ESE 505/MEAM 513 2014-04-09 MODERN CONTROL

RECAP $AQ = Q\Lambda$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$A q_k = \lambda_k q_k$ RIGHT EIGENVECTORS

$$Q^{-1} A = \Lambda Q^{-1}$$

$$Q^{-1} = \begin{bmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_n^T \end{bmatrix} \qquad p_k^T A = \lambda_k p_k^T \quad \text{LEFT EIGENVECTORS}$$

ODE $\dot{\underline{x}} = A\underline{x}, \quad \underline{x}(0) = \underline{x}_0$

$$\underline{x}(t) = Q\underline{z}(t) \Rightarrow \dot{\underline{z}} = \Lambda \underline{z}, \quad \underline{z}(0) = \underline{z}_0 = Q^{-1}\underline{x}_0$$

$$\dot{z}_k = \lambda_k z_k \qquad z_k(0) = p_k^T \underline{x}_0$$

$$z_k(t) = e^{\lambda_k t} z_k(0)$$

$$\Rightarrow \underline{x}(t) = Q \underbrace{e^{\Lambda t} \underbrace{Q^{-1}\underline{x}_0}_{\underline{z}_0}}_{\underline{z}(t)} = e^{At}\underline{x}_0$$

$$\underbrace{e^{At} = Q e^{\Lambda t} Q^{-1}}$$

IN GENERAL, $f(t) \rightarrow f(At)$

$$f(At) = Q f(\Lambda t) Q^{-1}$$

$$= Q \begin{bmatrix} f(\lambda_1 t) & & & \\ & f(\lambda_2 t) & & \\ & & \ddots & \\ & & & f(\lambda_n t) \end{bmatrix} Q^{-1}$$

CAUTION: DOESN'T ALWAYS WORK TO FIND Λ! EX/ $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ HAS ONLY $\lambda = 2$, $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. JORDAN FORM

e^{At} HAS $te^{\lambda t}$!

EXAMPLE $A = \begin{pmatrix} 2 & -2 & 11 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix}$ $Q = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ $Q^{-1} = \begin{pmatrix} 0 & 1 & -4 \\ 1 & -2 & 5 \\ 0 & 0 & 1 \end{pmatrix}$

SOLVE $\dot{x} = Ax$ WITH $\underline{x}(0) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ $\underline{z}(0) = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$$\Rightarrow \underline{x}(t) = 0 \cdot e^{t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + (-2) e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot e^{3t} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

USE MATLAB'S "expm" FUNCTION TO CONFIM $\underline{x}(t) = e^{At} \underline{x}_0$

A COUPLE MORE USEFUL THINGS

$$\Delta(\lambda) = |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0$$

"CHARACTERISTIC POLYNOMIAL" $\Delta(\lambda_k) = 0$

$$\Delta(A) = A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$$

$$A^n = (Q\Lambda Q^{-1})(Q\Lambda Q^{-1}) \cdots (Q\Lambda Q^{-1}) = Q\Lambda^n Q^{-1}$$

$$\Rightarrow \Delta(A) = Q\,\Delta(\Lambda)\,Q^{-1} = Q \begin{bmatrix} \Delta(\lambda_1) & & \\ & \Delta(\lambda_2) & \\ & & \ddots \; \Delta(\lambda_n) \end{bmatrix} Q^{-1} = 0 \,!$$

ALSO TRUE FOR JORDAN FORM $\neq \Lambda$

CAYLEY-HAMILTON THEOREM: $\Delta(A) = 0$

TAYLOR SERIES (USE e^{tA} TO AVOID CONFUSION)

$$e^{tA} = I + tA + \frac{t^2}{2} A^2 + \dots$$

BUT $A^n = -a_{n-1} A^{n-1} - a_{n-2} A^{n-2} - \dots - a_0 I$

CAN ELIMINATE ALL POWERS OF A BIGGER THAN $n-1$

LET'S THINK ABOUT CONTROL...

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \Rightarrow \underline{x}(t) = e^{tA}\underline{x}_0 + \int_0^t e^{(t-\tau)A} B\underline{u}(\tau)\, d\tau$$

$\underline{x}(0) = \underline{x}_0$

τ = "DUMMY VARIABLE"

"PROOF" OF $\underline{x}(t)$ AS SOLUTION

$\underline{x}(0) = \underline{x}_0$ ✓

$$\dot{\underline{x}} = Ae^{tA}\underline{x}_0 + e^{(t-\tau)A} B\underline{u}(\tau)\Big|_{\tau=t} + \int_0^t \frac{\partial}{\partial t}\left[e^{(t-\tau)A} B\underline{u}(\tau)\right] d\tau$$

$\frac{d}{dt} \to \frac{\partial}{\partial t}$ BECAUSE τ IS FIXED

$$= \underline{Ae^{tA}\underline{x}_0} + B\underline{u}(t) + \underline{\int_0^t Ae^{(t-\tau)A} B\underline{u}(\tau)\, d\tau}$$

$$= A\underline{x}(t) + B\underline{u}(t) \checkmark$$

WHAT CAN I ACHIEVE WITH CONTROL?

"DEFINITION OF CONTROLLABILITY"

CAN I MAKE $\underline{x}(t_f) = \underline{x}_f$ ANYTHING I WANT FOR $t_f > 0$?

ROUGH PROOF

$$\underbrace{\left[\underline{x}(t_f) - e^{t_f A}\underline{x}_0\right]}_{\text{ARBITRARY VECTOR}} = \underbrace{e^{tA}}_{\text{ALWAYS NONSINGULAR}} \int_0^t \underbrace{e^{-\tau A} B\underline{u}(\tau)}\, d\tau$$

THIS IS A VECTOR $\underline{v}(\tau)$ MADE BY COMBINING COLUMNS OF $e^{-\tau A}B$

CAREFUL! $e^{tX}e^{tY} \neq e^{t(X+Y)}$ UNLESS $XY = YX$!

BUT $e^{-\tau A}$ CONTAINS ONLY $I, A, A^2 \ldots A^{n-1}$

SO WE CAN ONLY GET MULTIPLES OF COLUMNS OF $B, AB, A^2B, \ldots, A^{n-1}B$

$\Rightarrow \mathcal{C} \triangleq \left[B \;\; AB \;\; A^2B \;\ldots\; A^nB\right]$ MUST HAVE FULL RANK! THIS TURNS OUT TO BE NECESSARY & SUFFICIENT CONDITION FOR CONTROLLABILITY

"CONTROLLABILITY MATRIX"

RECAP $\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad \underline{x}(0) = \underline{x}_0$

$$\underline{y} = C\underline{x} + D\underline{u}$$

SHOWED $\underline{x}(t) = e^{tA}\underline{x}_0 + \int_0^t e^{(t-\tau)A} B\underline{u}(\tau)\,d\tau$

RANK $\mathcal{C}$ = RANK $\left[B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B\right] = n \iff$ CONTROLLABLE $\Rightarrow \exists\, \underline{u}(t)$ s.t. $\underline{x}(t_f) = \underline{x}_f$

SKIP AHEAD

WHAT ABOUT OBSERVATIONS?

FORMAL QUESTION ($\underline{u} = 0$) $\quad \dot{\underline{x}} = A\underline{x}, \quad \underline{x}(0) = \underline{x}_0$

$$y = C\underline{x}$$

GIVEN $y(t)$, $0 \le t \le t_f$, CAN WE DETERMINE $\underline{x}(0)$?

KNOWING $y(t)$ IS LIKE KNOWING $y(0), \dot{y}(0), \ldots y^{(k)}(0), \ldots$
BECAUSE $y(t)$ COULD BE WRITTEN WITH TAYLOR SERIES.

k^{TH} DERIVATIVE $y^{(k)}(0) = CA^k x_0 \qquad y^{(n)}(0) = CA^n x_0 = \underbrace{C\left[-a_1 A^{n-1} - a_2 A^{n-2} \ldots\right] x_0}_{\text{LINEAR COMBO OF } y(0) \text{ UP TO } y^{(n-1)}(0)}$

$$\Rightarrow \underline{y} = \begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = \mathcal{O} x_0$$

$x_0 = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T \underline{y}$ UNIQUE IF $(\mathcal{O}^T\mathcal{O})^{-1}$ EXISTS $\Rightarrow$ RANK $\mathcal{O} = n$

$\boxed{\text{DUALITY}}$

ESE 505 CONTROLLABILITY EXAMPLE 2014-04-14

$$A = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 4 & 4 & 4 \\ 5 & 10 & 20 \\ 0 & 0 & 0 \end{bmatrix}$$

$+3x$ $\quad -2x$

RANK $\mathcal{C} = 2$

x_3 UNCONTROLLABLE!

IN GENERAL, CAN IDENTIFY UNCONTROLLABLE MODE(S)

WHAT IF $A = \begin{bmatrix} 1 & 0 & x \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

$\mathcal{C}$ UNCHANGED!

WHAT IF $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ x & 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

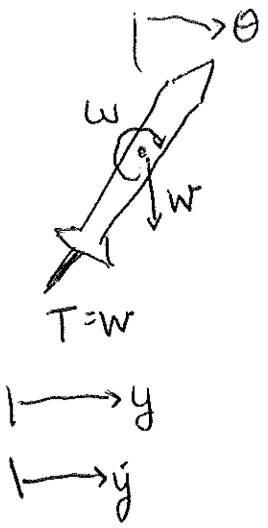
NOW $\mathcal{C} = \begin{bmatrix} 4 & 4 & 4 \\ 5 & 10 & 20 \\ 0 & 4x & 10x \end{bmatrix}$

ANY NON-ZERO $x \Rightarrow$ CONTROLLABLE.

(*) USE CONTROL TO INFLUENCE x_1 & x_1 TO INFLUENCE x_3.

SPACE-X GRASSHOPPER ROCKET

$\alpha^3 = g c_1$



$$\dot{y} = \dot{y}$$
$$\ddot{y} = g\theta$$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = u + \underbrace{c_1 \dot{y}}$$

ROCKET TIPS INTO WIND DUE TO FINS

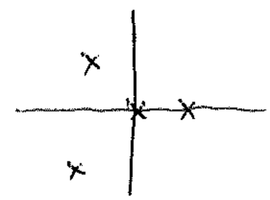
LET $x_4 = \frac{1}{g} y$, $x_3 = \frac{1}{g}\dot{y}$, $x_2 = \theta$, $x_1 = \omega$

$$\Rightarrow \dot{\underline{x}} = \begin{bmatrix} 0 & 0 & \alpha^3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \underline{u}$$

SPECIAL FORM CALLED "CONTROL CANONICAL FORM" (7.86 IN FRANKLIN)

TOP ROW IS $[-a_1 \ -a_2 \ \cdots \ -a_n]$

WHERE $\Delta(s) = s^n + a_1 s^{n-1} + \cdots + a_n$



HERE $\Delta(s) = s(s^3 - \alpha^3)$

⊛ FOR CONTROLLABLE SYSTEM, WE CAN <u>ALWAYS</u> FIND A STATE VARIABLE CHANGE TO GET A & B IN THIS FORM! ⊛

STATE VARIABLE CHANGE RELIES ON RANK $\mathcal{C} = n$! $\{\mathcal{C}^{-1}$ FOR SCALAR $u\}$

STATE FEEDBACK: $\underline{u} = -K\underline{x} = [-k_1 \ -k_2 \ -k_3 \ -k_4]\underline{x}$

$$\dot{\underline{x}} = [A - BK]\underline{x} = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & \cdots & -a_n - k_n \\ 1 & & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

NEW CHAR POLY IS

$$\Delta_{CL} = s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_n + k_n)$$

PICK ARBITRARY $\Delta_{CL}(s)$ USING STATE FEEDBACK!