My abstract algebra exercises

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Introduction to Groups

3.1 An Important Example

3.1.1

In S_4, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, *and* $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. *Calculate* $\sigma\tau$, $\tau\sigma$ *and* σ^{-1}.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

3.1.2

In S_5, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$ *and* $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$ *calculate* $\sigma\tau\sigma$, $\sigma\sigma\tau$, σ^{-1}.

$$\sigma\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

$$\sigma\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

3.1.3

How mamy permutations are there in S_n? In S_5, how many permutations α satisfy $\alpha(2) = 2$?

We can follow that there are $n!$ permutations total, and if we've got a restriction $\alpha(2) = 2$, then we've got $(n-1)!$ permutation. For the case S_5 it means that there are $4! = 24$ such permutations.

3.1.4

Let H be the set of all permutations $\alpha \in S_5$ satisfying $\alpha(2) = 2$. Which of the properties of closure, associativity, identit, inverses does H enjoy under composition?

All of them

3.1.5

Consider the set of all functions from 6 to 6. Which of the ...

Everything other then inverse

3.1.6

Let G be the set of all ...

All of them

3.2 Groups

3.2.1

Give group tables for following additive grops: Z_3, $Z_3 \times Z_2$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

last one is ommited

3.2.2

Give group tables for the following groups: $U(12), S_3$

We follow that $U(12) = \{1, 5, 7, 11\}$. THus

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

One of the programs in progs folder produces desired table for S_3 (and can produce one for any S_n for that matter).

3.2.3

Show that $G \times H$ is abelian iff G and H are both abelian

Was proven in dummit and foote, check 1.1.29

Rest of the exercises in this section were either already proven in D&F, are trivial, or could be solved at a later time if I encounter some gaps in the theory.

3.3

3.4

3.5

3.6 Cyclic Groups

3.6.1

Let $G = \langle a \rangle$ be a cyclic group of order 12. List every subgroup of G. List every group of Z_{12}

12's divisors are $\{1, 2, 3, 4, 6, 12\}$, therefore subgroups of G are $\langle a^i \rangle$ for $i \in \{0, 1, 2, 3, 4, 6\}$

Since Z_{12} is cyclic, we follow that $\langle [0, 1, 2, 3, 4, 6] \rangle$ are the subgroups of Z_{12}.

3.6.2

Let $G = \langle a \rangle$ be a cyclic group of order 120. List all of the groups of order 120. List all of the elements of order 12 in G.

Divisors of 120 are $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 24, 60, 120\}$, thus we can state that subgroups of a cyclic group are a to powers of those numbers

According to the theorems, there should be $\phi(12) = 4$ elements of order 12. All of them lie in a subgroup $\langle a^{120/12} \rangle = \langle a^{10} \rangle$ and are in form $(a^{10})^k$ where $k \in 1, 5, 7, 11$.

How many element of order 12 are there in a cyclic group of order 1200?

Also 4.

3.6.3

Let p be a prime and n a positive integer. Show that $\phi(p^n) = p^n - p^{n-1}$

If $j \in Z_+$ is such that $j = pi$ for some $i \in Z_+$, then we follow that $(p^n, j) = p$, therefore they are not relatively prime. Suppose that $(p^n, j) = 1$ for some $j \in Z_+$. Let S be a multiset of prime divisors of $p^n N$ and T be a multiset of divisors of j. Then we follow that $S \cap T = \emptyset$, since otherwise we would've had that j is a multiple of p, which is not relatively prime to p^n. Thus we follow that the set of not relatively prime numbers to p^n is equal to the set of multiples of p.

We can follow that there are pricicely p^{n-1} of multiples of p that are less or equal to p^n (don't think that we need to prove that), therefore the total amount of numbers that are less or equal to p^n, which are relatively prime to p^n is $p^n - p^{n-1}$, as desired.

3.6.4

Find all positive integers n such that $|U(n)| = 24$.

We can follow that $\phi(n)$ is an function that tends to infinity (i.e. for every $n \in Z_+$ there exists $j \in Z_+$ such that $m > n$ implies that $\phi(m) > j$) since $\phi(n)$ is larger than the number of prime numbers that is in the set $Z_+ \cap [1, n)$. Therefore we conclude that there is an upper bound for a number of numbers n such that $\phi(n) = 24$.

Brute-force shows that those numbers are

$$35, 39, 45, 52, 56, 70, 72, 78, 84, 90$$

Can't come up with a better answer than that, but I'm sure that it's there.

3.6.5

Let G be a nonabelian group. If H and K are cyclic subgroups of G, does it follow that $H \cap K$ is also a cyclyc subgroup? Prove that it does, or provide a counterexample.

We follow that every subgroup has an identity in it, thus $e \in H \cap K$. Suppose that $j \in H \cap K$. We follow that $j \in H \wedge j \in K$. Since H and K are both subgroups, we follow that $j^{-1} \in H \wedge j^{-1} \in K$. Thus $j^{-1} \in H \cap K$. Therefore $H \cap K$ is closed under inverses. We can follow also by the same logic that $j, l \in H \cap K$ implies that $jl \in H \cap K$. Therefore we can conclude that $H \cap K$ is a subgroup.

We can follow that if $H \cap K = \{e\}$, then it's cyclic. We can follow that $H \cap K$ can be not only a trivial subgroup by setting $H = K$. Suppose that $H \cap K \neq \{e\}$. By the fact that both H and K are cyclic we follow that $H \cap K = \{a^i : i \in \text{ some subset of } Z_+\}$. Since $H \cap K \neq \{e\}$, we follow that there exists an element $a \in G$ and two sets $H', K' \in \mathcal{P}(Z_+)$ such that $H = \{a^i : i \in H'\}$ and $K = \{a^i : i \in K'\}$. Since both H and K are cyclic we follow that both H' and K' are the sets of multiples of some number. Thus $H' \cap K'$ is a set of multiples of some number as well (proof ommited). Thus we follow that $H \cap K = \{a^i : i \in H' \cap K'\}$ is a cyclic group as well.

3.6.6

Let $G = \langle a \rangle$ be an infinite cyclic. If m and n are positive integers, find a generator for $\langle a^m \rangle \cap \langle a^n \rangle$.

We can follow pretty easily that $\langle a^m \rangle \cap \langle a^n \rangle = \langle a^{lcm(m,n)} \rangle$

3.6.7

Let n be a positive integer and let T be the set of positive integers that divide n. Show that $\sum_{k \in T} \phi(k) = n$.

For 12 we've got

$$T = \{1, 2, 3, 4, 6, 12\}$$

$$\phi(1) = 1$$

$$\phi(2) = 1$$

$$\phi(3) = 2$$

$$\phi(4) = 2$$

$$\phi(6) = 2$$

$$\phi(12) = 4$$

and we follow that result works.

3.7 Cosets and Lagrange's Theorem

3.7.1

For each group G and subgroup H, find all the left cosets and right cosets of H in G.

1. $G = Z, H = 4Z$.

We follow thaht $0 + H = 4Z = H$, $1 + H = \{1 + x : x \in Z\}$, and so on for $3 + H$. Since the group is abelian, we follow that right cosets are the same.

3.7.2

Let G be a group whose order is the product of two (not necessarily distinct) primes. Show that every proper subgroup of G is cyclic

We follow that order of any given proper subgroup is equal to one of those primes, or 1. This implies that this subgroup is cyclic, as desired.

3.7.3

Let G be a group of order p^n for some prime p and positive integer n. Show that G has an element of order p.

We follow that any proper subgroup is some power of p. By induction we can conclude that such an element exists.

3.7.4

Let G be a group having a subgroup H of order 28 and a subgroup K of order 65. Show that $H \cap K = \{e\}$.