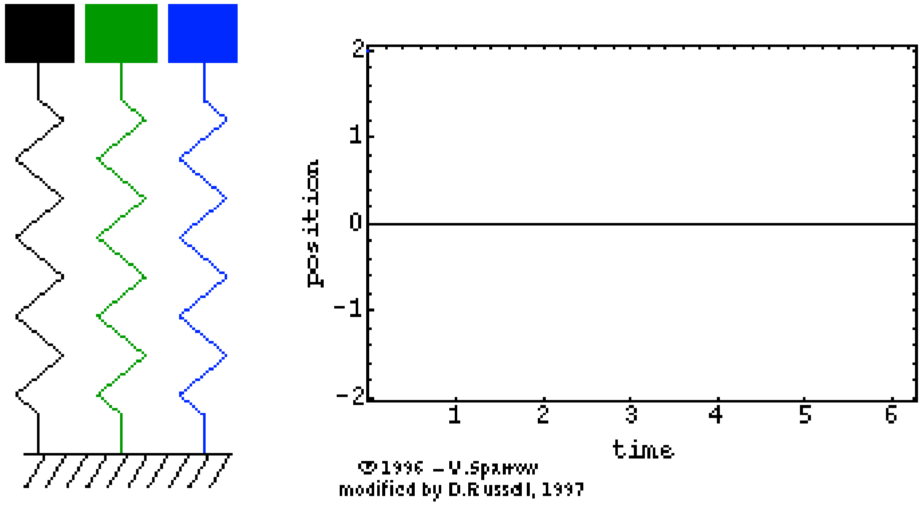
Pole Locations & Time Response



ESE 505 & MEAM 513

Bruce D. Kothmann

2014-02-12

Penn Engineering

System Response to Step Input

Recall General Form of Transfer Function (Rational Polynomial)

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_o s^m + b_1 s^{m-1} + \cdots + b_m}{a_o s^n + a_1 s^{n-1} + \cdots + a_n}$$

Factor Using Poles & Zeros & Gain

$$H(s) = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

Step Response Magnitude=A

$$Y(s) = H(s)U(s) = H(s)\frac{A}{s} = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} \frac{A}{s}$$

Partial-Fraction Expansion of Step Response

H(0) From "Cover Up Rule" Applied to (s=0)

$$Y(s) = \frac{AH(0)}{s} + \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}$$

$$y(t) = AH(0) + \underbrace{C_1 e^{p_1 t} + C_2 e^{p_2 t} + \cdots + C_n e^{p_n t}}_{}$$

→0 as t→∞ for Stable System

H(0) = "Steady State Gain"
("DC Gain")

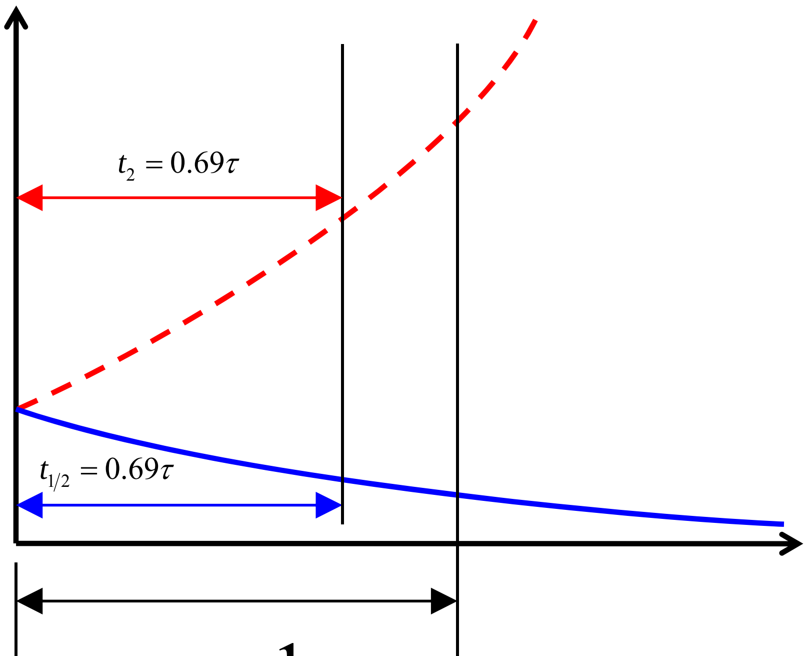
$$\frac{y(t)}{AH(0)} =$$ Dynamic Response Normalized by Steady Response

We Often Consider A=1 & Scale H(s) So That H(0)=1

With Linear Systems, Amplitudes Are Easy to Scale, so We Often Simplify at First

Dealing with Real Poles

$$p_k = -\sigma \;\Rightarrow\; \frac{C_k}{s - p_k} = \frac{C_k}{s + \sigma} \;\Rightarrow\; y(t) = \cdots + C_k e^{-\sigma t} + \cdots$$



$$\tau = \frac{1}{|\sigma|}$$

Stable System ($\boldsymbol{\sigma} > 0$)
$\boldsymbol{\tau}$ = Time for Response to Decay to 37% (1/e) of Initial Value

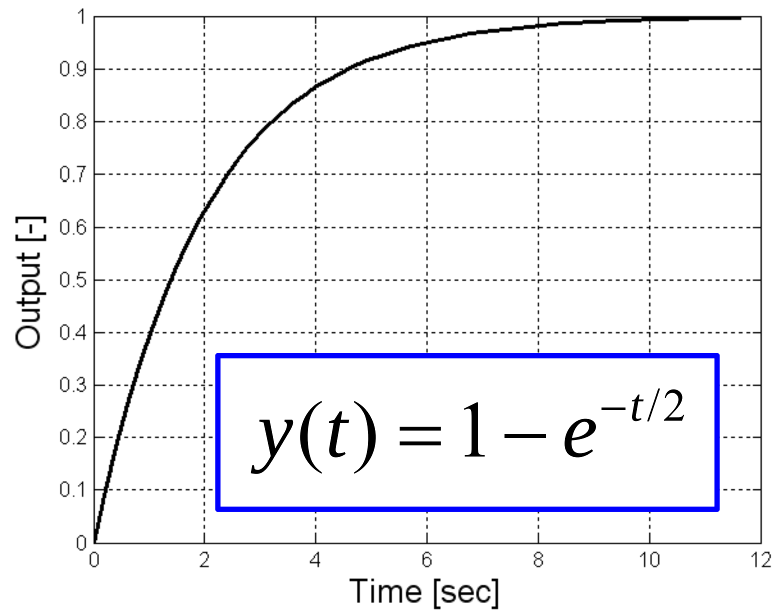
Untable System ($\boldsymbol{\sigma} < 0$)
$\boldsymbol{\tau}$ = Time for Response to Increase to 272% (e) of Initial Value

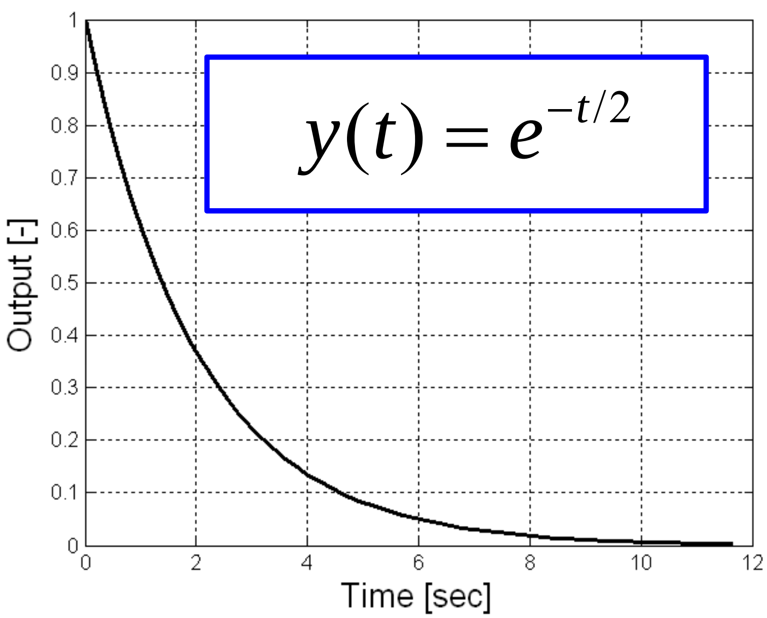
"Time to Half Amplitude" ($t_{1/2}$) & "Time to Double Amplitude" (t_2) Also Commonly Used to Describe First-Order Poles

Two Examples of First-Order Step Responses

$$H(s) = \frac{1}{2s+1} \quad Y(s) = \frac{1}{s} - \frac{1}{s+0.5}$$

Lag = “Low Pass”





Washout = “High Pass”

$$H(s) = \frac{2s}{2s+1} \quad Y(s) = \frac{0}{s} + \frac{1}{s+0.5}$$

Dealing with Complex Poles (with Damping)

$$\frac{C_k}{s-p_k}+\frac{\bar{C}_k}{s-\bar{p}_k}=\frac{C_k\left(s-\bar{p}_k\right)+\bar{C}_k\left(s-p_k\right)}{\left(s-p_k\right)\left(s-\bar{p}_k\right)}$$

$$C_k = A + jB$$

$$p_k = -\sigma + j\omega_d$$

$$=\frac{\left(A+jB\right)\left(s+\sigma+j\omega_d\right)+\left(A-jB\right)\left(s+\sigma-j\omega_d\right)}{\left(s+\sigma-j\omega_d\right)\left(s+\sigma+j\omega_d\right)}$$

$$=\frac{2A\left(s+\sigma\right)-2B\omega_d}{\left(s+\sigma\right)^2+\omega_d^2}=\frac{2As+\left(2A\sigma-2B\omega_d\right)}{s^2+2\sigma s+\left(\sigma^2+\omega_d^2\right)}$$

$$y(t)=\cdots+e^{-\sigma t}\left[2A\cos\left(\omega_d t\right)-2B\sin\left(\omega_d t\right)\right]+\cdots$$

Two Ways of Representing Complex Poles

Denominator of
Second-Order
Partial-Fraction
Expansion

$$\left(s+\sigma\right)^2+\omega_d^2$$

$$s^2+2\sigma s+\left(\sigma^2+\omega_d^2\right)$$

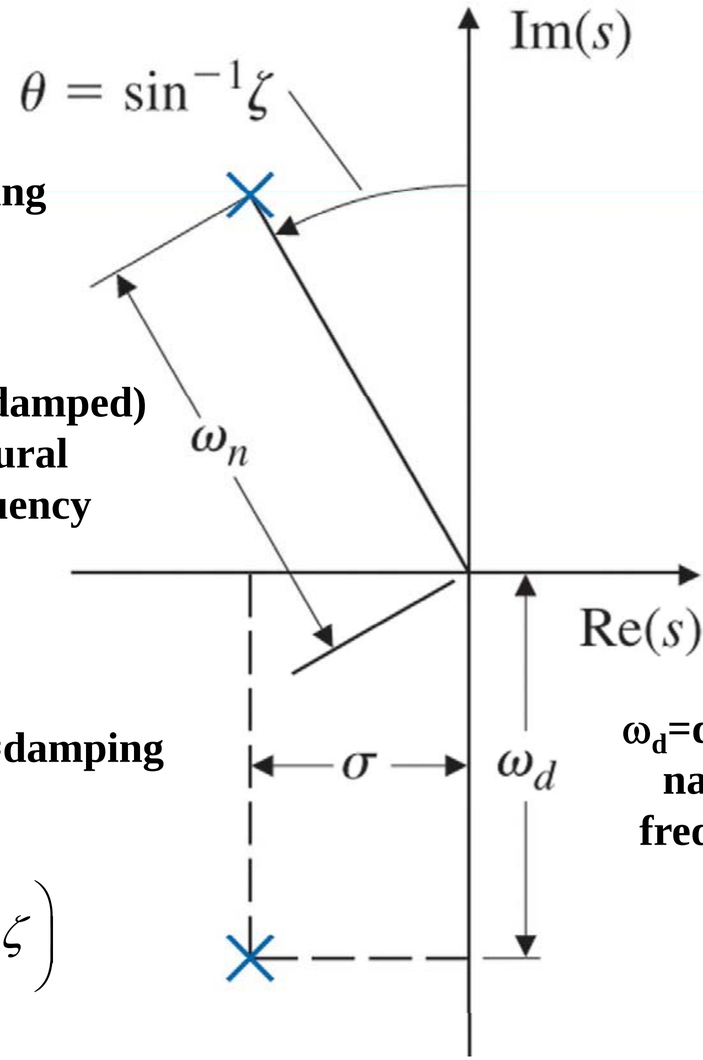
$$s^2+2\zeta\omega_n+\omega_n^2$$

$$p_k=-\sigma\pm j\omega_d=\omega_n e^{\pm j\left(\frac{\pi}{2}+\sin^{-1}\zeta\right)}$$

ζ=damping ratio

ω_n=(undamped) natural frequency

σ=damping



ω_d=damped natural frequency

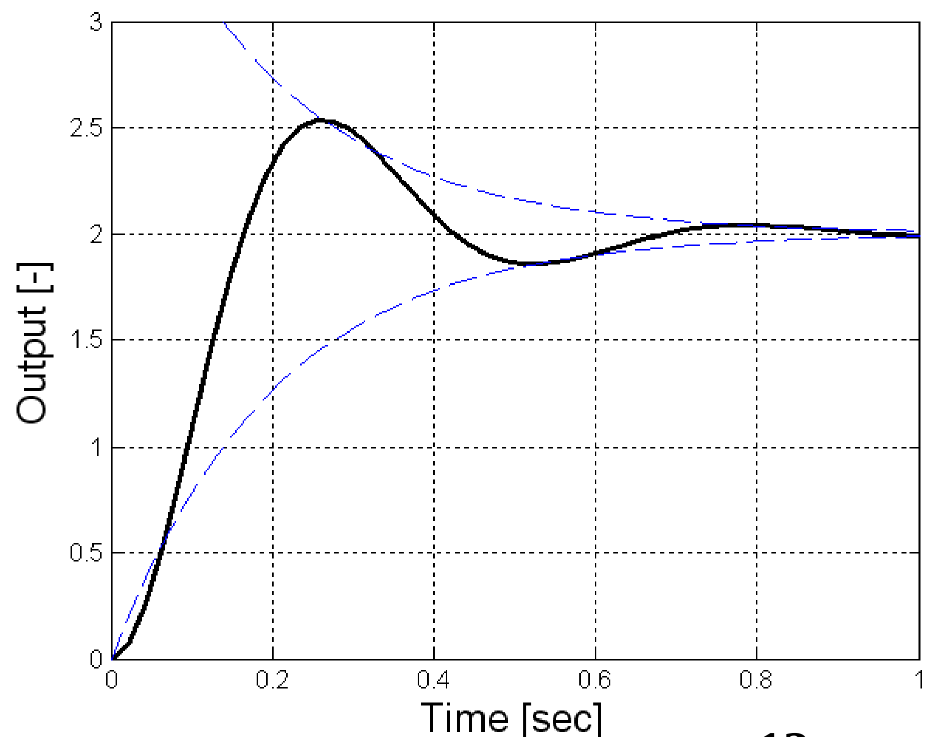
A Second-Order Example

$$H(s) = \frac{338}{s^2 + 10s + 169}$$

$$Y(s) = \frac{2}{s} - \frac{2s + 20}{s^2 + 10s + 169}$$

$$Y(s) = \frac{2}{s} - \frac{2(s+5) + 10}{(s+5)^2 + 12^2}$$

$$y(t) = 2 - 2e^{-5t}\cos(12t) - \frac{10}{12}e^{-5t}\sin(12t)$$



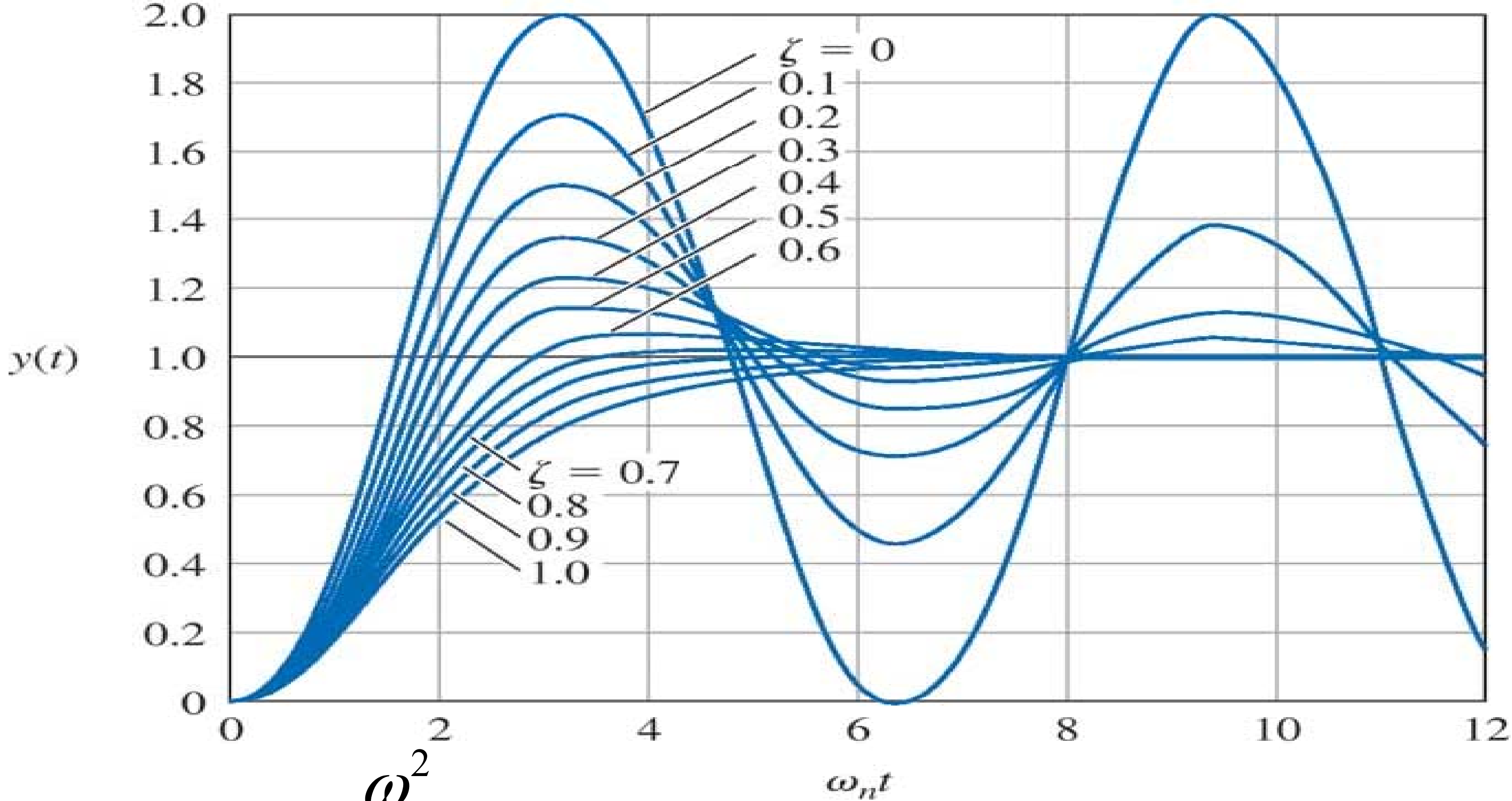
ω_d = 12 rps

σ = 5 rps

ζ = 5/13

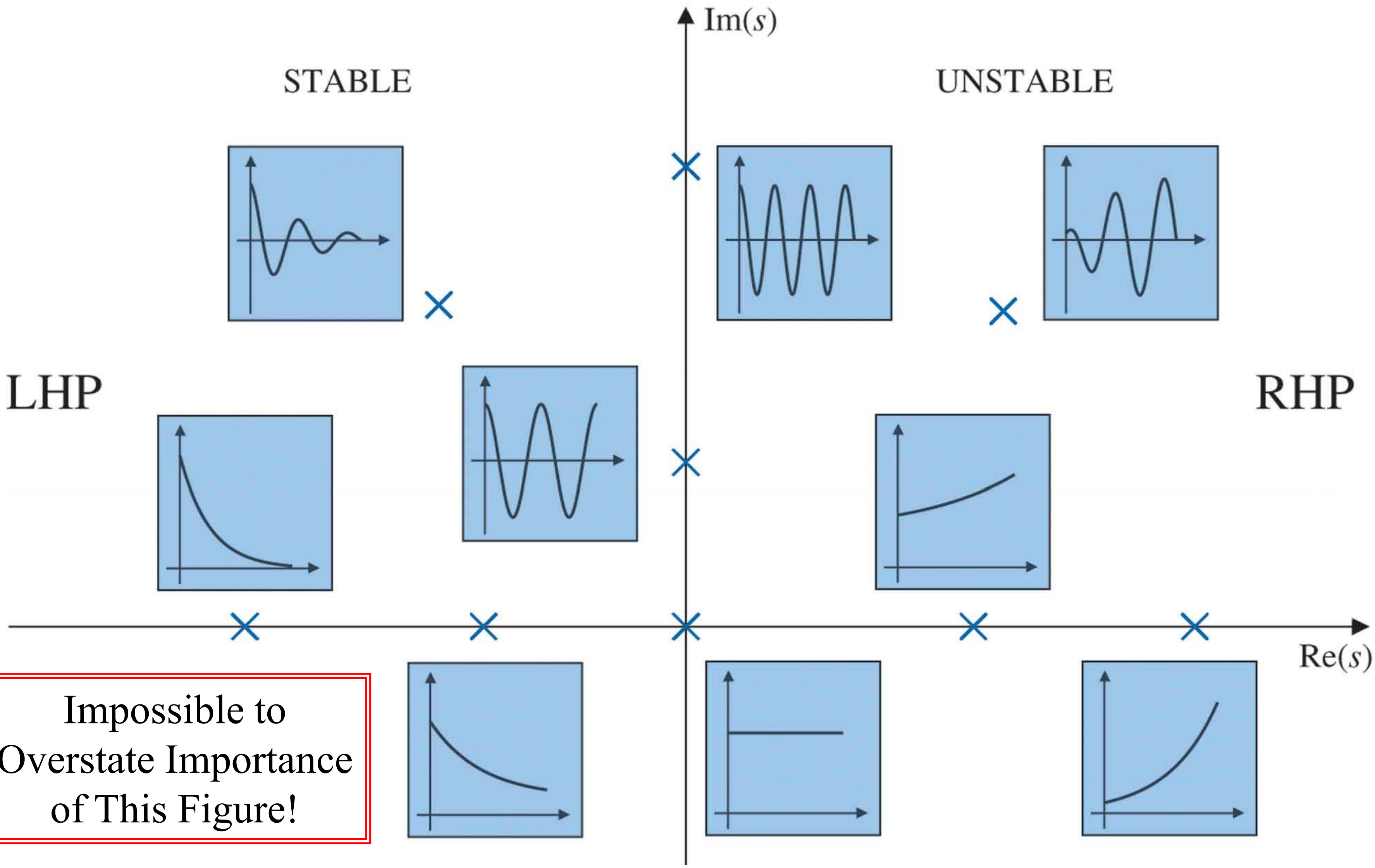
ω_n = 13 rps

Second-Order Step Responses



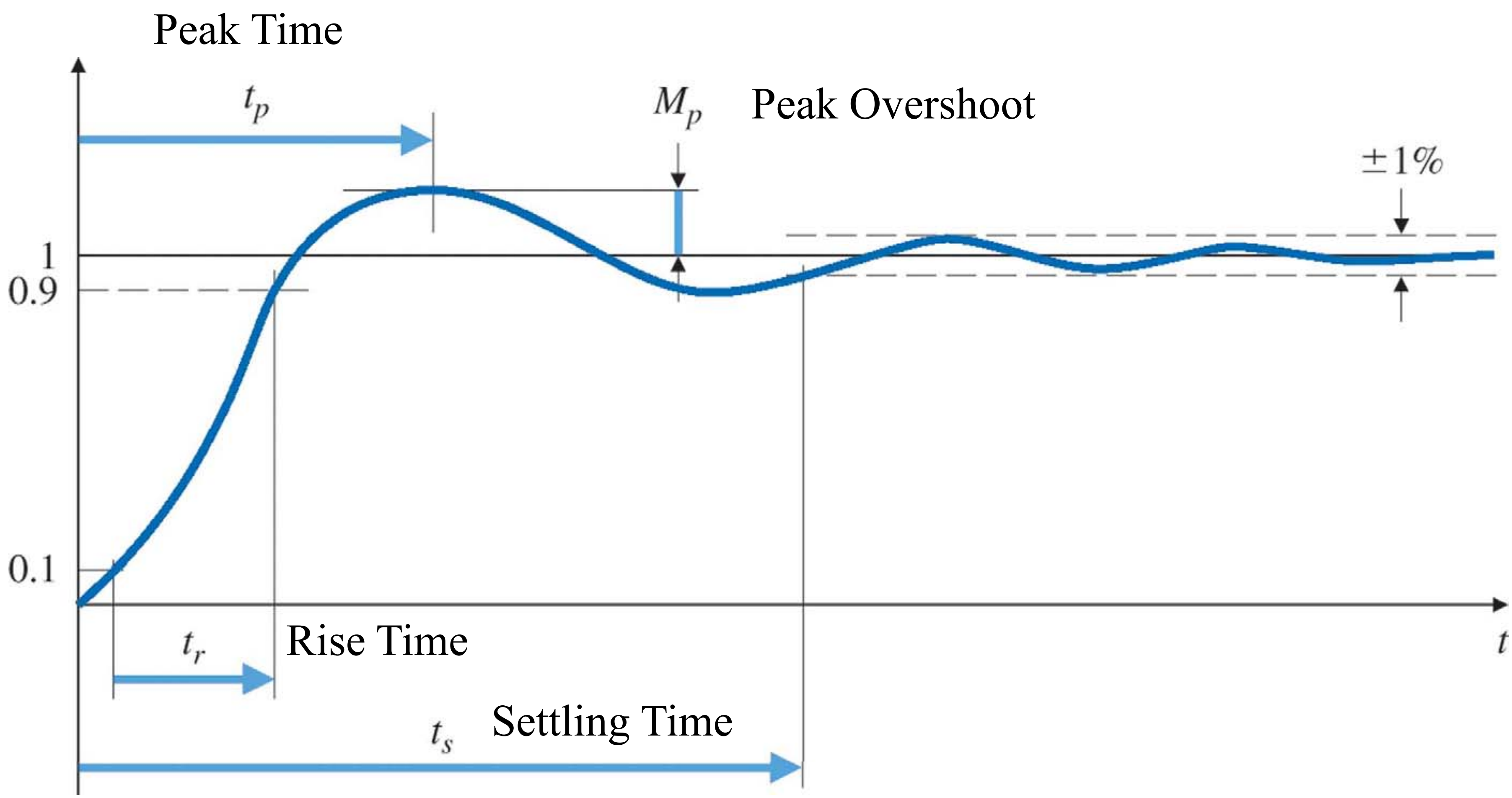
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Summary : Pole Locations & Impulse Response

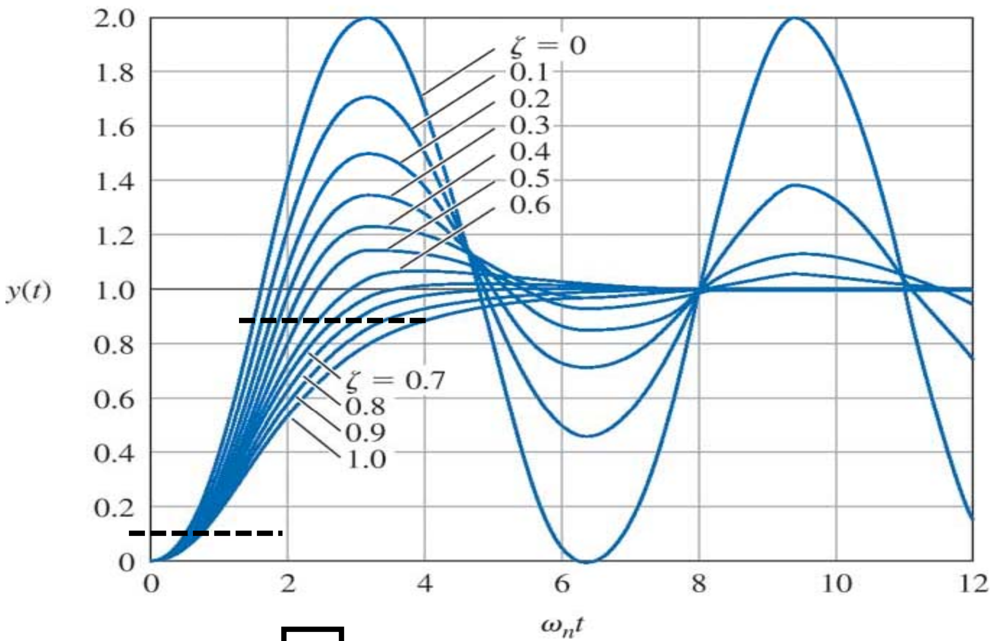


Impossible to
Overstate Importance
of This Figure!

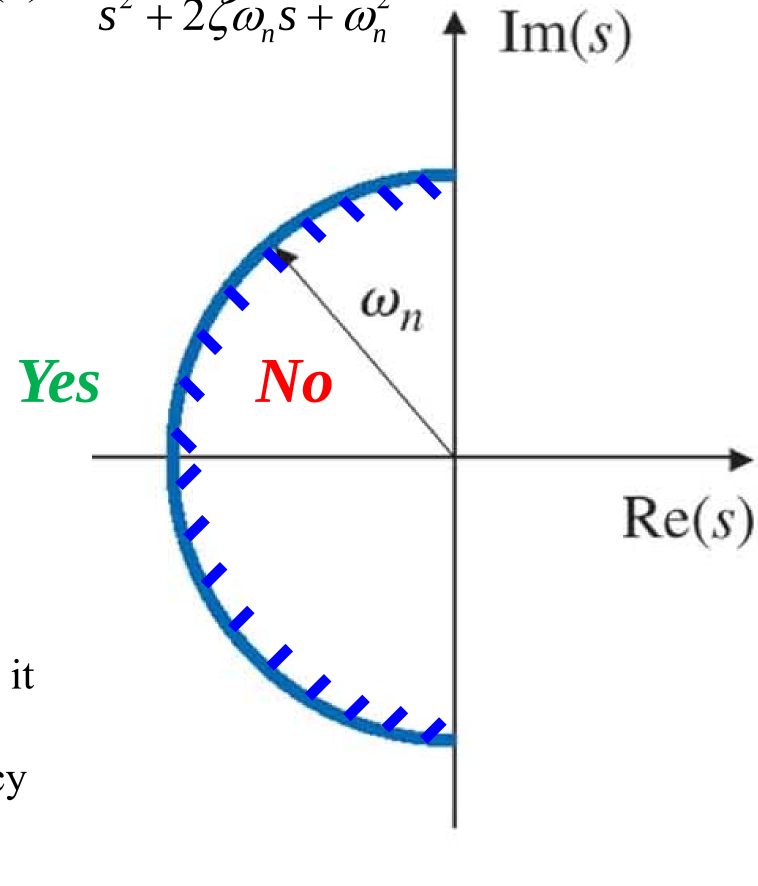
Specifications : Unit Step Response



Pole Location Limits : Rise Time



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



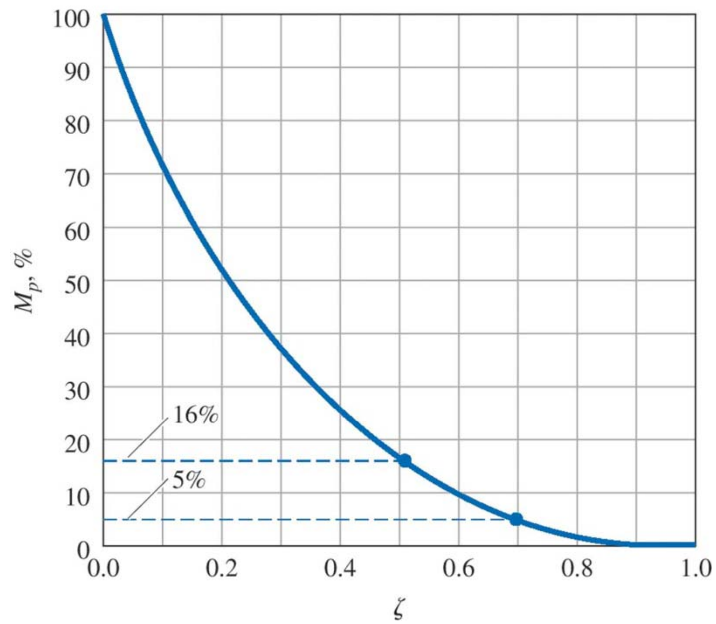
(a)

?

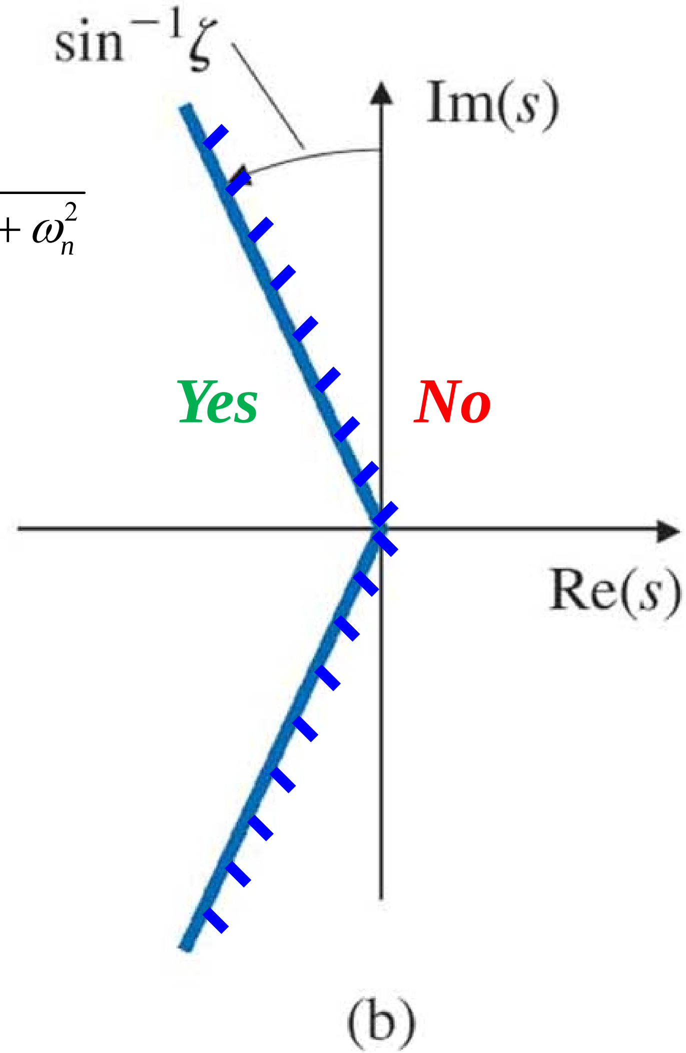
$$t_r \approx \frac{1.8}{\omega_n}$$

This isn't really true at all! But it is kind of useful. Roughly speaking, high natural frequency implies fast initial response…

Pole Location Limits : Peak Overshoot



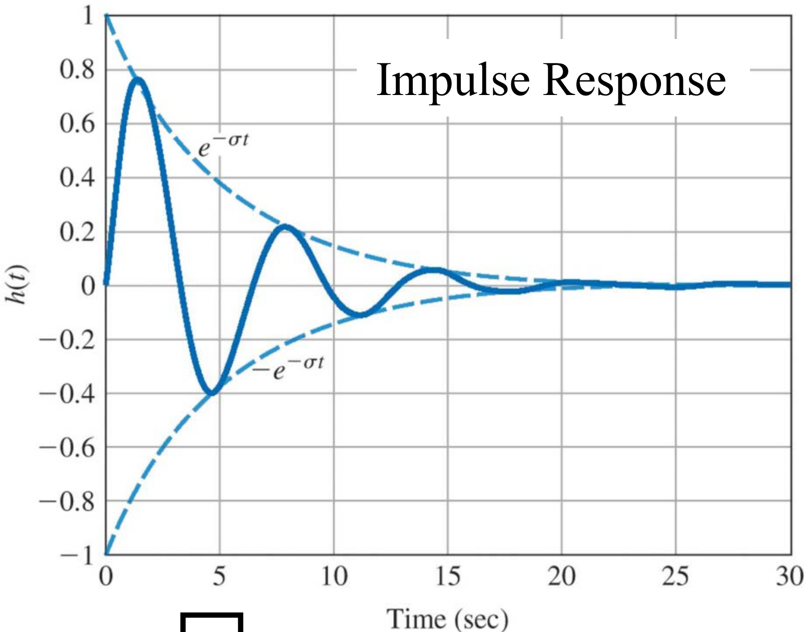
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$M_p = e^{-\pi\varsigma/\sqrt{1-\varsigma^2}}$$

This is formally true for a second-order system with no zeros. But zeroes can increase overshoot. Damping ratio is a good specification in general to prevent too many “wiggles” in the response.

Pole Location Limits : Settling Time

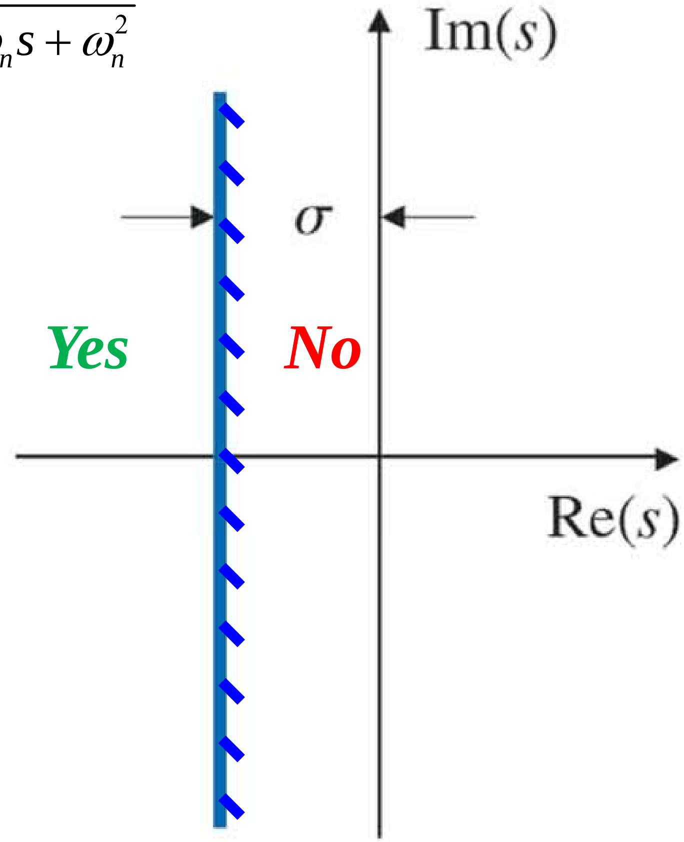


$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

?

$$t_s = \frac{4.6}{\sigma}$$

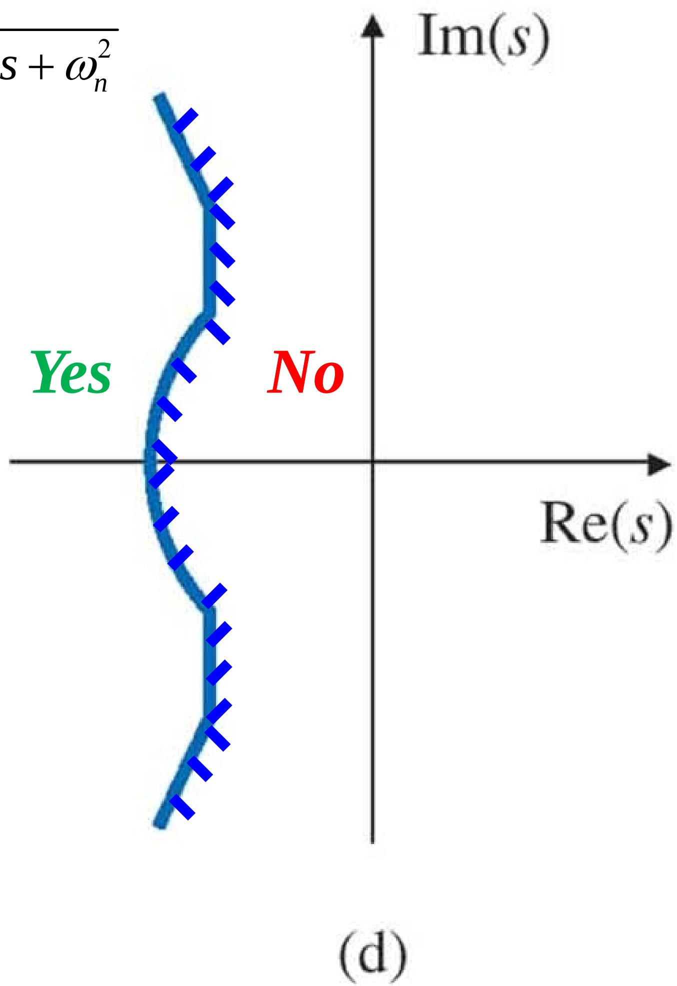
This is formally true and reasonable even for systems with zeros. Time it takes a system to settle to a steady state is strongly related to how far left the poles are in the complex plane.



(c)

Pole Location Limits : Composite Requirements

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Rotorcraft Requirements (ADS-33E) : Near Hover

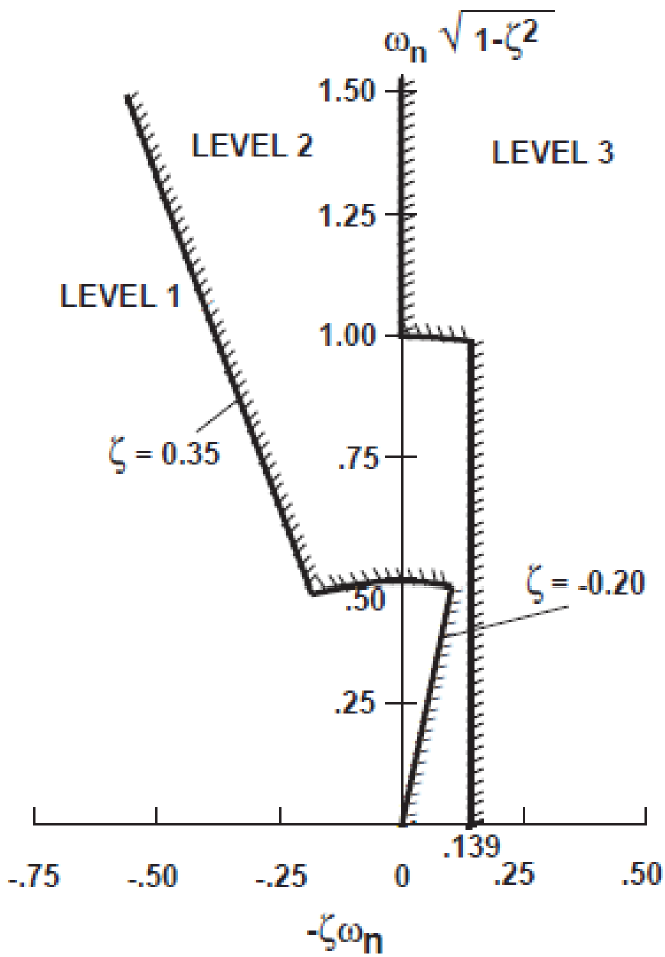


Figure 7. Limits on pitch (roll) oscillations – hover and low speed

Rotorcraft Requirements – Forward Flight

- Difficulty of Completing Task Governed by Pole Locations (Boundaries Determined Using Simulation & Flight-Test Experiments)
- Pole Location Requirements Depend on Tasks to be Completed by Pilot

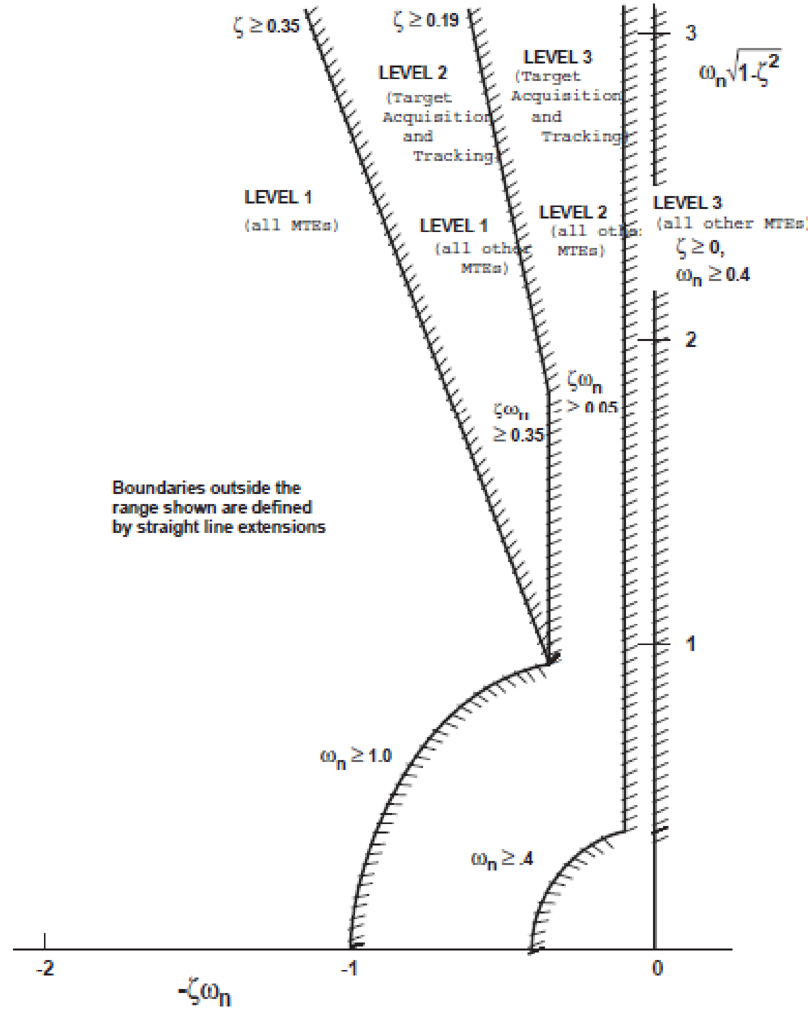


Figure 23. Lateral-directional oscillatory requirements

Appendix

More Details & Comments

Math for 2nd-Order Response Specs

Assume Second-Order Transfer Function with Unity DC Gain

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Response to Unit Step

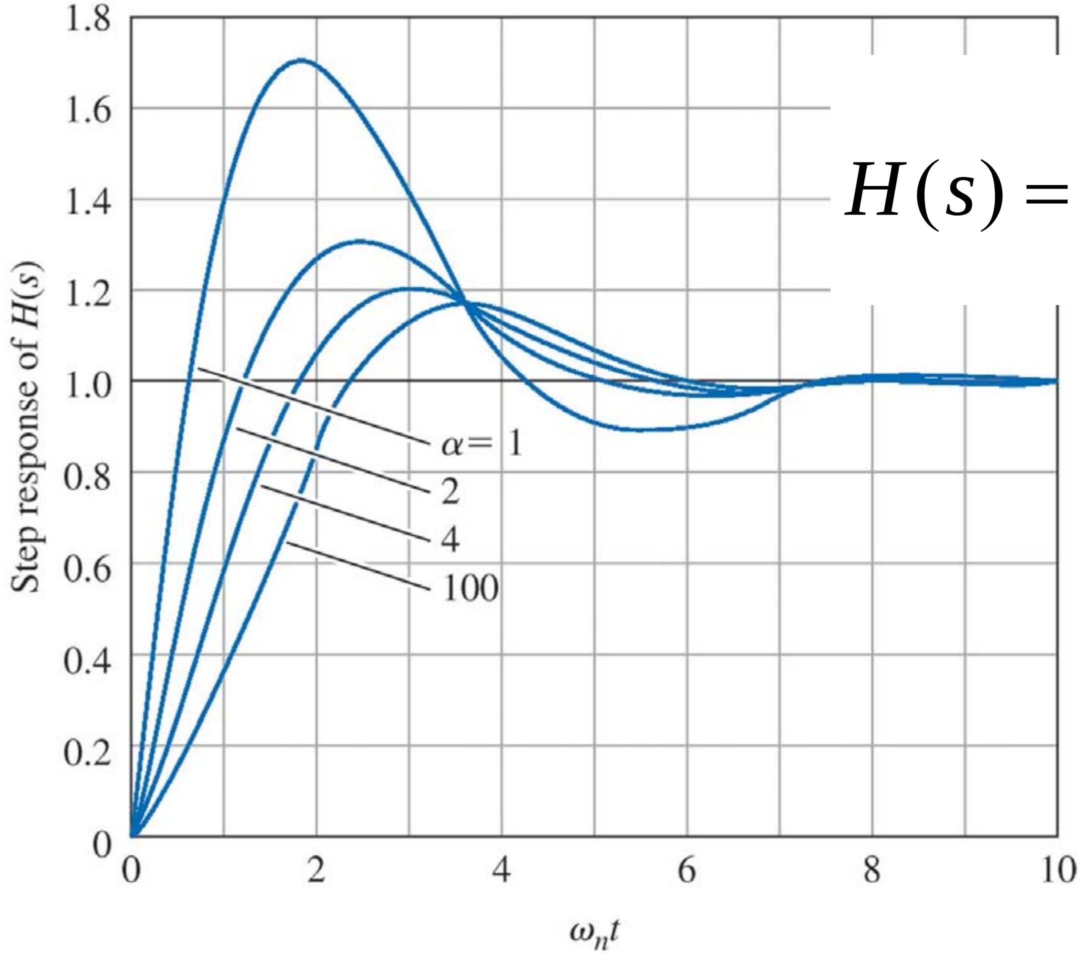
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}$$

$$\boxed{y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right]}$$

y(0)=0 ✓
dy/dt(0)=0 ✓

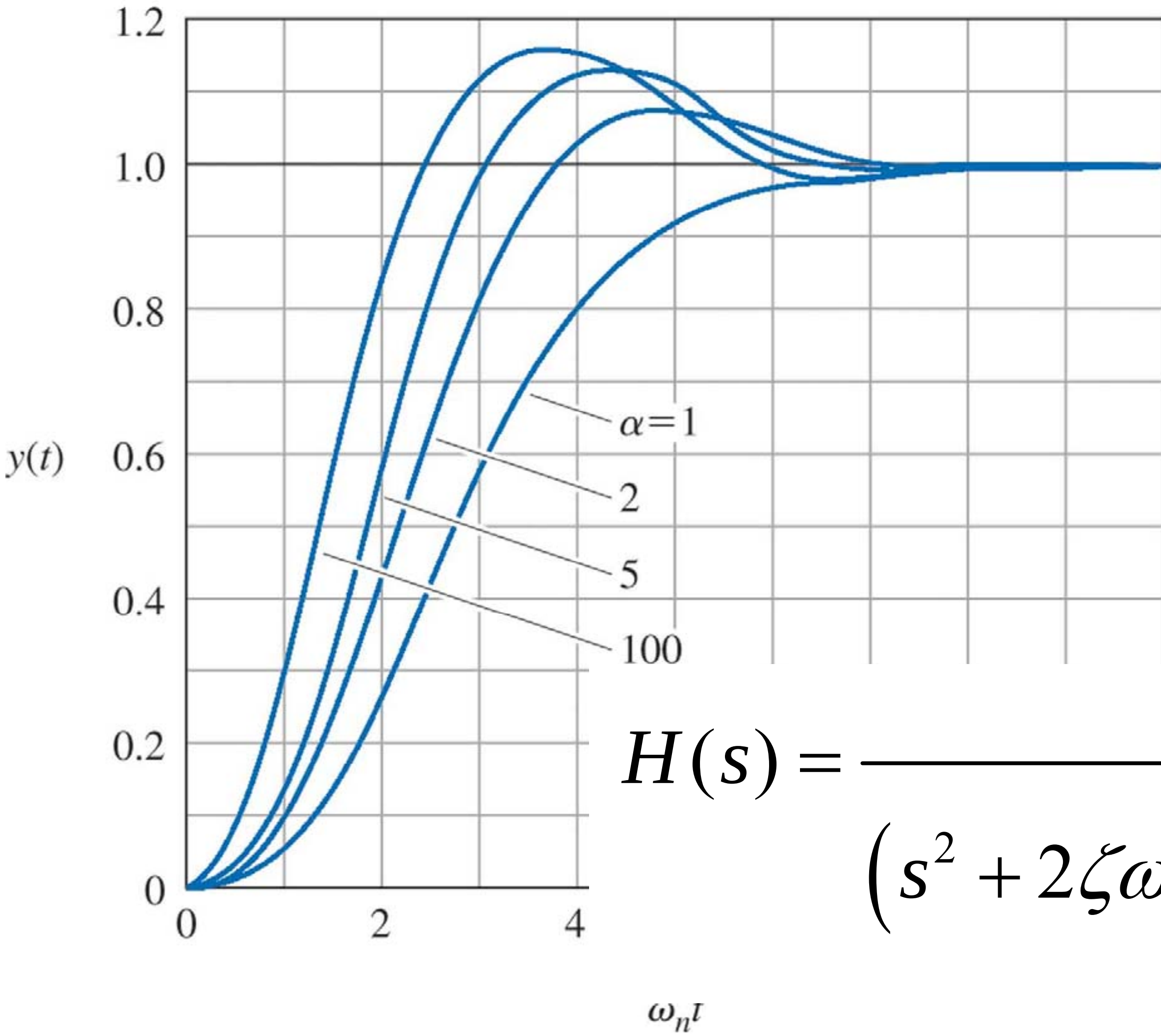
Now Do Math to Figure Out Relationship of Time Specs to Pole Locations (Section 4.3 in Textbook)

Effect of Zero on Second-Order Response



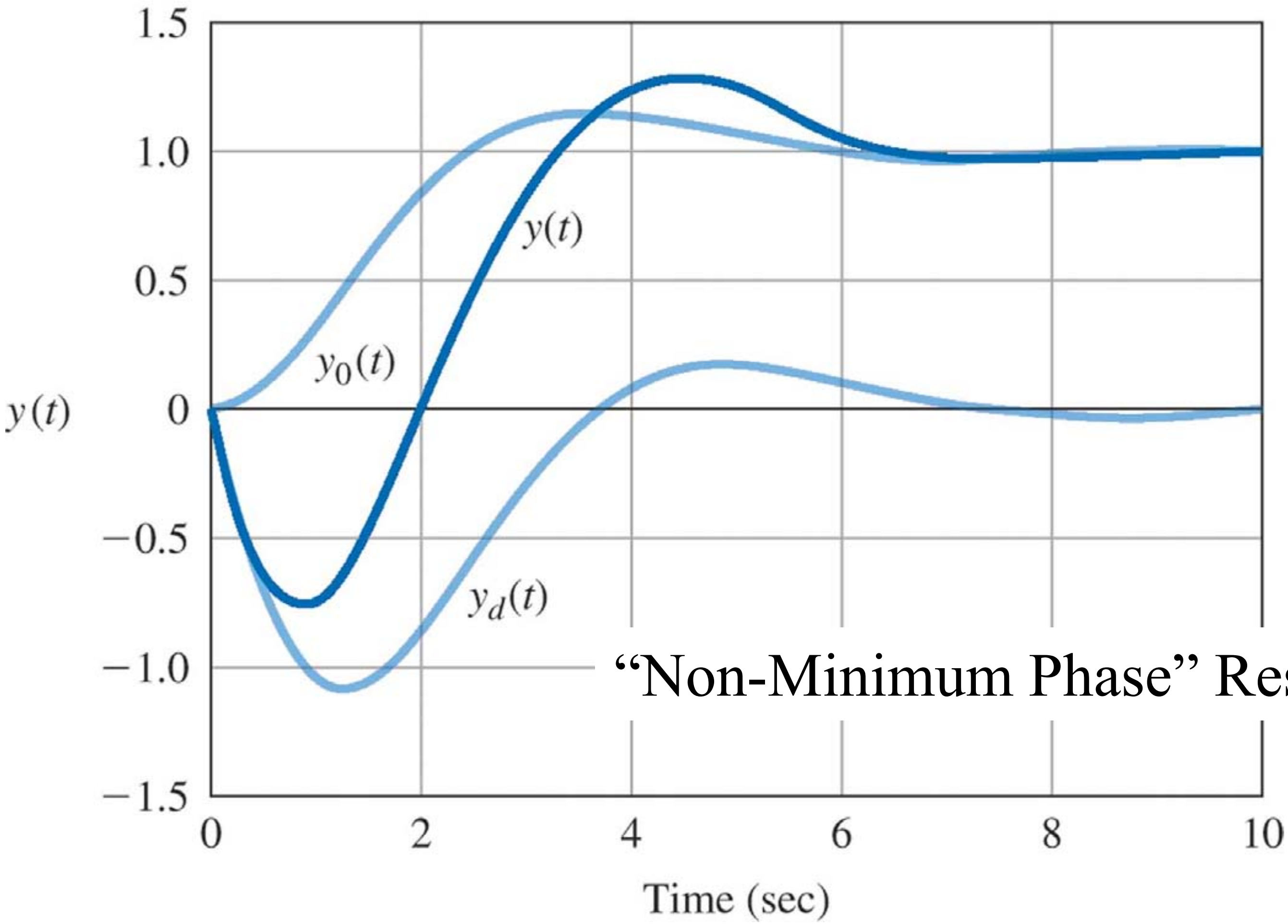
$$H(s) = \frac{(\omega_n s)/(\alpha\zeta) + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Third-Order System Step Response



$$H(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)\left(\dfrac{s}{\alpha\zeta\omega_n} + 1\right)}$$

System with Zero in Right-Half Plane



"Non-Minimum Phase" Response

Important Things to Remember

- Relationship Between Time-Domain Specifications & Pole Locations Strictly Applies Only for Simple Second-Order Systems
- Zeros in Numerator Affect Response
  - Decrease Rise Time (Get Going Faster)
  - Increase Overshoot
- Additional Poles Affect Response in Variety of Ways
- Feed-Forward Compensation Profoundly Alters Transient Response to Command Inputs
- So...Connection Between Time Response to Command Inputs & Pole Locations is Actually Very Loose
- But...Pole Location Specs Are Very Widely Used!

More Typical Requirements

- "High-Frequency Modes" Must Have Damping Ratio of At Least 0.5% (But More Like 2% is Preferred)
  - Aircraft Structural Vibration
  - Rotor Dynamics

- No Objectionable Vibrations
  - Generally Avoid Putting Structural Modes Near Harmonics of Rotor Frequency to Avoid Excitation
  - Note That This is Distinct from Stability...Don't Confuse These!

- Stability Robustness = System Must Be Stable Even if Actual Plant Dynamics Different From Nominal or Expected (We Need Frequency Response Tools to Get Precise with This Idea...After Spring Break)