Input-Output Linearization

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1 Definitions

We consider the state, $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, and the output $y \in \mathbb{R}^m$ we wish to regulate. We assume that the state equations are affine:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $f(x)$ is a smooth vector fields on $\mathbb{R}^n$, $g(x)$ is a $n \times m$ matrix of smooth functions, and the output is given by a smooth vector field $h(x)$:

$$y = h(x).$$

In order to evaluate changes of the output, we will need the definition of a *Lie derivative* of a function $f : \mathbb{R}^n \to R$ along a vector field X defined on $\mathbb{R}^n$. The Lie derivative $\mathcal{L}_X$ fof a function $f(x)$ along a vector field X is the rate of change of $f(x)$ along the flow of X. In other words, this is the directional derivative of f along X:

$$\mathcal{L}_X f = X \cdot \nabla f \tag{2}$$

In component form, if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \ X(x) = \begin{bmatrix} X_1(x) \\ X_2(x) \\ \dots \\ X_n(x) \end{bmatrix}$$

and similarly if the vector field X has components X_i that are functions of x,

$$\mathcal{L}_X = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{bmatrix}.$$

We will primarily be interested in the rate of change of the output function $h(x)$ along the vector fields $f(x)$ and $g(x)$.

2 Single-Input-Single-Output System

Let us consider the SISO case when $m = 1$. Our goal is to drive y to the desired trajectory $y^{\mathrm{des}}(t)$, and to guarantee exponential convergence to the desired trajectory.

The rate of change of the output is given by

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f(x) + g(x)u)$$

We can see that this expression can be written in terms of two Lie derivatives:

$$\dot{y} = \mathcal{L}_f h + \mathcal{L}_g h u \tag{3}$$

where

$$\mathcal{L}_f h = \frac{\partial h}{\partial x} f(x), \quad \mathcal{L}_g h = \frac{\partial h}{\partial x} g(x). \tag{4}$$

First Order System If the system is first order, exponential convergence is achieved if the output satisfies the following first order differential equation.

$$\dot{y} - \dot{y}^{\text{des}} + k(y - y^{\text{des}}) = 0 \tag{5}$$

We can rewrite this as follows:

$$\mathcal{L}_f h + \mathcal{L}_g h u - \dot{y}^{\text{des}} + k(y - y^{\text{des}}) = 0$$

This gives us the control law:

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\text{des}} + k(y^{\text{des}} - y)\right) \tag{6}$$

provided $\mathcal{L}_g h \neq 0$. Note that if this Lie derivative were zero, the system would not be a first order system. You can verify that for second and higher order systems, this derivative is zero.

Second Order System For a second order system, $\mathcal{L}_g h = 0$. Further, exponential convergence of y to the desired trajectory $y^{des}(t)$ is achieved if the following equation to be satisfied.

$$\ddot{y} - \ddot{y}^{\text{des}} + k_1(\dot{y} - \dot{y}^{\text{des}}) + k_2(y - y^{\text{des}}) = 0 \tag{7}$$

By differentiating (3), we obtain the expression for $\ddot{y}$:

$$\begin{aligned} \ddot{y} &= \frac{\partial}{\partial x}\left[\mathcal{L}_f h + \mathcal{L}_g h u\right](f(x) + g(x)u) \\ &= \mathcal{L}_f \mathcal{L}_f h + \mathcal{L}_g \mathcal{L}_f h u \end{aligned} \tag{8}$$

Substituting in (7), we get

$$\mathcal{L}_f \mathcal{L}_f h + \mathcal{L}_g \mathcal{L}_f h u - \ddot{y}^{\text{des}} + k_1(\dot{y} - \dot{y}^{\text{des}}) + k_2(y - y^{\text{des}}) = 0$$

Thus, if $\mathcal{L}_g \mathcal{L}_f h \neq 0$, the following controller guarantees exponential convergence.

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f h}\left(-\mathcal{L}_f \mathcal{L}_f h + \ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y)\right) \tag{9}$$

Third Order System For a third order system, $\mathcal{L}_g h = 0, \mathcal{L}_g \mathcal{L}_f h = 0$. We desire

$$\dddot{y} - \dddot{y}^{\text{des}} + k_1(\ddot{y} - \ddot{y}^{\text{des}}) + k_2(\dot{y} - \dot{y}^{\text{des}}) + k_3(y - y^{\text{des}}) = 0$$

If $\mathcal{L}_g \mathcal{L}_f^2 h \neq 0$, the controller that guarantees the satisfaction of the above equation is:

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f^2 h} \left(-\mathcal{L}_f^3 h + \dddot{y}^{\text{des}} + k_1(\ddot{y}^{\text{des}} - \ddot{y}) + k_2(\dot{y}^{\text{des}} - \dot{y}) + k_3(y^{\text{des}} - y) \right) \tag{10}$$

Relative degree, r Clearly, the number of times one has to differentiate the output $y(t)$ in order to have an explicit dependence on $u(t)$ is important and is related to the order of the system. We define the relative degree as $r = k + 1$ where k is the index in the sequence:

$$\mathcal{L}_g h,\ \mathcal{L}_g \mathcal{L}_f h,\ \mathcal{L}_g \mathcal{L}_f^2 h, \ldots, \mathcal{L}_g \mathcal{L}_f^k h, \ldots$$

such that all the terms preceding the k^{th} function are zero. In other words, r is the index of the first nonzero function in this sequence. Formally, the first, second, and the third order systems have relative degrees 1, 2, and 3 respectively.

General form of feedback law By reviewing the forms of the feedback laws (6, 9, 10), we can write the general form of the nonlinear feedback law to be:

$$u = \alpha(x) + \beta(x) v \tag{11}$$

where

$$\alpha(x) = \frac{-\mathcal{L}_f^r h}{\mathcal{L}_g \mathcal{L}_f^{r-1} h}, \quad \beta(x) = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{r-1} h},$$

which reduces the original system to a system of r integrators:

$$y^{(r)} = v. \tag{12}$$

Thus the control in Equation (11) reduces the affine, nonlinear system to the linear system in (12) with the input v. This is called *input-output linearization*. The nonlinear system has effectively been reduced to a linear system for which the control system design is trivial and has the form:

$$v = y^{(r),\text{des}} - k_r(y^{(r-1)} - y^{(r-1),\text{des}}) + \ldots - k_2(\dot{y} - \dot{y}^{\text{des}}) - k_1(y - y^{\text{des}}) \tag{13}$$

3 Extension to Multiple-Input-Multiple-Output Systems

We treat the special case of MIMO systems in which the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}^m$, and each output y_i has the same relative degree. In this special case, we can apply the theory above by simply stacking up the expressions for each output component. Of course, directional derivatives of vectors along vector fields will now be vectors. For example, $\mathcal{L}_f h$ is a $m \times 1$ vector field. Further, we will have m Lie derivatives along the vector fields $\{g_1(x), g_2(x), \ldots, g_m(x)\}$. Accordingly, we must modify (11) so that:

$$\alpha(x) = \left(\mathcal{L}_g \mathcal{L}_f^{r-1} h \right)^{-1} (-\mathcal{L}_f^r h), \quad \beta(x) = \left(\mathcal{L}_g \mathcal{L}_f^{r-1} h \right)^{-1}.$$

4 Planar Quadrotor

The state equations for a planar quadrotor are given by:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{14}$$

The two outputs of interest are:

$$\mathbf{y} = \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}$$

However, as a quick calculation will reveal, the system is *not* input output linearizable. In other words, there is no nonzero Lie derivative to define the relative degree for either output function in $y(x)$.

For quadrotors, we must use *dynamic state feedback*, a feedback policy that uses higher order derivatives of the state. Rather than explain this concept in its full generality, we develop the basic theory for planar quadrotors. Define a new state vector:

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \dddot{y} \\ \dddot{z} \end{bmatrix} = \begin{bmatrix} y^{(iii)} \\ z^{(iii)} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} -u_1\dot{\phi}\cos\phi - \dot{u}\sin\phi \\ -u_1\dot{\phi}\sin\phi + \dot{u}\cos\phi \end{bmatrix} \tag{15}$$

It is not too hard to show that

$$\dot{\mathbf{z}}_4 = \begin{bmatrix} y^{(iv)} \\ z^{(iv)} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} -\sin\phi & -\frac{u_1}{I_{xx}}\cos\phi \\ \cos\phi & -\frac{u_1}{I_{xx}}\sin\phi \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} + \frac{1}{m} \begin{bmatrix} -2\dot{u}_1\dot{\phi}\cos\phi + u_1\dot{\phi}^2\sin\phi \\ -2\dot{u}_1\dot{\phi}\sin\phi - u_1\dot{\phi}^2\cos\phi \end{bmatrix} \tag{16}$$

Accordingly define a new input vector $\bar{\mathbf{u}}$:

$$\bar{\mathbf{u}} = \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}.$$

If we define a vector of fictitious inputs

$$\mathbf{v} = [v_1, v_2]^{\mathrm{T}},$$

so that,

$$\dot{\mathbf{z}}_4 = \mathbf{v},$$

these inputs are related to the inputs $\bar{\mathbf{u}}$ by a nonlinear transformation, from (16), of the form:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{p}(\mathbf{z}) \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} + \mathbf{q}(\mathbf{z}) \tag{17}$$

This reduces the state equations to the canonical form:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v} \tag{18}$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} \\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} \\ \mathbf{I}_{2\times2} \end{bmatrix},$$

for which the control system design problem is quite simple. Of course the real inputs are obtained from the virtual inputs through a nonlinear transformation:

$$\begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} = \alpha(\mathbf{z}) + \beta(\mathbf{z}) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{19}$$

which is obtained through a matrix inversion from (17).