My linear algebra exercises

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Preface

Exercises are from "Linear algebra done right" by Sheldon Axler, 3rd ed. I've already read this book before and completed some exercises from it. Right now I want to brush up the material once again, put all the proofs on a more durable material than paper and to prepare myself to what's gonna happen afterwards.

Glossary

FTLM - Fundamental Theorem of Linear Maps

Chapter 1

Vector Spaces

1.1 R^n and C^n

1.1.1

Suppose a and b are real numbers, not both 0. Find real nuber c and d such that

$$1/(a+bi) = c + di$$

$$\frac{1}{a+bi} = c + di$$
$$\frac{1}{a+bi} - c - di = 0$$
$$\frac{a-bi}{(a+bi)(a-bi)} = c + di$$
$$\frac{a-bi}{(a^2+b^2)} = c + di$$
$$\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = c + di$$

Thus $c = \frac{a}{a^2+b^2}$ and $d = -\frac{b}{a^2+b^2}$

1.1.2

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 *(meaning that its cube equals 1)*

$$(\frac{-1+\sqrt{3}i}{2})^3 = \frac{(-1+\sqrt{3}i)^3}{8} = \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)^2}{8} = \frac{(-1+\sqrt{3}i)(1-2\sqrt{3}i-3)}{8} =$$

$$= \frac{(-1+\sqrt{3}i)(-2-2\sqrt{3}i)}{8} = \frac{2+2\sqrt{3}i-2\sqrt{3}i+6}{8} = \frac{8}{8} = 1$$

as desired.

1.1.3

Find two distinct square roots of i

Square root of i, I assume, is a number, whose square is equal to i. Suppose that $(a+bi)^2 = i$. It follows that

$$(a+bi)^2 = a^2 + 2abi - b^2$$

So if we set

$$a = b = 1/\sqrt{2}$$

this equation holds. Also it holds for

$$a = b = -1/\sqrt{2}$$

maxima seems to agree with me on this one

1.1.4

Show that $\alpha+\beta = \beta+\alpha$ for all $\alpha, \beta \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$ and $\beta = a_2 + b_2 i$. It follows

$$\alpha+\beta = a_1+b_1i+a_2+b_2i = a_2+b_2i+a_1+b_1i = \beta+\alpha$$

as desired.

1.1.5

Show that $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ for all $\alpha, \beta, \lambda \in \boldsymbol{C}$

Let $\alpha = a_1 + b_1 i$, $\beta = a_2 + b_2 i$, $\lambda = a_3 + b_3 i$. It follows that

$$\alpha+(\beta+\lambda) = a_1+b_1i+(a_2+b_2i+a_3+b_3i) = (a_1+b_1i+a_2+b_2i)+a_3+b_3i = (\alpha+\beta)+\lambda$$

1.1.6

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

$$\alpha+(\beta+\lambda) = (a_1+b_1i)((a_2+b_2i)+(a_3+b_3i)) = ((a_1+b_1i)(a_2+b_2i))+(a_3+b_3i) = (\alpha\beta)\lambda$$

1.1.7

Show that for every $\alpha \in \boldsymbol{C}$ *there exists a unique* $\beta \in \boldsymbol{C}$ *such that* $\alpha + \beta = 0$

Suppose that there exist two different $\beta_1 \neq \beta_2$ such that $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$. It follows that

$$\beta_1 = \beta_1 + 0 = \beta_1 + \alpha + \beta_2 = \alpha + \beta_1 + \beta_2 = 0 + \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

1.1.8

Show that for every $\alpha \in \boldsymbol{C}$ *with* $\alpha \neq 0$ *there exists a unique* $\beta \in \boldsymbol{C}$ *such that* $\alpha\beta = 1$

Suppose that it is not true and there exist two different $\beta_1 \neq \beta_2$ such that

$$\alpha\beta_1 = 1 \textit{ and } \alpha\beta_2 = 1$$

it follows then that

$$\beta_1 = 1 * \beta_1 = \alpha\beta_2\beta_1 = \alpha\beta_1\beta_2 = 1 * \beta_2 = \beta_2$$

which is a contradiction. Therefore there exists only one unique β.

1.1.9

The rest of the section is the repetition of this kind of stuff. That is a lot of writing, and not a lot of thinking, so I'll skip it. I don't ususally like to skip sections, but I have aa feeling, that I've completed this thing on paper somewhere, and there is not much reason to rewrite it here.

1.2 Definition of Vector Space

1.2.1

Prove that $-(-v) = v$ *for every* $v \in V$*.*

For v there exists only one $-v$. For $-v$ there exists only one $-(-v))$.

Thus

$$v = v + 0 = v + (-v) + (-(-v)) = 0 + (-(-v)) = -(-v)$$

as desired (idk if it's true, I'm not good at axioms and stuff)

1.2.2

Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Suppose that $a \neq 0$, $v \neq 0$ but $av = 0$. It follows that there exist $1/a$ - multiplicative inverse of a. It follows that

$$1/a * av = 1/a * 0$$

$$1v = 0$$

$$v = 0$$

which is a contradiction. Thus either $a = 0$ or $v = 0$.

1.2.3

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Suppose that there exists $x_1 \neq x_2$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$. Thus

$$3x_1 = w - v = 3x_2$$

$$x_1 = \frac{1}{3}(w - v) = x_2$$

which is a contradiction.

Same can be stated from the fact that x is a unique additive inverse of $\frac{1}{3}(v - w)$.

1.2.4

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Additive indentity. Empty set does not have zero element in it. BTW $\{0\}$ is a vector space.

1.2.5

Show that n the defintition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \textit{ for all } v \in V$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

$$0v = 0$$

$$(1 - 1)v = 0$$

$$1v - 1v = 0$$
$$v - v = 0$$
$$v + (-v) = 0$$

1.2.6

Let ∞ and $-\infty$ denote two distinct object, neither of which is in R. Define an addition and multiplication on $R \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and the product of two real numbers is as usual, and for $t \in R$ define

$$t\infty = \begin{cases} -\infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ \infty \text{ if } t > 0 \end{cases}$$

$$t(-\infty) = \begin{cases} \infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ -\infty \text{ if } t > 0 \end{cases}$$

$$t + \infty = \infty + t = \infty$$
$$t + (-\infty) = (-\infty) + t = (-\infty)$$
$$\infty + \infty = \infty$$
$$(-\infty) + (-\infty) = (-\infty)$$
$$\infty + (-\infty) = 0$$

Is $R \cup \{\infty\} \cup \{-\infty\}$ a vector space over R? Explain.

I don't think that it is.

$$(t + \infty) - \infty = \infty - \infty = 0$$
$$t + (\infty - \infty) = t + 0 = t$$

thus

$$t + (\infty - \infty) \neq (t + \infty) - \infty$$

thus $R \cup \{\infty\} \cup \{-\infty\}$ is not associative, therefore it is not a vector space.

1.3 Subspaces

1.3.1

For each of the following subsets of F^3, determine whether it is a subspace of F^3:

(a) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$

Yes, it is. 0 is contained within it.

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

therefore

$$x_1 + y_1 + 2(x_2 + y_2) + 3(x_3 + y_3) = x_1 + 2x_2 + 3x_3 + y_1 + 2y_2 + 3y_3 = 0 + 0 = 0$$

therefore it is closed under addition

$$n(x_1, x_2, x_3) = (nx_1, nx_2, nx_3)$$

$$nx_1 + 2nx_2 + 3nx_3 = n(x_1 + 2x_2 + 3x_3) = 0n = 0$$

therefore it is closed under multiplication.

(b) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$

It's not a subspace, because it does not contain zero.

(c) $\{(x_1, x_2, x_3) \in F^3 : x_1x_2x_3 = 0\}$

It's not a subspace, because

$$(0, 1, 1) + (1, 0, 0) = (1, 1, 1)$$

therefore it's not closed under addition.

(d) $\{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}$

It's a subspace, proof is the same as in (a), can be seen more clearly when we rewrite constraint as

$$x_1 = 5x_3 \to x_1 + 0x_2 - 5x_3 = 0$$

1.3.2

Verify all the assertions in Example 1.35

(a) if $b \in F$, then

$$\{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

is a subspace of F^4 if and only if $b = 0$

If $b \neq 0$, then 0 is not an element of this set.

Proving that it's a subspace when $b = 0$ is trivial

(b) The set of continous real-valued functions on the interval $[0, 1]$ is a subspace of $R^{[0,1]}$.

$(kf) = kf$ by algebraic properties of continous functions. If f and g are continous, then $(f+g)$ is continous as well by the same property. $f(x) = 0$ is continous because it's a constant functions.

By the way, same (probably) applies to a set of uniformly continous functions.

(c) The set of differentiable real-valued functions on R is a subspace of R^R.

Same deal, algebraic proerties imply linearity, adn zero is included.

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $R^{(0,3)}$ if and only if $b = 0$.

Same deal as in previous one, $f'(2)$ needs to be equal to zero in order to include zero. Previous part does not include it, because it does not have specific restrictions on derivatives being particular values at particular places.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of C^∞.

Here we can take zero to be $(x_n) = 0$. Linearity is implied by aldebraic properties of limits of sequences.

1.3.3

Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(1) = 3f(2)$ is a subspace of $R^{[-4,4]}$.

Zero is included here. Suppose that f and g are functions in given set. It follows that

$$f'(1) + g'(1) = 3f(2) + 3g(2)$$

$$f'(1) + g'(1) = 3(f(2) + g(2))$$

$$(f+g)'(1) = 3(f+g)(2)$$

thus it's closed under addition.

$$(kf)'(1) = 3(kf)(2)$$

implies

$$kf'(1) = 3kf(2)$$

therefore it's closed under multiplication by scalar. Therefore we can state that given subset is a vector subspace.

1.3.4

analogous to previous

1.3.5

Is R^2 a subspace of the complex vector space C^2?

No, it's not closed under scalar multiplication.

1.3.6

(a) Is

$$\{(a, b, c) \in R^3 : a^3 = b^3\}$$

a subspace if R^3?

Yes. it is. $a^3 = b^3 \to a = b \to a - b = 0$, the rest of proof is trivial.

(b) Is

$$\{(a, b, c) \in C^3 : a^3 = b^3\}$$

a subspace if C^3?

I want to say no to this one, example is

$$(1/2 + i\frac{\sqrt{3}}{2}, -1, 0) + (1/2 - i\frac{\sqrt{3}}{2}, -1, 0) = (1, -1, 0)$$

thus it's not closed under additon.

1.3.7

Give an example of a nonemplty subset U of R^2 such that U is closed under addition and under additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of R^2

Q^2. On the other thouhgh, Z will do as well.

1.3.8

Give an example of a nonempty subset U of R^2 such that U is closed under scalar multiplication, but U is not a subspace of R^2.

Two lines through origin.

1.3.9

A function is called periodic if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in R$. Is the set of periodic functions from R to R a subspace of R^R? Explain.

Zero is a periodic function. Set is certainly closed under scalar multiplication.

Suppose that f and g are both periodic and f has a period of $p1$ and g has a period of $p2$. Thus if $p2/p1 \in I$, then functions will be constantly out of phase, therefore the set is not closed under addition. Thus this subset is not a subspace.

1.3.10

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Zero is included in any subspace, therefore zero is included.

Suppose that $u_1, u_2 \in U_1 \cap U_2$. It follows that for $z \in F$ $zu_1 \in U_1$ and $zu_1 \in U_2$ by closure of those two subspaces. Therefore $zu_1 \in U_1 \cap U_2$ for any scalar, thus the set is closed under scalar multiplication.

$u_1 + u_2 \in U_1$ and $u_1 + u_2 \in U_2$ by closure under addition for both subspaces. Thus $u_1 + u_2 \in U_1 \cap U_2$ for any such vectors. Therefore the set is closed under addition.

Thus the set satisfies all requirements to be a subspace. Therefore it is a subspace.

1.3.11

Prove that the interseection of every collection of subspace of V is a subspace of V

Intersection of two subspaces is a subspace. Therefore by induction intersection of any finite collection of subspaces is a subspace.

Suppose that Λ is an arbitrary collection of subspaces. Every subspace contains a zero element, therefore

$$0 \in \cap\Lambda$$

Any vector in $\cap\Lambda$ will be closed under scalar multiplication for every $U \in \Lambda$. Thus, it will be contained in every $U \in \Lambda$. Therefore it is contained in $\cap\Lambda$.

Any two vectors in $\cap\Lambda$ will be closed under addition, for every $U \in \Lambda$. Thus, their sum will be contained in every $U \in \Lambda$. Therefore it is contained in $\cap\Lambda$.

Thus $\cap\Lambda$ is a vector space.

1.3.12

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Suppose that a union of two subspaces $U_1 \cup U_2$ is a subspace of V.

Zero is included in eveery subspace, so in case of the union we don't worry about it. Scalar multiplication is also trivial, as we are working only with one vector.

Now for the interesting part: addition. Let $u_1, u_2 \in U_1 \cup U_2$. In case when u_1, u_2 are contained only in one subspace we've got a trivial case. Interesting part comes when $u_1 \in U_1$ and $u_2 \in U_2$.

What we want to prove is that it is impossible to have $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$ and we're going to use contradiction. Suppose that $u_1 \in U_1 \setminus U_2$, $u_2 \in U_2 \setminus U_1$ and $u_1 + u_2 \in U_1 \cup U_2$. Thus it must be the case that $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$. Suppose that the former is true; then it follows that $u_1 + u_2 - u_1 = u_2 \in U_1$, which is a contradiction (same thing happens if we assume the latter). Thus given case is impossible. Therefore there cannot exist $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. Thus $U_1 = U_1 \cup U_2$ or $U_2 = U_1 \cup U_2$.

The reverse case is trivial: if we have two subspaces and one of it is a subset of another, then larger subspace is is subspace.

1.3.13

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Same thing applies as in previous exercise: zero and multiplication are trivial.

We are going to proceed with a proof by contradiction, but firstly we want to state precisely what we want to prove in a first place. We want to state, that if a union of three subspaces is a subspace, then this union is equal to one of the subpsaces. So let us start: suppose that the union of three subspaces is not equal to one of the subspaces.

Firstly, we can eliminate the case, when one of the subspaces is a subset of another subspace, but third isn't, because it will mean that union of first two subspaces constitues a subspace, and thus we'll default to result in the previous exercise.

Thus let us assume that none of the subspaces is a subset of another subspace. Now we've got two cases to sort out: suppose that if we take $u_2 \in U_2$ and $u_3 \in U_3$ we get that

$$u_2 + u_3 \in U_1$$

for every $u_2 \in U_2$ and $u_3 \in U_3$. Then we can follow, by setting $u_2 = 0$ to the case that

$$\forall u_3 \in U_3 \to u_3 + u_2 \in U_1 \to u_3 + 0 \in U_1 \to u_3 \in U_1$$

thus U_3 is a subset of U_1, which raises a contradiction (in our assumptions that U_3 is not a subset of U_1 and by extension for the default 2-subspace case).

The case when $u_2 \in U_2$, $u_3 \in U_3$ and $u_2 + u_3 \notin U_1 \cup U_2 \cup U_3$ implies that $U_1 \cup U_2 \cup U_3$ is not a vector space, thus it cannot happen.

The case when $u_2 \in U_2$, $u_3 \in U_3$ and $u_2 + u_3 \notin U_1$ implies that $u_2 + u_3$ is in $U_2 \cup U_3$. This raises the case that U_2 is a subspace of U_3, which is a contradiction.

Thus we can follow that there exists $u_1 \in U_1$ such that it cannot be represented in terms of vectors from U_2 and U_3. Thus we can follow that analogous vectors $u_2 \in U_2$ and $u_3 \in U_3$ also exist.

Because we are still assuming that $U_1 \cup U_2 \cup U_3$ we can follow that

$$u_1 + u_2 + u_3 \in U_1 \cup U_2 \cup U_3$$

Thus this sum is bound to be located in one of the U_1, U_2 or U_3. Let us assume for simplicity of notation that it is located in U_1. Then we can follow that

$$u_1 + u_2 + u_3 - u_1 = u_2 + u_3 \in U_1$$

Suppose that we take $u_2 \in U_2 \setminus (U_3 \cup U_1)$ and $u_3 \in U_3 \setminus (U_1 \cup U_2)$. It follows that $u_2 + u_3$ cannot be in either U_2 nor in U_3 because in this case we have that

$$u_2 + u_3 - u_2 = u_3 \in U_2$$

which is a contradiction. Thus

$$u_2 + u_3 \in U_1 \setminus (U_2 \cup U_3)$$

let us call it u_1'. In the same fashion we can define u_2' and u_3'.

Thus $u_1' + u_2' + u_3' \in U_1 \cup U_2 \cup U_3$. Thus it needs to be in one of U_1, U_2 or U_3. Suppose that it is included in U_1. Then we can follow that

$$u_1' + u_2' + u_3' \in U_1$$

$$u_2' + u_3' \in U_1$$

$$u_1 + u_3 + u_1 + u_2 \in U_1$$

$$2u_1 + u_3 + u_2 \in U_1$$

$$u_3 + u_2 \in U_1$$

TODO

1.3.14

Verify the assertion in Example 1.38

1.38 states that

Suppose that $U = \{(x, x, y, y) \in F^4 : x, y \in F\}$ *and* $W = \{(x, x, x, y) \in F^4 : x, y \in F\}$. *Then*

$$U + W = \{(x, x, y, z) \in F^4 : x, y, z \in F\}$$

as you should verify

Let $u \in U$ and $w \in W$. It follows that

$$u = (x_1, x_1, x_1, y_1)$$

$$w = (x_2, x_2, y_2, y_2)$$

Suppose that $q \in U + W$. It follows that

$$q = (x_1 + x_2, x_1 + x_2, x_1 + y_2, y_1 + y_2)$$

thus we can set $x = x_1 + x_2$, $y = x_1 + y_2$ and $z = y_1 + y_2$ and call it a day.

1.3.15

Suppose U is a subspace of V. What is $U + U$.

By properties of vector space, if we take $u_1, u_2 \in U$ then

$$u_1 + u_2 \in U$$

for every $u_1, u_2 \in U$. Thus we can follow that

$$U + U = U$$

1.3.16

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

If $q \in U + W$ it follows that there exists $u \in U$ and $w\imath W$ such that

$$q = v + w = w + v = q'$$

where $q' \in W + U$. Thus we can follow that $W + U = U + W$.

1.3.17

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Yes it is. We can apply the same logic as in the previous exercise and it'll do the job.

1.3.18

Does the operation of addition on the subspaces of V have an additive identity? Which subspace have additive inverces?

Every subspace contains zero, therefore

$$U + 0 = U$$

thus we've got additive identity.

By adding two subspaces together we get a larger subspace, thus we can follow that the only way to get 0 vector space as the result of addition of two subspaces is to add

$$0 + 0 = 0$$

thus the only subspace that contains additive inverse is 0.

Prove or give counterexample: if U_1, U_2, W are subspaces of V, such that

$$U_1 + W = U_2 + W$$

then $U_1 = U_2$

This is wrong: suppose that U_2 is a nonzero subspace of W and $U_1 = 0$. Then it follows that

$$U_1 + W = 0 + W = W = W + U_2$$

and

$$U_1 \neq U_2$$

as desired.

1.3.19

Suppose

$$U = \{(x, x, y, y) \in F^4 : x, y \in F\}$$

Find a subspace W *of* F^4 *such that* $F^4 = U \bigoplus W$

$$W = \{(0, x, y, 0) \in F^4 : x, y \in F\}$$

1.3.20

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in F^5 : x, y \in F\}$$

Find a subspace W *of* F^5 *such that* $F^5 = U \oplus W$

$$W = \{(0, 0, x, y, z) \in F^5 : x, y, z \in F\}$$

1.3.21

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in F^5 : x, y \in F\}$$

Find a thee subspaces W_1, W_2, W_3 *of* F^5 *such that* $F^5 = U \oplus W_1 \oplus W_2 \oplus W_3$

$$W_1 = \{(0, 0, x, 0, 0) \in F^5 : x \in F\}$$
$$W_2 = \{(0, 0, 0, y, 0) \in F^5 : y \in F\}$$
$$W_3 = \{(0, 0, 0, 0, z) \in F^5 : z \in F\}$$

1.3.22

Prove or give a counterexample: if U_1, U_2, W *are subspaces of* V *such that*

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

then $U_1 = U_2$

This one is false;

$$U_1 = \{(x, x) \in F^2 : x \in F\}$$
$$U_2 = \{(x, 0) \in F^2 : x \in F\}$$
$$W = \{(0, y) \in F^2 : y \in F\}$$

1.3.23

A function $f : R \to R$ is called even if

$$f(-x) = f(x)$$

for all $x \in R$. A function $f : R \to R$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in R$. Let U_e denote the set of real-valued even functions on R and let U_o denote the set of real-valued odd functions on R. Show that

$$R^R = U_e \oplus U_o$$

Let $f : R \to R$ be arbitrary. It follows that

$$f_e(x) = \begin{cases} 2f(x) - f(-x) \text{ if } x \geq 0 \\ f(x) \text{ if } x = 0 \\ 2f(-x) - f(x) \text{ if } x < 0 \end{cases}$$

Every odd function satisfies $f(0) = 0$. Therefore for even function we've got to have $f_e(0) = f(0)$

$$f_e(x) = \begin{cases} a_1 f(x) + b_1 f(-x) \text{ if } x > 0 \\ a_1 f(-x) + b_1 f(x) \text{ if } x < 0 \end{cases}$$

$$f_o(x) = \begin{cases} a_2 f(x) + b_2 f(-x) \text{ if } x > 0 \\ -a_2 f(-x) - b_2 f(x) \text{ if } x < 0 \end{cases}$$

$$\begin{cases} a_1 + a_2 = 1 \\ b_1 + b_2 = 0 \\ a_1 - a_2 = 0 \\ b_1 - b_2 = 1 \end{cases}$$

$$\begin{cases} a_1 = 0.5 \\ b_1 = 0.5 \end{cases}$$

$$f_e(x) = \begin{cases} 1/2 f(x) + 1/2 f(-x) \text{ if } x > 0 \\ f(x) \text{ if } x = 0 \\ 1/2 f(x) + 1/2 f(-x) \text{ if } x < 0 \end{cases}$$

$$f_o(x) = \begin{cases} 1/2f(x) - 1/2f(-x) \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1/2f(-x) + 1/2f(x) \text{ if } x < 0 \end{cases}$$

Thus

$$f_e(x) = f_e(-x)$$

$$f_o(-x) = -f_o(x)$$

and

$$f_e(x) + f_o(x) = f(x)$$

as desired.

Also, the only function that is odd and even at the same time is 0, therefore we've got a direct sum, as desired.

Chapter 2

Finite-Dimentional Vector Spaces

2.1 Span and Linear Independence

2.1.1

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v4$$

also spans V.

Let $v \in V$ be represented as

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$$

then we can follow that

$$v = a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4$$

therefore any $v \in V$ can be represented using given list, therefore given list spans V, as desired.

2.1.2

Verify the assertion in Example 2.18

Suppose that $v \in V$. Then it follows from some exercise in previous chapter that $a_1 v = 0$ iff $a_1 = 0$ or $v = 0$. Thus if $v \neq 0$ we can follow that the only way to represent zero is to set a_1 to 0. Thus list is linearly independent.

Suppose that we've got linearly independent list of two vectors. We therefore can follow that the only way to represent 0 is to set $a_1 = 0$ and $a_2 = 0$. Thus vectors are not a scalar multiples of each other. In other directon we've got a trivial case.

For the list

$$v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,01,0)$$

we've got that

$$v = a_1v_1 + a_2v_2 + a_3v_3 = (a_1, a_2, a_3, 0)$$

therefore the only way to represent zero is to set all of a's into 0.

Same case applies for the last one.

2.1.3

Find a number t such that

$$(3,1,4), (2,-3,5), (5,9,t)$$

is not linearly inependent in R^3

The only way that this list is not linearly independent is if we can represent last vector as a linear combiination of the other two. Thus

$$\begin{cases} 3a_1 + 2a_2 = 51a_1 - 3a_2 = 9 \end{cases}$$

$$\begin{cases} 3a_1 + 2a_2 = 5a_1 = 9 + 3a_2 \end{cases}$$

$$3(9 + 3a_3) + 2a_2 = 5$$

$$27 + 9a_2 + 2a_2 = 5$$

$$11a_2 = -22$$

$$a_2 = -2$$

thus

$$a_1 = 3$$

therefore

$$3 * 4 - 5 * 2 = t$$

$$t = 2$$

2.1.4

Verify the assertion in the second bullet point in Example 2.20

$c = 8$ is the only solution such that third vector is a scalar multipe of first vector plus scalal multiple of second. Thus we can follow that the last vector is not in the span of first two, therefore the list is linearly independent.

2.1.5

(a) Show that if we think of C as a vector space over R, then the list $(1+i, 1-i)$ is linearly independent.

$$(1+i+1-i)/2 = 1$$
$$(1+i-1+i)/2 = i$$

thus the only way to represent 0 is to set all of a's to zero

(b) Show that if we think of C as a vector space over C, then the list $(1+i, 1-i)$ is linearly dependent

List (1) spans C, and its length is less that the length of given set. THus given set is linearly dependent.

2.1.6

Suppose v_1, v_2, v_3, v_4 is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v4$$

is also linearly independent.

As we've shown before, spans of two sets are equal, therefore the only way to represent 0 is to put all a's to 0.

2.1.7

Prove or give counterexample: If $v_1, v_2, ...v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, ...v_m$$

is linearly independent

Both sets span the same space and have the same length, therefore they are both linearly independent.

2.1.8

Trivial, equivalent to previous

2.1.9

Prove or give counterexample: If $v_1, ...v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

False: set $w_1 = -v_1$ and get the desired result.

2.1.10

Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, v_2 + w, ...v_m + w$ is linearly dependent, then $w \in span(v_1, v_2, ...v_m)$.

Suppose that resulting list is linearly dependent. It follows that there exists a way to represent

$$\sum_{n=1}^{m} a_n(v_n + w) = 0$$

such that not all a's are zeroes. Thus

$$\sum_{n=1}^{m} a_n(v_1 + w) = \sum_{n=1}^{m} (a_n w + a_n v_n) = \sum_{n=1}^{m} a_n w + \sum_{n=1}^{m} a_n v_n = w \sum_{n=1}^{m} a_n + \sum_{n=1}^{m} a_n v_n = 0$$

$$-w \sum_{n=1}^{m} a_n = \sum_{n=1}^{m} a_n v_n$$

$\sum_{n=1}^{m} a_n \neq 0$, because otherwise left side is zero and therefore right side is zero, which is not assumed.

$$w = \sum_{n=1}^{m} -\frac{a_n}{\sum_{j=1}^{m} a_j} v_n$$

thus $w \in span(v_1, v_2, ...v_m)$, as desired.

2.1.11

Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin span(v_1, ..., v_m)$$

Because otherwise we've got a bigger linearly independent list, that spans V.

2.1.12

Explain why there does not exist a list of six polinomials that is linearly independent of $\mathcal{P}_\triangle(F)$.

Because the list of length 5 spans this space.

2.1.13

Explain why no list of four polynomials spans $\mathcal{P}_\triangle(F)$.

Because the list of length 5 spans this space.

2.1.14

Prove that V is infinite-dimentional if and only if there is a sequence $v_1, v_2, ...$ of vectors in V such that $v_1, ...v_m$ is linearly independent for every possible integer m.

Forward is coming from the fact that we can always add new vectors to a given linearly independent list of vectors, that are outside of span of given list.

Because there always exists list that is bigger than given list and is linearly independent in V we can follow that no final list of vectors spans V, therefore it is infinite-dimentional.

2.1.15

Prove that F^∞ is infinite-dimentional.

Infinite list

$$(1, 0, ...), (0, 1, 0, ...), ...$$

is all linearly indepenent, therefore no finite set spans the space.

2.1.16

PRove that the real vector space of all continous real-valued functions on the interval $[0, 1]$ is infinite-dimentional.

We can create a countable sequence $(r_1, r_2, ...)$ of rationals in this space, and correspod each one of them with some number, thus creating a infinite linearly inedependent list.

2.1.17

Suppose $p_0, p_1, ...p_n$ are polynomials in $\mathcal{P}_{\Updownarrow}(F)$ such that $p_j(2) = 0$ for each j. Prove tat $p_0, p_1, ...p_m$ is not linearly independent in $\mathcal{P}_{\Updownarrow}(F)$.

Because it has the same length as $1, x, x^2...$, but doesn't span the same space.

2.2 Bases

There are no challenging exercises in this section, just a recap of the material. Looked them over, brushed up the material, not gonna waste my time writing them down.

2.3 Dimention

2.3.1

Suppose V is finite-dimentional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$

They have the same length of basis, thus basis of U is a basis of V.

2.3.2

Show that the subspaces of R^2 *are precisely* $\{0\}, R^2$ *and all lines through the origin*

For 0 dimention we've got null

For dimention 1 we've got scalar multiple of any vector, which are lines through the origin

For dimention 2 we've got the space itself

2.3.3

Show that the subspaces of R^3 *are precisely* $\{0\}, R^3$*, all lines through the origin, and all planes through the origin*

Same idea as in previos exericise, but list of length 2 defines a plane through the origin and 3 defined space itself

2.3.4

(a) Let $U = \{p \in P_4(F) : p(6) = 0$*. Find a basis of* U*.*

$$(x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, (x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$\{c : c \in F\}$$

2.3.5

(a) Let $U = \{p \in P_4(F) : p''(6) = 0$*. Find a basis of* U*.*

$$1, (x-6), (x-6)^3, (x-6)^4$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, (x-6), (x-6)^2, (x-6)^3, (x-6)^4$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$(x-6)^2$$

2.3.6

(a) Let $U = \{p \in P_4(F) : p(2) = p(5)$. *Find a basis of* U.

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$x$$

2.3.7

(a) Let $U = \{p \in P_4(F) : p(2) = p(5) = p(6)$. *Find a basis of* U.

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, x^2, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$x, x^2$$

2.3.8

(a) Let $U = \{p \in P_4(F) : \int\_ 1^1 p = 0\}$. *Find a basis of* U.

$$x, x^3$$

(b) Extend the basis in part (a) to a basis of $P_4(F)$

$$1, x, x^2, x^3, x^4$$

Find a subspace W *of* $P_4(F)$ *such that* $P_4(F) = U \oplus W$

$$1, x^2, x^4$$

2.3.9

Suppose $v_1, ...v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim span(v_1 + w, ..., v_m + w) \geq m - 1$$

Because $v_1, ...v_m$ is linearly independent we can follow that w is either in $span(v_1, ..., v_m)$ or not. In the latter case we've got that the case that we increase the span. In the former we've got by linear independence of $v_1, ...v_m$ that the maximum decline of degree is 1. Thus

$$\dim span(v_1 + w, ..., v_m + w) \geq m - 1$$

as desired.

2.3.10

Suppose $p_0, p_1, ..., p_m \in P(F)$ are such taht each p_j has degree j. Prove that $p_0, ...p_m$ is a basis of $P_m(F)$.

Suppose that $p \in P_m(f)$. Because each p_n has a degree of n we can follow that there exists only 1 $a_m \in F$ such that of p_m such that

$$p - a_m p_m \in P_{m-1}(F)$$

.

Bu applying the same procedure again repeatedly we get unique $a_m, ..., a_0$ such that

$$\sum a_n p_m = p$$

for every $p \in P_m(f)$. Thus we can follow that given list spans $P_m(F)$ and by unique representation we get that this list is linearly independent. Thus we can follow that given list is a basis of $P_m(F)$, as desired.

2.3.11

Suppose that U and W are subspaces of R^8 such that $\dim U = 3$, $\dim W = 5$, and $U + W = R^8$. Prove that $R^8 = U \oplus W$.

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Thus we can follow that in this particular case

$$\dim(R^8) = \dim U + \dim W - \dim(U \cap W)$$

$$8 = 3 + 5 - \dim(U \cap W)$$

$$\dim(U \cap W) = 0$$

thus we can follow that $U \cap W = \{0\}$. Therefore

$$U + W = U \oplus W = R^8$$

as desired.

2.3.12

Suppose that U and W are both five-dimentional subspaces of R^9. Prove that $U \cap W \neq \{0\}$

Once again we get that

$$\dim R^9 = \dim U + \dim W - \dim(U \cap W)$$

$$9 = 5 + 5 - \dim(U \cap W)$$

$$\dim(U \cap W) = 1$$

thus

$$U \cap W \neq 0$$

as desired.

2.3.13

Suppose U and W are both 4-dimentional subspaces of C^6. Prove that there exists two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other

Goto previous exercise for concretee explanation if needed, but we can conclude that

$$\dim U \cap W = 2$$

thus there exists a linearly independent list of length 2 in $U \cap W$ (basis) so that neither of them is a scalar multiple of another by some exercise in 2.A

2.3.14

Suppose $U_1, ...U_m$ are finite-dimentional subspaces of V. Prove that $U_1 + ... + U_m$ is finite-dimentional and

$$\dim(\sum U_n) \leq \sum \dim U_n$$

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

given that $\dim W \geq 0$ for any vector space W we follow that

$$\dim(U_1 + U_2) \leq \dim U_1 + \dim U_2$$

Thus by induction

$$\dim(\sum U_n) \leq \sum \dim U_n$$

which in presented case get us desired result.

2.3.15

Suppose V is finite-dimentional, with $\dim V = n \geq 1$. Prove that there exist 1-dimentional subspaces $U_1, ...U_n$ of V such that

$$V = U_1 \oplus ... \oplus U_n$$

For V there exists a basis of length n. Thus by setting

$$U_j = \{cv_j : c \in F\}$$

we get desired result.

2.3.16

Suppose $U_1, ..., U_m$ are finite-dimentional subspaces of V such that $U_1 + .. + U_m$ is a direct sum. Prove that $U_1 + ... + U_m$ is finite dimentional and that

$$\dim \sum U_n = \sum \dim U_n$$

We can just go by induction on the case that

$$\dim(U \oplus W) = \dim U + \dim W + \dim(U \cap W) = \dim U + \dim W + 0$$

Or we can use the fact, that we can combine all bases of subspaces together in one mega-basis for their sum. Both will suffice.

2.3.17

You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of finite-dimentional vector space, then

$$\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) -$$

$$- \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$$

We know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

and

$$U_1 + U_2 + U_3 = (U_1 + U_2) + U_3$$

thus

$$\dim(U_1+U_2+U_3) = \dim((U_1+U_2)+U_3) = \dim(U_1+U_2)+\dim U_3-\dim((U_1+U_2)\cap U_3) =$$

$$= \dim U_1 + \dim U_2 - \dim U_1 \cap U_2 + \dim U_3 - \dim((U_1 + U_2) \cap U_3) =$$

here we get a little problem because we don't know how to reduce $(U_1 + U_2) \cap U_3$ to some managable pieces. After this discovery one might even glance over the equation once again in order to try to disprove the theorem. And indeed we've found a counterexample: suppose that U_1, U_2, U_3 are lines through the origin in R^3 such that they are located on the same plane. Then it follows that left-hand side becomes 2, and the right side is equal to 3. Thus we've got a contradiction (which is a shame, because the formula looks nice :( ).

Chapter 3

Linear maps

3.1 The Vector Space of Linear Maps

3.1.1

Suppose $b, c \in R$. *Define* $T : R^3 \to R^2$ *by*

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that T *is linear if and only of* $b = c = 0$.

Suppose that T is linear. Then it follows that

$$T(0) = 0 = (0 + b, 0)$$

thus we can follow that $b = 0$.

Also,

$$T((1, 1, 1) + (2, 2, 2)) = (6 - 12 + 9, 18 + 27c) = (3, 18 + 27c) =$$

$$= T((1, 1, 1)) + T(2, 2, 2) = (2-4+3, 6+c) + (4-8+6, 12+8c) = (1, 6+c) + (2, 12+8c) = (3, 18+9c)$$

Thus

$$27c = 9c$$

$$3c = c$$

$$c = 0$$

as desired.

Reverse implication is trivial, thus we get the desired result.

3.1.2

Suppose $b, c \in R$. Define $T : (P)(R) \to R^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c\sin p(0)\right)$$

Show that T is linear if and only if $b = c = 0$.

Suppose that T is linear. Then it follows that if $p(0) = \pi/2$, then latter part of resulting vector has additive property only when $c = 0$. For the former we've got result that

$$\lambda^2 b = b$$

for all $\lambda \in R$, which happens only if $b = 0$. Thus $b = c = 0$.

Reverse implication is trivial, thus we have the desired result.

3.1.3

Suppose $T \in \mathcal{L}(F^n, F^m)$. Show that there exists scalars $A_{j,k} \in F$ for $j = 1, ..., m$ and $K = 1, ..., n$ such that

$$T(x_1, ..., x_n) = (A_{1,1}x_1 + ... + A_{1,n}x_n, ..., A_{m,1}x_1 + ... + A_{m,n}x_n)$$

for every $(x_1, ..., x_n) \in F^n$.

Because $(1, 0, ...), (0, 1, ...), ...$ is a basis of F^n we can follow that there vector in F^m, such that $T(v) \in F^m$. Thus let us denote

$$T(1, 0, ...) = (A_{1,1}, A_{2,1}, ..., A_{m,1})$$

$$T(0, 1, ...) = (A_{1,2}, A_{2,2}, ..., A_{m,2})$$

$$...$$

Thus given given arbitrary vector $v = (x_1, x_2, ..., x_n) \in T^n$ we get that

$$T(v) = T(x_1, x_2, ...) = T(x_1, 0, 0, ...) + T(0, x_2, 0, ...) + ... = x_1 T(1, 0, 0, ...) + x_2 T(0, 1, 0, ...) + ... =$$

$$= (x_1 A_{1_1}, x_1 A_{2,1}, ...) + (x_2 A_{1_2}, x_2 A_{2,2}, ...) = (x_1 A_{1,1} + x_2 A_{1,2} + ..., x_1 A_{2,1} + x_2 A_{2,2} + ...)$$

as desired.

3.1.4

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, v_2, ...v_m$ is a list of vectors in V such that $Tv_1, ..., , Tv_m$ is a linearly inndependent list in W. Prove that $v_1, v_2, ..., v_m$ is linearly independent.

Suppose that it isn't. Then we can follow that there exist $w_1 \in W$ such that

$$w_1 = \sum a_j v_j = 0$$

and not all of a_j's are zeroes. Thus we can follow that

$$T(w) = T(\sum a_j v_j) = \sum T(a_j v_j) = \sum a_j T(v_j) = 0$$

But $T(v_j)$ is a list of linearly independent vectors, and therefore their sum is equal to zero iff all a_j's are zeroes, which is false. Thus we've got a contradiction.

3.1.5

Prove the assertion in 3.7

Let $T_1 = T, T_2 = S, T_3 \in L(V, W)$. Then it follows that

(1)

$$(T_1 + T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2 + T_1)(v)$$

(2)

$$(T_1 + (T_2 + T_3))(v) = T_1(v) + (T_2 + T_3)(v) = T_1(v) + T_2(v) + T_3(v) =$$

$$= (T_1 + T_2)(v) + T_3(v) = ((T_1 + T_2) + T_3)(v)$$

(3)

$$\lambda((S + T)(v)) = \lambda(S(v) + T(v)) = \lambda S(v) + \lambda T(v) = (\lambda S + \lambda T)(v)$$

(4)

$$T + 0 = T$$

(5)

$$1T = T$$

(6)

$$T + -1T = (1 - 1)T = 0T = 0$$

Thus $L(V, W)$ satisfies all reqirements of a vector space, as desired.

3.1.6

Prove the assertion in 3.9

Let $v \in V$.

(1) Then it follows that

$$((T_1T_2)T_3)(v) = (T_1T_2)(T_3(v)) = T_1(T_2(T_3(v))) = T_1((T_2T_3)(v)) = (T_1(T_2T_3))(v)$$

directly from definition. (I wonder if it's true in general for all functions; it probably is).

(2)

$$TIv = T(I(v)) = T(v) = I(T(v))$$

(3)

$$(S_1 + S_2)T(v) = (S_1 + S_2)(T(v)) = S_1(T(v)) + S_2(T(v)) = S_1Tv + S_2Tv$$

$$S(T_1 + T_2)(v) = S((T_1 + T_2)(v)) = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v)) = ST_1v + ST_2v$$

as desired.

3.1.7

Show that every linear map from a 1-dimentional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ *and* $T \in L(V, V)$*, then there exists* $\lambda \in F$ *such that* $Tv = \lambda v$ *for all* $v \in V$*.*

Because we've got a 1-dimentional space, it follows that there exists a basis of V - v_1. For this vector we've got that

$$Tv_1 = v_2 = \lambda v_1$$

Thus we can follow that if $u \in V$ then

$$Tu = T\sigma v_1 = \sigma T v_1 = \sigma\lambda v_1 = \lambda\sigma v_1 = \lambda u$$

as desired.

3.1.8

Give an example of a function $\phi : R^2 \to R$ *such that*

$$\phi(av) = a\phi(v)$$

for all $a \in R$ *and all* $v \in R^2$ *but* ϕ *is not linear.*

$$\phi(x, y) = \begin{cases} x \text{ if } x \neq y \\ 0 \text{ otherwise} \end{cases}$$

3.1.9

Give an example of a function $\phi : C \to C$ such that

$$\phi(w+z) = \phi(w) + \phi(z)$$

for all $w, z \in C$ but ϕ is not linear.

Let us define

$$\phi(a+bi) = b + ai$$

Thus

$$\phi(a+bi+c+di) = ai + ci + b + d = \phi(a+bi) + \phi(c+di)$$

but

$$i\phi(a+bi) = -a + bi$$

$$\phi(i(a+bi)) = \phi(ai - b) = -bi + a \neq i\phi(a+bi)$$

3.1.10

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in L(V, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv \text{ if } v \in U \\ 0 \text{ if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that T is not a linear map on V.

Let $u \neq 0 \in U$ such that $Su \neq 0$ and $v \in V \setminus U$. Then it follows that

$$v + u \notin U$$

(because otherwise $-(v+u)$ is in U, therefore $u - (v+u) = -v \in U$ and thus $v \in U$, which is a contradiction) Thus we can follow that

$$T(v+u) = 0$$

but

$$T(v) + T(u) = 0 + Su = Su \neq 0 = T(v+u)$$

therefore the function is not linear, as desired.

3.1.11

Suppose V is finite-dimentional. Prove that every linear map on a subspace of V can be extended to a lineaer map on V. In other words, show that if U is a subspace of V and S is a subspace of V and $S = L(V, W)$, then there exists $T \in L(V, W)$ such that $Tu = Su$ for all $u \in U$.

Because V is finite-dimentional and U is a subspace of V, we can follow that U is finite-dimentional as well. Thus we can follow that there exists $u_1, ..., u_m$ - basis of U. As we know, we can extend this basis to a basis of V - $u_1, ..., u_m, v_1, ...v_n$. Therefore we can define a map $P \in L(V, U)$ by

$$P(x_1, x_2, ...) = (x_1, x_2, ...x_m, 0, 0, ...)$$

(basically trim every element of basis that is not in U). Thus we can follow that $P(u) = u$ if $u \in U$. Proof that P is linear is trivial. Thus if $S \in L(U, W)$, then $T = SP \in (V, W)$ with the desired properties.

3.1.12

Suppose V is finite-dimentional with $\dim V > 0$, and suppose W is infinite-dimentional. Prove that $L(V, W)$ is infinite-dimentional.

Let $v_1, ..., v_m$ be a basis of V and let $w_1, w_2, ...$ be a list of linearly independent vectors in W. Now let us look at $T_n : V \to W$

$$T_n((x_1, x_2, ...)) = x_1 w_n$$

Then it follows that by linear independence of w_n there does not exist a linear combination of T_m such that

$$\sum_{m \neq n} a_m T_m \neq T_n$$

Thus we can follow that list T_n is linearly independent. Because list is not finite we can follow that the space $L(V, W)$ is infinite-dimentional, as desired.

3.1.13

Suppose $v_1, ..., v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, ...w_m \in W$ such that no $T \in L(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, ..., m$.

Because $v_1, ..., v_m$ is linearly dependent we can reduce it to a linearly independent list $v'_1, ..., v'_n$. Thus resulting list will span some subspace of V and will be its basis.

Thus we can take vector v_j from the original list, that does not appear in basis. Then take some vectors $w_1, ...w_n$ in W. We know that there exists a unique map

$$Tv'_n = w_n$$

thus by adding to list $w_1, ...w_n$ any vectors from W, apart from $T(v_j)$ we create desired list.

3.1.14

Suppose V is finite-dimentional with $\dim V \geq 2$. Prove that there exists $S, T, \in L(V, V)$ such that $ST \neq TS$

Let v_1, v_2 be a basis of V and let

$$S(x, y) = (y, x)$$
$$T(x, y) = (x, 0)$$

Then

$$ST = (0, x)$$

and

$$TS = (y, 0)$$

as desired.

3.2 Null Spaces and Ranges

3.2.1

Give an example of a linear map T such that $\dim nullT = 3$ and $\dim rangeT = 2$.

$T(x, y, z) = (x, y)$

3.2.2

Suppose V is a vector space and $S, T \in L(V, V)$ are such that

$$rangeS \subset nullT$$

Prove that $(ST)^2 = 0$.

Let $v \in V$. Then it follows that $S(T(v)) \in rangeS$. Thus $ST(v) \in nullT$. Therefore $TST(v) = 0$. And thus $STST = (ST)^2 = 0$, as desired.

3.2.3

Suppose $v_1, ..., v_m$ is a list of vectors in V. Define $T \in L(F^m, V)$ by

$$T(z_1, ..., z_m) = z_1 v_1 + ... + z_m v_m$$

(a) What property of T corresponds to $v_1, ..., v_m$ spanning V?

Surjectivity

(b) What property of T corresponds to $v_1, ..., v_m$ being linearly independent?

Injectivity

3.2.4

Show that

$$\{T \in L(R^5, R^4) : \dim nullT > 2\}$$

is not a subspace of $L(R^5, R^4)$.

We can set

$$T_1(x, y, z, w, q) = (x, 0, 0, 0)$$
$$T_2(x, y, z, w, q) = (0, y, 0, 0)$$
$$T_3(x, y, z, w, q) = (0, 0, z, 0)$$
$$T_4(x, y, z, w, q) = (0, 0, 0, w)$$

all of which are in the desired subset, but their sum is

$$T(x, y, z, w, q) = (x, y, z, w, 0)$$

which has $\dim null = 1$. Thus this subset is not closed under addition and therefore it is not a subspace.

3.2.5

Give an example of a linear map $T : R^4 \to R^4$ *such that*

$$rangeT = nullT$$

$$T(x, y, z, w) = (z, w, 0, 0)$$

.

3.2.6

Prove that there does not exist a linear map $TR^5 \to R^5$ *such that*

$$rangeT = nullT$$

dim is always an integer, therefore for $\dim rangeT = \dim nullT = n$ and

$$\dim T = 2n = 5$$

which is impossible.

3.2.7

Suppose V and W are finite-dimentional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in L(V,W) : T$ is not injective$\}$ is not a subspace of $L(V,W)$.

Suppose that $v_1, ..., v_m$ is a basis for V and $w_1, ..., w_n$ is a basis of W. We can follow that there exist, which maps v_1 to w_1 and so on. By adding all of those maps together we get an injective map. Thus we can follow that given set is not closed under addition and therefore is not a subspace.

3.2.8

Suppose V and W are finite-dimentional with $2 \leq \dim W \leq \dim V$. Show that $\{T \in L(V,W) : T$ is not surjective$\}$ is not a subspace of $L(V,W)$.

By following the simular logic as in previous exercise, we get a desired result.

3.2.9

Suppose $T \in L(V,W)$ is injective and $v_1, ..., v_n$ is linearly independent in V. Prove that $Tv_1, ..., Tv_n$ is linearly independent in W.

Suppose that it is not the case. Then it follows that there exists $a_1, ...a_n \in F$ such that not all of them are equal to zero and

$$\sum a_n T v_n = 0$$

Thus we can follow that

$$T \sum a_n v_n = 0$$

Thus $\sum a_n v_n \in nullT$. Because T is injective we can follow that

$$\sum a_n v_n = 0$$

and some of a_n's are not equal to zero. But $v_1, ..., v_n$ is linearly independent, thus we get a contradiction.

3.2.10

Suppose $v_1, ..., v_n$ spans V and $T \in L(V,W)$. Prove that the list $Tv_1, ...Tv_n$ spans range T.

Suppose $w \in rangeT$. Thus we can follow that there exists $v \in V$ such that

$$Tv = w$$

Given that $v_1, ..., v_n$ spans V we can follow that there exists $a_1, ...a_n$ such that

$$v = \sum a_n v_n$$

and thus

$$w = T \sum a_n v_n$$

$$w = \sum T a_n v_n$$

thus we can follow that $v_1, ..., v_n$ spans the range of T, as desired.

3.2.11

Suppose $S_1, ..., S_n$ are injective linear maps such that $S_1 S_2 ... S_n$ makes sense. Prove that $S_1 S_2 ... S_n$ is injective.

Suppose that T and S are injective such that ST makes sence. Suppose that

$$STv = 0$$

Then by injectivity of S we get that $Tv \in nullS$ and thus $Tv = 0$. Thus, by injectivity of T we get that $v = 0$. Therefore $nullST = 0$. Therefore ST is injective.

The case in the exercise is derived from induction on presented argument.

3.2.12

Suppose that V is finite-dimentional and that $T \in L(V, W)$. Prove that there exists a subspace U of V such that $U \cap nullT = 0$ and $rangeT = \{Tu : u \in U\}$.

Let N be a nullspace of T. It follows that it is a subspace of V. Now let $n_1, ..., n_m$ be a basis of N and extend it to a basis of V: $n_1, ..., n_m, v_1, ..., v_n$. Then if follows that $span(v_1, ...v_n) \cap N = 0$ (because otherwise the vector is in nullspace) and if $w \in rangeT$, then there exists $u \in span(v_1, ...v_n)$ such that $Tu = w$. Thus $span(v_1, ...v_n)$ is the desired subspace.

3.2.13

Suppose T is a linear map from F^4 to F^2 such that

$$nullT = \{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 5x_2, x_3 = 7x_4\}$$

Prove that T is surjective.

$\dim nullT = 2$, thus $\dim rangeT = 2$, therefore T is surjective, as desired.

3.2.14

Suppose U is a 3-dimentional subspace of R^8 and that T is a linear map from R^8 to R^5 such that $nullT = U$. Prove that T is surjcetive.

We can follow that $\dim rangeT = 5$, and therefore T is surjective, as desired.

3.2.15

Very similar to previous one

3.2.16

Same

3.2.17

Same

3.2.18

Same

3.2.19

Same

3.2.20

Suppose W is finite-dimentional and $T \in L(V, W)$. Prove that T is injective if and only if there exists $S \in L(W, V)$ such that ST is the identity map on V.

I don't know why it isn't stated explicitly, but by existence of injective T we can follow that $\dim V \leq \dim W$, and thus V is finite-dimentional.

In forward direction:

Suppose that T is injective. Now let $v_1, ..., v_m$ be a basis of V. Then we can follow that $Tv_1, ..., Tv_n$ is a basis of range T. Thus, extend this basis to a basis of W: $Tv_1, ..., Tv_n, w_1, ..., w_m$. Now let us define $S \in L(W, V)$ such that

$$STv_n = v_n$$

$$Sw_n = 0$$

Which will exist, and by the way, will be unique because we're pairing basis of W with a list of vectors in V. Thus we can follow that if $v \in V$ then

$$STv = ST \sum a_n v_n = S \sum T a_n v_n = S \sum a_n T v_n = \sum a_n v_n = v$$

thus $ST = I$, as desired.

In reverse dierction:

Suppose that there exists $S \in L(W,V)$ such that ST is an identity map on V. Suppose that T is not injective. Then we follow that $nullT \neq 0$. Then let $v_1 \in nullT \neq 0$. Then we can follow that

$$STv_1 = S(Tv_1) = S(0) = 0 \neq Iv_1$$

which is a contradiction. Thus we can conclude that T is injective, as desired.

3.2.21

Suppose W is finite-dimentional and $T \in L(V,W)$. Prove that T is surjective if and only if there exists $S \in L(W,V)$ such that TS is the identity map on W.

In forward direction:

Suppose that T is surjective and let $w_1, ..., w_n$ be a basis of W. Then we can follow that there exists $v_1, ..., v_m$ such that $Tv_1 = w_1, ...Tv_m = w_m$. Thus we can follow that there exists a map in $L(W,V)$ such that it maps

$$Sw_1 = v_1$$

$$Sw_n = v_n$$

Thus if $w \in W$, then we can follow that

$$TSw = TW(\sum a_n w_n) = T(\sum a_n W w_n) = T(\sum a_n v_n) = \sum a_n T v_n = \sum a_n w_n = w$$

for every $w \in W$. Thus we can follow that $TS = I$, as desired.

In reverse direction:

Suppose that there exists a map $S \in L(W,V)$ such that TS is an identity map on W.

Suppose now that T is not surjective. Then we can follow that there exists $w \in W$ such that there is no $v \in V$ such that $Tv = w$. But we've got that

$$TSw = T(Sw) = w$$

thus we've got a contradiction.

3.2.22

Suppose U and V are finite-dimentional vector spaces and $S \in L(V,W)$ andd $T \in L(U,V)$. Prove that

$$\dim nullST \leq \dim nullS + \dim nullT.$$

We know that if T maps a vector to zero, then $STv = S(Tv) = S0 = 0$. Thus we can follow that

$$\text{null } T \subseteq \text{null } ST$$

Suppose that $STv = 0$. Then we can follow that $Tv \in nullS$. Thus $nullST$ exhaustively decomposes into two sets: $nullT$ and $\{u \in U : Tu \in rangeT \cap nullS\}$. We know that

$$\dim(rangeT \cap nullS) \leq \dim nullS$$

. thus we can follow that

$$\dim nullST = \dim nullT + \dim(rangeT \cap nullS) \leq \dim nullS + \dim nullT$$

as desired.

3.2.23

Suppsoe U and V are finite-dimentional vector spaces and $S \in L(V, W)$ and $T \in L(U, V)$. Prove that

$$\dim rangeST \leq \min\{\dim rangeS, \dim rangeT\}$$

Given that $rangeST \subseteq rangeS$ we can follow that

$$\dim rangeST \leq \dim rangeS$$

Suppose that U' is a preimage of range of ST. Then we can follow that if $u' \in U'$, then u' is also in preimage of range of T. Thus we can follow that preimage of ST is a subset of preimage of T, and thus

$$\dim rangeST \leq \dim rangeT$$

Because both equations must hold, in follows that we get out desired inequality.

3.2.24

Suppose W is finite-dimentional and $T_1, T_2 \in L(V, W)$. Prove that $nullT_1 \subset nullT_2$ if and only if there exists $S \in L(W, W)$ such that $T_2 = ST_1$.

Firstly I should state that proposition in the exercise holds if we state that $\subset$ does not donote a proper subset, but a regular subset.

In forward direction: Suppose $nullT_1 \subset nullT_2$. This implies that $\dim rangeT_1 \geq \dim rangeT_2$. Let $v_1, ..., v_n, u_1, ...u_n, r_1, ..., r_n$ be a basis of V such that $r_1, ...r_n$ is a basis of $nullT_1$, $u_1, ..., u_n, r_1, ...r_n$ is a basis of T_2. Then we can follow that $T_2v_1, ..., T_2v_n$ is a basis of range of T_2 and $T_1v_1, ...T_1v_n$ is a basis of a subspace of range of T_1. Thus we can create a map $S : W \to W$ such that $ST_1v_n = T_2v_n$ Suppose that $v \in V$. Then it follows that

$$ST_1v = ST_1 \sum a_nv_n = \sum a_nST_1v_n = \sum a_nT_2v_n = T_2v$$

Thus we get our desired result.

In reverse direction: Suppose that there exists $S \in L(W, W)$ such that $T_2 = ST_1$. Suppose that $v \in nullT_1$. Thus $T_1v = 0 = ST_1v = T_2v$. Thus $v \in nullT_2$. Therefore $nullT_1 \subset T_2$, as desired.

3.2.25

Suppose W is finite-dimentional and $T_1, T_2 \in L(V, W)$. Prove that $rangeT_1 \subset rangeT_2$ if and only if there exists $S \in L(V, V)$ such that $T_2 = T_1 S$.

In forward direction:

Suppose that $rangeT_1 \subset rangeT_2$. Then let $q_1, ...q_n$ be a basis of range of T_1. Thus we can extend it to be a basis of range of T_2 be adding $w_1, ..., w_m, q_1, ..., q_n$. Thus we can follow that there exist $v_1, ...v_k \in V$ such that

$$T_1 v_n = w_n$$

and $v_1', ..., v_k' \in V$ such that

$$T_2 v_n' = w_n$$

Thus we can create a map $S \in L(V, V)$ such that

$$S v_n = v_n'$$

and thus

$$T_2 S v = T_2 S \sum a_n v_n = T_2 \sum a_n S v_n = T_2 \sum a_n v_n' = \sum a_n T_1 v_n = T_1 \sum a_n v_n = T_1 v$$

as desired.

In reverse direction:

Suppose that there exists S such that $T_1 = T_2 S$. Then it follows that if $u \in rangeT_1$, then $u \in T_2$ as well. Thus $rangeT_1 \subset T_2$, as desired.

3.2.26

Suppose $D \in L(P(R), P(R))$ is such that $\deg Dp = (degp) - 1$ for every nonconstant polynomial $p \in P(R)$. Prove that D is surjective.

Let us define a list of polynomials p_n such that $\deg(p_n) = n$. Then it follows that the list $D(p_n)$ is a list of polynomials such that $\deg(D(p_n)) = n - 1$, thus it spans the space of polynomials. Thus we can follow that D is surjective.

3.2.27

Suppose $p \in P(R)$. Prove that there exists a polynomial $q \in P(R)$ such that $5q'' + 3q' = p$.

By the exercise above we can state that differentiation is surjective. Thus double differentiation is also surjective. Thus there exists $k \in P(R)$ such that $q'' = k'$, therefore $5q'' = 5k'$. Thus by surjectivity of differentiation we've got the desired result.

3.2.28

Suppose $T \in L(V,W)$. and $w_1, ..., w_m$ is a basis of range T. Prove that there exist $\phi_1, ...\phi_m \in L(V,F)$ such that

$$T(v) = \phi_1(v)w_1 + ...\phi_n(v)w_n$$

for every $v \in V$

Suppose that $v_1, ..., v_n$ is a basis of V. It follows that we can get coefficients

$$Tv_j = A_{j,1}w_1 + ... + A_{j,n}w_n$$

thus if we set

$$\phi_j(v) = \phi_j(\sum a_n v_n) = \sum a_j A_{j,n}$$

then we get that

$$Tv = T\sum a_n v_n = \sum a_n T v_n = \sum a_n \sum A_{n,j} w_j = \sum\sum a_n A_{n,j} w_j = \sum \phi_n(v) w_n$$

as desired.

3.2.29

Suppose $\phi \in L(V,F)$. Suppose $u \in B$ is not in null ϕ. Prove that

$$V = null\phi \oplus \{au : a \in F\}$$

ϕ maps into a space of dimention one. Thus we can follow that its range is either 1 or 0. In this case there exists $u \in V$, such that it is not in null space of ϕ, therefore we can follow that $\dim range\phi = 1$. Thus the space, that is not in $nullT$ has dimention 1. Thus we can follow that this space is scalar multiples of u. Therefore

$$null\phi + \{au : a \in F\} = V$$

because $u \notin null\phi$ we follow that

$$null\phi \cap \{au : a \in F\} = 0$$

and thus we can state that

$$null\phi \oplus \{au : a \in F\} = V$$

as desired.

3.2.30

Suppose ϕ_1 and ϕ_2 are linear maps from V to F that have the same null space. Show that there exists a constant $c \in F$ such that $\phi_1 = c\phi_2$.

If $\dim range\phi = 0$, then the case is trivial. Thus let us assume that $\dim range\phi = 1$. Because they have the same null space we can follow that they have the same preimage of the range. Thus we follow that if $v_1, ..., v_n$ is a basis of nullspace, then $v_1, ..., v_n, w$ is a basis of V, therefore w is a basis of a preimage. Thus

$$\phi_1 v = a_{n+1}\phi_1 w = a_{n+1}c_1$$

$$\phi_2 v = a_{n+1}\phi_2 w = a_{n+1}c_2$$

thus

$$\phi_1 = c_2/c_1\phi_2$$

as desired.

3.2.31

Give an example of two linear maps T_1 and T_2 from R^5 to R^2 that have the same null space but are such that T_1 is not a scalar multiple of T_2

$$T_1(x, y, z, w, q) = (x, y)$$
$$T_2(x, y, z, w, q) = (y, x)$$

3.3 Matrices

3.3.1

Suppose V and W are finite-dimentional and $T \in L(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim rangeT$ *nonzero entries.*

Suppose that we've there exists a choice of bases of V and W, such that matrix of this linear map has less nonzero entries, then $\dim rangeT$. Then it follows, that $\dim rangeT$ is spanned by list of vectors, that has length less than $\dim rangeT$, which is impossible.

3.3.2

Suppose $D \in L(P_3(R), P_2(R))$ is the differentiation map defined by $Dp = p'$. Find a basis of $P_3(R)$ and a basis of $P_2(R)$ such that the matrix of D with resprect to these bases is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

I think that we can use standart basis for $P_3(R)$, and for $P_2(R)$ we gotta use basis $1, 2x, 3x^2$.

3.3.3

Suppose V and W are finite-dimentional and $T \in L(V,W)$. Prove that there exist a basis of V and a basis of W. such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim rangeT$.

We can create a basis out of preimage of range of T. Thus if we set $v_1, ..., v_n$ to be a basis of preimage and $Tv_1, ..., Tv_n$ to be the basis of range. Thus if we extend those lists to be a bases of V and W respectively, we get the desired result.

3.3.4

Suppose $v_1, ..., v_m$ is a basis of V and W is finite-dimentional. Suppose $T \in L(V,W)$. Prove that there exists a basis $w_1, ..., w_n$ of W such that all the entries in the first column of $M(T)$ (with respect to the bases $v_1, ..., v_m$ and $w_1, ..., w_m$) are 0 except for possibly a 1 in the first row, first column.

We can plug in v_1 into T to get Tv_1. If $Tv_1 = 0$, then v_1 is in the nullspace and any basis will do. Otherwise we can extend Tv_1 to a basis of W and get the desired result.

3.3.5

Suppose $w_1, ..., w_n$ is a basis of W and V is finite-dimentional. Suppose $T \in L(V,W)$. Prove that there exists a basis $v_1, ..., v_m$ of V such that all the entries in the first row of $M(T)$ (with respect to the bases $v_1, ..., v_m$ and $w_1, ..., w_n$) are 0 except for possibly a 1 in the first row, first column.

Suppose that we've got a random basis $v_1, ..., v_n$ of V and then map it through T. Then pick a vector v_j such that $Tv_j = a_1w_1 + ... + a_mw_m$ such that $a_1 \neq 0$. If there is no such vector, then we're set. Otherwise go through all the other vecrors v_k and look at the representation

$$v_k = a_1'w_1 + ... + a_m'w_m$$

and set

$$v_k' = v_k - b_nv_n$$

where b_n satisfies a_1/a_1' (or vice versa) such that v_k' represented as

$$Tv_k' = 0w_1 + ... + a_mw_m$$

Thus by linear independence of $v_1, ..., v_n$ we've got that $v_j, ..., v_1', ..., v_n'$ is also linearly independent. Then, by plugging this vector into a matrix in this order, we get the desired result.

3.3.6

Suppose V and W are finite-dimentional and $T = L(V, W)$. Prove that $\dim rangeT = 1$ *if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ equal 1.*

In forward direction: Suppose that we've got T and $\dim rangeT = 1$. Suppose that $v_1, ..., v_n$ is the resulting basis of V. Thus we can follow that $Tv_1 = Tv_2 = ... = Tv_n = w \neq 0$. Thus we can follow that $w_1 + ... + w_m = w$ is the basis of range of T.

Thus we can create a vector v_1 such that $Tv_1 = w$, then expand it to a basis of V $v_1, ..., v_n$ and then it'll follow that $v_2, ..., v_n$ is a basis of a nullspace. Thus we can make a list $v_1, v_2 + v_1, v_3 + v_1, ..., v_n + v_1$, that will also a basis of V and for it it'll follow that

$$Tv'_j = T(v_j + v_1) = 0 + w = w$$

Thus the only thing that is left is to create a basis of W such that

$$w_1 + ... + w_n = w$$

we can actually do it by expanding w to a basis of W, and getting $w, w_1, ..., w_n$. Then we can set the first vector to be $w - w_1 - w_2 - ... - w_n$ and we'll get the desired property. Thus we can construct the bases such that we have the desired property.

In reverse direction: Suppose that there exist a basis of V and basis of W such that all entries of $M(T)$ are equal to 1. Then we can follow that if we plug any vector into T, then we'll get the constant multiple of the vector $w_1 + ... + w_n$. Thus we can follow that $\dim rangeT = 1$, as desired.

3.3.7

Verify 3.36

3.36 states that if $S, T \in L(V, W)$, then $M(S + T) = M(S) + M(T)$.

Suppose that $S, T \in L(V, W)$, $v_1, ..., v_n$ is a basis of V and $w_1, ..., w_m$ is a basis of W. Then we can follow that values at j'th column of $M(S + T)$ are obtained through

$$(S + T)(v_j) = S(v_j) + T(v_j) = (a_1 + a'_1)w_1 + ... + (a_m + a'_m)w_m$$

where $a_1, ...a_m$ will be numbers in j'th row of $M(S)$ and $a'_1, ..., a'_m$ will be numbers in j'th row of $M(T)$. Thus we can follow that $M(S + T) = M(S) + M(T)$.

3.3.8

Verify 3.38

3.38 states that if $\lambda \in F$ and $T \in L(V, W)$, then $M(\lambda T) = \lambda M(T)$

$$(\lambda T)v_j = \lambda a_1 w_1 + ... + \lambda a_m w_m = \lambda(a_1 w_1 + ... + a_m w_m) = \lambda(Tv_j)$$

thus by the same reasoning as in previous exercise we've got that $M(\lambda T) = \lambda M(T)$, as desired.

3.3.9

Verify 3.52

Follows directrly from the definition.

3.3.10

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

$$(AC)_{j,k} = A_{j,\cdot}C_{k,\cdot}$$

thus

$$(AC)_{j,\cdot} = (A_{j,\cdot}C_{1,\cdot}, A_{j,\cdot}C_{2,\cdot}, ..., A_{j,\cdot}C_{k,\cdot}) = A_{j,\cdot}C$$

3.3.11

Suppose $a = (a_1, ..., a_n)$ is a 1-by-n matrix and C is an n-by-p matrix. Prove that

$$aC = a_1C_{1,\cdot} + ... + a_nC_{n,\cdot}$$

In other words, show that aC is a linear combination of the rows of C, with the scalars that multiply the rows coming from a.

This follows directly from a definition of matrix multiplication.

3.3.12

Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices A and C such that $AC \neq CA$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$CA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The rest of the exercises are just basic applications of definitions, and equating them rigorously. Nothing interesting in there

3.4 Invertibility and Isomorphic Vector Spaces

3.4.1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$

Let $u \in U$. Then

$$u = Iu = (T^{-1}T)u = T^{-1}(Tu) = T^{-1}I(Tu) = T^{-1}(S^{-1}S)(Tu) = (T^{-1}S^{-1})(ST)u$$

thus we can follow that (ST) is invertible and $(T^{-1}S^{-1}) = (ST)^{-1}$, as desired.

3.4.2

Suppose V is finite-dimentional and $\dim V > 1$*. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.*

We can have

$$S_1(a_1v_1 + ...a_n + v_n) = a_1v_1 + ...a_j + v_j$$

$$S_2(a_1v_1 + ...a_n + v_n) = a_{j+1}v_{j+1} + ...a_n + v_n$$

for some $1 \leq j < n$. (you can interpret it as an upper part of the identity and a lower part). Both of them are non-invertible (by non-surjectivity), but their sum is the identity, which is invertible. Thus the subset is not closed under addition and therefore it is not a subspace.

3.4.3

Suppose V is finite-dimentional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T(u) = S(u)$ for every $u \in U$ if and only if S is injective.

In forward direction: Suppose that there exists such an operator and S is not injective. Thus there exists u_1 such that $u_1 \neq 0$ and $Su_1 = 0$. Thus $Tu_1 = Su_1 = 0$, therefore T is not injective, therefore it is not invertible, which is a contradiction.

In reverse direction: Suppose that S is injective. Let $u_1, ..., u_n, v_1, ..., v_m$ be a basis of V such that $u_1, ..., u_n$ is a basis of U. Thus, by FTLM we've got that

$$\dim U = \dim \operatorname{range} S + \dim \operatorname{null} S$$

Given that S is injective, we can follow that $\dim \operatorname{null} S = 0$. Thus

$$\dim U = \dim \operatorname{range} S$$

Thus, because $\operatorname{range} S$ is a subspace, we can create a basis of it $u_1', ..., u_n'$, which will have the same length as the basis of U. By expanding this basis to a basis of V we can create

$u'_1, ..., u'_n, v'_1, ..., v'_m$. Then if we map $u_1, ..., u_n, v_1, ..., v_m$ to $u'_1, ..., u'_n, v'_1, ..., v'_m$, we'll get $T \in \mathcal{L}(V)$, which by uniqueness of representation that will have

$$Tu = Su$$

and because $u'_1, ..., u'_n, v'_1, ..., v'_m$ is a basis of V we'll get that T is surjective, and thus invertible, as desired.

3.4.4

Suppose W is finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ *if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.*

In forward direction: Suppose that $\operatorname{null} T_1 = \operatorname{null} T_2$. Then we can follow that $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2$. We can also follow that there exists $v_1, ...v_n$ - basis of preimage of T_1 and T_2. Thus, $T_1 v_1, ...T_1 v_n$ is a basis of $\operatorname{range} T_1$ and $T_2 v_1, ..., T_2 v_n$ is a basis of $\operatorname{range} T_2$. Thus we can create a unique map $S \in \mathcal{L}(\operatorname{range} T_2, \operatorname{range} T_1)$

$$S'(a_1 T_2 v_1 + ... + a_n T_2 v_n) = a_1 T_1 v_1 + ... + a_n T_1 v_n$$

Thus, if $v \in V$, then

$$ST_2 v = S(a_1 T_2 v_1 + ... + a_n T_2 v_n) = a_1 T_1 v_1 + ... + a_n T_1 v_n = T_1(a_1 v_1 + ... + a_n v_n) = T_1 v$$

Given that S' is injective and $S' \in (\operatorname{range} T_2, \operatorname{range} T_1) \to S' \in (\operatorname{range} T_2, W)$ (because $\operatorname{range} T_2 \subseteq W$) , we can follow by results of our previous exercise, that there exists invertible $S \in \mathcal{L}(W)$ such that

$$S(w) = S'(w)$$

and therefore

$$T_1 = ST_2$$

as desired.

In reverse direction: Suppose that there exists invertible $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. Thus if $v \in \operatorname{null} T_1$, then we can follow that $ST_2 v = 0$. By invertability of S we've got that there exists $S^{\text{-}1}$ and therefore

$$ST_2 v = 0$$

$$S^{\text{-}1}\, ST_2 v = S^{\text{-}1}\, 0$$

$$T_2 v = 0$$

Thus, $v \in \operatorname{null} T_2$. Therefore we can follow that $\operatorname{null} T_1 \subseteq \operatorname{null} T_2$. By the same logic, but in other direction we've got that $\operatorname{null} T_2 \subseteq \operatorname{null} T_1$, thus

$$\operatorname{null} T_1 = \operatorname{null} T_2$$

as desired.

3.4.5

Suppose that V is finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ *if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = T_2 S$.*

In forward direction: Let $w_1, ...w_n$ be a basis of $\operatorname{range} T_1 = \operatorname{range} T_2$. Thus we can follow that there exists basis $v_1, ...v_n \in V$ of preimage of T_1 and $v_1', ...v_n' \in V$ - basis of preimage of T_2 such that $T_1 v_j = w_j = T_2 v_j'$. Because those lists are linearly independent and have the same length, we can follow that there exists an isomorphy $S \in \mathcal{L}(V)$ such that

$$S(a_1 v_1 + ...a_n v_n) = a_1 v_1' + ...a_n v_n'$$

Thus we can follow that for $v \in V$

$$T_2 S v = T_2 S(a_1 v_1 + ...a_n v_n) = a_1 w_1 + ... + a_n w_n = T_1 v$$

as desired.

In reverse direction: Suppose that there exists $S \in \mathcal{L}(W)$ such that $T_1 = T_2 S$. Then it is obvious that $\operatorname{range} T_1 = \operatorname{range} T_2$.

3.4.6

Suppose that V and W are finite-dimentional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exists invertible operators $R \in \mathcal{L}\, L(V)$ and $S \in L(W)$ such that $T_1 = S T_2 R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

In forward direction: By resullts of previous exercise we can follow that $\operatorname{null} T_1 = \operatorname{null} T_2 R$. Thus by injectivity of R we've got that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

In reverse direction: Suppose that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$. We can follow that there exists isomorphism $S \in \mathcal{L}(W)$ such that $\operatorname{null} T_1 = \operatorname{null} T_2 R$. Thus, by results of previous exercises we can folllow that there exists $S \in \mathcal{L}(V)$ such that $T_1 = S T_2 R$, as desired.

3.4.7

Suppose V and W are finite-dimentional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

Suppose that $S, T \in E$. Then for $v \in V$ we've got that

$$(S + T)v = S(v) + T(v) = 0$$

thus $S + T \in E$ and E is closed under addition.

Let $\lambda \in F$. Then

$$(\lambda T)v = \lambda(Tv) = \lambda 0 = 0$$

thus $(\lambda T) \in E$, therefore E is closed under scalar multiplication. Given that $\mathcal{L}(V, W)$ is a vector space we can follow that E is a subspace, as desired.

(b) Suppose $v \neq 0$. What is $\dim E$*?*

Extend v to a basis $v, v_1 ... v_{n-1}$ of V and let $w_1, ..., w_n$ be arbitrary basis of W. Because $Tv = 0$, we require that first column of $M(T)$ will be zeroes. Then we can follow that $\mathcal{M}$ is an isomorphism between E and $F^{m,(n-1)}$. Thus

$$\dim E = (\dim V - 1)(\dim W)$$

3.4.8

Suppose V is finite-dimentional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of W such that $T|_U$ is an isomorphism of U onto W.

Let $v_1, ..., v_n$ be a basis of $\operatorname{null} V$. Then we can extend it to $v_1, ..., v_n, u_1, ..., u_m$, which will be a basis of V. Let $U = \operatorname{span}(u_1, ..., u_m)$. We can follow by FTLM that $\dim U = \dim \operatorname{range} T$. Also, because U is surjective, we can follow that it is invertible and therefore is isomorphism, as desired.

3.4.9

Suppose V is finite-dimentional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

In forward direction:

Suppose that ST is invertible. Suppose that T is not invertible. Then it is not injective. Therefore theree exists $v \in V \neq 0$ such that $Tv = 0$. Therefore $STv = S0 = 0$, which is a contradiction. Thus T is injective and therefore invertible.

Suppose that S is not invertible. Then we can follow that there exists $v \in V \neq 0$ such that $Sv = 0$. Given that T must be invertible we can follow that there exists $w \in V \neq 0$ such that $Tw = v$. Thus $STw = Sv = 0$. Therefore ST is not injective, which is a contradiction.

Thus we can follow that in order for ST to be invertible, both S and T must be invertible as well, as desired.

In reverse direction: Suppose that S and T are invertible. Then we can follow that both of them are injective. Thus by some exercise in this chapter (looked it up, it's 3.2.11) ST is injective as well. Thus ST is invertible, as desired.

3.4.10

Suppose V is finite0dimentional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

From previous exercise we can follow that both S and T, as well as ST and TS are invertible.

All of the following are equivalences and not implications, therefore we can prove everything in one go.

$$ST = I$$

$$STS = IS$$

$$STS = SI$$

$$TS = I$$

3.4.11

Suppose V is finite-dimentional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Firstly I should state that we assume here that S and U are invertible, otherwise we default to one of the previous exrecises.

Suppose that T is not invertible. Then it isn't injective, and therefore there exists $v \in V \neq 0$ such that $Tv = 0$. Because U is invertible we can follow that there exists $w \in V \neq 0$ such that $Uw = v$. Thus

$$STUw = STv = S0 = 0 \neq w$$

Therefore $STU \neq I$, which is a contradiction.

Now we can use some algebra in here

$$STU = I$$

$$TU = S^{-1} I$$

$$TU = S^{-1} U^{-1}$$

$$T = S^{-1} U^{-1}$$

and by first exercise in this chapter

$$T^{-1} = (S^{-1} U^{-1})^{-1} = US$$

as desired.

3.4.12

Show that the result in the previous exercise can fail withoud the hypothesis that V is finite-dimentional

We can set U to be I,

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...)$$

$$S(x_1, x_2, ...) = (x_2, x_3, x_4, ...)$$

and so on. Then T will not be surjective, and therefore will not be invertible (which doesn't follow from our usual equivalence, but by the fact that there does not exist a map such that $TS = I$, because there is no way to represent (1, 1, ...))

3.4.13

Suppose V is a finite-dimentional vector space and $R, S, T \in \mathcal{L}(V)$. are such that RST is surjective. Prove that S is injective.

Suppose that it isn't. We can follow that RST is invertible. Then we can follow that there exits $v \in V \neq 0$ such that $Sv = 0$. By invertability of T we follow that there exists $w \in V \neq 0$ such that $Tw = v$. thus

$$RSTw = RSv = R0 = 0$$

Thus RST is not invertible and isn't surjective, which is a contradiction.

3.4.14

Suppose $v_1, ..., v_n$ is a basis of V. Prove that the map $T : V \to F^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $F^{n,1}$.

Only way to represent 0 in $M(v)$ is that if $v = 0$, therefore the map is injective. Also, by common sense, map is surjective. Thus it is invertible and therefore it is an isomorphism.

3.4.15

Trivial, seen simular in previous chapter

3.4.16

Suppose V is finite-dimentional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Forward direction is trivial.

Suppose that $ST = TS$ for every $S \in \mathcal{L}(V)$. Suppose that $Tv \neq \lambda v$ for some $v \neq 0$ and $\lambda \in F$. Then we can follow that there exsits S such that

$$S(Tv) = v$$

and

$$S(v) = 0$$

Thus

$$S(Tv) = v \neq 0 = S(v) = T(Sv)$$

Thus we can follow that for every $v \in V$ there exists $a \in F$ such that $Tv = av$.

Now suppose that $v, w \in V$. If $v \neq \lambda w$, then

$$T(v+w) = a_{v+w}(v+w) = a_{v+w}v + a_{v+w}w = T(v) + T(w) = a_v v + a_w w$$

thus

$$a_v = a_w = a_{v+w}$$

by the unique representation of zero.

If $v = \lambda w$, then

$$Tv = a_v v = T(\lambda w) = a_w \lambda w$$

$$a_v v = a_w \lambda w$$

$$a_v v = a_w v$$

thus $a_v = a_w$. Therefore for any given v, w we follow that $a_v = a_w$. Thus, T is a scalar multiple of I, as desired. (Proof aquired after reading another proof in supplimentary material).

3.4.17

Suppose V is finite-dimentional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

It's trivial to proof that $\{0\}$ and $\mathcal{L}(V)$ are such supspaces. We also should mention that in order for it not to be a trivial case we assume that $\dim V > 1$ (otherwise we don't have any other subspaces other than $\mathcal{L}(V)$ and $\{0\}$).

Our strategy with this proof is to show that there exists an invertible map in $\mathcal{E}$. If we've got that, then the rest of the proof is trivial.

Thus suppose that $E \neq \{0\}$ and $E \neq \mathcal{L}(V)$. Then we can follow that there exists a map $T \in E$ such that $T \neq 0$. Because we want to prove that there exists an invertible map in E, suppose that T is not invertible (otherwise $TT^{-1} = I \in \mathcal{E}$, and we can skip to the later part). Thus there exists a vector $v \in V \neq 0$ such that $Tv \neq 0$ and $w \in V \neq 0$ such that $Tw = 0$. Therefore extend v to a basis $v, v_1, ..., v_n$ of V and make a map

$$S(v) = v$$

.

$$S(v_k) = 0$$

. Then it follows that there exist maps $TS \in E_j$ such that

$$TS(a_0v + a_1v_1 + ...a_nv_n) = Ta_0v = a_0Tv$$

Thus there exists a map

$$Q_j(a_0v + ... + a_nv_n) = a_jv_j$$

also in $\mathcal{E}$. Thus we can follow that $I \in \mathcal{E}$. Thus we can follow that for every $S \in \mathcal{L}(V)$ we've got that

$$SI = S$$

is also in $\mathcal{E}$. Thus $\mathcal{L}(V) \subseteq \mathcal{E}$. Thus we can follow that $\mathcal{L}(V) = \mathcal{E}$, which is a contradiction.

Thus we can follow that $\mathcal{E}$ is either $\mathcal{L}(V)$ and $\{0\}$, as desired.

3.4.18

Show that V and $\mathcal{L}(F, V)$ are isomorphic vector spaces.

For finite-dimentional spaces we've got that

$$\dim \mathcal{L}(F, V) = (\dim F)(\dim V) = 1(\dim V) = \dim V$$

Thus they are isomorphic.

Otherwise we suppose that $v_1, ...$ is a basis of V. Given that

$$v = \sum a_jv_j$$

we can follow that we've got bijectivity between $\mathcal{L}(F, V)$ and V, as desired.

3.4.19

Suppose $T \in \mathcal{L}(P(R))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in P(R)$.

(a) Prove that T is surjective.

Because T is injective, we can follow that $\operatorname{null} T = \{0\}$ (looked it up, this one applies to any vector space).

Now let's try to prove that $\deg Tp = \deg p$. We probably don't need the induction here, but we'll use it anyways.

For $p = 0$ we've got that $Tp = 0$. Let us prove that this is the case also for $\deg p = 0$, just in case. Suppose that $\deg p = 0$ and $\deg Tp < 0$, then $\operatorname{null} T$ is not equal to zero, which is a contradiction.

For inductive step let us assume that $\deg Tp = \deg p$ for p such that $\deg p = n - 1$. Now suppose that $\deg p = n$ and $\deg Tp < n$. By our assumption, it follows that there exists a basis $p_0, ..., p_{n-1}$ such that $\deg Tp_j = j$. Thus we can follow that $Tp_0, ...Tp_n$ spans $P_{n-1}(R)$. Thus we follow that there exists $a_0, ...a_{n-1}$ such that

$$a_0Tp_0 + ...a_{n-1}Tp_{n-1} = Tp$$

then it follows that T is not injective, which is a contradiction.

This concludes the proof that $\deg p = \deg Tp$. By this we can follow that given $p \in P(R)$ with $\deg p = n$ there exists $p_0, ...p_n$ and by extension $Tp_0, ...Tp_n$ such that $p \in \text{range}(Tp_0, ...Tp_n)$. Thus there exists $p' \in P(R)$ such that $Tp' = p$. Thus T is surjective, as desired (this dragged on for waaay too long).

3.4.20

not gonna repeat the text of the exercise, but it basically reduces to our usual eqivalence of surjectivity/injectivity on finite-dimentional operators.

3.5 Products and Quotients of Vector Spaces

3.5.1

Suppose T is a function from V to W. The graph of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Firstly, $G(T)$ denotes the graph of T.

In forward direction: Suppose that T is a linear map. Suppose that $v, w \in G(T)$. Then it follows that

$$v + w = (v + w, T(v + w))$$

Thus $G(T)$ is closed under addition.

$$\lambda v = (\lambda v, \lambda Tv)$$

thus it is also closed under scalar multiplication. Given that $G(T)$ is a subset of a product of vector spaces, which is a vector space, we follow that $G(T)$ is a subspace, as desired.

In reverse direction:

Suppose that $G(T)$ is a subspace of $V \times W$. Suppose that $v, w \in V$. Then

$$T(v + w) = T(v) + T(w)$$

and

$$\lambda T(v) = T(\lambda v)$$

by properties of a product of vector spaces. Thus we can follows that T is linear, as desired.

3.5.2

Suppose $V_1, ..., V_m$ are vector spaces such that $V_1 \times ... \times V_m$ is finite-dimentional. Prove that V_j is finite-dimentional for each $j = 1, ..., m$.

We can follow that the combined list of bases of V_n's spans $V_1 \times ... \times V_m$ and is linearly independent. Given that this list is finite, we can follow that $V_1 \times ... \times V_m$ is finite-dimentional, as desired.

3.5.3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$, but $U_1 + U_2$ is not a direct sum.

$$U_1 = (0, x_1, 0, x_2, 0, x_3, ...)$$
$$U_2 = (x_1, x_2, x_3, 0, x_4...)$$

3.5.4

Suppose $V_1, ...V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times ... \times V_m, W)$ and $\mathcal{L}(V_1, W) \times ... \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

For finite-dimentional spaces we can just follow the standard equations for calculating the dimentions of given spaces and produce the desired result.

Conversely, if any of the spaces is infinite-dimentional, then we gotta produce the bijective map between the two spaces.

Our strategy here will be to prove that there exists a bijective map between the two. Let $T : \mathcal{L}(V_1 \times ... \times V_m, W) \to \mathcal{L}(V_1, W) \times ... \times \mathcal{L}(V_m, W)$ be defined by

$$T(S) = S_1 \times ... \times S_n$$

where

$$S_j(v_j) = S(0, ..., v_j, .., 0) = w_j$$

thus

$$T(\lambda S) = \lambda S_1 \times ... \times \lambda S_n$$

and

$$T(S + M) = (S + M)_1 \times ... \times (S + M)_n = S_1 \times ... \times S_n + M_1 \times ... \times M_n$$

thus it is linear.

Injectivity and surjectivity are given with this one. Thus we follow that there exists a linear bijectivity between the two, thus they are isomorphic.

We can actually follow here that if the space is infinite-dimentional, then they are isomorhic by default.

3.5.5

Suppose $W_1, ...W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times ...W_n)$ and $\mathcal{L}(V, W_1) \times ... \times \mathcal{L}(V, W_n)$ are isomorphic

Same logic as in previous ones works on this oen too.

3.5.6

For a posisitivee integer n, define V^n by

$$V^n = V \times ... \text{ n times } ...V$$

Prove that V^n and $\mathcal{L}(F^n, V)$ are isomorphic vector spaces.

$$\dim \mathcal{L}(F^n, V) = (\dim F^n)(\dim V) = n \dim V = \dim V^n$$

as desired.

3.5.7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Let $v' = x - v$. It follows that $x = v + v'$. Given that $v + U = x + W$ we follow that there exists $-w \in W$ such that $x - w = v + 0$. Thus

$$x - w = v + 0$$

$$v + v' - w = v$$

$$v' - w = 0$$

$$v' = w$$

Thus we follow that $v' \in W$. Therefore we've got that

$$x + W = v + v' + W = v + W = v + U$$

thus we follow that

$$W = U$$

as desired.

3.5.8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in F$.

In forward direction:

Suppose that A is an affine subset of V. Then we follow that there exists vector $x \in V$ and subspace $U \subseteq V$ such that

$$A = x + U$$

Now let $v, w \in A$. Then we can follow that there exist $v', w' \in U$ such that

$$v = x + v'$$

$$w = x + w'$$

Thus we follow that

$$\lambda v + (1-\lambda)w = \lambda x + \lambda v' + x + w' - \lambda x - \lambda w' = \lambda v' + x + w' - \lambda w'$$

Because $v', w' \in U$ we follow that

$$\lambda v' + w' - \lambda w' \in U$$

thus

$$\lambda v' + x + w' - \lambda w' \in A$$

as desired.

In reverse direction:

Suppose that for every $\lambda \in F$ and $v, w \in A$ we've got that

$$\lambda v + (1-\lambda)w \in A$$

Fix $x \in A$ and suppose that $v, w \in A$. Then it follows that

$$\lambda(v - x) = \lambda v - \lambda x = \lambda v - \lambda x + x - x = (\lambda v - (1-\lambda)x) - x$$

Thus space $A - x$ is closed under scalar multiplication. And

$$(v - x) + (w - x) = 2(v/2 + w/2 - x) = 2(\frac{1}{2}v + (1 - \frac{1}{2})w - x)$$

by the fact that $v, w \in A \to \frac{1}{2}v + (1-\frac{1}{2})w \in A$ we follow that $(\frac{1}{2}v + (1-\frac{1}{2})w - x) \in A - x$. Thus, by the fact that $A - x$ is close under scalar multiplication we follow that

$$2(\frac{1}{2}v + (1 - \frac{1}{2})w - x) \in A - x$$

thus we can conclude that $A - x$ is a subspace of V. Thus, $A - x + x = A$ is an affine subset of V, as desired.

3.5.9

Suppose A_1 and A_2 aare affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Fisrtly, let us denote that

$$A_1 = x_1 + U_1$$

$$A_2 = x_2 + U_2$$

Firstly, if A_1 and A_2 are parallel, then they are either equal to each other (in which case their intersection is an affine subset), or empty. Thus we can follow that intersection of two affine subsets can be empty.

Suppose that their intersection is nonempty and let $x \in A_1 \cap A_2$. Then it follows that $U_1 = A_1 - x$ and $U_2 = A_2 + x$. Intersection of two vector spaces is a vector space, thus we follow that

$$x + U_1 \cap U_2 = A_1 \cap A_2$$

is an affine space, as desired.

3.5.10

Prove that the intersection of every collection of affine subsets of V is either an affine subset of V or the empty set

It follows by induction from the previous exercise.

3.5.11

Suppose $v_1, ..., v_m \in V$. Let

$$A = \{\lambda_1 v_1 + ... + \lambda_m v_m : \lambda_{[1,m]} \in F \text{ and } \sum \lambda_j = 1\}$$

(a) Prove that A is an affine subset of V.

Let $v, m \in A$. Then it follows that

$$\kappa v + (1-\kappa)w = \kappa(\lambda_1 v_1 + ... + \lambda_m v_m) + (1-\kappa)(\lambda'_1 v_1 + ... + \lambda'_m v_m) =$$

$$= \kappa(\lambda_1 v_1 + ... + \lambda_m v_m) + (1-\kappa)(\lambda'_1 v_1 + ... + \lambda'_m v_m) = (\kappa\lambda_1 + (1-\kappa)\lambda'_1)v_1 + ...(\kappa\lambda_n + (1-\kappa)\lambda'_n)v_n$$

Thus the sum of the coefficients is equal to

$$\sum(\kappa\lambda_n + (1-\kappa)\lambda'_n) = \sum(\kappa\lambda_n) + \sum[(1-\kappa)\lambda'_n] = \kappa\sum(\lambda_n) + (1-\kappa)\sum[\lambda'_n] = \kappa + (1-\kappa) =$$

$$= 1$$

Thus we follow that A is an affine subset by equivalence, that was proven in a couple of exercises above.

(b) Prove that every affine subspae of V that contains $v_1, ..., v_m$ also contains A

Suppose that some affine subset B contains $v_1, ..., v_m$. Let $a \in A$.

We know that there exists $U \subseteq V$ such that $v_1 + U = A$ and $W \in V$ such that $B = v_1 + W$. Thus we follow that $v_2 - v_1 \in B - v_1$, ..., $v_n - v_1 \in B - v_1$. Thus

$$\text{span}(v_2 - v_1, ...v_n - v_1) \subseteq B - v_1$$

Suppose that $a \in A$. Then

$$a - v_1 \in \text{span}(v_2 - v_1, ...v_n - v_1)$$

Thus

$$a - v_1 \in B$$

as desired.

(c) Prove that $A = v+U$ for some $v \in V$ and some subspace U of V with $\dim U \leq m-1$

We know, that the list $(v_2 - v_1, ...v_m - v_1)$ spans $A - v_1$. Therefore we can follow that $\dim U \leq m - 1$.

3.5.12

Suppose U is a subspace of V such that V/U is finite-dimentional. Prove that V is isomorphic to $U \times (V/U)$.

For finite-dimentional case is trivial, for infinite-dimentional we have isomorphism by default.

3.5.13

Suppose U is a subspace of V and $v_1 + U, ..., v_m + U$ is a basis of V/U and $u_1, ..., u_n$ is a basis of U. Prove that $v_1, ..., v_m, u_1, ..., u_m$ is a basis of V.

Because $v_1 + U, ..., v_m + U$ is a basis of V/U, we can follow that $v_1, ..., v_m$ is linearly independent in V.

If $v_j \in U$, then $v_j + U = U = 0 + U$, therefore we follow that $v_j \notin U$.

Thus we can follows that $v_1, ..., v_n, u_1, ..., u_m$ is a linearly independent list of vectors.

Given that $v_1, ..., v_n, u_1, ..., u_m$ is a list of vectors in V, we can follow that $\text{span}(v_1, ..., v_n, u_1, ..., u_m)$ is a subspace of V.

Now suppose that $v \in V$ and $v \notin \text{span}(v_1, ..., v_n, u_1, ..., u_m)$. We can follow that $v \notin U$. Thus we can follow that $v + U$ is an element of V/U. In this case we follow that $v \in \text{span}(v_1, ..., v_n)$, which is a contradiction. Thus we conclude that such a vector does not exist and therefore $V = \text{span}(v_1, ..., v_n, u_1, ..., u_m)$. Given that $v_1, ..., v_n, u_1, ..., u_m$ is linearly independent, we follow that it is a basis of V, as desired.

3.5.14

Suppose $U = \{(x_1, x_2, ...) \in F^\infty : x_j \neq 0$ *for only finitely many* $j\}$

(a) Show that U *is a subspace of* F^∞

Let $v, u \in U$. We can follow, that sinse finitely many x's are not zero, then we follow that for $v + u$

$$\{j_1, ..., j_n\} \cup \{j'_1, ..., j'_n\}$$

is a set of positions, in which the $v + u$ might not be zero. Since union of finite sets is finite, we follow that U is closed undeer addition. The same reasoning, but applied to intersection, rather then union, can be applied to get closure for multiplication. Thus we follow that U is a subspace of F^∞.

(b) Prove that F^∞/U *is infinite-dimentional*

We can follow that there exist

$$x_j = (0, 0, 0, ... \text{ j times } ...0, 1, 1, 1...)$$

such that each x_j is linearly independent from one another. Since $x_j \notin U$, we can follow that there is no basis of F^∞/U, therefore it is infinite-dimentional, as desired.

3.5.15

Suppose $\phi \in \mathcal{L}(V, F)$ *and* $\phi \neq 0$. *Prove that* $\dim V/(\operatorname{null} \phi) = 1$

Because $\phi \neq 0$, we can follow that there exists $v \in V$ such that

$$\phi v \neq 0$$

thus we follow that $\dim \operatorname{range} \phi = 1$. Given that $V/\operatorname{null} \phi$ is isomorphic to $\operatorname{range} \phi$, we follow that its dimention is also one, as desired.

3.5.16

Suppose U *is a subspace of* V *such that* $\dim V/U = 1$. *Prove that there exists* $\phi \in \mathcal{L}(V, F)$ *such that* $\phi = U$.

Given that $\dim V/U = 1$ we can follow that there exists $v \in V$ such that $v \notin U$. Thus v is a basis for V/U. Thus we can define $g : V/U \to F$

$$g(kv + U) = k$$

Then, by plugging $\pi : V \to V/U$ such that $\pi(v) = v + U$ and making

$$\phi(v) = g(\pi(v))$$

we get $\phi : V \to F$ such that $u \in U \to \phi(u) = 0$. Thus $U \subseteq \operatorname{null} \phi$.

Then suppose that $v \neq 0$ and $v \notin U$. Then it follows that $\phi(v) \neq 0$. Thus we can conclude that $\operatorname{null} \phi = U$, as desired.

3.5.17

Suppose U is a subspace of V such that V/U is finite-dimentional. Prove that there exists a subspace W of V such that $\dim W = \dim V/U$ *and* $V = U \oplus W$.

Because V/U is finite-dimentional, we can follow that there exists a basis of V.

$$v_1 + U, ..., v_n + U$$

We can follow that $v_1, ..., v_n$ is linearly independent. It is also will be helpful later to mention that none of v_j's are in U. Then, define

$$W = \operatorname{span}(v_1, ..., v_n)$$

Suppose that $v \in V$. Then we can follow that $v \in v' + U$ for some v. Thus we follow that $v' = a_1 v_1 + ... a_n v_n$ and therefore there exists $u \in U$ and $v' \in W$ such that $v = u + v'$. Thus $V = W + U$.

Suppose that $w \in W$ and $w \in U$. Then it follows that

$$w = a_1 v_1 + ... a_n v_n$$

given that none of $v_1, ... v_n$ are in U, we follow that the only way that it is possible is when $a_1, ..., a_n = 0$. Thus we follow that

$$W \cap U = \{0\}$$

. Thus we can conclude that

$$V = W \oplus V$$

as desired.

3.5.18

Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in (V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \operatorname{null} T$.

In forward direction:

Suppose that there exists S such that $T = S \circ \pi$. Let $u \in U$. Then it follows that

$$\pi(u) = u + U = 0 + U$$

Because S is a linear function, we follow that $S(0) = S(U) = 0$. Thus we follow that $u \in \operatorname{null} T$. Thus we can conclude that $U \subseteq T$

In reverse direction:

Suppose that $U \subseteq \operatorname{null} T$. Then we follow that $\pi(U) = 0$. Create a map $S' : V/U \to V/U$ such that $\operatorname{null}(S' \circ \pi) = \operatorname{null} T$. Then there exists an invertible operator R such that $RS' \circ \pi = T$. Thus we follow that there exists a map $RS' = S$ such that $S \circ \pi = T$, as desired.

3.5.19

Find a correct statement analogous to 3.78 that is applicable to finite sets, with unions analogous to sums of subspaaces and disjoint unions analagous to direct sums

Suppose A is a finite set and $U_1, ..., U_m$ are subsets of A. Then

$$U_1 \cap ... \cap U_n = A$$

is a disjoint union if and only if

$$\sum |U_j| = |A|$$

3.5.20

Suppose U is a subspace of V. Define $\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$$\Gamma(S) = S \circ \pi$$

(a) Show that Γ is a linear map

Let $S, T \in \mathcal{L}(V/U, W)$. Then

$$\Gamma(S+T) = (S+T) \circ \pi = S \circ \pi + T \circ \pi = \Gamma(S) + \Gamma(T)$$

If $\lambda \in F$, them

$$\Gamma(\lambda S) = (\lambda S) \circ \pi = \lambda S \circ \pi = \lambda \Gamma(S)$$

Therefore we can follow that Γ is linear

(b) Show that Γ is injective

Let $\Gamma(T) = 0$. It follows that

$$T \circ \pi = 0$$

Then we can follow that for $v \in V$

$$T \circ \pi(v) = 0 = T(v + U)$$

Thus $T = 0$. Therefore we follow that $\operatorname{null} \Gamma = \{0\}$, therefore it is injective.

(c) Show that $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0$ *for every* $u \in U\}$

Suppose that $T \in \operatorname{range} \Gamma$. Then it follows that there exists $S \in \mathcal{L}(V/U, W)$ such that

$$S \circ \pi = T$$

thus if $u \in U$ then

$$S \circ \pi(u) = S(u + U) = S(U) = 0$$

Suppose that $S \in \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$. Then it follows by the results in exercise 18, that there exists $T \in \mathcal{L}(V/U, W)$ such that

$$T = S \circ \pi$$

therefore by double inclusion we've got that $\operatorname{range} \Gamma = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$ as desired.

3.6 Duality

3.6.1

Explain why every linear functional is either surjective or the zero map

Dimention of its codomain is 1, therefore we've got that range is either 1 or 0. In former case it is surjective, in latter it's a null map.

3.6.2

Give three distinct examples of linear functionals on $R^{[0,1]}$

$$f(1) - f(0)$$
$$f(0.5)$$
$$f(0.2) + f(0.3)$$

3.6.3

Suppose V *is finite-dimentional and* $v \in V$ *with* $v \neq 0$. *Prove that there exists* $\phi \in V'$ *such that* $\phi(v) = 1$

We can extend v to a basis $v, v_1, ..., v_n$ of V, then define the dual basis $\phi, \phi_1, ..., \phi_n$ on this basis, and then get the desired function.

3.6.4

Suppose V *is finite-dimentional and* U *is a subspace of* V *such that* $U \neq V$. *Prove that there exists* $\phi \in V'$ *such that* $\phi = 0$ *for every* $u \in U$ *but* $\phi \neq 0$.

Simular to the previous one, we define $u_1, ..., u_n$ to be the basis of U, then expand it to a basis of V - $v_1, ..., v_m, u_1, ..., u_n$, and define dual basis on this basis.

The property that $U \neq V$ guaranteed that there exists v_1, therefore there exists ϕ_1 such that

$$\phi_1(v_1) = 1$$

thus $\phi_1 \neq 0$. And by definition of dual basis we get that $\phi_1(u_j) = 0$. Thus we follow that

$$u \in U \to u \in \text{span}(u_1, ..., u_n) \to \phi(u) = \phi(a_1 u_1) + ... + \phi(a_n u_n) = 0$$

as desired.

3.6.5

Suppose $V_1, ..., V_m$ *are vector spaces. Prove that* $(V_1 \times ... V_m)'$ *and* $V_1' \times ... \times V_m'$ *are isomorphic vector spaces.*

We follow it from the exercise 3.5.4.

3.6.6

Suppose V is finite-dimentional and $v_1, ..., v_m \in V$. Define a linear map $\Gamma : V' \to F^m$ by

$$\Gamma(\phi) = (\phi(v_1), ..., \phi(v_m))$$

(a) Prove that $v_1, ..., v_m$ spans V if and only if Γ is injective.

In forward direction: Suppose that $v_1, ...v_m$ spans V. Let $\phi \in V'$ be such that

$$\Gamma(\phi) = 0$$

Then we follow that

$$\Gamma(\phi) = (0, 0, ..., 0)$$

thus for $1 \leq j \leq m$ we've got that

$$\phi(v_j) = 0$$

Because $v_1, ..., v_m$ spans V we follow that if $v \in V$ then there exist $a_1, ..., a_m$ such that

$$v = \sum a_j v_j$$

thus we follow that

$$\phi v = \sum \phi v_j = 0$$

for any $v \in V$. Thus $\phi = 0$. Therefore we've got that

$$\Gamma\phi = 0 \to \phi = 0$$

thus we can follow that Γ is injective.

In reverse direction: Suppose that Γ is injective. We've going to proceed with a proof by contradiction on this one.

Suppose that $v_1, ..., v_m$ does not span V. Then we follow that we can create linearly independent list $v_1, ..., v_n$ by removing some elements from $v_1, ..., v_m$, if it is not already linearly independent. Then we can expand this list to $v_1, ..., v_n, v_1', ...$ - basis of V. Then we can create a dual basis for this basis and get our ϕ', which will be a dual basis for v_1'. Given that $\phi'(v_j) = 0$ by definition of the dual basis, we follow that

$$\Gamma(\phi') = 0$$

Given that $\phi'(v_1') \neq 0$ we follow that $\phi' \neq 0$. Thus $\operatorname{null} \Gamma \neq \{0\}$, therefore it is not injective, which is a contradiction.

(b) Prove that $v_1, ..., v_m$ is linearly independent if and only if Γ is surjective

In forward direction: Suppose that $v_1, ..., v_m$ is linearly independent. Then we follow that there exists a dual basis $\phi_1, ..., \phi_m$ of $v_1, ..., v_m$. Thus we follow that range of Γ has

$$(\phi_1(v_1), ..., \phi_1(v_m)) = (1, 0, ..., 0)$$

Therefore we can conclude that range of Γ contains a list of length m of linearly independent vectors. Given that $\dim(F^m) = m$, we follow that $F^m = \operatorname{range}\Gamma$. Thus Γ is surjective, as desired.

In reverse direction: Suppose that $v_1, ..., v_m$ are linearly independent. Then we follow that there exists a space $\operatorname{span}(v_1, ..., v_m)$, for which $v_1, ..., v_m$ is a basis. Thus we follow that for this basis there exists a dual basis $\phi_1, ..., \phi_m$, where $\phi_j \in V'$. Thus we follow that

$$\Gamma(\phi_1) = (\phi_1(v_1), ..., \phi_1(v_m)) = (1, 0, ...0) \in \operatorname{range}\Gamma$$

$$\Gamma(\phi_j) = (\phi_j(v_1), ..., \phi_j(v_j), ..., \phi_1(v_m)) = (0, ..., 1, ...0) \in \operatorname{range}\Gamma$$

Thus we follow that there exist a linearly independent list of length m in $\operatorname{range}\Gamma$. Thus we can follow that the dimention of $\operatorname{range}\Gamma$ is at least m. Then we can state, that because $\dim F^m = m$, $\operatorname{range}\Gamma$ is a subspace of F^m and the fact that a subspace has a dimention less or equal to the original space, we can follow that $\operatorname{range}\Gamma = F^m$. Thus we follow that Γ is surjective, as desired.

3.6.7

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, x^2, ..., x^m$ of $P_m(R)$ is $\phi_0, ..., \phi_m$ where $\phi_j(p) = \frac{p^{(j)}(0)}{j!}$.

It's straightforward to check that $(x^j)^{(j)} = j!$ (proof by induction will be useful in this case). Thus we follow that

$$\phi_j(x^j) = 1$$

If $k < j$, then we can follow that

$$\phi_j(x^k) = 0$$

and if $k > j$, then

$$\phi_j(x^k) = \frac{k!}{j!}0^{k-j} = 0$$

thus we get that

$$\phi_j(p_k) = \begin{cases} 0 \text{ if } k \neq j \\ 1 \text{ if } k = j \end{cases}$$

thus it is a dual basis by definition.

3.6.8

Suppose m is a positive

(a) Show that $1, x - 5, ..., (x - 5)^m$ is a basis for $P_m(R)$.

All of them are linearly independent, since all of them have different degrees, and the fact that length of the list is equal to the dimention of the space that it is in, implies that it is a basis for this space.

(b) What is a dual basis of the basis in part (a)

If we try to create this basis by drawing inspiration from the previous exercise, then we'll get

$$\phi_j(p_k) = \frac{p_k^{(j)}(5)}{j!}$$

some basic implication will show that it is indeed the dual basis for given basis.

3.6.9

Suppose $v_1, ..., v_m$ isi a basis of V and $\phi_1, ..., \phi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \sum \psi(v_j)\phi_j$$

We know that beacause $\phi_1, ..., \phi_m$ is a basis of V' that

$$\psi = \sum a_j\phi_j$$

for some $a_1, ..., a_m \in F$. Thus we follow that

$$\psi(v_j) = \sum a_j\phi_j(v_j) = \sum_{k \neq j} [a_k\phi_k(v_j)] + a_j\phi_j(v_j) = 0 + a_j * 1 = a_j$$

thus

$$\psi = \sum a_j\phi_j = \sum \psi(v_j)\phi_j$$

as desired.

3.6.10

Prove the first two bullet points in 3.101

Specifically we want to prove that

$$(S + T)' = S' + T'$$

$$(\lambda T)' = \lambda T'$$

$$(S + T)'(\phi) = \phi(S + T) = \phi S + \phi T = S' + T'$$

where the second equality comes from distributive properties of linear maps.

$$(\lambda T)'(\phi) = \phi(\lambda T) = \lambda\phi T = \lambda T'$$

3.6.11

Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, ..., c_m) \in F^m$ and $(d_1, ..., d_n) \in F^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, ..., m$ and every $k = 1, ..., n$.

In forward direction: Suppose that rank of A is equal to 1. Then we follow that the dimension of span of columns is 1. Thus we follow that every column is a scalar multiple of the first non-zero column. Thus we can set

$$v_j = a_j v_m$$

where v_m is the first non-zero column and a_j is corresponding coefficient. Thus we can create $(1, a_2, ..., a_m)$ - vector of corresponding multiplicities. Thus we can conclude that there exists vectors v_1 and $(1, a_2, ..., a_m)$ with the desired properties.

In reverse direction: Suppose that there exist two vectors, as defined in the exercise. Then we can follow that every column is a scalar multiple of the first non-zero column. Therefore we follow that the column rank of given matrix is 1, as desired.

3.6.12

Show that the dual map of the identity map on V is the identity map on V'

Let I' be the dual map of the identity. Then we can follow that for every $\phi \in V'$

$$I'(\phi) = \phi I = \phi$$

thus I' is the identity on V', as desired.

3.6.13

Define $T : R^3 \to R^2$ by $T(x, y, z) = (4x+5y+6z, 7x+8y+9z)$. Suppose ϕ_1, ϕ_2 denotes the dual basis of the standart basis of R^2 and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of R^3.

(a) Describe the linear functions $T'(\phi_1)$ and $T(\psi_2)$

We can follow that

$$M(T) = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

thus

$$M(T') = (M(T))^t = \begin{pmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{pmatrix}$$

also, ϕ_1 is represented by $(1, 0)$ and ϕ_2 is represented by $(0, 1)$. Therefore

$$M(T'(\phi_1)) = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

and

$$M(T'(\phi_2)) = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$$

thus

$$T'(\phi_1) = 4x + 5y + 6z$$

$$T'(\phi_2) = 7x + 8y + 9z$$

(b) Write $T'(\phi_1)$ and $T'(\phi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.
Given that

$$\psi_1 = (1, 0, 0)$$

$$\psi_2 = (0, 1, 0)$$

$$\psi_3 = (0, 0, 1)$$

we follow that

$$T'(\phi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$$

$$T'(\phi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$$

3.6.14

Define $T : P(R) \to P(R)$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in R$.

(a) Suppose $\phi \in P(R)'$ is defined by $\phi(p) = p'(4)$. Describe the linear functional $T'(\psi)$ on $P(R)$.

$$T'(\psi)(p) = \psi(Tp) = \phi(x^2 p + p''(x)) = 2xp + x^2 p + p'''(x)$$

(b) Suppose $\phi \in P(R)'$ is defined by $\phi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\phi))(x^3)$

$$T'(\phi)(p) = \phi(Tp) = \phi(x^2 p + p''(x)) = \phi(x^2 x^3 + 6x) = \phi(x^5 + 6x) =$$

$$= \int_0^1 x^5 + 6x = [x^6/6 + 3x^2]_0^1 = 1/6 + 3 = 3\frac{1}{6}$$

3.6.15

Suppose W is finite-dimentional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

In forward direction: Suppose that $T' = 0$. Then we can follow that $\phi T(v) = 0$ for every $\phi \in W'$ and $v \in v$. Suppose that $T \neq 0$. Then there exists $v \in V$ such that $Tv \neq 0$. Thus we can create a basis $Tv, w_1, ..., w_m$ of W and define the dual basis of this basis. Thus we follow that there exists ϕ_1 such that

$$\phi_1(Tv) = 1$$

Thus

$$\phi_1(Tv) = T'(\phi)(v) \neq 0$$

therefore we follow that $T' \neq 0$, which is a contradiction. Thus we can conclude that $T' = 0$ implies that $T = 0$.

In reverse direction: Suppose that $T = 0$. Then we follow that $\phi Tv = T'(\phi)(v) = \phi 0 = 0$. Thus we conclude that for any $\phi \in V$ $T'(\phi) = 0$. Thus $T' = 0$.

3.6.16

Suppose V and W are finite-dimentional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Let us denote this map as $\pi : \mathcal{L}(V, W) \to \mathcal{L}(W', V')$

We know that

$$\pi(S + T) = (S + T)' = S' + T' = \pi S + \pi T$$

and

$$\pi(\lambda T) = (\lambda T)' = \lambda(T') = \lambda(\pi T)$$

thus we can follow that π is a linear map.

We've already proven that $T = 0$ if and only if $T' = 0$, thus we can follow that given map is injective.

Because V and W are finite-dimentional, we can follow that $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$ are finite dimentional. Thus we can make bases of those vector spaces

$$\kappa_1, ..., \kappa_m \in \mathcal{L}(V, W)$$

$$\gamma_1, ..., \gamma_n \in \mathcal{L}(W', V)$$

and create an invertible map $\zeta \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(W', V'))$ such that

$$\zeta(\sum a_j \kappa_j) = \sum a_j \gamma_j$$

therefore we can follow that $\pi\zeta^{-1}$ is an operator on $\mathcal{L}(W', V')$. By invertibility of ζ we follow that ζ^{-1} is invertible as well and therefore it is injective. Thus $\pi\zeta^{-1}$ is a composition of injective linear maps, and therefore it is itself injective. Given that $\pi\zeta^{-1}$ is an operator on a finite-dimentional vector space, we follow that its injectivity implies inversibility. Therefore we can follow that π is also inversible, as desired. (I just wanted to go this road to get surjectivity, but got the whole inversibility from it.)

3.6.17

Suppose $U \subseteq V$. Explain why $U^0 = \{\phi \in V' : U \subseteq \operatorname{null} \phi\}$

Suppose that $\phi \in U^0$. Then we can follow that $\phi(u) = 0$, therefore $u \in \operatorname{null} \phi$ for every $u \in U$. Thus $U \subseteq \operatorname{null} \phi$. Thus $U^0 \subseteq \{\phi \in V' : U \subseteq \operatorname{null} \phi\}$.

Now let ϕ be such that $U \subseteq \operatorname{null} \phi$. Then we can follow that for all $u \in U$

$$\phi(u) = 0$$

thus $\phi \in U^0$. Therefore $U^0 \supseteq \{\phi \in V' : U \subseteq \operatorname{null} \phi\}$ Therefore by double inclusion we've got the desired equality.

3.6.18

Suppose V is finite-dimentional and $U \subseteq V$. Show that $U = \{0\}$ is and only if $U^0 = V'$.

In forward direction: Suppose that $U = \{0\}$. Then we can follow that

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}$$

U^0 is a subspace of V', therefore we've got that $U^0 \subseteq V'$. Suppose that $\phi \in V'$. Then it follows that

$$\phi(0) = 0$$

because it's a linear function. Thus we follow that $V \subseteq U^0$. Thus by double inclusion we've got that $V' = U^0$.

In reverse direction: Suppose that $U^0 = V'$. Now suppose that $u \in U$. Then we can follow that there exist $a_1, ..., a_m$ such that

$$u = \sum a_j v_j$$

for some basis $v_1, ..., v_m$ of V. For this basis there exists a dual basis $\phi_1, ..., \phi_m$. Thus we can follow that

$$0 = \phi_j(u) = \phi(\sum a_k v_k) = \phi_j(\sum_{k \neq j} a_k v_k) + \phi(a_j v_j) = a_j$$

Thus $a_1 = ... = a_m = 0$. Therefore $u = 0$. Thus $U = \{0\}$, as desired.

3.6.19

Suppose V is finite-dimentional and U is a subpace of V. Show that $U = V$ if and only if $U^0 = \{0\}$.

In forward direction: Suppose that $U = V$. Then it follows that $\dim U = \dim V$. Thus

$$\dim V = \dim U + \dim U^0$$

$$\dim V = \dim V + \dim U^0$$
$$\dim U^0 = 0$$

thus $U^0 = \{0\}$.

In reverse direction:

Suppose that $U^0 = 0$. Then

$$\dim V = \dim U + \dim U^0$$

$$\dim V = \dim U$$

Thus $U = V$.

3.6.20

Suppose U and W are subsets of V with $U \subseteq W$. Prove that $W^0 \subseteq U^0$.

Suppose that $\phi \in W^0$. Then we follow that if $u \in U$ then $u \in W$, and therefore $\phi(u) = 0$ for any $u \in U$. Thus $\phi \in U^0$. Therefore $\phi \in W^0 \to \phi \in U^0$. Thus $W^0 \subseteq U^0$, as desired.

3.6.21

Suppose V is finite-dimetionmal and U and W are subspaces of V with $W^0 \subseteq U^0$. Prove that $U \subseteq W$.

Let $w_1, ..., w_m$ be a basis of W and let us extend this basis to basis $w_1, ..., w_m, ..., w_n$ of V. Then we can follow that there exists a dual basis $\phi_1, ..., \phi_m, ..., \phi_n$ of V. Therefore we can follow that if $\psi \in \text{span}(\phi_{m+1}, ..., \phi_n)$, then

$$\psi(w) = 0$$

therefore $\text{span}(\phi_{m+1}, ..., \phi_n) \subseteq W^0$. Conversely if $\psi \in W^0$, then we can follow that

$$\psi = 0\phi_1 + ... + 0\phi_m + a_{m+1}\phi_{m+1} + ...$$

(otherwise $\psi(w_j) \neq 0$.) Therefore we can follow that $W^0 \subseteq \text{span}(\phi_{m+1}, ..., \phi_n)$. Thus we've got that $W^0 = \text{span}(\phi_{m+1}, ..., \phi_n)$. Therefore $\phi_{m+1}, ..., \phi_n$ is a basis for W^0.

Suppose now that there exists $u \in U$ such that $u \notin W$. Then we can follow that

$$u = \sum a_j w_j$$

Because $u \notin W$ we can follow that there exists $k > m$ such that $a_k \neq 0$. Thus $\phi_k(u) = a_k \neq 0$. Given that $k > m$, we follow that $\phi_k \in W^0$. Thus we follow that $\phi_k \in U^0$. Therefore $\phi_k(u) = 0$, which is a contradiction.

Therefore we follow that there does not exist $u \in U$ such that $u \notin W$. Thus we can follow that $u \in U \to u \in W$. Therefore $U \subseteq W$, as desired.

3.6.22

Suppose U, W are subspaces of V. Show that $(U+W)^0 = U^0 \cap W^0$.

Suppose that $\phi \in (U+W)^0$. Then we can follow that for every $u \in U$, $\phi(u) = 0$. therefore $\phi \in U^0$. By the same logic we have that $\phi \in W^0$. Thus we can follow that

$$\phi \in (U+W)^0 \to \phi \in U^0 \wedge \phi \in W^0$$

$$\phi \in (U+W)^0 \to \phi \in U^0 \cap W^0$$
$$(U+W)^0 \subseteq U^0 \cap W^0$$

Conversely, suppose that $\phi \in U^0 \cap W^0$. Then we follow that if $v = u + w \in U + W$, then

$$\phi(v) = \phi(u+w) = \phi(u) + \phi(w) = 0 + 0 = 0$$

Thus we follow that

$$\phi \in U^0 \cap W^0 \to \phi \in (U+W)^0$$
$$U^0 \cap W^0 \subseteq (U+W)^0$$

thus by double inclusion we've got the desired equality.

3.6.23

Suppose V is finite-dimentional and U and W are subsets of V. Prove that $(U \cap W)^0 = U^0 + W^0$

Because V is finite-dimentional we can follow that U and W are both finite-dimentional, and therefore $U \cap W$ is finite dimentional (the fact that it is a subspace was proven in the exercises before). We can therefore make a basis $u_1, ..., u_n, ..., u_m, ..., u_l, ..., u_k$ where $u_n, ..., u_m$ is a basis of $U \cap W$ (empty in case if the dimention of intersection is zero), $u_1, ..., u_{n-1}$ is the basis of U, $u_{m+1}, ..., u_l$ is the rest of the basis of W and the rest is the basis of V. Then we can create a dual basis on this basis and get that

$$U^0 + W^0 = \operatorname{span}(\phi_1, ..., \phi_{n-1}, ..., \phi_{m+1}, ..., \phi_k) = (U \cap W)^0$$

as desired.

3.6.24

Prove 3.106 using the ideas scketched in the discussion before the statement of 3.106

We've outlined this proof in 3.6.21.

3.6.25

Suppose V is finite-dimentional and U is a subspace of V. Show that

$$U = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

Let $u_1, ..., u_n$ be a basis of U, and extend this basis to $u_1, ..., u_n, ..., u_m$ - a basis of V. Then let us define dual basis on this basis $\phi_1, ..., \phi_m$, and it'll follow that if $v \notin U$, then there will exist ϕ_k for some $k > n$ such that

$$\phi_k(v) \neq 0$$

Thus we can follow that

$$v \notin U \rightarrow v \notin \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

and therefore

$$v \in \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\} \rightarrow v \in U$$

thus

$$\{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\} \subseteq U$$

Conversely, suppose that $u \in U$. Then it follows that $\phi \in U^0 \rightarrow \phi(u) = 0$. Thus we've got that $U \subseteq \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$.

Thus by double inclusion we've got our desired equality.

3.6.26

Suppose V is finite-dimentional and Γ is a subspace of V'. Show that

$$\Gamma = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

Suppose that Γ is a subspace of V'. Then we can follow that there exists a basis of Γ - $\phi_1, ..., \phi_n$, and that we can extend this basis to a basis of V' - $\phi_1, ..., \phi_m$.

For every ϕ_j we've got that $\phi_j \neq 0$ and therefore by FTOLM we've got that

$$\dim V' = \dim \operatorname{null} \phi_j + \dim \operatorname{range} \phi_j$$

$$\dim V' = \dim \operatorname{null} \phi_j + 1$$

$$\dim V' - 1 = \dim \operatorname{null} \phi_j$$

thus we can follow that if we take a basis of $\operatorname{null} \phi_j$, extend it to a basis of V by adding one vector v'_j, then

$$\phi_j v'_j \neq 0$$

then we can define

$$v_j = v'_j * \frac{1}{\phi_j v_j}$$

so that

$$\phi_j(v_j) = 1$$

By making v_j for each corresponding ϕ_j we can get the list $v_1, ..., v_m$. Because every ϕ_m has a diffenent nullspace (otherwise they are scalar multiple of each other) we can follow that they have different preimage, and therefore $v_1, ..., v_m$ is linearly independent. By length of this linearly independent list we can follow that it is a basis of V.

Then we can follow that there exist vectors $v_1, ..., v_m$ such that

$$\phi_j(v_j) = 1$$

and for $k \neq j$

$$\phi_k(v_j) = 0$$

Therefore we'll have a subspace $U = \operatorname{span}(v_{n+1}, ..., v_m)$. Then it follows that

$$\Gamma = U^0$$

and by results of our previous exercise we'll have that

$$U = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}$$

$$U^0 = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in U^0\}^0$$

$$\Gamma = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

as desired.

3.6.27

Suppose $T \in \mathcal{L}(P_5(R), P_5(R))$ and $\operatorname{null} T' = \operatorname{span}(\phi)$, where ϕ is the linear functional on $P_5(R)$ defined by $\phi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in P_5(R) : p(8) = 0\}$.

By a theorem in the chapter we've got that

$$\operatorname{null} T' = (\operatorname{range} T)^0$$

and by results of the previous exercise we've got that

$$\operatorname{span}(\phi) = \{p \in P_5(R) : \phi(p) = 0\}^0$$

(we've shortened the right-hand side, since $\psi \in \operatorname{span}(\phi) \wedge \psi(v) = 0 \rightarrow \phi(v) = 0$). thus we can follow taht

$$\operatorname{null} T' = \operatorname{span}(\phi)$$

is equivalent to stating that

$$(\operatorname{range} T)^0 = (\{p \in P_5(R) : \phi(p) = 0\})^0$$

For the next implication we'll probably need a little lemma

Lemma: Equivalent annihilators implies equivalent subspaces

Suppose that V is finite-dimentional, U and W are subspaces of V. Then $U^0 = W^0$ implies $U = W$.

By exercise 21 in this chapter we've got that

$$W^0 \subseteq U^0 \to U \subseteq W$$

thus we follow that

$$W^0 = U^0 \to W^0 \subseteq U^0 \wedge U^0 \subseteq W^0 \to U \subseteq W \wedge W \subseteq U \to W = U$$

as desired. (If we think about it, exercise 20 gives us that this statement is actually an equivalence and not implicaton. For infinite-dimentional vector spaces we probably got just the implication)

Thus we can indeed follow that

$$(\operatorname{range} T)^0 = (\{p \in P_5(R) : \phi(p) = 0\})^0$$

implies that

$$\operatorname{range} T = \{p \in P_5(R) : \phi(p)$$

as desired

3.6.28

Suppose V and W are finite-dimentional, $T \in \mathcal{L}(V, W)$, and there exists $\phi \in W'$ such that $\operatorname{null} T' = \operatorname{span} \phi$. *Prove that* $\operatorname{range} T = \operatorname{null} \phi$

This is a generalization of the previous exercise

$$\{v \in V : \psi(v) = 0 \text{ for every } \psi \in \operatorname{span}(\phi)\} = \{v \in V : \lambda\phi(v) = 0\} = \{v \in V : \phi(v) = 0\} = \operatorname{null} \phi$$

Thus

$$\operatorname{null} T = (\operatorname{range} T)^0 = (\operatorname{null} \phi)^0$$

thus

$$\operatorname{range} T = \operatorname{null} \phi$$

as desired.

3.6.29

Suppose V and W are finite-dimentional, $T \in \mathcal{L}(V, W)$, and there exists $\phi \in V'$ such that $\operatorname{range} T' = \operatorname{span} \phi$. *Prove that* $\operatorname{null} T = \operatorname{null} \phi$

$$\operatorname{range} T' = (\operatorname{null} T)^0 = \operatorname{span} \phi = (\operatorname{null} \phi)^0$$

thus

$$(\operatorname{null} T)^0 = (\operatorname{null} \phi)^0$$

$$\operatorname{null} T = \operatorname{null} \phi$$

as desired.

3.6.30

Suppose V is finite-dimentional and $\phi_1, ..., \phi_m$ is a linearly independent list in V'. Prove that

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) = (\dim V) - m$$

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) + \dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = (\dim V)$$

Thus what we probably are intended to prove is that

$$((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = \operatorname{span}(\phi_1, ..., \phi_m)$$

By exercise 26 we can get that

$$\operatorname{span}(\phi_1, ..., \phi_m) = \{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\}^0$$

Since $\phi \in \operatorname{span}(\phi_1, ..., \phi_m) \to \phi = \sum a_j \phi_j$, we follow that

$$\operatorname{span}(\phi_1, ..., \phi_m) = \{v \in V : \phi_j(v) = 0 \text{ for every } \phi_j\}^0$$

If $\phi_j(v) = 0$ for every ϕ_j, then $v \in \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$ du definition of nullspace. If $v \in \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$, then it is obliously true that $\phi_j(v) = 0$ by the same definition. Thus by double inclusion we get that

$$\{v \in V : \phi(v) = 0 \text{ for every } \phi \in \Gamma\} = \operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m$$

thus

$$\operatorname{span}(\phi_1, ..., \phi_m) = (\operatorname{null} \phi_1 \cap ... \cap \operatorname{null} \phi_m)^0$$

and therefore

$$\dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m)) + \dim((\operatorname{null} \phi_1) \cap ... \cap (\operatorname{null} \phi_m))^0 = (\dim V)$$

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + \dim \operatorname{span}(\phi_1, ..., \phi_m) = (\dim V)$$

since $\phi_1, ..., \phi_m$ are linearly independent, we can follow that

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + m = (\dim V)$$

$$\dim((\operatorname{null}\phi_1) \cap ... \cap (\operatorname{null}\phi_m)) + = (\dim V) - m$$

as desired.

3.6.31

Suppose V is finite-dimentional and $\phi_1, ..., \phi_n$ is a basis of V'. Show that there exists a basis of V, whose dual basis is $\phi_1, ..., \phi_n$.

From previous exercice we can follow that there for given basis there exists a space

$$U_k = \cap_{i \neq k} \operatorname{null}\phi_i$$

such that

$$\dim U_k = \dim V - (n-1) = n - (n-1) = 1$$

From the definition we can follow that

$$u \in U_k \to u \in \operatorname{null}\phi_{i \neq k} \to \phi_{i \neq k}(u) = 0$$

Suppose that $u \neq 0$. Then we can follow that we can extend this vector to the basis of V. Thus we can follow that we can create a map $\psi \in \mathcal{L}(V, F) \iff \psi \in V'$ such that

$$\psi(u) = 1$$

Given that $\phi_1, ..., \phi_n$ spans V', we can follow that there exist $a_1, ..., a_n$ such that

$$\psi = \sum a_j \phi_j$$

from this we can follow that

$$\psi(u) = \sum a_j \phi_j(u) = \sum_{j \neq k} a_j \phi_j(u) + a_k \phi_k(u) = 0 + a_k \phi(u) = a_k \phi_k(u)$$

Thus we can follow that

$$a_k \phi(u) = \psi(u) = 1$$

$$a_k \phi(u) = 1$$

thus we follow that $a_k \neq 0$ and $\phi(u) \neq 0$. Thus we follow that $\phi(a_k u) = 1$. Set $v_k = a_k u$.

In this fashion we can create list $v_1, ..., v_n$ with the property that

$$\phi_k(v_k) = 1$$

$$\phi_{i\neq k}(v_k) = 0$$

The only thing that is left is to show that this list is linearly independent. Suppose that it isn't. Then it follows that

$$v_k = \sum_{j\neq k} a_j v_j$$

thus

$$\phi_k(v_k) = \phi_k(\sum_{j\neq k} a_j v_j)$$
$$1 = 0$$

which is false. Thus we follow that $v_1, ..., v_n$ is linearly independent list with desired properties in V. Given that its length is the dimention of the space, we follow that it is a basis, for which $\phi_1, ..., \phi_n$ is dual basis, as desired.

3.6.32

Suppose $T \in \mathcal{L}(V)$, and $u_1, ..., u_n$ and $v_1, ..., v_n$ are bases of V. Prove that the following are equivalent:

(a) T is invertible

(b) The columnns of $M(T)$ are linearly independent in $F^{n,1}$.

(c) The columns of $M(T)$ span $F^{n,1}$

Because T is invertible, we can follow that the dimention of its range is equal to n. Thus we can follow that rank of its matrix is n. Therefore its row rank and column rank is n. Thus (a) implies (b). By the size of the mateix we get that (b) implies (c). And (c) implies that with given bases the function is invertible, therefore (c) implies (a).

Because T is invertible if and only if T' is invertible, we get the (d) and (e) for free.

3.6.33

Suppose m and n are positive integers. Prove that teh function that takes A to A^t is a linear map from $F^{m,n}$ to $F^{n,m}$. Furtermore, prove that this linear maps is invertible.

Linearity follows directly from definitions of the transpose, matrix addition and scalar multiplication. Invertability follows directily from injectifity and surjectivity of this transformation.

3.6.34

The double dual space of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda(v))(\phi) = \phi(v)$$

for $v \in V$ and $\phi \in V'$

Since I haven't fully understood the definition, I'll try to dumb it down a notch. Suppose that $v \in V$. Then we follow that there exists $\kappa \in V''$ such that

$$\Lambda(v) = \kappa$$

Thus

$$\kappa \in V'' \to \kappa \in (V')' \to \kappa \in \mathcal{L}(V', F) \to \kappa \in \mathcal{L}(\mathcal{L}(V, F), F)$$

Thus we can plug in some $\phi \in V'$ into κ, and get some number from it.

What could be an example of such a map? Suppose that $p \in P(R)$. Then we can define $\phi(p) = \int_0^1 p$. And thus we can define Λ to be a function, that inputs a polynomial, and returns a function, that inputs a linear functional on a polynomial, and returns the result of applying inputed polynomial into the linear functional.

That kind of made a dent in the understanding of what the hell is going on, but nothing major had happened.

(a) Show that Λ is a linear map from V to V''.

$$(\Lambda(\lambda v))(\phi) = \phi(\lambda v) = \lambda(\phi(v)) = \lambda((\Lambda(v))(\phi))$$
$$(\Lambda(v + w))(\phi) = \phi(v + w) = \phi(v) + \phi(w) = (\Lambda(w))(\phi) + (\Lambda(w))(\phi)$$

Thus we follow that Λ is linear, as desired.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

Let $v \in V$ and $\phi \in V'$. Then we can follow that

$$(T'' \circ (\Lambda v))(\phi) = (\Lambda v)(T'(f)) = (T' f)(v) = f(T(v)) = \Lambda(Tv)(f)$$

as desired.

(c) Show that if V is finite-dimentional, then Λ is an isomorphism from V onto V''

If V is finite-dimentional, then V'' is finite-dimentional, and their dimentions are the same. Thus injectivity of Λ implies the invertibility. Suppose that $\Lambda v = 0$. Then we follow that $\phi(v) = 0$ for any $\phi \in V'$. By exercise 19 we've got that it happens if and only if $v = 0$. Thus Λ is injective, and therefore invertible, as desired.

3.6.35

Show that $(P(R))'$ and R^∞ are isomorphic.

$P(R)$ and $(P(R))'$ are isomorphic, $P(R)$ and R^∞ are also isomorphic, therefore $(P(R))'$ and R^∞ are isomorphic, as desired.

3.6.36

Suppose U is a subspace of V. Let $i : U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$

$$\operatorname{null} i' = (\operatorname{range} i)^0 = U^0$$

(b) Prove that if V is finite-dimentional, then $\operatorname{range} i' = U'$.

Let $v_1, ..., v_n, ..., v_m$ be a basis of V, where $v_1, ..., v_n$ is a basis of U. Define dual basis of it. Then we follow that $v_{n+1}, ..., v_n$ is a basis of $\operatorname{null} i$, and therefore $\psi_1, ...\psi_n$ is a basis of $(\operatorname{null} i)^0$, which is equal to basis of U'. THus

$$(\operatorname{null} i)^0 = U'$$

and since

$$\operatorname{range} i' = (\operatorname{null} i)^0$$

we follow that

$$\operatorname{range} i' = U'$$

as desired.

(c) Prove that if V is finite-dimentional, then $\overline{i'}$ is an isophormism from V'/U^0 onto U'.

$$\dim V'/U^0 = \dim V' - \dim U^0 = \dim V - (\dim V - \dim U) = \dim U$$

and

$$\dim U' = \dim U$$

thus we follow that the dimentions of those vector spaces are the same.

Since $\operatorname{range} i' = U'$, we can follow that $\operatorname{range} \overline{i'} = \operatorname{range} i' = U'$. Thus we follow that this map is sujective, and therefore by their identical dimention we follow that $\overline{i'}$ is an isomorphism, as desired.

3.6.37

Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the usual quotent map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

Sinse π is surjective (suppose that $a \in (V/U)$, then $a = v + U$, then there exists $v \in V$, then $\pi(v) = v + U$, therefore it is surjective.) we can follow that π' is injective, as desired.

(b) Show that $\operatorname{range} \pi' = U^0$.

$$\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$$

(the fact that $\operatorname{null}\pi = U$ is somewhat followed in the proof of 3.89, where it is derived from 3.85)

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

π' is inhective and is surjective, if we restrict its codomain to U^0. Thus it is invertible, therefore it is an isomorphism.

Chapter 4

Polynomials

4.1 Polynomials

4.1.1

Verify all assertions in 4.5 except the last one.

Firstly, let us state that $z, w \in C$ and

$$z = a + bi$$

$$w = c + di$$

Then we follow that

$$z + \overline{z} = a + bi + a - bi = 2a = 2Re(z)$$

$$z - \overline{z} = a + bi - a + bi = 2bi = 2Im(z)i$$

$$z\overline{z} = (a+bi)(a-bi) = a^2+abi-abi-b^2 = a^2+b^2 = (\sqrt{a^2+b^2})^2 = (\sqrt{Re(z)^2+Im(z)^2})^2 = |z|^2$$

$$\overline{w+z} = \bar{(}a + bi + c + di) = \bar{(}a + c + (b + d)i) = a + c - (b + d)i = a - bi + c - di = \overline{z} + \overline{w}$$

$$\overline{\overline{z}} = \overline{\overline{a+bi}} = \overline{a-bi} = a + bi = z$$

$$|a^2| = a^2 \le a^2 + b^2 = |a^2 + b^2| \to |a|^2 \le a^2 + b^2 \to |a| \le \sqrt{a^2+b^2} \to |Re(z)| \le |z|$$

$$|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2+b^2} = |z|$$

$$|wz| = |(a+bi)(c+di)| = |ac+adi+cbi-bd| = |ac-bd+(ad+cb)i| = \sqrt{(ac-bd)^2+(ad+cb)^2} =$$

$$= \sqrt{(ac)^2 - 2abcd + (bd)^2 + (ad)^2 + 2abcd + (cb)^2} = \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (cb)^2} =$$

$$= \sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)} = \sqrt{(a^2+b^2)(c^2+d^2))} = |z||w| = |w||z|$$

4.1.2

Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(F) : \deg p = m\}$$

a subspace of $\mathcal{P}(F)$?

Suppose $m \neq 1$ Let $p_1 = x^m + 1$ and $p_2 = -x^m + 1$. Then we follow that

$$\deg(p_1 + p_2) = \deg(2) = 0 \neq m$$

(we use the fact that m must be positive). Thus given set is not closed under addition, therefore it is not a subspace.

4.1.3

Is the set

$$\{0\} \cup \{p \in \mathcal{P}(F) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(F)$?

Let $p_1 = x^2 + x$ and $p_2 = -x^2 + x$. Then we follow that

$$\deg(p_1 + p_2) = \deg(2x) = 1$$

thus the space is not closed under addition, therefore it is not a subspace.

4.1.4

Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, ..., \lambda_m \in F$. Prove that there exists a polynomial $p \in P(F)$ with $\deg(p) = n$ such that $0 = p(\lambda_1) = ... = p(\lambda_n)$ and such that p has no other zeroes.

Let

$$p = (z - \lambda_1)...(z - \lambda_m)^{m-n+1}$$

Then we can follow that the only zeroes of p are precisely zeroes of

$$p_1 = (z - \lambda_1)...(z - \lambda_m)$$

which are $\lambda_1, ..., \lambda_m$ Then we follow that zeroes of

$$p_2 = (z - \lambda_1)...(z - \lambda_m)^2 = p_1(z - \lambda_m)$$

are zeroes of p_1 and λ_m. Given that zeroes of p_1 already have λ_m, we follow that p_1 and p_2 have the same zeroes. Then by induction we follow that zeroes of p are zeroes of p_1, which are $\lambda_1, ..., \lambda_m$, as desired.

4.1.5

Suppose m is nonnegative integer, $z_1, ..., z_{m+1}$ are distinct elements of F, and $w_1, ..., w_{m+1} \in F$. Prove that there exists a unique polynomial $p \in P_m(F)$ such that

$$p(z_j) = w_j$$

for $j = 1, ...m + 1$.

Let $T : P_m(F) \to F^{m+1}$ be defined as

$$T(p) = (p(z_1), p(z_2), ..., p(z_{m+1}))$$

We can follow that if

$$T(p) = 0$$

then

$$p(z_1) = p(z_2) = ... = p(z_{m+1}) = 0$$

But if $p \neq 0$, then it has at most m roots. Thus we follow that $p = 0$. Therefore T is injective. By FTLM we have that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Therefore for this case we have

$$\dim P_m(F) = \dim \operatorname{null} T + \dim \operatorname{range} T$$

$$m + 1 = \dim \operatorname{range} T$$

given that $\dim F^{m+1} = m + 1$ we follow that $\operatorname{range} T = F^{m+1}$. Therefore we follow that it is surjective. Thus for any vector $(w_1, ..., w_{m+1}) \in F^{m+1}$ there exists $p \in P_m(F)$ such that

$$p(z_j) = w_j$$

as desired.

4.1.6

Suppose $p \in \mathcal{P}(C)$ has degree m. Prove that p has m distinct zeroes if and only if p and its derivative p' have no zeroes in common

Suppose that p has m distinct zeroes and suppose that $p'(\lambda) = p(\lambda) = 0$ for some $\lambda \in F$. Then we follow that