My abstract algebra exercises

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Contents

Chapter 1

Groups

1.1 Symmetries of a Regular Polygon

Content of this section was pretty much taken care of in a previous try at an abstract algebra coutse

1.2 Introduction to Groups

For the next 14 exercises decide whether or not hte given pair forms a group.

1.2.1

The pair $(N, +)$

No, since there are no inverses for nonzero elements

1.2.2

The pair $(Q \setminus \{-1\}, \star)$, *where* $a \star b = a + b + ab$

$$a \star (b \star c) = a \star (b + c + bc) = a + (b + c + bc) + ab + ac + abc$$

so associativity checks out.

We can follow that 0 is an identity, since

$$a \star 0 = a + 0 + a0 = a$$

Suppose that $a \in Q \setminus \{-1\}$. We follow that

$$a + b + ab = 0$$

$$b = -a(1 + b)$$

$$b/(1+b) = -a$$
$$-b/(1+b) = a$$

since $b \in Q \setminus \{-1\}$, we follow that $b = m/n$, and thus

$$-\frac{m/n}{1+m/n} = a$$
$$-\frac{m/n}{(n+m)/n} = a$$
$$-\frac{m}{n+m} = a$$

since $a \in Q \setminus \{-1\}$ we follow that $a = k/l$, and thus

$$-\frac{m}{n+m} = k/l$$
$$\frac{-m}{n+m} = \frac{k}{l}$$
$$\begin{cases} m = -k \\ n = l+k \end{cases}$$

thus we follow that as long as $n \neq 0$, a will have an inverse. $n = 0 \iff l = -k \iff a = -1$, and since $a \neq -1$, we conclude that any given element in the given set is an inverse, and thus the given set satisfies all the axioms of a group.

1.2.3

The pair $\langle Q \setminus \{0\}, / \rangle$

We follow that if $a \in lhs$, then $a = m/n$, and thus n/m is the inverse, thus every element got an inverse ($a \neq 0$, thus $m \neq 0$).

$$a/(b/c) = a/\frac{b}{c} = a\frac{c}{b} = \frac{ac}{b}$$
$$(a/b)/c = \frac{a}{b}/c = \frac{a}{b}\frac{1}{c} = \frac{a}{bc}$$

nonzero a, b, c ($\langle 1, 2, 3 \rangle$ should do the trick) will give us a concrete proof that $/$ is not associative, which means that there's no group

1.2.4

The pair $\langle A, + \rangle$ *where* $A = \{x \in Q : |x| < 1\}$

Assuming that $|\star|$ means absolute value, we follow that $+$ won't be a binary operation on A.

The rest of the exercises are left for better times

1.3 Properties of Group Elements

1.3.1

Find the orders of $\overline{5}$ *and* $\overline{6}$ *in* $(Z/21Z, +)$

We follow that order of $\overline{5}$ is 21 and 7 for $\overline{6}$.

1.3.2

Find the orders of $\overline{21}$ *in* $Z/52$

It's' 13

1.3.3

Calculate the order of $\overline{285}$ *in the group* $Z/360Z$

$$(285 * 24)/360 = 19$$

thus the order is 19

The rest of the exercises (or exercises similar to those given in a book) were taken care of previously in previous books

1.3.18

Prove that $(Q, +)$ *is not a cyclic group.*

We can follow that $q \in Q$ is either positive, negative or zero. Thus q^n is either positive, negative or zero respectively for all $n \in \omega$, thus proving that no element of Q can be a generator, which means that Q has no generator.

1.4 Concept of a Classification Theorem