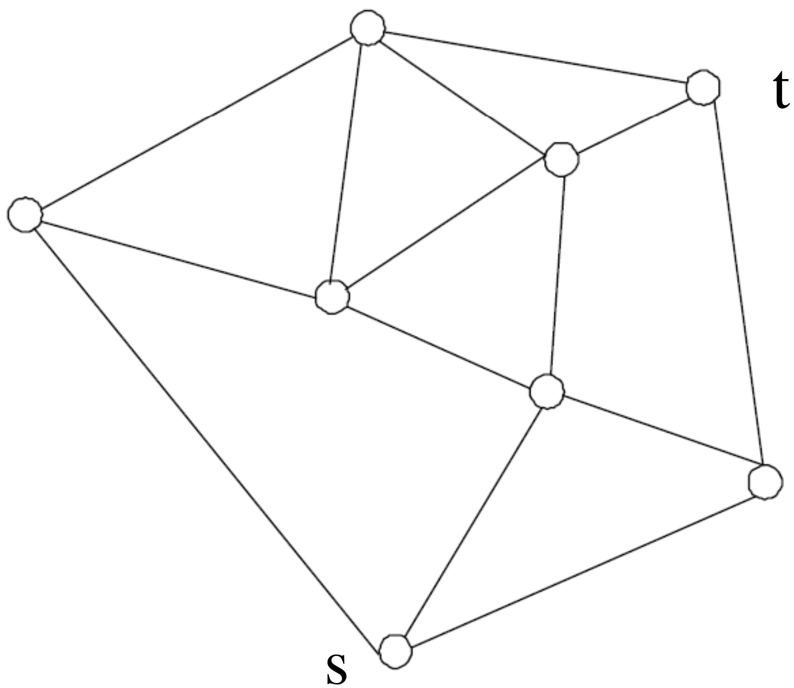
Given a graph, find an embedding s.t. greedy routing works

Greedy embedding of a graph

Greedy embedding

- Given a graph G, find an embedding of the vertices in R^d, s.t. for each pair of nodes s, t, there is a neighbor of s closer to t than s itself.



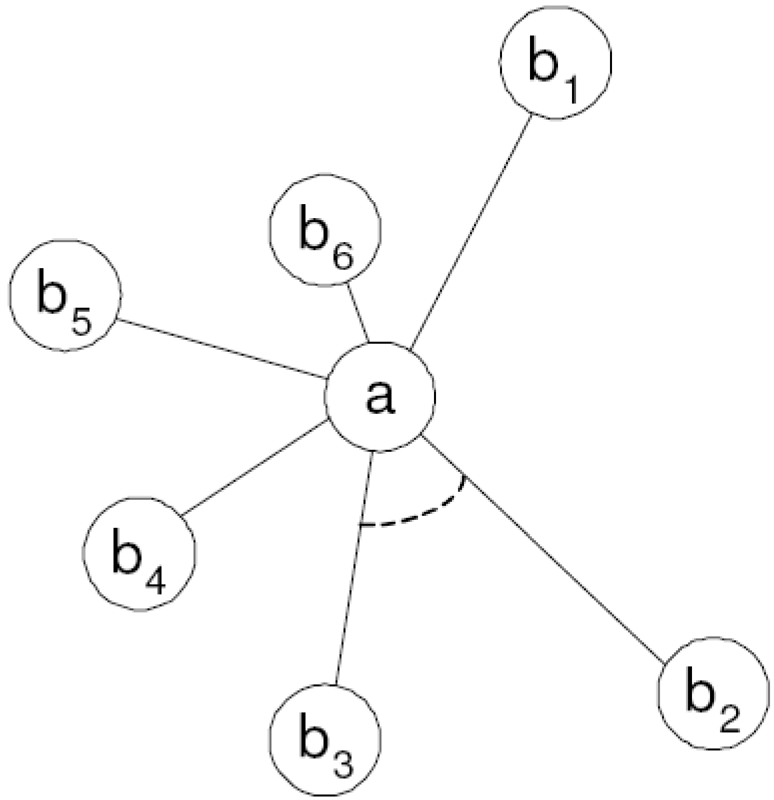
Questions to ask

- We want to find a virtual coordinates such that greedy routing always works.

- Does there exist such a greedy embedding in R^2?
- in R^3?
- in Euclidean metric? Hyperbolic space?
- If it exists, how to compute?

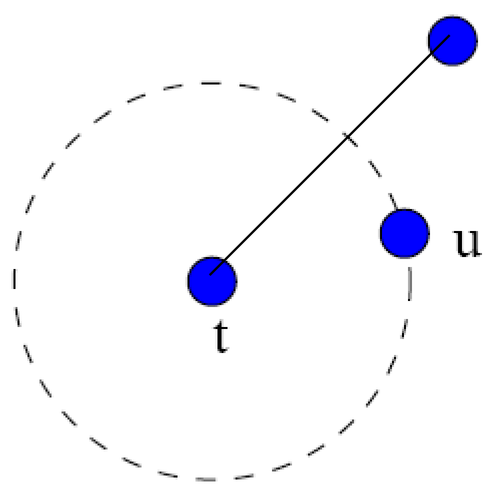
Greedy embedding does not always exist

- $K_{1,6}$ does not have a greedy embedding in R^2



A lemma

- Lemma: each node t must have an edge to its closest (in terms of Euclidean distance) node u.
- Otherwise, u has no neighbor that is closer to t than itself.

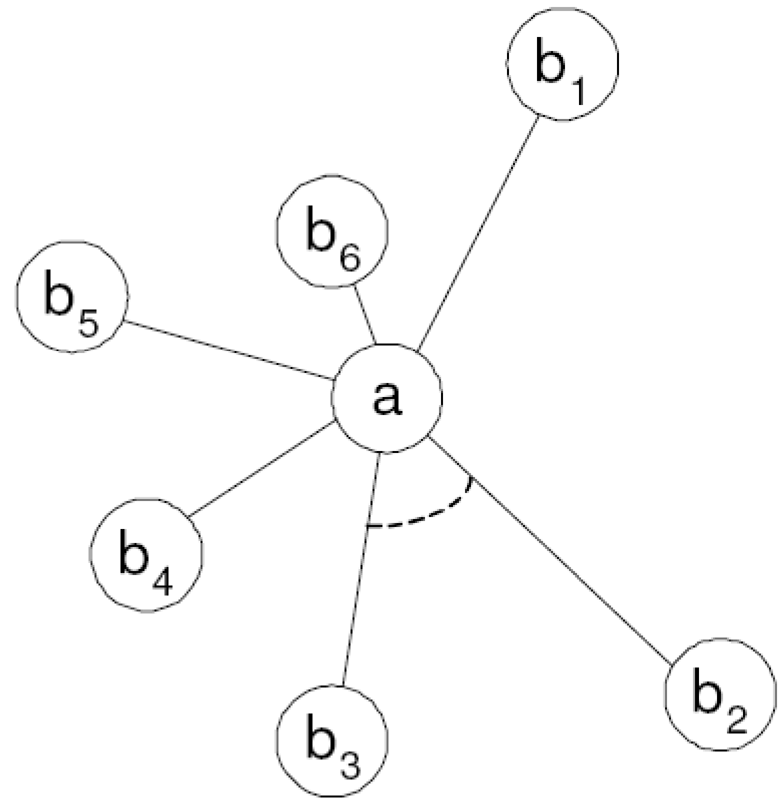


Proof

- $K_{1,6}$ does not have a greedy embedding in R^2

Proof:

1. One of the angles is less than $\pi/3$.
2. One of ab_2 and ab_3, say, ab_2, is longer than b_2b_3.
3. Then b_2 does not have edge with its closest point b_3.



A conjecture

- Corollary: $K_{k,\ 5k+1}$ does not have a greedy embedding in R^2.
- Conjecture: Any planar 3-connected graph has a greedy embedding R^2.

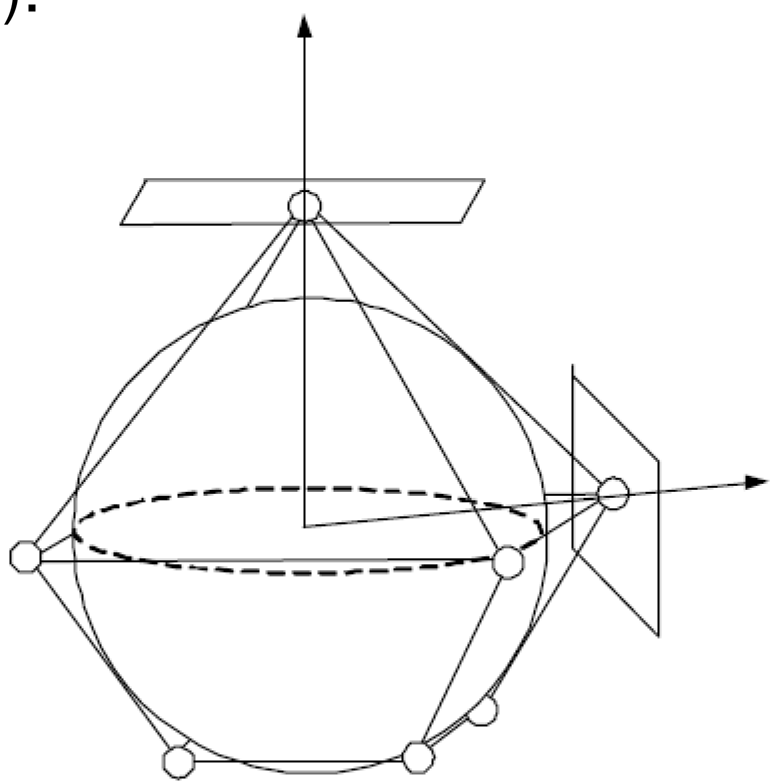
- Hint: this is tight.
- $K_{2,11}$ is planar but not 3-connected.
- $K_{3.16}$ is 3-connected but not planar. (it has $K_{3.3}$ minor).
- Planar 3-connected graph has a greedy embedding in R^3

Polyhedral routing

Theorem: Any 3-connected planar graph has a greedy embedding **e** in R^3, where the distance function is defined as $d(u, v) = -\mathbf{e}(u)\cdot\mathbf{e}(v)$.

Proof:

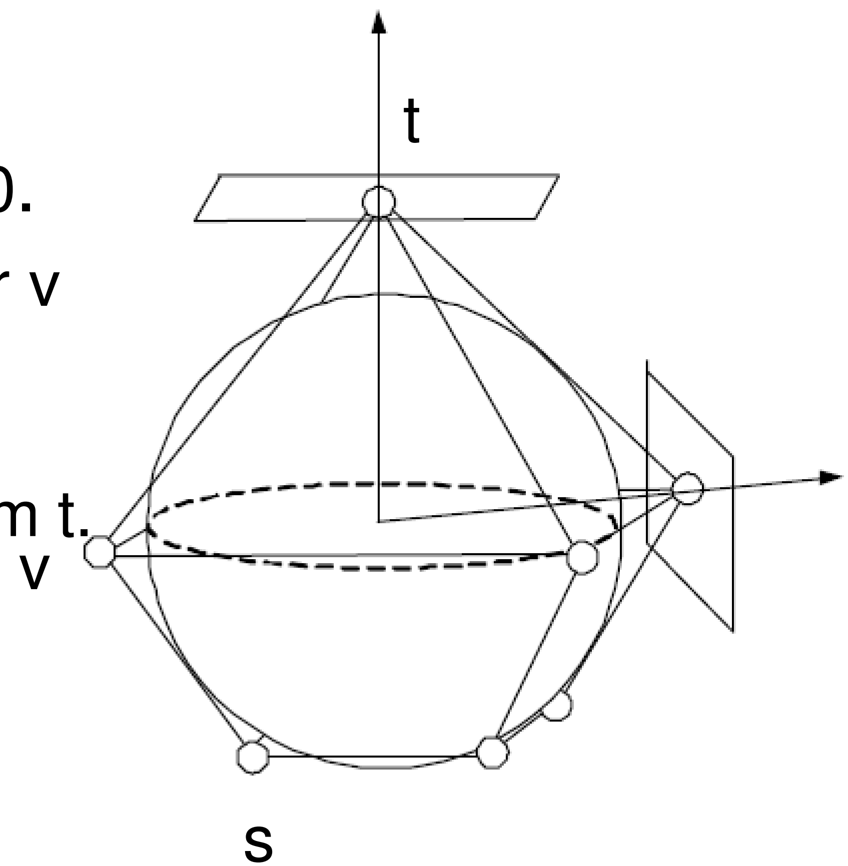
1. Any 3-connected planar graph is the edge graph of a 3D convex polytope, with edges tangent to a sphere. [Steinitz 1922].
2. Each vertex has a supporting hyperplane with the normal being the 3D coordinate of the vertex.



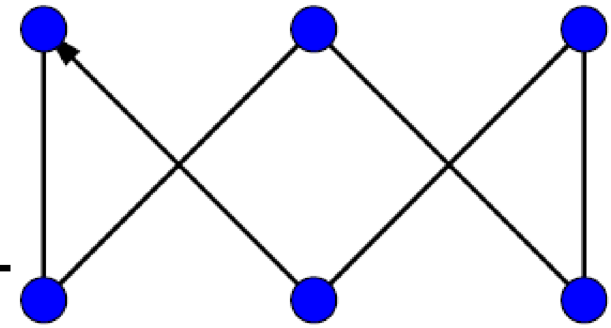
Polyhedral routing

Proof: For any s, t, there is a neighbor v of s, d(v,t)<d(s,t).

1. d(s,t)-d(v,t)=[**e**(v)-**e**(s)]·**e**(t)>0.
2. Now suppose such neighbor v does not exist, then s is a reflex vertex, with all the neighbors pointing away from t.
3. This contradicts with the convexity of the polytope.



Discussions



- Papadimitriou's conjecture: Any planar 3-connected graph has a greedy embedding R^2. ← has been proved!
- The theorem only gives a sufficient condition, not necessary.
  - $K_{3.3}$ has a greedy embedding.
  - A graph with a Hamiltonian cycle has a greedy embedding on a line.
- Given a graph, can we tell whether it has a greedy embedding in R^2? Is this problem hard? (Recall that many such embedding problems are hard…)
- More understanding of greedy embedding in R^2, R^3…

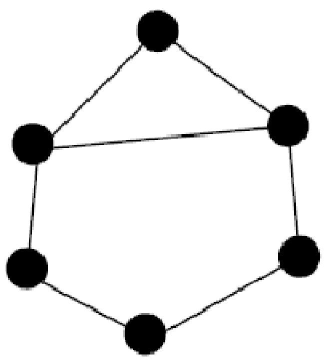
Follow-up work

- Dhandapani proved that any triangulation admits a greedy embedding (SODA'08).
- Leighton and Moitra proved the conjecture (FOCS'08).
- Independently, Angelini et al. also proved it (Graph Drawing'08).
- Goodrich and D. Strash improved the coordinates to be of size O(log n) (under submission).

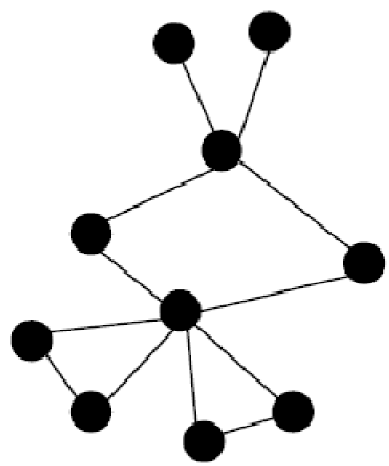
- We briefly introduce the main idea.

Leighton and Moitra

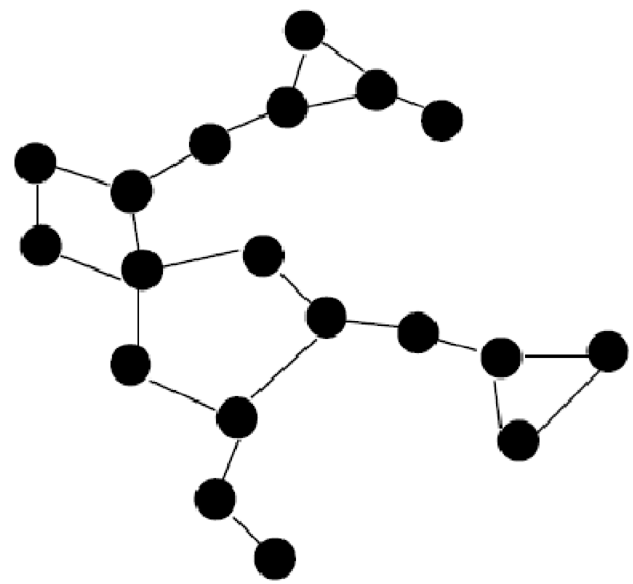
- All 3-connected planar graph contain a spanning Christmas Cactus graph.
- All Christmas Cactus graphs admit a greedy embedding in the plane.



not cactus



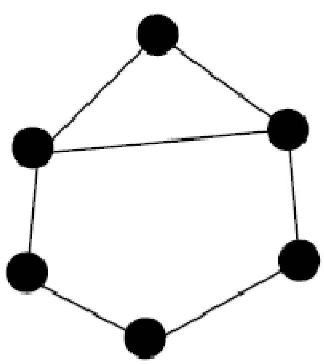
cactus, not Christams cactus

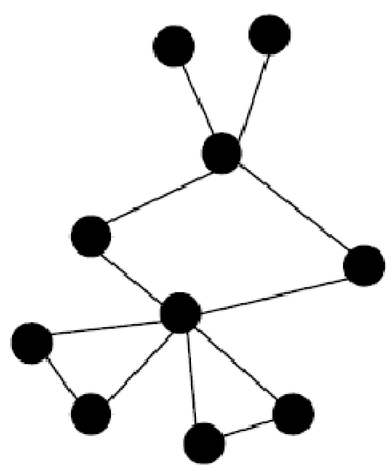


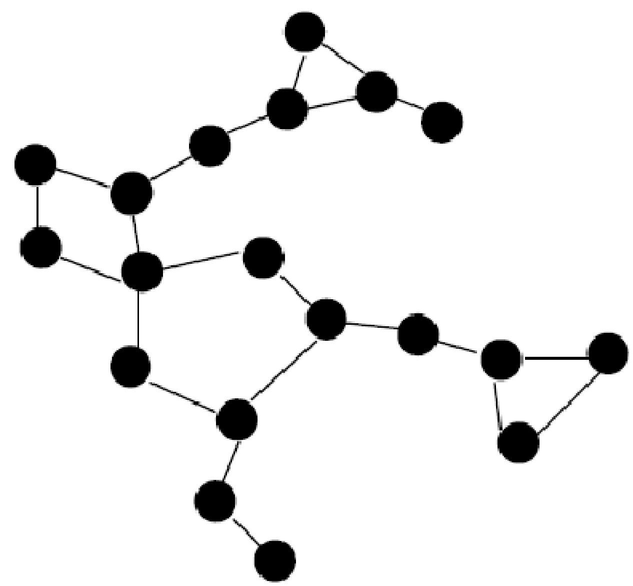
Christmas cactus

Leighton and Moitra

- A cactus graph is connected, each edge is in at most one simple cycle.
- A Christmas Cactus graph is a cactus graph for which the removal of any node disconnects into at most 2 pieces.


not cactus


cactus, not Christams cactus


Christmas cactus

A Christmas Cactus

Example

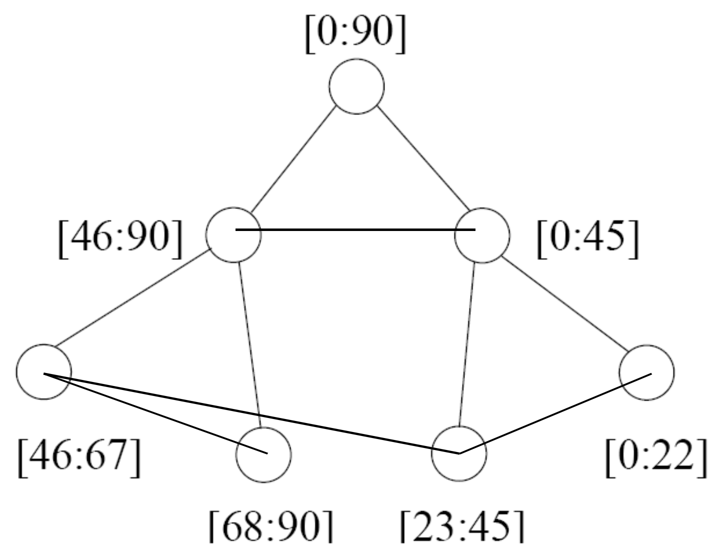
Connection to graph labeling

- Given a graph, find a labeling of the nodes such that one can compute the (approximate) shortest path distance between any two vertices from their labels only.
- Tradeoff between approximation ratio and the label size.
- For shortest path distance, the maximum label size is $\Theta(n)$ for general graph, $O(n^{1/2})$ ($\Omega(n^{1/3})$) for planar graphs, and $\Theta(\log^2 n)$ for trees.
- General graph: $\exists$ a scheme with label size $O(kn^{1/k})$ and approximation ratio 2k-1.
- Google "distance labeling" for the literature.

Approach II:
Embed a spanning tree in polar coordinate system

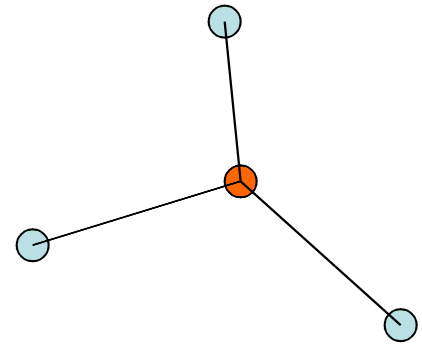
Embed a tree in polar coordinate system

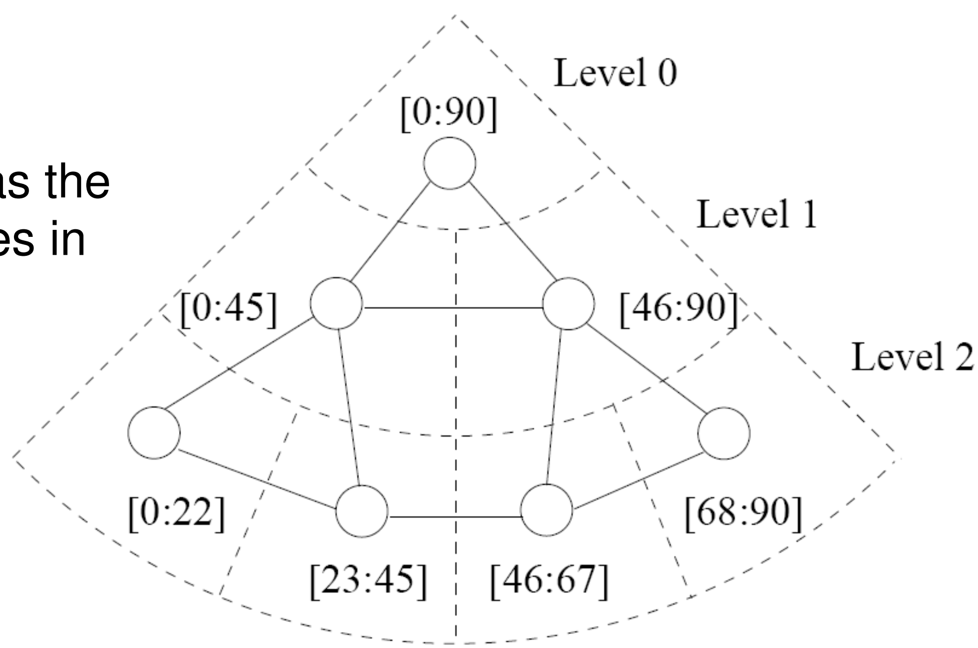
- Start from any node as root, flood to find the shortest path tree.
- Assign polar ranges to each node in the tree.
  - The range of a node is divided among its children.
  - The size of the range is proportional to the size of its subtree.
- Order the subtrees that align with the sensor connectivity.



Embed a tree in polar coordinate system

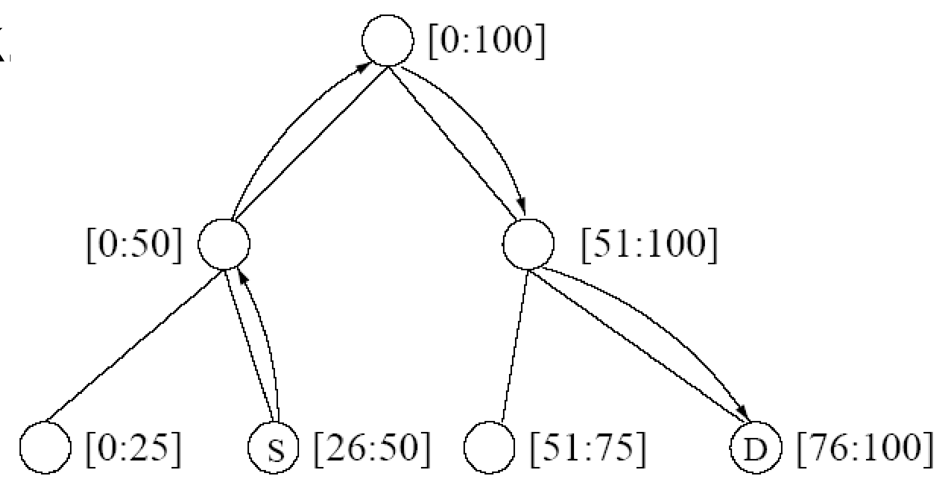
- Order the subtrees that align with the sensor connectivity.
  - Three reference nodes flood the network. Each node knows the hop count to each reference.
  - Each node embed itself with respect to the references. (trilateration with hop counts)
  - A node's position is defined as the center of mass of all the nodes in its subtree.
  - This will provide an angular ordering of all the children.





Routing on a tree

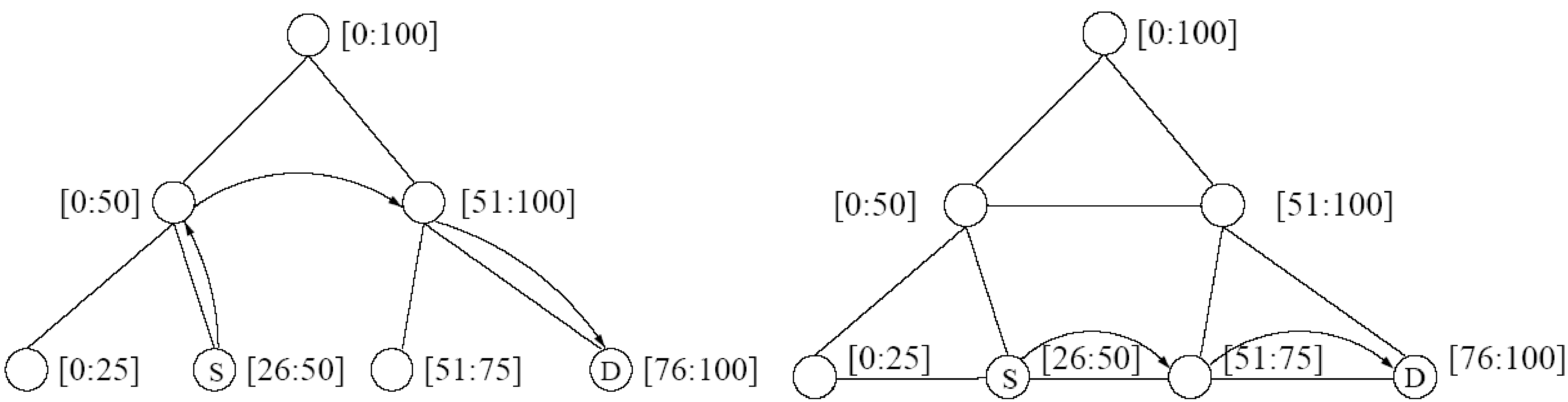
- Route to the common ancestor of the source and destination.
  - Check whether the destination range is included in the range of the current node.
  - If not, go to the parent.
  - Otherwise go to the corresponding child.
- Root is the bottleneck.
- Path may be long.



Routing on a tree

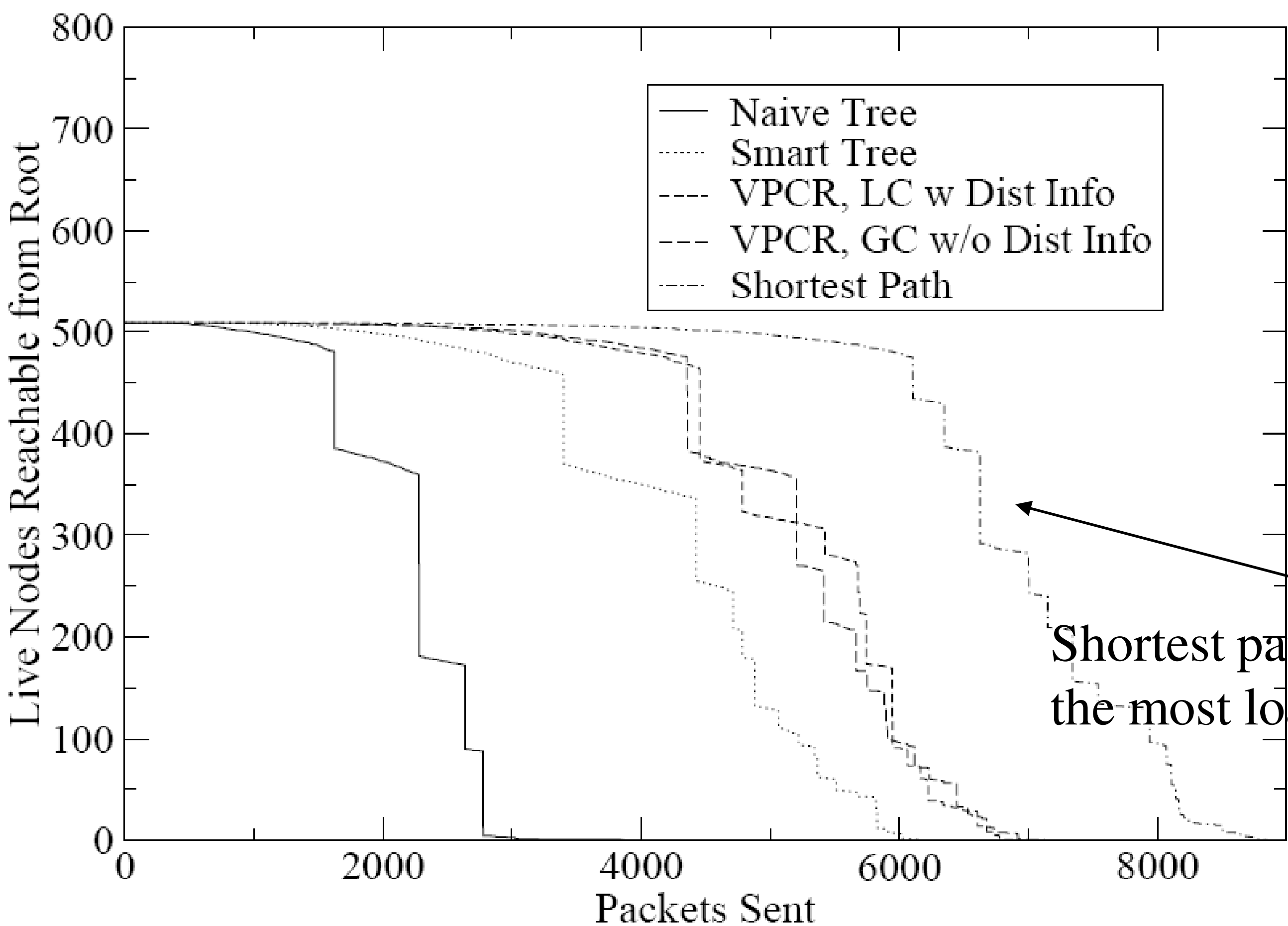
- Be a little smarter: store a local routing table that keeps the ranges of up to k-hop neighbors. ➔ find shortcuts.
- Virtual Polar Coordinate Routing: check the neighborhood, find the node that is closer to the destination. 🡐 greedy forwarding in polar coordinates.

If the upper/lower bound is closer to the destination.



Load balancing

- Root is still the bottleneck even for smart routing.



Shortest path routing, still not the most load balanced routing

Routing on spanning trees – in theory and in practice

- For any graph G there is a spanning tree T, s.t. the average stretch of the shortest paths on T, compared with G, is $O((\log n \log\log n)^2)$.