CS3000: Algorithms & Data
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Recitation 1:
- Proof by Induction

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Proof by Induction

- We will prove our claim using **induction**

- **Induction:**
  - Used to prove a claim *H* is true for every natural number *i* starting at a first value (usually *0* or *1*) – *H(i)* is true ∀ *i*
  - How:
    - 1. Base case – prove directly for *H(1)* (or whatever the base case(s) is/are)
    - 2. Inductive step – For general *k*, show that if *H(k-1)* is true, then *H(k)* is true. The assumption that *H(k-1)* is true is the **inductive hypothesis (IH).**

BASE

$H(1) \xrightarrow{\text{ind.}} H(2) \xrightarrow{\text{ind.}} H(3) \rightarrow \dots \rightarrow H(100)$

  - Why:
    - Suppose we want to prove *H(100)*. First, we can use the base case to show *H(1)* holds. Then, because *H(1)* is true, *H(2)* is true via inductive step, and then *H(3)* is true, and so on, all the way to *H(100)* (or whatever value!).

Problem 1

given

Prove by induction that, for every $n \geq 4$, $\boxed{n^2 \leq 2^n}$.

$H(k): k^2 \leq 2^k$ $(k \geq 4)$

Base: $H(4) \to 4^2 \leq 2^4$

$16 \leq 16$ ✓

Ind. $H(k-1) \to H(k)$

assume true (IH)

alt. $H(k) \to H(k+1)$

IH: $\boxed{(k-1)^2 \leq 2^{k-1}}$

WTS: $k^2 \leq 2^k$

H(k)

$2(k-1)^2 \leq 2^{k-1} \cdot 2 = 2^k$

Suffices to show: $k^2 \leq 2(k-1)^2 \leq 2^k$

$k^2 \leq 2^k$ $\square$

(work on next slide)

$$k^2 \le 2(k-1)^2 = 2\underbrace{(k-1)(k-1)} = 2(k^2-2k+1)$$

(use algebra to solve)

$$k^2 \le 2(k^2-2k+1)$$

$$0 \le 2(k^2-2k+1) - k^2 = 2k^2-4k+2-k^2$$

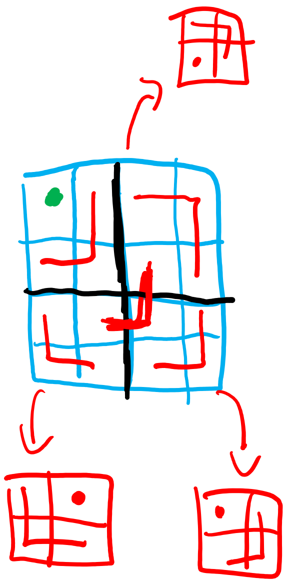
$$0 \le k^2-4k+2$$

$$4k \le k^2 + 2 \qquad (k \ge 4)$$

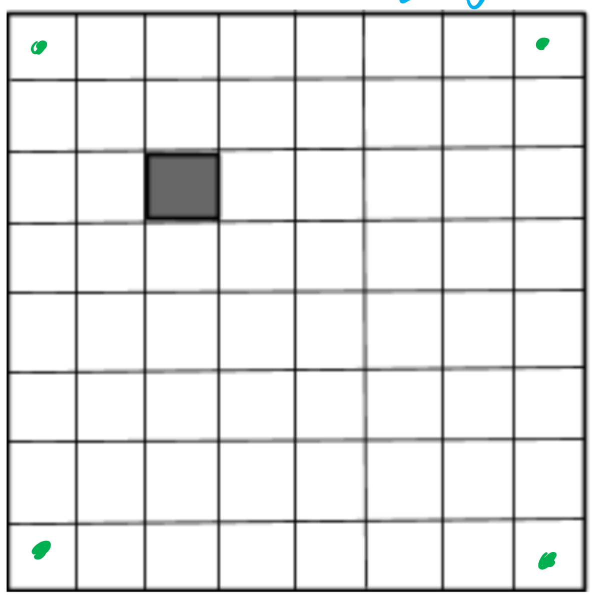
$$4 \le \underline{\underline{k}} + \underline{\underline{2/k}}$$

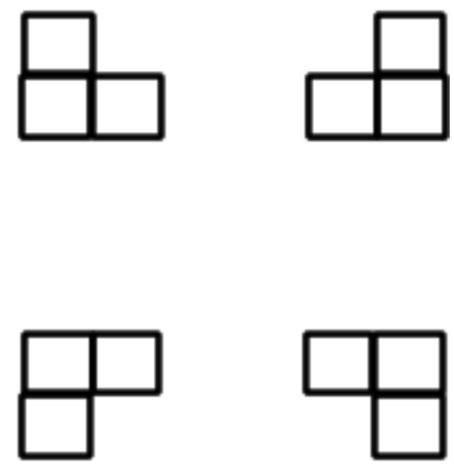
Problem 2

An $m \times m$ grid is the set $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq m\}$, where we refer to each element as a *cell*. An L-shaped tromino is a set of three adjacent cells, forming an L shape, and can be placed in any of four orientations. A $2^3 \times 2^3$ grid with shaded cell $(3, 3)$ and the four orientations of a tromino are shown in the accompanying figure.



cells

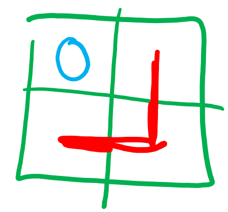


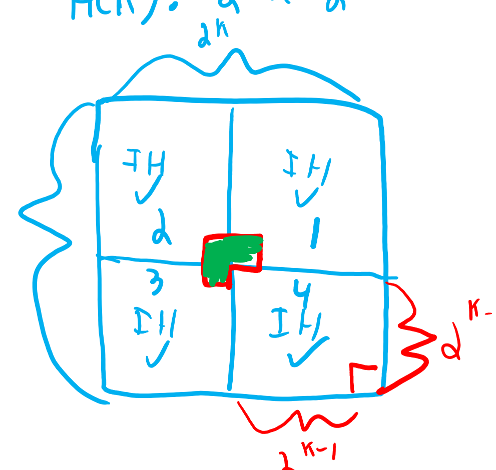


(a) Using induction, prove that for every integer $n \geq \cancel{0}\,1$, a $2^n \times 2^n$ grid can be tiled with non-overlapping L-shaped trominoes such that every grid cell, except some corner cell, is covered by a tromino.

$H(k)$: $\underbrace{2^k \times 2^k}$ grid can be covered accordingly by trominoes

Base: $\underline{H(1) \to 2^1 \times 2^1 = 2 \times 2}$





Ind. $H(k-1) \longrightarrow H(\underline{k})$

↳ $\underline{2^{k-1} \times 2^{k-1}}$ grid can be covered accordingly by trominoes

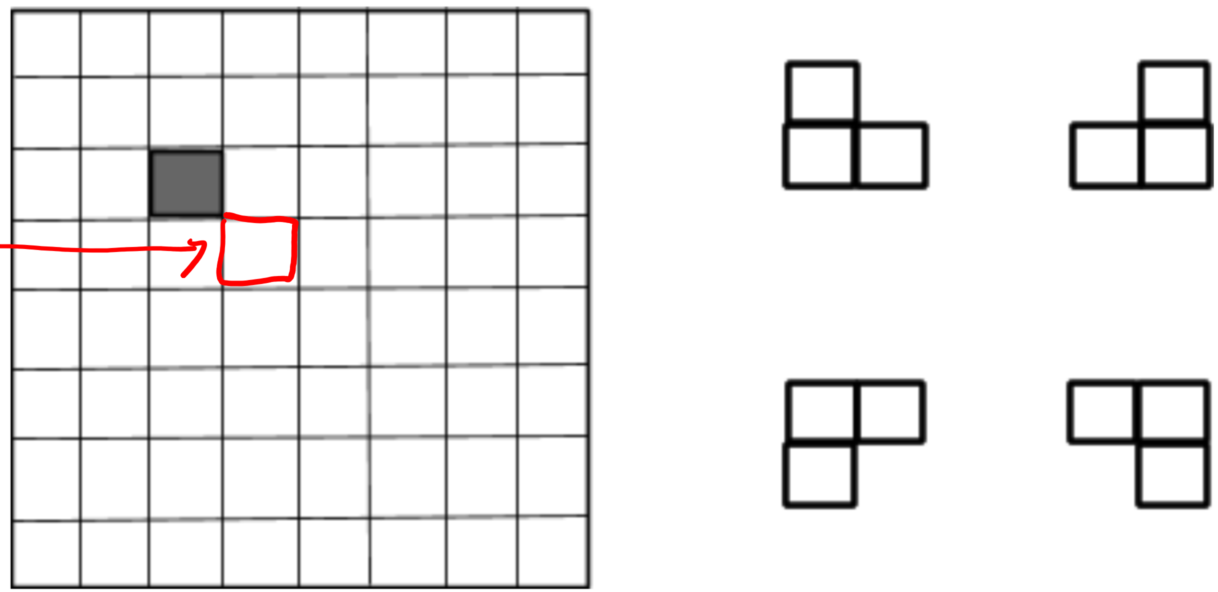
Split $2^k \times 2^k$ grid into 4 $2^{k-1} \times 2^{k-1}$ quadrants.

Leave bottom right corner uncovered, can cover quad 4 by IH.
Place tromino that covers 1 cell in quads. 1, 2, 3. Can cover remaining cells by IH.

Problem 2 ctd.

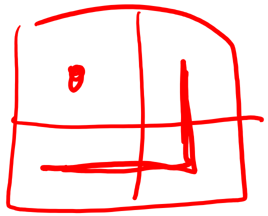
An $m \times m$ grid is the set $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq m\}$, where we refer to each element as a *cell*. An L-shaped tromino is a set of three adjacent cells, forming an L shape, and can be placed in any of four orientations. A $2^3 \times 2^3$ grid with shaded cell $(3, 3)$ and the four orientations of a tromino are shown in the accompanying figure.

middle cell



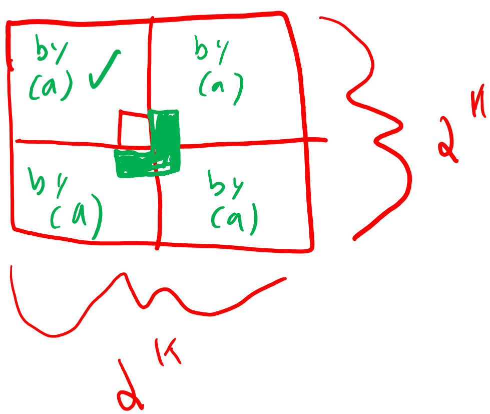
(b) Using (a), prove that for every integer $n \geq \cancel{0}\,1$, a $2^n \times 2^n$ grid can be tiled with non-overlapping L-shaped trominoes such that every grid cell, except the middle cell $(\lceil 2^{n-1} \rceil, \lceil 2^{n-1} \rceil)$, is covered by a tromino.

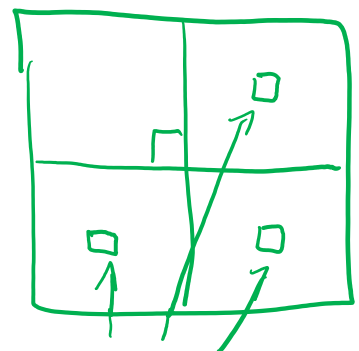
Base:



✓

Ind.: $H(k-1) \longrightarrow H(k)$

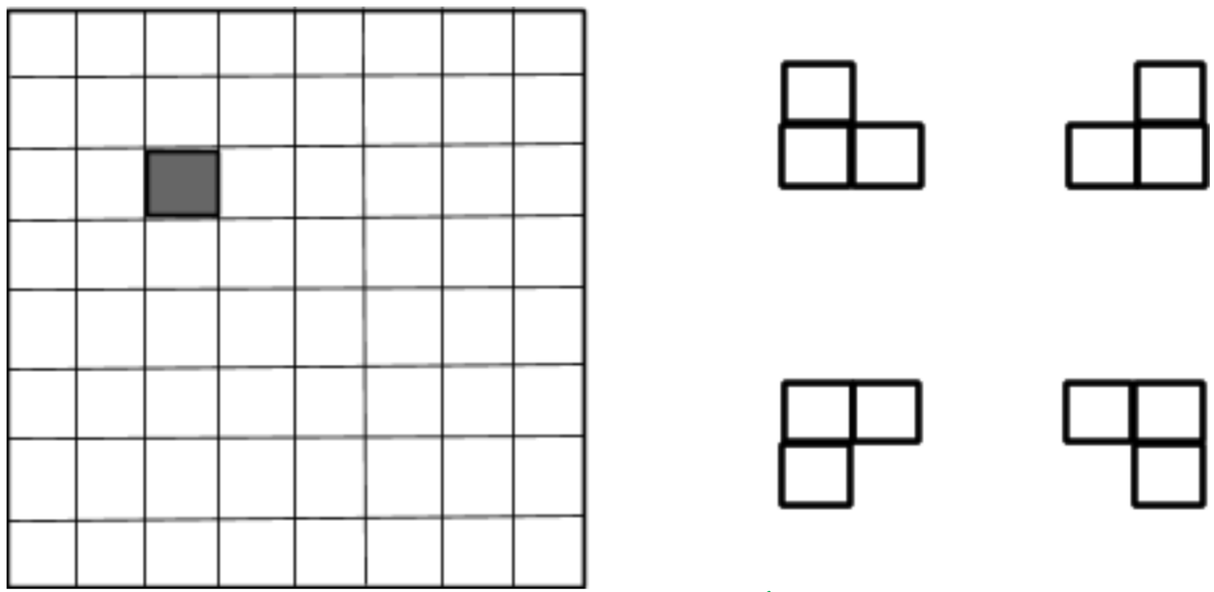




No way to cover all w/ 1 tromino!

Problem 2 ctd.

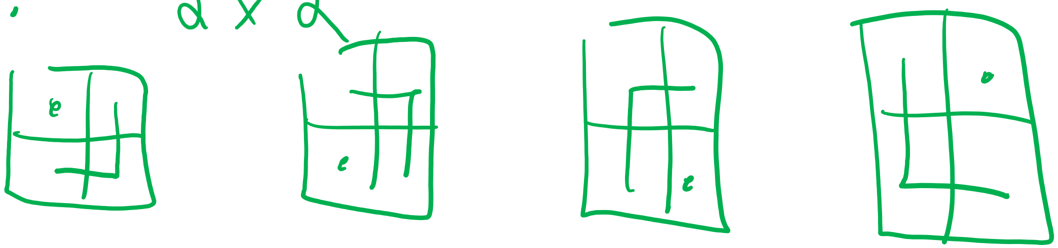
An $m \times m$ grid is the set $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq m\}$, where we refer to each element as a *cell*. An L-shaped tromino is a set of three adjacent cells, forming an L shape, and can be placed in any of four orientations. A $2^3 \times 2^3$ grid with shaded cell $(3, 3)$ and the four orientations of a tromino are shown in the accompanying figure.



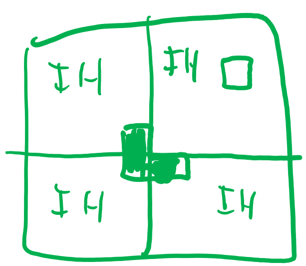
(c) Using induction, prove that for every integer $n \geq \overset{1}{\cancel{0}}$ and *any cell* (x, y) with $1 \leq x \leq 2^n, 1 \leq y \leq 2^n$, a $2^n \times 2^n$ grid can be tiled with non-overlapping L-shaped trominoes such that every grid cell, except (x, y), is covered by a tromino.

H(k): any cell in a $2^k \times 2^k$ grid can be the uncovered one!

Base: 2×2



Ind.: $H(k-1) \rightarrow H(k)$



- Place tromino that covers 1 cell in each of the 3 quads that don't contain special uncovered cell
- use IH on all 4 quads.