My set theory exercises

Evgeny Markin

2023

Contents

Chapter 1

Introduction

1.1 Elementary Set Theory

Let A, B, C be sets

1.1.1

If $a \notin A \setminus B$ and $a \in A$, show that $a \in B$

Because $a \notin A \setminus B$, we follow that $x \in B$ or $x \notin A$. Since $x \in A$, we follow that $x \in B$, as desired.

1.1.2

Show that if $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$

Let $c \in C \setminus B$. Then we follow that $c \in C$ or $c \notin B$. Since $A \subseteq B$, we follow that $c \notin B$ implies that $c \notin A$. Thus we follow that $c \in C \setminus B$ implies that $c \in C \setminus A$. Therefore $C \setminus B \subseteq C \setminus A$.

1.1.3

Suppose $A \setminus B \subseteq C$. Show that $A \setminus C \subseteq B$.

Suppose that $a \in A \setminus C$. Then we follow that $a \in A$ and $a \notin C$.

Given that $A \setminus B \subseteq C$ and $A \notin C$, we follow that $a \notin A \setminus B$. Thus $a \notin A$ or $a \in B$. Since $a \in A$, we follow that $a \in B$. Thus

$$a \in A \setminus C \to a \in B$$

$$A \setminus C \subseteq B$$

as desired.

1.1.4

Suppose $A \subseteq B$ and $A \subseteq C$. Show that $A \subseteq B \cap C$

Suppose that $a \in A$. Then we follow that $a \in B$ and $a \in C$. Thus $a \in B \cap C$. Therefore we follow that $A \subseteq B \cap C$.

1.1.5

Suppose $A \subseteq B$ and $B \cap C = \emptyset$. Show that $A \in B \setminus C$

Suppose that $a \in A$. Then we follow that $a \in B$ and since $B \cap C = \emptyset$, we follow that $a \notin C$. Thus $a \in B \setminus C$ by definition. Therefore $A \subseteq B \setminus C$.

1.1.6

Show that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$. Suppose that $a \in A \setminus (B \setminus C)$. Then we follow that $a \in A$ and $a \notin B \setminus C$. Thus $a \notin B$ and $a \in C$. Thus we follow that $a \in A \setminus B$ or $a \in C$. Thus $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$ as desired.

1.1.7

Let $P(x)$ be the property $x > \frac{1}{x}$. Are the assertions $P(2)$, $P(-2)$, $P(\frac{1}{2})$ $P(\frac{-1}{2})$ true or false .

$$2 > \frac{1}{2} \to P(2) = true$$
$$-2 < \frac{-1}{2} \to P(-2) = false$$

last two are reversed.

1.1.8

Sow that each of the following sets can be expressed as an interval

$$a)(-3,3)$$
$$b)(-3,\infty)$$
$$c)(-3,3)$$

all of them follow from order properties of real numbers.

1.1.9

Express the following sets as truth sets

$$A = \{1, 4, 9, 16, 25, ...\} \iff A = \{x \in N : x = y^2 \textit{ for some } y \in N\}$$
$$B = \{..., -15, -10, -5, 0, 5, ...\} \iff A = \{x \in N : x = 5y \textit{ for some } y \in N\}$$

Rest are also trivial, not gonna go deep here

1.2 Logical Notation

1.2.1

Using truth tables, show that $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$
false	false	true	false	true	false
false	true	true	false	false	false
true	false	false	true	true	true
true	true	true	false	false	false

from this we can see that they are equqivalent.
Following 4 exercises are the same as this one, so I'm skipping them

1.2.5

Show that $(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow P \Rightarrow (Q \wedge R)$*, using logic laws*

$$(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \Leftrightarrow \neg P \vee (R \wedge Q) \Leftrightarrow P \Rightarrow (R \wedge Q)$$

Laws used:

$$CL \rightarrow DIST \rightarrow CL$$

1.2.6

Show that $(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$*, using logic laws*

$$(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (\neg P \vee R) \vee (\neg Q \vee R) \Leftrightarrow \neg P \vee R \vee \neg Q \vee R \Leftrightarrow (\neg Q \vee \neg P) \vee R \Leftrightarrow$$
$$\Leftrightarrow \neg(Q \wedge P) \vee R \Leftrightarrow (Q \wedge R) \Rightarrow R$$

Laws used:

$$CL \rightarrow ASC \rightarrow ID, ASC \rightarrow DML \rightarrow CL$$

1.2.7

Show that $P \Rightarrow (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$*, using logic laws*

$$P \Rightarrow (Q \Rightarrow R) \Leftrightarrow \neg P \vee (Q \Rightarrow R) \Leftrightarrow \neg P \vee (\neg Q \vee R) \Leftrightarrow (\neg P \vee \neg Q) \vee R \Leftrightarrow \neg (P \wedge Q) \vee R \Leftrightarrow (P \wedge Q) \Rightarrow R$$

Laws used:

$$CL \rightarrow CL \rightarrow ASC \rightarrow DML \rightarrow CL$$

1.2.8

Show that $(P \Rightarrow Q) \Rightarrow R$ *and* $P \Rightarrow (Q \Rightarrow R)$ *are not logically equivalent*

We're gonna show that $q \wedge w \Leftrightarrow false$

$$((P \Rightarrow Q) \Rightarrow R) \wedge (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow (\neg(\neg P \vee Q) \vee R) \wedge (\neg P \vee (\neg Q \vee R)) \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge \neg Q) \vee R) \wedge (\neg P \vee \neg Q \vee R) \Leftrightarrow ((P \wedge Q) \wedge (\neg P \vee \neg Q)) \vee R \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge Q) \wedge \neg (P \wedge Q)) \vee R \Leftrightarrow false \vee R \Leftrightarrow false$$