Control of Affine Systems

State x

Input u

State equations

$$\dot{x} = f(x) + g(x)u$$

Output

$$y = h(x)$$

Lie Derivatives

Want

$$\dot{y} - \dot{y}^{\text{des}} + k(y - y^{\text{des}}) = 0$$

or

$$\ddot{y} - \ddot{y}^{\text{des}} + k_1(\dot{y} - \dot{y}^{\text{des}}) + k_2(y - y^{\text{des}}) = 0$$

Need derivative of the output function

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f(x) + g(x)u)$$

Lie Derivatives

$$\mathcal{L}_f h = \frac{\partial h}{\partial x} f(x)$$

$$\mathcal{L}_g h = \frac{\partial h}{\partial x} g(x)$$

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Input Output Linearization

State equations

$$\dot{x} = f(x) + g(x)u$$

Output

$$y = h(x)$$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Control law $\boxed{\text{if } \mathcal{L}_g h \neq 0}$

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\mathrm{des}} + k(y^{\mathrm{des}} - y)\right)$$

Closed loop system

$$\dot{y} - \dot{y}^{\mathrm{des}} + k(y - y^{\mathrm{des}}) = 0$$

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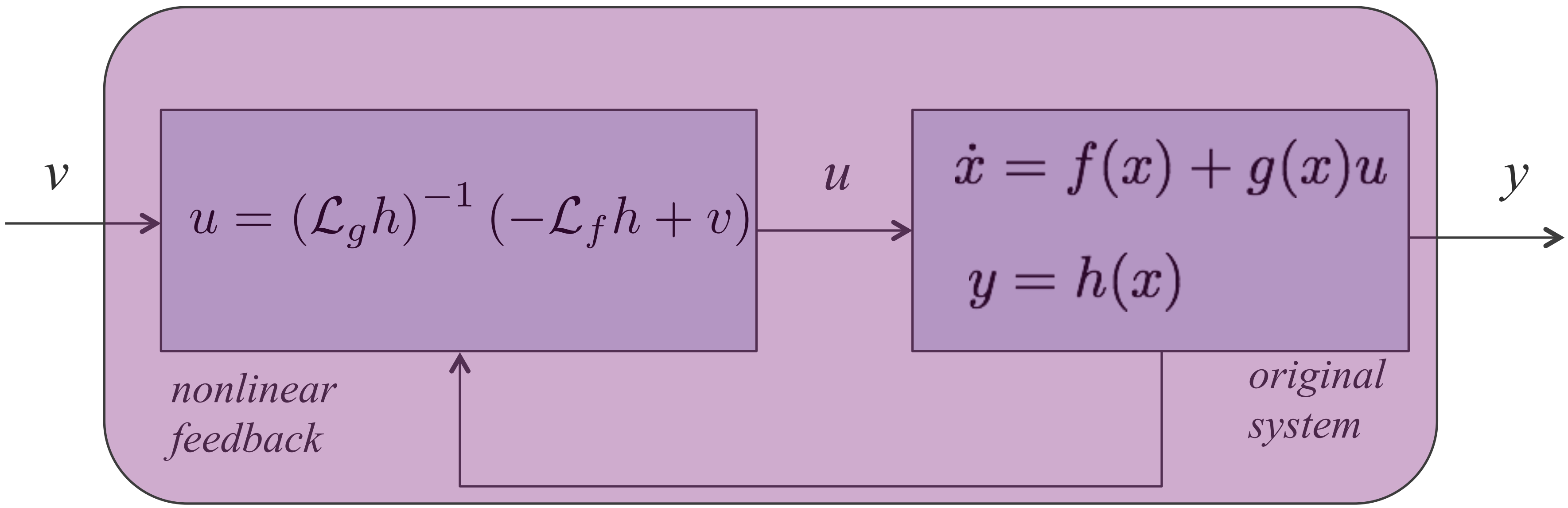
Input Output Linearization



new system $\dot{y} = v$

Nonlinear feedback transforms the original nonlinear system to a new linear system

Linearization is exact (distinct from linear approximations to nonlinear systems)

Affine, Single Input Single Output

State x $\quad x \in R^n$

Input u $\quad u \in R$

State equations $\quad \dot{x} = f(x) + g(x)u$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Output $\quad y = h(x) \in R$

Control law

if $\mathcal{L}_g h \neq 0$

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\text{des}} + k(y^{\text{des}} - y)\right)$$

if $\mathcal{L}_g h = 0$ $\quad \dot{y} = \mathcal{L}_f h \quad$ (rate of change of output is independent of u)

Explore higher order derivatives of output

nonzero ?

$$\ddot{y} = \mathcal{L}_f \mathcal{L}_f h + \boxed{(\mathcal{L}_g \mathcal{L}_f h)}\, u$$

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Affine, Single Input Single Output

State x $\quad x \in R^n$

Input u $\quad u \in R$

State equations

$$\dot{x} = f(x) + g(x)u$$

Output

$$y = h(x) \in R$$

$$\mathcal{L}_f^2 h = \mathcal{L}_f\left(\mathcal{L}_f h\right)$$
$$\mathcal{L}_f^3 h = \mathcal{L}_f\left(\mathcal{L}_f\left(\mathcal{L}_f h\right)\right)$$
$$\ldots$$

Relative degree, r *The index of the first nonzero term in the sequence*

$r=k+1$

$$\mathcal{L}_g h, \mathcal{L}_g \mathcal{L}_f h, \mathcal{L}_g \mathcal{L}_f^2 h, \ldots, \mathcal{L}_g \mathcal{L}_f^k h, \ldots,$$

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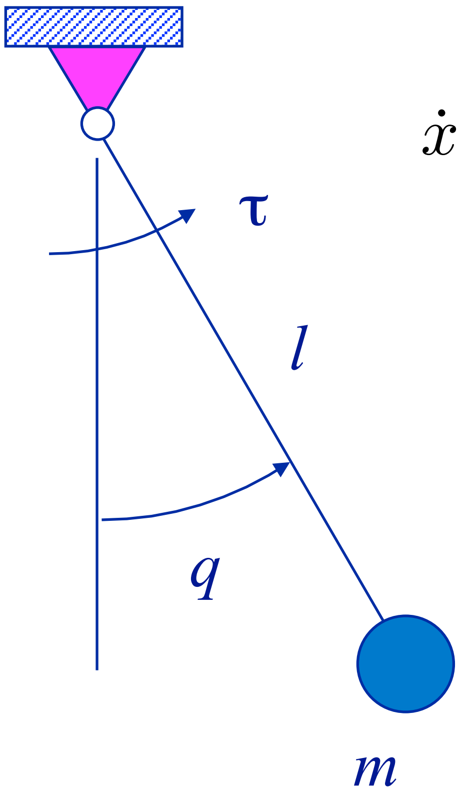
Example 1. Single degree of freedom arm

$$ml^2\ddot{q} + \frac{1}{2}mgl\sin q = \tau \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$



$$\dot{x} = \underbrace{\boxed{\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}}}_{f(x)} + \underbrace{\boxed{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}}_{g(x)} u \qquad h = x_1$$

$$\mathcal{L}_g h = 0 \qquad \mathcal{L}_f h = x_2$$

$$\boxed{\mathcal{L}_g \mathcal{L}_f h = \frac{1}{ml^2}} \qquad \mathcal{L}_f^2 h = -\frac{g}{l}\sin x_1$$

r=2

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Affine, SISO

Linear control, model independent

feed forward

feedback

r=1

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \boxed{\dot{y}^{\text{des}} + k(y^{\text{des}} - y)}\right)$$

r=2

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f h}\left(-\mathcal{L}_f \mathcal{L}_f h + \boxed{\ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y)}\right)$$

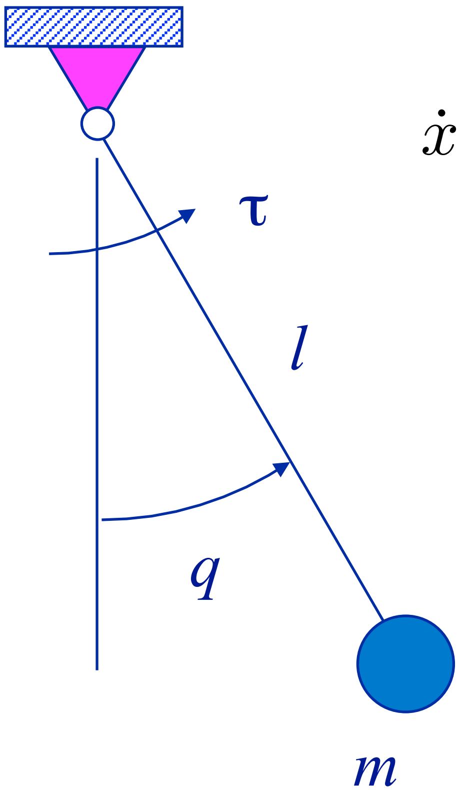
r=3

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f^2 h}\left(-\mathcal{L}_f^3 h + \boxed{\dddot{y}^{\text{des}} + k_1(\ddot{y}^{\text{des}} - \ddot{y}) + k_2(\dot{y}^{\text{des}} - \dot{y}) + k_3(y^{\text{des}} - y)}\right)$$

General form of control law

$$u = \alpha(x) + \beta(x)\boxed{v}$$

Single degree of freedom arm



$$ml^2\ddot{q} + \frac{1}{2}mgl\sin q = \tau \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$\dot{x} = \underbrace{\boxed{\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}}}_{f(x)} + \underbrace{\boxed{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}}_{g(x)} u \qquad h = x_1$$

$$\mathcal{L}_g\mathcal{L}_f h = \frac{1}{ml^2} \qquad \mathcal{L}_f^2 h = -\frac{g}{l}\sin x_1$$

$$\boxed{u = \frac{1}{\mathcal{L}_g\mathcal{L}_f h}\left(-\mathcal{L}_f\mathcal{L}_f h + \ddot{y}^{\mathrm{des}} + k_1(\dot{y}^{\mathrm{des}} - \dot{y}) + k_2(y^{\mathrm{des}} - y)\right)}$$

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Multiple Input Multiple Output Systems

State x $\quad x \in \mathbb{R}^n$

Input u $\quad u \in \mathbb{R}^m$

Output $\quad y = h(x) \in \mathbb{R}^m$

$$\dot{x} = \underbrace{f(x)}_{n \times 1} + \underbrace{g(x)}_{n \times m} u$$

Assume each output has relative degree r

Nonlinear feedback law

$$u = \left(\mathcal{L}_g \mathcal{L}_f^{r-1} h\right)^{-1} \left(-\mathcal{L}_f^r + v\right)$$

leads to the equivalent system

$$y^{(r)} = v$$

Example 2: Fully-actuated robot (*n* joints, *n* actuators)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

Dynamic model

- *M* is the positive definite, *n* by *n* inertia matrix
- *C* is the *n*-dimensional vector of Coriolis and centripetal forces
- *N* is the *n*-dimensional vector of gravitational forces
- τ is the *n*-dimensional vector of actuator forces and torques
- $\dot{M} - 2C$ is skew symmetric

Key: *M* is non singular

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Fully-actuated robot (continued)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$u = \tau \in \mathbb{R}^n$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ M(x_1)^{-1} \end{bmatrix} u$$

$$y = q_{des} - q \in \mathbb{R}^n$$

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Fully-actuated robot (continued)

$$h(x) = x_1$$

$$f(x) = \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2) \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ M(x_1)^{-1} \end{bmatrix}$$

$$\mathcal{L}_g h = 0, \; \mathcal{L}_g \mathcal{L}_f h \neq 0$$

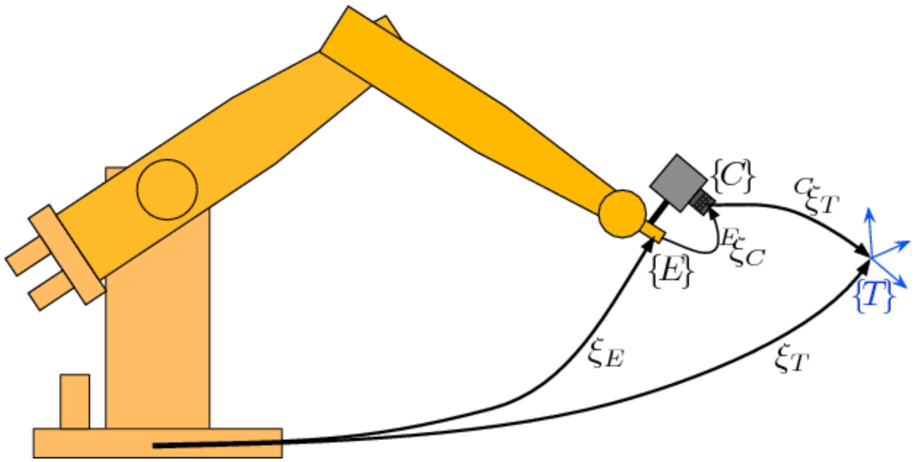
Relative degree is 2

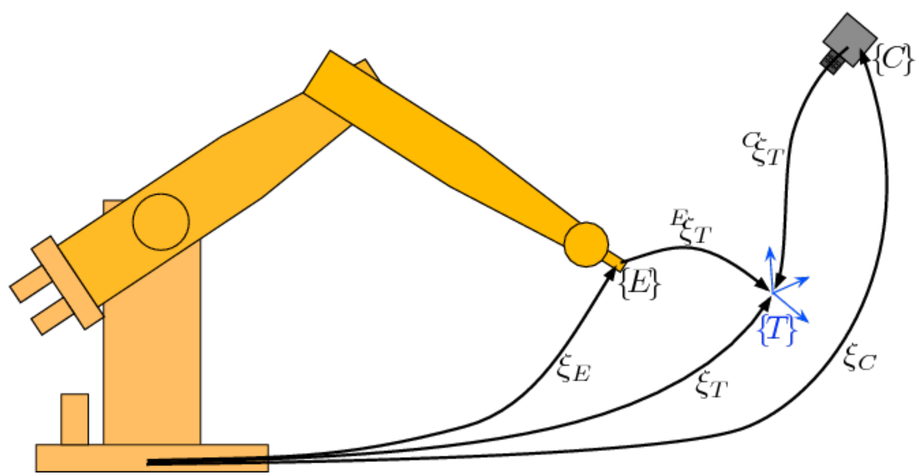
$$u = \boxed{(\mathcal{L}_g \mathcal{L}_f h)^{-1}} \left(\boxed{-\mathcal{L}_f \mathcal{L}_f h} + \boxed{\ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y)} \right)$$

Control law

$$u = \boxed{M(x_1)} \left(\boxed{(C(x_1, x_2)x_2 + N(x_1))} + \boxed{(\ddot{y}^{des} + k_1(\dot{y}^{des} - \dot{y}) + k_2(y^{des} - y))} \right)$$

Visual Servoing





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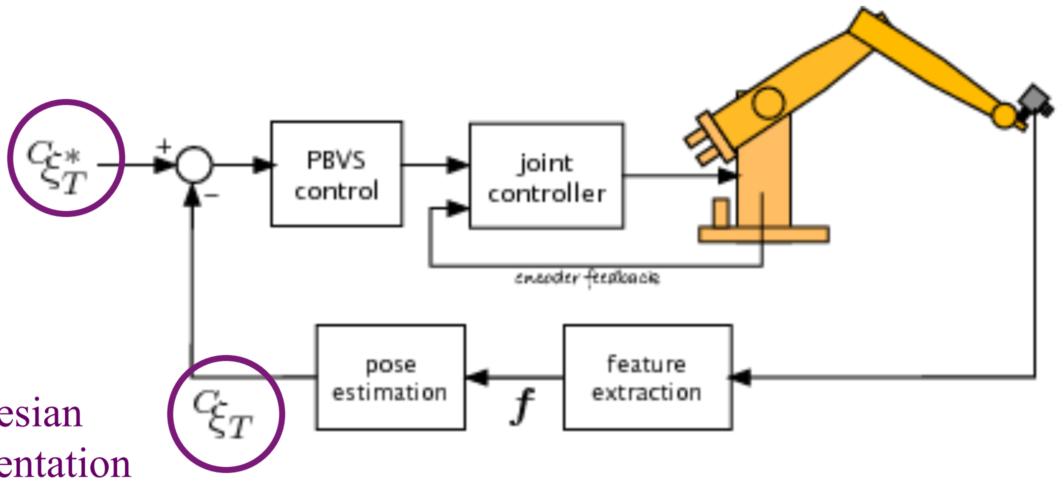
Position-Based Visual Servoing

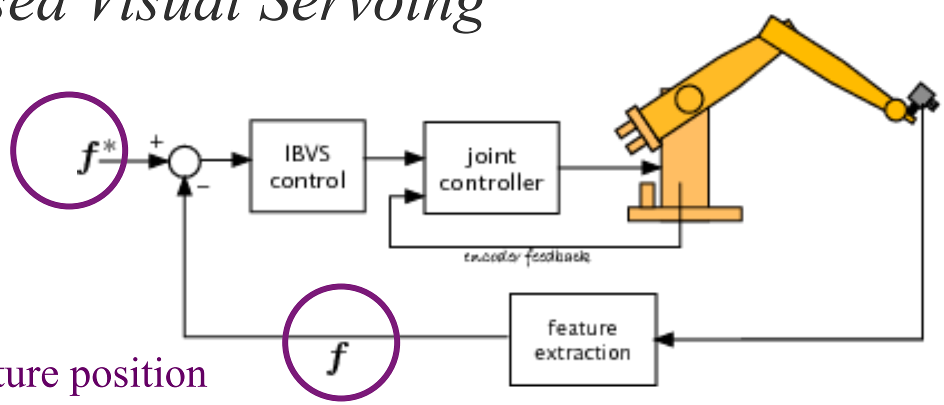
Desired Cartesian position/ orientation



Actual Cartesian position/orientation

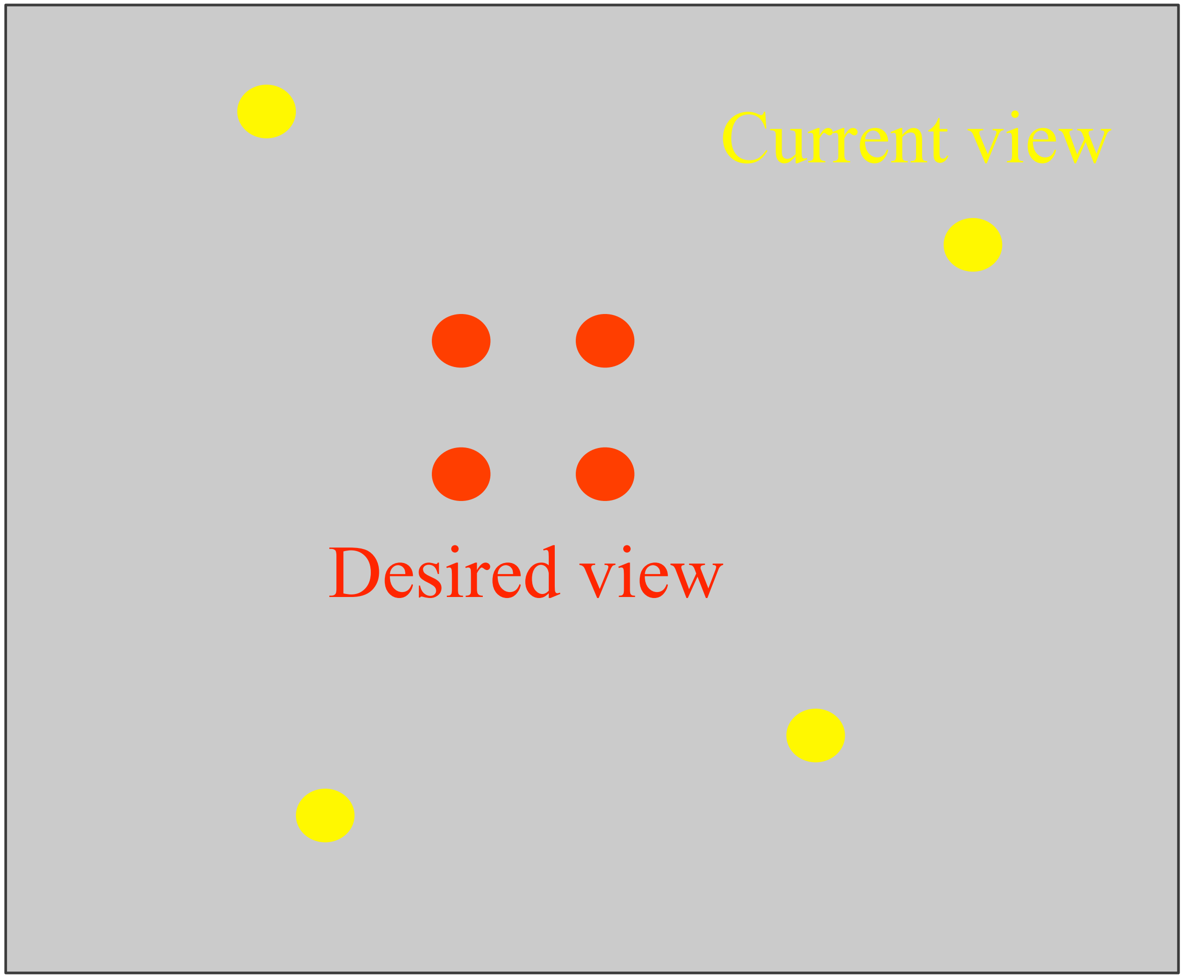
Image Based Visual Servoing

Desired position of feature(s) in image



Actual feature position

Image-Based Visual Servoing



8 outputs, 6 inputs

Image plane motion

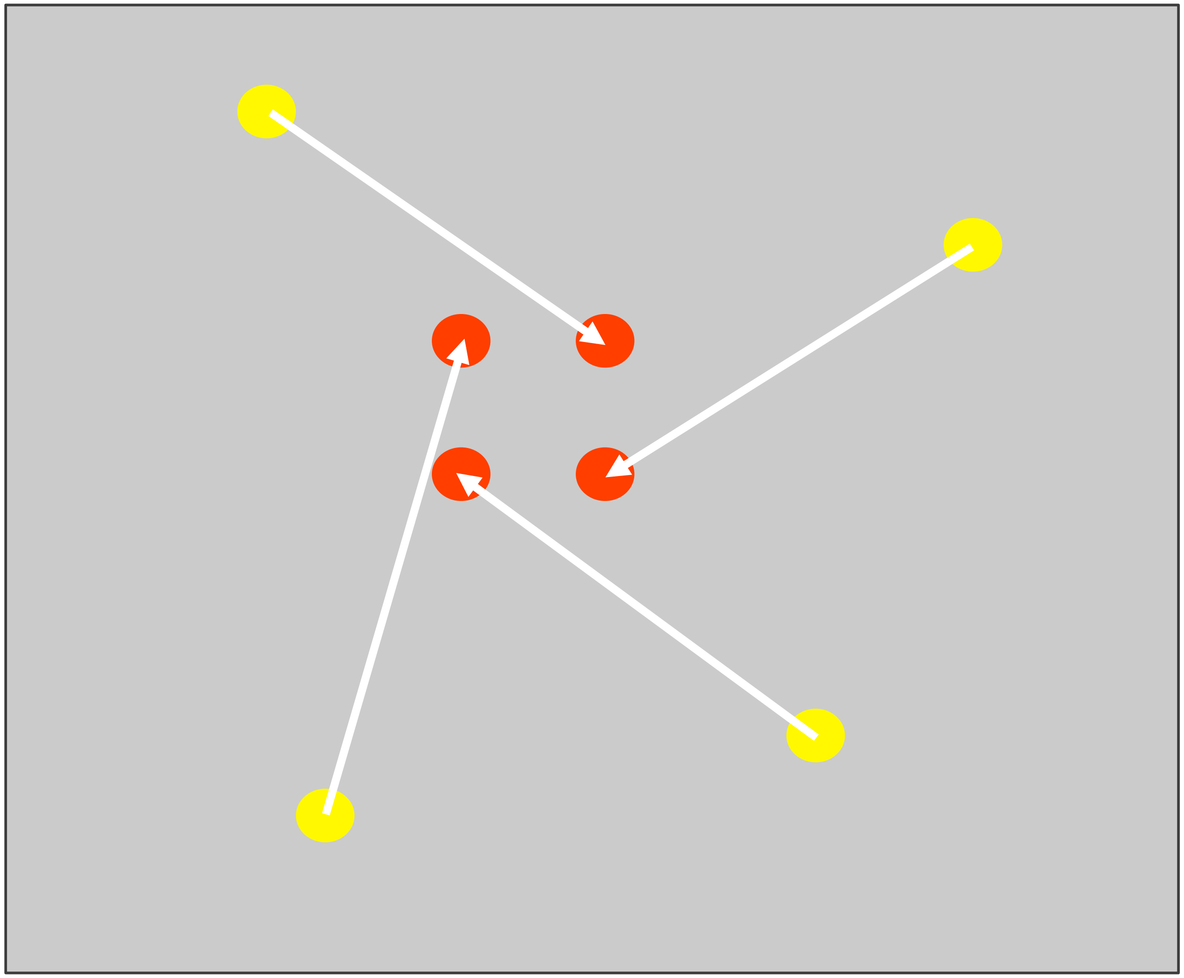


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Image plane motion

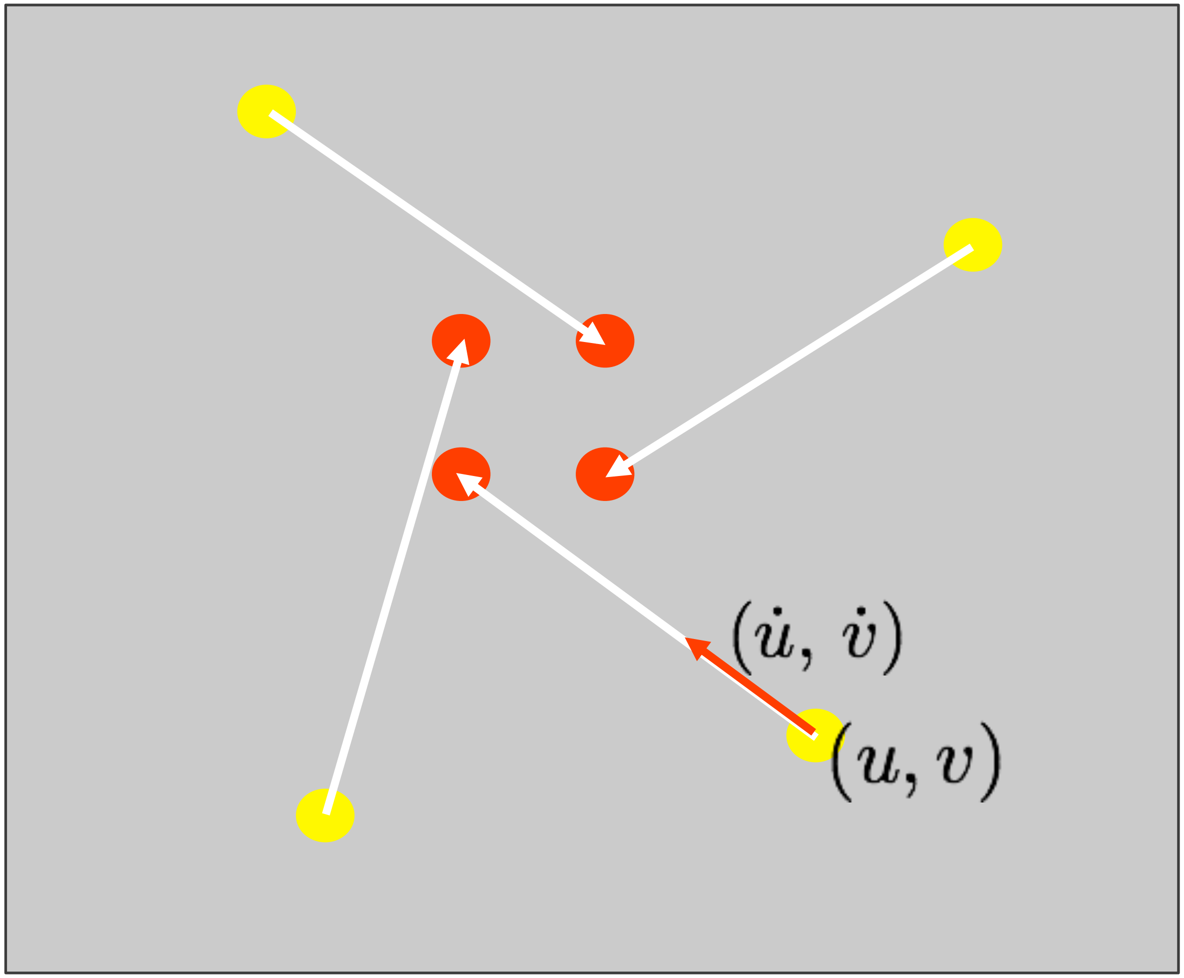


Image Based Visual Servoing

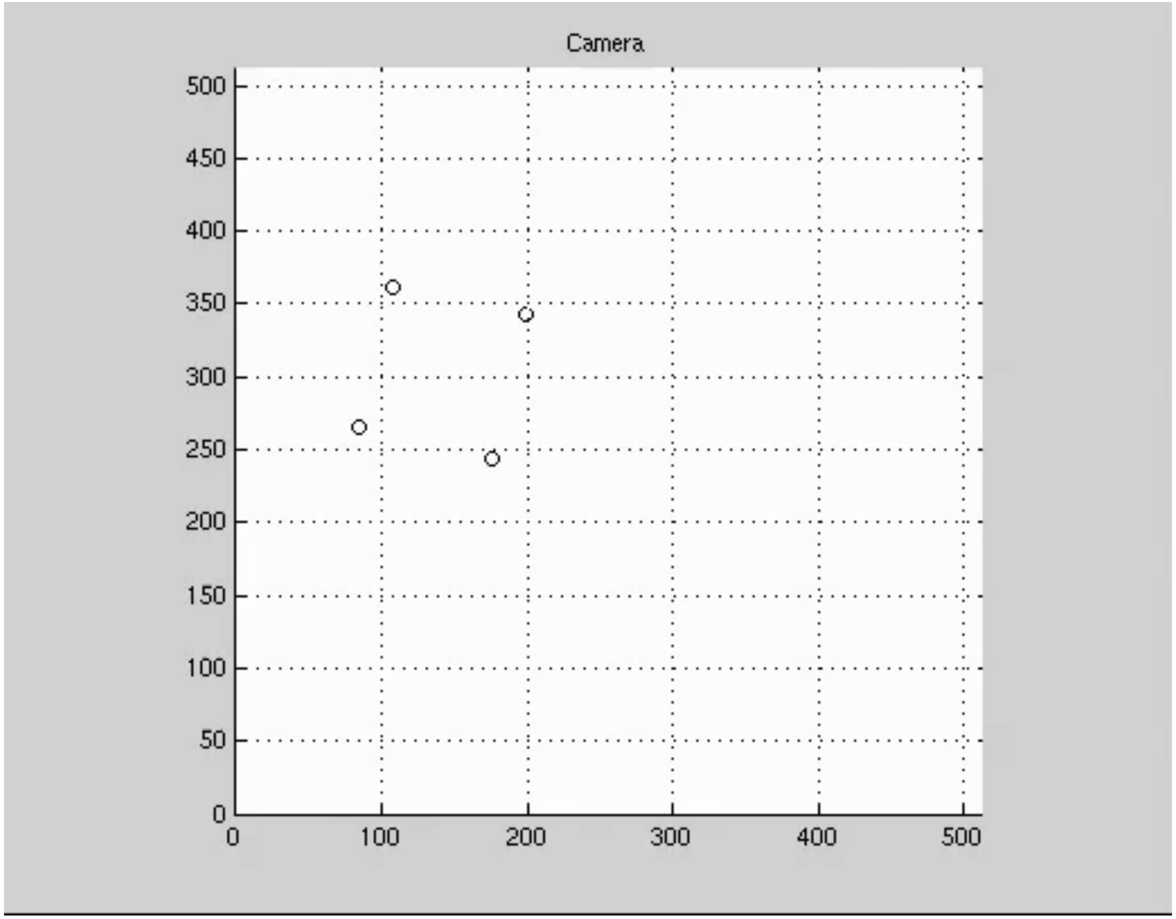


Image-Based Visual Servoing for 3 Features

$$h(x) = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\bar{u}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} -\frac{f}{\rho_u Z} & 0 & \frac{\bar{u}}{Z} & \frac{\rho_u \bar{u}\bar{v}}{f} & -\frac{f^2+\rho_u^2 \bar{u}^2}{\rho_u f} & \bar{v} \\ 0 & -\frac{f}{\rho_v Z} & \frac{\bar{v}}{Z} & \frac{f^2+\rho_v^2 \bar{v}^2}{\rho_v f} & -\frac{\rho_v \bar{u}\bar{v}}{f} & -\bar{u} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$g(x)$$

input

$$u \in \mathbb{R}^6$$

Relative degree is 1

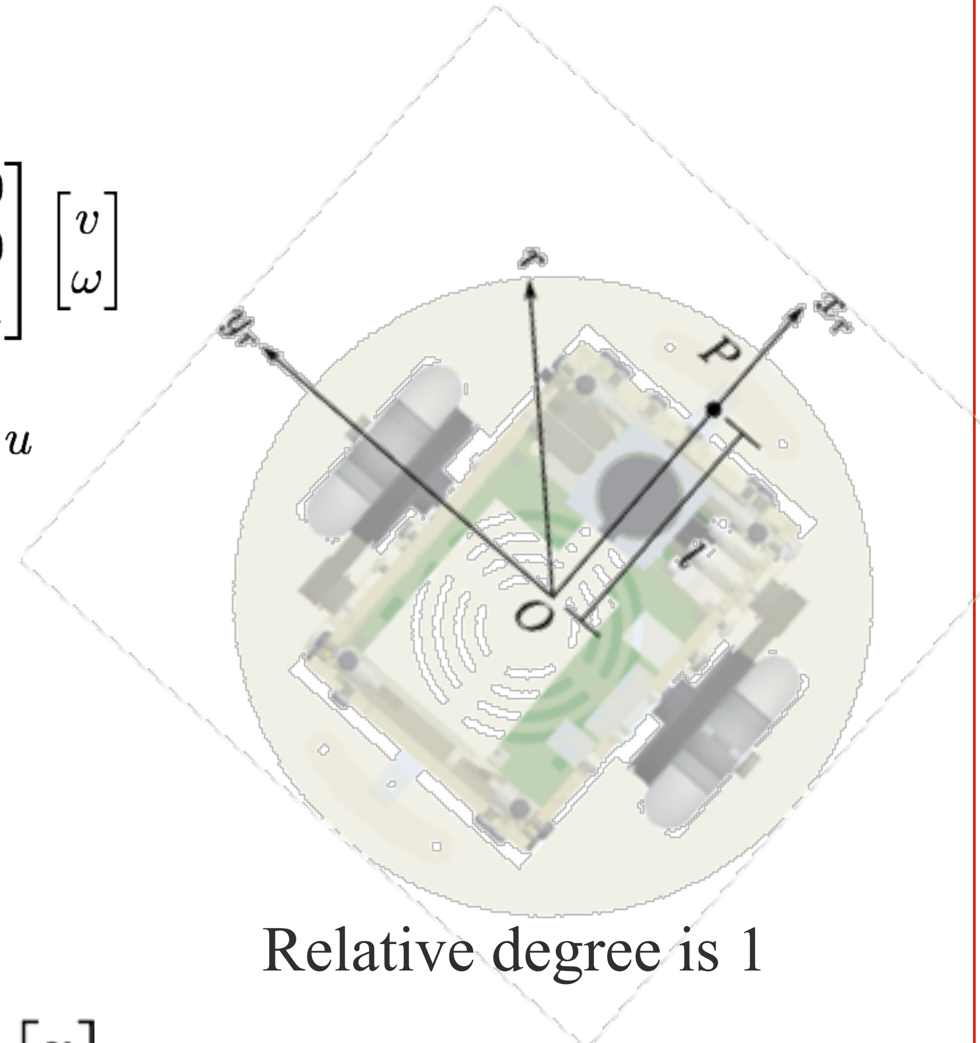
Kinematic planar cart

State equations, inputs

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$

$$\dot{X} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\dot{X} = g(X)u$$



Outputs

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x + L\cos\theta \\ y + L\sin\theta \end{bmatrix}$$

$$y = h(x) = \begin{bmatrix} x + L\cos\theta \\ y + L\sin\theta \end{bmatrix}$$

Relative degree is 1

$$\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \end{bmatrix} = \begin{bmatrix} \cos\theta & -L\sin\theta \\ \sin\theta & L\cos\theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

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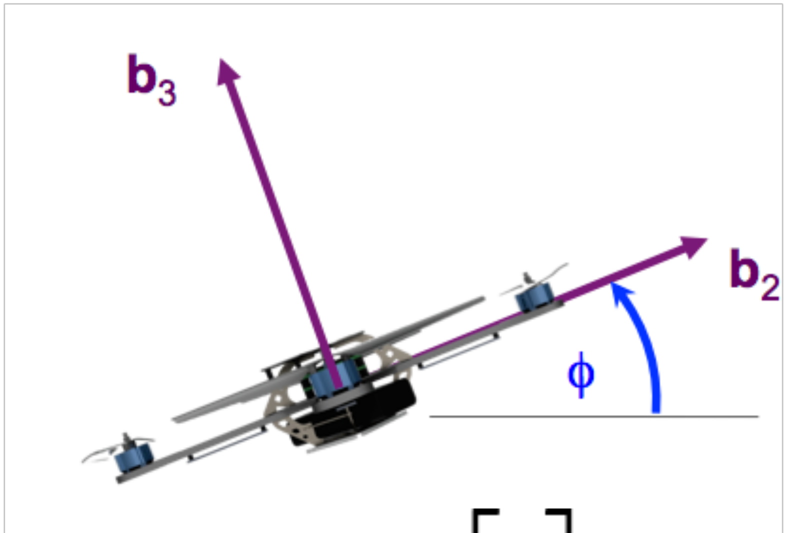
Quadrotor

The quadrotor is underactuated!

Planar Quadrotor



$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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Planar Quadrotor

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = h(x) = \begin{bmatrix} y \\ z \end{bmatrix}$$

Repeated differentiation of $h(x)$ does not yield explicit dependence on u

Must extend state with higher order derivatives of input

New extended state

$$\bar{x} = \begin{bmatrix} y & z & \phi & \dot{y} & \dot{z} & \dot{\phi} & u_1 & \dot{u}_1 \end{bmatrix}^T$$

New input

$$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}$$

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Planar Quadrotor

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = h(x) = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ -\frac{u_1}{m}\sin\phi \\ \frac{u_1}{m}\cos\phi - g \\ 0 \\ \dot{u}_1 \\ 0 \end{bmatrix}}_{\bar{f}(\bar{x})} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{I_{xx}} \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\bar{g}(\bar{x})} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$$

Verify $\mathcal{L}_{\bar{g}}\mathcal{L}_{\bar{f}}^{3}h$ is full rank ($r = 4$)

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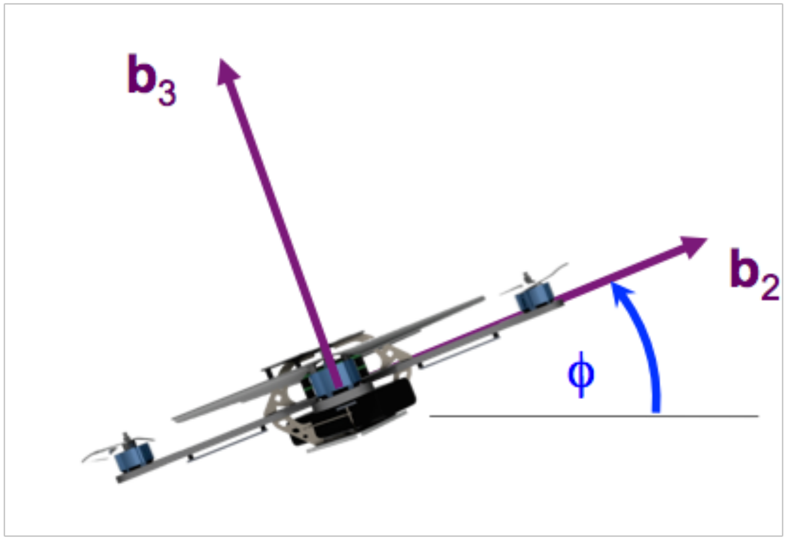
Relative Degree of Freedom is 4

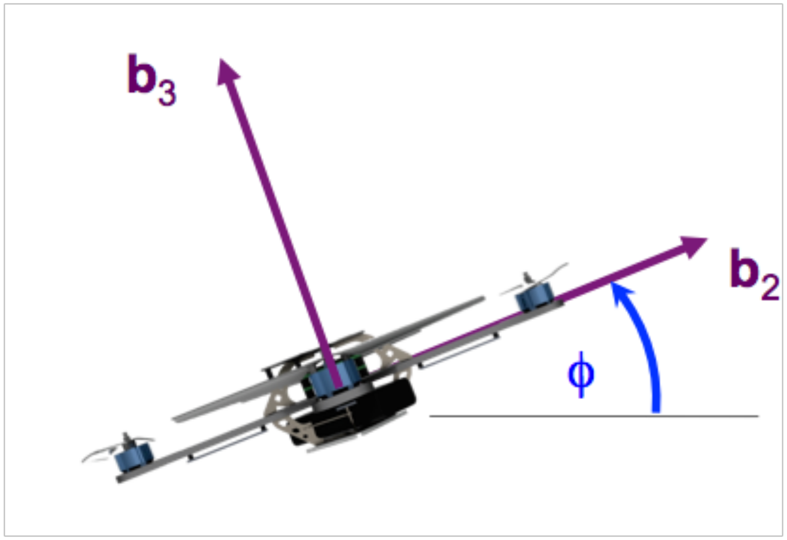


$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} y^{(iv)} \\ z^{(iv)} \end{bmatrix}} = \boxed{\frac{1}{m}\begin{bmatrix} -\sin\phi & -\frac{u_1}{I_{zz}}\cos\phi \\ -\cos\phi & -\frac{u_1}{I_{zz}}\sin\phi \end{bmatrix}} \boxed{\begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}} + \frac{1}{m}\begin{bmatrix} -2\dot{u}_1\cos\phi\dot{\phi} + u_1\dot{\phi}^2\sin\phi \\ -2\dot{u}_1\sin\phi\dot{\phi} - u_1\dot{\phi}^2\cos\phi \end{bmatrix}$$

$h^{(iv)}$ $\qquad$ $\mathcal{L}_{\bar{g}}\mathcal{L}_f^3 h$ $\qquad$ $\bar{u}$

Dynamic State Feedback



$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \end{bmatrix}$$

$$\begin{bmatrix} y^{(iv)} \\ z^{(iv)} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} -\sin\phi & -\frac{u_1}{I_{zz}}\cos\phi \\ -\cos\phi & -\frac{u_1}{I_{zz}}\sin\phi \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} + \frac{1}{m} \begin{bmatrix} -2\dot{u}_1 \cos\phi\dot{\phi} + u_1\dot{\phi}^2 \sin\phi \\ -2\dot{u}_1 \sin\phi\dot{\phi} - u_1\dot{\phi}^2 \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} \\ \mathbf{I}_{2\times2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

v=Kz

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Differential Flatness

All state variables and the inputs can be written as smooth functions of $\boxed{\textit{flat outputs}}$ and their derivatives

$$\begin{bmatrix} \boxed{\begin{matrix} y \\ z \end{matrix}} \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \\ y^{(iv)} \\ z^{(iv)} \end{bmatrix} \xleftrightarrow{\text{diffeomorphism}} \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ u_2 \end{bmatrix}$$

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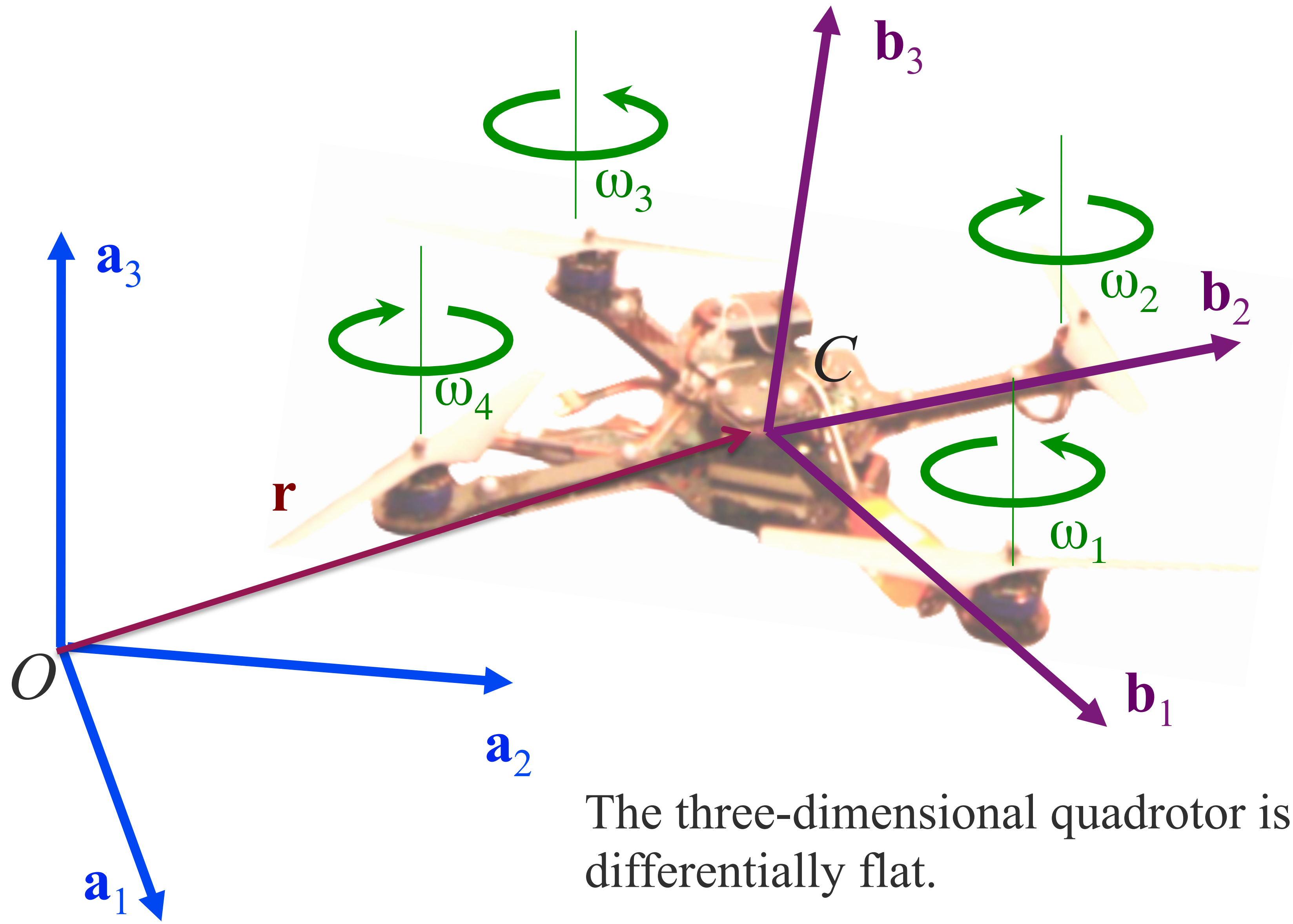

The three-dimensional quadrotor is differentially flat.

Differential Flatness (3-D Quadrotor)

Inputs

$u_1, \mathbf{u}_2$

$$u_1 = \sum_{i=1}^{4} F_i \qquad \mathbf{u}_2 = L \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \mu & -\mu & \mu & -\mu \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

State

$(\mathbf{x}, \dot{\mathbf{x}})$

σ

$$\begin{array}{r l} & \left[\begin{array}{c} \boxed{\mathbf{r}} \\ \mathbf{v} \\ \mathbf{a} \\ \mathbf{j} \\ \mathbf{s} \\ \boxed{\psi} \\ \dot{\psi} \\ \ddot{\psi} \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \textit{jerk} \\ \textit{snap} \\ \textit{yaw} \\ \\ \\ \end{array} \longleftrightarrow \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

$$u_1 = m(a_3 - \mathbf{g}.\mathbf{b}_3)$$

$$\dot{u}_1 = m j_3$$

$$\ddot{u}_1 = m s_3 + u_1(q^2 + rp)$$

$$p = \frac{-m j_2}{u_1}$$

$$q = \frac{m j_1}{u_1}$$

[Mellinger and Kumar, ICRA 2011]

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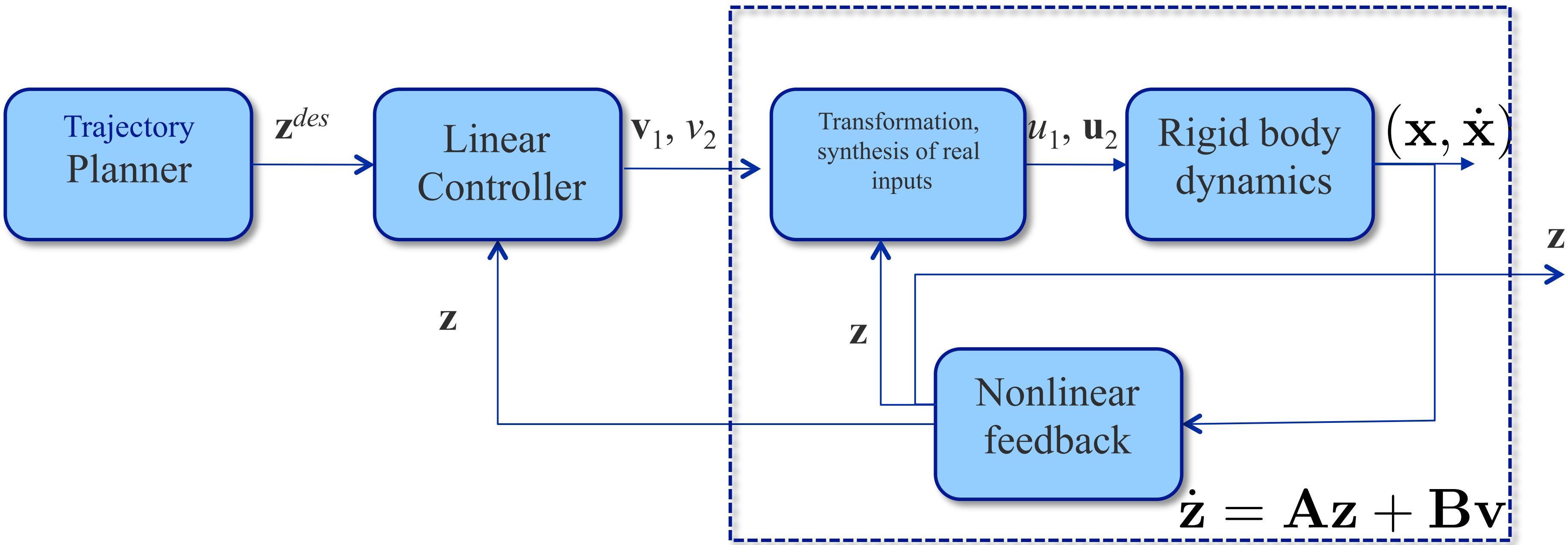
Summary


Trajectory Planner
$\mathbf{z}^{des}$
Linear Controller
$\mathbf{v}_1, v_2$
Transformation, synthesis of real inputs
$u_1, \mathbf{u}_2$
Rigid body dynamics
$(\mathbf{x}, \dot{\mathbf{x}})$
$\mathbf{z}$
$\mathbf{z}$
$\mathbf{z}$
Nonlinear feedback
$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v}$