"MODERN CONTROL" USUALLY REFERS TO STATE-SPACE METHODS

WE BEGIN WITH

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

$$y = C\underline{x} + D\underline{u}$$

SOLUTION OF HOMOGENEOUS EQUATION

CONSIDER

$$\underline{x}(0) = \underline{x}_0, \quad \underline{u} \equiv \underline{0}$$

EIGENSTRUCTURE $\quad A\underline{q} = \lambda\underline{q} \quad \underline{q} \neq \underline{0} \Rightarrow \underline{q}$ = EIGENVECTOR

λ = EIGENVALUE

$$(\lambda I - A)\underline{q} = \underline{0}$$

$$\Downarrow$$

$$\Delta(\lambda) = |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

$\exists$ n EIGENVALUES OF A $\Rightarrow$ THESE ARE OUR POLES.

EACH λ HAS AT LEAST ONE $\underline{q}$.

REPEATED ROOTS OF $\Delta(\lambda)$ MAY NOT HAVE MULTIPLE $\underline{q}$.

EXAMPLE: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \Delta(\lambda) = (\lambda-2)^2$

$\lambda = 2, 2$ REPEATED

$(A - \lambda I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \underline{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ONLY 1 EIGENVECTOR!

WE'LL ASSUME n EIGENVECTORS (GOOGLE: "GENERALIZED EIGENVECTOR" & "JORDAN FORM")

LET $\underline{x} = \sum_{j=1}^{n} c_j(t)\, \underline{q}_j$ WITH $A\underline{q}_j = \lambda_j \underline{q}_j$

$$\dot{\underline{x}} = \sum_j \dot{c}_j \underline{q}_j = \sum_j c_j A\underline{q}_j = \sum_j c_j \lambda_j \underline{q}_j$$

$$\Rightarrow \dot{c}_j = \lambda_j c_j \qquad c_j(t) = e^{\lambda_j t} c_j(0)$$

DYNAMICS UNCOUPLE!

NOTE
$AQ = Q\Lambda$
$A = Q\Lambda Q^{-1}$

$$\underline{x} = Q\underline{c}(t) \qquad Q = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_n \end{bmatrix}$$

$$\underline{x}(0) = Q\underline{c}(0) = \underline{x}_0$$

$$\underline{c}(0) = Q^{-1}\underline{x}_0$$

$$\underline{c}(t) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \underline{c}(0)$$

So
$$\underline{x}(t) = Q \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} Q^{-1}\underline{x}_0$$

$$\underline{x}(t) \triangleq e^{tA}\underline{x}_0 \qquad e^{tA} = Q e^{t\Lambda} Q^{-1}$$

COMPARE TO SCALAR CASE: $\dot{x} = ax \Rightarrow x = e^{ta}x_0$ ✓

BY ANALOGY TO SCALAR CASE, WE CAN GUESS SOLUTION WHEN $\underline{u} \neq \underline{0}$:

$$\boxed{\underline{x}(t) = e^{tA}\underline{x}_0 + \int_0^t e^{(t-\tau)A} B\underline{u}(\tau)\, d\tau}$$

WE CAN ALSO WRITE e^{tA} USING A TAYLOR SERIES:

$$e^{tA} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

NOW OBSERVE WHAT HAPPENS IF WE SUBSTITUTE A INTO $\Delta(\lambda)$:

$$\Delta(A) = A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I$$

SINCE $A = Q\Lambda Q^{-1}$, $A^2 = Q\Lambda Q^{-1} Q \Lambda Q^{-1} = Q\Lambda^2 Q^{-1}$, ETC...

SO $A^k = Q\Lambda^k Q^{-1}$

THUS, $Q^{-1}\Delta(A)Q = \Lambda^n + a_1 \Lambda^{n-1} + \dots + a_{n-1}\Lambda + a_n I = 0$

(EACH MATRIX IS DIAGONAL & CONTAINS $\Delta(\lambda)$)

* CAYLEY-HAMILTON THEOREM *

MATRIX SATISFIES ITS OWN CHARACTERISTIC EQUATION

<u>CONSEQUENCE</u>:

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \dots - a_n I$$

ALL POWERS OF $A \geq n$ CAN BE WRITTEN AS SUM OF POWERS 0 TO $n-1$!

WRITE AS TAYLOR SERIES & REPLACE HIGHER POWERS.

THUS, $e^{(t-\tau)A}B$ ALWAYS HAS COLUMNS THAT ARE MULTIPLES OF COLUMNS OF $B, AB, A^2B, \dots A^{n-1}B$.

SO, CONTROL CAN ONLY CHANGE STATE BY MULTIPLES OF THE COLUMNS OF

$$\mathcal{C} = [B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B]$$

"CONTROLLABILITY MATRIX"

RANK $\mathcal{C} = n \Rightarrow \mathcal{C}$ HAS n LINEARLY INDEPENDENT COLUMNS

"CONTROLLABLE" SYSTEM. $\Rightarrow \underline{u}(t)$ CAN BE USED TO ACHIEVE ANY DESIRED LOCATION $\underline{x}(t_f)$, $t_f > 0$.

EIGENSTRUCTURE ASSIGNMENT

LETTING $\underline{u} = -K\underline{x}$ "FULL STATE FEEDBACK"

WE FIND

$$\dot{\underline{x}} = (A - BK)\underline{x}$$

CLOSED-LOOP POLES ARE ROOTS OF ..

$$\Delta_{CL}(s) = |\lambda I - (A-BK)| = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_{n-1}\lambda + \alpha_n = 0$$

(*) IF RANK $\mathcal{C} = n$, CAN CHOOSE K TO ACHIEVE ARBITRARY REAL α_i $\Rightarrow$ CAN PLACE POLES!

IF $\underline{u}$ HAS m ELEMENTS, $K \in \mathbb{R}^{m \times n}$ $\Rightarrow$ WE CAN ALSO CHOOSE SOME OF THE CLOSED-LOOP EIGENVECTORS!

IF $\underline{u} = u$ $(m=1)$, THEN $K \in \mathbb{R}^{1 \times n}$ IS A ROW VECTOR (n GAINS)

WE CAN CHOOSE K ONLY TO PLACE POLES.

LET $\underline{x} = T\underline{z}$ $\Rightarrow$ $\dot{\underline{x}} = T\dot{\underline{z}} = A\underline{x} + B\underline{u} = AT\underline{z} + B\underline{u}$

$$\left.\begin{aligned} \dot{\underline{z}} &= T^{-1}AT\underline{z} + T^{-1}B\underline{u} \\ y &= CT\underline{z} + D\underline{u} \end{aligned}\right\} \text{EQUIVALENT SYSTEM}$$

$\tilde{A} = T^{-1}AT$
SIMILARITY TRANSFORMATION.
$\tilde{A}$, A SAME λ!

CHOOSE $T = \mathcal{C}W$, $W = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & \ddots & & \vdots \\ & 0 & & & a_1 \\ & & & & 1 \end{bmatrix}$

ONLY WORKS IF $\mathcal{C}^{-1}$ EXISTS!

"TOEPLITZ MATRIX"

THEN $\tilde{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix}$

$\tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ CONTROLLABLE CANONICAL FORM!

$$u = -K\underline{x} = -KT\underline{z} = -\tilde{K}\underline{z}$$

CLOSED-LOOP

$$\dot{\underline{z}} = (\tilde{A} - \tilde{B}\tilde{K})\underline{z}$$

$$= \begin{bmatrix} -a_1-\tilde{k}_1 & -a_2-\tilde{k}_2 & \cdots & -a_n-\tilde{k}_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ & 0 & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}$$

SO, $-\alpha_i = -a_i - \tilde{k}_i \Rightarrow \boxed{\tilde{k}_i = \alpha_i - a_i}$

GAINS COMPUTED DIRECTLY FROM DIFFERENCE BETWEEN DESIRED AND ACTUAL CHARACTERISTIC EQUATION COEFFICIENTS!

NOW, $K = \tilde{K}T^{-1} \Rightarrow$ GAINS IN ORIGINAL SYSTEM.

DESIGN PROCESS:

- FORM STATE-SPACE DESCRIPTION, A, B, C, D
- COMPUTE $\mathcal{C} = [B \ AB \ \dots \ A^{n-1}B]$
  CHECK FOR CONTROLLABILITY.
- COMPUTE W (TOEPLITZ MATRIX) FROM COEFFICIENTS a_i OF $\Delta(\lambda)$
- CHOOSE DESIRED $\Delta_{CL}(\lambda) \Rightarrow \alpha_i$
- COMPUTE $\tilde{K}$ FROM $\tilde{K} = \alpha_i - a_i$
- COMPUTE K FROM $\tilde{K}T^{-1}$, WHERE $T = \mathcal{C}W$

IN REALITY, WE DON'T KNOW $\underline{x}(t)$, BECAUSE WE HAVE MEASUREMENTS, $\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$

IF $\underline{u} = \underline{0}$, WE HAVE $\dot{\underline{x}} = A\underline{x}$

$$\underline{y} = C\underline{x}$$

IF $\underline{x}(0) = \underline{x}_0$, CAN WE DETERMINE $\underline{x}_0$ FROM $\underline{y}(t)$ FOR $0 \le t \le t_f$? * OBSERVABILITY *

THIS IS "MATHEMATICAL DUAL" OF CONTROLLABILITY.

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

IF $\mathcal{O}$ HAS n LINEARLY INDEP. ROWS, THEN SYSTEM IS OBSERVABLE.

WE CAN TRANSFORM TO "OBSERVABLE CANONICAL FORM".

BUILDING A STATE ESTIMATOR:

LET $\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}})$

$$\hat{\underline{y}} = C\underline{x} + D\underline{u}$$

$$\Rightarrow \dot{\hat{\underline{x}}} = (A - LC)\hat{\underline{x}} + B\underline{u} + LC\underline{x}$$

DEFINE $\underline{e} = \underline{x} - \hat{\underline{x}} \Rightarrow \dot{\underline{e}} = A\underline{x} + B\underline{u} - (A-LC)\hat{\underline{x}} - B\underline{u} - LC\underline{x}$

$$= (A-LC)\underline{e}$$

IF (A, C) IS OBSERVABLE ($\mathcal{O}$ HAS RANK n)

THEN WE CAN PLACE POLES OF $(A-LC)$ ARBITRARILY.

GENERALLY TRY TO MAKE ESTIMATOR POLES FASTER THAN CLOSED-LOOP POLES IN PLANT. (POLES OF $A-BK$)

NOW, CAN WE USE $\hat{\underline{x}}$ FOR FEEDBACK?

$$\underline{u} = -K\hat{\underline{x}} \Rightarrow \dot{\underline{x}} = A\underline{x} - BK\hat{\underline{x}}$$

BUT $\hat{\underline{x}} = \underline{x} - \underline{e} \Rightarrow$

$$\dot{\underline{x}} = (A - BK)\underline{x} + BK\underline{e}$$

$$\dot{\underline{e}} = (A - LC)\underline{e}$$

OR

$$\begin{pmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{pmatrix} = \underbrace{\begin{bmatrix} (A-BK) & BK \\ 0 & (A-LC) \end{bmatrix}}_{\substack{\text{EIGENVALUES ARE} \\ \text{UNION OF EIGENVALUES} \\ \text{OF } (A-BK) \text{ \& } (A-LC)}} \begin{pmatrix} x \\ e \end{pmatrix}$$

} "SEPARATION PRINCIPLE"

(*) DESIGN K ASSUMING FULL STATE FEEDBACK + USE $\hat{\underline{x}}$ INSTEAD OF $\underline{x}$ AND YOU WILL STILL GET THE RIGHT CLOSED-LOOP POLES!

AT THIS POINT, THINGS SHOULD START TO SMELL FUNNY... POLES OF ESTIMATOR DON'T SHOW UP AT ALL IN THE CLOSED-LOOP TRANSFER FUNCTION!? WHAT IS GOING ON?

WE HAVE A <u>PERFECT</u> POLE-ZERO CANCELLATION THAT DEPENDS ON <u>PERFECT</u> KNOWLEDGE OF SYSTEM DYNAMICS $\Rightarrow$ THAT IS, WE HAVE TO KNOW A, B, C, D EXACTLY! IN FACT, ARBITRARILY SMALL ERRORS CAN CAUSE CLOSED-LOOP INSTABILITY... CAUTION