ESE 406 - SPRING 2011
HOMEWORK #5
DUE 23-Feb-2011 (Monday, 28-Feb with late pass)

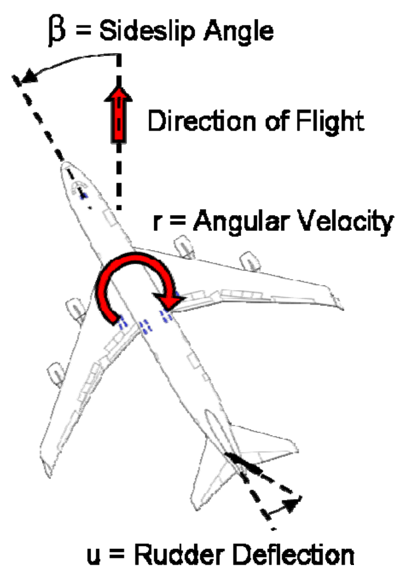
Problem 1 You should look over problems 3.33 through 3.40, as well as Example 3.26. Submit your solution (a hand sketch) to problem 3.35. If you don't have graph paper, you should "make" some by printing the figure you get from the following commands in Matlab:

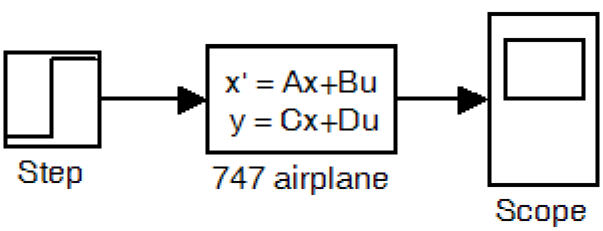
```
step(tf(1,1)); grid on; axis([0 4 0 2]);
```

Problem 2 In this problem, we will study the development of a yaw damper for the 747 aircraft in high-altitude, high-speed cruise[1]. By walking through a real design problem in considerable detail, you will see how the vocabulary and analytical tools you have learned so far enable you to understand almost all of the key considerations in the design. The following specify the state-space linearized model of the dynamics:

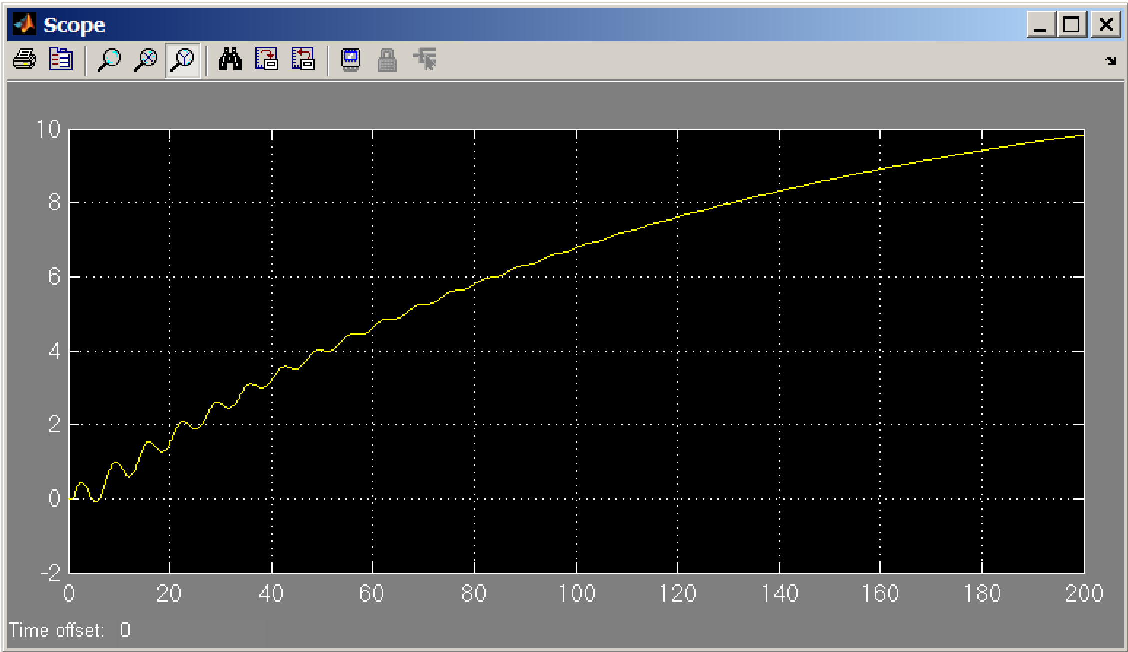
$$A=\begin{bmatrix} -0.05 & -1 & 0 & 0.04 \\ 0.84 & -0.15 & -0.01 & 0 \\ -3.00 & 0.41 & -0.43 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B=\begin{bmatrix} -0.01 \\ 0.46 \\ -0.11 \\ 0 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad D=\begin{bmatrix} 0 \end{bmatrix}$$

The control input is the deflection of the rudder, which is a small flap at the back of the vertical tail. The state vector is $\underline{x}=\begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}$, where β is the sideslip angle, r is yaw rate, p is roll rate, and ϕ is roll angle. The scalar output is the yaw rate. Let's set up a SIMULINK model and see what the step response looks like (the "747 airplane" is a "state space" block from the "Continuous" library; the "step" is in the "sources" library; the "scope" is in the "sinks" library):





[1] Etkin and Reid, *Dynamics of Flight*, 3rd Edition, 1996. Numbers simplified slightly from original values. This system was also discussed at the end of the lecture on higher-order systems, but the numerical values are slightly different.



We know our 4-state system has to have 4 poles. In this response, it looks like we can see 3 poles: one very slow first-order mode and one very lightly damped oscillatory mode (2 poles, called the "dutch roll") with a period of about 6 or 7 seconds (just count cycles between 20-second grid intervals). Evidently, the 4th pole has a nearby zero in the yaw-rate transfer function, making it difficult to see in the response. While you aren't expected to have an intuition for the magnitudes or character of the response for this complicated example, you should recognize that it is not a simple second-order system. Use the results shown above to confirm that your SIMULINK model is implemented correctly.

Virtually all airplanes and helicopters have modes that are qualitatively very much like those seen here. The very slow mode, called the "spiral mode" turns out to be not very important to the pilot—it is quite easy for her to compensate for that one herself[2]. But the oscillatory mode is very objectionable and would create serious ride comfort issues for passengers, as well as problems with control in crosswinds. As a result, perhaps the highest priority element of a "stability and control augmentation system" (SCAS) is a "yaw damper" that will improve the damping of this "dutch roll" mode.

Before we dive into a detailed design, let's get smarter about the nature of this "dutch roll" mode. If we ignore roll completely[3], we can write two *approximate* equations for sideslip and yaw rate:

$$\frac{d\beta}{dt} = -0.05\beta - r - 0.01u$$
$$\frac{dr}{dt} = 0.84\beta - 0.15r + 0.46u$$

Use Laplace transforms on these equations to find the transfer function $\frac{R(s)}{U(s)}$ from rudder inputs to the approximate yaw rate response. You should have a second-order denominator. Double check that the damped frequency is close to what you expect (from the response of the full model shown above) and that the damping ratio is low.

---

[2] The pilot can compensate for the slow "spiral" mode when her attention is properly focused on flying the airplane. This mode can be dangerous if the pilot is distracted or flying in conditions with poor visibility. An inadvertent rudder input was evidently the primary cause of the crash of the private airplane being flown by singer John Denver that resulted in his death.

[3] This is a somewhat ironic choice, given that the mode is called the dutch *roll*, but it turns out we get a qualitatively useful approximation to the characteristics of the mode from this simple approximation. We'll do our final design using the complete system.

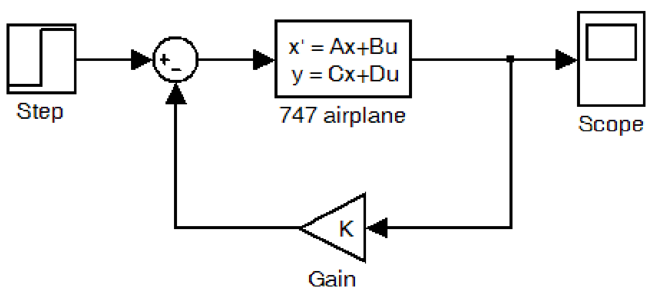
Notice that there is a zero (a non-zero "s term") in the numerator of $\dfrac{R(s)}{U(s)}$. From our study of the initial-value theorem, you know that this means there will be a non-zero initial slope in the yaw rate response to a step input from the rudder[4]. Looking at the approximate equation above, we expect this initial slope to be 0.46 for a unit step. Use the initial value theorem, combined with your transfer function, to confirm this value. You may double-check by using Matlab to compute the step response if you like. You don't have to submit anything for this part, but you should be sure you understand what is going on here.

The simplest yaw damper uses a measurement of the yaw rate (using a device called a "rate gyro" which is inexpensive and quite reliable) to generate a proportional rudder command in the opposite direction, which is summed with the pilot input, $u = u_{PILOT} - Kr$. Find the closed-loop transfer function $\dfrac{R(s)}{U_{PILOT}(s)}$, again using the approximate dynamics written above, but this time with feedback. The poles of this transfer function depend on K.

<u>SUBMIT</u>*: *Make a pretty graph showing how the poles move in the complex plane as we vary K. That is, plot 2 points (the pole locations) in the complex plane for each value of K. Choose a range of K that will take the poles from the open-loop values to the point of critical damping (unity damping ratio). Be sure to choose equal scaling for the real and imaginary axes, so that a curve of constant natural frequency will be a circle on your graph. What value of K will give a closed-loop damping ratio of 0.3? How about 0.5?

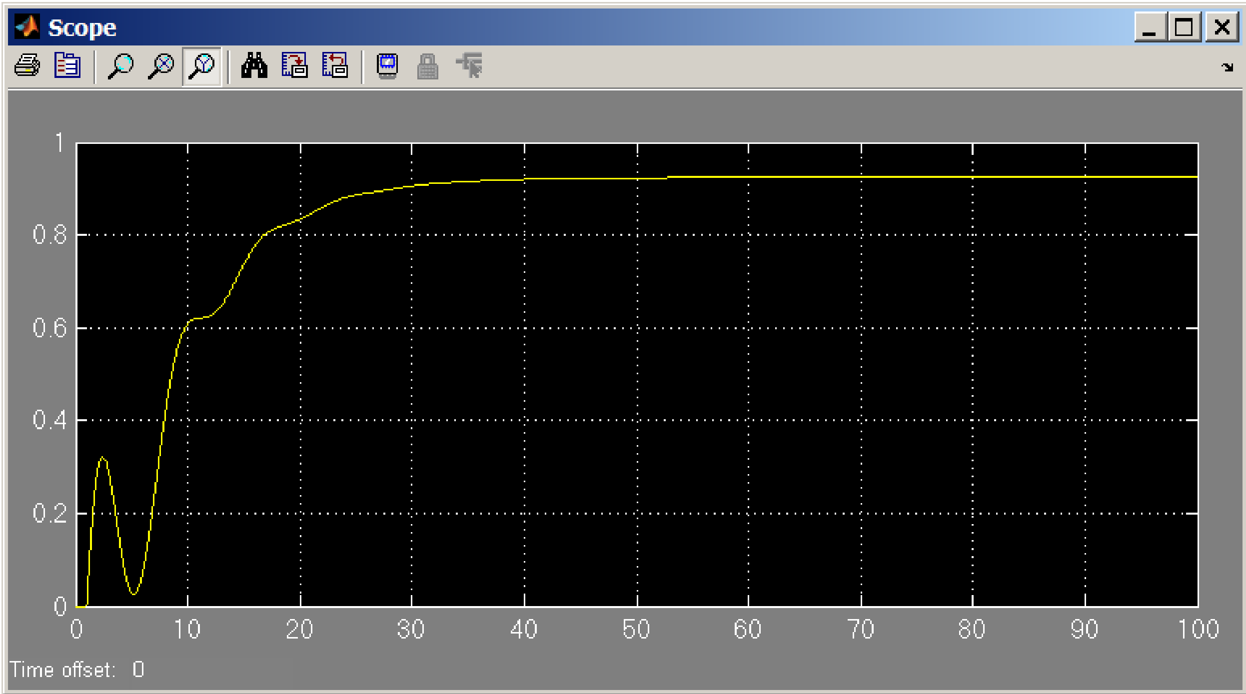
The figure that you made showing the effects of a feedback gain on the closed-loop pole locations is called a "root locus" plot. In the second half of the course, we will learn how to do these plots very easily and systematically (as discussed in Chapter 5 of the textbook).

Now that we have some reason to think that yaw rate feedback can dramatically increase the damping of the dutch roll mode, we want to do some development on the complete system. Let's go back to our SIMULINK model and add the feedback path:



And now the step response looks something like this (depending on the value of K)

[4] It makes good physical sense that a rudder deflection will result in a non-zero yaw acceleration. The rudder applies an aerodynamic moment to the airplane, so the response is a corresponding angular acceleration.



Note that the steady-state response to the step input is much smaller than it was before. This isn't good, because it suggests that our damper is taking away the pilot's steady-state control authority. Also, when the pilot wants the airplane to be turning, our yaw rate feedback will be force the pilot to hold a large input, which will be uncomfortable. (The pilot controls the rudder with pedals at her feet; the forces required to move the pedals can be quite large, by design, to prevent unintended inputs.) We need to do something other than just proportional feedback.

Solving this problem is straight-forward, as we discussed in class: we need a high-pass, or washout, filter. It turns out that we also want to use a low-pass filter so that high-frequency noise and aircraft structural vibrations[5] don't get into the feedback signal. Therefore, the actual 747 SCAS used a "yaw damper" that looked like this (once we include dynamics in our feedback, it is much easier to describe the design using transfer functions):

$$U(s) = U_{PILOT}(s) - KG_F(s)R(s)$$

where the filter transfer function is

$$G_F(s) = \frac{\tau_1 s}{\tau_1 s + 1}\frac{1}{\tau_2 s + 1}$$

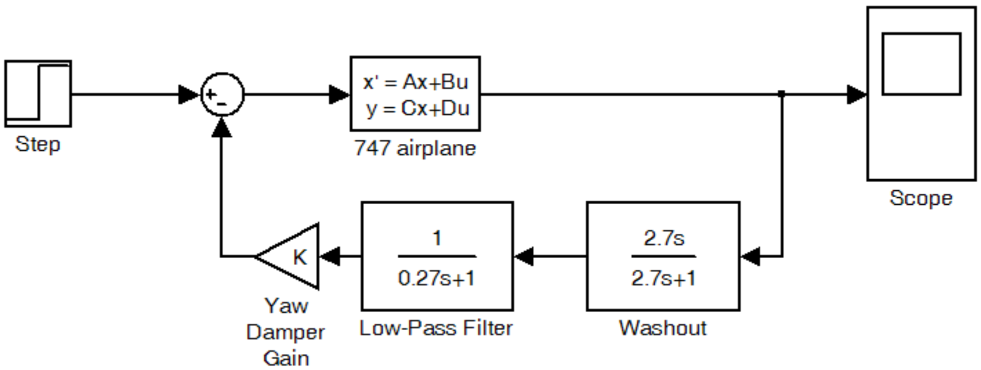
In the filter, the washout time constant, τ_1 , is 2.7 seconds and the structural filter time constant, τ_2 is 0.27 seconds. To better understand the effects of this filter, we can use the frequency response analysis from this week's lecture. In particular, let's look at the magnitude of the filter as a function of the frequency of a sinusoidal input.

<u>SUBMIT</u>: Make a pretty graph showing the magnitude of the filter as a function of the frequency of the input. Note that when we write u(t) = sin(ωt), ω has units of rad/sec, so this should be the label on the x-axis. Based on the time response of the original open-loop, place a vertical line on your graph, showing the location of the dutch-roll mode. (Be careful converting the period of the oscillation to frequency in rad/sec.) Do you see why this filter is called a "bandpass filter"? Note that it won't significantly change the effect of feedback on the dutch-roll mode, because that mode is near the center of the "pass band" of the filter.

In SIMULINK, our system now looks like this (the filters use the "transfer function" block from the "continuous" library):

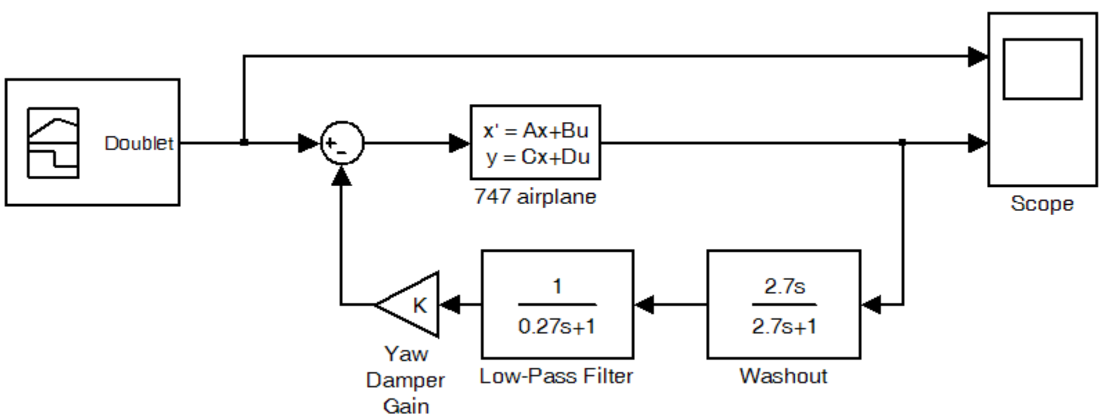
---

[5] Structural vibrations have lower frequencies on bigger airplanes. On a very large airplane, the fuselage might have bending modes in the range of 1 to 2 Hz, so it might be easy for the pilot or the feedback to excite these modes.

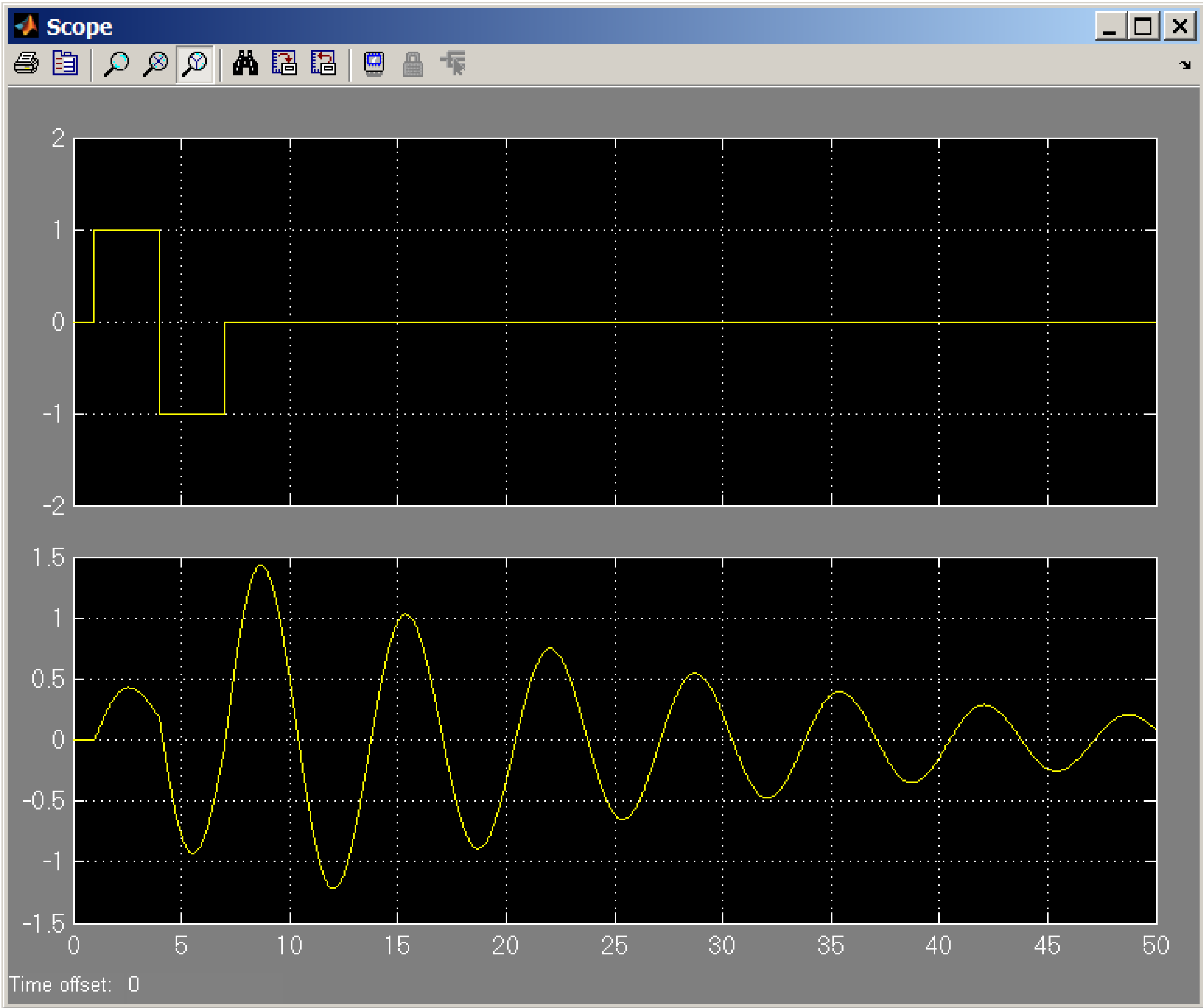


You should look at the step response and confirm that we have solved the steady-state response problem—we get the same steady-state response that we got with the open-loop system. Our yaw rate feedback certainly improved the damping of the mode. But it is difficult to quantify how much, because of the slow first-order response of the spiral mode that is super-imposed. We could update our state-space equations to include the effects of feedback, but for now let's pretend that we are doing flight tests, and we need to confirm the increased damping from our "test data." If we want to get a quantitative measure of dutch roll damping, we are going to have to do something more clever with the input than a step

We need to figure out how to excite primarily the dutch roll mode, without exciting the slower mode, so that we can get accurate quantitative estimates of how the pole is moving as we vary the gain. Our closed-loop system can be written as $R(s) = H(s)U_{PILOT}(s)$. $H(s)$ will now have 6 poles, because we have added two poles in the feedback path. For any $U_{PILOT}(s)$, we could do a partial-fraction expansion on $R(s)$ and we would find terms for each mode in $H(s)$. We want to pick a $U_{PILOT}(s)$ that will be very large when s is equal to the dutch roll root, but small at the small value of s that corresponds to the slow mode. A typical input to choose is called a "doublet", which we can build with the "signal builder" element in the SIMULINK "sources" library. A doublet is basically a squared-off, single cycle sine wave. Our system now looks like this:



For a small value of the feedback gain, the input and the system response looks like this:



Notice that the doublet input has excited a strong dutch roll response, but almost none of the slow spiral mode. Now we just have to figure out how to get an accurate quantitative measure of frequency and damping ratio from this time response.

Determining the damped natural frequency, ω_d, is very straight-forward: $\omega_d = \frac{2\pi}{T}$, where T is the period of the oscillation. Of course, we want to measure the period after the doublet has ended, as that best reflects the system modes. The damping ratio can be found from a technique called "logarithmic decrement" which is described on Wikipedia as follows[6]:

[6] The math behind the logarithmic decrement is the subject of problem 3.32, on page 163, of Franklin, et al. You are encouraged to have a look at that problem and reconcile the math with the equations given here. Searching for "logarithmic decrement" on google yields thousands of hits, too.

Logarithmic decrement, δ, is used to find the damping ratio of an underdamped system in the time domain. The logarithmic decrement is the natural log of the amplitudes of any two peaks:

$$\delta = \frac{1}{n} \ln \frac{x_0}{x_n},$$

where x_0 is the greater of the two amplitudes and x_n is the amplitude of a peak n periods away. The damping ratio is then found from the logarithmic decrement:

$$\zeta = \frac{1}{\sqrt{1 + (\frac{2\pi}{\delta})^2}}.$$

The **damping ratio** can then be used to find the undamped natural frequency ω_n of vibration of the system from the damped natural frequency ω_d:

$$\omega_d = \frac{2\pi}{T},$$
$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}},$$

where T, the period of the waveform, is the time between two successive amplitude peaks.

Applying this technique to the data shown above, taking the first peak to be slightly larger than 1.0 at about 15.5 seconds (after the input has ended), the amplitude 4 peaks later, at about 42.5 seconds, is about 0.25. Thus, we have

$$\delta = \frac{1}{4} \ln \frac{1}{0.25} = 0.3466$$

$$\varsigma = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{.3466}\right)^2}} = 0.055$$

$$\omega_n = \frac{2\pi}{T\sqrt{1 - \varsigma^2}} = 0.932 rps$$

Natural frequency of 0.932 rps and damping ratio of 0.055 correspond to a pole at $-0.051 \pm 0.931 j$. The exact answer for this configuration is $-0.047 \pm 0.941 j$. You should make more accurate measurements of the peak amplitudes and the modal period.

<u>SUBMIT</u>**:** ***As you did for the simple analytical model, make a pretty plot (a "root locus") showing how the dutch roll mode moves in the complex plane as the gain K is increased away from zero (open-loop) to at least 3.0. The actual value of the gain on the 747 yaw damper is about 1.4. Do you see why? What happens at large values of the gain?***