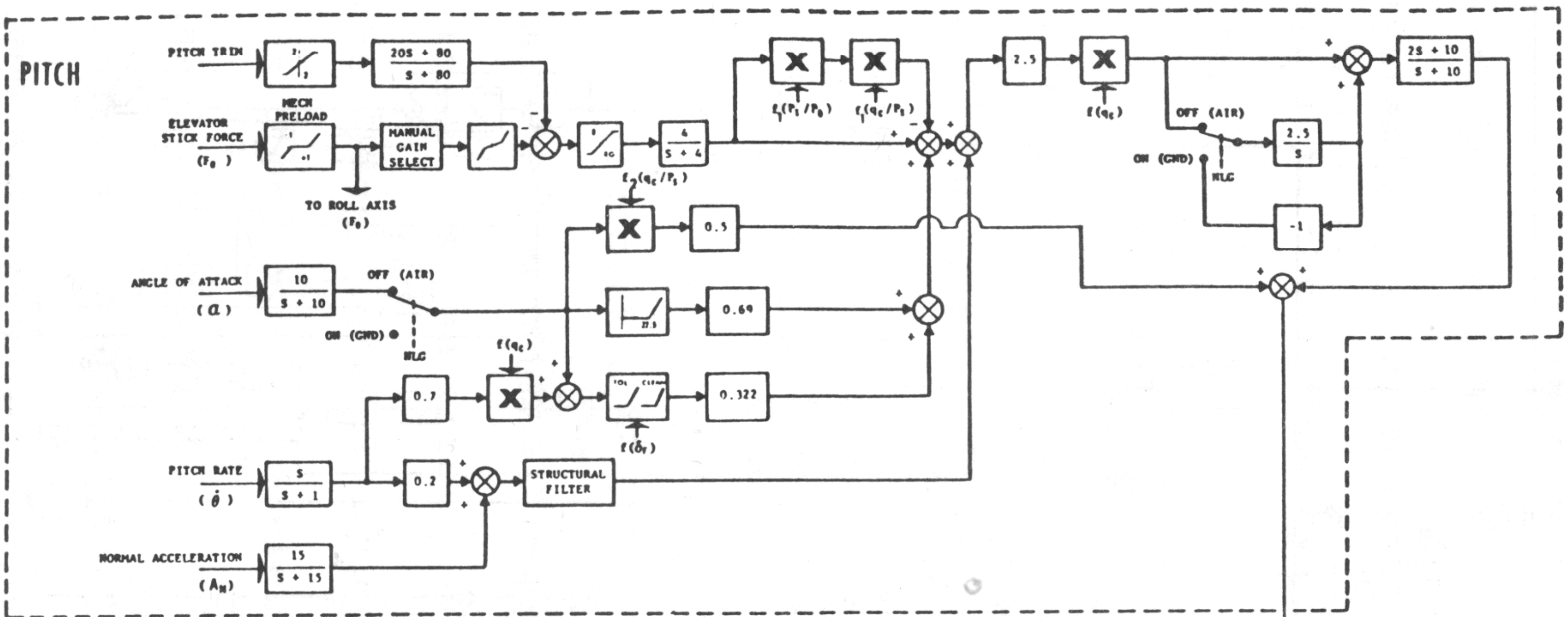
Transfer Functions & Block Diagrams



ESE 505 & MEAM 513

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Penn Engineering

What Are We Talking About ?

- Step 1 = Systems Composed of Elements
  - Input → Output = Dynamic Relationship!
  - Draw Cartoon Block Diagrams to Represent System
- Step 2 = System Modeling
  - F = m*a & T = I*alpha are ODEs!
  - L-C-R Circuits & Op-amps Described by ODEs!
- Step 3 = State Space = Generic Form For Writing ODE System Descriptions
  - Nonlinear State-Space Models
  - Linearization = Trim + Small Perturbations
- Step 4 = Laplace Transforms
  - Convert Linear ODEs into Linear Algebraic Equations!
  - Partial-Fraction Expansion to Find Solutions in Terms of Exponential Functions [exp(λt)]

What Are We Talking About (Equations)?

- Write Equations Describing System Dynamics
  - State Equations = n Coupled Nonlinear ODEs with Input

$$\dot{\underline{x}} = \underline{f}\left(\underline{x}, u\right)$$

  - Output Equation = Nonlinear Algebraic Equation

$$y = h(\underline{x}, u)$$

This Can Be Mechanics (F=ma) or Electronics (V=iR) or Other Stuff (Chapter 2 of Franklin)

- Find "Trim" & Linearized Equations Governing Dynamics of Small Perturbations from Trim

$$\underline{f}\left(\underline{x}_o, u_o\right) = \underline{0} \qquad \Delta\dot{\underline{x}} \approx A\Delta\underline{x} + B\Delta u \qquad \Delta y \approx C\Delta\underline{x} + D\Delta u$$

- Apply Laplace Transform to Linearized Equations

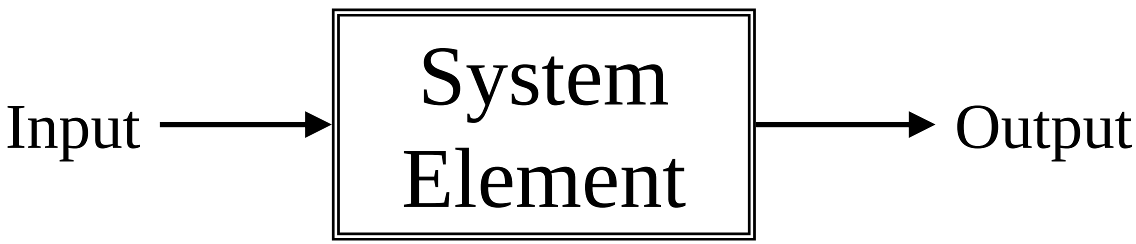
$$\Delta Y(s) = \left[C\left(sI - A\right)^{-1}B + D\right]\Delta U(s) + C\left(sI - A\right)^{-1}\Delta\underline{x}(0)$$

Our Primary Interest = “Zero-State Response”

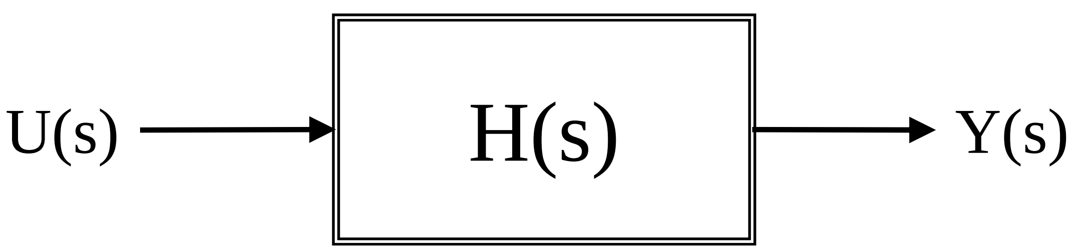
We Don’t Really Care, *At All*, About Solving ODE’s!
We Want to Understand Dynamic Input-Output Relationships

Ignore Initial Conditions ➔ $$\Delta \underline{x}(0) = \underline{0} \Rightarrow Y(s) = \left[C\left(sI - A\right)^{-1} B + D \right] U(s)$$

$$H(s) \triangleq \left[C\left(sI - A\right)^{-1} B + D \right] =$$ “Transfer Function”



Simply Multiply Input by Transfer Function to Get Output!



$$\frac{Y(s)}{U(s)} = H(s)$$

We’ll Often *Start* Our Problems with H(s) Given

- Virtually ALL Controls Textbook Problems Begin with “Consider Such-and-Such a System with the Following Transfer Function…”

- But Good To Remember That Transfer Functions Come From Stuff We Did in First Few Lectures
  - Simple Planar Translation & Rotation Equations
  - Should Also Know Simple Circuits (L-C-R)
  - Should Know Key Ideas & Equations for DC Motor

- Note: Students Trying to Apply Ideas Usually Get Stuck at the Beginning = How to Use “s” to Describe an Actual System. (Modeling is Hard Work!)

What Do Transfer Functions Look Like?

- Linear State-Space Models *Always* Result in Rational Polynomial Transfer Functions:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_o s^m + b_1 s^{m-1} + \cdots + b_m}{a_o s^n + a_1 s^{n-1} + \cdots + a_n}$$

  - Usual Case (Control Output Matrix = D = 0) → m < n
  - If Control Output Matrix = D != 0, m = n

- Numerator & Denominator Can Be Factored:

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Watch Sign Convention on z & p!

  - z_i = Zeros (May Be Real or Complex)
  - p_i = Poles (May Be Real or Complex) [System Behavior ~ exp(p_i t)]

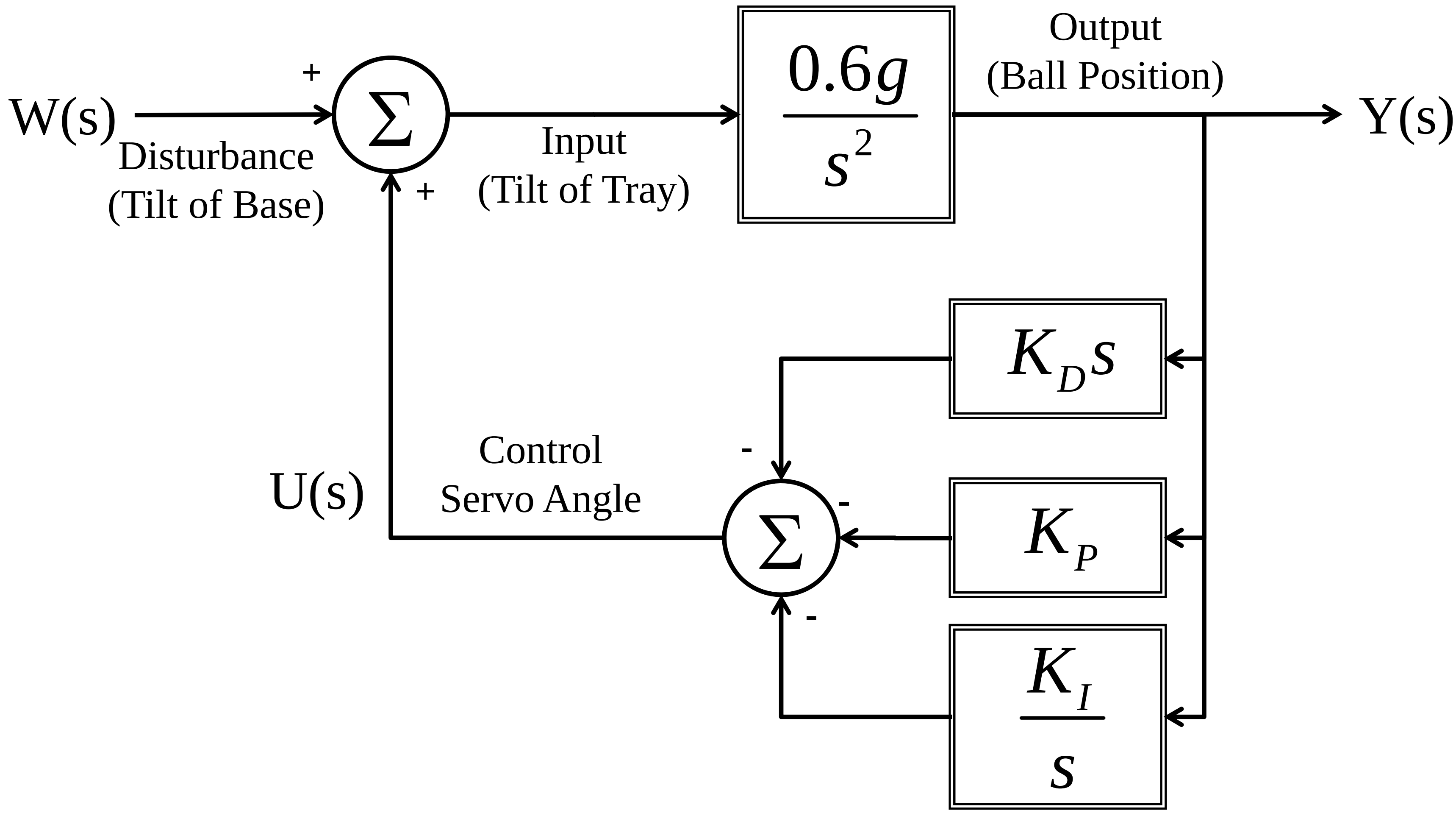
More About Transfer Functions

- "Compensators" = Dynamic System Elements Inside our Controllers Also Represented by Transfer Functions—For Example, PID Control Given By:

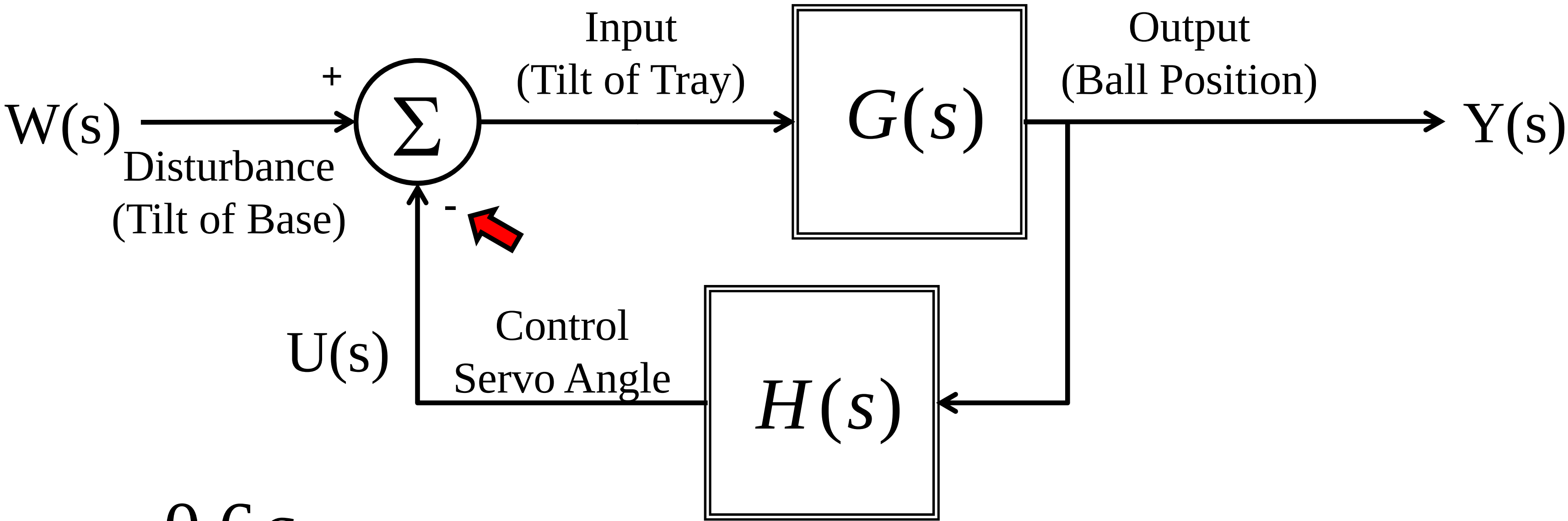
$$H(s) = \left(K_P + \frac{K_I}{s} + K_D s \right)$$

- We Also Frequently Use G(s) To Represent a Transfer Function (Particularly for Plant & Overall Loop)
- Subscripts on Transfer Function Often Used to Indicate Which System Element is Represented
  - $G_P(s)$ = "Plant"
  - $G_C(s)$ = "Compensator"

Block Diagrams : Ping-Pong Poise Example



Block Diagrams : Ping-Pong Poise Example



$$G(s) = \frac{0.6g}{s^2}$$

$$H(s) = \left(K_P + \frac{K_I}{s} + K_D s \right)$$

$$Y(s) = G(s)\big(W(s) - U(s)\big)$$

$$U(s) = H(s)Y(s)$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} W(s)$$

We Found Closed-Loop Dynamics!

$$\frac{Y(s)}{W(s)} = \frac{\frac{0.6g}{s^2}}{1 + \frac{0.6g}{s^2}\left(K_P + \frac{K_I}{s} + K_D s\right)}$$

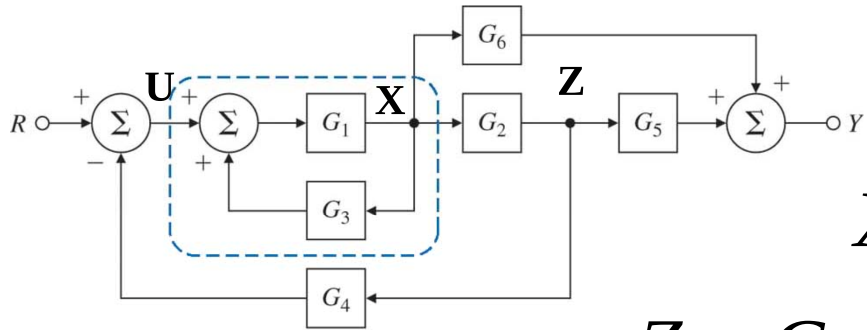
$$\frac{Y(s)}{W(s)} = \frac{0.6gs}{s^3 + 0.6gK_D s^2 + 0.6gK_P s + 0.6gK_I}$$

Closed-Loop Poles Determined by Values of Feedback Gains!

$$\Delta_{CL}(s) = s^3 + 0.6gK_D s^2 + 0.6gK_P s + 0.6gK_I$$

Recall Poles Determine Character of Closed-Loop Response [~exp(pt)]

Label Internal Signals & Solve Linear Algebraic Equations



$$Y = G_5 Z + G_6 X \quad 1$$

$$X = G_1 \left(G_3 X + U \right) \quad 2$$

$$Z = G_2 X \quad 3$$

$$U = R - G_4 Z \quad 4$$

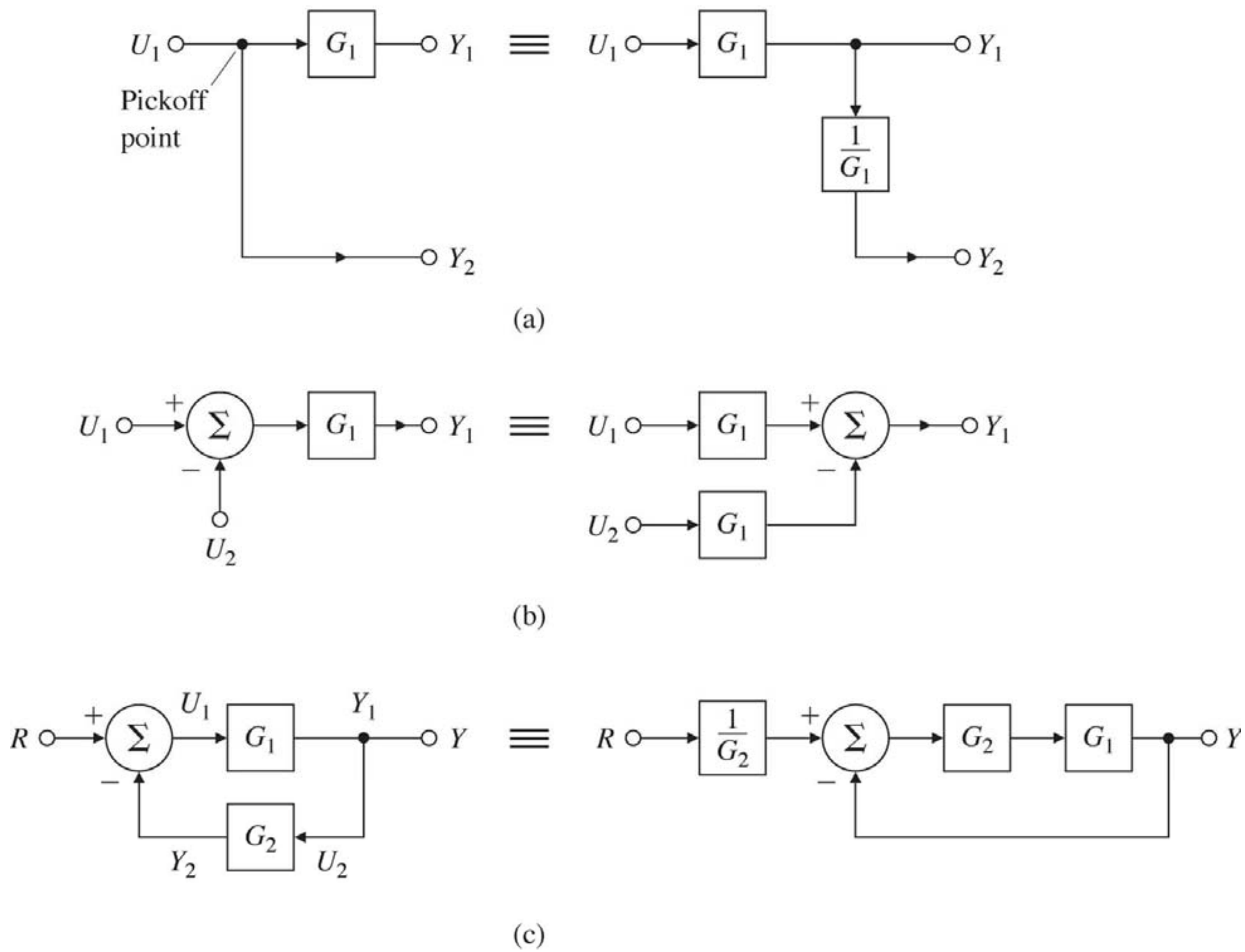
This is Just Algebra!

$$3,\ 4 \rightarrow 2 \implies X = G_1 G_3 X + G_1 R - G_1 G_4 G_2 X$$

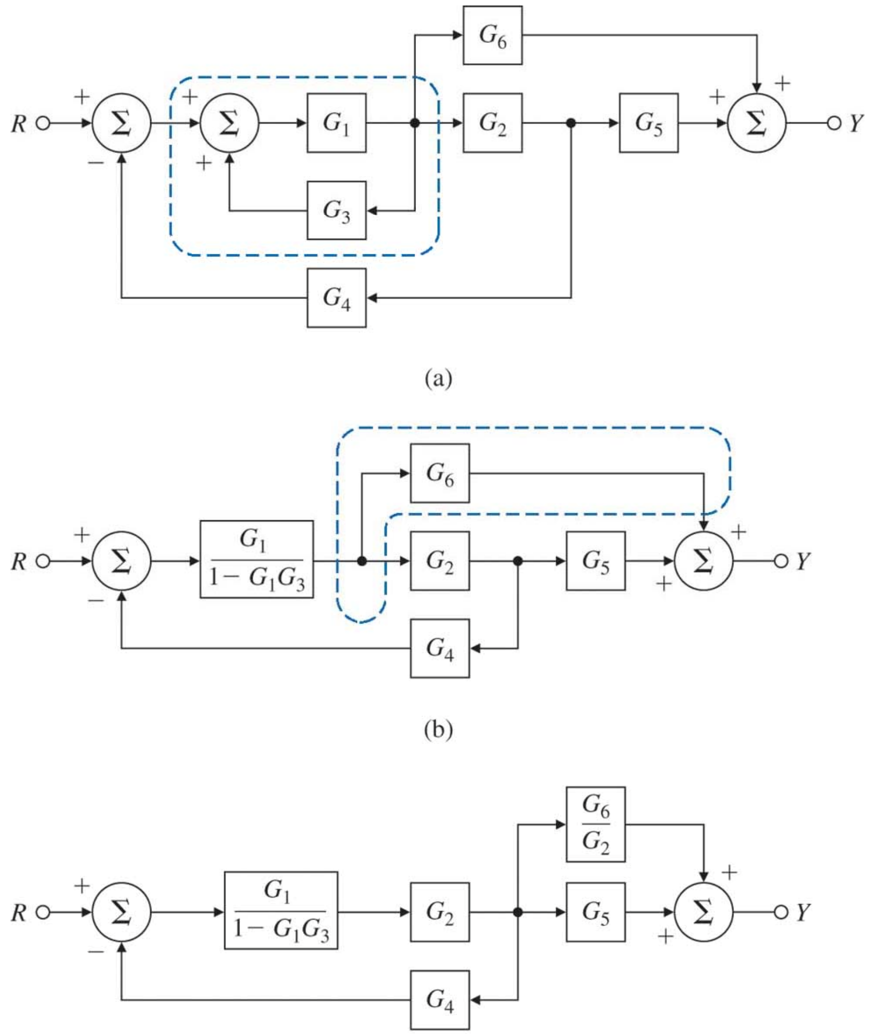
$$X = \frac{G_1}{1 - G_1 G_3 + G_1 G_4 G_2} R$$

$$1,\ 3 \implies Y = \frac{G_1 \left(G_2 G_5 + G_6 \right)}{1 - G_1 G_3 + G_1 G_4 G_2} R$$

Block Diagram Equivalence



Block Diagram Manipulation Using Equivalence



(a)

(b)

(c)

Transfer Functions & Block Diagrams in Matlab

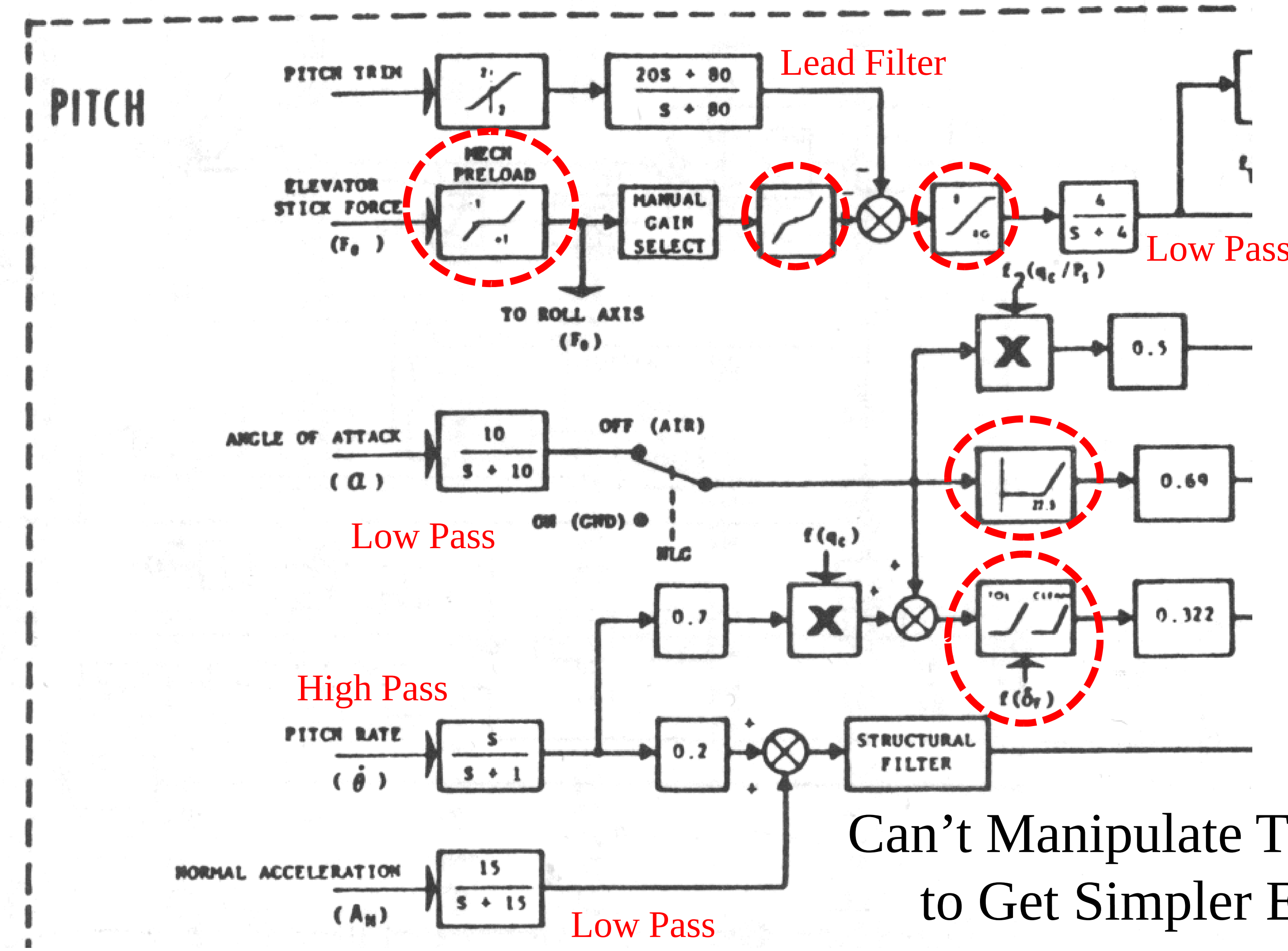
- Polynomials Defined Using Coefficients of “s”
- Transfer Function “System Objects” Defined with Numerator & Denominator Using “`tf`” Command

```
num = [2 10];
den = [1 5 25];
G = tf(num,den)
```

$$G(s) = \frac{2s+10}{s^2+5s+25}$$

- Block Diagram Manipulation Also Possible (see MATLAB Help)
  - `Series`
  - `Parallel`
  - `Feedback`

Danger! Block Diagrams Often Have Nonlinear Elements



Can’t Manipulate These Diagrams to Get Simpler Equivalents!