Root Locus

ESE 505 & MEAM 513

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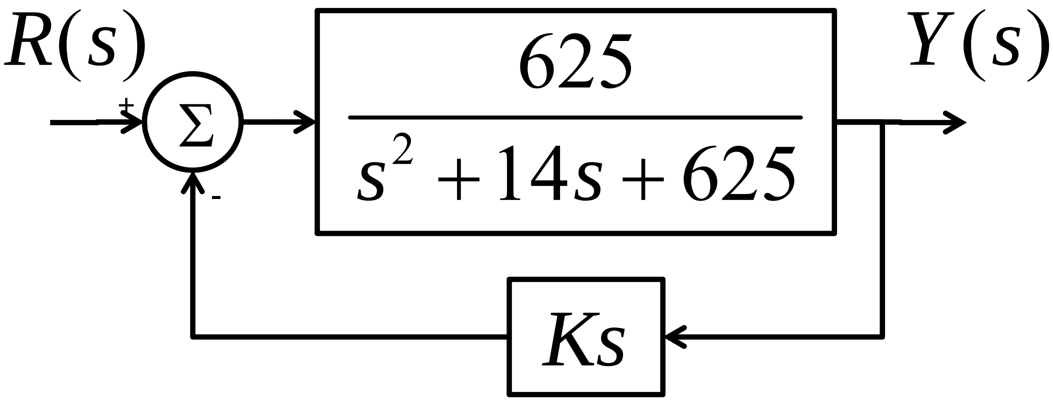
Root Locus Details

Additional Examples

Penn Engineering

Example Design Problem



$$\frac{Y(s)}{R(s)} = \frac{625}{s^2 + 14s + 625 + 625Ks}$$

$$\Delta_{CL}(s) = \left(s^2 + 14s + 625\right) + K\left(625s\right) = 0$$

Equation for Closed-Loop Poles

What is a "Root Locus"

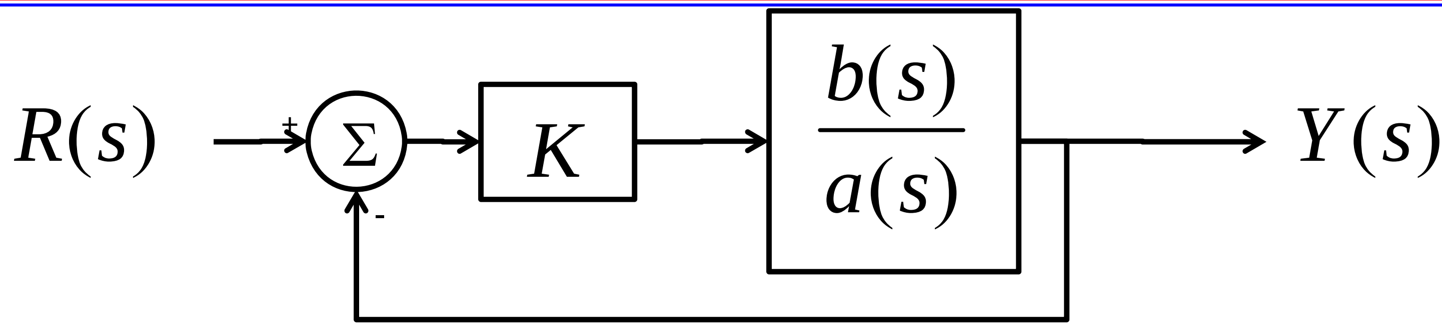
We Often Encounter Closed-Loop Characteristic Equations Which Can Be Written in the Following Form:

$$\Delta_{CL}(s) = a(s) + Kb(s) = 0$$

Where a(s) and b(s) Are Polynomials in s & K is a Some System Parameter (Typically a Compensator Gain)

A "Root Locus is a Graphical Presentation of How the Closed-Loop Poles (Roots of $\Delta_{CL}(s)=0$) Vary with $0 \leq K < \infty$.

The Quintessential Example System



$$\frac{Y(s)}{R(s)} = \frac{Kb(s)}{a(s)+Kb(s)} \qquad \Delta_{CL}(s) = a(s) + Kb(s) = 0$$

- Terminology Derived From This Quintessential Case
  - a(s) Called "Open Loop Denominator" & Roots of a(s)=0 Called "Open Loop Poles"
  - b(s) Called "Open Loop Numerator" & Roots of b(s)=0 Called "Open-Loop Zeros"
- Actual System May Have Different Architecture But Δ_{CL}(s) Always Has Form a(s)+Kb(s)

We Factor a(s) & b(s) Using Poles & Zeros

$$L(s) \triangleq \frac{b(s)}{a(s)} = \frac{A(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Note that We Choose "A" So That

$$\lim_{s\to\infty} L(s) \sim \frac{A}{s^{n-m}}$$

$$\Delta_{CL}(s) = 0 \Rightarrow 1 + KL(s) = 0$$

$$\Delta_{CL}(s) = (s-p_1)\cdots(s-p_n) + KA(s-z_1)\cdots(s-z_m) = 0$$

- For All K, There are n Closed-Loop Poles
- K=0 → Closed-Loop Poles = Open-Loop Poles
- "Root Locus" Has n "Branches" Showing Movement of Poles for K>0

Limiting Cases (K→0, K→∞)

$$\left[(s-p_1)\cdots(s-p_n)\right]+K\left[A(s-z_1)\cdots(s-z_m)\right]=0$$

$$K\to 0 \Rightarrow \left[(s-p_1)\cdots(s-p_n)\right]=0$$

Each Branch of the Locus Begins (K=0) @ Open-Loop Poles

$$\left.\begin{array}{l} K\to\infty \\ s\not\to\infty \end{array}\right\}\Rightarrow\left[A(s-z_1)\cdots(s-z_m)\right]=0$$

m Branches Approach Open-Loop Zeros as K→∞

$$\left.\begin{array}{l} K\to\infty \\ s\to\infty \end{array}\right\}\Rightarrow s^{n-m}+KA=0$$

$$s=(-KA)^{\frac{1}{n-m}}$$

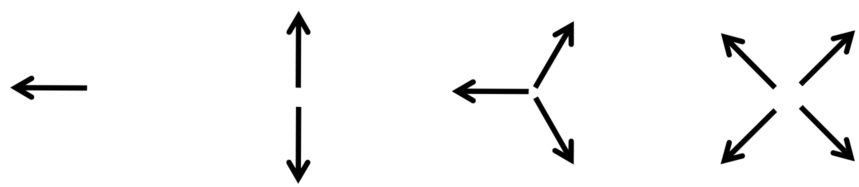
(n-m) Branches Approach ∞ Along Asymptotes as K→∞

Rules for Sketching Asymptotes

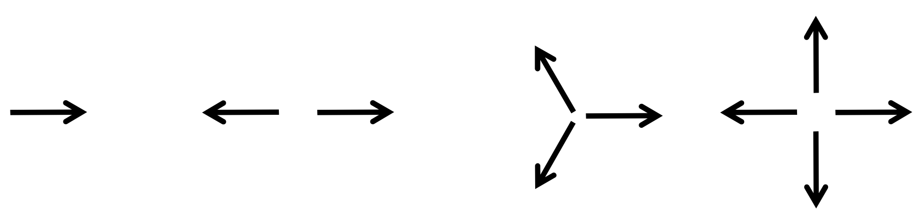
- Asymptotes Originate on Real Axis at “Center of Gravity” of Poles (Zeros Have Negative Mass)

$$x_{CG} = \frac{\sum_{k=1}^{k=n} p_k - \sum_{k=1}^{k=m} z_k}{n-m}$$

- Asymptote Patterns (A>0)



- Asymptote Patterns (A<0)



One More Helpful Rule

- (A>0) : Real Axis on Locus "Left of Odd Number of Poles & Zeros"

- (A<0) : Real Axis on Locus "Right of Odd Number of Poles & Zeros"

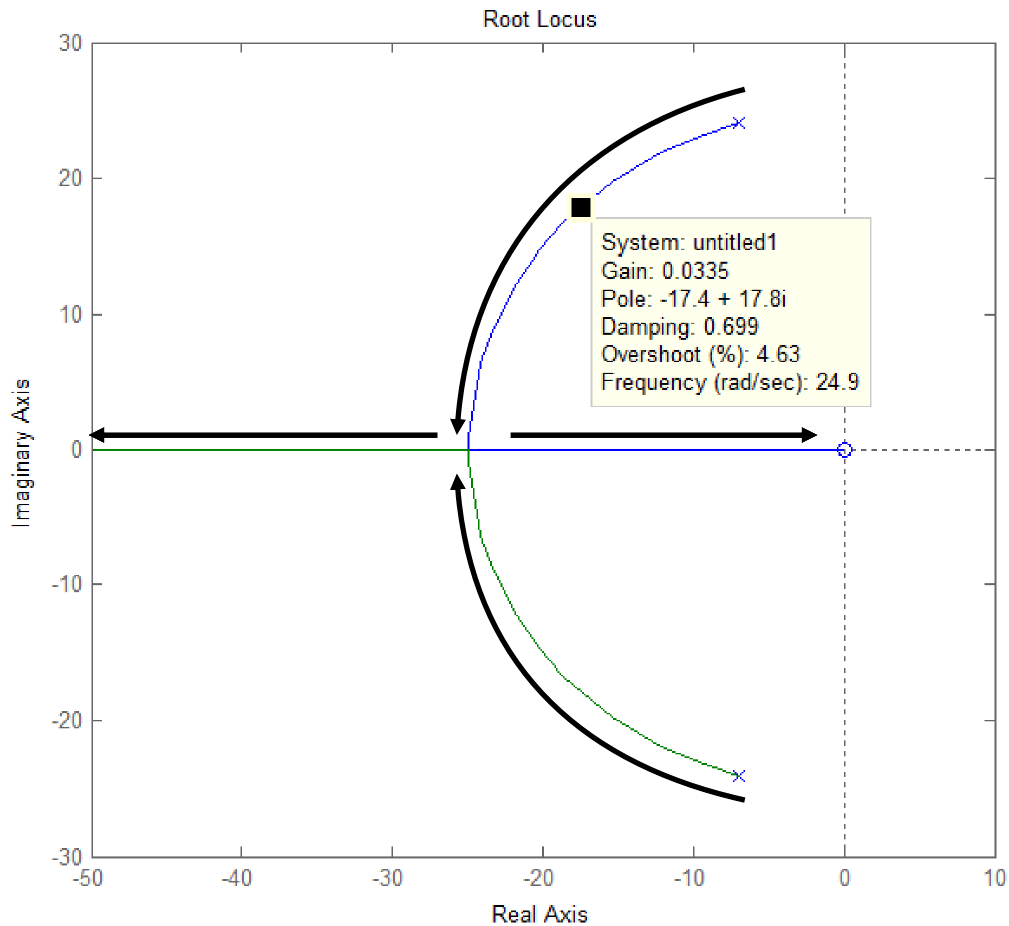
Summary : Sketching Root Loci (A>0)

$$\Delta_{CL}(s) = a(s) + Kb(s)$$

- Place Open-Loop Poles & Zeros on Plot
  - n = # Open-Loop Poles
  - m = # Open-Loop Zeros
- K=0 ➔ Poles Start @ Open-Loop Poles
- K→∞ ➔ m Poles Go to Open-Loop Zeros
- K→∞ ➔ (n-m) Poles Go to "Asymptotes to Infinity"
  - Asymptotes Always Equally Spaced in Angle
  - (n-m) Odd ➔ First Asymptote on Negative Real Axis
  - (n-m) Even ➔ Asymptotes Symmetric Around Real Axis
- 0<K<∞ ➔ Poles on Real Axis "Left of Odd # Open-Loop Poles + Open-Loop Zeros"

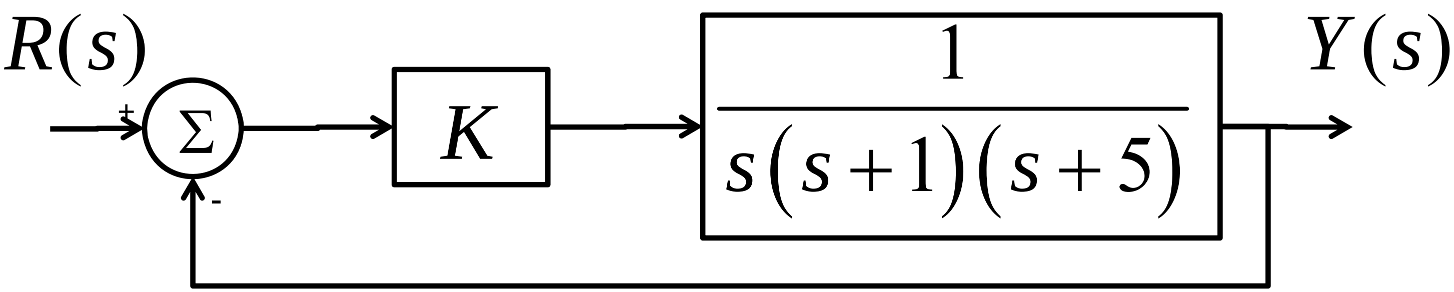
Root Locus for Example System

$$\Delta_{CL}(s) = \left(s^2 + 14s + 625\right) + K\left(625s\right) = 0$$



Typical Closed-Loop Pole Location (Damping Ratio ~ 0.7) Shown

Another Simple Example

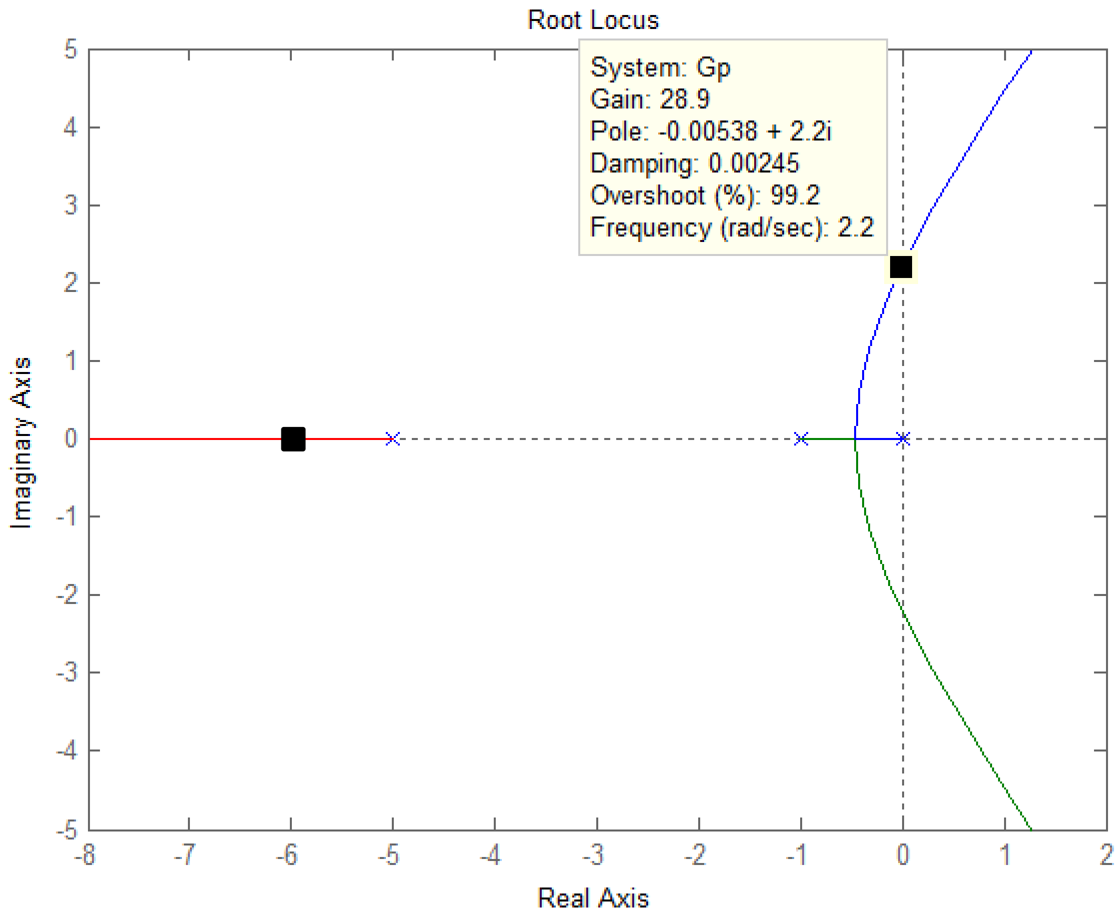


$$\frac{Y(s)}{R(s)} = \frac{K}{s(s+1)(s+5)+K}$$

$$\Delta_{CL}(s) = s(s+1)(s+5)+K = 0$$

Root Locus

$$\Delta_{CL}(s) = s(s+1)(s+5) + K = 0$$



Neutral Stability

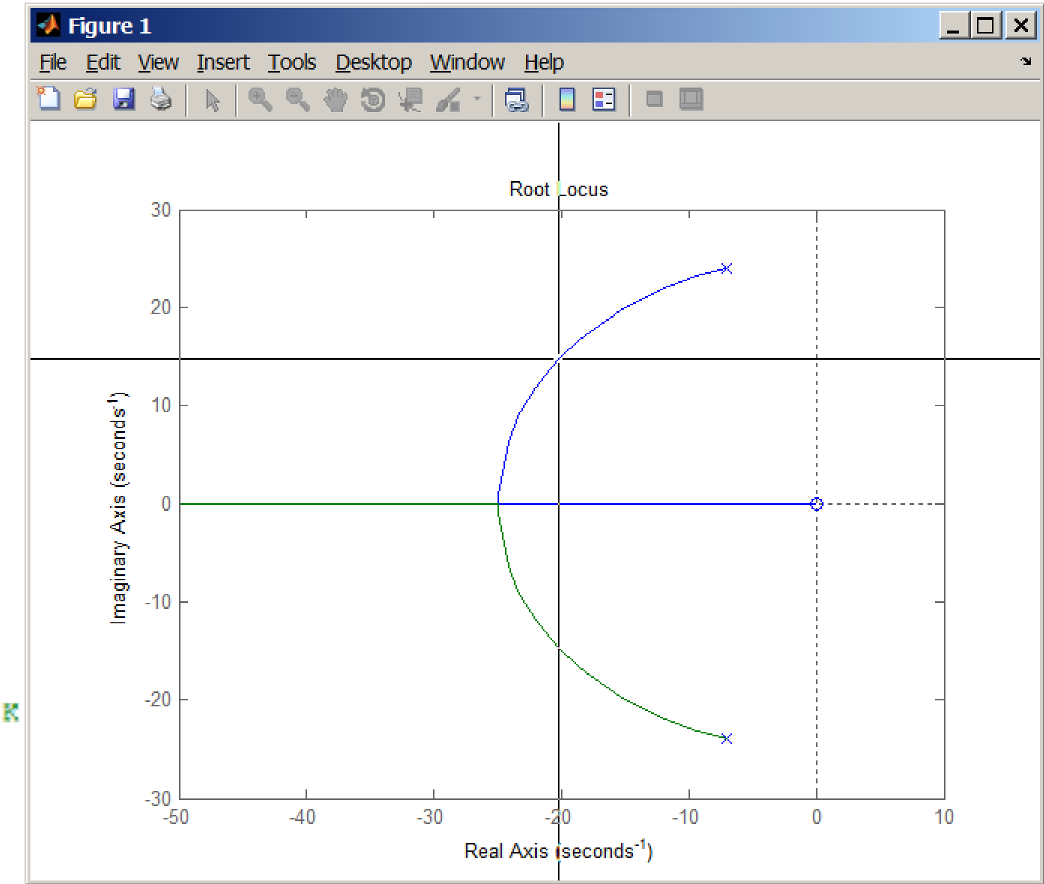
$$K = 30$$

$$\Delta_{CL}(s) = s(s+1)(s+5) + 30$$

$$\Delta_{CL}(s) = (s+6)(s^2+5)$$

MATLAB : `rlocus` & `rlocfind`

```
%
% simple root locus example
%
den = [1 14 625];
num = 625
Gp = tf(num,den);
%
% root locus with derivative control
%
Gc = tf([1 0],[1]);
figure(1);
rlocus(Gp*Gc);
axis([-50 10 -30 30]); set(gcf,'Color','w');
%
% now find the closed-loop root
%
rlocfind(Gp*Gc)
%
% another way is to find the roots for specified K
%
Kp = [0:0.001:1];
[p]=rlocus(Gp*Gc,Kp);
figure(2);
plot(real(p),imag(p),'xr');
axis([-50 10 -30 30]); set(gcf,'Color','w');
grid on;
```



Finding Poles for all K & Plotting Yourself

```
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Gc = tf([1 0],[1]);
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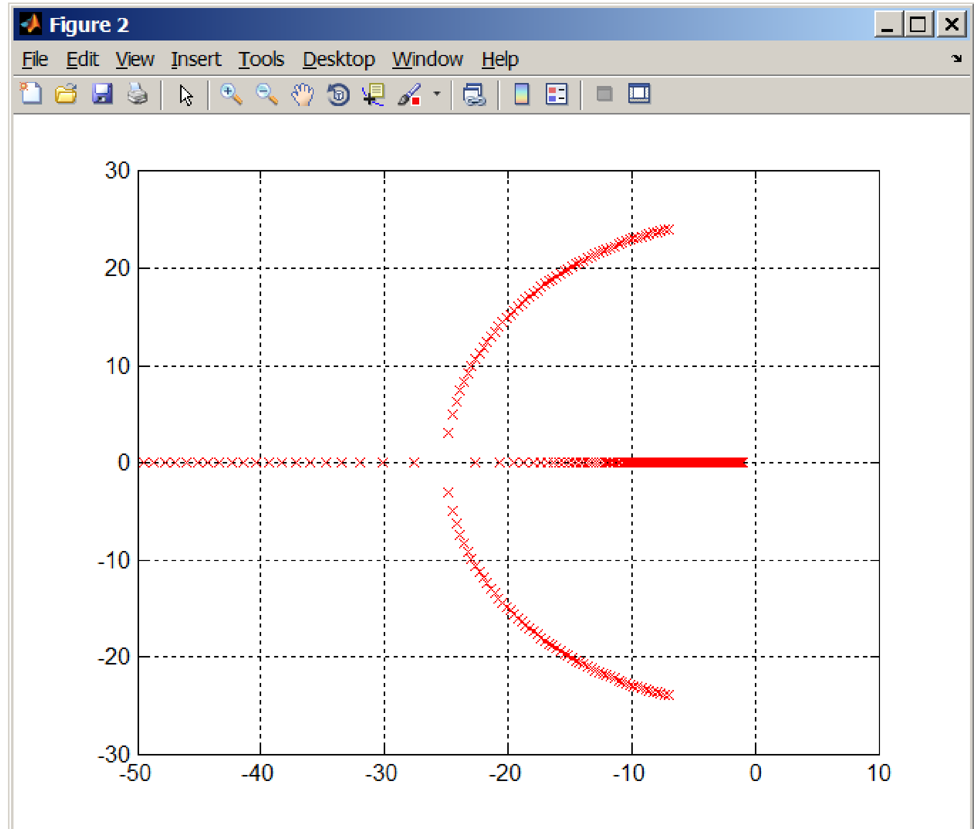

More Examples

System with 0.2-sec Time Delay (Using 1st-Order Pade):

$$\Delta_{CL}(s) = \left(s+1\right)\left(s+5\right)(0.1s+1) + K(-0.1s+1) = 0$$

Systems with Complex Poles & Zeros

$$\Delta_{CL}(s) = \left(s+1\right)\left(s+5\right)(s^2+2s+36) + K(s^2+2s+49) = 0$$

$$\Delta_{CL}(s) = \left(s+1\right)\left(s+5\right)(s^2+2s+49) + K(s^2+2s+36) = 0$$

Similar Complex Systems (Pole-Zero Cancellation Changes Topology)

$$\Delta_{CL}(s) = \left(s-3\right)\left(s+5\right)\left(s+10\right)(s^2+3s+36) + K(s+8) = 0$$

$$\Delta_{CL}(s) = \left(s-3\right)\left(s+5\right)\left(s+10\right)(s^2+3s+36) + K(s+5) = 0$$