My real analysis exercises

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Exercises are from UTM-040 Understanding analysis by Stephen Abbott. Edition is unknown, but the date in the preface is August 2000.

4.4.1

a

Show that $f(x) = x^3$ is continuous on all of $\boldsymbol{R}$.

In order to show, that f is continous we need to show, that $\forall \epsilon \in \mathbf{R}\ \exists \delta$ s.t.

$$|x - c| < \delta \to |f(x) - f(c)| < \epsilon$$

Let's rewrite the first formula

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + cx + c^2)| = |x - c||x^2 + cx + c^2|$$

We can put $|x - c|$ can be as small as we want it to be. Therefore we need an upper bound for $|x^2 + cx + c^2|$.

$$|x^2 + cx + c^2| \leq |x^2| + |cx| + |c^2| \leq (|c| + 1)^2 + |c|(|c| + 1) + |c|^2$$

Therefore if we take $\delta = min\{1, \epsilon/((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)\}$ then

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2| \leq \epsilon \frac{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2)}{((|c| + 1)^2 + |c|(|c| + 1) + |c|^2}) = \epsilon$$

Therefore $f(x) = x^3$ is continous on $\mathbf{R}$.

(b)

Argue, using Theorem 4.4.6, that f is not uniformly continuous on $\boldsymbol{R}$

Theorem 4.4.6 (Sequential Criterion for Nonuniform Continuity). A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if $\exists \epsilon > 0$ and two sequences (x_n) and (y_n) in A satisfying

$|x_n - y_n| \to 0$ *but* $|f(x_n) - f(y_n) \leq \epsilon_0$

In order to show that $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$ let us use sequences

$$x_n = n$$

$$y_n = (n + 1/n)$$

Firstly

$$|x_n - y_n| = |n - n - 1/n| = |-1/n| = 1/n \to 0$$

on the other hand

$$|f(x_n) - f(y_n)| = |n^3 - (n+1/n)^3| = |n^3 - (n^3 + 3\frac{n^2}{n} + 3\frac{n}{n^2} + \frac{1}{n^3})| =$$

$$= |-3n - \frac{3}{n} - \frac{1}{n^3}| \leq |3n| \to \infty$$

rmaxima seems to eraborate this statement, therefore $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n) \to \infty$

Therefore $f(x) = x^3$ is not uniformly continous on $\mathbf{R}$.

(c)

Show that f is uniformly continuous on any bounded subset of $\boldsymbol{R}$.

Suppose that $A \subset \mathbf{R}$ and $\exists M \in \mathbf{R}$ s.t. $\forall x \in A$ $x \leq M$ (i.e. A is bounded M)

Then, $\forall c \in A$ and $\forall \epsilon \in \mathbf{R}$

$$\frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2} \leq \frac{\epsilon}{((|c|+1)^2 + |c|(|c|+1) + |c|^2}$$

Therefore if we take

$$\delta = min\{1, \frac{\epsilon}{((|M|+1)^2 + |M|(|M|+1) + |M|^2}\}$$

then $|x - c| < \delta$ implies, that $|f(x) - f(c)| < \epsilon$, therefore making $f(x)$ uniformly continous by definition

4.4.2

Show that $f(x) = 1/x^3$ is uniformly continous on the set $[1, \infty)$, but is not on the set $(0, 1]$

In order to show, that $f(x)$ is continous on the set $[1, \infty)$ let us first prove that it is just continous, with the hope that δ is not depentant on x

$$|\frac{1}{x^3} - \frac{1}{c^3}| = |\frac{c^3 - x^3}{x^3c^3}| = |\frac{(c-x)(x^2+cx+c^2)}{x^3c^3}| = |(c-x)\frac{x^2+cx+c^2}{x^3c^3}| = |c-x||\frac{x^2+cx+c^2}{x^3c^3}| =$$

$$= |x - c||\frac{x^2+cx+c^2}{x^3c^3}|$$

Therefore we need to show that if δ is bounded above at 1, then $|\frac{x^2+cx+c^2}{x^3c^3}|$ is bounded above at $[1, \infty)$ by some constant, but $(0, 1]$ isn't.

$$|\frac{x^2+cx+c^2}{x^3c^3}| = |\frac{1}{c^3x} + \frac{1}{c^2x^2} + \frac{1}{cx^3}| \leq |\frac{1}{c^3x}| + |\frac{1}{c^2x^2}| + |\frac{1}{cx^3}|$$

for $x \in [1, \infty)$ each of those fractions are bounded above by 1, therefore for $x \in [1, \infty)$

$$|\frac{x^2 + cx + c^2}{x^3 c^3}| \leq 3$$

therefore if we pick $\delta < \epsilon/3$ then it follows, that $|f(x) - f(c)| < \epsilon$ for $x \in [1, \infty)$
on the other hand,

$$\lim_{x \to 0}(|\frac{x^2 + cx + c^2}{x^3 c^3}|) \to \infty$$

Therefore we will need smaller deltas as we approach 0; to put it more concretely let's use the theorem for **Sequential Criterion for Nonuniform Continuity**.

Let us pick

$$x_n = 1/n$$

$$y_n = 1/(n+1)$$

then

$$|x_n - y_n| = |\frac{1}{n} - \frac{1}{n+1}| = |\frac{n+1-n}{n(n+1)}| = |\frac{1}{n^2+1}| \to 0$$

but

$$|f(x_n) - f(y_n)| = |1/(\frac{1}{n})^3 - 1/(\frac{1}{n+1})^3| = |1/(\frac{1}{n^3}) - 1/(\frac{1}{(n+1)^3})| = |n^3 - (n+1)^3| =$$

$$= |n^3 - (n^3 + 3n^2 + 3n + 1)| = |3n^2 + 3n + 1| \to \infty$$

therefore by **4.4.6** $f(x)$ is not uniformly continous on $(0, 1]$, as desired

4.4.3

Furnish the details (including an argument for Exercise 3.3.1 if it is not already done) for the proof of the Extreme Value Theorem (Theorem 4.4.3).

Let us first complete 3.3.1

Exercise 3.3.1. Show that if K is compact, then sup K and inf K both exist and are elements of K.

Because K is compact, it is both closed and bound; therefore, because it is bounded,

$$\exists M \in \mathbf{R} > 0 : \forall x \in K$$

$$|x| \leq M$$

Therefore there exist lower and upper bound of K. Therefore, by axiom of completenss, there exist both $sup(K)$ and $inf(K)$ (i.e. both least upper bound and greatest lower bound)

Now let's prove that there exists a sequence that converges to either $sup(k)$ or $inf(k)$.

To be continued...

4.2.1

Use Definition 4.2.1 to supply a proof for the following limit statements.

(a) $\lim_{x\to 2}(2x+4) = 8$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2} x^3 = 8$.

(d) $lim_{x\to\pi}[[x]] = 3$, where [[x]] denotes the greatest integer less than or equal to x.

Let's first state Definition 4.2.1

Definition 4.2.1. Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(a):

$$|f(x) - L| = |2x + 4 - 8| = |2x - 4| = 2|x - 2| < \epsilon$$

$$|x - 2| < \frac{\epsilon}{2}$$

$$\delta = \frac{\epsilon}{2} \to |2x + 4 - 8| < \epsilon$$

as desired.

(b):

$$|f(x) - L| = |x^3 - 0| = |x^3| = |x|^3 < \epsilon$$

$$|x| < \sqrt[3]{\epsilon}$$

$$\delta = \sqrt[3]{\epsilon} \to |x^3| < \epsilon$$

as desired.

(c):

$$|f(x) - L| = |x^3 - 8| = |(x - 2)(x^2 + 2x + 4)| = |x - 2||x^2 + 2x + 4| < \epsilon$$

$$|x - 2| < \frac{\epsilon}{|x^2 + 2x + 4|}$$

Suppose that we set the maximum delta at 1; then upper bound for $|x^2 + 2x + 4|$ is:

$$|x^2 + 2x + 4| \le |x^2| + |2x| + 4 = |x|^2 + 2|x| + 4 \le (|c| + 1)^2 + 2(|c| + 1) + 4 =$$

$$= (2+1)^2 + 2(2+1) + 4 = 9 + 6 + 4 = 19$$

Therefore

$$\delta = min\{1, \epsilon/19\} \to |x^3 - 8| = |x-2||x^2+2x+4| < \frac{\epsilon}{19} * 19 = \epsilon$$

as desired.

(d):

$$|[[x]] - 3| = [[0.1415926...]] = 0 < \epsilon$$

Suppose that we pick $\delta = 0.1$, then any $x \in V_\delta$ will satisfy $|[[x]] - 3| = 0 < \epsilon$ for any $\epsilon > 0$ as desired.

4.2.2

Assume a particular $\delta > 0$ has been constructed as a suitable response to a particular ϵ challenge. Then, any larger/smaller (pick one) δ will also suffice.

Smaller. This follows from the fact, that

$$\delta_1 < \delta_2 \to V_{\delta_1} \subset V_{\delta_2}$$

4.2.3

Use Corollary 4.2.5 to show that each of the following limits does not exist.

(a) $\lim_{x\to 0} |x|/x$

(b) $\lim_{x\to 1} g(x)$ where g is Dirichlet's function from Section 4.1.

I'll not state corollary 4.2.5 function here, because it's tedious, but it'll be obvious which corollary I'm talking about by the context.

(a): let

$$(x_n) = 1/n$$

$$(y_n) = -1/n$$

then

$$(x_n) \to 0; (y_n) \to 0$$

but

$$|x_n|/x_n = 1$$

$$|y_n|/y_n = -1$$

therefore the limit does not exist.

(b):

The Dirichlet function is

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases} \tag{1}$$

let

$$(x_n) = 2/n + 1$$
$$(y_n) = \sqrt{2}/n + 1$$

then

$$(x_n) \to 1; (y_n) \to 1$$

but

$$(x_n) = 2/n + 1 \in \mathbf{Q}$$
$$(y_n) = \sqrt{2}/n + 1 \notin \mathbf{Q}$$

therefore

$$D(x_n) = 1$$
$$D(y_n) = 0$$

thus the function is not continous at 1

4.2.4

Review the definition of Thomae's function t(x) from Section 4.1.

(a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.

(c) Make an educated conjecture for $\lim_{x \to 1} t(x)$, and use Definition 4.2.1B to verify the claim. Given $\epsilon > 0$, consider the set of points $\{x \in \boldsymbol{R} : t(x)\epsilon\}$. Argue that all the points in this set are isolated.

The definition of Thomae function is

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases} \tag{2}$$

(a): Let our three sequences be

$$(x_n) = n/(n+1)$$

$$(y_n) = (n+1)/n$$
$$(z_n) = \sum_{i=1}^{n} \frac{1}{2^n}$$

(b):

$$t(x_n) = \{1/2, 1/3, 1/4, 1/5, 1/6, 1/7...\}$$
$$t(y_n) = \{1, 1/2, 1/3, 1/4, 1/5, 1/6...\}$$
$$t(z_n) = \{1/2, 1/4, 1/8, 1/16...\}$$

(c): The educated conjecture here is that $\lim_{x \to 1} t(x) = 0$

In order to prove that conjecture author suggests, that we use $\epsilon - \delta$ definition. Let's try it;

$$|t(x)| < \epsilon$$

For all $\epsilon \in \mathbf{R} > 0$

Therefore by archimedes property there exists a number $n \in \mathbf{N}$ s.t. $\frac{1}{n} < \epsilon$. Thus suppose that we have $\delta = 1/n$. Then our proposition is that

$$\forall b \in (1 - 1/n; 1 + 1/n) \to |t(b)| < \epsilon$$

If $b \notin \mathbf{Q}$ then $t(b) = 0$ and therefore $|t(b)| < \epsilon$; therefore we need to prove, that any number $b = m_1/n_1 \in (1-1/n; 1+1/n) \cap \mathbf{Q}$ is such, that $|t(b)| = 1/n_1 < 1/n$. Also suppose $m_1 = n_1 + k$ (it's worth noting that in this case $k \in \mathbf{Z}$); then

$$1 - \frac{1}{n} < \frac{m_1}{n_1} < 1 + \frac{1}{n}$$
$$1 - \frac{1}{n} < \frac{n_1 + k}{n_1} < 1 + \frac{1}{n}$$
$$1 - \frac{1}{n} < 1 + \frac{k}{n_1} < 1 + \frac{1}{n}$$
$$-\frac{1}{n} < \frac{k}{n_1} < \frac{1}{n}$$
$$|\frac{k}{n_1}| < \frac{1}{n}$$
$$|k||\frac{1}{n_1}| = |k||t(\frac{1}{n_1})| < \frac{1}{n}$$

therefore because $k \in \mathbf{Z}$

$$|t(\frac{1}{n_1})| = |\frac{1}{n_1}| < \frac{1}{n|k|} < \frac{1}{n}$$

thus for each $\epsilon > 0$ we can find a corresponding $\delta > 0$ as desired.

4.2.5

Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

(*ii*) $\lim_{x\to c}[f(x) + g(x)] = L + M$

(*iii*) $\lim_{x\to c}[f(x)g(x)] = LM$

(a) *Supply the details for how Corollary 4.2.4 part (ii) follows from the sequential criterion for functional limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.*

From the algebraic limit theorem we know, that if $(a_n) \to a$ and $(b_n) \to b$ then

$$(a_n) + (b_n) = a + b$$

We also know, that for any sequence $(c_n) \to c$ it is true, that $f(c_n) \to L$ and $g(c_n) \to M$; therefore by the algebraic limit theorem

$$f(c_n) + g(c_n) = L + M$$

for any sequence $(c_n) \to c$. Therefore we can state that

$$\lim_{x\to c}(f(x) + g(x)) = L + M$$

as desired

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$; therefore for any $\epsilon_1 > 0$ we can find $\delta_1 > 0$ s.t.

$$|x - c| < \delta_1 \to |f(x) - L| < \epsilon_1$$

Also for the same ϵ_1 there exist $\delta_2 > 0$ s.t.

$$|x - c| < \delta_2 \to |g(x) - M| < \epsilon_1$$

let $\delta_3 = min\{\delta_1, \delta_2\}$; then it is true that

$$|x - c| < \delta_3 \to |f(x) - L| < \epsilon_1$$

$$|x - c| < \delta_3 \to |g(x) - M| < \epsilon_1$$

because $V_{\delta_1} \subseteq V_{\delta_3}$ and $V_{\delta_2} \subseteq V_{\delta_3}$
therefore

$$|f(x) - L| + |g(x) - M| < 2\epsilon_1$$

Therefore

$$|f(x)+g(x)-L-M| = |f(x)-L+g(x)-M| \leq |f(x)-L|+|g(x)-M| < 2\epsilon_1$$

Thus for any $\epsilon > 0$ there exist corresponding $\epsilon_1 = \frac{\epsilon}{2}$ for which there exist corresponding $\delta = min\{\delta_1, \delta_2\}$ (where δ_1 is a delta for $f(x)$ and δ_2 is a delta for $g(x)$) which satisfies

$$|x-c| < \delta \to |f(x)+g(x)-(L+M)| < \epsilon$$

therefore $\lim_{x\to c}(f(x)+g(x)) = L+M$ as desired.

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

(a):

From the algebraic limit theorem we know, that if $(a_n) \to a$ and $(b_n) \to b$ then

$$(a_n)(b_n) = ab$$

We also know, that for any sequence $(c_n) \to c$ it is true, that $f(c_n) \to L$ and $g(c_n) \to M$; therefore by the algebraic limit theorem

$$f(c_n)g(c_n) = LM$$

for any sequence $(c_n) \to c$. Therefore we can state that

$$\lim_{x\to c}(f(x)g(x)) = LM$$

as desired

(b):

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$;

In order to prove the needed limit let's first use some algebra

$$|f(x)g(x)-LM| =$$

$$|f(x)g(x)+f(x)M-f(x)M-LM| =$$

$$|f(x)(g(x)-M)+M(f(x)-L)| \leq |f(x)(g(x)-M)|+|M(f(x)-L)| =$$

$$|f(x)||g(x)-M|+|M||f(x)-L|$$

our strategy is to show that both elements of the last sum are less or equal to $\epsilon/2$

Let $\epsilon > 0$.

$$|M||f(x)-L| < \frac{\epsilon}{2}$$

If $M = 0$ then the abovementioned inequality always holds and we are free to choose any δ_1;

Otherwise tet us pick δ_1 such that inequality

$$|f(x) - L| < \frac{\epsilon}{2|M|}$$

holds.

The next step is a little bit more complicated because we need to work with $f(x)$; let us pick y = 1; then because $\lim_{x\to c} f(x) = L$ we know that there exists δ_2 s.t. $|x-c| < \delta_2 \to |f(x) - L| < 1$.

Therefore

$$|f(x) - L| < 1$$

Little sidenote: let's prove that

$$|a - b| < c \to |a| < |b| + c$$

Firstly some preliminary stuff

$$|a - b| \geq 0 \to c > |a - b| > 0 \to c > 0$$

$$|a - b| < c \to -c < a - b < c$$

$$b - c < a < b + c$$

Now let's see all the cases for $a, b \in \mathbf{R}$

if $a \geq 0$ and $b \geq 0$ then

$$a < b + c$$

$$|a| < |b| + c$$

if $a < 0$ and $b \geq 0$ then

$$b + c \geq 0 > a$$

$$a < b + c$$

$$|a| < |b| + c$$

if $a \geq 0$ and $b < 0$ then

$$b - c < a < b + c$$

$$-b + c > -a > -b - c$$

$$|b| + c > -a > |b| - c$$

$$-|b| - c < a < c - |b|$$

$$|a| < c - |b| \leq c + |b|$$

$$|a| < c + |b|$$

if $a < 0$ and $b < 0$ then

$$b - c < a < b + c$$

$$-b + c > -a > -b - c$$

$$|b| + c > |a| > |b| - c$$

$$|b| + c > |a|$$

$$|a| < |b| + c$$

Therefore $\forall a, b \in \mathbf{R}$

$$|a - b| < c \to |a| < |b| + c$$

as desired.

Back to our exercise:

$$|f(x) - L| < 1$$

$$|f(x)| < |L| + 1$$

Therefore we can state that upper bound for our $|f(x)|$ with $\epsilon = 1$ is $|L| + 1$

Thus if we pick δ_2 sufficient for

$$|g(x) - M| < \frac{\epsilon}{2(|L| + 1)}$$

therefore if we pick $\delta = min\{\delta_1, \delta_2\}$ then

$$|x - c| < \delta \to$$

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

therefore $\lim_{x \to c}[f(x)g(x)] = LM$ as desired

4.2.6

Let $g : A \to \boldsymbol{R}$ and assume that f is bounded function on $A \subseteq \boldsymbol{R}$ (i.e. there exists $M > 0$ satisfying $|f(x| \leq M$ for all $x \in A$). Show that if $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} g(x)f(x) = 0$ as well.

Here we can't use an intuitive approach of just using algebraic limit theorem because $f(x)$ may not hav limit at c. Anyways we proceed by $\epsilon - \delta$ approach.

Therefore we need to show that

$$\exists \delta : |f(x)g(x)| < \epsilon$$

First of all,

$$|f(x)g(x)| = |f(x)||g(x)|$$

Then we notice, that because $f(x)$ is bounded

$$\exists M \in \mathbf{R} > 0 : |f(x)| \leq M$$

therefore

$$|f(x)||g(x)| < |M||g(x)| = M|g(x)|$$

therefore if we pick δ sufficient for $|g(x)| < \frac{\epsilon}{M}$ then it follows that

$$|f(x)g(x)| \leq M|g(x)| < \epsilon$$

therefore

$$\forall \epsilon \in \mathbf{R} \exists \delta : |x - c| < \delta \to |f(x)g(x)| < \epsilon$$

therefore

$$\lim_{x \to c} [f(x)g(x)] = 0$$

as desired.

4.2.7

(a) The statement $\lim_{x \to 0} 1/x^2 = \infty$ *certainly makes intuitive sense. Construct a rigirius definition in the "challenge-response" style of Definition 4.2.1 for a limit statement of the form* $\lim_{x \to c} f(x) = \infty$ *and use it to prove the previous statement*

Definition of limit to infinity Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = \infty$ provided that, for all $\epsilon \in \mathbf{R}$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

Now we need to show that for $f(x) = 1/x^2$

$$\lim_{x \to 0} f(x) = \infty$$

First

$$f(x) > \epsilon$$

$$\frac{1}{x^2} > \epsilon$$

$$x^2 < \frac{1}{\epsilon}$$

$$x < \sqrt{\frac{1}{\epsilon}}$$

therefore if we pick $\delta = \sqrt{\frac{1}{\epsilon}}$, then it follows that

$$f(x) > \epsilon$$

as desired.

Quick (and insufficient) test in Python seems to corraborate this statement

(b) Now construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. *Show* $\lim_{x\to\infty} 1/x = 0$

Definition of infinite limit Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to\infty} f(x) = L$ provided that, for all $\epsilon \in \mathbf{R} > 0$, there exists a δ such that whenever $x > \delta$ (and $x \in A$) it follows that $|f(x) - c| < \epsilon$.

We start as ususal at the ϵ

$$|f(x) - 0| < \epsilon$$
$$|1/x| < \epsilon$$

Given that we can pick any δ as we want, we can pick it at the very least at 0 to get rid of the absolute value

$$1/x < \epsilon$$
$$x > 1/\epsilon$$

therefore $\delta = 1/\epsilon$ then it follows that

$$|f(x) - 0| < \epsilon$$

as desired.

(c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ *would look like? Give an example of such a limit*

Definition of infinite limit to infinity Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to\infty} f(x) = \infty$ provided that, for all $\epsilon \in \mathbf{R}$, there exists a δ such that whenever $x > \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

The corresponding example of such a limit is $f(x) = x$.

4.2.8

Assume $f(x) \geq g(x)$ for all x in some set A on which f and g are defined. Show that for any limit point c of A we must have

$$\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$$

I'm gonna do it by using contradiction; suppose that $f(x)$ and $g(x)$ are defined as in the exercise, but

$$\lim_{x\to c} f(x) < \lim_{x\to c} g(x)$$

then it follows that there exist a sequence $(a_n) \to c$ such that $f(a_n) \geq g(a_n)$ for all $n \in \mathbf{N}$; Therefore $\lim(f(a_n)) \geq \lim(g(a_n))$ and but it contradicts our initial assumption.

4.2.9 (Squeeze Theorem)

Let f, g and h satisfy $f(x) \geq g(x) \geq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well

As proven in the previous exercise

$$\forall x \in A : f(x) > g(x) \to \lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$$

therefore

$$\lim_{x\to c} f(x) = L \geq \lim_{x\to c} g(x)$$

and

$$\lim_{x\to c} g(x) \geq \lim_{x\to c} h(x) = L$$

Thus

$$L \geq \lim_{x\to c} g(x) \geq L$$

therefore

$$\lim_{x\to c} g(x) = L$$

as desired.

4.3.1

Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continous at $c = 0$

We're gonna use $\epsilon - \delta$ definition. First of all, let's state that $g(0) = 0$. Therefore

$$|f(x) - f(c)| = |\sqrt[3]{x} - 0| < \epsilon$$

$$|\sqrt[3]{x}| < \epsilon$$

Here I would like to proof that $\forall x \in \mathbf{R} : |\sqrt[3]{x}| = \sqrt[3]{|x|}$: if $x \geq 0$ then $|\sqrt[3]{x}| = \sqrt[3]{x} = \sqrt[3]{|x|}$; if $x < 0$ then $|\sqrt[3]{x}| = \sqrt[3]{-x} = \sqrt[3]{|x|}$. Therefore

$$|\sqrt[3]{x}| = \sqrt[3]{|x|} =< \epsilon$$

is justified.

Therefore we can state that

$$|x| =< \epsilon^3$$

Thus if we pick $\delta = \epsilon^3$ then

$$|x-c| = |x| < \delta \to |f(x) - f(c)| = |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\epsilon^3} = \epsilon$$

Therefore g is continous at 0

(b) Prove that g is continous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ will be helpful)

We're gonna use $\epsilon - \delta$ definition once again.

$$|f(x) - f(c)| = |\sqrt[3]{x} - \sqrt[3]{c}| < \epsilon$$

First, let's use a little algebra

$$|\sqrt[3]{x} - \sqrt[3]{c}| = |\sqrt[3]{x} - \sqrt[3]{c}| * 1 = |\sqrt[3]{x} - \sqrt[3]{c}| \frac{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)} = \frac{|\sqrt[3]{x} - \sqrt[3]{c}|(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)}$$

Let's look now at the sum $\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2$: $\sqrt[3]{x}^2 \geq 0$ because it is a square. For $\sqrt[3]{x} + \sqrt[3]{x}^2$ we need to be able to articulate δ so that both x and c are the same sign; if we fo that then it becomes nonnegative. $\sqrt[3]{c}^2 \geq 0$ because it is a square

Therefore if right now we pinky-promise that we will account for unusual delta in the future, then we are able to say that

$$\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2 \geq 0$$

And therefore

$$\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2 = |\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2|$$

Continuing with our initial algebra

$$\frac{|\sqrt[3]{x} - \sqrt[3]{c}|(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)} = \frac{|\sqrt[3]{x} - \sqrt[3]{c}||\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2|}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)} = \frac{|x-c|}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)} =$$

$$|x-c| \frac{1}{(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)} < \epsilon$$

As we disussed earlier $(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2) \geq 0$ and therefore

$$|x-c| < \epsilon(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2)$$

Thus, if we pick $\delta = min\{\epsilon(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2), |x-0|\}$ (we need the second value because we need the sum to be equal to its absolute value;) then

$$|x-c| < \delta \to |f(x) - f(c)| < \epsilon$$

Therefore $f(x) = \sqrt[3]{x}$ is continous on $\mathbf{R}$.

4.3.2

(a) Supply a proof for Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.

First, let's state the theorem

Theorem 4.3.9 (Composition of Continuous Functions). *Given $f : A \to \boldsymbol{R}$ and $g : B \to \boldsymbol{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f = g(f(x))$ is well-defined on A.*

If f is continous ac $c \in A$, and if g is continous at $f(c) \in B$, then $g \circ f$ is continous at c.

Firstly, the fact that both f and g are continous tells that

$$\forall \epsilon_1 \in \mathbf{R} : \exists \delta_1 : |x - c| < \delta_1 \to |f(x) - f(c)| < \epsilon_1$$
$$\forall \epsilon_2 \in \mathbf{R} : \exists \delta_2 : |x - c| < \delta_2 \to |g(x) - g(c)| < \epsilon_2$$

And we need to prove that

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - c| < \delta \to |g(f(x)) - g(f(c))| < \epsilon$$

The main strategy for this one is to plug some delta into some epsilon (or vice versa), and get some use out of it.

Firstly, let's get some things out of the way: let us fix particular $c \in A$ and $\epsilon \in \mathbf{R} > 0$. Then, let's plug this ϵ at $f(c)$ into the continuity of $g(x)$ so we can get a $\delta_g > 0$. Therefore we will have

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |x - f(c)| < \delta_g \to |g(x) - g(f(c))| < \epsilon$$

which is kinda close to the thing, that we're trying to prove.

We also know that

$$\forall \epsilon_f \in \mathbf{R} : \exists \delta_f : |x - c| < \delta_f : |f(x) - f(c)| < \epsilon_f$$

therefore it is true that

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |y - f(c)| < \delta_g \to |g(y) - g(f(c))| < \epsilon$$
$$\exists \delta_f : |x - c| < \delta_f \to |f(x) - f(c)| < \delta_g$$

From this we can state that

$$\forall \epsilon \in \mathbf{R} > 0 : \exists \delta_f : |x - c| < \delta_f \to |f(x) - f(c)| < \delta_g \to |g(f(x)) - g(f(c))| < \epsilon$$

This doesn't sound too persuasive for me, so I probably need to explore it a little but more.

Suppose that with all the present assumptions, we get the given ϵ. If we plug it into definition of continuity for $g(x)$ at $g(f(c))$, then we'll get the neccesary δ_g. If we plug δ_g as an ϵ for the definition of continuity of $f(x)$, then we'll get δ_f.

We can probably prove it with a little bit more clarity. We need to prove that

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - f(c)| < \delta \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

Firstly, definition of contonuity of $g(x)$ gives us the fact, that

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |x - f(c)| < \delta_g \rightarrow |g(x) - g(f(c))| < \epsilon$$

then if $x \in f(A)$ then $\exists y \in A$ s.t. $f(y) = x$ therefore

$$\forall \epsilon \in \mathbf{R} : \exists \delta_g : |f(y) - f(c)| < \delta_g \rightarrow |g(f(y)) - g(f(c))| < \epsilon$$

From the definition of continuity of f we know that

$$\forall \epsilon_f \in \mathbf{R} : \exists \delta_f : |x - c| < \delta_f : |f(x) - f(c)| < \epsilon_f$$

Therefore

$$\forall \epsilon \in \mathbf{R} : \exists \delta : |x - f(c)| < \delta \rightarrow |g(f(x)) - g(f(c))| < \epsilon$$

as desired.

(b) Give another proof of this theorem using the sequantial characterization of continuity (from Theorem 4.3.2 (iv))

Theorem 4.3.2 (iv) states that if $(x_n) \rightarrow c$ (with $x_n \in A$), then $f(x_n) \rightarrow f(c)$.

Because $f(x)$ is continous we can state that for every sequence $(x_n) \rightarrow c$ it is true that $f(x_n) \rightarrow f(c)$. Therefore because $f(x_n)$ is a sequence itself, we can state that $g(f(x_n)) \rightarrow g(f(c))$. Therefore it is true, that for every sequence $(x_n) \rightarrow c$ it follows, that $g(f(x_n)) \rightarrow g(f(c))$. Therefore $g(f(x))$ is continous, as desired.

4.3.3

Using the $\epsilon - \delta$ characteriation of continuity (and tus using no previous results anbout the sequences), show that the linear function $f(x) = ax + b$ is continous at every poinnt of $\boldsymbol{R}$.

Let's start with our usual stuff

$$|f(x) - f(c)| < \epsilon$$

$$|ax + b - (ac + b)| = |ax + b - ac - b| = |a||x - c| < \epsilon$$

$$|x - c| < \epsilon/a$$

Therefore if we pick $\delta = \epsilon/a$ then it follows that $|f(x) - f(c)| < \epsilon$, as desired.

4.3.4

(a) Show using Definition 4.3.1 that any function f with domain $\boldsymbol{Z}$ with necessarily be continous at every point in its domain.

Suppose that $f : Z \to R$. We need to prove that

$$\forall \epsilon : \exists \delta : |x - c| < \delta \to |f(x) - f(c)| < \epsilon$$

Suppose that we pick $\delta = 0.1$ (or any other value, such that the only one of the domain values will be in the needed neighborhood). Then there will be only one number in the domain neighhborhood, and because of that we can state that

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

Therefore the fucntion is continous, as desired.

(b) Show in general that if c is an isolated point of $A \subseteq \boldsymbol{R}$, then $f : A \to \boldsymbol{R}$ is continous at c.

In this particular case we can't just set δ at some number, so we gotta be a little more creative. To be distract myself from getting any unproductive ideas, I should state here that $\mathbf{Q}$ is a set of isolated points.

To be continued

3.2.1

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite get used

Theorem 3.2.3 states that

(ii) The intersection of a finite collection of open sets is open.

The assumption of the finality of the set is used in the fact, that we need the minimum of the epsilons.

(b) Give an example of an infinite collection of nested open sets

$$O_1 \supseteq O_2 \supseteq O_3 \supseteq O_4 \supseteq ...$$

whose intersection $\cap_{n=1}^{\infty} O_n$ is closed and nonempty.

First of all, we should state that open is not an opposite of closed in this context. We can get $O_n = (-\infty; \infty)$. Then this definition (technically) fits into the requrement of qs

Let $O_n = (1 - 1/n, 2 + 1/n)$. Let us also define

$$A = \cap_{n=1}^{\infty} O_n$$

Suppose that $x \in A$. To be continued...

1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a simular argument work to show $\sqrt{6}$ is irrational?

Suppose that $\sqrt{3}$ is a rational number; then it is true that

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{3}$$

where m and n are at their lowest possible terms. Then

$$\sqrt{3}n = m$$
$$3n^2 = m^2$$

Therefore we can state, that $m\%3 = 0$. Therefore $\exists k : 3k = m$. Thus we can reformulate formula as

$$3n^2 = (3k)^2$$
$$n^2 = 3k^2$$

Therefore $n\%3 = 0$ as well. Therefore n and k are not in their possible terms, which conradicts our initial assumtions. Therefore we can state that $\sqrt{3} \notin \mathbf{Q}$.

Let's try the same argument for $\sqrt{6}$.

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{6}$$
$$\sqrt{6}n = m$$
$$6n^2 = m^2$$

then m has as their dividers both 2 and 3. Therefore $m\%2 = 0$ and $m\%3 = 0$. Therefore we can proceed with the same argument as earlier

$$6n^2 = (6k)^2$$
$$n^2 = 6k^2$$

Therefore n is divided by 6, etc., etc., $\sqrt{6} \notin \mathbf{Q}$.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Suppose that $\sqrt{4}$ is a rational number; then it is true that

$$\exists m \in \mathbf{Z}, n \in \mathbf{N} : \frac{m}{n} = \sqrt{2}$$

where m and n are at their lowest possible terms. Then

$$\sqrt{4}n = m$$
$$4n^2 = m^2$$

n can still be odd and m can still be even. In other words, m is divisible by a prime, and the number under the radical consists of two primes. Therefore if a number decomposes to two equal sets of primes, then its square root is a rational number. Otherwise it isn't.

1.2.2

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific exapmple where the statement in question does not hold.

(a) if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well. -

$\cap_{n=1}^{\infty} A_n = (0, 1/n)$ has no numbers in it.

Proof is easy -

$$\forall x \in \mathbf{R} > 0 : \exists n \in N : 1/n < x$$

(b) if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty. -

True.

There is no need for the proof, but I'll supply one anyways. If all A_n are finite and nonempty, then $\exists j \in \mathbf{N} : |A_1| = j$. Therefore, because of the same reasons, there are only $j - 1$ times when

$$A_k \supset A_{k+1}$$

can happen, because after $j - 1$ times the set will be empty. Therefore, because it is finite, their intersection will have finite number of elements and will be non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

False: let

$$x \notin A, x \notin B, x \in C$$

Then

$$x \in A \cap (B \cup C); x \notin = (A \cap B) \cup C$$

(c) $A \cap (B \cap C) = (A \cap B) \cap C$

True. Kinda goes without a proof; if you imagine a Vien diagram, then it's obvious.

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

True. For the same reason as before.

I'm sure that there exist more concrete versions of those proofs, but I'm not required to provide any. My suspition on why is it so, is because it's a little more complicated and requires more knowlege in set theory and/or logic.

1.2.3 (De Morgan's Laws).

Let A and B be subsets of $\mathbf{R}$

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq (A \cap B)^c$.

If we have two sets A and B, then $\mathbf{R}$ desintegrates into 4 different sets: A, B, A^c, B^c.

Therefore there must exists sets

$$S_1 = A \cap B$$

$$S_2 = A^c \cap B$$

$$S_3 = A \cap B^c$$

$$S_4 = A^c \cap B^c$$

An element cannot be in the set and not in the set at the same time. Therefore, there does not exist an element, which is in two of S_n's.

For any $x \in \mathbf{R} \rightarrow xinA$ or $x \notin A$. Therefore an element of $\mathbf{R}$ needs to be in at least one of those sets. It is easily seen by

$$A \cap \mathbf{R} = A$$

$$A \cap (B \cup B^c) = A$$

$$(A \cap B) \cup (A \cap B^c)) = A$$

Therefore $\cup_{n=1}^{4} S_n = \mathbf{R}$ and $\cap_{n=1}^{4} S_n = \emptyset$.

Suppose $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Therefore $x \in S_2 \cup S_3 \cup S_4$.

Suppose that $x \in A^c \cup B^c$. Then $x \in S_2 \cup S_3 \cup S_4$.

Therefore $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion

I didn't find a good axiomatic way to settle this; proof in part (a) will suffice for both of thosse

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

$$(A \cup B)^c = S_4 = A^c \cap B^c$$

I spent too much time on this one, so this will suffice.

1.2.4

Verify the triangle inequality in the special cases where

(a) a and b have the same sign

Suppose $a \geq 0$, $b \geq 0$. Then $|a| = a$ and $|b| = b$. Therefore

$$|a + b| = a + b = |a| + |b| \leq |a| + |b|$$

Suppose $a < 0$, $b < 0$. Then $|a| = -a$ and $|b| = -b$; also $a + b < 0 \rightarrow |a + b| = -(a + b) = -a - b$. Therefore

$$|a + b| = -a + (-b) = |a| + |b| \leq |a| + |b|$$

.

(b) $a \geq 0$, $b < 0$ *and* $a + b \geq 0$.

$$a + b \geq 0 \rightarrow a + b = |a + b|$$

Also, $|a| = a$ and $|b| = -b$. Therefore

$$a + b \geq 0 \rightarrow a \geq -b \rightarrow a \geq |b| \rightarrow |a| \geq |b|$$

$$b < 0$$

$$b \leq 0$$

$$2b \leq 0$$

$$b + b \leq 0$$

$$b \leq (-b)$$

$$a + b \leq a + (-b)$$

$$|a + b| \leq |a| + |b|$$

1.2.5

Use the triangle inequality ti establish the inequalities

(a) $|a - b| \leq |a| + |b|$;

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

(b) $||a| - |b|| \leq |a - b|$*;*

let $a = a + b - b$. Then

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|a| - |b| \leq |a - b|$$

$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$$

$$|b| - |a| \leq |a - b|$$

$$|a| - |b| \geq -|a - b|$$

$$-|a - b| \leq |a| - |b| \leq |a - b| \rightarrow ||a| - |b|| \leq |a - b|$$

1.2.6

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. if $A = [0,2]$ (the closed interval $\{x \in \boldsymbol{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

First things first: $f(A) = [0,4]$; $f(B) = [1,16]$ (without any proof because if we don't go with axiomatic stuff, then it is obvious).

$$f(A \cap B) = f([1,2]) = [1,4]$$

$$f(A) \cap f(B) = [1,4]$$

Therefore in this case $f(A) \cap f(B) = f(A \cap B)$.

$$f(A \cup B) = f([0,4]) = [0,16] = f(A) \cup f(B)$$

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

Let $A = [-1,0]$ and $B = [0,1]$. Then

$$f(A \cap B) = f(\{0\}) = \{0\}$$

$$f(A) \cap f(B) = [0,1] \cap [0,1] = [0,1] \neq f(A \cap B)$$

(c) Show that, for an arbitrary function $g : \boldsymbol{R} \to \boldsymbol{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \boldsymbol{R}$.

$$x \in g(A \cap B) \to x \in g(A)$$

$$x \in g(A \cap B) \to x \in g(B)$$

Therefore

$$x \in g(A \cap B) \to x \in g(A) \cap g(B)$$

Thus

$$g(A \cap B) \subseteq g(A) \cap g(B)$$

(d) Form and prove a conjecture aout the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

$$x \in g(A) \to x \in g(A \cup B)$$

$$x \in g(B) \to x \in g(A \cup B)$$

Therefore

$$x \in g(A) \cup g(B) \to x \in g(A \cup B)$$

Thus

$$g(A) \cup g(B) \subseteq g(A \cup B)$$

Suppose that

$$\exists y \in \mathbf{R} : y \in g(A \cup B); y \notin g(A) \cup g(B)$$

Then $\exists q_1 \in A \cup B : g(q_1) = y$ but

$$\forall q_2 \in A, q_3 \in B : g(q_2) \neq y; g(q_3) \neq y$$

Therefore $q_1 \notin A$ and $q_1 \notin B$. Therefore $q_1 \in A^c \cap B^c$. Using De Morgan rule

$$q_1 \in A^c \cap B^c \rightarrow q_1 \in (A \cup B)^c$$

therefore

$$q_1 \notin g(A \cup B)$$

which is a contradiction. Therefore

$$y \in g(A \cup B) \rightarrow g(A) \cup g(B)$$

Thus

$$g(A \cup B) \subseteq g(A) \cup g(B)$$

Therefore if we take into account previous conclusion

$$g(A \cup B) = g(A) \cup g(B)$$

for any g.

1.2.7

Given a function $f : D \rightarrow \boldsymbol{R}$ and a subset $B \subseteq \boldsymbol{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This is called the preimage of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

$$f^{-1}(A) = [-2, 2]$$

$$f^{-1}(B) = [-1, 1]$$

$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$$

$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$$

(b) The good behaviour of preimages demonstated in (a) is completely general. Show that for an arbitrary function $g : \boldsymbol{R} \to \boldsymbol{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \boldsymbol{R}$.

By definition

$$x \in g^{-1}(A \cap B) \to \exists y \in A \cap B : y = g(x)$$

Therefore if we we use the fact $y \in A \cap B \to y \in A$ and $y \in A \cap B \to y \in B$

$$x \in g^{-1}(A \cap B) \to \exists y \in A : y = g(x) \to x \in g^{-1}(A)$$

$$x \in g^{-1}(A \cap B) \to \exists y \in B : y = g(x) \to x \in g^{-1}(B)$$

therefore $x \in g^{-1}(A \cap B)$ implies that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, or in other words

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$$

In other direction:

$$x \in g^{-1}(A) \to \exists y_1 \in A : y_1 = g(x)$$

$$x \in g^{-1}(B) \to \exists y_2 \in B : y_2 = g(x)$$

$x \in g^{-1}(A) \cap g^{-1}(B)$ implies that $\exists y_1 \in A : g(x) = y_1$ and $\exists y_2 \in B : y_2 = g(x)$. Because g is a function we know, that for every x there exists only one $y = g(x)$. Therefore $y_1 = y_2 = g(x)$. Thus we can state that $y \in A \cap B$.

thus

$$x \in g^{-1}(A) \cap g^{-1}(B) \to \exists y \in A \cap B : y = g(x) \to x \in g^{-1}(A \cap B)$$

Therefore

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$$

If we take previous conclusion into account, then it follows that

$$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$$

as desired.

Now let's prove that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$:

If $x \in g^{-1}(A) \cup g^{-1}(B))$ then $\exists y \in A : y = g(x)$ or $\exists y \in B : y = g(x)$. If we take into account that $y \in A \to y \in A \cup B$ then we can conclude that

$$x \in g^{-1}(A) \cup g^{-1}(B)) \to \exists y \in A \cup B : y = g(x) \to x \in g^{-1}(A \cup B)$$

Thus

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B)$$

In other direction:

$$x \in g^{-1}(A \cup B) \to \exists y \in A \cup B : y = g(x) \to y = g(x)$$