

My algorithms exercises

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Preface

Exercises for Introduction to Algorithms by Cormen et al., 4th ed. It has exercises, that should be written down, mostly in math and whatnot.

Some of the exercises require that you code something (sometimes it's not explicitly required, but that would be nice to code it anyways), and this code is presented in the progs folder. Everything is written in C, because I'm most familiar with it.

Pseudocode is written by using package **algorithm2e**, which does not really correspond to the one, that is used in the book, but it still does the job.

Same rules as usual apply, if you want to use this book for any reason – go right ahead, it's free, just be aware that it is full of mistakes.

Part I

Foundations

Chapter 1

The Role of Algorithms in Computing

1.1 Algorithms

1.1.1

Describe your own real-world example that required sorting. Describe one that required finding the shortest distance between two points.

I've needed both when I was creating 8-puzzle program

1.1.2

Other than speed, what other measures of efficiency might you need to consider in a real-world setting?

Memory and parallelability.

1.1.3

Select a data structure that you've seen, and discuss its strengths and limitations.

Linked lists. They are perfect in everything, apart from sorting; but even then you can define any data structure through linked lists, which makes them just perfect (especially omnidirectional ones).

1.1.4

Suggest a real-world problem in which only the best solution will do. Then come up with one in which "approximately" the best solution is good enough.

Sorting has to be perfect, otherwise it's borderline useless. Estimated time to complete the task can tolerate imperfections.

1.1.5

Describe a real-world problem in which sometimes the entire input is available before you need to solve the problem, but other times the input is not entirely available in advance and arrives over time

Traffic on maps does this thing. Sometimes you have all the input, sometimes it changes.

1.2 Algorithms as a technology**1.2.1**

Give an example of an application that requires algorithmic content at the application level, and discuss the function of the algorithms involved.

Path finding on maps will do. It requires to traverse graphs and whatnot.

1.2.2

Suppose that for inputs of size n on a particular computer, insertion sort runs in $8n^2$ steps and merge sort runs in $64n \lg n$ steps. For which values of n does insertion sort beat merge sort?

For

$$8n^2 < 64n \lg n$$

$$n < 8 \lg n$$

$$\frac{n}{\lg n} < 8$$

$$n \approx 44$$

cases.

1.2.3

What is the smallest value of n such that an algorithm whose running time is $100n^2$ runs faster than an algorithm whose running time is 2^n on the same machine?

Calculator says 15

Chapter 2

Getting Started

2.1 Insertion sort

2.1.1

Using Figure 2.2 as a model, illustrate the operation of Insertion-Sort on an array initially containing the sequence [31, 41, 59, 26, 41, 58]

[31, 41, 59, 26, 41, 58]

[26, 31, 41, 59, 41, 58]

[26, 31, 41, 41, 59, 58]

[26, 31, 41, 41, 58, 59]

2.1.2

State loop invariant for the Sum-Array procedure.

At the start of each iteration of the for loop, the *sum* variable contains the sum of elements in $A[1 : i]$.

Initialization:

Firstly, we've got 0 as the sum. Given that we've summed 0 elements so far, we can conclude that this is indeed a correct value to set it.

Maintenance:

For each iteration of i we've got that we add a i 'th element from the array to our sum and incrementing i . Thus before iterating through i we had a sum of all of the elements before i , and after iterating through it we create a sum of elements before i and the i 'th element as well. Thus the sum after iterating through i is correct.

Termination:

Given that the array is finite, we follow that because we are incrementing i at each iteration the algorithm will terminate. Because we increment through elements, we follow that we've added every element of the array to the sum at the point of termination.

2.1.3

Rewrite the Insertion-Sort procedure to sort into monotonically decreasing instead of monotonically increasing order.

Done it in the progs section; long story short: reverse the ordering function in the inner loop, replace $A[j] > key$ with $A[j] < key$.

2.1.4

Consider the searching problem

Input: A sequence of n numbers $[a_1, \dots, a_n]$ stored in array $A[1 : n]$ and a value x .

Output: An index i such that x equals $A[i]$ or the special value NIL if x does not appear in A .

Write pseudocode for linear search, which scans through the array from beginning to end, looking for x . Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the three necessary properties.

Function Linear-search(A, x)

```

1 for ( $i = 1 \rightarrow n$ ) do
2   | if  $A[i] = x$  then
3   |   | return  $i$  ;
4   | end
5 end
6 return  $NIL$ ;

```

At each iteration of the for loop, we've got that elements $A[1 : i]$ do not contain x .

Initialization: Null case is when we haven't gone through any of the elements yet; because we haven't searched anything, we can follow that we haven't found anything, thus the base case is correct

Maintenance: Suppose that $i = j + 1$. Then we follow that $A[1 : j]$ does not contain our element by our induction hypothesis (not sure that we can use this kind of language here, but it's my book and I can do whatever I want). Then i 'th element is checked for the necessary equality and returned in case of the equality; otherwise we increment i and make it so $A[1 : j + 1]$ is the array of processed values. Thus before and after the loop iteration we have a correct solution.

Termination: We terminate either after going through every element, or some time before it.

Example of this thing in C is presented in progs directory

2.1.5

Consider the problem of adding two n -bit binary integers a and b , stored in two n -element arrays $A[0 : n - 1]$ and $B[0 : n - 1]$, where each element is either 0 or 1, $a = \sum_{i=0}^{n-1} A[i] * 2^i$ and $b = \sum_{i=0}^{n-1} B[i] * 2^i$. The sum $c = a + b$ of the two integers should be stored in binary form in an $(n + 1)$ -element array $C[0 : n]$, where $c = \sum_{i=0}^{n-1} C[i] * 2^i$. Write a procedure *Add-Binary-Integers* that takes an input arrays A and B , along with the length n , and returns array C holding the sum.

```

Function Add-Binary-Integers(A, B, n)


---


1  define C[0: n + 1] and fill it with zeroes;
   /* carry stores overflow from the previous iteration */
2  carry  $\leftarrow$  0;
3  for ( $i = 0 \rightarrow n + 1$ ) do
   /* Initializing temporary variable with carry bit; we need to sum
      carry, A[i], and B[i], so we can just initialize temp with
      carry */
4  r  $\leftarrow$  carry ;
   /* if it is not the last bit, where A nor B are not defined; We
      can just define it to be zero with the same result */
5  if  $i \neq n + 1$  then
6  |   r  $\leftarrow$  A[i] + B[i] + r ;
7  end
   /* If we've got an overflow as the result, set carry bit and
      result of summation appropriately */
8  if  $r > 1$  then
9  |   carry  $\leftarrow$  1 ;
10 |   r  $\leftarrow$  r%2;
11 else
   /* otherwise zero the carry bit */
12 |   carry  $\leftarrow$  0;
13 end
   /* lastly, put the result of the partial summation into the
      resulting array */
14 C[i]  $\leftarrow$  r;
15 end
16 return C;


---



```

2.2 Analyzing algorithms

2.2.1

Express the function $n^3/1000 + 100n^2 - 100n + 3$ in terms of Θ -notation

$$n^3/1000 + 100n^2 - 100n + 3 \in \Theta(n^3)$$

2.2.2

Consider sorting n numbers stored in array $A[1 : n]$ by first finding the smallest element of $A[1 : n]$ and exchanging it with the element in $A[1]$. Continue in this manner for the first $n - 1$ elements of A . Write a pseudocode for this algorithm, which is known as selection sort. What loop invariant does this algorithm maintain? Why does it need to run for only the first $n - 1$ elements, rather than for all n elements? Give the worst-case running time for selection sort in Θ -notation. Is the best-case running time any better?

Function Selection-Sort(A, n)

```

1 for  $i = 1 \rightarrow n - 1$  do
2    $minPos \leftarrow i$ ;
3   for  $j = i + 1 \rightarrow n$  do
4     if  $A[j] < A[i]$  then
5        $minPos \leftarrow j$ 
6     end
7   end
8    $temp \leftarrow A[i]$ ;
9    $A[i] \leftarrow A[minPos]$ ;
10   $A[minPos] \leftarrow temp$ ;
11 end

```

(this function in C is located in progs)

Algorithm maintains that every $A[1 : i]$ is sorted and any element in $A[i + 1 : n]$ is greater or equal to every element of $A[1 : i]$. It needs to run only for $n - 1$ times, because the last element n will be greater or equal than any element in $A[1 : n - 1]$ and therefore will be places in its rightful place.

Worst-case running time of this algorithm is $\Theta(n^2)$, because we compute that coefficient for c_4 is $\sum_{i=1}^n i = \Theta(n^2)$, while all the other coefficients are linear.

Given coefficient for c_4 is not changed depending on the input, we follow that the best running time and worst running time are the same

2.2.3

Consider linear search again. How many elements of the input array need to be checked on the average, assuming that the element being searched for is equally likely to be any element in the array? How about the worst case? Using Θ -notation, give the average-case and worst-case running times of linear search. Justify your answers.

The low-hanging answer here is that we need to check the first half of the array in order to find the needed input. It is wrong though.

Assuming that $A[i] = x$ with probability p , we follow that we need to check first $1/p$ elements in order to get the desired element (I can be wrong here, I'm not good with probabilities).

Assuming that I'm right about it, we follow that $pn \in \Theta(n)$ for the average case.

The worst case is if A does not contain x , in which case we've got the running time of n . It is also in $\Theta(n)$.

2.2.4

How can you modify any sorting algorithm to have a good best-case running time?

We can throw a check for the case if A is sorted from the start in the start of the sorting algorithm. Thus we can follow, that the best case for any such algorithm is when we get the sorted input, and we get linear best-case running time.

2.3 Designing algorithms**2.3.1**

Using Figure 2.4 as a model, illustrate the operation of merge sort on an array initially containing the sequence

[3, 41, 52, 26, 38, 57, 9, 49]

[3, 41, 52, 26][38, 57, 9, 49]

[3, 41][52, 26][38, 57][9, 49]

[3][41][52][26][38][57][9][49]

[3, 41][26, 52][38, 57][9, 49]

[3, 26, 41, 52][9, 38, 49, 57]

[3, 9, 26, 38, 41, 49, 52, 57]

2.3.2

The test in line 1 of the Merge-Sort procedure reads "if $p \geq r$ " rather than "if $p \neq r$ ". If Merge-Sort is called with $p > r$, then the subarray $A[p : r]$ is empty. Argue that as long as the initial call of Merge-Sort($A, 1, n$) has $n \geq 1$, the test "if $p \neq r$ " suffices to ensure that no recursive call has $p > r$.

For a general case when $p \leq r$ we've got that

$$p = 2p/p = (p + p)/2 \leq \lfloor (p + r)/2 \rfloor = q$$

thus $p \leq q$ and call on line 4 will be done with the condition that $p \leq q$.

Focusing our attention on r , we get that by condition on lines 1 and 2 we follow that the function returns if $p = r$. Thus we follow that $p \neq r$, and by our assumption we get that by the line 3 we get if $p < r$. Given that both p and r are integers, we follow that

$$(p + r)/2 \leq (p + p + 1)/2 = p + 1/2$$

thus

$$\lfloor (p + r)/2 \rfloor \leq \lfloor p + 1/2 \rfloor = p$$

Therefore we follow that

$$q \leq r - 1$$

$$q + 1 \leq r$$

thus we can follow that the call to the Merge-Sort on line 5 happens also with the condition that $p \leq r$.

Thus we can state that given that $p \leq r$ we can follow that all the other calls to Merge-Sort, that happen inside of it will have the same restriction. Therefore we follow that if initial call to Merge-Sort happens with $n \geq 1$, then the test "if $p \geq r$ " is equivalent to the test "if $p \neq r$ ".

2.3.3

State a loop invariant for the while loop of lines 12-18 of the Merge procedure. Show how to use it, along with the while loops 20-23 and 24-27, to prove that the Merge procedure is correct.

At the start of each iteration of the while loop of lines 12-18, the subarray $A[1 : k]$ consists of the elements, originally in $L[1 : i]$ or $R[1 : j]$ in sorted order and any element in $A[1 : k]$ is less or equal to any element of $L[i + 1, n_L]$ and $R[i + 1, n_R]$.

Base case is trivial, since none of the subarrays contain any elements.

For maintenance we've got that we take the lowest element of $L[i : n_L]$ or $R[j : n_R]$ (which happens to be the lowest of the first elements of those subarrays, given that both of those arrays are sorted), and placing them at $A[k]$. Because of our restrictions on $L[i + 1, n_L]$

and $R[+1, n_R]$ we follow that $A[k]$ will be less or equal then any element of $L[i + 1, n_L]$ and $R[j + 1, n_R]$. And given that it is originally from $L[i + 1, n_L]$ or $R[i + 1, n_R]$, we follow that $A[k]$ is greater or equal then any of $A[1 : k - 1]$. Thus we can follow that $A[1 : k]$ is sorted. Thus at the end of the loop we've got our loop invariant.

Termination happens whenever we run out of elements in L or R , at which point we still have our invariant.

We follow then that after our loop, one of the arrays L or R will be empty, and will contain elements, that are greater or equal then $A[1 : k]$. Thus we just append remaining elements of either of the arrays to A .

Given that L and R are two subarrays of $A[k]$, whose disjoint union is A , we follow that merge indeed merges two subarrays of $A[k]$ into one sorted array, as desired.

2.3.4

Use mathematical induction to show that when $n \geq 2$ is an exact power of 2, the solution of the recurrence

$$T(n) = \begin{cases} 2 & \text{if } n = 2 \\ 2T(n/2) + n & \text{if } n > 2 \end{cases}$$

is $T(n) = n \lg(n)$.

Our domain here is $(n_k) = 2^k$

For base case $k = 1 \rightarrow n = 2$ we've got

$$T(2) = 2 = 2 * 1 = 2 \lg(2)$$

Our hypothesis is that for $k - 1 < n$ and n_{k-1} we've got that $T(n_{k-1}) = n_{k-1} \lg(n_{k-1})$

Thus we follow that if n is an exact power of 2 and $n \geq 2 > 0$ we've got that $n/2 < n$. Thus we follow that

$$\begin{aligned} T(n) &= 2T(n/2) + n = 2n/2 \lg(n/2) = n \lg(n/2) + n = n(\lg(n/2) + 1) = \\ &= n(\lg(n * 2^{-1}) + 1) = n(\lg(n) - 1 + 1) = n \lg(n) \end{aligned}$$

Therefore for $x \in (n_k)$ we've got that $T(x) = x \lg(x)$, as desired.

2.3.5

You can also think of insertion sort as a recursive algorithm. In order to sort $A[1 : n]$, recursively sort the subarray $A[1 : n - 1]$ and then insert $A[n]$ into the sorted subarray $A[1 : n - 1]$. Write a pseudocode for this recursive version of insertion sort. Give a recurrence for its worst-case running time.

For the worst-case we've got

$$T(n) = T(n - 1) + n$$

```

Function Recursive-insertion-Sort(A, n)
  /* Ensure termination and sane inputs */
  1 if  $n \leq 1$  then
  2   | return;
  3 end
  /* Sort everything in A[1:n - 1] */
  4 Recursive-insertion-Sort (A, n - 1);
  5  $i \leftarrow n - 1$  ;
  6 while  $A[i] > A[n]$  and  $i \geq 1$  do
  7   |  $i \leftarrow i - 1$ 
  8 end
  /* Swapping  $A[n]$  and  $A[i]$  */
  9  $temp \leftarrow A[i]$  ;
 10  $A[i] \leftarrow A[n]$  ;
 11  $A[n] \leftarrow temp$  ;

```

which reduces to

$$T(n) = n(n + 1)/2$$

2.3.6

Referring back to the searching problem (see Exercise 2.1-4), observe that if the subarray being searched is already sorted, the searching algorithm can check the midpoint of the subarray against v and eliminate half of the subarray from further consideration. The binary search algorithm repeats this procedure, halving the size of the remaining portion of the subarray each time. Write pseudocode, either iterative or recursive, for binary search. Argue that the worst-case running time of binary search is $\Theta(\lg(n))$

For this one we conclude that its worst-case is $\Theta(\lg n)$ by the same logic as in merge, but we go through only one branch.

2.3.7

The while loop of lines 5-7 of the Insertion-Sort procedure in Section 2.1 uses a linear search to scan (backward) through the sorted subarray $A[1 : j - 1]$. What if insertion sort used a binary search (see Exercise 2.3-6) instead of a linear search? Would that improve the overall worst-case running time of insertion sort to $\Theta(n \lg(n))$?

If we use binary search instead of the linear search, then we'll undoubtedly have a better running time on a real machine, given that the the binary search uses less instructions than the linear search.

```

Function Binary-Search( $A, x, s, f$ )
1 if  $s = f$  then
2   | return  $NIL$ ;
3 end
4  $q \leftarrow \lfloor (s + f)/2 \rfloor$ ;
5 if  $A[q] = x$  then
6   | return  $q$ ;
7 else
8   | if  $A[q] < x$  then
9     | return  $\text{Binary-Search}(A, x, q + 1, f)$  ;
10  | else
11    | return  $\text{Binary-Search}(A, x, s, q - 1)$  ;
12  | end
13 end

```

But in order to put the required value in the needed place, we'll still have to go through the entire array in a worst-case scenario (array sorted in reverse order). Thus it can be argued that the overall asymptotic approximation will be the same.

2.3.8

Describe an algorithm that, given a set S of n integers and another integer x , determines whether S contains two elements that sum exactly to x . Your algorithm should take $\Theta(n \lg n)$ time in the worst case.

The obvious case with just going through the whole array will not do, as it will give us worst-case scenario of $\Theta(n^2)$.

So the main strategy here will be to sort the whole array and then make a linear search for the desired values.

Thus we'll get something that looks like Sum-of-ints

The last while iterates once through the array, therefore we can (non-rigorously) follow that everything except for the search runs at $\Theta(n)$. Thus the whole thing goes through the array at time $\Theta(n \lg(n))$ (or at the very least I think so; once again we've got non-rigorous approach here, but I'm sure that in subsequent chapters we'll have some more sophisticated tools at our hands to prove me right).

Function Sum-of-ints(A, n, x, a, b)

```
1 Merge-Sort(A) ;
2  $i \leftarrow 1$ ;
3  $j \leftarrow n$ ;
4 while  $i \neq j$  do
5   if  $A[i] + A[j] > x$  then
6      $j \leftarrow j - 1$  ;
7   else
8     if  $A[i] + A[j] > x$  then
9        $i \leftarrow i + 1$  ;
10    else
11      return  $[A[i], A[j]]$ ;
12    end
13  end
14 end
```

Part II

Appendix: Mathematical Background

Chapter 3

Summations

1-1

Prove that $\sum_{k=1}^n O(f_k(i)) = O(\sum_{k=1}^n f_k(i))$

Short answer:

$$\sum cg(x) = c \sum g(x)$$

Long answer:

Suppose that $g \in O(f_k(i))$. It follows that there exists n_i and c_i such that $0 \leq g(n) \leq cf_i(n)$. Thus we can pick $n = \max\{n_0, n_1, \dots\}$ and $c = \max\{c_0, c_1, \dots\}$. We know that both n and c will work all of functions f_k . Therefore by linearity of summations

$$\sum_{k=1}^n O(f_k(i)) = \sum_{k=1}^n cf_k(i) == c \sum_{k=1}^n f_k(i) == O(\sum_{k=1}^n f_k(i))$$

(notation is a little abused and there is nothing is rigorously proven, but it'll do).

1-2

Find a simple formula for $\sum_{k=1}^n (2k - 1)$.

$$\sum_{k=1}^n (2k - 1) = \sum_{k=1}^n (2k) - \sum_{k=1}^n (1) = 2 \sum_{k=1}^n (k) - n = 2 \frac{n(n+1)}{2} - n = n(n+1) - n = n^2$$

1-3

Interpret the decimal number 111,111,111 in light of equation A.6

$$111, 111, 111 = \sum_{k=0}^9 10^k = \frac{10^{10} - 1}{10 - 1}$$

1-4

Evaluate the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$

The series converges absolutely to 2, so we are free to do anything with it.

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots &= \sum_{k=0}^{\infty} \frac{1^{2k}}{2} - \sum_{k=0}^{\infty} \frac{1^{1+2k}}{2} = \sum_{k=0}^{\infty} \frac{1^{2k}}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{1^{2k}}{2} = \left(1 - \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{1^{2k}}{2} = \\ &= \left(1 - \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{1^k}{4} = \left(1 - \frac{1}{2}\right) \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} * \frac{4}{3} = \frac{2}{3} \end{aligned}$$

1-5

Let $c \geq 0$ be a constant. Show that $\sum_{k=1}^n k^c = \Theta(n^{c+1})$

$$\sum_{k=1}^n k^c = \sum_{k=1}^{n-1} k^c + n^c = n^c \sum_{k=1}^n \frac{k^c}{n^c} =$$

Let $f(n) = n^c$. It can be seen from the graph that

$$\int_0^n f(x)dx \leq \sum_{k=1}^n k^c \leq \int_0^n f(x+1)dx$$

Thus

$$\begin{aligned} \int_0^n f(x)dx &= \int_0^n x^c = \frac{n^{c+1}}{c+1} \in \\ \int_0^n f(x+1)dx &= \int_0^n (x+1)^c = \frac{(n+1)^{c+1}}{c+1} \end{aligned}$$

Thus we can state that $\sum_{k=1}^n k^c = \Theta(n^{c+1})$ (I'm not good enough yet to show that $\frac{(n+1)^{c+1}}{c+1} \in \Theta(n^{c+1})$, but I'm pretty sure that it's true TODO).

1-6

Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $|x| < 1$

We know that for $|x| < 1$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

thus if we differentiate both sides we get

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

and then if we multiply all of it by x we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

thus if we factor all of this jazz we'll get

$$\sum_{k=0}^{\infty} k^2 x^k = -\frac{x(x+1)}{(x-1)^3}$$

and if we tuck this minus into denominator we'll get (which we can do because the power is odd)

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(x+1)}{(1-x)^3}$$

as desired.

1-7

Prove that $\sum_{k=1}^n \sqrt{k \lg k} = \Theta(n^{3/2} \lg^{1/2} n)$

$$\int \sqrt{k \lg k} =$$

1-9

Show that

$$\sum_{k=0}^{\infty} (k-1)/2^k = 0$$

$$\sum_{k=0}^{\infty} (k-1)/2^k = \sum_{k=0}^{\infty} k/2^k - \sum_{k=0}^{\infty} 1/2^k = \sum_{k=0}^{\infty} k/2^k - 2 = 0$$

$$\sum_{k=0}^{\infty} k/2^k - 2 = 0$$

$$\sum_{k=0}^{\infty} k/2^k = 2$$

$$\sum_{k=0}^{\infty} k/2^k - 2 = 0$$

$$\sum_{k=0}^{\infty} \frac{k - 2^{k+1}}{2^k} = 0$$

Chapter 4

Sets, Etc.

1-1

Draw Venn diagrams that illustrate the first of the distributive laws (B.1)

TODO, add picture here

1-2

Prove the generalization of DeMorgan's laws to any finite collection of sets

Copy from real analysis exercises

Suppose that $x \in (\cup_{\lambda \in \Lambda} E_{\lambda})^c$. It follows, that x is not in the union of given sets. Therefore there is no set E_n such that $x \in E_n$ (because if there would be such a set, then x wouldn't be in $(\cup_{\lambda \in \Lambda} E_{\lambda})^c$). Therefore $x \in \cap_{\lambda \in \Lambda} E_{\lambda}^c$. Therefore

$$(\cup_{\lambda \in \Lambda} E_{\lambda})^c \subseteq \cap_{\lambda \in \Lambda} E_{\lambda}^c$$

The proof of reverse inclusion is the same as with the forward, but in reverse order.

$x \in (\cap_{\lambda \in \Lambda} E_{\lambda})^c$ implies that x is not in every E_n . Therefore there exists $x \in E_n^c$ for some E_n . therefore it is in $\cup_{\lambda \in \Lambda} E_{\lambda}^c$. The proof of reverse inclusion uses the same argument, but in other direction.

1-3

TODO

1-4

Show that the set of odd natural numbers is countable.

Let us set a function $f : A \rightarrow N$, where A denotes the set of odd natural numbers

$$f(n) = (n + 1)/2$$

for this function we've got

$$f^{-1}(n) = 2n - 1$$

Both functions are injective and therefore f is bijective. Therefore we've got a bijective function between A and N , therefore $A \sim N$, therefore it is countable, as desired.

1-5

Show that for any finite set S , the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

Another copy from real analysis

This proof is dumb, but intuitive:

Every subset is corresponding to a number in binary system: 0 for excluded, 1 for included. Therefore there exist 2^n possible combinations.

For a more concrete proof let's resort to induction.

Base case(s): subsets of \emptyset are \emptyset itself ($2^0 = 1$ in total). Subsets of set with one element are \emptyset and set itself ($2^1 = 2$ in total).

Proposition is that set with n elements has 2^n subsets.

Inductive step is that for set with $n+1$ elements can either have or not have the $n+1$ 'th element. Therefore there exist $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets, as desired.

1-6

Give an inductive definition for an n -tuple by extending the set-theoretic definition for an ordered pair.

The tuple is actually just a re-writing of particular set

$$(a_1, a_2, \dots, a_n) = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\} \dots \{a_1, a_2, a_3, \dots, a_n\}\}$$