Nonlinear Systems
Lyapunov Theory

ESE 505 & MEAM 513

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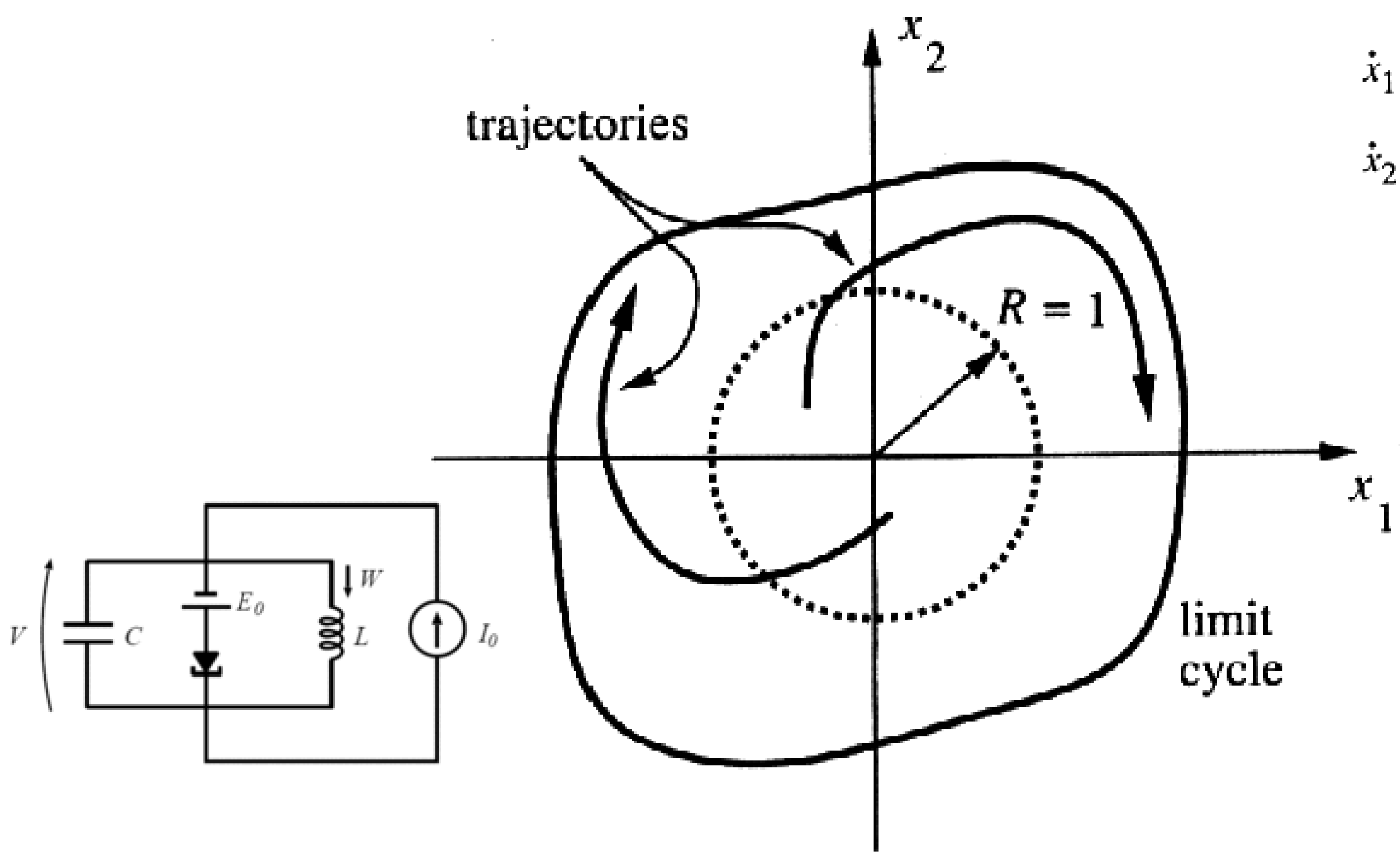
Notes Taken From: *Applied Nonlinear Control*, Slotine & Li, 1991

Penn Engineering

Outline

- Phase Portraits for Analysis of Nonlinear Systems
- Lyapunov's First Method (Already Know This)
  - Linearization!
  - Hartmann-Grobman Theorem
- Stability of an Equilibrium (Fixed Point)
- Lyapunov's Second (Direct) Method
  - Find Lyapunov Function V(x)
  - Never Solve the ODE to Ascertain Stability!
- Many Extensions of Basic Idea
- Application to Feedback Control

Phase Portrait Example

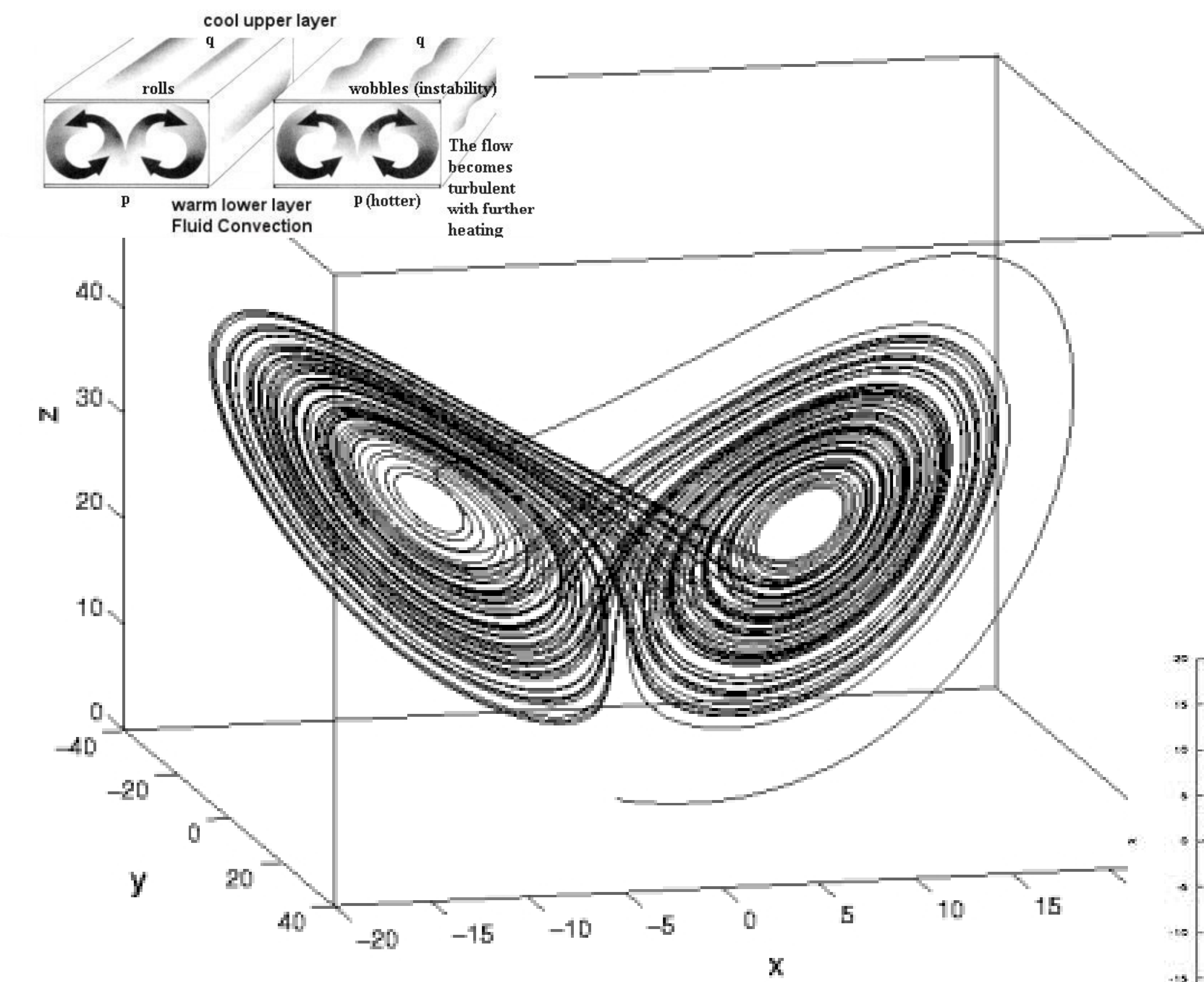


$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2$$

Figure 3.4 : Unstable origin of the Van der Pol Oscillator

Lorenz Equations Phase Portrait

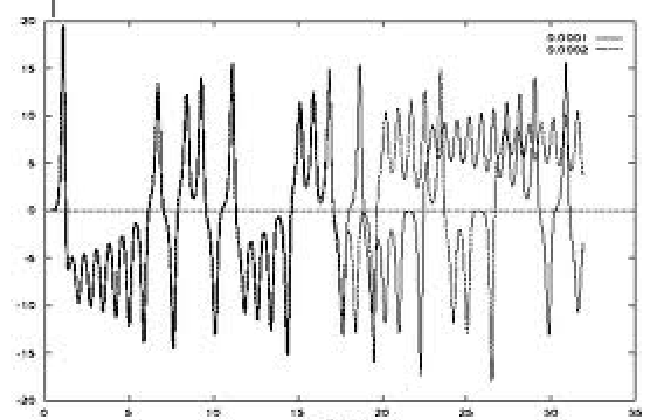


$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

Extreme Sensitivity to Initial Conditions



CHAOS!

Recall : Linearization

$$\underline{\dot{x}} = \underline{f}\left(\underline{x}, u\right)$$

Complete Nonlinear Dynamics

$$\underline{f}\left(\underline{x}_o, u_o\right) = \underline{0}$$

"Fixed Point" = Steady Condition (Called "Trim" in Airplane World)

$$\underline{f}\left(\underline{x}, u\right) = \underbrace{\underline{f}\left(\underline{x}_o, u_o\right)}_{=0} + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_o \left(\underline{x} - \underline{x}_o\right) + \left.\frac{\partial \underline{f}}{\partial u}\right|_o \left(u - u_o\right) + \ldots$$

$$h\left(\underline{x}, u\right) = \underbrace{h\left(\underline{x}_o, u_o\right)}_{y_o} + \left.\frac{\partial h}{\partial \underline{x}}\right|_o \left(\underline{x} - \underline{x}_o\right) + \left.\frac{\partial h}{\partial u}\right|_o \left(u - u_o\right) + \ldots$$

$$\Delta \underline{x}(t) \triangleq \underline{x}(t) - \underline{x}_o$$

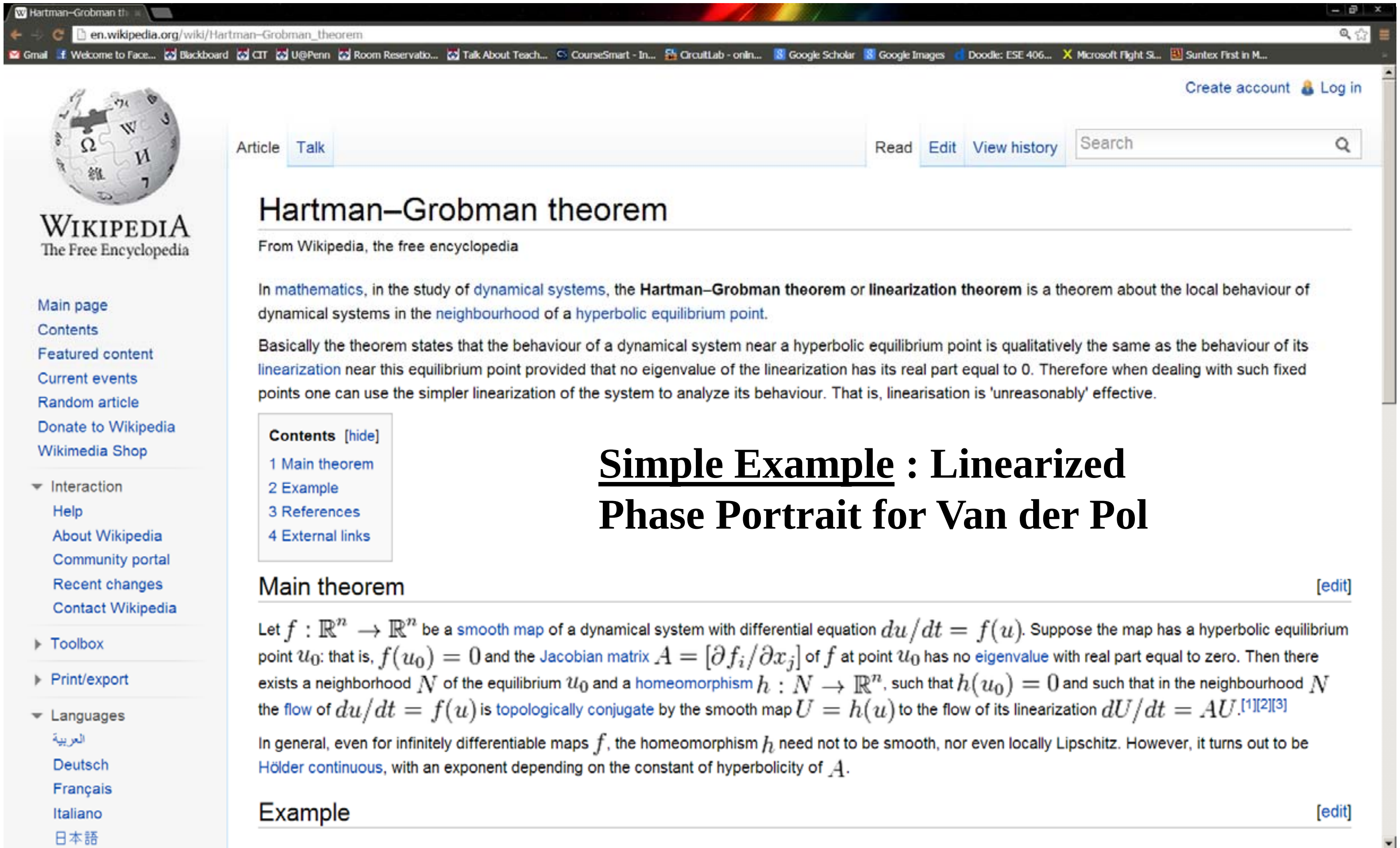
$$\Delta u(t) \triangleq u(t) - u_o$$

$$\Delta y(t) \triangleq y(t) - y_o$$

$$\left.\begin{array}{l} \boxed{\Delta \underline{\dot{x}} \approx A \Delta \underline{x} + B \Delta u} \\ \boxed{\Delta y \approx C \Delta \underline{x} + D \Delta u} \end{array}\right\}$$

Linearized System

Linearization & Hartmann-Grobman Theorem



Hartman–Grobman theorem

From Wikipedia, the free encyclopedia

In mathematics, in the study of dynamical systems, the **Hartman–Grobman theorem** or **linearization theorem** is a theorem about the local behaviour of dynamical systems in the neighbourhood of a hyperbolic equilibrium point.

Basically the theorem states that the behaviour of a dynamical system near a hyperbolic equilibrium point is qualitatively the same as the behaviour of its linearization near this equilibrium point provided that no eigenvalue of the linearization has its real part equal to 0. Therefore when dealing with such fixed points one can use the simpler linearization of the system to analyze its behaviour. That is, linearisation is 'unreasonably' effective.

Contents [hide]

Main theorem [edit]

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map of a dynamical system with differential equation $du/dt = f(u)$. Suppose the map has a hyperbolic equilibrium point u_0: that is, $f(u_0) = 0$ and the Jacobian matrix $A = [\partial f_i / \partial x_j]$ of f at point u_0 has no eigenvalue with real part equal to zero. Then there exists a neighborhood N of the equilibrium u_0 and a homeomorphism $h : N \to \mathbb{R}^n$, such that $h(u_0) = 0$ and such that in the neighbourhood N the flow of $du/dt = f(u)$ is topologically conjugate by the smooth map $U = h(u)$ to the flow of its linearization $dU/dt = AU$.[1][2][3]

In general, even for infinitely differentiable maps f, the homeomorphism h need not to be smooth, nor even locally Lipschitz. However, it turns out to be Hölder continuous, with an exponent depending on the constant of hyperbolicity of A.

Example [edit]

<u>Simple Example</u> : Linearized Phase Portrait for Van der Pol

Linearized Dynamics Are Locally Right Except Neutral Stability Case

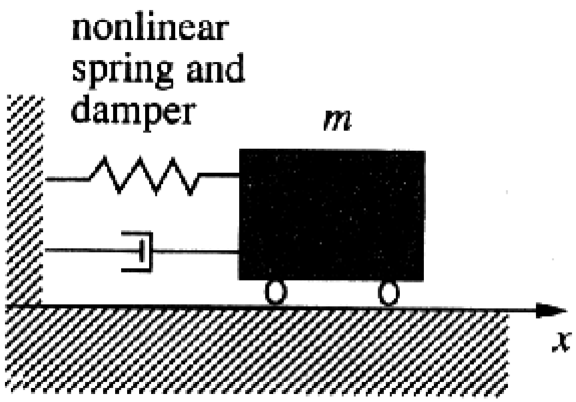
Definitions of Stability (Note Formal Style)

Definition 3.3 *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *is said to be* <u>*stable*</u> *if, for any* $R > 0$, *there exists* $r > 0$, *such that if* $\|\mathbf{x}(0)\| < r$, *then* $\|\mathbf{x}(t)\| < R$ *for all* $t \geq 0$. *Otherwise, the equilibrium point is* <u>*unstable*</u>.

Definition 3.4 *An equilibrium point* $\mathbf{0}$ *is* <u>*asymptotically stable*</u> *if it is stable, and if in addition there exists some* $r > 0$ *such that* $\| \mathbf{x}(0) \| < r$ *implies that* $\mathbf{x}(t) \rightarrow \mathbf{0}$ *as* $t \rightarrow \infty$.

Motivation for Lyapunov Direct Method

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0$$



$$V(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \int_o^x (k_o x + k_1 x^3)\, dx \;=\; \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o x^2 + \frac{1}{4} k_1 x^4$$

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_o x + k_1 x^3)\,\dot{x} = \;\dot{x}\,(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

Lyapunov Theorem for Local Stability (Formality)

LYAPUNOV THEOREM FOR LOCAL STABILITY

Theorem 3.2 (Local Stability) *If, in a ball* $\mathbf{B}_{R_o}$*, there exists a scalar function* $V(\mathbf{x})$ *with continuous first partial derivatives such that*

- $V(\mathbf{x})$ *is positive definite (locally in* $\mathbf{B}_{R_o}$*)*
- $\dot{V}(\mathbf{x})$ *is negative semi-definite (locally in* $\mathbf{B}_{R_o}$*)*

then the equilibrium point $\mathbf{0}$ *is stable. If, actually, the derivative* $\dot{V}(\mathbf{x})$ *is locally negative definite in* $\mathbf{B}_{R_o}$*, then the stability is asymptotic.*

Example 3.8: Asymptotic stability

Let us study the stability of the nonlinear system defined by

$$\dot{x}_1 = x_1\,(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = 4x_1^2x_2 + x_2\,(x_1^2 + x_2^2 - 2)$$

around its equilibrium point at the origin. Given the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

its derivative $\dot{V}$ along any system trajectory is

$$\dot{V} = 2\,(x_1^2 + x_2^2)\,(x_1^2 + x_2^2 - 2)$$

Thus, $\dot{V}$ is locally negative definite in the 2-dimensional ball $\mathbf{B}_2$, *i.e.*, in the region defined by $x_1^2 + x_2^2 < 2$. Therefore, the above theorem indicates that the origin is asymptotically stable.

Global Stability Theorem

Theorem 3.3 (Global Stability) *Assume that there exists a scalar function V of the state $\mathbf{x}$, with continuous first order derivatives such that*

- $V(\mathbf{x})$ *is positive definite*
- $\dot{V}(\mathbf{x})$ *is negative definite*
- $V(\mathbf{x}) \to \infty \quad as \quad \|\mathbf{x}\| \to \infty$

then the equilibrium at the origin is globally asymptotically stable.

Comment: Many Simplifying Assumptions Often Required to Get a Lyapunov Function to Work…Conclusions Only Valid if Model is Right!

Invariant Set Theorem Gives Very Useful Results

Definition 3.9 *A set* **G** *is an* <u>*invariant set*</u> *for a dynamic system if every system trajectory which starts from a point in* **G** *remains in* **G** *for all future time.*

Theorem 3.4 (Local Invariant Set Theorem) *Consider an autonomous system of the form (3.2), with* **f** *continuous, and let* $V(\mathbf{x})$ *be a scalar function with continuous first partial derivatives. Assume that*

- *for some* $l > 0$, *the region* Ω_l *defined by* $V(\mathbf{x}) < l$ *is bounded*
- $\dot{V}(\mathbf{x}) \leq 0$ *for all* $\mathbf{x}$ *in* Ω_l

Let **R** *be the set of all points within* Ω_l *where* $\dot{V}(\mathbf{x}) = 0$, *and* **M** *be the largest invariant set in* **R**. *Then, every solution* $\mathbf{x}(t)$ *originating in* Ω_l *tends to* **M** *as* $t \to \infty$.

Example 3.12: Domain of Attraction

Consider again the system in Example 3.8. For $l = 2$, the region Ω_2, defined by $V(\mathbf{x}) = x_1^2 + x_2^2 < 2$, is bounded. The set **R** is simply the origin **0**, which is an invariant set (since it is an equilibrium point). All the conditions of the local invariant set theorem are satisfied and, therefore, any trajectory starting within the circle converges to the origin. Thus, a domain of attraction is explicitly determined by the invariant set theorem. □

Invariant Set Theorem → "Attractive" Limit Cycle

Example 3.13: Attractive Limit Cycle

Consider the system

$$\dot{x}_1 = x_2 - x_1^7 [x_1^4 + 2 x_2^2 - 10]$$

$$\dot{x}_2 = - x_1^3 - 3 x_2^5 [x_1^4 + 2 x_2^2 - 10]$$

Notice first that the set defined by $x_1^4 + 2 x_2^2 = 10$ is invariant, since

$$\frac{\mathrm{d}}{\mathrm{d}t} (x_1^4 + 2 x_2^2 - 10) = - (4 x_1^{10} + 12 x_2^6) (x_1^4 + 2 x_2^2 - 10)$$

which is zero on the set. The motion *on* this invariant set is described (equivalently) by *either* of the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - x_1^3$$

Therefore, we see that the invariant set actually represents a *limit cycle*, along which the state vector moves clockwise.

Attractive Limit Cycle (Continued…)

Is this limit cycle actually attractive? Let us define as a Lyapunov function candidate

$$V = (x_1^{\,4} + 2\,x_2^{\,2} - 10)^2$$

which represents a measure of the "distance" to the limit cycle. For any arbitrary positive number l, the region $\mathbf{\Omega}_l$, which surrounds the limit cycle, is bounded. Using our earlier calculation, we immediately obtain

$$\dot{V} \; = \; -8\,(x_1^{\,10} + 3\,x_2^{\,6})\,(x_1^{\,4} + 2\,x_2^{\,2} - 10)^2$$

Thus $\dot{V}$ is strictly negative, except if

$$x_1^{\,4} + 2\,x_2^{\,2} = 10 \qquad \text{or} \qquad x_1^{\,10} + 3\,x_2^{\,6} \; = \; 0$$

in which case $\dot{V} = 0$. The first equation is simply that defining the limit cycle, while the second equation is verified only at the origin. Since both the limit cycle and the origin are invariant sets, the set $\mathbf{M}$ simply consists of their union. Thus, all system trajectories starting in $\mathbf{\Omega}_l$ converge either to the limit cycle, or to the origin (Figure 3.15).

Attractive Limit Cycle (…Concluded)

Moreover, the equilibrium point at the origin can actually be shown to be *unstable*. However, this result cannot be obtained from linearization, since the linearized system ($\dot{x}_1 = x_2$, $\dot{x}_2 = 0$) is only marginally stable. Instead, and more astutely, consider the region $\mathbf{\Omega}_{100}$, and note that while the origin $\mathbf{0}$ does not belong to $\mathbf{\Omega}_{100}$, every other point in the region enclosed by the limit cycle is in $\mathbf{\Omega}_{100}$ (in other words, the origin corresponds to a local *maximum* of V). Thus, while the expression of $\dot{V}$ is the same as before, *now the set* **M** *is just the limit cycle.* Therefore, reapplication of the invariant set theorem shows that any state trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular, this implies that the equilibrium point at the origin is unstable. □

Good Example of a Result That CANNOT Be Obtained from Linear Analysis…Not Even Close!

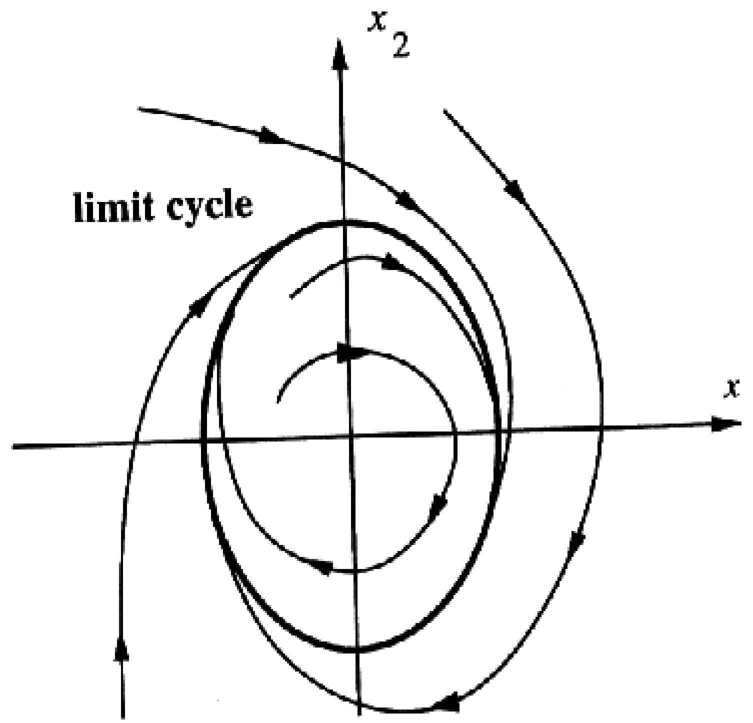


Figure 3.15 : Convergence to a limit cycle

Applications to Control

It is important to note that although equation (3.1) does not explicitly contair the control input as a variable, it is directly applicable to feedback control systems The reason is that equation (3.1) can represent the *closed-loop* dynamics of a feedbacl control system, with the control input being a function of state **x** and time t, anc therefore disappearing in the closed-loop dynamics. Specifically, if the plant dynamic: is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

and some control law has been selected

$$\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$$

then the closed-loop dynamics is

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}, \mathbf{g}(\mathbf{x}, t), t]$$

which can be rewritten in the form (3.1). Of course, equation (3.1) can also represer dynamic systems where no control signals are involved, such as a freely swingin pendulum.

Course Feedback

- Please Take Time & Be Thoughtful – I Read These & Make Changes Based on What You Say!
- Not All Feedback is Especially Useful…
  - “Everyone Knows Bruce is the Man.”
  - “Worst Class I Have Taken at Penn”
  - “I love Bruce Kothmann!”
  - “Kothmann is difficult to follow and he refuses or is unable to explain things simply.”
- What I Think Was Good in 2014 ESE 505 & MEAM 513
  - Project (Actually, This was Too Late & Too Low Quality)
  - Weekly Homework
  - Curriculum Relevant to Real Professional Problems & Student Projects
- What I Think Should Be Better
  - More Project(s) !
  - More Patient Lectures
  - Faster Turnaround on Grading
- Good Luck in Whatever Comes Next!