WHY OPTIMAL CONTROL?

→ WE DON'T REALLY KNOW WHERE WE WANT ALL POLES OF A COMPLICATED SYSTEM TO GO. MAY ONLY KNOW APPROXIMATELY WHERE "DOMINANT" POLES SHOULD GO.

→ WE MIGHT WANT TO MINIMIZE CONTROL USE

(a) MINIMUM FUEL BURN ON SPACECRAFT

(b) AVOID ACTUATOR SATURATION ⇒ NONLINEAR RESPONSE

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LQR: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ WITH $\underline{x}(0) = \underline{x}_0$

LET $J = \int_0^{\infty} (\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u})\, dt$

Q & R ARE STATE & CONTROL WEIGHTING MATRICES.

FIND $\underline{u}(t)$ $0 \le t < \infty$ TO MINIMIZE $J$.

(*) DON'T TAKE IT TOO LITERALLY!

- WE DON'T REALLY KNOW WHAT TO PICK FOR Q & R
- WE CAN'T REALLY EXPRESS THE OPTIMAL THIS SIMPLY.
- THIS IS A TOOL TO COME UP WITH A DESIGN.

YOU CAN TWEAK IT (LIKE ZIEGLER NICHOLS).

NOTE: QUADRATIC FORMS:

$$R = \underbrace{\tfrac{1}{2}(R + R^T)}_{R_s} + \underbrace{\tfrac{1}{2}(R - R^T)}_{R_u}$$

$$R = R_s + R_u$$

$u^T R u = (u^T R u)^T$

SCALAR ⇒ $u^T R_u u = 0$

WE ASSUME $R = R^T$ $(R_u = 0)$

REQUIRE $\underline{u}^T R \underline{u} > 0$ $\forall\, \underline{u}$ ⇒ R "POSITIVE DEFINITE"

GUARANTEES FINITE CONTROL.

$\underline{x}^T Q \underline{x} \ge 0$ ⇒ Q "POSITIVE SEMI-DEFINITE"

→ WE'LL PICK $Q = C^T C \ge 0$

⇒ $\underline{x}^T Q \underline{x} = \underline{y}^T \underline{y}$

WHAT KIND OF MATH PROBLEM IS THIS?

FIND FUNCTION (m-DIMENSIONAL) $\underline{u}(t)$

TO MINIMIZE <u>SCALAR</u> J (J CALLED "FUNCTIONAL")

SUBJECT TO "DYNAMIC CONSTRAINT" $\dot{\underline{x}} = A\underline{x} + B\underline{u}$

<u>GOOGLE</u>: "CALCULUS OF VARIATIONS"

HILDEBRAND, "METHODS OF APPLIED MATHEMATICS" (DOVER)

THIS PROBLEM ALSO IMPORTANT FOR <u>LINEAR SYSTEMS THEORY</u> (ESE 500)

FOR OUR PURPOSES WE JUST NEED TO KNOW:

1. (A, B) CONTROLLABLE $\Rightarrow$ WE CAN GET TO $\underline{x} = \underline{0}$ IN FINITE TIME, SO WE KNOW J CAN BE MADE FINITE. THUS, WE KNOW SOL'N EXISTS. TURNS OUT $\exists$ UNIQUE GLOBAL OPTIMUM!

2. (A, C) OBSERVABLE: J FINITE $\Rightarrow \underline{x} \to \underline{0}$ AS $t \to \infty$.

3. OPTIMAL SOLUTION IS OF FORM $\underline{u} = -K\underline{x}$

   WITH $K = R^{-1}B^TP$

   WHERE P IS UNIQUE POSITIVE-DEFINITE SOLUTION OF THE <u>ALGEBRAIC RICCATI EQUATION</u>:

   $$PA + A^TP - PBR^{-1}B^TP + Q = 0$$

   <u>NOTE</u>: THIS IS A NONLINEAR ALGEBRAIC EQUATION

   MATLAB WILL SOLVE WITH "LQR COMMAND"

IF ACTUAL STATE AVAILABLE, LQR K YIELDS GOOD STABILITY ROBUSTNESS (SYSTEM STABLE FOR LARGE PLANT MODEL ERRORS)

CONSIDER $\dot{\underline{x}} = A\underline{x} + B\underline{u} + \underline{v}$

DISTURBANCE

$y = C\underline{x} + \underline{w}$

NOISE

$$\left.\begin{aligned} E\{\underline{v}(t)\,\underline{v}^T(\tau)\} &= V\,\delta(t-\tau) \\ E\{\underline{w}(t)\,\underline{w}^T(\tau)\} &= W\,\delta(t-\tau) \\ E\{\underline{v}(t)\,\underline{w}(\tau)\} &= \emptyset \end{aligned}\right\}$$

$\underline{v}$ & $\underline{w}$ ARE WHITE NOISE RANDOM VARIABLES WITH COVARIANCES $V$ & $W$, RESPECTIVELY.

FOR OUR PURPOSES, $V$ IS A MEASURE OF MAGNITUDE OF DISTURBANCES AND $W$ IS A MEASURE OF MAGNITUDE OF NOISE.

WE WILL BUILD A LINEAR ESTIMATOR:

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(y - C\hat{\underline{x}})$$

THE ERROR, $\underline{e} = \underline{x} - \hat{\underline{x}}$, IS ALSO A RANDOM VARIABLE, AND ITS COVARIANCE MATRIX IS $P$.

UNDER CONDITIONS OF INTEREST ($(A, C)$ OBSERVABLE & $(A, V^{1/2})$ CONTROLLABLE)

$P$ WILL APPROACH A STEADY VALUE AS $t \to \infty$. THEN

$$AP + PA^T - PC^T W^{-1} C P + V = \emptyset$$

__NOTE__: SAME AS LQR ALGEBRAIC RICCATI WITH

$A \to A^T$

$B \to C^T$

$R \to W$

$Q \to V$

__NOTE__: $V$ LARGE
$\Downarrow$
$P$ LARGE
$\Downarrow$
$L$ LARGE
(USE MEASUREMENTS)

OPTIMAL GAIN IS $L = PC^T W^{-1}$

$W$ LARGE $\Rightarrow$ $P$ SMALL $\Rightarrow$ $L$ SMALL
(USE PROJECTED STATE)