My set theory exercises

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Contents

Useful things

I think that it is pretty straightforward to define some function based on axioms that we get. For example pairing axiom allows us to define $PA : S \times S \to S$ by

$$PA(u, v) = \{u, v\}$$

same goes for union axiom

$$UA(u) = \{\text{elements of elements of U}\}$$

Later some other function might be defined in the same manner.

In logic notation, I denote tautology as 'true' and contradiction as 'false'

Chapter 1

Introduction

1.1 Elementary Set Theory

Let A, B, C be sets

1.1.1

If $a \notin A \setminus B$ *and* $a \in A$*, show that* $a \in B$

Because $a \notin A \setminus B$, we follow that $x \in B$ or $x \notin A$. Since $x \in A$, we follow that $x \in B$, as desired.

1.1.2

Show that if $A \subseteq B$*, then* $C \setminus B \subseteq C \setminus A$

Let $c \in C \setminus B$. Then we follow that $c \in C$ or $c \notin B$. Since $A \subseteq B$, we follow that $c \notin B$ implies that $c \notin A$. Thus we follow that $c \in C \setminus B$ implies that $c \in C \setminus A$. Therefore $C \setminus B \subseteq C \setminus A$.

1.1.3

Suppose $A \setminus B \subseteq C$*. Show that* $A \setminus C \subseteq B$*.*

Suppose that $a \in A \setminus C$. Then we follow that $a \in A$ and $a \notin C$.

Given that $A \setminus B \subseteq C$ and $A \notin C$, we follow that $a \notin A \setminus B$. Thus $a \notin A$ or $a \in B$. Since $a \in A$, we follow that $a \in B$. Thus

$$a \in A \setminus C \to a \in B$$

$$A \setminus C \subseteq B$$

as desired.

1.1.4

Suppose $A \subseteq B$ and $A \subseteq C$. Show that $A \subseteq B \cap C$

Suppose that $a \in A$. Then we follow that $a \in B$ and $a \in C$. Thus $a \in B \cap C$. Therefore we follow that $A \subseteq B \cap C$.

1.1.5

Suppose $A \subseteq B$ and $B \cap C = \emptyset$. Show that $A \in B \setminus C$

Suppose that $a \in A$. Then we follow that $a \in B$ and since $B \cap C = \emptyset$, we follow that $a \notin C$. Thus $a \in B \setminus C$ by definition. Therefore $A \subseteq B \setminus C$.

1.1.6

Show that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$. Suppose that $a \in A \setminus (B \setminus C)$. Then we follow that $a \in A$ and $a \notin B \setminus C$. Thus $a \notin B$ and $a \in C$. Thus we follow that $a \in A \setminus B$ or $a \in C$. Thus $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$ as desired.

1.1.7

Let $P(x)$ be the property $x > \frac{1}{x}$. Are the assertions $P(2)$, $P(-2)$, $P(\frac{1}{2})$ $P(\frac{-1}{2})$ true or false
.

$$2 > \frac{1}{2} \rightarrow P(2) = true$$
$$-2 < \frac{-1}{2} \rightarrow P(-2) = false$$

last two are reversed.

1.1.8

Sow that each of the following sets can be expressed as an interval

$$a)(-3,3)$$
$$b)(-3,\infty)$$
$$c)(-3,3)$$

all of them follow from order properties of real numbers.

1.1.9

Express the following sets as truth sets

$$A = \{1, 4, 9, 16, 25, ...\} \iff A = \{x \in N : x = y^2 \textit{ for some } y \in N\}$$
$$B = \{..., -15, -10, -5, 0, 5, ...\} \iff A = \{x \in N : x = 5y \textit{ for some } y \in N\}$$

Rest are also trivial, not gonna go deep here

1.2 Logical Notation

1.2.1

Using truth tables, show that $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$
false	false	true	false	true	false
false	true	true	false	false	false
true	false	false	true	true	true
true	true	true	false	false	false

from this we can see that they are equqivalent.
Following 4 exercises are the same as this one, so I'm skipping them

1.2.5

Show that $(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow P \Rightarrow (Q \wedge R)$*, using logic laws*

$$(P \Rightarrow Q) \wedge (P \Rightarrow R) \Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \Leftrightarrow \neg P \vee (R \wedge Q) \Leftrightarrow P \Rightarrow (R \wedge Q)$$

Laws used:

$$CL \rightarrow DIST \rightarrow CL$$

1.2.6

Show that $(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$*, using logic laws*

$$(P \Rightarrow R) \vee (Q \Rightarrow R) \Leftrightarrow (\neg P \vee R) \vee (\neg Q \vee R) \Leftrightarrow \neg P \vee R \vee \neg Q \vee R \Leftrightarrow (\neg Q \vee \neg P) \vee R \Leftrightarrow$$
$$\Leftrightarrow \neg(Q \wedge P) \vee R \Leftrightarrow (Q \wedge R) \Rightarrow R$$

Laws used:

$$CL \rightarrow ASC \rightarrow ID, ASC \rightarrow DML \rightarrow CL$$

1.2.7

Show that $P \Rightarrow (Q \Rightarrow R) \Leftrightarrow (P \wedge Q) \Rightarrow R$, using logic laws

$P \Rightarrow (Q \Rightarrow R) \Leftrightarrow \neg P \vee (Q \Rightarrow R) \Leftrightarrow \neg P \vee (\neg Q \vee R) \Leftrightarrow (\neg P \vee \neg Q) \vee R \Leftrightarrow \neg (P \wedge Q) \vee R \Leftrightarrow (P \wedge Q) \Rightarrow R$

Laws used:

$$CL \rightarrow CL \rightarrow ASC \rightarrow DML \rightarrow CL$$

1.2.8

Show that $(P \Rightarrow Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ are not logically equivalent

We're gonna show that $q \wedge w \Leftrightarrow false$

$$((P \Rightarrow Q) \Rightarrow R) \wedge (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow (\neg(\neg P \vee Q) \vee R) \wedge (\neg P \vee (\neg Q \vee R)) \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge \neg Q) \vee R) \wedge (\neg P \vee \neg Q \vee R) \Leftrightarrow ((P \wedge Q) \wedge (\neg P \vee \neg Q)) \vee R \Leftrightarrow$$

$$\Leftrightarrow ((P \wedge Q) \wedge \neg(P \wedge Q)) \vee R \Leftrightarrow false \vee R \Leftrightarrow false$$

1.3 Predicates and Quantifiers

1.4 A Formal Language for Set Theory

1.4.1

What does the formula $\exists x \forall y (x \notin y)$ say in English?

There exists x such that for every y we've got that x is not in y. In other ways, there exists an empty set.

1.4.2

What does the formula $\forall y \exists x (y \notin x)$ say in English?

For every y there exists set x such that y is not in x.

1.4.3

What does the formula $\forall y \exists x (x \notin y)$ say in English?

For every y there exists x such that x is not in y.

1.4.4

What does the formula $\forall y \neg \exists x (x \notin y)$ say in English?

For every y there does not exist an x such that x is not in y.

1.4.5

What does the formula $\forall z \exists x \exists y (x \in y \land y \in z)$ say in English?

For every z there exists x and y such that x is in y and y is in z

1.4.6

Let $\phi(x)$ be a formula. What does $\forall z \forall y ((\phi(x) \land \phi(y)) \to z = y$

For every z and y, $\phi(x)$ and $\phi(y)$ implies that z = y.

1.4.7

Translate each of the following into the language of set theory.

(a) x is the union of a and b

$$\forall (y \in x)(y \in a \land y \in b)$$

(b) x is not a subset of y

$$\exists (z \in x)(\neg z \in y)$$

(c) x is the intersection of a and b

$$\forall (y \in x)(y \in a \lor y \in b)$$

(d) a and b have no elements in common

$$\forall (x \in a) \forall (y \in b)(\neg x = y)$$

1.4.8

Let a, b, C and D be sets. Show that the relationship

$$y = \begin{cases} a \text{ if } x \in C \setminus D \\ b \text{ if } x \notin C \setminus D \end{cases}$$

$$((x \in C \land \neg x \in D) \to (y = a)) \land ((\neg x \in C \land \neg x \in D) \to (y = a))$$

1.5 The Zermelo-Fraenkel Axioms

1.5.1

Let u, v, w be sets. By pairing axiom, the sets $\{u\}$ and $\{v, w\}$ exist. Using the pairing and union axioms, show that the set $\{u, v, w\}$ exists.

By pairing axiom we've got that

$$PA(u, u) = \{u\}$$

$$PA(v, w) = \{v, w\}$$

thus

$$PA(\{u\}, \{v, w\}) = \{\{u\}, \{v, w\}\}$$

and therefore by union axiom we follow that

$$UA(\{\{u\}, \{v, w\}\}) = \{u, v, w\}$$

as desired.

1.5.2

Let A be a set. Show that the pairing axiom implies that the set $\{A\}$ exists

$$PA(A, A) = \{A, A\}$$

which by extension axiom is equal to $\{A\}$, as desired.

1.5.3

Let A be a set. The pairing axiom implies that the set $\{A\}$ exists. Using the regularity axiom, show that $A \cap \{A\} = 0$. Conclude that $A \notin A$.

Since $\{A\} \neq \emptyset$, we follow that there exists x such that $x \in \{A\}$ and $x \cap \{A\} = \emptyset$. Since A is the only element of $\{A\}$, we follow that $A \cap \{A\} = \emptyset$, as desired.

1.5.4

For sets A, B, the set $\{A, B\}$ exists by the pairing axiom. Let $A \in B$. Using the regularity axiom, show that $A \cap \{A, B\} = \emptyset$, and thus $B \notin A$.

$\{A, B\}$ consists of sets A and B, thus it is not empty and therefore there exists $x \in \{A, B\}$ such that $x \in \{A, B\} \wedge x \cap \{A, B\} = \emptyset$. For B we've got that $B \in \{A, B\}$. Since $A \in B$ and $A \in \{A, B\}$, we can follow that $A \in (B \cap \{A, B\})$. By pairing axiom we follow that the element with desired property must exists, and given that the only other choice is A, we conclude that $A \cap \{A, B\} = \emptyset$. Therefore we can follow that $B \notin A$, as desired.

1.5.5

Let A, B, C be sets. Suppoes that $A \in B$ and $B \in C$. Using the regularity axiom, show that $C \notin A$.

This is an expantion of previous exercise. We can follow that

$$B \in \{A, B, C\} \wedge B \in C \Rightarrow B \in C \cap \{A, B, C\} \Rightarrow C \cap \{A, B, C\} \neq \emptyset$$

$$A \in \{A, B, C\} \wedge A \in B \Rightarrow A \in B \cap \{A, B, C\} \Rightarrow B \cap \{A, B, C\} \neq \emptyset$$

thus the only other choice is A, and therefore $A \cap \{A, B, C\} = \emptyset$. Therefore $C \notin A$, as desired.

1.5.6

Let A, B be sets. Using the subset and power set axioms, show that the set $\mathcal{P}(A) \cap B$ exists.

Because set A exists we follow that $\mathcal{P}(A)$ exists. By setting $\phi(x) : x \in B$ and subset axiom we follow that there exists a subset of $\mathcal{P}(A)$ such that $x \in S \Leftrightarrow x \in \mathcal{P}(A) \wedge x \in B$. Thus we follow by Extensionality axiom that $S = \mathcal{P}(A) \cap B$. Thus $\mathcal{P}(A) \cap B$ exists.

1.5.7

Let A, B be sets. Using the subset axiom, show that the set $A \setminus B$ exists.

$$\phi(x) : \neg x \in B$$

thus by subset axiom

$$x \in S \Leftrightarrow x \in A \wedge \neg x \in B$$

thus $A \setminus B$ exists.

1.5.8

Show that no two of the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are equal to each other.

I had a little confusion with this one at first because I thought that every set has empty set in it, which is false. Every set has an empty set as a subset, but it might be so that empty set is not in the set itself.

$$\emptyset \notin \emptyset \wedge \emptyset \in \{\emptyset\} \Rightarrow \emptyset \neq \{\emptyset\}$$

$$\emptyset \notin \emptyset \wedge \emptyset \in \{\emptyset, \{\emptyset\}\} \Rightarrow \emptyset \neq \{\emptyset, \{\emptyset\}\}$$

$$\{\emptyset\} \notin \{\emptyset\} \wedge \{\emptyset\} \in \{\emptyset, \{\emptyset\}\} \Rightarrow \{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$$

all of the implication follow from extensionality axiom.

1.5.9

Let A be a set with no elements. Show that for all x, we have that $x \in A$ if and only if $x \in \emptyset$. Using the extensionality axiom, conclude that $A = \emptyset$.

Suppose that $\neg x \in A$. Then we follow that x is an element, therefore $\neg x \in \emptyset$. Thus

$$\neg x \in A \Rightarrow \neg x \in \emptyset \iff x \in \emptyset \Rightarrow x \in A$$

Suppose that $\neg x \in \emptyset$. Then we follow that x is an element. Thus $\neg x \in A$. Thus

$$\neg x \in \emptyset \Rightarrow \neg x \in A \iff x \in A \Rightarrow x \in \emptyset$$

thus we follow that

$$x \in \emptyset \Leftrightarrow x \in A$$

thus by extensionality axiom we follow that

$$\emptyset = A$$

which gives us nice follow-up that

$$\emptyset = \{\}$$

1.5.10

Let $\phi(x, y)$ be the formula $\forall z(z \in y \Leftrightarrow z = x)$ which asserts that $y = \{x\}$. For all x the set $\{x\}$ exists. So $\forall x \exists! y \phi(x, y)$. Let A be a set. Show that the collection $\{\{x\} : x \in A\}$ is a set.

We know that A is a set and therefore $\mathcal{P}(A)$ is also a set. Thus by subset axiom there exists a set

$$\exists S(x \in S \Leftrightarrow x \in \mathcal{P}(A) \wedge \exists(y \in A)(\phi(x, y)))$$

which is precisely our collection.

Chapter 2

Basic Set-Building Axioms and Operations

2.1 The First Six Axioms

Prove the following theorems, where A, B, C, D are sets.

2.1.1

$$A \subseteq B \rightarrow (A \subseteq A \cup B \wedge A \cap B \subseteq A)$$

$$\forall x(x \in A \rightarrow x \in B) \rightarrow ((\forall x(x \in A \Rightarrow x \in A \vee x \in B)) \wedge (\forall (x \in A \wedge x \in B \Rightarrow x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \rightarrow x \in B) \rightarrow ((\forall x(\neg x \in A \vee x \in A \vee x \in B)) \wedge (\forall (\neg (x \in A \wedge x \in B) \vee x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \rightarrow x \in B) \rightarrow ((\forall x(\text{true} \vee x \in B)) \wedge (\forall (\neg x \in A \vee \neg x \in B \vee x \in A))) \Leftrightarrow$$

$$\Leftrightarrow \forall x(x \in A \rightarrow x \in B) \rightarrow (\text{true} \wedge (\forall (true \vee \neg x \in B))) \Leftrightarrow$$

$$\Leftrightarrow \neg \forall x(x \in A \rightarrow x \in B) \vee (\text{true} \wedge \text{true}) \Leftrightarrow$$

$$\Leftrightarrow \neg \forall x(x \in A \rightarrow x \in B) \vee \text{true} \Leftrightarrow$$

$$\text{true}$$

2.1.2

$$A \subseteq B \land B \subseteq C \to A \subseteq C$$

$$(\forall x(x \in A \Rightarrow x \in B)) \land (\forall x(x \in B \Rightarrow x \in C)) \to \forall x(x \in A \Rightarrow x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x(\neg x \in A \lor x \in B)) \land (\forall x(\neg x \in B \lor x \in C)) \to \forall x(\neg x \in A \lor x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \lor x \in B) \land (\neg x \in B \lor x \in C))) \to \forall x(\neg x \in A \lor x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \land (\neg x \in B \lor x \in C)) \lor (x \in B \land (\neg x \in B \lor x \in C)))) \to \forall x(\neg x \in A \lor x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x(\neg x \in A \land (\neg x \in B \lor x \in C)) \lor ((x \in B \land \neg x \in B) \lor (x \in B \land x \in C)))) \to \forall x(\neg x \in A \lor x \in C) \Leftrightarrow$$

$$\Leftrightarrow (\forall x((\neg x \in A \land \neg x \in B) \lor (\neg x \in A \land x \in C) \lor (x \in B \land x \in C)) \to \forall x(\neg x \in A \lor x \in C) \Leftrightarrow ...$$

So this thing is tedious as hell and should be left to computers.

Suppose that $x \in A$. Then we follow by $A \subseteq B$ that $x \in B$. Thus by $B \subseteq C$ we follow that$x \in C$. Therefore $x \in A \to x \in C$. Therefore $A \subseteq C$, as desired.

2.1.3

$$B \subseteq C \Rightarrow A \setminus C \subseteq A \setminus B$$

Suppose that $x \in A \setminus C$. Then we follow that $x \in A$ and $x \notin C$. Therefore $x \in A$ and $x \notin B$ since $B \subseteq C$. Thus $x \in A \setminus B$. Therefore we follow that $B \subseteq C$ implies that $A \setminus C \subseteq A \setminus B$, as desired.

2.1.4

$$C \subseteq A \land C \subseteq B \iff C \subseteq A \cap B$$

Suppose that $x \in C$. Then we follow that $x \in A$ and $x \in B$. Thus $x \in A \cap B$. Therefore $C \subseteq A \cap B$. Thus we follow that $C \subseteq A \land C \subseteq B \Rightarrow C \subseteq A \cap B$

Suppose that $x \in C$. Then we follow that $x \in A \cap B$. Thus $x \in A$ and $x \in B$. Therefore $C \subseteq A \land C \subseteq B$. Therefore $C \subseteq A \cap B \Rightarrow C \subseteq A \land C \subseteq B$ thus we follow that

$$C \subseteq A \land C \subseteq B \iff C \subseteq A \cap B$$

as desired.

2.1.5

There exists an x such that $x \notin A$

Suppose that there does not exist x such that $x \notin A$. Then we follow that every set is a member of A, which is impossible.

2.1.6

$$A \cap B = B \cap A$$

$$x \in A \cap B \iff x \in A \wedge x \in B \iff x \in B \wedge x \in A \iff x \in B \cap A$$

2.1.7

$$A \cup B = B \cup A$$

$$x \in A \cup B \iff x \in A \vee x \in B \iff x \in B \vee x \in A \iff x \in B \cup A$$

2.1.8

$$A \cap (B \cup C) = (A \cup C) \cap (A \cup B)$$

$$x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in (B \cup C) \Leftrightarrow x \in A \wedge (x \in B \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in A \vee x \in C) \Leftrightarrow (x \in A \cup B) \vee (x \in A \cup C) \Leftrightarrow x \in ((A \cup B) \cap (A \cup C))$$

2.1.31

$$A \subseteq \mathcal{P}(\cup(A))$$

Let $x \in A$. Then we follow that $x \subseteq \cup(A)$. Thus $x \in \mathcal{P}(A)$. Thus $A \subseteq \mathcal{P}(\cup(A))$.

2.1.32

Let $C \in F$. Then $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\cup F))$

Suppose that $C \in F$. Then we follow that $C \subseteq \cup F$. Therefore $C \in \mathcal{P}(\cup F)$. Thus $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\cup F))$.

the rest of the exercises for this section are more of the same.

2.2 Operations on Sets

Prove the following theorems

2.2.1

Let A be a set and $F \neq \emptyset$. Then

$$A \setminus \cap F = \cup\{A \setminus C : C \in F\}$$

$$x \in A \setminus \cap F \Leftrightarrow x \in A \wedge x \notin \cap F \Leftrightarrow x \in A \wedge \neg x \in \cap F \Leftrightarrow x \in A \wedge \neg(\forall(C \in F)(x \in C)) \Leftrightarrow$$

$$\Leftrightarrow x \in A \wedge \exists(C \in F)(x \notin C) \Leftrightarrow \exists(C \in F)(x \notin C \wedge x \in A) \Leftrightarrow \exists(C \in F)(x \in A \setminus C) \Leftrightarrow x \in \cup\{A \setminus C : C \in F\}$$

which seems to hold.