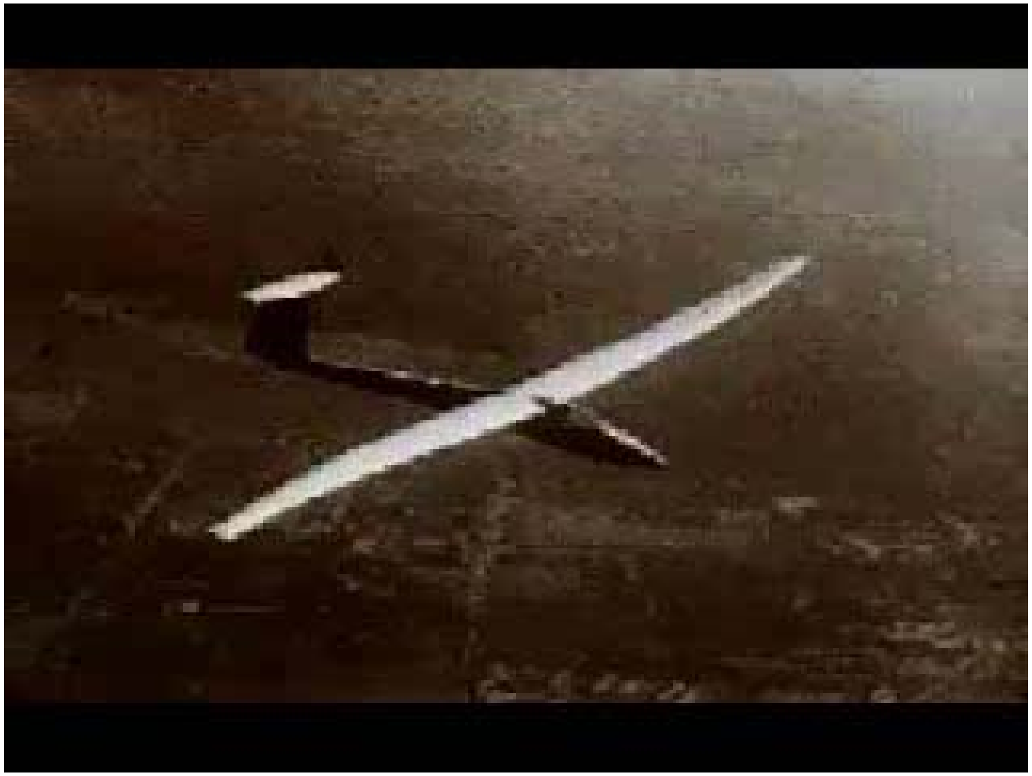
Frequency Response



ESE 505 & MEAM 513

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Penn Engineering

Response to Sinusoidal Input

Step Response $$Y(s) = G(s)U(s) = G(s)\frac{\omega}{s^2+\omega^2}$$

Partial Fraction Expansion $$Y(s) = \frac{C_0}{s-j\omega} + \frac{\bar{C}_0}{s+j\omega} + \underbrace{\frac{C_1}{(s-p_1)} + \cdots + \frac{C_n}{(s-p_n)}}$$

$$C_0 = A + jB$$

$$Y(s) = \frac{A+jB}{s-j\omega} + \frac{A-jB}{s+j\omega} + \cdots$$

Transient Response (→0 in Steady State for Stable System)

$$Y(s) = \frac{2As - 2B\omega}{s^2+\omega^2} + \cdots$$

$$y(t) = \left[2A\cos(\omega t) - 2B\sin(\omega t)\right] + \overbrace{C_1 e^{p_1 t} + \cdots + C_n e^{p_n t}}$$

Response to Sinusoidal Input (Continued)

"Cover Up" Rule $$C_0 = \lim_{s \to j\omega} \left[\left(s - j\omega \right) Y(s) \right] = G(j\omega) \frac{\omega}{2 j\omega}$$

$$G(j\omega) = Me^{j\phi} \quad \Longrightarrow \quad C_0 = \frac{Me^{j\phi}}{2j} = \frac{M}{2} \left[-j\cos\phi + \sin\phi \right]$$

$$A = \frac{M}{2}\sin\phi \qquad B = -\frac{M}{2}\cos\phi$$

$$y(t) = M\left[\cos(\omega t)\sin\phi + \sin(\omega t)\cos\phi \right] = M\sin(\omega t + \phi)$$

Response to Unit Sinusoidal Input of Frequency ω is Sinusoidal Output of Frequency ω with Magnitude M and Phase Shift ϕ, where $G(j\omega)=Me^{j\phi}$

Long Journey to Get Here…Still More to Go!

- Develop ODE Model of Dynamic System

- Linearize ODE Model
  – Trim
  – Small Perturbations From Trim

- G(s) = Transfer Function (Zero Initial Perturbation)

- Evaluate $G(j\omega)$ to Find Magnitude & Phase of Response to Sinusoidal Input

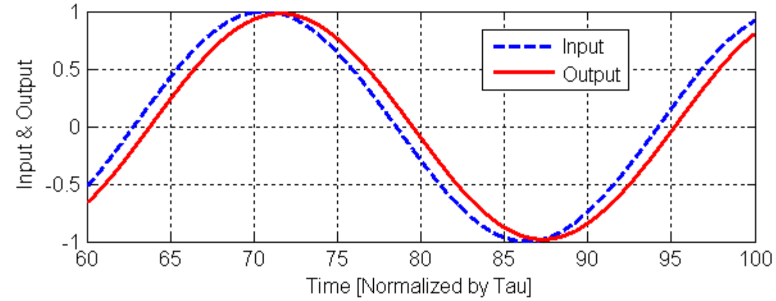
First-Order Example : Low-Pass Filter

$$G(s)=\frac{1}{\tau s+1}=\frac{\sigma}{s+\sigma}$$

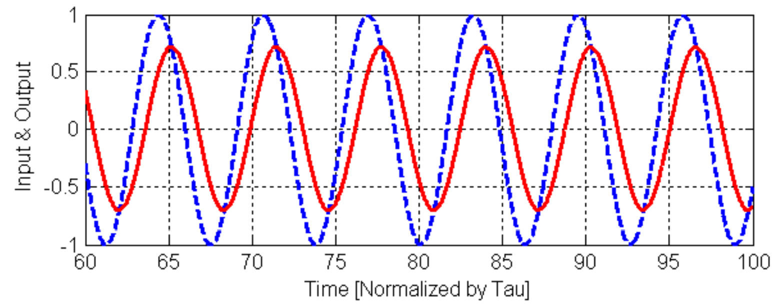
$$G(j\omega)=\frac{\sigma}{j\omega+\sigma}$$

$$M=\frac{\sigma}{\sqrt{\sigma^2+\omega^2}}=\frac{1}{\sqrt{1+(\tau\omega)^2}}$$

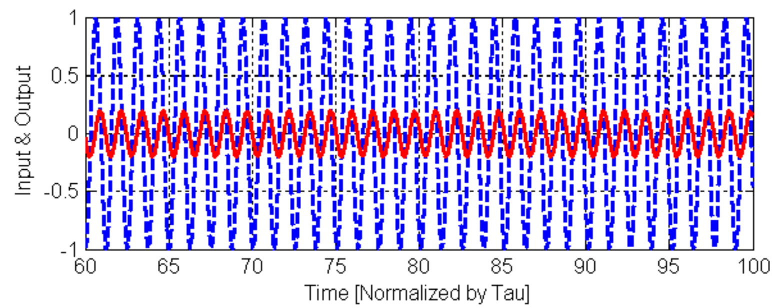
$$\phi=-\tan^{-1}\left(\frac{\omega}{\sigma}\right)=-\tan^{-1}(\tau\omega)$$



$\tau\omega=0.2 \quad M=0.98 \quad \phi=-11.3°$



$\tau\omega=1.0 \quad M=0.707 \quad \phi=-45°$



$\tau\omega=5.0 \quad M=0.20 \quad \phi=-79°$

First-Order Example : High-Pass Filter

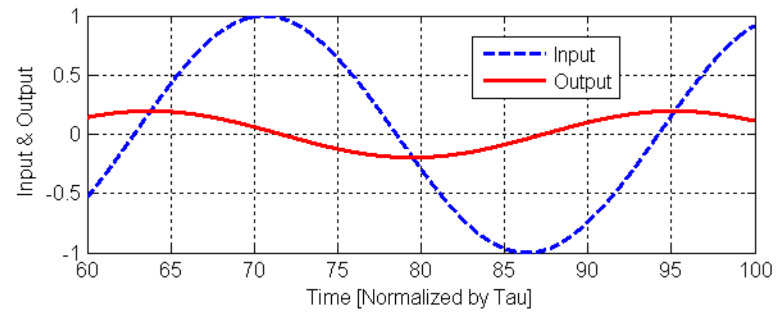
$$G(s)=\frac{\tau s}{\tau s+1}=\frac{s}{s+\sigma}$$

$$G(j\omega)=\frac{j\omega}{j\omega+\sigma}$$

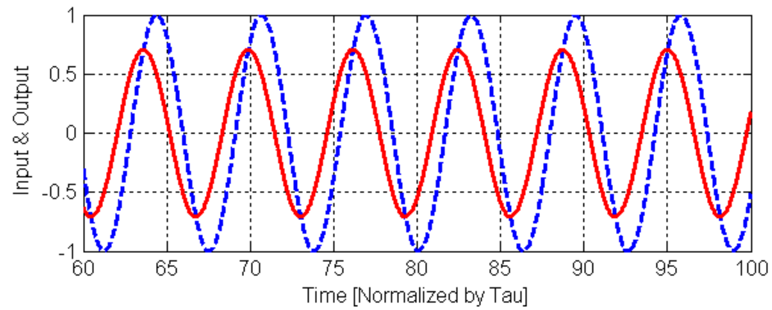
$$M=\frac{\omega}{\sqrt{\sigma^2+\omega^2}}=\frac{\tau\omega}{\sqrt{1+(\tau\omega)^2}}$$

$$\phi=90^\circ-\tan^{-1}(\tau\omega)$$

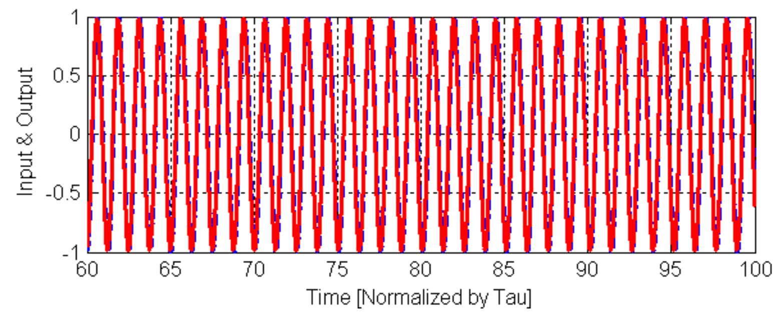
Output Leads Input!



$\tau\omega=0.2$ $\quad M=0.20$ $\quad \phi=79^\circ$

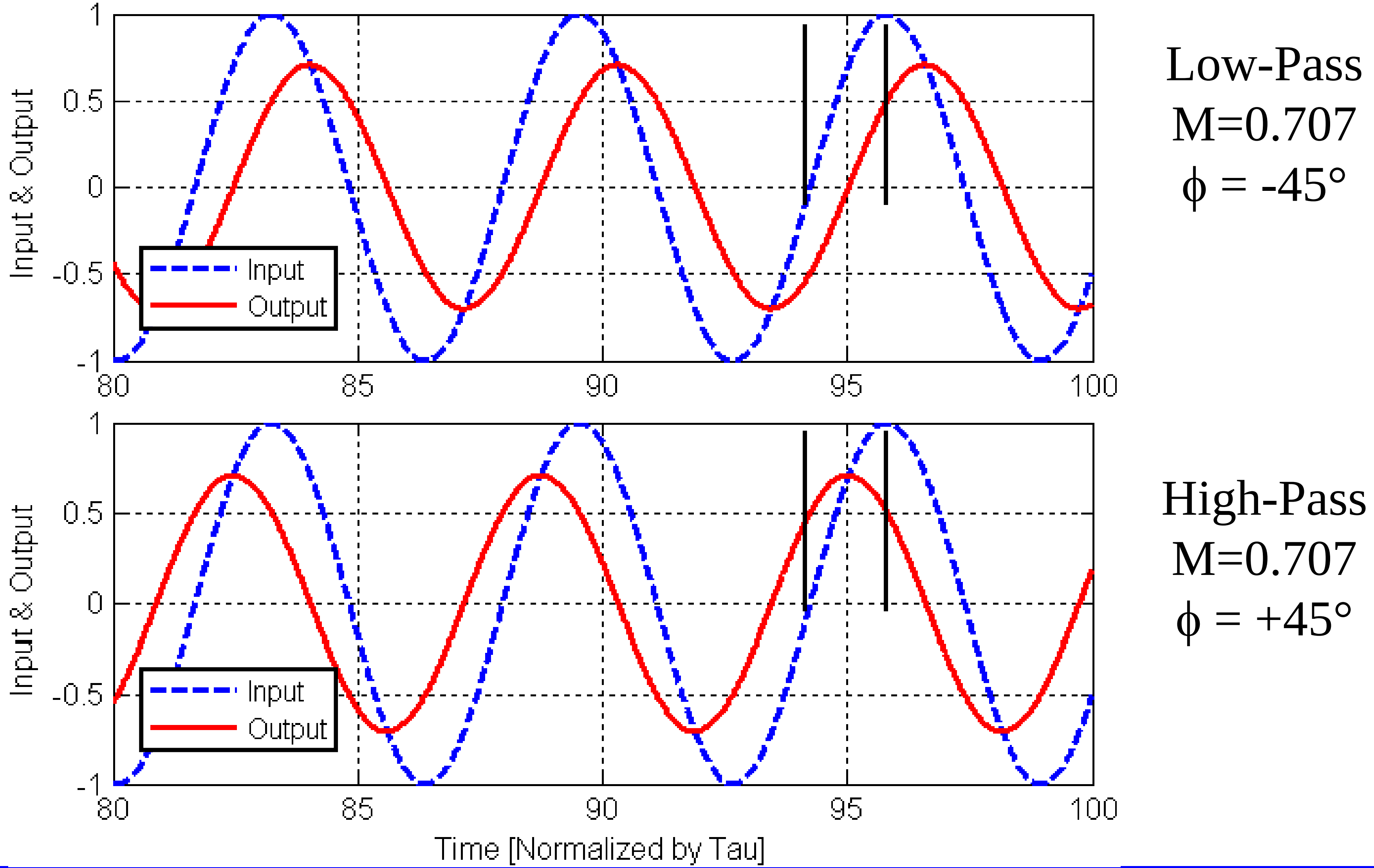


$\tau\omega=1.0$ $\quad M=0.707$ $\quad \phi=45^\circ$



$\tau\omega=5.0$ $\quad M=0.98$ $\quad \phi=11.3^\circ$

Low-Pass vs. High-Pass @ $\omega = 1/\tau$



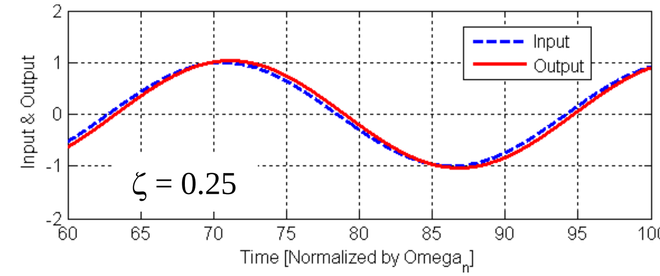
Second-Order Example

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

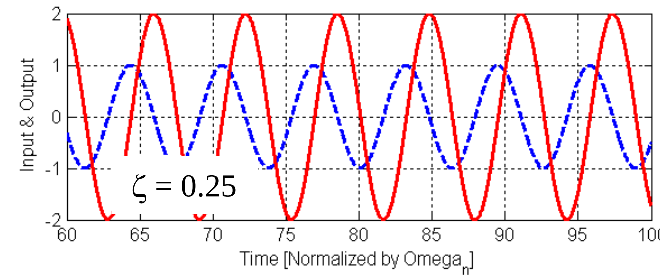
$$G(j\omega)=\frac{\omega_n^2}{\omega_n^2-\omega^2+j2\zeta\omega_n\omega}$$

$$M=\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{-1/2}$$

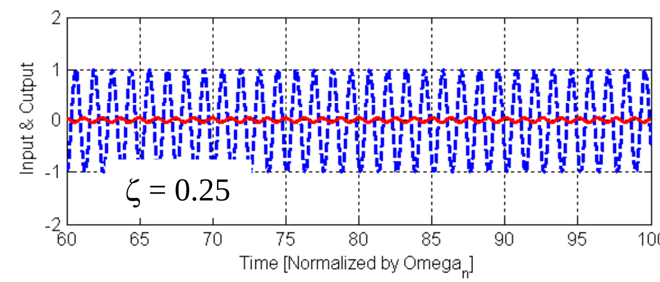
$$\phi=-\tan^{-1}\left(2\varsigma\frac{\omega}{\omega_n}\Big/\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\right)$$



$\omega/\omega_n=0.2 \qquad M=1.0 \qquad \phi=-5.9^\circ$



$\omega/\omega_n=1.0 \qquad M=2.0 \qquad \phi=-90^\circ$



$\omega/\omega_n=5.0 \qquad M=0.04 \qquad \phi=-174^\circ$

Second-Order System "At Resonance"

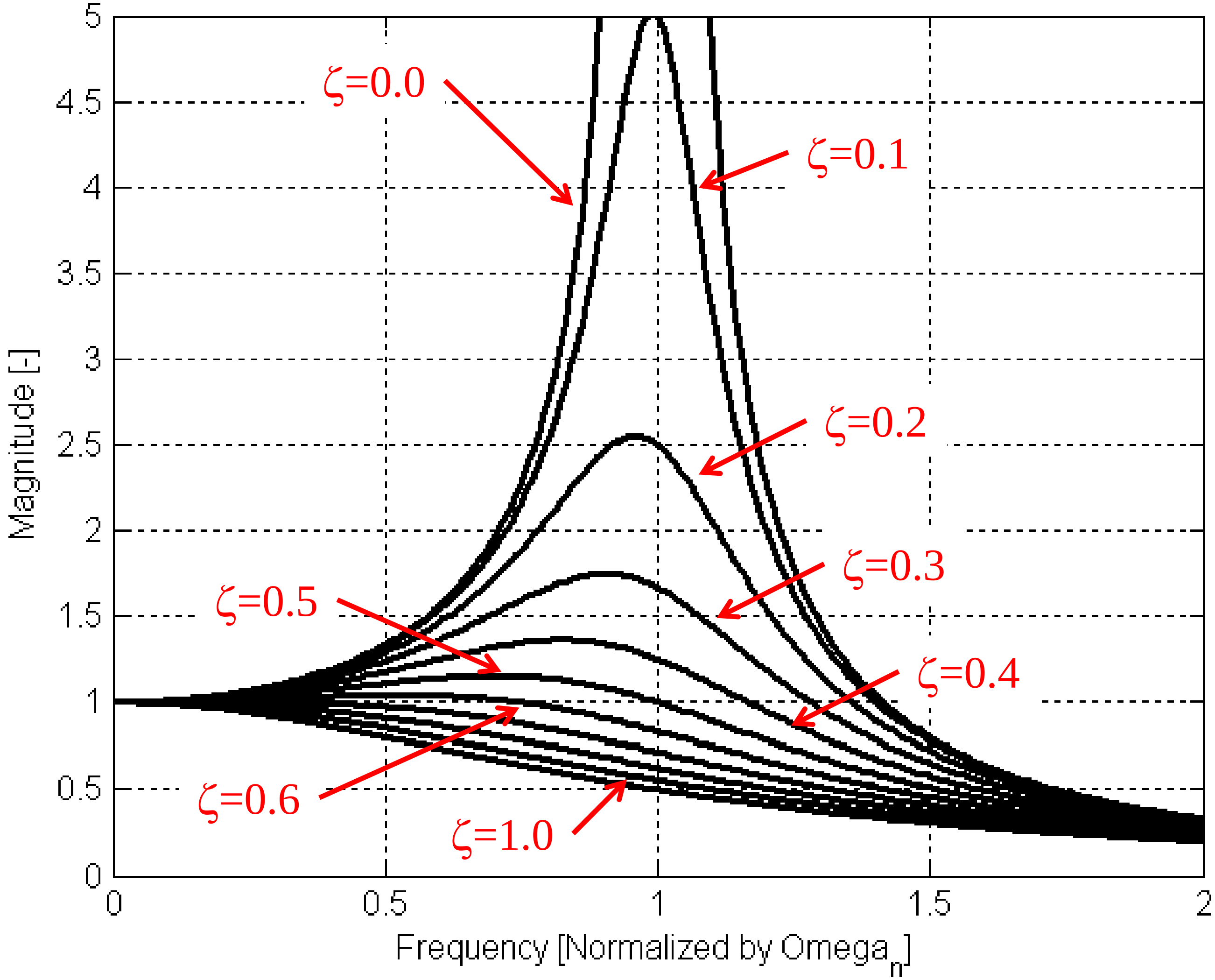
$$G(j\omega_n) = \frac{\omega_n^2}{\omega_n^2 - \omega_n^2 + j2\zeta\omega_n^2} = \frac{1}{j2\zeta}$$

$$M = \frac{1}{2\varsigma} \qquad \phi = -90^\circ$$

Second-Order System Forced at Natural Frequency → "Mass" & "Spring" Cancel → Only "Damper" Balances Applied Force!

Second-Order System Forced at Natural Frequency → ALWAYS 90 Degrees Phase Lag!

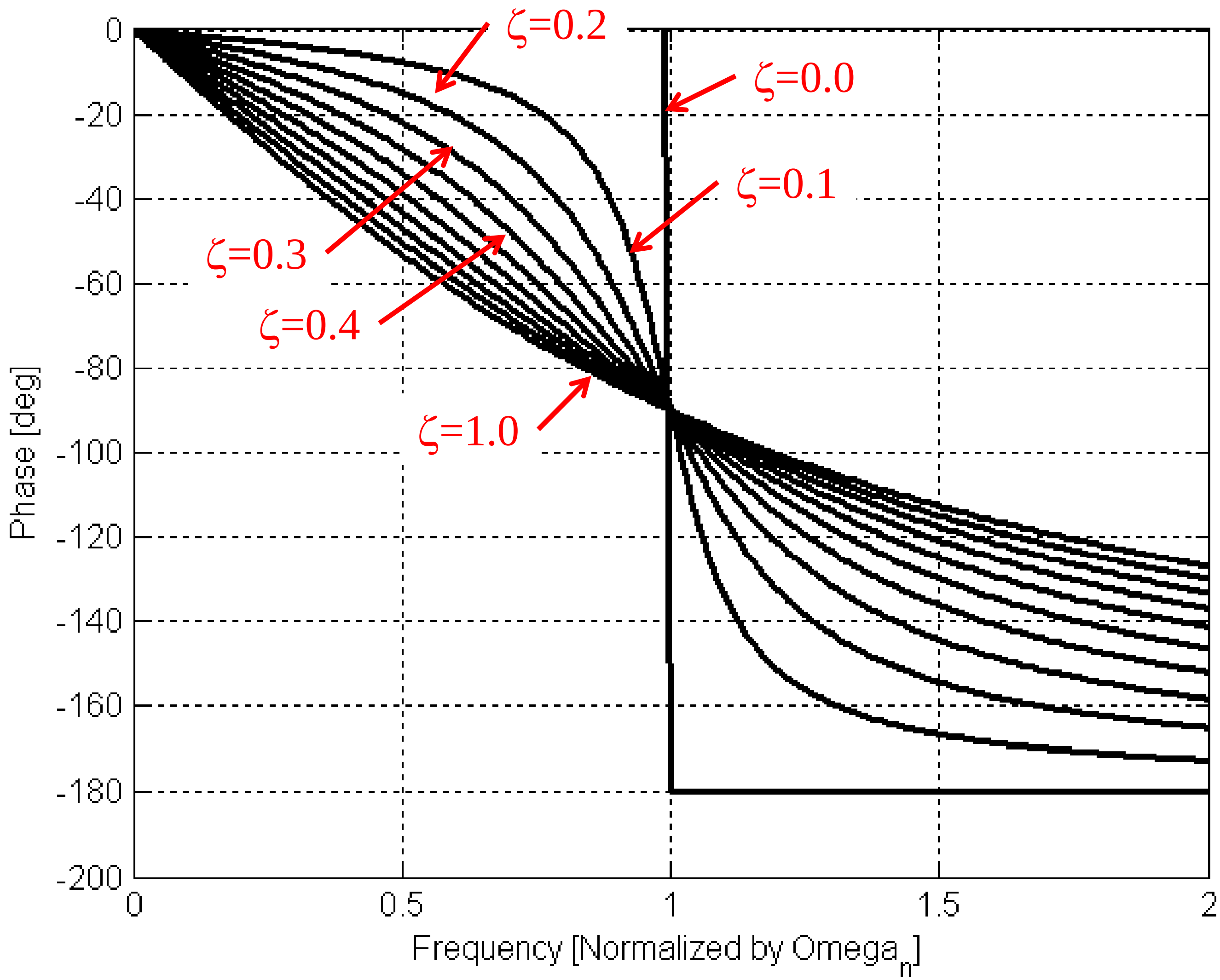
2nd-Order Frequency Response Magnitude



This is How Most Books on Vibration Plot Second-Order System Response

We'll Do It Differently Soon…

2^{nd}-Order Frequency Response Phase



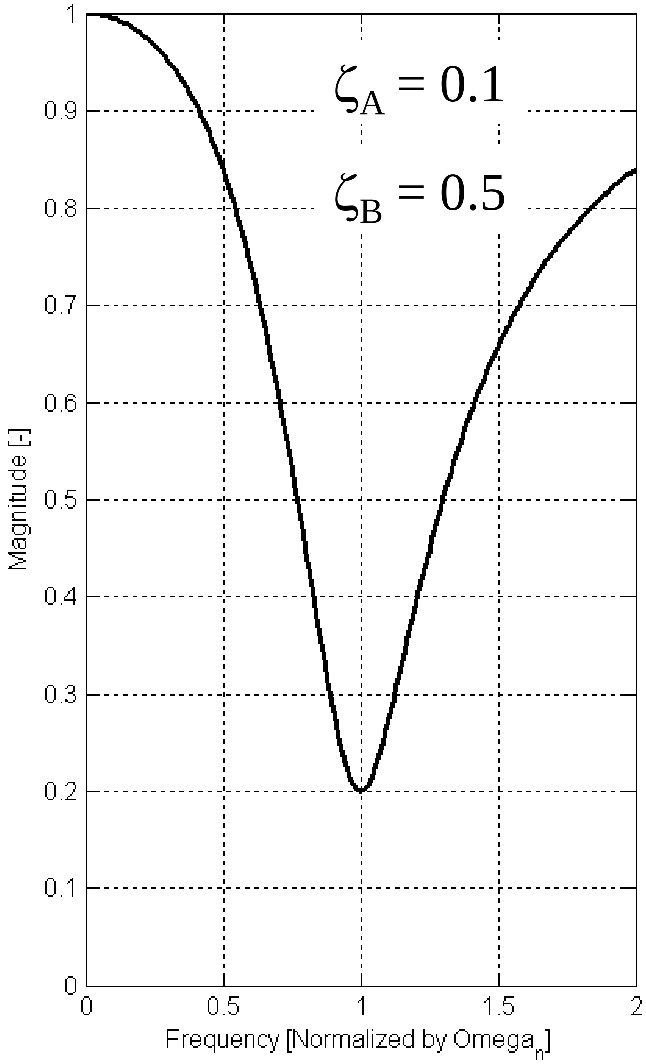
We'll Do This Differently Soon, Too…

Second-Order Notch Filter

$$G(s) = \frac{s^2 + 2\zeta_A \omega_n s + \omega_n^2}{s^2 + 2\zeta_B \omega_n s + \omega_n^2}$$

$$G(j\omega) = \frac{\omega_n^2 - \omega^2 + j2\zeta_A \omega_n \omega}{\omega_n^2 - \omega^2 + j2\zeta_B \omega_n \omega}$$

$$M = \sqrt{\frac{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta_B \omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta_A \omega}{\omega_n}\right)^2}}$$



Higher-Order Systems…

$$G(s) = G(0)G_1(s)G_2(s)\cdots$$

$$G_k(s) = \frac{\tau_{A_k} s + 1}{\tau_{B_k} s + 1} \quad \textbf{\textit{...OR...}} \quad G_k(s) = \frac{\left(\frac{s}{\omega_{A_k}}\right)^2 + 2\zeta_{A_k}\left(\frac{s}{\omega_{A_k}}\right)s + 1}{\left(\frac{s}{\omega_{B_k}}\right)^2 + 2\zeta_{B_k}\left(\frac{s}{\omega_{B_k}}\right)s + 1}$$

$$G(j\omega) = G(0)G_1(j\omega)G_2(j\omega)\cdots$$

$$M = G(0)M_1 M_2 \cdots$$

$$\phi = \phi_1 + \phi_2 + \cdots$$

We'll Make This Easy
Next Lecture…