{1}{z}) = \prod (\frac{1}{z} - \lambda)$$

$$z^{\dim V}p(\frac{1}{z}) = z^{\dim V}\prod(\frac{1}{z} - \lambda)$$
$$z^{\dim V}p(\frac{1}{z}) = \prod[z(\frac{1}{z} - \lambda)]$$
$$z^{\dim V}p(\frac{1}{z}) = \prod[1 - \lambda z]$$
$$z^{\dim V}p(\frac{1}{z}) = \prod -[\lambda z - 1]$$
$$z^{\dim V}p(\frac{1}{z}) = \prod -\lambda[z - \frac{1}{\lambda}]$$
$$z^{\dim V}p(\frac{1}{z}) = \prod[-\lambda]\prod[z - \frac{1}{\lambda}]$$
$$z^{\dim V}p(\frac{1}{z}) = \prod[0 - \lambda]\prod[z - \frac{1}{\lambda}]$$
$$z^{\dim V}p(\frac{1}{z}) = p(0)\prod[z - \frac{1}{\lambda}]$$
$$\frac{1}{p(0)}z^{\dim V}p(\frac{1}{z}) = \prod[z - \frac{1}{\lambda}]$$
$$\frac{1}{p(0)}z^{\dim V}p(\frac{1}{z}) = q(z)$$

as desired.

8.3.11

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that there exists a polynomial $p \in P(F)$ such that $T^{-1} = p(T)$.

Suppose that p' is a minimal polynomial of T. Since T is not invertible, we follow that its contant term is not equal to zero. Let us denote this term with a_0. We follow that polynomial

$$p'(z) - a_0$$

has 0 as its constant term. Therefore there exists $q \in P(F)$ such that

$$zq(z) = p'(z) - a_0$$

Let us plug T into previous polynomial and get

$$p'(T) - a_0I = 0 - a_0I = -a_0I$$
$$p'(T) - a_0I = -a_0I$$

$$\frac{1}{-a_0}(p'(T) - a_0 I) = I$$

$$T^{-1}(\frac{1}{-a_0}p'(T) - a_0 I) = T^{-1}$$

$$\frac{1}{-a_0}T^{-1}(p'(T) - a_0 I) = T^{-1}$$

$$\frac{1}{-a_0}T^{-1}Tq(T) = T^{-1}$$

$$\frac{1}{-a_0}q(T) = T^{-1}$$

Since $a_0 \neq 0$ we follow that $-\frac{1}{a_0} \in F$ is legal, therefore

$$\frac{1}{-a_0}q(z) \in P(F)$$

is the desired polynomial.

8.3.12

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated zeroes.

Repeated zeroes in this sense means that if q is a minimal polynomial of T, then

$$q(z) = \prod_{\lambda \in eig(T)} (z - \lambda)$$

where each $\lambda \in eig(T)$ is distinct.

We follow that if V has a basis consisting of eigenvectors of T, then

$$V = E(\lambda_0, T) \oplus E(\lambda_1, T) \oplus ... \oplus E(\lambda_n, T)$$

where each λ_n is an eigenvalue of T. Then we follow that

$$(T - \lambda_j I)|_{E(\lambda_j, T)} = 0$$

therefore for any $v \in V$ such that

$$v = \sum a_j e_j$$

where e_j is an eigenvector, corresponding to eigenvalue λ_j we've got that

$$\prod (T - \lambda_j I) a_j e_j = 0$$

because $(T-\lambda_j I)$ in the product commute and because $T-\kappa I$ is invariant on every $E(\lambda_j, T)$ for every $\kappa \in F$. Thus we follow that $\prod(T-\lambda_j I)=0$, therefore we conclude that it's a polynomial multiple of the minimal polynomial. Since it doesn't have repeated zeroes, we follow that the minimal polynomial does not have them either.

Conversely, suppose that minimal polynomial of T does not have repeated zeroes. Then we follow that every generalized eigenvector of T is an eigenvector of T, which by exercise 8.2.5 gives us that T is has a basis consisting of eigenvectors of T, as desired.

8.3.13

Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is normal. Prove that the minimal polynomial of T has no repeated zeroes.

If $F=C$, then we follow that T is diagonalizable, thus we've got the desired result by previous exercise.

Suppose that $F=R$. Suppose that minimal polynomial of T has repeated zeroes and therefore there exists $v \in V$ such that for some $\lambda \in R$ we've got that

$$(T-\lambda I)v \neq 0$$

but there exists $j \in N$ such that

$$(T-\lambda I)^j v \neq 0$$

thus v is a generalized eigenvector for the value λ, therefore $v \in G(\lambda, T)$

Because T is normal, we follow that T and T^* commute, therefore we can follow that $p(T)q(T^*)=q(T^*)p(T)$ for $p,q \in P(F)$, therefore $(T-\lambda I)$ and $(T-\lambda I)^*$ commute, therefore they are normal as well.

From the proof of 8.21 we can follow that parts (b) and (c) do not depend on the F, thus we've got that $(T-\lambda I)|_{G(\lambda,T)}$ is nilpotent. As proven before, $(T-\lambda I)$ is normal, therefore $(T-\lambda I)|_{G(\lambda,T)}$ is also normal, and because it's nilpotent we follow by exercise 8.1.13 that

$$(T-\lambda I)|_{G(\lambda,T)} = 0|_{G(\lambda,T)}$$

thus

$$T|_{G(\lambda,T)} = \lambda I|_{G(\lambda,T)}$$

Thus we follow that for our initial v

$$(T-\lambda I)v = Tv-\lambda v = T|_{G(\lambda,T)}v-\lambda v = \lambda v-\lambda v = 0$$

therefore we've got a contradiction.

8.3.14

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$ is an isometry. Prove that the constant term in the characteristic polynomial of S has absolute value 1.

Since S is an isometry, we follow that each eigenvalue of S has absolute value of 1. Suppose that p is a characteristic polynomial of S. Thus

$$p(z) = \prod (z - \lambda)$$

where λ is an eigenvalue of S. We then follow that the constant term (i.e. p(0)) is equal to

$$\prod (-\lambda)$$

thus

$$|p(0)| = |\prod (-\lambda)| = \prod |(-\lambda)| = \prod |\lambda| = \prod 1 = 1$$

as desired.

8.3.15

Suppose $T \in \mathcal{L}(V)$ and $v \in V$.

(a) Prove that there exists a unique monic polynomial p of smallest degree such that $p(T)v = 0$

We can follow the proof of the existence and uniqueness of minimal polynomial of T, but substituting T^j for $T^j v$ to get the desired result

(b) Prove that p divides the minimal polynomial of T.

We follow it the same way as in 8.46

8.3.16

Suppose V is an inner product spacce and $T \in \mathcal{L}(V)$. Suppose

$$a^0 + a_1 z + a_2 z^2 + ... + a_{m-1} z^{m-1} + z^m$$

is the minimal polynomial of T. Prove that

$$\overline{a^0} + \overline{a_1} z + \overline{a_2} z^2 + ... + \overline{a_{m-1}} z^{m-1} + z^m$$

We can follow that if

$$a^0 + a_1 z + a_2 z^2 + ... + a_{m-1} z^{m-1} + z^m$$

is a minimal polynomial of T, the

$$a_0 I + a_1 T + a_2 T^2 + ... + a_{m-1} T^{m-1} + T^m = 0$$

and by taking the adjoint of both sides we get

$$\overline{a_0}I^* + \overline{a_1}T^* + \overline{a_2}(T^*)^2 + ... + \overline{a_{m-1}}(T^*)^{m-1} + (T^*)^m = 0$$

thus

$$\overline{a^0} + \overline{a_1}z + \overline{a_2}z^2 + ... + \overline{a_{m-1}}z^{m-1} + z^m$$

is a polynomial multiple of minimal polynomial of T^*. Suppose that there exists a polynomial

$$\overline{b_0} + \overline{b_1}z + \overline{b_2}z^2 + ... + \overline{b_{m-1}}z^{n-1} + z^n$$

such that $n < m$ and that has lesser degree then our derived polynomial. Then we follow that its adjoint is a minimal polynomial for T with lesser degreee then on our given polynomial, which is a contradiction.

8.3.17

Suppose $F = C$ and $T \in \mathcal{L}(V)$. Suppose the minimal polynomial of T ahs degree $\dim V$. Prove that the characteristic polynomial of T equals the minimal polynomial.

We follow that the characteristic polynomial is a polynomial multiple of minimal polynomial, and if both of them have degree $\dim V$, then characteristic polynomial is indeed a minimal polynomial by the uniqeness of minimal polynomial and the fact that characteristic polynomial is monic.

last 3 exercises are left for the better days.

8.4 Jordan Form

8.4.1

Find the charateristic polynomial and the minimal polynomial of the operator N defined by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$z^4$$

will do

8.4.2

Simular to previous, generalized in next exercise, the answer is z^3.

8.4.3

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that the minimal polynomial of N is $z^(m+1)$, where m is the length of the longest consecutive string of 1's that appears on the line directly above the diagonal in the matrix of N with respect to any Jordan basis for N.

We can follow that the characteristic polynomial of N is $z^{\dim V}$, since every eigenvalue of N is 0. Since characteristic polynomial is the polynomial multiple of the minimal polynomial, we follow that the minimal polynomial is in the form z^j for some $j \in Z^+$.

Since the product of the block diagonal matrix is equal to the block-diagonal matrix of the respective products of the blocks, we can follow that the minimal j that so that $N^j = 0$ is the length if the longest consecutive string of 1's that appear of the line directly above the diagonal, as desired.

8.4.4

Suppose $T \in \mathcal{L}(V)$ adn $v_1, ..., v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of T with respect to the basis $v_n, ..., v_1$ obtained by reversing the order of the v's

We can follow that produced matrix is the product of given matrix, mirrored along its center in both vertical and horisontal orientations, thus we follow that this matrix is in the form where 1's appear only below the diagonal.

8.4.5

Suppose $T \in \mathcal{L}(V)$ and $v_1, ..., v_n$ is a basis of V that is Jordan basis for T. Describe the matrix of T^2 with respect to this basis.

It is a matrix with squares of original numbers along the diagonal, above the diagonal there are 0's where the're used to be zeroes, but instead of 1's there are sums of numbers in preceeding rows and columns (i.e. if $a_{j,j-1} = 1$, then $a_{j,j-1} = a_{j,j} + aj - 1, j - 1$ in the square). Above those numbers there may be 1's as well, and every other number is zero.

8.4.6

Suppose $N \in \mathcal{L}(V)$ is nilpotent and $v_1, ..., v_n$ and $m_1, ..., m_n$ are as in 8.55. Prove taht $N^{m_1}v_1, ..., N^{m_n}v_n$ is a basis of $\operatorname{null} N$.

We follow that

$$N^{m_1+1}v_1 = ... = N^{m_n+1}v_n = 0$$

thus we can follow that

$$N^{m_1}v_1, ..., N^{m_n}v_n \in \operatorname{null} N$$

Suppose that $v \in \operatorname{null} T$. Then we follow that

$$v = \sum a_j N^{m_j} v_j$$

and

$$Nv = \sum a_j N^{m_j+1} v_j = 0$$

thus we follow that every a_j that correspond to non-zero $N^{m_j+1}v_j$ is equal to zero because $N^{m_1}v_1, ..., Nv_1, v_1, ..., N^{m_n}v_n, ..., Nv_n, v_n$ is a basis and therefore linearly independent. Thus we follow that the only a's that may not be equal to zero correspond to $N^{m_1}v_1, ..., N^{m_n}v_n$, for which $N^{m_j+1}v_j = 0$. Thus we follow that

$$v \in \operatorname{span}(N^{m_1}v_1, ..., N^{m_n}v_n)$$

thus we follow that $N^{m_1}v_1, ..., N^{m_n}v_n$ spans $\operatorname{null} T$ and is linearly independent, therefore it is a basis of $\operatorname{null} T$, as desired.

8.4.7

Suppose $p, q \in P(C)$ are monic polynomials with the same zeroes and q is a polynomial multiple of p. Prove that there exists $T \in \mathcal{L}(C^{\deg q})$ such that the characteristic polynomial of T is q and the minimal polynomial of T is p.

Suppose that $\lambda_1, lambda_2, ...,$ are zeroes of p, according to their multiplicities. Then we follow that we can create an operator $T \in \mathcal{L}(C^{\deg q})$, whose matrix with respect to some basis consists of those values on the main diagonal, where equal numbers are clumped together. Then we can follow that q is a characteeisric polynomial of T'. By adjusing the number of 1's above the diagonal we can adjust the power of each of the $(z - \lambda)$ divisors of minimal polynomial to achieve the desired result.

8.4.8

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that there does not exist a direct sum decomposition of V into two proper subspaces invariant under T if and only if the minimal polynomial of T is of the form $(z - \lambda)^{\dim V}$ for some $\lambda \in C$.

Latter is the case if and only if we've got full row of 1's above the diagonal, which happens if and only if V cannot be split into two subspaces invariant under T.

Chapter 9

Operators on Real Vector Spaces

9.1 Complexification

9.1.1

Prove 9.3

Suppose that V is a real vector space. Then we follow that

$$0 + i0 + u + iv = 0$$

$$1(u + iv) = u + iv$$

thus we follow that 0 is the additive identity and 1 is the multiplicative identity

$$((u+iv)+(u'+iv'))+(u''+iv'') = (u+u'+u'')+i(v+v'+v'') = (u+iv)+((u'+iv')+(u''+iv''))$$

$$a((u + iv) + (u' + iv')) = a(u + iv) + a(u' + iv)$$

$$(u + iv) + (-u + -iv) = 0$$

$$(u + iv) + (u' + iv') = u + iv + u' + iv' = (u' + iv') + (u + iv)$$

thus we've got associativity, distributivity, commutativity and additive inverse.

Thus we follow that V_C is a vector field.

9.1.2

Verify that if V is a real vector space and $T \in \mathcal{L}(V)$, then $T_C \in \mathcal{L}(V_C)$.

Suppose that V is finite-dimentional field over R. Then we follow that

$$T_C((u + iv) + \lambda(u' + iv')) = T_C((u + \lambda u) + (iv + \lambda iv')) =$$

$$= T(u + \lambda u') + i(T(v + \lambda v')) = Tu + \lambda Tu' + iTv + \lambda iTv' =$$

$$= T_C(u + iv) + \lambda T_C(u' + iv')$$

thus we can follow that $T_C \in \mathcal{L}(V_C)$.

9.1.3

Suppose that V is a real vector space and $v_1, ..., v_m \in V$. Prove that $v_1, ..., v_m$ is linearly independent in V_c if and only if $v_1, ..., v_m$ is linearly independent in V spans V

In forward direction: suppose that $v_1, ..., v_m$ is linearly independent in V_C. Then we follow that $v_1, ..., v_m$ is not linearly independent in V. Then we follow that there exist $a_1, ..., a_m \in R$ such that

$$\sum a_j v_j = 0$$

but not all of the $a_1, ..., a_m$ are zeroes. Then we follow that

$$\sum (a_j + ia_j)v_j = 0$$

which contradicts the fact that $v_1, ..., v_m$ is linearly independent in V_C.

In reverse direction:

Suppose that $v_1, ..., v_m$ is linearly independent in V. Then we follow that

$$\sum a_j v_j = 0$$

if and only if all of the $a_j = 0$. Thus we follow that

$$\sum (a_j + ia'_j)v_j = 0$$

if and only if $(a_j + ia_j) = 0$, from which we follow that $v_1, ..., v_m$ is linearly independent in V_C.

9.1.4

Suppose that V is a real vector space and $v_1, ..., v_m \in V$. Prove that $v_1, ..., v_m$ spans V_C if and only if $v_1, ..., v_m$ spans V

Suppose that $v_1, ..., v_m$ spans V_C. Suppose that $u \in V$. We follow that for every $u + i0 \in V_C$ we have $a_1, ..., a_m \in C$ such that

$$u + iv = \sum a_j v_j$$

let $a_j = b_j + ib'_j$, where $b_j, b'_j \in R$. Thus we follow that

$$u + i0 = \sum (b_j + ib'_j)v_j$$

$$u + i0 = \sum b_j v_j + i \sum b'_j v_j$$

thus

$$u = \sum b_j v_j$$

thus we follow that $v_1, ..., v_m$ spans V.

Conversely, assume that $v_1, ..., v_m$ spans V. Let $u+iv \in V_C$. Then we follow that there exist $b_1, ..., b_m$ and $b'_1, ..., b'_m$ such that

$$u = \sum b_j v_j$$

$$v = \sum b'_j v_j$$

thus

$$u + iv = \sum b_j v_j + i \sum b'_j v_j$$
$$u + iv = \sum (b_j + ib'_j) v_j$$

thus if we set $a_j = b_j + ib'_j \in C$ we have that

$$u + iv = \sum a_j v_j$$

thus $v_1, ..., v_m$ spans V_C.

Thus we follow that $v_1, ..., v_m$ spans V_C if and only if $v_1, ..., v_m$ spans V, as desired.

9.1.5

Suppose that V is a real vector space and $S, T \in \mathcal{L}(V)$. Show that

$$(S + T)_C = S_C + T_C$$

and that

$$(\lambda T_C) = \lambda T_C$$

For all $u + iv \in V_C$ we've got

$$(S+T)_C(u+iv) = (S+T)u + i(S+T)v = Su + iSv + Tu + iTv = S_C(u+iv) + T_C(u+iv)$$

$$(\lambda S)_C(u + iv) = (\lambda S)u + i(\lambda S)v =$$
$$= \lambda(Su + iSv) = \lambda S_C(u + iv)$$

9.1.6

Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Prove that T_C is invertible if and only if T is invertible

In forward direction: Suppose that T_C is invertible and suppose that T is not invertible. Then we follow that there exists $v \neq 0 \in \operatorname{null} T$ such that

$$Tv = 0$$

thus

$$Tv + iTv = 0$$

$$T_C v = 0$$

which is a contradiction.

In reverse direction: Suppose that T is invertible and T_C is not invertible. Then we follow that there exists $u + iv \neq 0$ such that

$$T_C(u + iv) = 0$$

$$Tu + iTv = 0$$

Since $u + iv \neq 0$ we follow that $u \neq 0$ or $v \neq 0$, both of which give us a contradiction.

9.1.7

Suppose V is a real vector space and $N \in \mathcal{L}(V)$. Prove that N_C is nilpotent if and only if N is nilpotent.

Suppose that N_C is nilpontent and N is not nilpotent. Then we follow that there exists v suc

$$N^{\dim V} v \neq 0$$

thus

$$N^{\dim V} v + iN^{\dim V} v + \neq 0$$

$$N_C^{\dim V} v \neq 0$$

which is a contradiction.

Suppose that N is nilpotent. Then we follow that

$$N^{\dim V} + iN^{\dim V} + = 0$$

$$N_C = 0$$

as desired.

9.1.8

Suppose $T \in \mathcal{L}(R^3)$ and $5, 7$ are eigenvalues of T. Prove that T_C has no nonreal eigenvalues.

Suppose that it does have a non-real eigenvalue λ. Then we follow that $\lambda \neq \overline{\lambda}$ and thus $\overline{\lambda}$ is an eigenvalue of T as well. Thus T has 4 distinct eigenvalues, but it's located in space with dimetion 3, which is a contradiction.

9.1.9

Prove there does not exist an operator $T \in \mathcal{L}(R^7)$ such that $T^2 + T + I$ is nilpotent.