My real analysis exercises

Evgeny Markin

2022

Contents

Chapter 1

Differentiation

1.1 Standard differentiation

1.1.1

$$g(x) = 4x + 7$$
$$g'(x) = 4$$

1.1.2

$$g(x) = 5x + 4x^2$$
$$g'(x) = 5 + 8x$$

1.1.3

$$f(x) = x^{75} - x + 3$$
$$f'(x) = 75x^{74} - 1$$

1.1.4

$$g(x) = 7/4x^2 - 3x + 12$$
$$g'(x) = 7/2x - 3$$

1.1.5

$$f(t) = -2e^t$$
$$f'(t) = -2te^t$$

1.1.6

$$F(t) = t^2 + e^3$$
$$F(t) = 2t + 3e^3$$

1.1.7

$$y = (4x^2 + 3)(2x + 5)$$
$$y' = 8x(2x + 5) + 2(4x^2 + 3)$$

1.1.8

$$y = x^3 e^x$$
$$y = 3x^2 e^x + x^3 e^x$$

1.1.9

$$f(x) = (3x^2 - 5x)e^x$$
$$f'(x) = (6x - 5)e^x + (3x^2 - 5x)e^x$$

1.1.10

$$f(x) = x \cos x \sin x$$
$$f'(x) = \cos x \sin x - x \sin^2 x + x \cos^2 x$$

1.2 Inverse trigonometric and logarythmic integration

1.2.1

$$f(x) = \ln(3 + x^2)$$
$$f'(x) = \frac{2x}{3 + x^2}$$

1.2.2

$$f(x) = \ln(x^2 + 3x + 5)$$
$$f'(x) = \frac{2x + 3}{x^2 + 3x + 5}$$

1.2.3

$$f(x) = x \ln x - x$$
$$f'(x) = \ln x + 1 - 1 = \ln x$$

1.2.4

$$f(x) = \sin(\ln x)$$
$$f'(x) = \cos(\ln x)\frac{1}{x}$$

1.2.5

$$f(x) = \ln(\sin^2 x)$$
$$f'(x) = \frac{2 \sin x \cos x}{\sin^2 x} = \frac{2 \cos x}{\sin x}$$

1.2.6

$$f(x) = \ln \frac{1}{x}$$
$$f'(x) = x(-x^{-2}) = -\frac{1}{x}$$

1.2.7

$$f(x) = \frac{1}{\ln x}$$
$$f'(x) = -\frac{1}{(\ln x)^2 x}$$

1.2.8

$$g(x) = \ln(xe^{-2x})$$
$$g'(x) = \frac{e^{-2x} - 2xe^{-2x}}{xe^{-2x}}$$

1.2.9

$$g(t) = \sqrt{1 + \ln t}$$
$$g'(t) = \frac{1}{2t\sqrt{1 + \ln t}}$$

1.2.10

$$f(t) = (\ln t)^2 \sin t$$
$$f'(t) = 2\ln t \frac{1}{t}\sin t + \cos t(\ln t)^2$$

1.2.11

$$f(t) = \ln(\sqrt{t^2+1})$$
$$f'(t) = \frac{2t}{2(t^2+1)} = \frac{t}{t^2+1}$$

1.2.12

$$f(x) = \log_8{(x^2+3x)}$$
$$f'(x) = \frac{2x+3}{\ln(8)(x^2+3x)}$$

1.2.13

$$f(x) = \sin^{-1}(5x)$$
$$f'(x) = \frac{5}{\sqrt{1-(5x)^2}}$$

1.2.14

$$f(x) = \sec^{-1}(e^x)$$
$$f'(x) = \frac{2e^{2x}}{e^x\sqrt{e^{2x}-1}}$$

1.2.15

$$f(x) = \tan^{-1}(\sqrt{x-1})$$
$$f'(x) = \frac{1}{2\sqrt{x-1}(1+|x-1|)}$$

1.2.16

$$f(x) = \tan^{-1}(x^2)$$
$$f'(x) = \frac{2x}{1+x^4}$$

1.2.17

$$f(x) = (\tan^{-1} x)^2$$
$$f'(x) = \frac{2(\tan^{-1} x)}{1 + x^2}$$

1.2.18

$$f(x) = \arccos \sqrt{x}$$
$$f'(x) = \frac{1}{2\sqrt{x}\sqrt{1 - |x|}}$$

1.2.19

$$f(x) = (\arcsin x) \ln x$$
$$f'(x) = \frac{\ln x}{\sqrt{1 - x^2}} + \frac{\arcsin x}{x}$$

1.2.20

$$f(x) = \ln(\arctan(x^4))$$
$$f'(x) = \frac{4x^3}{(1 + x^8) \arctan x^4}$$

1.2.21

$$f(x) = e^{\arcsin(x^2)}$$
$$f'(x) = e^{\arcsin(x^2)} \frac{2x}{\sqrt{1 - x^4}}$$

Chapter 2

Basic integration

2.1 Basic integrals

2.1.1

$$\int_1^3 (x^2+2x-4)dx = [\frac{x^3}{3}+x^2-4x]_1^3 = (9+9-12)-(1/3+1-4) = 6+\frac{8}{3} = \frac{26}{3}$$

2.1.2

The error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

(a) Show that

$$\int_a^b e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}[\text{erf}(b)-\text{erf}(a)]$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\frac{\sqrt{\pi}}{2}\,\text{erf}(x) = \int_0^x e^{-t^2}dt$$

$$\frac{\sqrt{\pi}}{2}\,\text{erf}(a) = \int_0^a e^{-t^2}dt$$

$$\frac{\sqrt{\pi}}{2}\,\text{erf}(b) - \frac{\sqrt{\pi}}{2}\,\text{erf}(a) = \int_0^b e^{-t^2}dt - \int_0^a e^{-t^2}dt$$

$$\frac{\sqrt{\pi}}{2}\,\text{erf}(b) - \frac{\sqrt{\pi}}{2}\,\text{erf}(a) = \int_0^b e^{-t^2}dt + \int_a^0 e^{-t^2}dt$$

$$\frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \int_a^b e^{-t^2} dt$$
$$\frac{1}{2}\sqrt{\pi}[\operatorname{erf}(b) - \operatorname{erf}(a)] = \int_a^b e^{-t^2} dt$$

(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation

$$y' = 2xy + 2/\sqrt{\pi}$$

$$y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\frac{2}{\pi}e^{-x^2} = 2xe^{x^2}\operatorname{erf}(x) + \frac{2}{\pi}$$

$$2xy + 2/\sqrt{\pi} = 2xe^{x^2}\operatorname{erf}(x) + 2/\sqrt{\pi}$$

as desired.

2.2 Substitution

2.2.1

$$\int \cos 2x dx$$
$$u = 2x$$
$$\int \cos 2x dx = \int \frac{1}{2} 2\cos 2x dx = \int \frac{1}{2}\cos u du = \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u = \frac{1}{2}\sin 2x$$

2.2.2

$$\int xe^{-x^2}dx = \int (-\frac{1}{2}) - 2xe^{-x^2}dx = -\frac{1}{2}\int e^u dx = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$$

2.2.3

$$\int x^2\sqrt{x^3+1}dx = \int \frac{1}{3}3x^2\sqrt{x^3+1}dx = \frac{1}{3}\int \sqrt{u}dx = \frac{1}{3}u^{3/2}\frac{2}{3} = \frac{2}{9}(x^3+1)^{3/2}$$

2.2.4

$$\int \sin^2 x \cos x = \int u^2 = \frac{u^3}{3} = \frac{\sin x}{3}$$

2.3 Integration by parts

2.3.1

$$\int xe^{2x} = x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{2}e^{2x}$$