My set theory exercises

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2023

Contents

Useful things

I think that it is pretty straightforward to define some function based on axioms that we get. For example pairing axiom allows us to define $PA : S \times S \to S$ by

$$PA(u, v) = \{u, v\}$$

same goes for union axiom

$$UA(u) = \{\text{elements of elements of U}\}$$

Later some other function might be defined in the same manner.

In logic notation, I denote tautology as 'true' and contradiction as 'false'

There is a rule that I've used

$$a \wedge (b \vee \neg a) \Leftrightarrow (a \wedge b) \vee (a \wedge \neg a)) \Leftrightarrow (a \wedge b) \vee (\text{false}) \Leftrightarrow a \wedge b$$

which I don't remember seeing in the book, but it's pretty useful.

Chapter 1

Introduction

1.1 Elementary Set Theory

Let A, B, C be

1.1.1

If $a \notin A \setminus B$ and $a \in A$, show that $a \in B$

Because $a \notin A \setminus B$, we follow that $x \in B$ or $x \notin A$. Since $x \in A$, we follow that $x \in B$, as desired.

1.1.2

Show that if $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$

Let $c \in C \setminus B$. Then we follow that $c \in C$ or $c \notin B$. Since $A \subseteq B$, we follow that $c \notin B$ implies that $c \notin A$. Thus we follow that $c \in C \setminus B$ implies that $c \in C \setminus A$. Therefore $C \setminus B \subseteq C \setminus A$.

1.1.3

Suppose $A \setminus B \subseteq C$. Show that $A \setminus C \subseteq B$.

Suppose that $a \in A \setminus C$. Then we follow that $a \in A$ and $a \notin C$.

Given that $A \setminus B \subseteq C$ and $A \notin C$, we follow that $a \notin A \setminus B$. Thus $a \notin A$ or $a \in B$. Since $a \in A$, we follow that $a \in B$. Thus

$$a \in A \setminus C \to a \in B$$

$$A \setminus C \subseteq B$$

as desired.

1.1.4

Suppose $A \subseteq B$ and $A \subseteq C$. Show that $A \subseteq B \cap C$

Suppose that $a \in A$. Then we follow that $a \in B$ and $a \in C$. Thus $a \in B \cap C$. Therefore we follow that $A \subseteq B \cap C$.

1.1.5

Suppose $A \subseteq B$ and $B \cap C = \emptyset$. Show that $A \in B \setminus C$

Suppose that $a \in A$. Then we follow that $a \in B$ and since $B \cap C = \emptyset$, we follow that $a \notin C$. Thus $a \in B \setminus C$ by definition. Therefore $A \subseteq B \setminus C$.

1.1.6

Show that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$. Suppose that $a \in A \setminus (B \setminus C)$. Then we follow that $a \in A$ and $a \notin B \setminus C$. Thus $a \notin B$ and $a \in C$. Thus we follow that $a \in A \setminus B$ or $a \in C$. Thus $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$ as desired.

1.1.7

Let $P(x)$ be the property $x > \frac{1}{x}$. Are the assertions $P(2)$, $P(-2)$, $P(\frac{1}{2})$ $P(\frac{-1}{2})$ true or false

.

$$2 > \frac{1}{2} \to P(2) = true$$
$$-2 < \frac{-1}{2} \to P(-2) = false$$

last two are reversed.

1.1.8

Sow that each of the following sets can be expressed as an interval

$$a)(-3, 3)$$
$$b)(-3, \infty)$$
$$c)(-3, 3)$$

all of them follow from order properties of real numbers.

1.1.9

Express the following sets as truth sets

$$A = \{1, 4, 9, 16, 25, ...\} \iff A = \{x \in N : x = y^2 \textit{ for some } y \in N\}$$
$$B = \{..., -15, -10, -5, 0, 5, ...\} \iff A = \{x \in N : x = 5y \textit{ for some } y \in N\}$$

Rest are also trivial, not gonna go deep here

1.2 Logical Notation

1.2.1

Using truth tables, show that $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$
false	false	true	false	true	false
false	true	true	false	false	false
true	false	false	true	true	true
true	true	true	false	false	false

from this we can see that they are equqivalent.

Following 4 exercises are the same as this one, so I'm skipping them

1.2.5

Show that $(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow P \Rightarrow (Q \wedge R)$*, using logic laws*

$$(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \Leftrightarrow \neg P \vee (R \wedge Q) \Leftrightarrow P \Rightarrow (R \wedge Q)$$

Laws used:

$$CL \rightarrow DIST \rightarrow CL$$

1.2.6

Show that $(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$*, using logic laws*

$$(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (\neg P \vee R) \vee (\neg Q \vee R) \Leftrightarrow \neg P \vee R \vee \neg Q \vee R \Leftrightarrow (\neg Q \vee \neg P) \vee R \Leftrightarrow$$
$$\Leftrightarrow \neg(Q \wedge P) \vee R \Leftrightarrow (Q \wedge R) \Rightarrow R$$

Laws used:

$$CL \rightarrow ASC \rightarrow ID, ASC \rightarrow DML \rightarrow CL$$

1.2.7

Show that $P \Rightarrow (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$*, using logic laws*

$$P \Rightarrow (Q \Rightarrow R) \Leftrightarrow \neg P \vee (Q \Rightarrow R) \Leftrightarrow \neg P \vee (\neg Q \vee R) \Leftrightarrow (\neg P \vee \neg Q) \vee R \Leftrightarrow \neg (P \wedge Q) \vee R \Leftrightarrow (P \wedge Q) \Rightarrow R$$

Laws used:

$$CL \rightarrow CL \rightarrow ASC \rightarrow DML \rightarrow CL$$

1.2.8

Show that $(P \Rightarrow Q) \Rightarrow R$ *and* $P \Rightarrow (Q \Rightarrow R)$ *are not logically equivalent*

We're gonna show that $q \wedge w \Leftrightarrow false$

$$((P \Rightarrow Q) \Rightarrow R) \wedge (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow (\neg(\neg P \vee Q) \vee R) \wedge (\neg P \vee (\neg Q \vee R)) \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge \neg Q) \vee R) \wedge (\neg P \vee \neg Q \vee R) \Leftrightarrow ((P \wedge Q) \wedge (\neg P \vee \neg Q)) \vee R \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge Q) \wedge \neg (P \wedge Q)) \vee R \Leftrightarrow false \vee R \Leftrightarrow false$$

1.3 Predicates and Quantifiers

1.4 A Formal Language for Set Theory

1.4.1

What does the formula $\exists x \forall y (x \notin y)$ *say in English?*

There exists x such that for every y we've got that x is not in y. In other ways, there exists an empty set.

1.4.2

What does the formula $\forall y \exists x (y \notin x)$ *say in English?*

For every y there exists set x such that y is not in x.

1.4.3

What does the formula $\forall y \exists x (x \notin y)$ *say in English?*

For every y there exists x such that x is not in y.

1.4.4

What does the formula $\forall y \neg \exists x (x \notin y)$ *say in English?*

For every y there does not exist an x such that x is not in y.

1.4.5

What does the formula $\forall z \exists x \exists y(x \in y \wedge y \in z)$ say in English?

For every z there exists x and y such that x is in y and y is in z

1.4.6

Let $\phi(x)$ be a formula. What does $\forall z \forall y((\phi(x) \wedge \phi(y)) \rightarrow z = y$

For every z and y, $\phi(x)$ and $\phi(y)$ implies that z = y.

1.4.7

Translate each of the following into the language of set theory.

(a) x is the union of a and b

$$\forall(y \in x)(y \in a \wedge y \in b)$$

(b) x is not a subset of y

$$\exists(z \in x)(\neg z \in y)$$

(c) x is the intersection of a and b

$$\forall(y \in x)(y \in a \vee y \in b)$$

(d) a and b have no elements in common

$$\forall(x \in a)\forall(y \in b)(\neg x = y)$$

1.4.8

Let a, b, C and D be sets. Show that the relationship

$$y = \begin{cases} a \text{ if } x \in C \setminus D \\ b \text{ if } x \notin C \setminus D \end{cases}$$

$$((x \in C \wedge \neg x \in D) \rightarrow (y = a)) \wedge ((\neg x \in C \wedge \neg x \in D) \rightarrow (y = a))$$

1.5 The Zermelo-Fraenkel Axioms

1.5.1

Let u, v, w be sets. By pairing axiom, the sets $\{u\}$ and $\{v, w\}$ exist. Using the pairing and union axioms, show that the set $\{u, v, w\}$ exists.

By pairing axiom we've got that

$$PA(u, u) = \{u\}$$

$$PA(v, w) = \{v, w\}$$

thus

$$PA(\{u\}, \{v, w\}) = \{\{u\}, \{v, w\}\}$$

and therefore by union axiom we follow that

$$UA(\{\{u\}, \{v, w\}\}) = \{u, v, w\}$$

as desired.

1.5.2

Let A be a set. Show that the pairing axiom implies that the set $\{A\}$ exists

$$PA(A, A) = \{A, A\}$$

which by extension axiom is equal to $\{A\}$, as desired.

1.5.3

Let A be a set. The pairing axiom implies that the set $\{A\}$ exists. Using the regularity axiom, show that $A \cap \{A\} = 0$. Conclude that $A \notin A$.

Since $\{A\} \neq \emptyset$, we follow that there exists x such that $x \in \{A\}$ and $x \cap \{A\} = \emptyset$. Since A is the only element of $\{A\}$, we follow that $A \cap \{A\} = \emptyset$, as desired.

1.5.4

For sets A, B, the set $\{A, B\}$ exists by the pairing axiom. Let $A \in B$. Using the regularity axiom, show that $A \cap \{A, B\} = \emptyset$, and thus $B \notin A$.

$\{A, B\}$ consists of sets A and B, thus it is not empty and therefore there exists $x \in \{A, B\}$ such that $x \in \{A, B\} \wedge x \cap \{A, B\} = \emptyset$. For B we've got that $B \in \{A, B\}$. Since $A \in B$ and $A \in \{A, B\}$, we can follow that $A \in (B \cap \{A, B\})$. By pairing axiom we follow that the element with desired property must exists, and given that the only other choice is A, we conclude that $A \cap \{A, B\} = \emptyset$. Therefore we can follow that $B \notin A$, as desired.

1.5.5

Let A, B, C be sets. Suppoes that $A \in B$ and $B \in C$. Using the regularity axiom, show that $C \notin A$.

This is an expantion of previous exercise. We can follow that

$$B \in \{A, B, C\} \land B \in C \Rightarrow B \in C \cap \{A, B, C\} \Rightarrow C \cap \{A, B, C\} \neq \emptyset$$

$$A \in \{A, B, C\} \land A \in B \Rightarrow A \in B \cap \{A, B, C\} \Rightarrow B \cap \{A, B, C\} \neq \emptyset$$

thus the only other choice is A, and therefore $A \cap \{A, B, C\} = \emptyset$. Therefore $C \notin A$, as desired.

1.5.6

Let A, B be sets. Using the subset and power set axioms, show that the set $\mathcal{P}(A) \cap B$ exists.

Because set A exists we follow that $\mathcal{P}(A)$ exists. By setting $\phi(x) : x \in B$ and subset axiom we follow that there exists a subset of $\mathcal{P}(A)$ such that $x \in S \Leftrightarrow x \in \mathcal{P}(A) \land x \in B$. Thus we follow by Extensionality axiom that $S = \mathcal{P}(A) \cap B$. Thus $\mathcal{P}(A) \cap B$ exists.

1.5.7

Let A, B be sets. Using the subset axiom, show that the set $A \setminus B$ exists.

$$\phi(x) : \neg x \in B$$

thus by subset axiom

$$x \in S \Leftrightarrow x \in A \land \neg x \in B$$

thus $A \setminus B$ exists.

1.5.8

Show that no two of the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are equal to each other.

I had a little confusion with this one at first because I thought that every set has empty set in it, which is false. Every set has an empty set as a subset, but it might be so that empty set is not in the set itself.

$$\emptyset \notin \emptyset \land \emptyset \in \{\emptyset\} \Rightarrow \emptyset \neq \{\emptyset\}$$

$$\emptyset \notin \emptyset \land \emptyset \in \{\emptyset, \{\emptyset\}\} \Rightarrow \emptyset \neq \{\emptyset, \{\emptyset\}\}$$

$$\{\emptyset\} \notin \{\emptyset\} \land \{\emptyset\} \in \{\emptyset, \{\emptyset\}\} \Rightarrow \{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$$

all of the implication follow from extensionality axiom.

1.5.9

Let A be a set with no elements. Show that for all x, we have that $x \in A$ if and only if $x \in \emptyset$. Using the extensionality axiom, conclude that $A = \emptyset$.

Suppose that $\neg x \in A$. Then we follow that x is an element, therefore $\neg x \in \emptyset$. Thus

$$\neg x \in A \Rightarrow \neg x \in \emptyset \iff x \in \emptyset \Rightarrow x \in A$$

Suppose that $\neg x \in \emptyset$. Then we follow that x is an element. Thus $\neg x \in A$. Thus

$$\neg x \in \emptyset \Rightarrow \neg x \in A \iff x \in A \Rightarrow x \in \emptyset$$

thus we follow that

$$x \in \emptyset \Leftrightarrow x \in A$$

thus by extensionality axiom we follow that

$$\emptyset = A$$

which gives us nice follow-up that

$$\emptyset = \{\}$$

1.5.10

Let $\phi(x, y)$ be the formula $\forall z(z \in y \Leftrightarrow z = x)$ which asserts that $y = \{x\}$. For all x the set $\{x\}$ exists. So $\forall x \exists! y \phi(x, y)$. Let A be a set. Show that the collection $\{\{x\} : x \in A\}$ is a set.

We know that A is a set and therefore $\mathcal{P}(A)$ is also a set. Thus by subset axiom there exists a set

$$\exists S(x \in S \Leftrightarrow x \in \mathcal{P}(A) \wedge \exists(y \in A)(\phi(x, y)))$$

which is precisely our collection.

Chapter 2

Basic Set-Building Axioms and Operations

2.1 The First Six Axioms

Prove the following theorems, where A, B, C, D are sets.

2.1.1

$$A \subseteq B \to (A \subseteq A \cup B \wedge A \cap B \subseteq A)$$

$$\forall x(x \in A \to x \in B) \to ((\forall x(x \in A \Rightarrow x \in A \vee x \in B)) \wedge (\forall (x \in A \wedge x \in B \Rightarrow x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \to x \in B) \to ((\forall x(\neg x \in A \vee x \in A \vee x \in B)) \wedge (\forall (\neg (x \in A \wedge x \in B) \vee x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \to x \in B) \to ((\forall x(\text{true} \vee x \in B)) \wedge (\forall (\neg x \in A \vee \neg x \in B \vee x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \to x \in B) \to (\text{true} \wedge (\forall (true \vee \neg x \in B))) \Leftrightarrow$$

$$\Leftrightarrow \neg \forall x(x \in A \to x \in B) \vee (\text{true} \wedge \text{true}) \Leftrightarrow$$

$$\Leftrightarrow \neg \forall x(x \in A \to x \in B) \vee \text{true} \Leftrightarrow$$

$$\text{true}$$

2.1.2

$$A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$$

$$(\forall x(x \in A \Rightarrow x \in B)) \wedge (\forall x(x \in B \Rightarrow x \in C)) \rightarrow \forall x(x \in A \Rightarrow x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x(\neg x \in A \vee x \in B)) \wedge (\forall x(\neg x \in B \vee x \in C)) \rightarrow \forall x(\neg x \in A \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \vee x \in B) \wedge (\neg x \in B \vee x \in C))) \rightarrow \forall x(\neg x \in A \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \wedge (\neg x \in B \vee x \in C)) \vee (x \in B \wedge (\neg x \in B \vee x \in C)))) \rightarrow \forall x(\neg x \in A \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x(\neg x \in A \wedge (\neg x \in B \vee x \in C)) \vee ((x \in B \wedge \neg x \in B) \vee (x \in B \wedge x \in C)))) \rightarrow \forall x(\neg x \in A \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \wedge \neg x \in B) \vee (\neg x \in A \wedge x \in C) \vee (x \in B \wedge x \in C)) \rightarrow \forall x(\neg x \in A \vee x \in C) \Leftrightarrow ...$$

So this thing is tedious as hell and should be left to computers.

Suppose that $x \in A$. Then we follow by $A \subseteq B$ that $x \in B$. Thus by $B \subseteq C$ we follow that$x \in C$. Therefore $x \in A \rightarrow x \in C$. Therefore $A \subseteq C$, as desired.

2.1.3

$$B \subseteq C \Rightarrow A \setminus C \subseteq A \setminus B$$

Suppose that $x \in A \setminus C$. Then we follow that $x \in A$ and $x \notin C$. Therefore $x \in A$ and $x \notin B$ since $B \subseteq C$. Thus $x \in A \setminus B$. Therefore we follow that $B \subseteq C$ implies that $A \setminus C \subseteq A \setminus B$, as desired.

2.1.4

$$C \subseteq A \wedge C \subseteq B \iff C \subseteq A \cap B$$

Suppose that $x \in C$. Then we follow that $x \in A$ and $x \in B$. Thus $x \in A \cap B$. Therefore $C \subseteq A \cap B$. Thus we follow that $C \subseteq A \wedge C \subseteq B \Rightarrow C \subseteq A \cap B$

Suppose that $x \in C$. Then we follow that $x \in A \cap B$. Thus $x \in A$ and $x \in B$. Therefore $C \subseteq A \wedge C \subseteq B$. Therefore $C \subseteq A \cap B \Rightarrow C \subseteq A \wedge C \subseteq B$ thus we follow that

$$C \subseteq A \wedge C \subseteq B \iff C \subseteq A \cap B$$

as desired.

2.1.5

There exists an x such that $x \notin A$

Suppose that there does not exist x such that $x \notin A$. Then we follow that every set is a member of A, which is impossible.

2.1.6

$$A \cap B = B \cap A$$

$$x \in A \cap B \iff x \in A \wedge x \in B \iff x \in B \wedge x \in A \iff x \in B \cap A$$

2.1.7

$$A \cup B = B \cup A$$

$$x \in A \cup B \iff x \in A \vee x \in B \iff x \in B \vee x \in A \iff x \in B \cup A$$

2.1.8

$$A \cap (B \cup C) = (A \cup C) \cap (A \cup B)$$

$$x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in (B \cup C) \Leftrightarrow x \in A \wedge (x \in B \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in A \vee x \in C) \Leftrightarrow (x \in A \cup B) \vee (x \in A \cup C) \Leftrightarrow x \in ((A \cup B) \cap (A \cup C))$$

2.1.31

$$A \subseteq \mathcal{P}(\cup(A))$$

Let $x \in A$. Then we follow that $x \subseteq \cup(A)$. Thus $x \in \mathcal{P}(A)$. Thus $A \subseteq \mathcal{P}(\cup(A))$.

2.1.32

Let $C \in F$. Then $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\cup F))$

Suppose that $C \in F$. Then we follow that $C \subseteq \cup F$. Therefore $C \in \mathcal{P}(\cup F)$. Thus $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\cup F))$.

the rest of the exercises for this section are more of the same.

2.2 Operations on Sets

Prove the following theorems

2.2.1

Let A be a set and $F \neq \emptyset$. Then

$$A \setminus \cap F = \cup\{A \setminus C : C \in F\}$$

$$x \in A \setminus \cap F \Leftrightarrow x \in A \wedge x \notin \cap F \Leftrightarrow x \in A \wedge \neg x \in \cap F \Leftrightarrow x \in A \wedge \neg(\forall(C \in F)(x \in C)) \Leftrightarrow$$

$$\Leftrightarrow x \in A \wedge \exists(C \in F)(x \notin C) \Leftrightarrow \exists(C \in F)(x \notin C \wedge x \in A) \Leftrightarrow \exists(C \in F)(x \in A \setminus C) \Leftrightarrow x \in \cup\{A \setminus C : C \in F\}$$

which seems to hold.

2.2.2

Let A, F be sets. Then $A \cup (\cup F) = \cup\{A \cup C : C \in F\}$

$$x \in A \cup (\cup F) \Leftrightarrow x \in A \vee x \in \cup F \Leftrightarrow x \in A \vee (\exists C \in F)(x \in C) \Leftrightarrow$$
$$\Leftrightarrow (\exists C \in F)(x \in A) \vee \exists(C \in F)(x \in C) \Leftrightarrow$$
$$\Leftrightarrow \exists(C \in F)(x \in A \vee x \in C) \Leftrightarrow \exists(C \in F)(x \in A \cup C) \Leftrightarrow x \in \cup\{A \cup C : C \in F\}$$

Where we've used the fact that

$$x \in A \Leftrightarrow x \in A \wedge \text{true} \Leftrightarrow x \in A \wedge (\exists C \in F)(\text{true}) \Leftrightarrow (\exists C \in F)(x \in A \wedge \text{true}) \Leftrightarrow (\exists C \in F)(x \in A)$$

don't know if we can use it, but I used it anyways.

2.2.3

Let A, F be sets. Then $A \cap (\cup F) = \cup\{A \cap C : C \in F\}$

$$x \in A \cap (\cup F) \Leftrightarrow x \in A \wedge x \in \cup F \Leftrightarrow x \in A \wedge (\exists C \in F)(x \in C) \Leftrightarrow$$
$$\Leftrightarrow \exists(C \in F)(x \in A \wedge x \in C) \Leftrightarrow \exists(C \in F)(x \in A \cap C) \Leftrightarrow x \in \cup\{A \cap C : C \in F\}$$

2.2.5

Let A and F be sets. Then there exists a unique set ϵ such that for all Y we have that $Y \in \epsilon$ if and only if $Y = A \cap C$ for some $C \in F$.

$\cup F$ exists by union axiom, $A \cap (\cup F)$ exists by subset axiom. Thus $\mathcal{P}(A \cap (\cup F))$ exists by power axiom. Since $Y = A \cap C \Rightarrow Y \subseteq A \cap (\cup F)$, we follow that Y is a subset of $\mathcal{P}(A \cap (\cup F))$, which exists by subset axiom. By extensionality axiom we follow that the set is unique.

2.2.12

If F and G are nonempty sets, then

$$\cap(F \cup G) = \cap(F) \cap \cap(G)$$

$$x \in \cap(F \cup G) \Leftrightarrow (\forall C \in F \cup G)(x \in C) \Leftrightarrow (\forall C \in F)(x \in C) \wedge (\forall C \in G)(x \in C) \Leftrightarrow$$

$$\Leftrightarrow x \in \cap(F) \wedge x \in \cap(G) \Leftrightarrow x \in (\cap(F)) \cap (\cap(G))$$

2.2.14

Let F be a nonempty set. Then

$$\mathcal{P}(\cap(F)) = \cap\{\mathcal{P}(C) : C \in F\}$$

$$x \in \mathcal{P}(\cap(F)) \Leftrightarrow x \subseteq \cap(F) \Leftrightarrow (\forall y \in x)(y \in \cap(F)) \Leftrightarrow (\forall y \in x)(\forall(C \in F)(y \in F)) \Leftrightarrow$$

$$\forall(C \in F)((\forall y \in x)y \in F) \Leftrightarrow \Leftrightarrow \forall(C \in F)(x \subseteq C) \Leftrightarrow \forall(C \in F)(x \in \mathcal{P}(C)) \Leftrightarrow x \in \cap\{\mathcal{P}(C) : C \in F\}$$

Chapter 3

Relations and Functions

3.1 Ordered Pairs in Set Theory

3.1.1

Define $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle$ for any sets a, b, c. Prove that this yields an ordered tuple; that is, prove tahat if $\langle x, y, z\rangle = \langle a, b, c\rangle$, then $x = a$, $y = b$, $z = c$.

Suppose that

$$\langle x_1, x_2, x_3\rangle = \langle y_1, y_2, y_3\rangle$$

then we follow that

$$\langle\langle x_1, x_2\rangle, x_3\rangle = \langle\langle y_1, y_2\rangle, y_3\rangle$$

from which we get that $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$ and $x_3 = y_3$. From $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$ we get that $x_1 = y_1$ and $x_2 = y_2$. In total we get that

$$\langle\langle x_1, x_2\rangle, x_3\rangle = \langle\langle y_1, y_2\rangle, y_3\rangle \Rightarrow x_1 = y_1 \wedge x_2 = y_2 \wedge x_3 = y_3$$

Thus we follow that given construction defines an ordered tuple, as desired.

3.1.2

Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$

$$x \in (A \cup B) \times C \Leftrightarrow x = \langle y, z\rangle \wedge y \in A \cup B \wedge z \in C \Leftrightarrow x = \langle y, z\rangle \wedge (y \in A \vee y \in B) \wedge z \in C$$

$$\Leftrightarrow (x = \langle y, z\rangle \wedge z \in C) \wedge (y \in A \vee y \in B) \Leftrightarrow$$

$$\Leftrightarrow (x = \langle y, z\rangle \wedge z \in C \wedge y \in A) \vee (x = \langle y, z\rangle \wedge z \in C \wedge y \in B) \Leftrightarrow$$

$$\Leftrightarrow (x \in A \times C) \vee (x \in B \times C) \Leftrightarrow x \in (A \times C) \cup (B \times C)$$

as desired.

3.1.3

Prove that $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$

$$x \in (A \setminus B) \times C \Leftrightarrow x = \langle y, z \rangle \wedge y \in A \setminus B \wedge z \in C \Leftrightarrow x = \langle y, z \rangle \wedge (y \in A \wedge y \notin B) \wedge z \in C$$
$$\Leftrightarrow (x = \langle y, z \rangle \wedge z \in C) \wedge (y \in A \wedge y \notin B) \Leftrightarrow$$
$$\Leftrightarrow x = \langle y, z \rangle \wedge z \in C \wedge y \in A \wedge y \notin B \Leftrightarrow$$
$$\Leftrightarrow (x = \langle y, z \rangle \wedge y \in A \wedge z \in C) \wedge (x \neq \langle y, z \rangle \vee y \notin B \vee z \notin C) \Leftrightarrow$$
$$\Leftrightarrow (x = \langle y, z \rangle \wedge y \in A \wedge z \in C) \wedge (x \neq \langle y, z \rangle \vee y \notin B \vee z \notin C) \Leftrightarrow$$
$$\Leftrightarrow (x = \langle y, z \rangle \wedge y \in A \wedge z \in C) \wedge \neg(x = \langle y, z \rangle \wedge y \in B \wedge z \in C)) \Leftrightarrow$$
$$\Leftrightarrow (x \in A \times C) \wedge \neg(x \in B \times C) \Leftrightarrow x \in (A \times C) \setminus (B \times C)$$

Used a biconditional defined in "useful things"

3.1.4

Prove that

$$(\cup F) \times C = \cup\{A \times C : A \in F\}$$

$$x \in (\cup F) \times C \Leftrightarrow x = \langle y, z \rangle \wedge y \in (\cup F) \wedge z \in C \Leftrightarrow x = \langle y, z \rangle \wedge (\exists A \in F)(y \in A) \wedge z \in C \Leftrightarrow$$
$$\Leftrightarrow (\exists A \in F)(y \in A \wedge x = \langle y, z \rangle \wedge z \in C) \Leftrightarrow (\exists A \in F)(x \in A \times C) \Leftrightarrow$$
$$\Leftrightarrow x \in \cup\{A \times C : A \in F\}$$

3.2 Relations

3.2.1

Explain why the empty set is a relation

Relation is defined to be a set of ordered pairs. That is, for every $x \in R$, x is an ordered pair. Since we haven't got any elements in the emptyset, we follow that the logical statement is true and therefore emptyset is a relation.

Other way to see it is to assume that it is not a relation. Then we follow that emptyset has an element that is not an ordered pair. Since emptyset does not have any elements, we follow that we have a contradiction.

3.2.2

Prove items 1-3 of Theorem 3.2.7

$$x \in \mathrm{dom}(R^{-1}) \Leftrightarrow \exists y(\langle x, y\rangle \in R^{-1}) \Leftrightarrow \exists y(\langle y, x\rangle \in R) \Leftrightarrow x \in \mathrm{ran}(R)$$

$$x \in \mathrm{ran}(R^{-1}) \Leftrightarrow \exists y(\langle y, x\rangle \in R^{-1}) \Leftrightarrow \exists y(\langle x, y\rangle \in R) \Leftrightarrow x \in \mathrm{dom}(R)$$

$$x \in (R^{-1})^{-1} \Leftrightarrow \exists y\exists z(\langle y, z\rangle \in (R^{-1})^{-1}) \wedge x = \langle y, z\rangle \Leftrightarrow \exists y\exists z(\langle z, y\rangle \in (R^{-1})) \wedge x = \langle y, z\rangle \Leftrightarrow$$

$$\Leftrightarrow \exists y\exists z(\langle y, z\rangle \in R) \wedge x = \langle y, z\rangle \Leftrightarrow x \in R$$

3.2.4

$$\mathrm{dom}(R) = \{0, 1, 2, 3, 4\}$$

$$\mathrm{ran}(R) = \{0, 1, 2, 3, 4\}$$

$$R \circ R = \{\langle 0, 2\rangle, \langle 0, 3\rangle, \langle 0, 0\rangle, \langle 0, 3\rangle, \langle 0, 4\rangle, \langle 1, 0\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 1, 3\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle,$$

$$\langle 2, 2\rangle, \langle 2, 3\rangle, \langle 2, 2\rangle, \langle 2, 4\rangle, \langle 3, 3\rangle, \langle 3, 2\rangle, \langle 4, 4\rangle\}$$

$$R|\{1\} = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$$

$$R^{-1}|\{1\} = \{\langle 1, 0\rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

3.2.5

Suppose that R is a relation. Prove that $R|(A \cup B) = (R|A) \cup (R|B)$ for any sets A, B

$$x \in R|(A \cup B) \Leftrightarrow (\exists y \in A \cup B)(\exists z \in \mathrm{ran}(R))(\langle y, z\rangle \in R \wedge x = \langle y, z\rangle) \Leftrightarrow$$

$$\Leftrightarrow (\exists y \in A)(\exists z \in \mathrm{ran}(R))(\langle y, z\rangle \in R \wedge x = \langle y, z\rangle) \vee (\exists y \in B)(\exists z \in \mathrm{ran}(R))(\langle y, z\rangle \in R \wedge x = \langle y, z\rangle) \Leftrightarrow$$

$$\Leftrightarrow x \in R|A \vee x \in R|B \Leftrightarrow x \in (R|A \cup R|B)$$

thus

$$R|(A \cup B) = (R|A) \cup (R|B)$$

as desired.

3.2.7

Let R and S be two relations and let A, B, C be sets. Prove that $R|A$, $R^{-1}[B]$, $R[C]$ and $R \circ S$ are sets.

Given that R and S are relation, we follow that both of them are sets, $\bigcup\bigcup R$ and $\bigcup\bigcup S$ are sets and $\operatorname{dom}(R), \operatorname{ran}(R), \operatorname{dom}(S), \operatorname{ran}(S)$ are sets. Thus we follow that $R|A$ is a subset of R, which is a set; $R^{-1}[B]$ and $R[C]$ are subsets of $\bigcup\bigcup R$, and $R \circ S$ are subsets of a set $\operatorname{dom}(R) \times \operatorname{ran}(S)$, which is a set.

3.2.8

Let R be a relation and G be a set. Prove that $\{R[C] : C \in G\}$ is a set. Prove that if G is nonempty, then $\{R[C] : C \in G\}$ is also nonempty

If R is a relation, then $\operatorname{ran}(R)$ is a set. Therefore $\mathcal{P}(\operatorname{ran}(R))$ is a set. Thus for any set C, $R[C] \subseteq \operatorname{ran}(R)$, therefore $R[C] \in \mathcal{P}(\operatorname{ran}(R))$. Thus $\{R[C] : C \in G\}$ is a subset of $\mathcal{P}(\mathcal{P}(\operatorname{ran}(R)))$, which is a set.

Suppose that G is nonempty. Then we follow that there exists $C \in G$. Thus $R[C]$ is a set. Thus $R[C] \in \{R[C] : C \in G\}$. Therefore $\{R[C] : C \in G\}$ is nonempty.

3.2.10

Let R be a relation on A. Prove that R is symmetric if and only if

$$R^{-1} \subseteq R$$

In forward direction: Suppose that R is symmetric. Let $y \in R^{-1}$. We follow that there exists u, v such that $y = \langle u, v\rangle$. Thus $\langle v, u\rangle \in R$ by the definition. Since R is symmetric, we follow that $\langle u, v\rangle \in R$. Therefore $y \in R^{-1} \Rightarrow y \in R$, as desired.

In reverse direction: Suppose that $R^{-1} \subseteq R$. Let $y \in R$. Then we follow that there exists u, v such that $y = \langle u, v\rangle$. Thus $\langle v, u\rangle \in R^{-1}$. Since $R^{-1} \subseteq R$, we follow that $\langle v, u\rangle \in R$. Thus we follow that $\langle u, v\rangle \in R \Rightarrow \langle u, v\rangle \in R$. Thus R is symmetric by definition, as desired.

3.2.19

Prove item (2) of Theorem 3.2.8

$$R[\bigcup G] = \bigcup R[C] : C \in G$$

$$x \in R[\bigcup G] \Leftrightarrow (\exists y \in \bigcup G)(\langle y, x\rangle \in R) \Leftrightarrow (\exists C \in G)(y \in C \wedge \langle y, x\rangle \in R) \Leftrightarrow$$

$$\Leftrightarrow (\exists C \in G)(x \in R[C]) \Leftrightarrow x \in \bigcup R[C] : C \in G$$

3.2.20

Prove item (4) fo Theorem 3.2.8

$$x \in R[\bigcap G] \Leftrightarrow (\exists y \in \bigcap G)(\langle y, x\rangle \in R) \Leftrightarrow \exists y(\forall C \in G)(y \in C \wedge \langle y, x\rangle \in R) \Rightarrow$$

$$\Rightarrow (\forall C \in G)(\exists y \in C)(\langle y, x\rangle \in R) \Leftrightarrow (\forall C \in G)(x \in R[C]) \Leftrightarrow x \in \bigcap\{R[C] : C \in G\}$$

3.3 Functions

3.3.1

Prove Lemma 3.3.5 and Lemma 3.3.13

Suppose that F and G are functions such that $\mathrm{dom}(F) = \mathrm{dom}(G)$. Lemma 3.3.5 states that $F = G$ iff $F(x) = G(x)$ for every $x \in \mathrm{dom}(F)$ If $F = G$, then

$$F(x) = y \Leftrightarrow \langle x, y\rangle \in F \Leftrightarrow \langle x, y\rangle \in G \Leftrightarrow G(x) = y$$

thus $F(x) = G(x)$ for every $x \in \mathrm{dom}(F)$.

Now suppose that $F(x) = G(x)$ for every $x \in \mathrm{dom}(F)$. Then we follow that

$$z \in F \Leftrightarrow z = \langle x, y\rangle \wedge F(x) = y \Leftrightarrow z = \langle x, y\rangle \wedge G(x) = y \Leftrightarrow z \in G$$

as desired.

Lemma 3.3.13 states that a function F is one-to-one if and only if F is single-rooted.

Suppose that F is one-to-one and F is not single rooted. Then we follow that there exists $x, y \in F$ such that $x = \langle u, w\rangle \in F, y = \langle j, w\rangle \in F$. Then we follow that $F(u) = w = F(j)$, which is a contradiction.

Proof of converse is extremely simular.

3.3.2

Let F be a function and let $A \subseteq B \subseteq \mathrm{dom}(F)$. Prove that $F[A] \subseteq F[B]$.

$$x \in F[A] \Leftrightarrow x = \langle u, v\rangle \wedge u \in A \wedge \langle u, v\rangle \in F \Rightarrow x = \langle u, v\rangle \wedge u \in B \wedge \langle u, v\rangle \in F \Leftrightarrow x \in F[B]$$

3.3.5

Let $g : C \to D$ be a one-to-one function, $A \subseteq C$ and $B \subseteq C$. Prove that if $A \cup B = \emptyset$, then $g[A] \cap g[B] = \emptyset$.

Suppose that $A \cap B = \emptyset$ and $g[A] \cap g[B] \neq \emptyset$. Then we follow that there exists $x \in g[A] \cap g[B]$. Thus

$$x \in g[A] \wedge x \in g[B] \Leftrightarrow (\exists y \in A)(\langle y, x\rangle \in g) \wedge (\exists z \in B)(\langle z, x\rangle \in g)$$

since g is one-to-one, we follow that $z = y$. Thus there exists $z \in A \cap B$, therefore $A \cap B \neq \emptyset$, which is a contradiction.

3.3.9

Suppose that $F : X \to Y$ is a function. Prove that if $C \subseteq Y$ and $D \subseteq Y$, then $F^{-1}[C \cap D] = F^{-1}[C] \cap F^{-1}[D]$.

Since F is a function, we follow that F^{-1} is a single-rooted relation. Thus we follow that

$$F^{-1}[C \cap D] = F^{-1}[C] \cap F^{-1}[D]$$

as desired.

3.3.10

Let F, G be functions from A to B. Suppose $F \subseteq G$. Prove that $F = G$.

Suppose that $x \in A$. Then we follow that

$$(\exists y \in B)(\langle x, y\rangle \in F) \Rightarrow (\exists y \in B)(\langle x, y\rangle \in G)$$

Thus we follow that for every $x \in A$ (where $\text{dom}(F) = A = \text{dom}(G)$)

$$F(x) = G(x)$$

thus by the lemma 3.3.5 we follow that

$$F = G$$

as desired.

3.3.11

Let C be a set of functions. Suppose that for all f and g in C, we have either $f \subseteq g$ or $g \subseteq f$.

(a) Prove that $\cup C$ is a function

Firstly, since C is a set of sets of ordered pairs, we follow that $\cup C$ is a set of ordered pairs, and therefore it is a relation. Suppose that $x \in \cup C$. Then we follow that there exist $f \in C$ such that $x \in f$ and $x = \langle u, v\rangle$. Suppose that there exists $y \in \cup C$, such that $y = \langle u, v'\rangle$, where $u' \neq u$. Since $y \in \cup C$, we follow that there exists $g \in C$ such that $y \in C$.

Because $u' \neq u$, we follow that $g \neq f$. Therefore either $g \subset f$, or $f \subset g$. In both cases we follow that we can't have the case that $u' \neq u$. Thus we follow that for $x, y \in \cup C$, whenever the first part of the x is equal to the first part of y, we follow that the last parts are also equal. Thus we follow that $\cup C$ is a single-valued relation, and therefore it is a function.

3.3.13

Assume $f : A \to B$ is onto B. Let $C \subseteq B$. Prove that $f[f^{-1}[C]] = C$

$$x \in f[f^{-1}[C]] \Leftrightarrow (\exists y \in f^{-1}[C])(f(y) = x) \Leftrightarrow (\exists z \in C)(\langle y, z\rangle \in f \wedge f(y) = x) \Leftrightarrow$$

$$\Leftrightarrow (\exists z \in C)(f(y) = z \wedge f(y) = x) \Leftrightarrow (\exists z \in C)(x = z) \Leftrightarrow x \in C$$

this notation may be a bit sloppy, but the result is derived faithfully.

3.3.15

Let $f : A \to B$ be a one-to-one function. Define $G : \mathcal{P}(A) \to \mathcal{P}(B)$ by $G(X) = f[X]$, for each $X \in \mathcal{P}(A)$. Prove that G is one-to-one.

Let $X_1, X_2 \in \mathcal{P}(A)$ be such that $G(X_1) = G(X_2)$. Then we follow that

$$f[X_1] = f[X_2]$$

thus

$$x \in X_1 \Leftrightarrow (\exists y \in f[X_1])(\langle x, y\rangle \in f) \Leftrightarrow (\exists y \in f[X_2])(\langle x, y\rangle \in f) \Leftrightarrow x \in X_2$$

thus we follow that $X_1 = X_2$. Therefore $G(X_1) = G(X_2) \to X_1 = X_2$, thus G is one-to-one, as desired.

3.3.21

Let $\langle A_i : i \in I\rangle$ be an indexed function with nonempty terms. Prove that there is an indexed function $x_i : i \in I$ is that $x_i \in A_i$ for all $i \in I$, using theorem 3.3.24

Let C be defined as

$$C = ran(A)$$

then by theorem 3.3.24 we follow that there exists a function $H : C \to \cup C$ such that

$$H(A_i) \in A_i$$

define $x = H \circ A$. Then we follow that

$$x(i) = H(A(i)) = H(A_i) \in A_i$$

thus we have the desired function.

3.4 Order Relations

3.4.1

Define a relation $\preceq$ on the set of intezers Z by $x \preceq y$ if and only if $x \leq y$ and $x + y$ is even for all $x, y \in Z$. Prove that $\preceq$ is a partial order on Z

Suppose that $x, y \in Z$. Poset requirements of $\leq$ are going to be ommited.

Then we follow that

$$x + x = 2x$$

, therefore we've got symmetry. If $x \leq y$, $y \leq z$, $x + y$ is even and $y + z$ is even, then we follow that $x + y + y + z$ is even, therefore $x + z - 2y$ is also even. Antisymmetry follows from $\leq$.

Then answer the following questions about the poset $(Z, \preceq)$

(a) Is $S = \{1, 2, 3, 4, 5, 6, ...,\}$ a chain in Z

No, since $1 + 2 = 3$ is not even, we follow that $1 \npreceq 2 \wedge 2 \npreceq 1$.

(b) Is $S = \{1, 3, 5, 7...,\}$ a chain in Z?

Suppose that $x, y \in Z$. Then we follow that there exist $m, n \in Z$ such that

$$x = 2m + 1$$

$$y = 2n + 1$$

thus

$$x + y = 2m + 2n + 2$$

thus $x + y$ is even for all cases. Since $\leq$ is a total order, we follow that S is indeed a chain in Z.

(c) Does the set $S = \{1, 2, 3, 4, 5\}$ have a lower bound or an upper bound?

No, since we've got both odd and even numbers in S, we follow that there is no $x \in Z$ such that $x + s$ is even for all $s \in S$, thus there does not exist an element such taht $x \preceq s$ or $s \preceq x$ for all $s \in S$

(d) Does $S = \{1, 2, 3, 4, 5\}$ have any maximal or minimal elements?

Yes, $1, 2$ are minimal elements and $4, 5$ are maximal elements.

3.4.2

Prove Lemma 3.4.5

Suppose that $(A, \preceq)$ is a poset and let $\prec$ be the strict order corresponding to $\preceq$. Let $x, y, z \in A$. Then we follow that:

(1)

Since $x = x$, we follow that $x \nprec x$ by definition of a strict order

(2)

Suppose that $x \prec y$. Then we follow that $x \preceq y$ and $y \neq x$. Thus by antisymmetry of $\preceq$ we follow that $y \npreceq x$, and therefore $y \nprec x$ by definition.

(3)

Suppose that $x \prec y$ and $y \prec z$. Therefore we follow that $x \preceq y$ and $y \preceq z$, which gives us that $x \preceq z$. Suppose that $x = z$. Then we follow that $z \preceq y$ and thus $z = y$, which is a contradiction. Thus we follow that $x \preceq z$ and $x \neq z$, therefore $x \prec z$, as desired.

(4)

If $x = y$ then $x \nprec y$ and $y \nprec x$ by definition of strict order

Both $x \prec y$ and $y \prec x$ imply that $x \neq y$, and by case (2) we follow that for given $x, y \in A$ only one of $x = y$, $x \prec y$ or $y \prec x$ hold.

3.4.3

Find the greatest lower bound of the set $S = \{15, 20, 30\}$ in the poset $(A, |)$, where $A = \{n \in N : n > 0\}$. Now find the least upper bound of S.

5 and 60 (gcd and lcm) respectively.

3.4.4

Let $(A, \preceq)$ be a poset and let $S \subseteq A$. Suppose that b is the largest element of S. Prove that b is also the least upper bound of S.

Suppose that s is an upper bound of S. Then from the definition of the lower bound we follow that $x \prec s$ for every $x \in A$. Since $b \in s$ we follow that $b \prec s$. Thus b is the least upper bound (can't we just call it supremum?) of A by definition.

3.4.11

Let $(B, \preceq')$ be a poset. Suppose $h : A \to B$ is an injective function. Define the retalion $\preceq$ on A by $x \preceq y$ if an only if $h(x) \preceq' h(y)$, for all $x, y \in A$. Prove that $\preceq$ is a parial order.

Let $x, y, z \in A$. Since h is a function, we follow that $x = x$ implies that $h(x) = h(x)$, therefore $h(x) \preceq' h(x)$, thus $x \preceq x$. Thus we've got reflexivity.

If $x \preceq y$ and $y \preceq z$ we follow that $h(x) \preceq' h(y)$ and $h(y) \preceq h(z)$, from which we follow that $h(x) \preceq' h(z)$ and therefore $x \preceq z$.

Suppose that $x \preceq y$ and $y \preceq x$. Then we follow that $h(x) \preceq' h(y)$ and $h(y) \preceq' h(x)$. Thus $h(y) = h(x)$. Because h is one-to-one, we follow that this implies that $x = y$. Thus we've got antisymmetry. Therefore $\preceq$ is indeed a poset.

3.4.12

Let $(B, \preceq')$ be a totally ordered. Suppose $h : A \to B$ is an injective function. Define the retalion $\preceq$ on A by $x \preceq y$ if an only if $h(x) \preceq' h(y)$, for all $x, y \in A$. Prove that $\preceq$ is a parial order.

We follow that $\preceq$ is a poset from previous exercise. Suppose that $x, y \in A$. then we follow that $h(x) \preceq' h(y) \vee h(y) \preceq' h(x)$. Thus $x \preceq y \vee y \preceq x$. Therefore $\preceq$ is a total order.

3.4.14

Analogous to 4

3.4.14

Let $\preceq$ be a partial order of A. Let $C \subseteq A$. Show that $\preceq_C$ is a partial order on C. Show that if $\preceq$ is a total order on A, then $\preceq_C$ is a total order on C.

Everything follows directly from definitions.

3.4.17

Let $\preceq$ be a partial order on A. Show that $fld(\preceq) = A$.

For every $x \in A$ we've got that $x \preceq x$, thus we follow that $\langle x, x\rangle \in \preceq$, thus $x \in fld(\preceq)$. Thus $A \subseteq fld(\preceq)$. Since $fld(\preceq) \subseteq A$ we follow that $A = fld(\preceq)$, as desired.