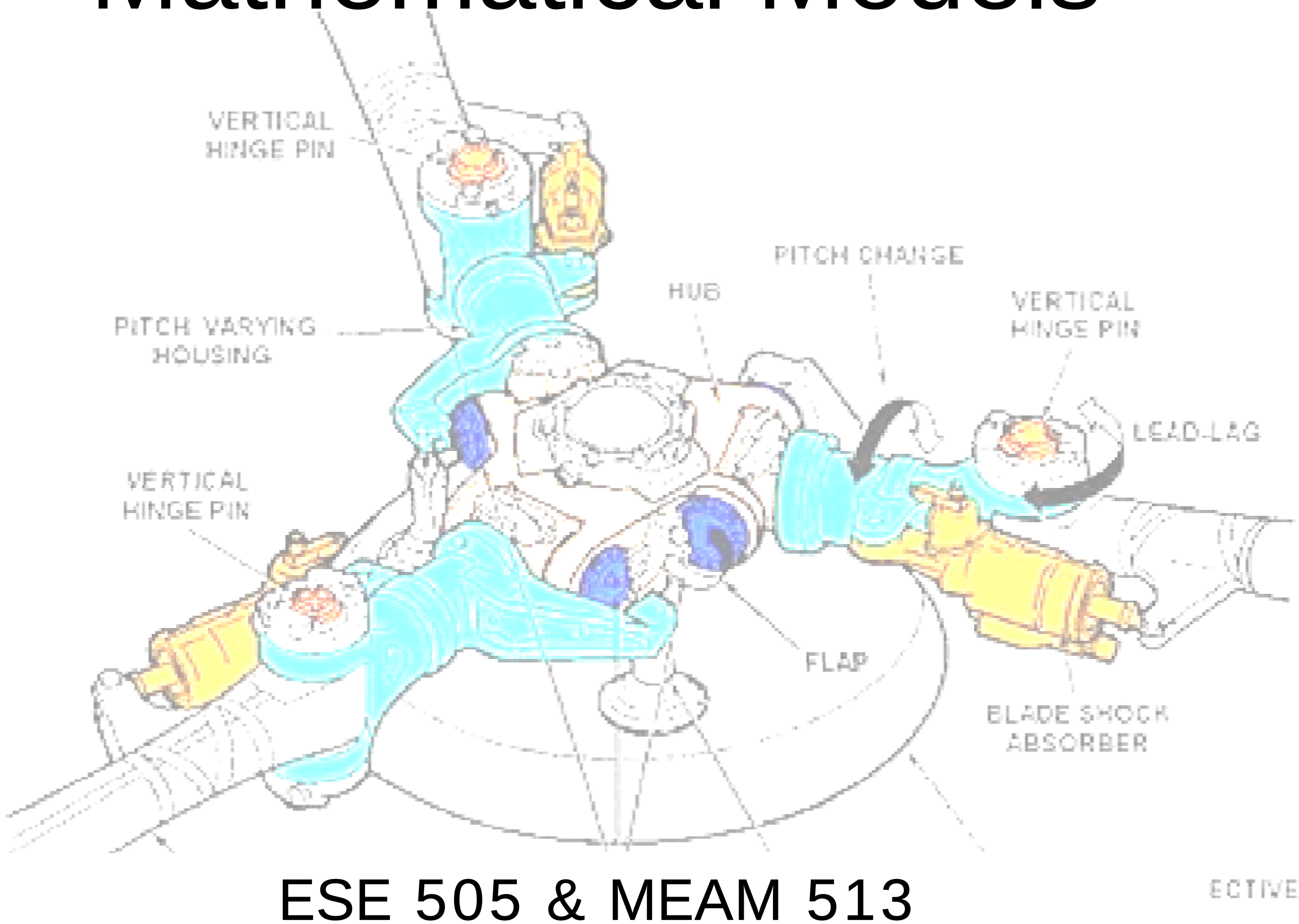
Mathematical Models



ESE 505 & MEAM 513

Bruce D. Kothmann

2014-01-22

Penn Engineering

http://www.chinook-helicopter.com/Fundamentals_of_Flight/Ground_Resonance/Chinook_helicopter_rotor_head.jpg

Common Systems We Want to Model

- Mechanical Systems
  - Often Idealized as Rigid Bodies with Linear Springs & Dampers
  - Planar Dynamics Generally in Scope for Undergraduates
  - 3D Rotational Dynamics are Difficult!
- Electrical Systems
  - Resistors, Capacitors & Inductors Analogous to Dampers, Springs & Masses, Respectively
  - Op-Amps Also Important
  - Easy to Analyze with "Complex Impedance" & Laplace Transforms
- DC Motors
  - Ubiquitous in Student Projects
  - Nice Example of Electro-Mechanical System
- Others : Heat & Fluid Flow, Robots, Stock Markets, Airplanes, Industrial Processes

Newton's Second Law of Motion

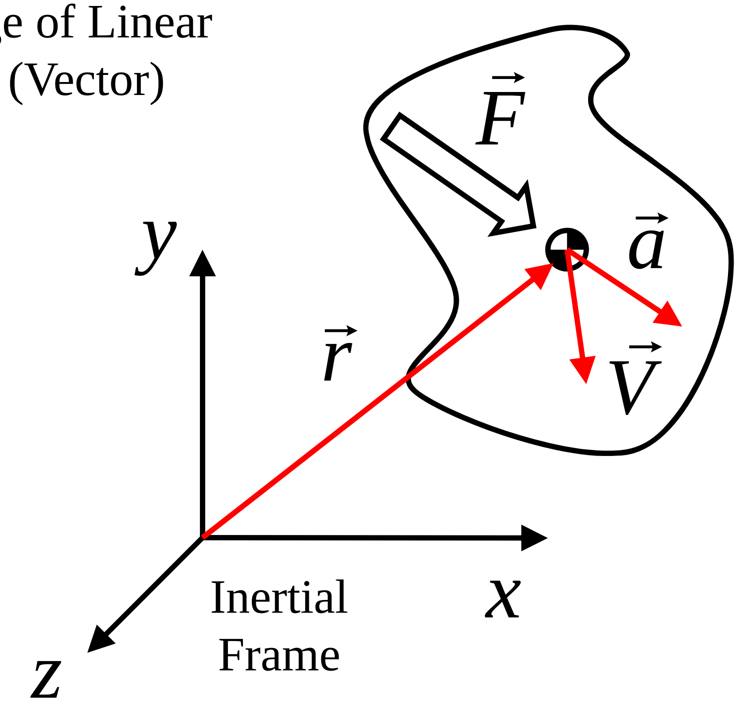
$$\vec{F} = \frac{d\vec{P}}{dt} = m\vec{a}$$

Rate of Change of Linear Momentum (Vector)

Total Force Acting on Body (Vector)

Mass of Body (Scalar)

Acceleration of CG of Body in Inertial Reference Frame (Vector)



$$\left.\begin{aligned} \vec{V} &= \frac{d\vec{r}}{dt} \\ \vec{a} &= \frac{d\vec{V}}{dt} \\ a &= \frac{d^2\vec{r}}{dt^2} \end{aligned}\right\}$$

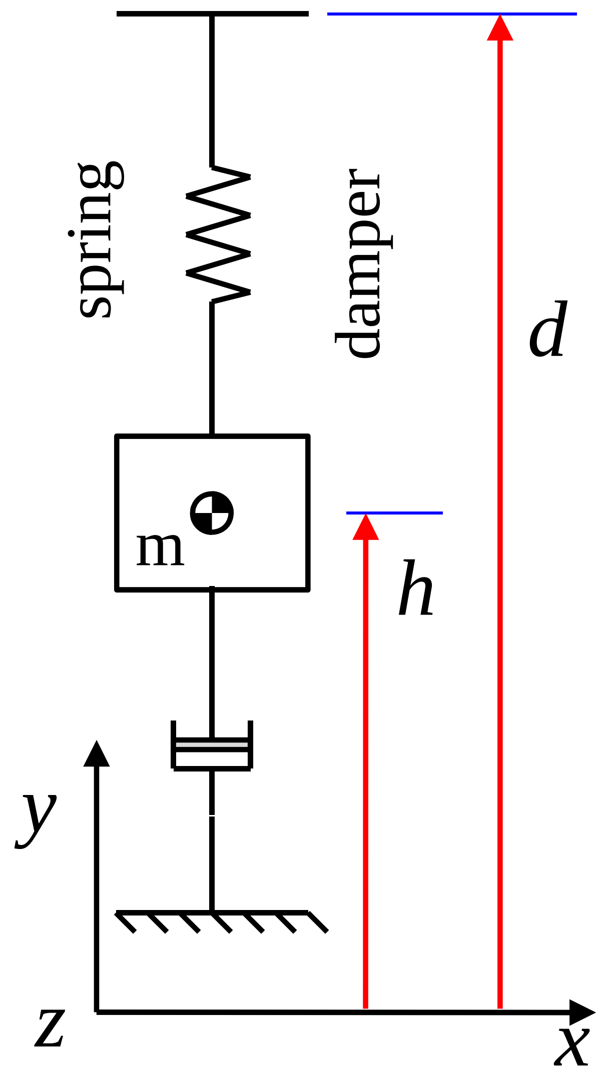
Kinematics (Relationships Between Position, Velocity, Acceleration)

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \frac{1}{m}\vec{F}\left(\vec{r}, \frac{d\vec{r}}{dt}\right)}$$

<u>MATH</u>

3 Coupled 2nd-Order Ordinary Differential Equations

Mass-Spring-Damper System : Kinematics



Consider a System with a Moving Upper Support & a Fixed Lower Support

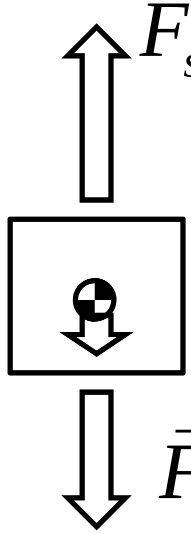
$$\vec{r} = 0\hat{i} + h\hat{j} + 0\hat{k}$$

$$\vec{V} = 0\hat{i} + \frac{dh}{dt}\hat{j} + 0\hat{k}$$

$$\vec{a} = 0\hat{i} + \frac{d^2h}{dt^2}\hat{j} + 0\hat{k}$$

Kinematics (Typically the Easy Part)

Mass-Spring-Damper System : Forces



$\vec{F}_{spring}$

$\vec{F}_{gravity} = -mg\hat{j}$

$\vec{F}_{damper}$

Always Draw "Free Body Diagram" (FBD) for Each Mass in the System

Writing Expressions for Forces is the Hard Part

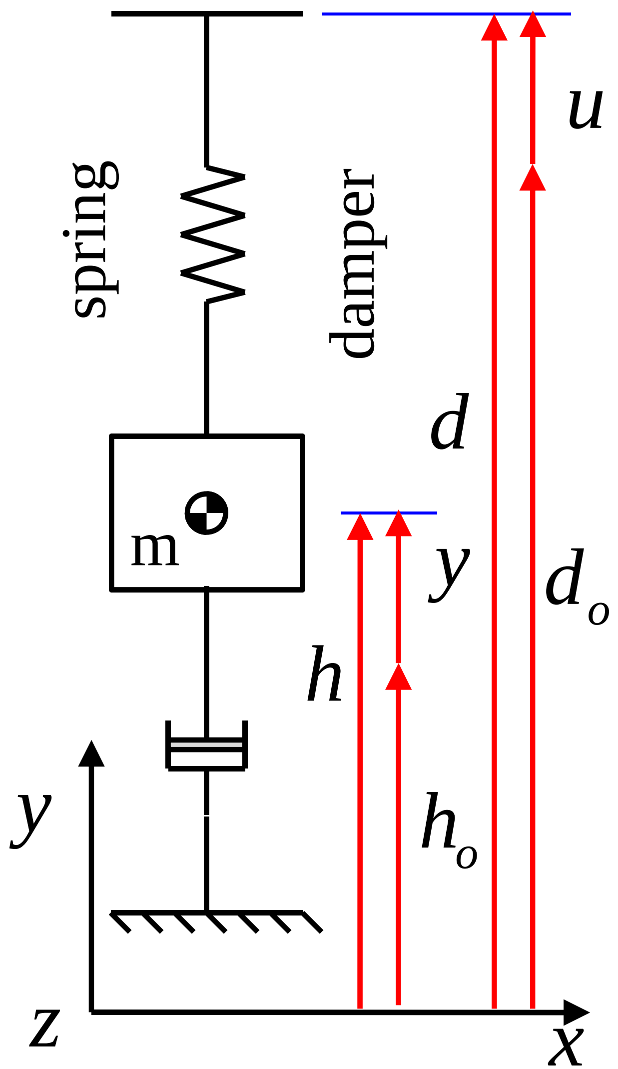
- Dependence of Forces on Velocity & Position Often Not Well Known
- Robustness = Controls Must Work When Force Model is Wrong
- Forces Often Non-Linear

<u>Idealized Spring & Damper</u>
(l = unstretched spring length)

$$\vec{F}_{spring} \approx 0\hat{i} + k(d - h - l)\,\hat{j} + 0\hat{k}$$

$$\vec{F}_{damper} \approx 0\hat{i} - c\frac{dh}{dt}\,\hat{j} + 0\hat{k}$$

Mass-Spring-Damper System : Final Equation



y-Component of Newton's 2nd Law

$$m\frac{d^2h}{dt^2}+c\frac{dh}{dt}+kh=k\left(d-l\right)-mg$$

Find Steady-State $kh_o=k\left(d_o-l\right)-mg$

$$\left.\begin{aligned} h&=h_o+y \\ d&=d_o+u \end{aligned}\right\}$$

Convenient to Define New Input & Output Relative to Steady State

$$\boxed{m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=ku}$$

Rearrange Equation into Standard Form

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{k}{m}u$$

$$\omega_n^2 \triangleq \frac{k}{m} \qquad\qquad 2\zeta\omega_n \triangleq \frac{c}{m}$$

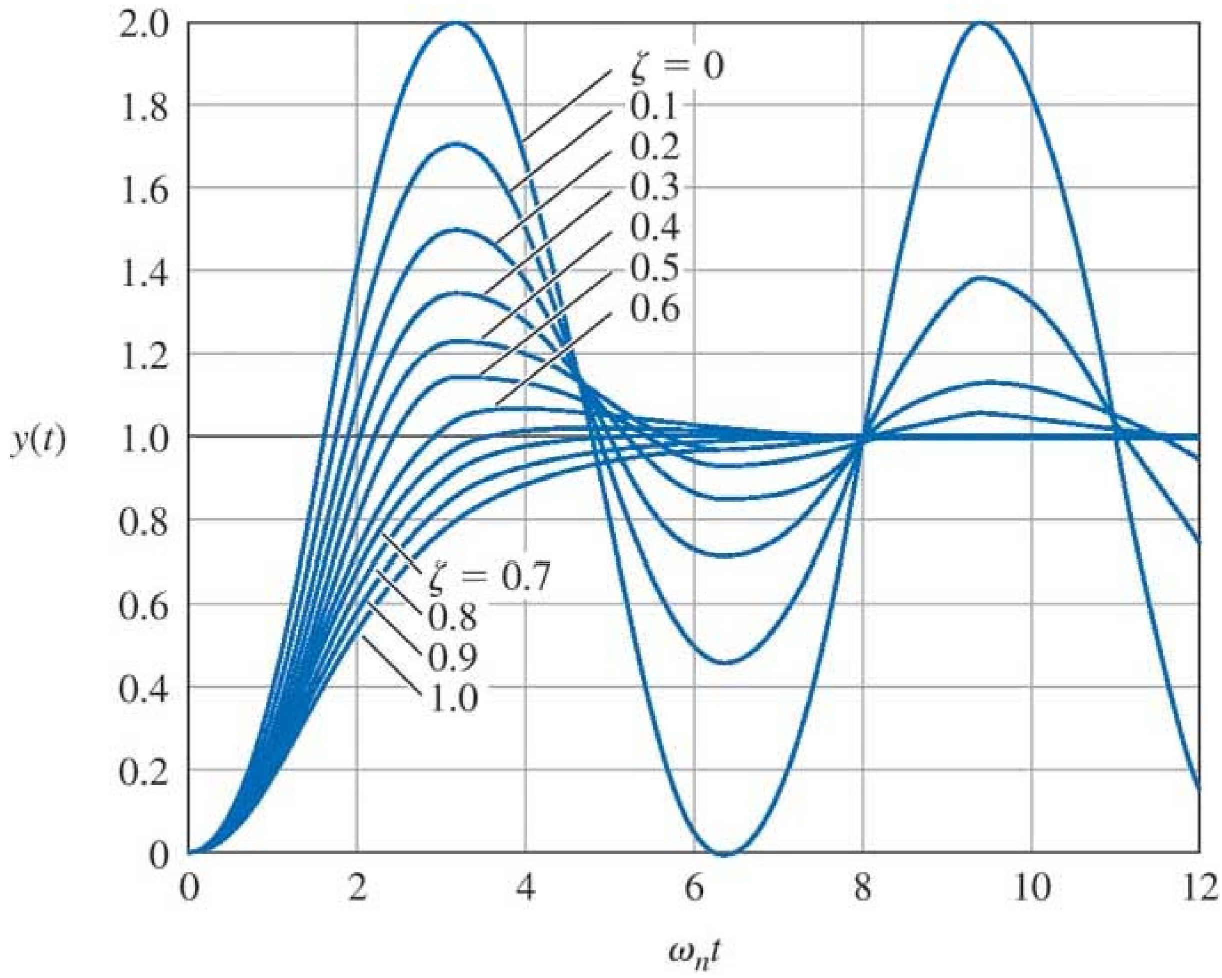
$$\boxed{\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = \omega_n^2 u}$$

We Will Encounter This System Often!

$\omega_n =$ Natural Frequency

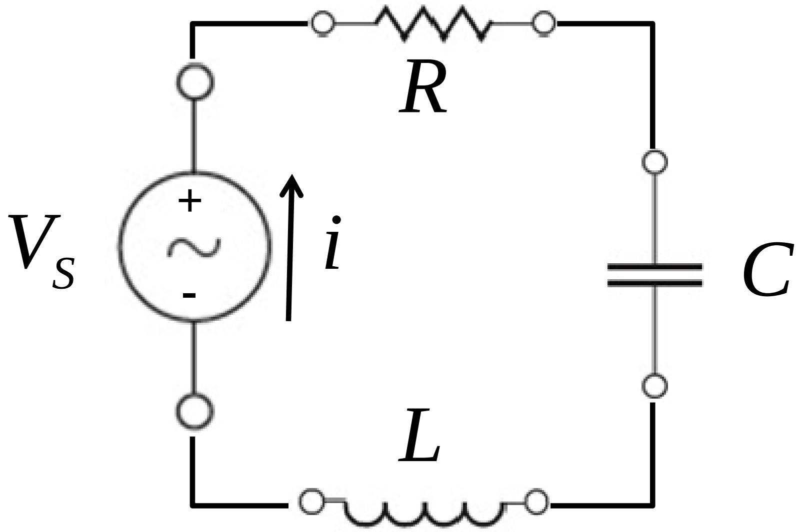
$\zeta =$ Damping Ratio

Unit Step Response of 2nd Order System



We Will Spend a Lot of Time Understanding Response to Unit Step Inputs

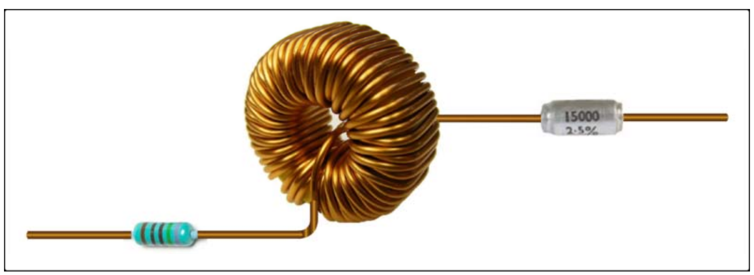
L-C-R Circuits Identical Equations



$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=V_s$$

If We Use Voltage on Capacitor as Output, Then We Find

$$\frac{d^2V_c}{dt^2}+\frac{R}{L}\frac{dV_c}{dt}+\frac{1}{LC}V_c=\frac{1}{LC}V_s$$



Standard Second-Order Form of Equation

Planar Rotation Problems (Easy)

$$M = I \frac{d^2\theta}{dt^2}$$

Total Moment About CG of Rigid Body OR About Fixed Point on Rigid Body

Moment of Inertia of Rigid Body About Same Point as Moments Are Summed

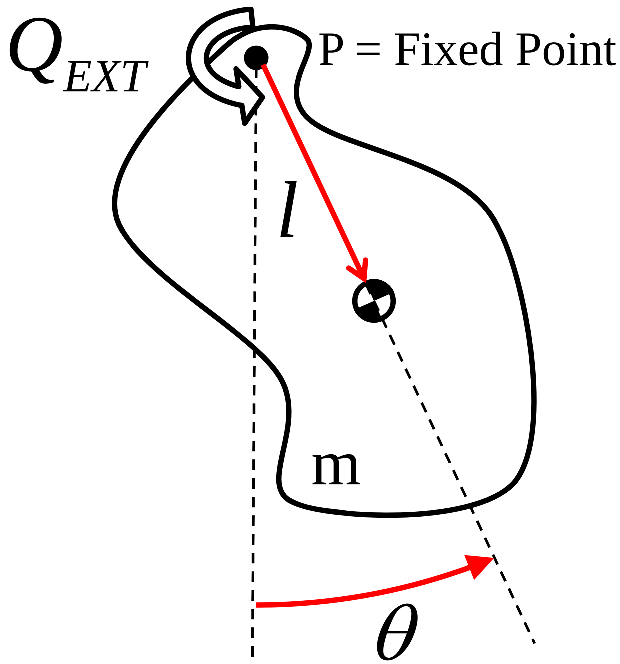
Angular Acceleration of Rigid Body

Planar Rotation Problems Look Just Like Single Component of Linear Translation Problems!

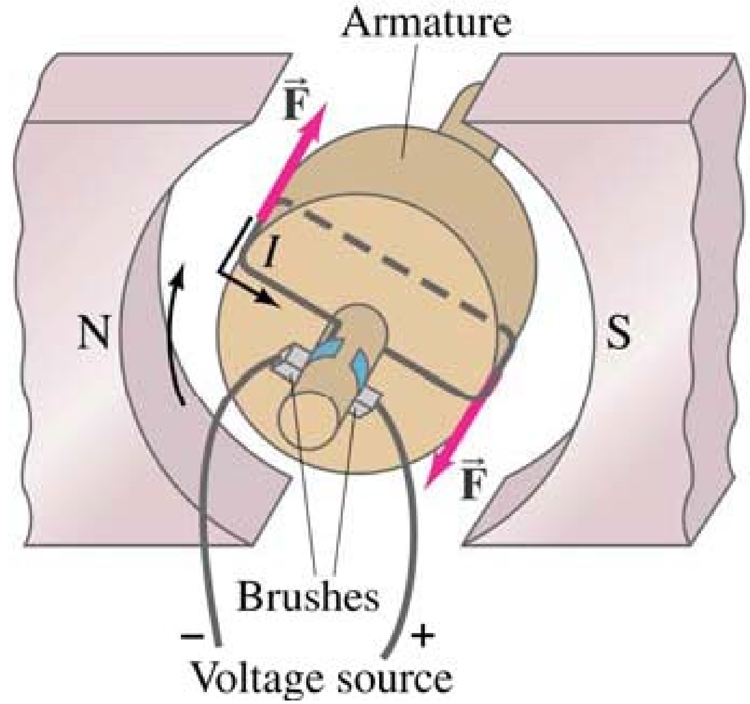
Simple Pendulum

$$I_P \frac{d^2\theta}{dt^2} = Q_{EXT} - mgl\sin\theta$$

This is a Non-Linear Equation!



Basic Physics & DC Motor



Lorenz Force

$$Q = Ki$$

Motor Torque
Proportional to Current

Faraday's Law

$$\varepsilon = K\Omega$$

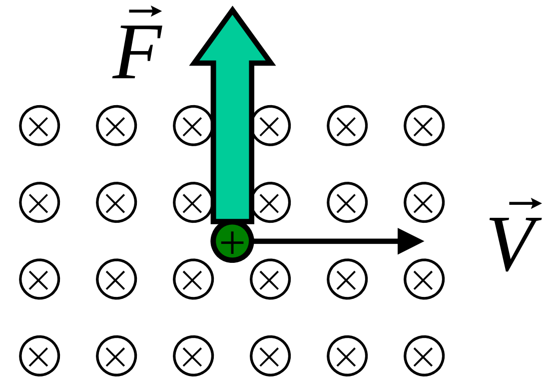
Motor "Back EMF"
Proportional to
Rotational Speed

Note: Same K in Both Equations
SI Units:
(N-m/amp) = (volts/[rad/sec])

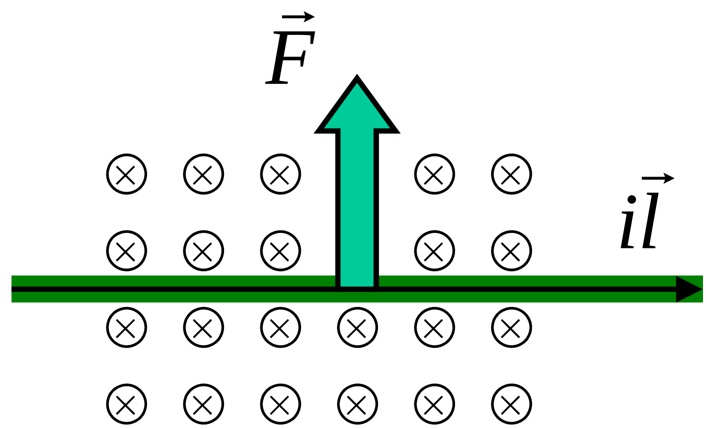
Basic Physics Part I : Lorenz Force

$$\vec{F} = q\vec{V} \times \vec{B}$$

Force on a Charged Particle Moving Through a Magnetic Field



$\vec{B}$ Into Page



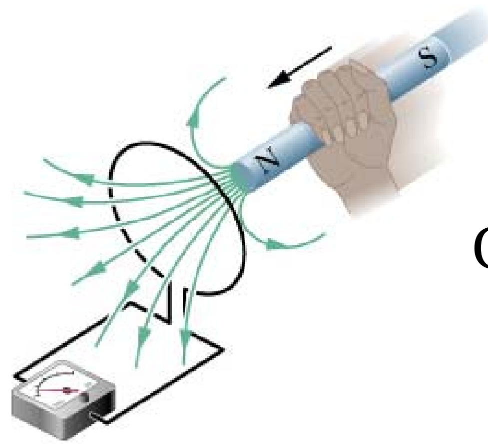
$$\vec{F} = i\vec{l} \times \vec{B}$$

Force on a Wire Carrying Current Through a Magnetic Field

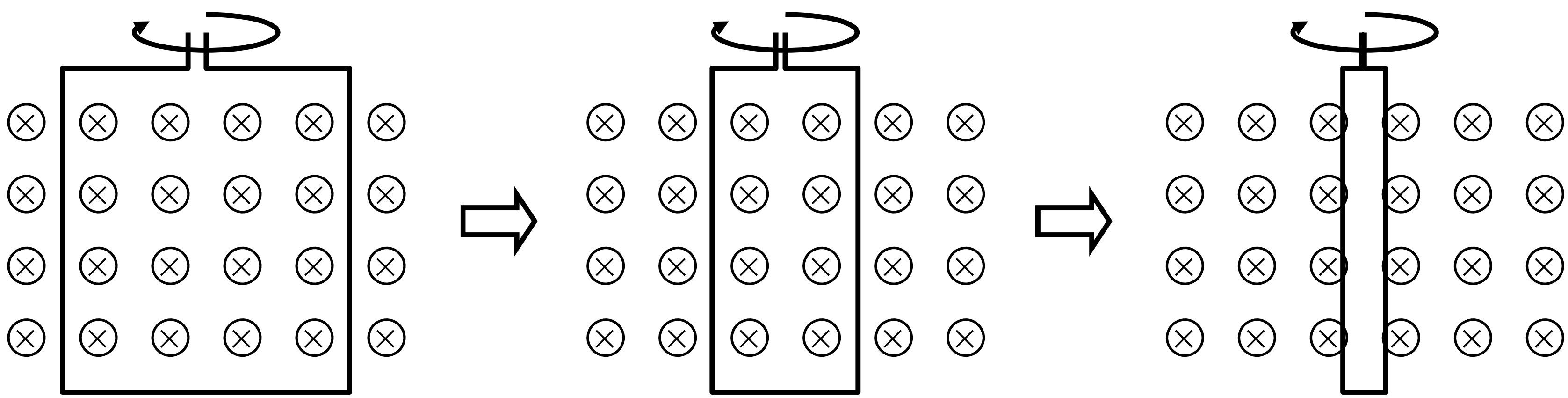
Basic Physics Part II : Faraday's Law

$$\varepsilon = -\frac{d}{dt}\int_A \vec{B}\cdot d\vec{A}$$

Electromotive Force (Voltage)
Generated by Rate of Change of
Magnetic Flux Enclosed by a Coil



Changing $\vec{B}$



$\vec{B}$ Into Page

$\vec{B}\cdot\vec{A}$ Changing

Kirchoff's Second Law

Summation of Voltages
Around a Closed Circuit
Must Equal Zero

$$e = \varepsilon + Ri + L\frac{di}{dt}$$

Voltage Applied	=	Back EMF	+	Coil Resistance	+	Coil Inductance

$$e = K\Omega + Ri + L\frac{di}{dt}$$

Newton's Second Law (Rotational Motion)

J = Rotational Inertia of Motor

$$J\frac{d\Omega}{dt} = -Q_{EXT} + Ki$$

Rate of Change of Angular Momentum	=	Torque Applied by Motor on Whatever Shaft Connected to	+	Torque Due to Motor Current

DC Motor Dynamic Model Summary

Sum Moments on Motor Shaft

$$J\frac{d\Omega}{dt} = -Q_{EXT} + Ki$$

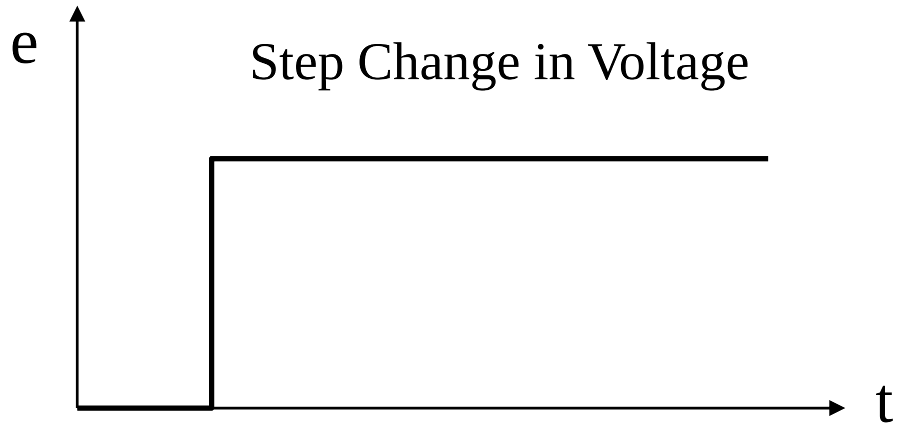
KVL Applied to Motor

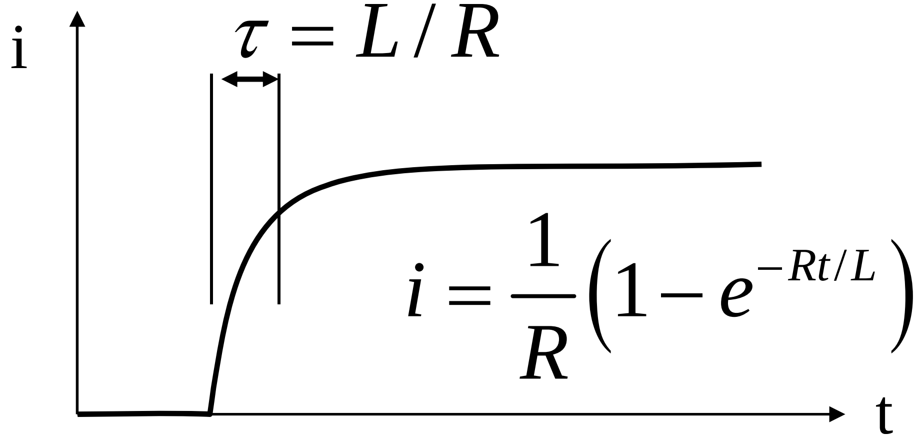
$$K\Omega + Ri + L\frac{di}{dt} = e$$

There is No Single Input-Output Relationship For a DC Motor!

You Have to Write Additional Equations For External Torque & These Depend What Motor Shaft is Connected to!

What Happens if Shaft Fixed (Ω=0)?





$$\tau = L / R$$

$$i = \frac{1}{R}\left(1 - e^{-Rt/L}\right)$$

Motor Current Lags Applied Voltage

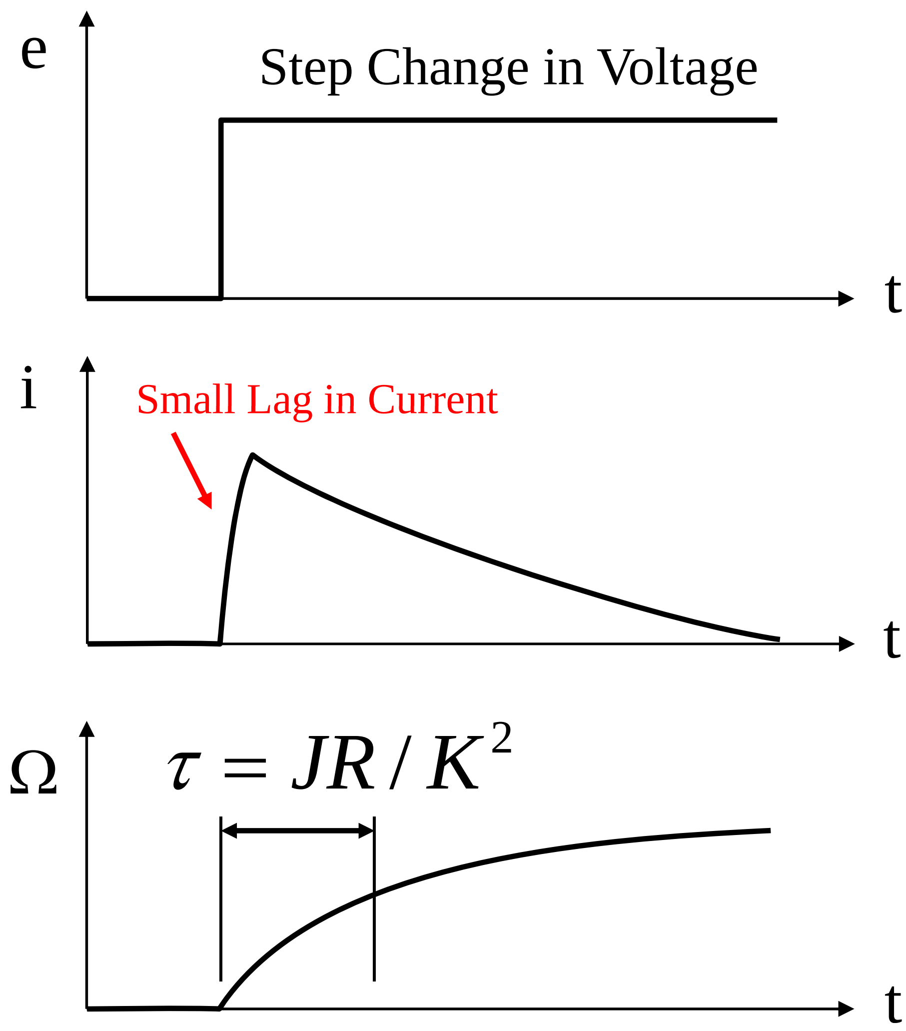
$$L\frac{di}{dt} + Ri = e$$

Time Constant is Typically Small Compared to Other System Dynamics & Often Ignored

Steady-State Response is Torque Proportional to Applied Voltage

$$Q_{EXT} = Ki = \frac{K}{R}e$$

What Happens with No External Torque?



Ignoring Motor Inductance in Equations for Simplicity

$$e = K\Omega + Ri$$

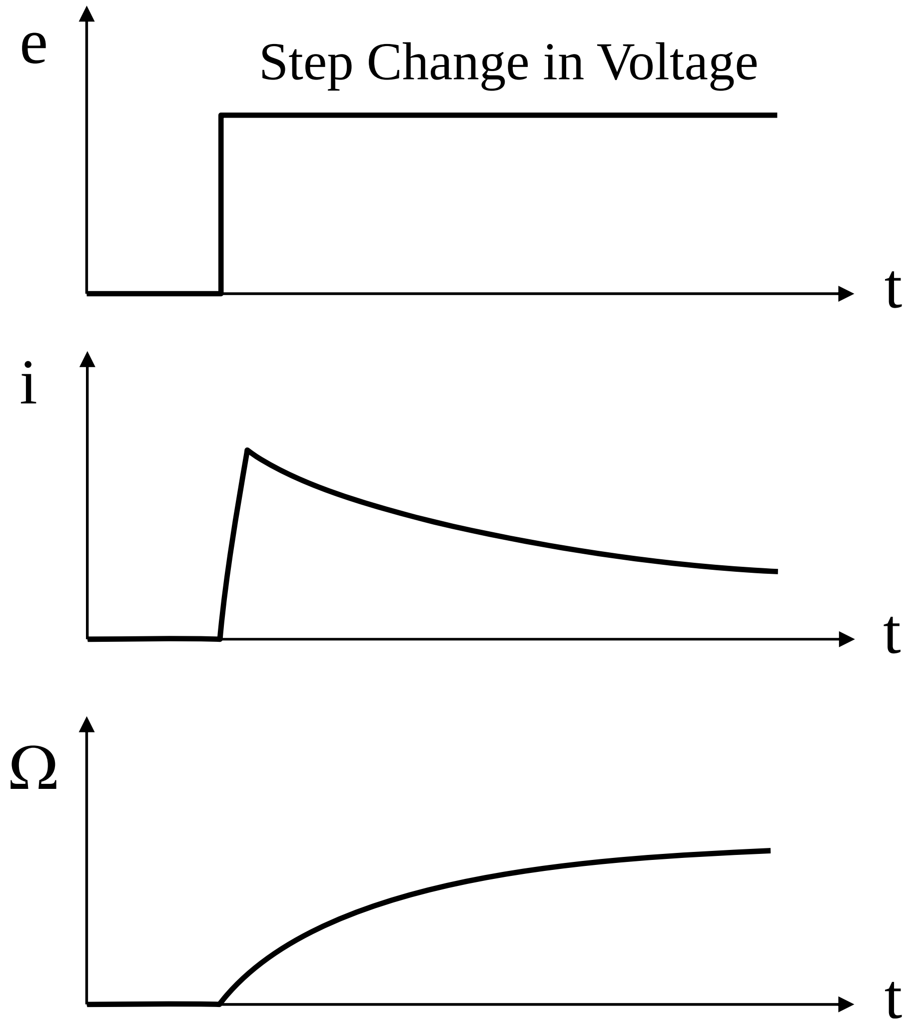
$$J\frac{d\Omega}{dt} = Ki = K\left(\frac{e - K\Omega}{R}\right)$$

Initial Response is Current ➔Torque Which Causes Rotational Acceleration of Motor Inertia

Steady-State Response is Rotational Speed Proportional to Applied Voltage with Zero Current

$$e = K\Omega$$

Common Case is Combination of Last Two…



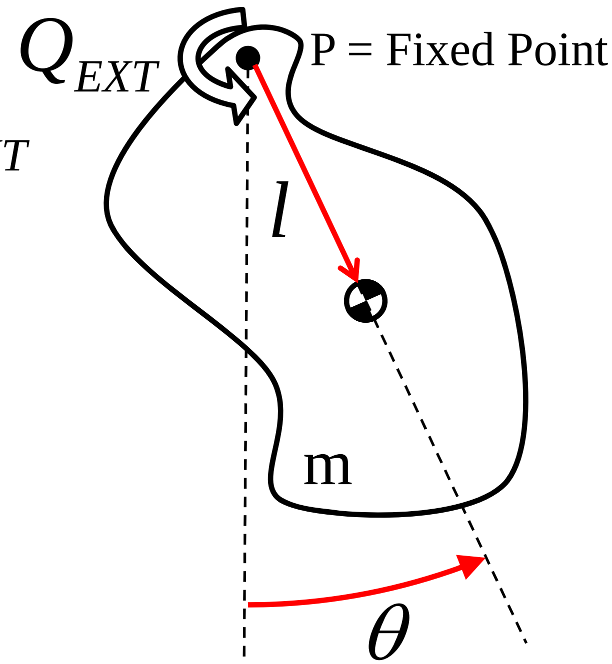
External Torque Includes Both Extra Inertia & Damping

Initial Response is Current ➔ Torque Which Causes Rotational Acceleration of Motor Inertia

Steady-State Response is Combination of Rotational Speed and Non-Zero Current as Required for Steady Torque to Sustain Rotation

Let's Attach DC Motor to Our Pendulum

Sum Moments on Pendulum
$$I_P \frac{d^2\theta}{dt^2} + mgl\sin\theta = Q_{EXT}$$



Sum Moments on Motor Shaft
$$J\frac{d\Omega}{dt} = -Q_{EXT} + Ki$$

KVL Applied to Motor
$$\boxed{K\Omega + Ri + L\frac{di}{dt} = e}$$

Kinematics of Shaft Rotation
$$\boxed{\frac{d\theta}{dt} = \Omega}$$

Combining Two Moment Equations Eliminates Q_{EXT}

$$\boxed{\left(I_P + J\right)\frac{d\Omega}{dt} = Ki - mgl\sin\theta}$$

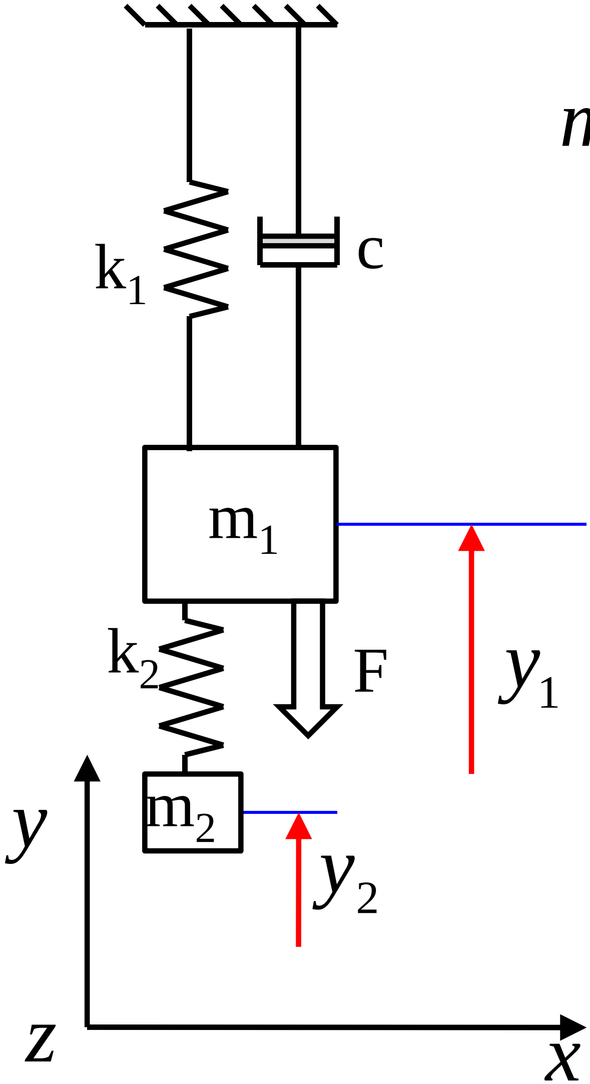
Final Model = System of ODE's

Appendix 1

More Stuff on Mechanical Systems

Vibration Absorber Motivation Video

Multiple-Mass Example : Vibration Absorber



$$m_1 \frac{d^2 y_1}{dt^2} + c\frac{dy_1}{dt} + (k_1 + k_2) y_1 - k_2 y_2 = F$$

$$m_2 \frac{d^2 y_2}{dt^2} + k_2 y_2 - k_2 y_1 = 0$$

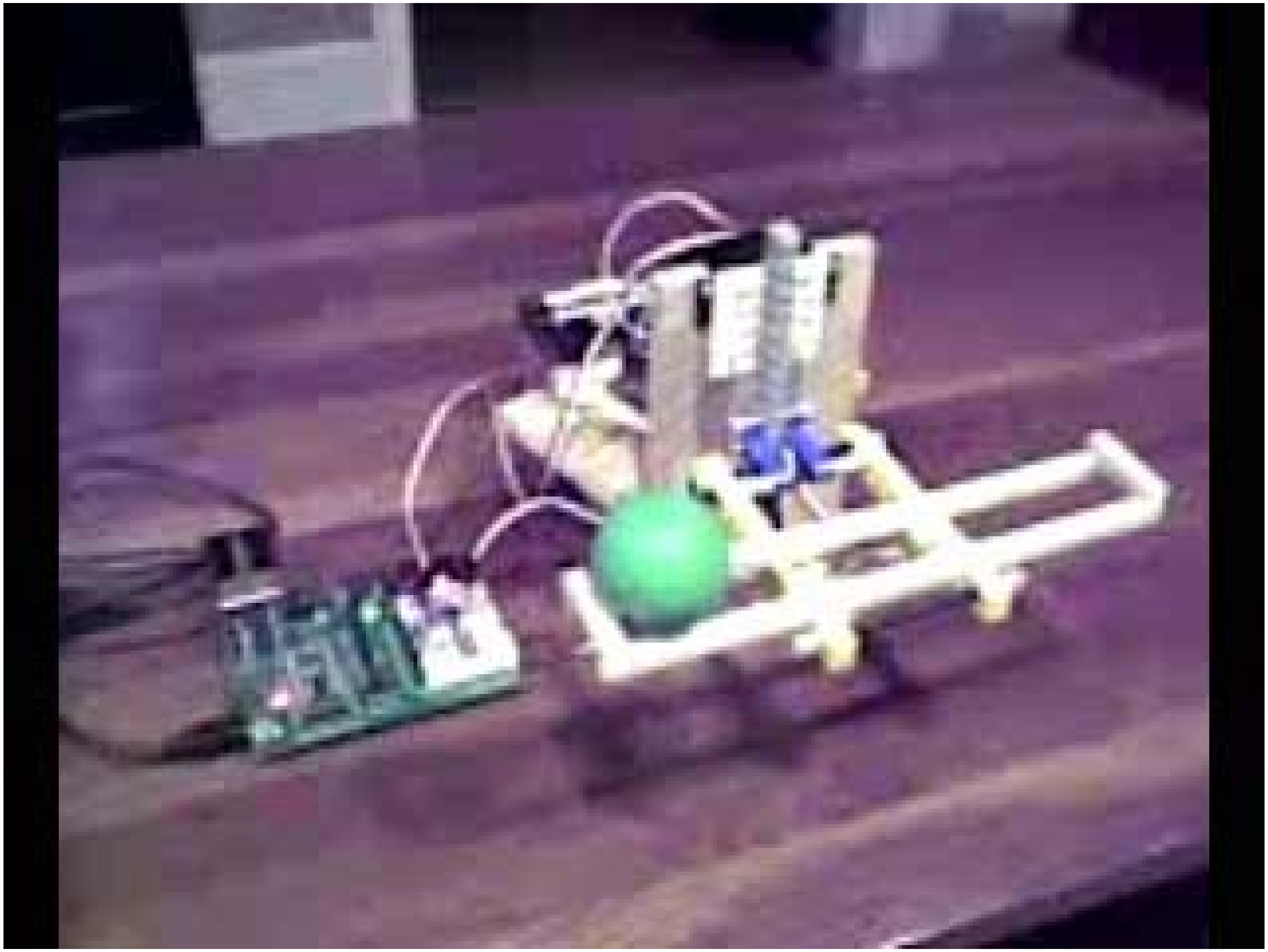
$$F = A\cos\left(\sqrt{\frac{k_2}{m_2}} t\right)$$

$$y_2 = -\frac{A}{k_2}\cos\left(\sqrt{\frac{k_2}{m_2}} t\right)$$

$$y_1 = 0 \;!$$

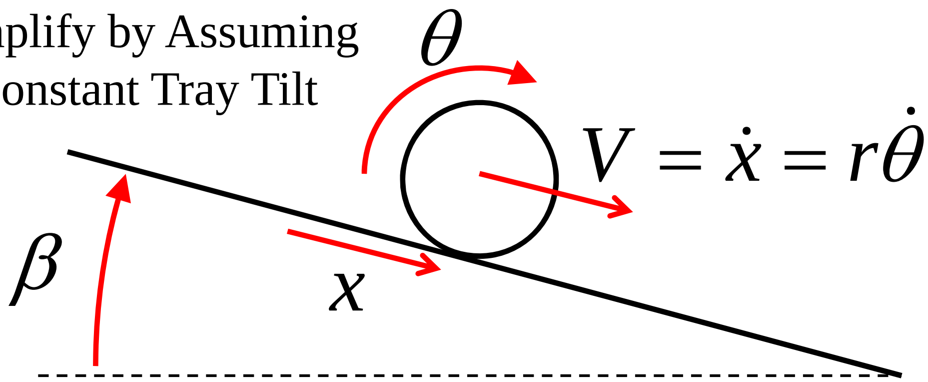
We Will Spend a Lot of Time Understanding Response to Sinusoidal Inputs (After Spring Break)

Recall : Ping Pong Poise

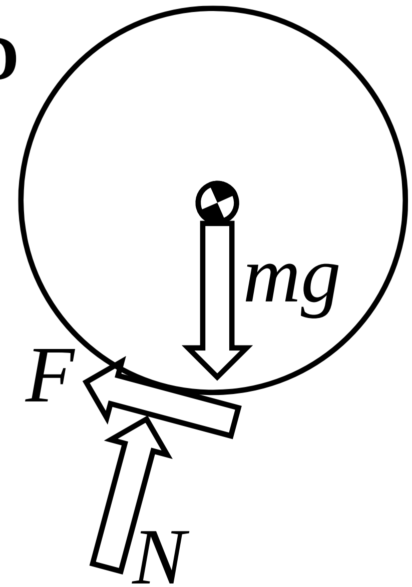


Equations for Ping-Pong Poise

Simplify by Assuming
Constant Tray Tilt



FBD



<u>Forces</u>
(Parallel to Tray)

$$m\ddot{x} = mg\sin\beta - F$$

<u>Moments</u>
(About CG)

$$I\ddot{\theta} = rF \Rightarrow I\ddot{x} = r^2 F$$

Small Angle Approximation on Tray Tilt

$$\boxed{\ddot{x} \approx \frac{g}{\left(1 + \frac{I}{mr^2}\right)}\beta}$$

Hollow Sphere $\dfrac{I}{mr^2} = \dfrac{2}{3}$

We'll Assume This Equation Applies Even When Tray Tilt Angle is Not Constant (We Use Tilt for Control).

Proper Treatment of Tray Tilt Dynamic Effects Gets Complicated…

Ping-Pong Poise & Energy Approach

Total Energy = Kinetic + Potential

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - mgx\sin\beta$$

Conservation of Total Energy

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + I\dot{\theta}\ddot{\theta} - mg\dot{x}\sin\beta = 0$$

Substitute Using Kinematics

$$\frac{dE}{dt} = \dot{x}\underbrace{\left(m\ddot{x} + \frac{I}{r^2}\ddot{x} - mg\sin\beta\right)}_{\textbf{Correct Equation!}} = 0$$

This Approach Eliminated the "Constraint Force" (F)
"Energy Methods" Can Be Very Useful for Complex Problems

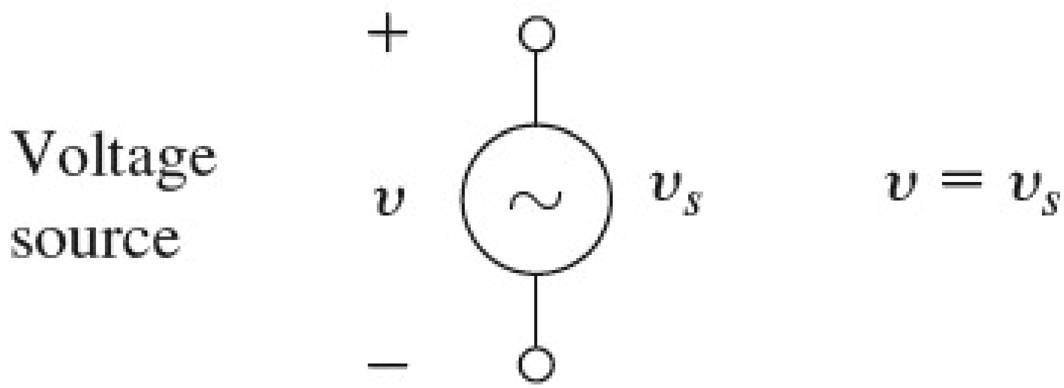
Appendix 2

Electric Circuits & Op-Amps

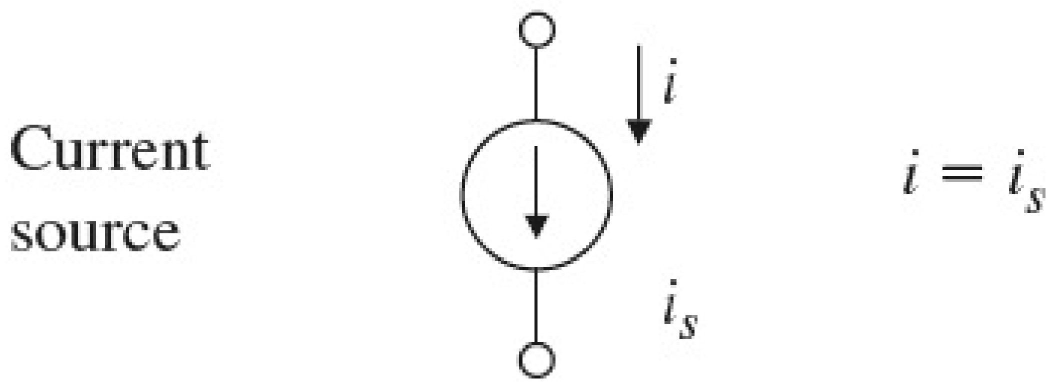
Circuit Fundamentals : Current & Voltage

- i = Electric Current = Flow of Charge
  - Sign Convention : Positive Current Opposite to Direction Electrons Actually Moving!
- v = Voltage = Potential Energy per Unit Charge
  - Also Sometimes Called “electromotive force” But Some People are Fussy About the Distinction
- Note: P = v*i = Power
- Electrical Circuit = Network of Nodes, Each Having a Voltage Relative to Some Reference Node (Typically “Ground”), Connected by Circuit Elements Through Which Current Flows.
- Circuit Element Analysis = Writing Equations Relating Voltage at Nodes to Current in Connections

Voltage & Current Sources (Power Supplies)



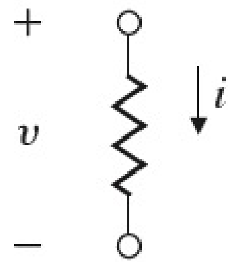
Current = Whatever is Required to Maintain Specified Voltage (In Practice, There are Finite Current Limits)



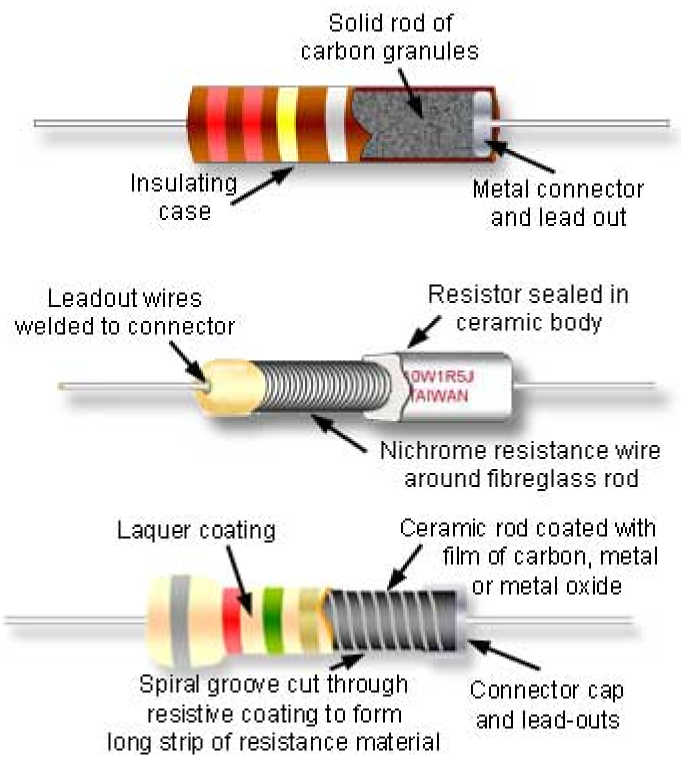
Voltage = Whatever is Required to Maintain Specified Current (In Practice, There are Finite Voltage Limits)

Resistor

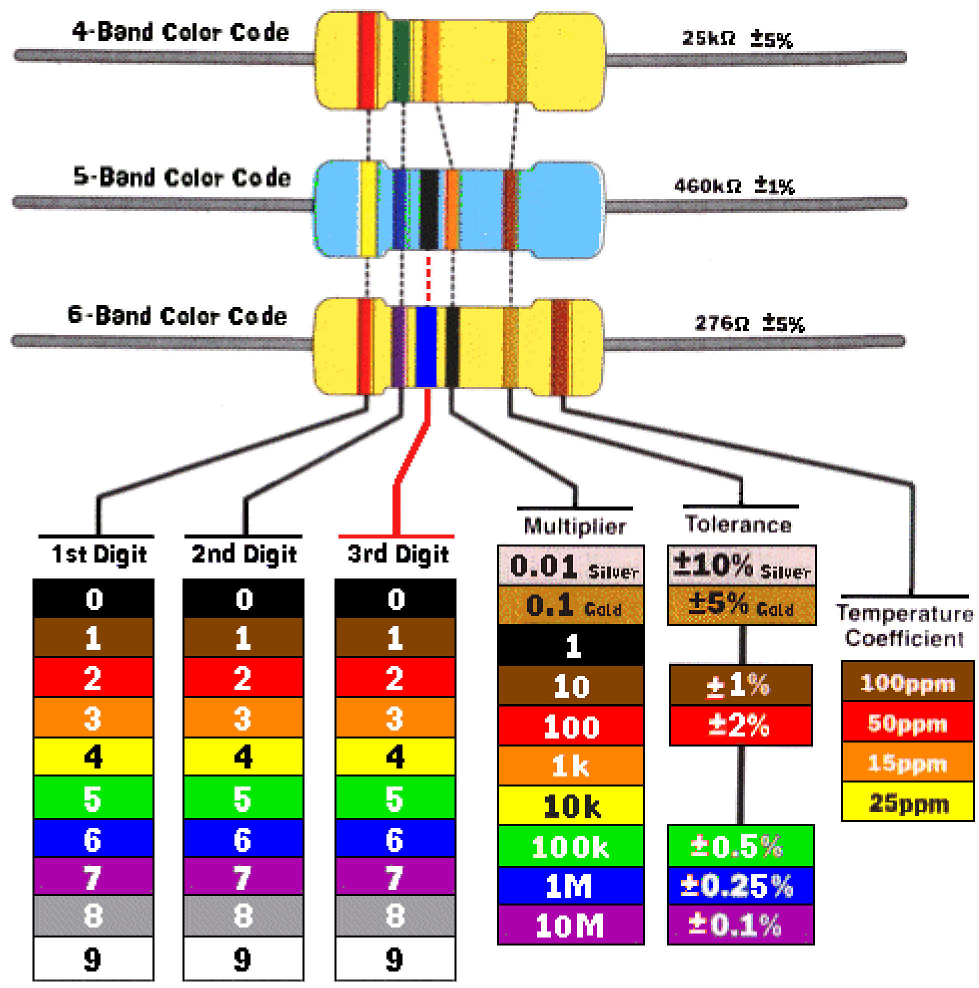
Resistor



$$v = Ri$$



http://www.learnabout-electronics.org/resistors_08.php

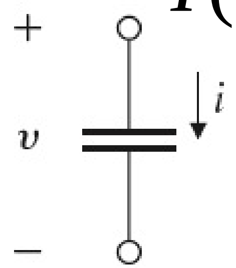


http://www.williamson-labs.com/images/resistor-color-code-all.gif

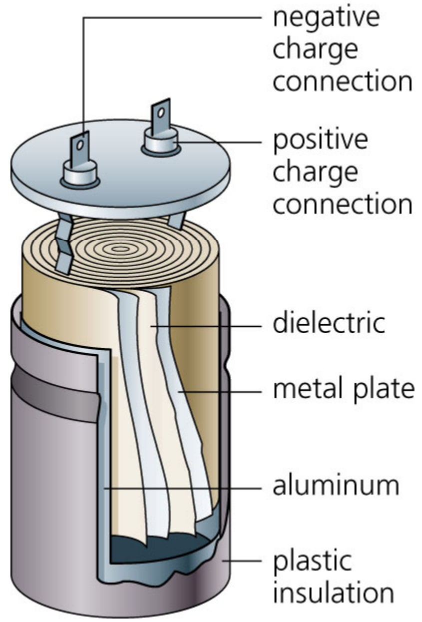
Capacitor

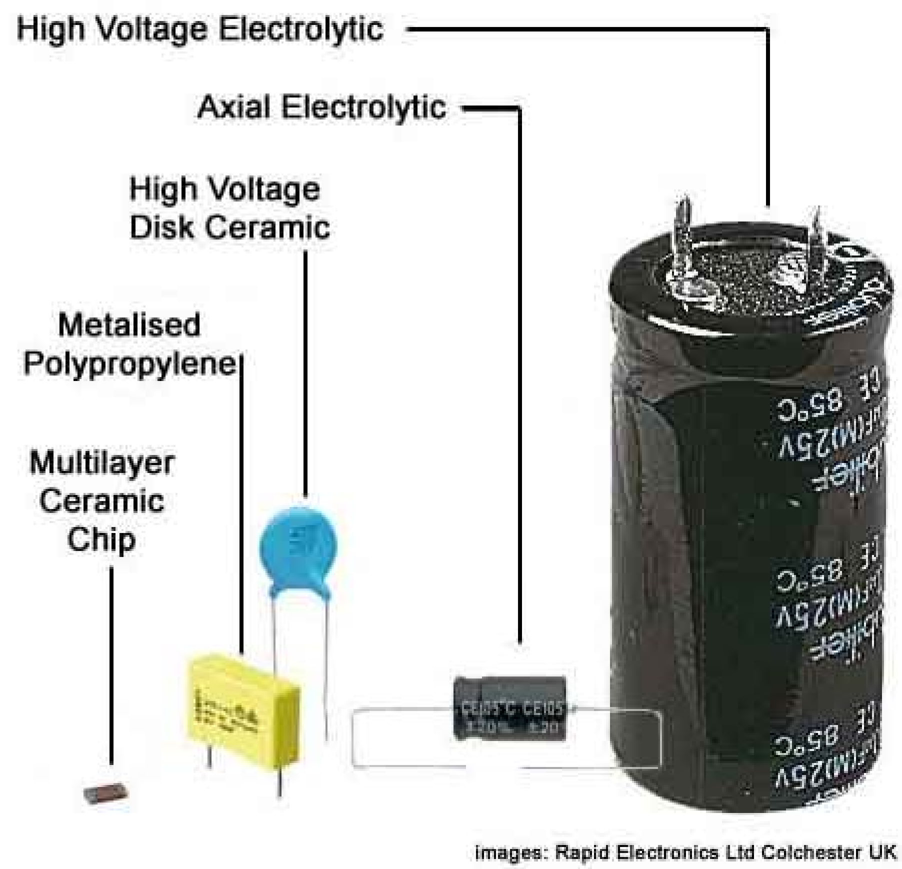
$$I(s) = CsV(s)$$

Capacitor



$$i = C\frac{dv}{dt}$$



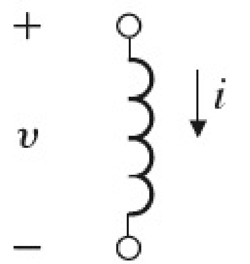


http://www.digitivity.com/articles/2008/11/choosing-the-right-capacitor.html

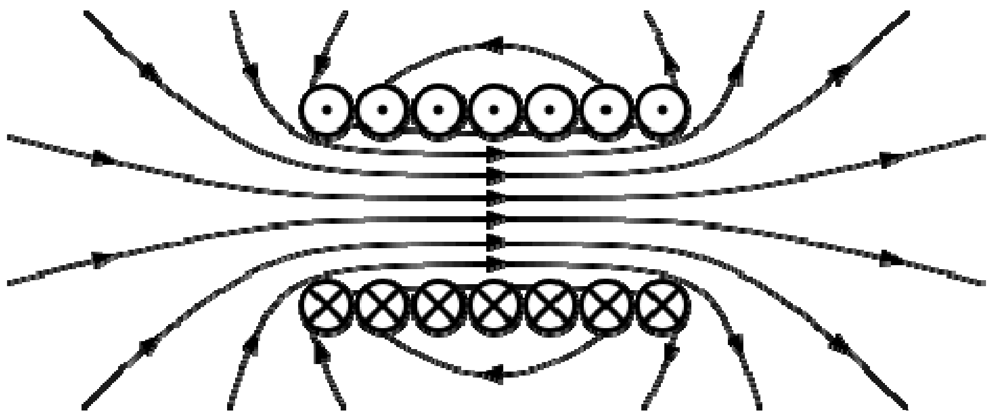
http://www.learnabout-electronics.org/ac_theory/capacitors01.php

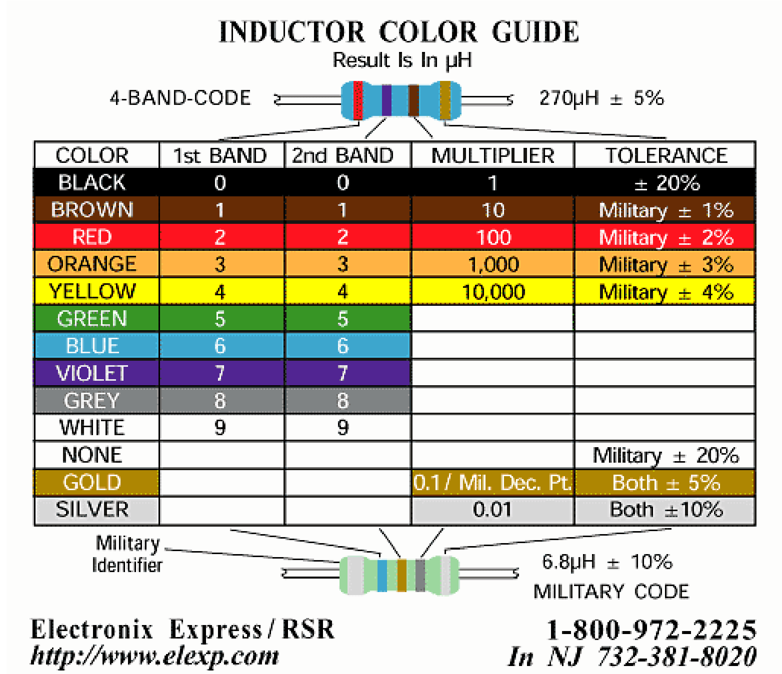
Inductor

$$V(s) = LsI(s)$$

Inductor 

$$v = L\frac{di}{dt}$$





COLOR	1st BAND	2nd BAND	MULTIPLIER	TOLERANCE
BLACK	0	0	1	± 20%
BROWN	1	1	10	Military ± 1%
RED	2	2	100	Military ± 2%
ORANGE	3	3	1,000	Military ± 3%
YELLOW	4	4	10,000	Military ± 4%
GREEN	5	5		
BLUE	6	6		
VIOLET	7	7		
GREY	8	8		
WHITE	9	9		
NONE				Military ± 20%
GOLD			0.1 / Mil. Dec. Pt.	Both ± 5%
SILVER			0.01	Both ±10%

Analyzing a Circuit

- KCL = Kirchoff's Current Law

  Sum of Currents Leaving a Node = Sum of Currents Entering the Node (Charge Does Not Accumulate At a Node)

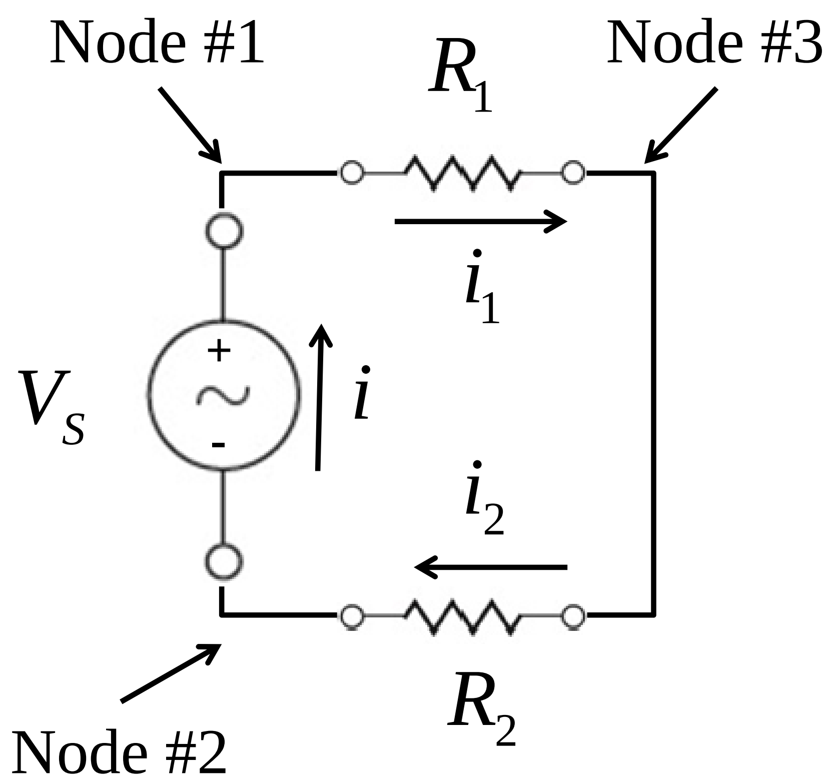
- KVL = Kirchoff's Voltge Law

  Sum of Voltage Changes Around a Closed Path in a Circuit is Zero

  ...OR...

  Voltage Change Between 2 Nodes Independent of Choice of Path in a Circuit

Resistors in Series



KVL Around Loop

$$V_s - i_1 R_1 - i_2 R_2 = 0$$

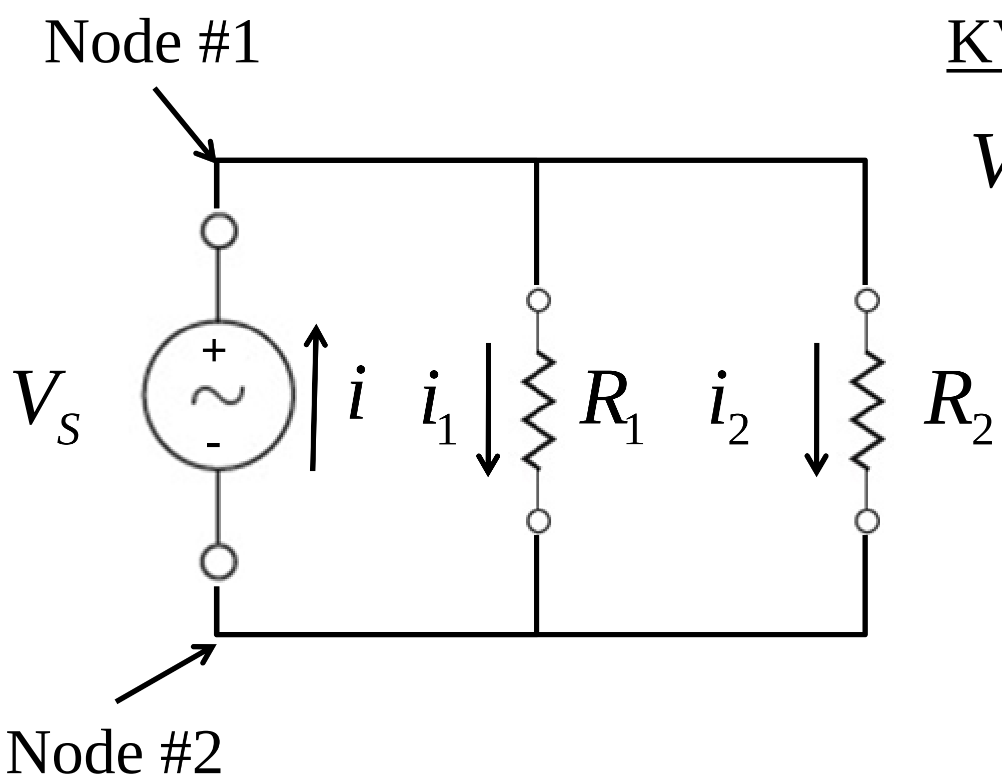
KCL @ #1 , #2, #3

$$i = i_1 = i_2 = i$$

$$V_s = i\left(R_1 + R_2\right)$$

Effective Resistance of Many Resistors in Series = Sum of Resistances

Resistors in Parallel



<u>KVL Between #1 & #2</u>

$$V_1 - V_2 = i_2 R_2 = i_1 R_1 = V_s$$

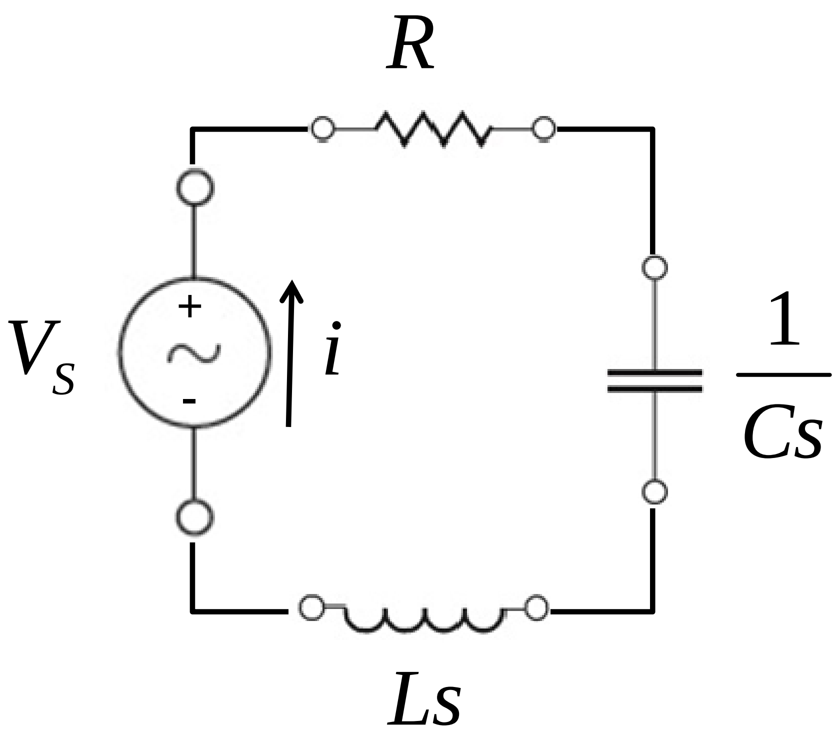
<u>KCL @ #1 or #2</u>

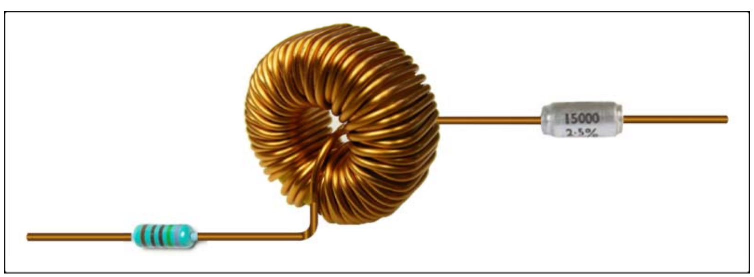
$$i = i_1 + i_2$$

$$i = V_s \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

Effective Resistance of Many Resistors in Parallel = Inverse of (Sum of Inverses)

Effective Resistance of Capacitors & Inductors





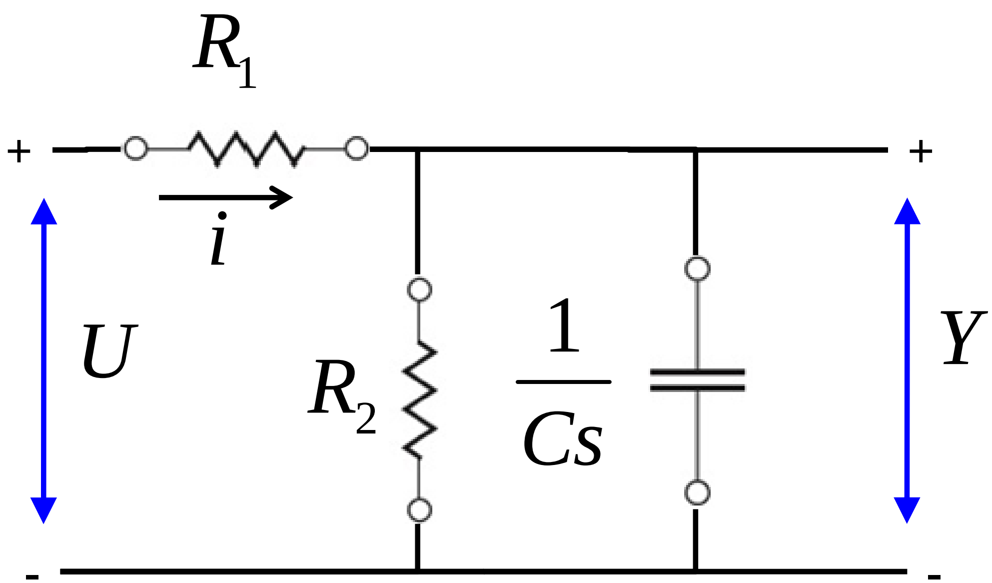
Treat Capacitor & Inductor as Equivalent Resistors in "s" Domain

$$V_s = i\left(R + \frac{1}{Cs} + Ls\right)$$

Use Inverse Laplace to Get EOM (if Desired / Needed)

$$V_s = Ri + C\int idt + L\frac{di}{dt}$$

Circuits = Filters (Dynamic Compensators)



$$U - Y = iR_1$$

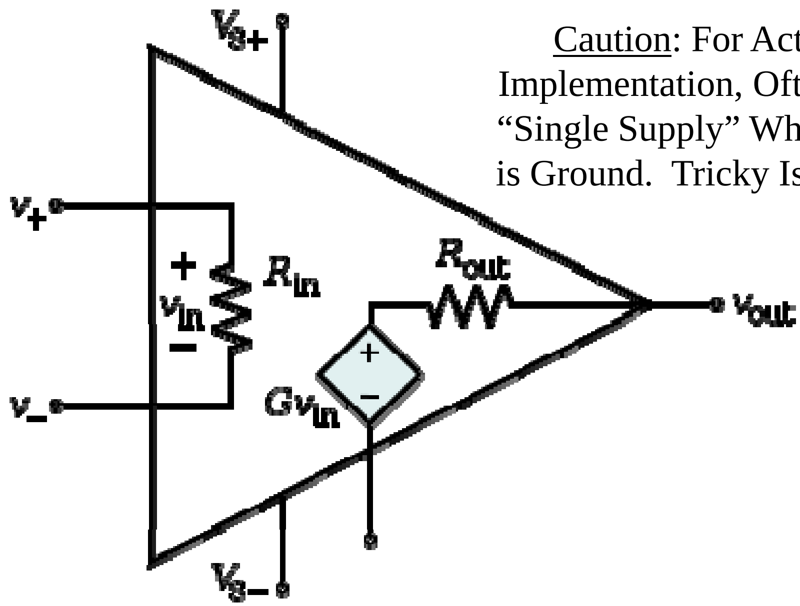
$$i = Y\left(\frac{1}{R_2} + Cs\right)$$

$$\frac{U-Y}{R_1} = Y\left(\frac{1}{R_2} + Cs\right)$$

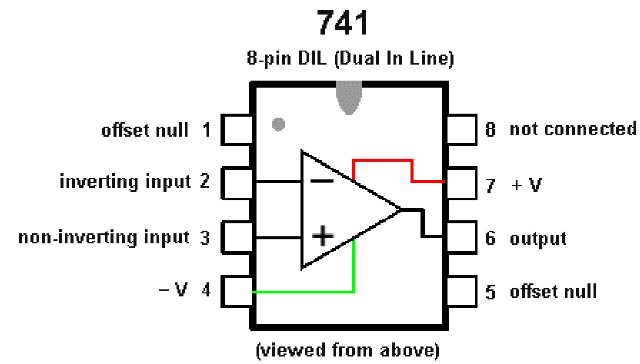
$$\boxed{\frac{Y}{U} = \frac{R_2}{R_1 + R_2 + R_1R_2Cs}} \Leftarrow U = Y\left[1 + R_1\left(\frac{1}{R_2} + Cs\right)\right]$$

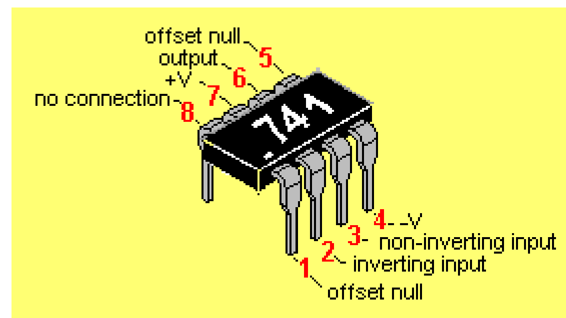
Operational Amplifier

http://en.wikipedia.org/wiki/File:Op-Amp_Internal.svg http://talkingelectronics.com/projects/OP-AMP/OP-AMP-1.html



<u>Caution</u>: For Actual Implementation, Often Use "Single Supply" Where V_{S-} is Ground. Tricky Issues…



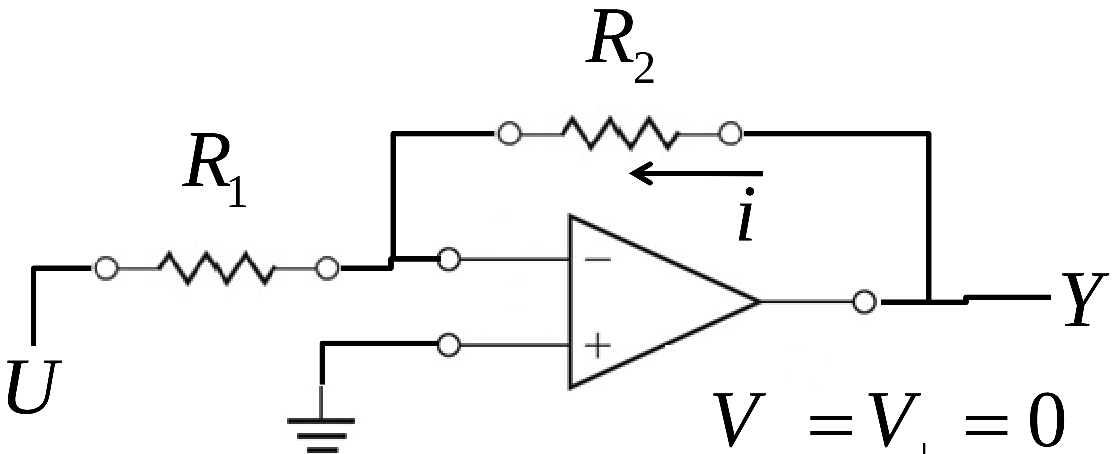


Simplified Analysis:

$$G \to \infty \qquad R_{IN} \to \infty \qquad R_{OUT} \to 0$$

$$\Rightarrow V_+ = V_- \qquad \Rightarrow i_+ = i_- = 0 \qquad \Rightarrow V_{out} \equiv GV_{in}$$

Op-Amp : Inverting Amplifier

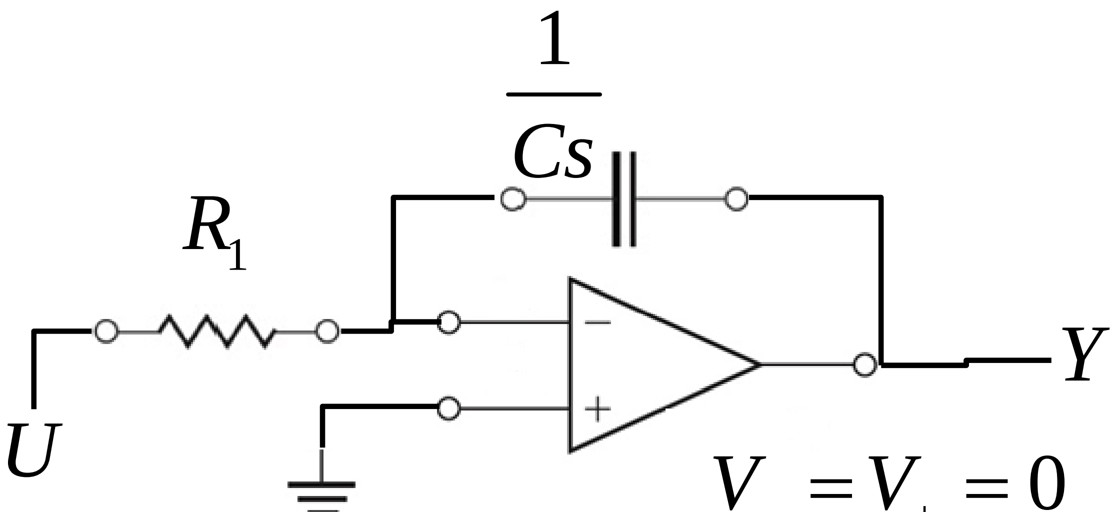


$$V_- = V_+ = 0$$

$$Y - U = i\left(R_2 + R_1\right)$$

$$Y - V_- = iR_2$$

$$\boxed{\frac{Y}{U} = -\frac{R_2}{R_1}}$$

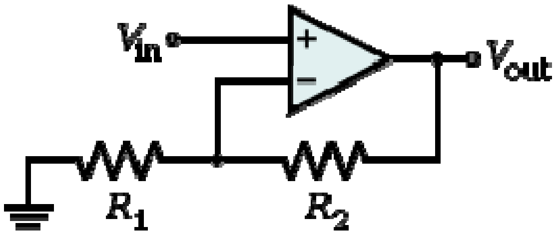


$$V_- = V_+ = 0$$

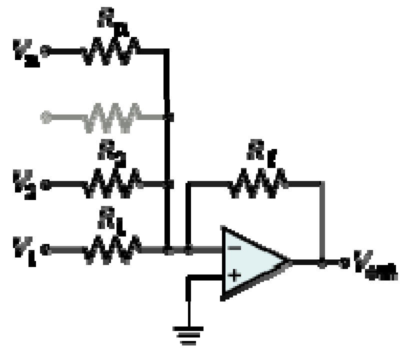
$$\boxed{\frac{V_{out}}{V_{in}} = -\left(\frac{1}{R_1 Cs}\right)}$$

Lots of Other Applications

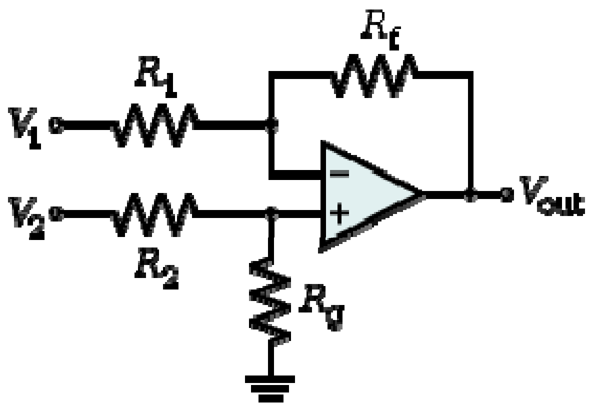
- Non-Inverting Amplifier



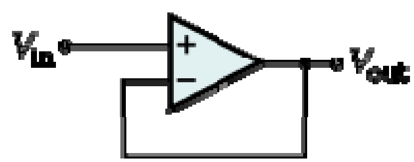
- Summing Amplifier



- Differential Amplifier



- Voltage Follower (Buffer)



http://en.wikipedia.org/wiki/Operational_amplifier_applications