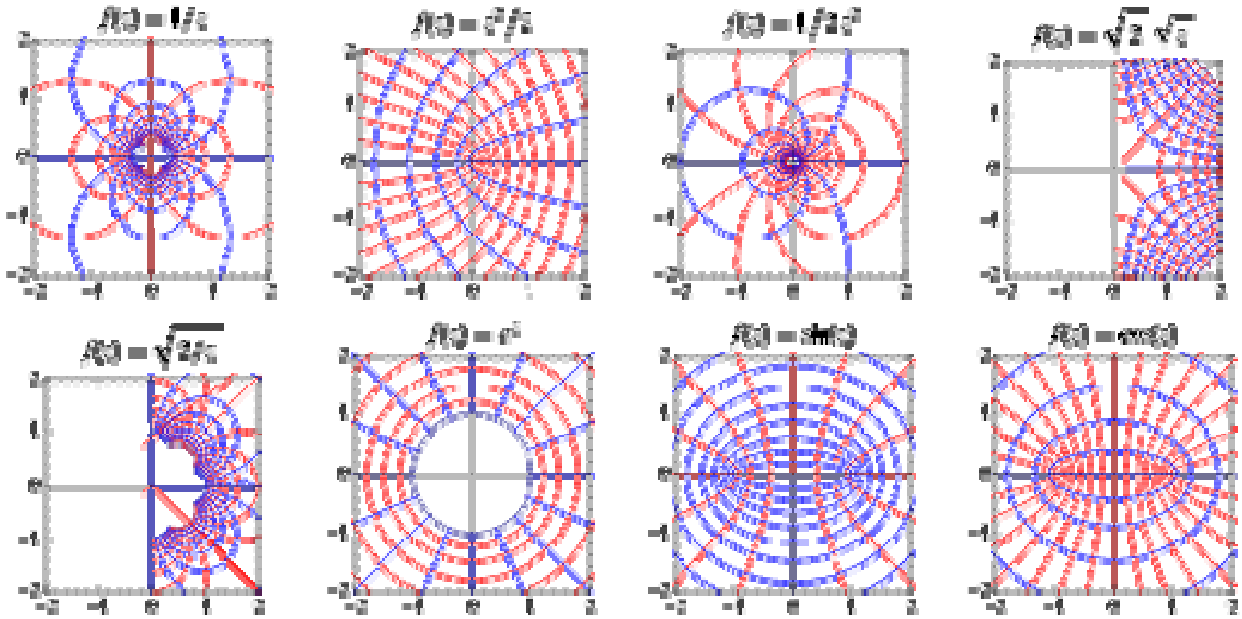
# Complex Variables & Laplace Transforms



ESE 505 & MEAM 513

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http://mathworld.wolfram.com/images/eps-gif/ConformalTransformations_700.gif

# Complex Variables

$$j \triangleq \sqrt{-1} \qquad z = re^{j\phi} = r\left(\cos\phi + j\sin\phi\right) = a + bj$$

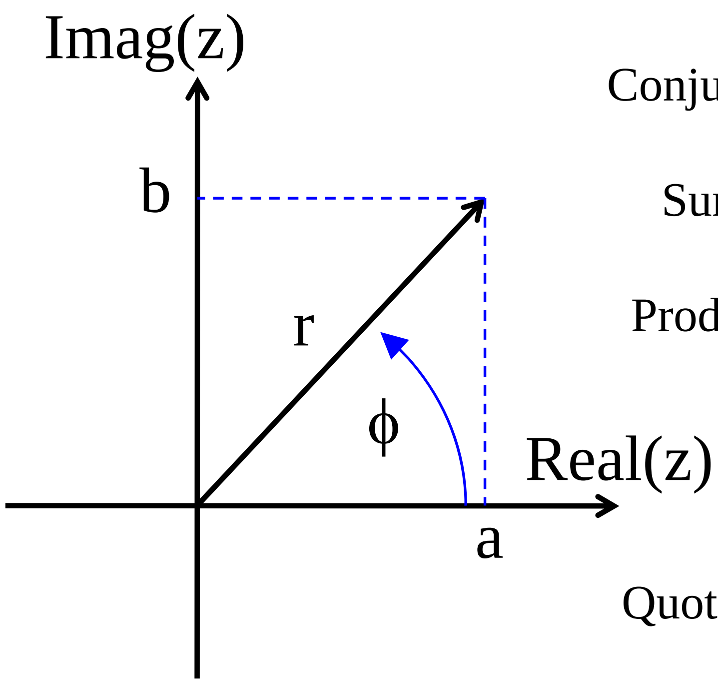
Complex Number

Magnitude

Phase

Real Part

Imaginary Part




Conjugate $\quad \bar{z} = a - bj = re^{-j\phi} \qquad z\bar{z} = a^2 + b^2 = r^2$

Sum $\quad z_1 + z_2 = \left(a_1 + a_2\right) + \left(b_1 + b_2\right) j$

Product $\quad z_1 z_2 = \left(a_1 a_2 - b_1 b_2\right) + \left(a_1 b_2 + a_2 b_1\right) j = r_1 r_2 e^{j(\phi_1 + \phi_2)}$

Quotient $\quad \dfrac{z_1}{z_2} = \dfrac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \dfrac{\left(a_1 a_2 + b_1 b_2\right) + \left(a_2 b_1 - a_1 b_2\right) j}{a_2^2 + b_2^2} = \dfrac{r_1}{r_2} e^{j(\phi_1 - \phi_2)}$

# More Basic Complex Variable Stuff

Exponential $$e^z = e^{a+bj} = e^a e^{bj} = e^a\left(\cos b + j\sin b\right)$$

Logarithm $$\ln z = \ln r + j\phi$$

Trig Functions $$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Analytic Function $$F(z) = \Phi(a,b) + j\Psi(a,b)$$

Cauchy-Riemann Equations $$\frac{\partial\Phi}{\partial a} = \frac{\partial\Psi}{\partial b} \qquad \frac{\partial\Phi}{\partial b} = -\frac{\partial\Psi}{\partial a}$$

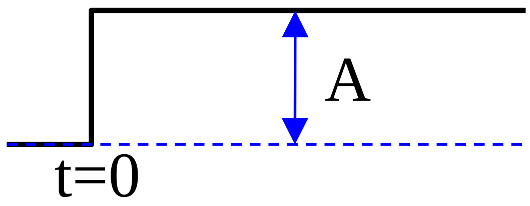
Taylor Series $$F(z) = F(z_o) + \left.\frac{dF}{dz}\right|_{z_o}\left(z - z_o\right) + \frac{1}{2!}\left.\frac{d^2F}{dz^2}\right|_{z_o}\left(z - z_o\right)^2 + \cdots$$

# (One-Sided) Laplace Transform

$$F(s) = L\{f(t)\} \triangleq \int_{t=0^-}^{t=\infty} f(t)e^{-st}dt$$

Examples

$$f(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases} \Rightarrow F(s) = \int_{t=0^-}^{t=\infty} Ae^{-st}dt = -\frac{A}{s}e^{-st}\Big|_{t=0^-}^{t=\infty} = \frac{A}{s}$$




$$f(t) = \begin{cases} 0 & t < 0 \\ Ae^{at} & t \geq 0 \end{cases} \Rightarrow F(s) = \int_{t=0^-}^{t=\infty} Ae^{-(s-a)t}dt = -\frac{A}{s-a}e^{-(s-a)t}\Big|_{t=0^-}^{t=\infty} = \frac{A}{s-a}$$

- Laplace Transform Changes Independent Variable (t→s)
- A Couple of Important Details We'll Usually Overlook
  - "$0^-$" Matters Only for Impulsive Behavior (More Later) So We Generally Omit the "-"
  - Technically, L{f(t)} Exists Only if Real(s) > (Some Threshold). But Everything Works Out Fine if We Pretend F(s) Always Exists!

# Laplace Transform = Linear Process

$$L\left\{ f_1(t) + f_2(t) \right\} = L\left\{ f_1(t) \right\} + L\left\{ f_2(t) \right\}$$

$$f(t) = \sin \Omega t = \frac{e^{j\Omega t} - e^{-j\Omega t}}{2} \Rightarrow F(s) = \frac{1}{2} L\left\{ e^{j\Omega t} \right\} - \frac{1}{2} L\left\{ e^{-j\Omega t} \right\} = \frac{1}{2}\left[ \frac{1}{s - j\Omega} - \frac{1}{s + j\Omega} \right]$$

$$L\left\{ \sin \Omega t \right\} = \frac{\Omega}{s^2 + \Omega^2} \qquad L\left\{ \cos \Omega t \right\} = \frac{s}{s^2 + \Omega^2}$$

**<u>Note</u>: Doesn't Work for Multiplication or Other Non-Linear Relationships**

$$L\left\{ f_1(t) f_2(t) \right\} \neq L\left\{ f_1(t) \right\} L\left\{ f_2(t) \right\}$$

# Why Do We Want to Learn Laplace Transforms?

$$\int_{t=0^-}^{t=\infty} \frac{df}{dt}(t)e^{-st}dt = f(t)e^{-st}\Big|_{t=0^-}^{t=\infty} + s\int_{t=0^-}^{t=\infty} f(t)e^{-st}dt$$

$$\boxed{L\left\{\frac{df}{dt}\right\} = sF(s) - f\left(0^-\right)}$$

**Laplace Transforms Convert Linear Constant-Coefficient Ordinary Differential Equations into Algebraic Equations!**

$$L\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f\left(0^-\right) - s^{n-2}\frac{df}{dt}\left(0^-\right) - \cdots - \frac{d^{n-1} f}{dt^{n-1}}\left(0^-\right)$$

# Important Properties of Laplace Transform

- Initial Value Theorem

$$y(t=0) = \lim_{s\to\infty} sY(s)$$

- Final Value Theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

  – Applies Only if Time Limit Exists !

$$y(t) = e^{2t} \Rightarrow Y(s) = \frac{1}{s-2} \qquad \lim_{s\to 0} s\frac{1}{s-2} = 0 \qquad \lim_{t\to\infty} y(t) = \infty$$

- Effect of Time Delay (f(t) = 0 for t<T)

$$L\left\{f(t+T)\right\} = \int_{t=0^-}^{t=\infty} f(t+T)e^{-st}dt = \int_{\eta=T}^{\eta=\infty} f(\eta)e^{-s(\eta-T)}d\eta = e^{-Ts}F(s)$$

# Laplace Transform of State-Space Equations

$$\dot{\underline{x}} = A\underline{x} + Bu$$

$$y = C\underline{x} + Du$$

We Get Tired of Writing "Δ" So We Often Just Drop it.

$$s\underline{X}(s) - x(0) = A\underline{X}(s) + BU(s)$$

$$Y(s) = C\underline{X}(s) + DU(s)$$

$$(sI - A)\underline{X}(s) = BU(s) + \underline{x}(0)$$

$$\underline{X}(s) = (sI - A)^{-1}\left(BU(s) + \underline{x}(0)\right)$$

$$\boxed{Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) + C(sI - A)^{-1}\underline{x}(0)}$$

# State Matrix Eigenvalues = System Poles

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\left|(sI - A)\right|} = \frac{adj(sI - A)}{\Delta(s)}$$

$adj(sI - A) =$ Adjugate Matrix

http://en.wikipedia.org/wiki/Adjugate_matrix

} Matrix Whose Elements are Polynominals in "s" with Maximum Degree of n-1

$\Delta(s) = \left|(sI - A)\right| =$ Characteristic Polynomial of A } Polynomial of Degree n Whose Roots Are Eigenvalues of A

$$\left[C(sI - A)^{-1}B + D\right] = \frac{1}{\Delta(s)}\left[\underbrace{\overbrace{C}^{1\times n}\,\overbrace{adj(sI - A)}^{n\times n}\,\overbrace{B}^{n\times 1}}_{1\times 1} + \Delta(s)\underbrace{D}_{1\times 1}\right]$$

# Let's Solve an ODE with Laplace Transforms

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+4y(t)=2\frac{du}{dt}+u(t)$$

Initial Conditions $\begin{cases} y(0)=0 \\ \dfrac{dy}{dt}(0)=3 \end{cases}$ Input $u(t)=\sin\left(2t\right)$

**Express in State-Space Form:**

$$\frac{d\underline{x}}{dt}=\underbrace{\begin{bmatrix}-5 & -4\\ 1 & 0\end{bmatrix}}_{A}\underline{x}+\underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{B}u \qquad y=\underbrace{\begin{bmatrix}2 & 1\end{bmatrix}}_{C}\underline{x}+\underbrace{\begin{bmatrix}0\end{bmatrix}}_{D}u$$

# How Did We Get That State-Space Representation?

**http://en.wikipedia.org/wiki/State_space_representation**

- Opposites of Coefficients of y and dy/dt from LHS → First Row of A Matrix
- Coefficients of u and du/dt from RHS → C Matrix
- B Matrix Always Has Only 1 as First Element
- Some Manipulation Required to Find Initial State

$$\left.\begin{aligned}
&y(0) = 2x_1(0) + x_2(0) \\
&\frac{dy}{dt}(0) = 2\frac{dx_1}{dt}(0) + \frac{dx_2}{dt}(0) \\
&= 2\left\{-5x_1(0) - 4x_2(0) + u(0)\right\} + x_1(0) \\
&= -9x_1(0) - 8x_2(0) + 2u(0)
\end{aligned}\right\} \quad \underline{x}(0) = \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix}$$

# Apply Laplace Transform…

$$s\underline{X}(s)-\underline{x}(0)=\begin{bmatrix}-5 & -4\\ 1 & 0\end{bmatrix}\underline{X}(s)+\begin{bmatrix}1\\ 0\end{bmatrix}U(s)$$

$$\begin{bmatrix}s+5 & 4\\ -1 & s\end{bmatrix}\underline{X}(s)=\begin{pmatrix}\frac{3}{7}\\ -\frac{6}{7}\end{pmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}\frac{2}{s^2+2}$$

**Linear Problem ➔ Separate "Zero Input" (or "Free") Response from "Zero Initial-Condition" (or "Forced") Response**

# Homogeneous (U=0) Response

$$\underline{X}_h(s) = \begin{bmatrix} s+5 & 4 \\ -1 & s \end{bmatrix}^{-1} \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix} = \frac{1}{s^2+5s+4} \begin{bmatrix} s & -4 \\ 1 & s+5 \end{bmatrix} \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix}$$

$$\underline{X}_h(s) = \frac{1}{s^2+5s+4} \begin{pmatrix} \frac{3}{7}s + \frac{24}{7} \\ -\frac{6}{7}s - \frac{27}{7} \end{pmatrix}$$

$$Y_h(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \underline{X}_h(s) = \frac{3}{s^2+5s+4}$$

# Can Get Homogeneous Response From ODE…

$$\left[s^2Y_h(s)-sy(0)-\dot{y}(0)\right]+5\left[sY_h(s)-y(0)\right]+4Y_h(s)=0$$

$$\left[s^2+4s+5\right]Y_h(s)=\dot{y}(0)+\left(s+4\right)y(0)$$

$$Y_h(s)=\frac{\dot{y}(0)+\left(s+4\right)y(0)}{\left[s^2+4s+5\right]}$$

**Mere Algebra Got Us to Here!**

$$Y_h(s)=\frac{3}{\left[s^2+4s+5\right]}$$

**Now We Just Need to Find Inverse Laplace Transform! (We Won't Do This with Hard Math, but Graduate Students Should Know that We Could!)**

# Partial-Fraction Expansion

$$Y_h(s) = \frac{3}{s^2 + 5s + 4} = \frac{3}{(s+4)(s+1)}$$ Factor Denominator

$$Y_h(s) = \frac{C_1}{(s+1)} + \frac{C_2}{(s+4)}$$ This is Partial-Fraction Expansion

$$\left.(s+1)Y_h(s)\right|_{s=-1} = C_1 + \left.\frac{C_2(s+1)}{(s+4)}\right|_{s=-1} = C_1$$

$$\left.(s+1)Y_h(s)\right|_{s=-1} = \left.\frac{3(s+1)}{(s+4)(s+1)}\right|_{s=-1} = \frac{3}{(-1+4)} = 1 = C_1$$

# "Cover-Up Method"

$$\frac{C_1}{(s+1)}+\frac{C_2}{(s+4)}=\frac{3}{(s+4)(s+1)}$$

$$C_1=\frac{3}{(\underset{-1}{\cancel{s}}+4)\,(s+1)}=1 \qquad C_2=\frac{3}{(s+4)\,(\underset{-4}{\cancel{s}}+1)}=-1$$

$$Y_h(s)=\frac{1}{(s+1)}+\frac{-1}{(s+4)} \qquad \boxed{y_h(t)=e^{-t}-e^{-4t}}$$

$$y_h(t=0)=0 \;\checkmark \qquad \frac{dy_h}{dt}(t=0)=\left[-e^{-t}+4e^{-4t}\right]_{t=0}=3 \;\checkmark$$

# Forced (x(0)=0) Response

$$\underline{X}_f(s) = \begin{bmatrix} s+5 & 4 \\ -1 & s \end{bmatrix}^{-1} \begin{pmatrix} \dfrac{2}{s^2+2} \\ 0 \end{pmatrix}$$

$$\underline{X}_f(s) = \frac{1}{s^2+5s+4} \begin{pmatrix} \dfrac{2s}{s^2+2} \\ \dfrac{2}{s^2+2} \end{pmatrix}$$

$$Y_f(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \underline{X}_f(s) = \frac{4s+2}{\left(s^2+5s+4\right)\left(s^2+2\right)}$$

# Can Get Forced Response From ODE…

$$\left(s^2+5s+4\right)Y_f(s)=\left(2s+1\right)U(s)=\left(2s+1\right)\frac{2}{s^2+4}$$

$$Y_f(s)=\left(2s+1\right)U(s)=\frac{2\left(2s+1\right)}{\left(s^2+5s+4\right)\left(s^2+4\right)}$$

# Partial-Fraction Expansion

Partial-Fraction Expansion

$$Y_f(s) = \frac{C_1}{(s+1)} + \frac{C_2}{(s+4)} + \frac{C_3}{(s-2j)} + \frac{C_4}{(s+2j)}$$

$$C_1 = \frac{-2}{15} \quad C_2 = \frac{7}{30} \quad C_3 = \frac{-1-4j}{20} \quad C_4 = \frac{-1+4j}{20} = \bar{C}_3$$

# Dealing with Complex Poles & Coefficients

$$\frac{C_3}{(s-2j)}+\frac{\bar{C}_3}{(s+2j)}=\frac{C_3(s+2j)+\bar{C}_3(s-2j)}{s^2+4}$$

$$=\frac{\left(C_3+\bar{C}_3\right)s}{s^2+4}+\frac{\left(C_3-\bar{C}_3\right)2j}{s^2+4}$$

$$=2\operatorname{Re}\left(C_3\right)\frac{s}{s^2+4}-2\operatorname{Im}\left(C_3\right)\frac{2}{s^2+4}$$

$$\boxed{y_f(t)=-\frac{2}{15}e^{-t}+\frac{7}{30}e^{-4t}-\frac{1}{10}\cos(2t)+\frac{4}{10}\sin(2t)}$$

$$y_f(t=0)=0 \checkmark \quad \frac{dy_f}{dt}(t=0)=0 \checkmark$$

# Dealing with Complex Poles (with Damping)

$$\frac{C_k}{s-p_k}+\frac{\bar{C}_k}{s-\bar{p}_k}=\frac{C_k\left(s-\bar{p}_k\right)+\bar{C}_k\left(s-p_k\right)}{\left(s-p_k\right)\left(s-\bar{p}_k\right)}$$

$$C_k = A + jB$$

$$p_k = -\sigma + j\omega_d$$

$$=\frac{\left(A+jB\right)\left(s+\sigma+j\omega_d\right)+\left(A-jB\right)\left(s+\sigma-j\omega_d\right)}{\left(s+\sigma-j\omega_d\right)\left(s+\sigma+j\omega_d\right)}$$

$$=\frac{2A\left(s+\sigma\right)-2B\omega_d}{\left(s+\sigma\right)^2+\omega_d^2}=\frac{2As+\left(2A\sigma-2B\omega_d\right)}{s^2+2\sigma s+\left(\sigma^2+\omega_d^2\right)}$$

$$y(t)=\cdots+e^{-\sigma t}\left[2A\cos\left(\omega_d t\right)-2B\sin\left(\omega_d t\right)\right]+\cdots$$

# Laplace Transform : Look Ahead

- Homework ➔ Basic Skills with Laplace Transforms
  - Taking Laplace Transforms of Simple Functions
  - Using Laplace Transforms to Solve ODEs
  - Complex Numbers, Exponential Decay, Sinusoidal Oscillation

- Then…We Won't Ever Do Detailed Calculations Like These Again!

- We Will Develop Understanding of Dynamic Systems Using "Transfer Functions"

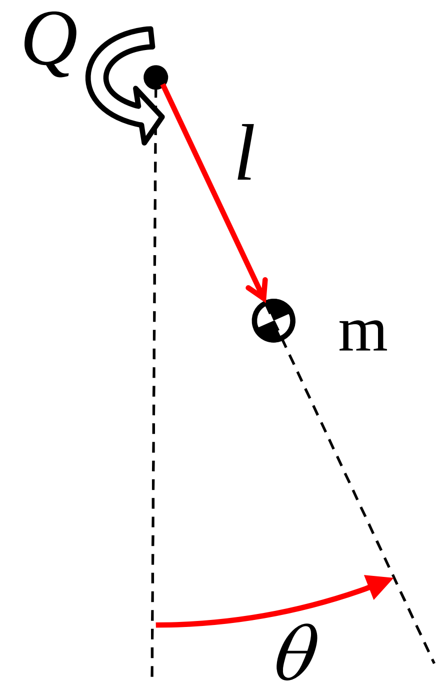
$$D(s)Y(s) = N(s)U(s) \Rightarrow \frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = H(s)$$

# Appendix

Detailed Example

Comparison of Linear & Nonlinear Simulations for Simple Pendulum

# State-Space Representation : Example




$$ml^2 \frac{d^2\theta}{dt^2} = Q - mgl\sin\theta$$

$$x_1 = \theta \qquad x_2 = \frac{d\theta}{dt} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$u = Q$$

$$\frac{dx_1}{dt} = \frac{d\theta}{dt} \Rightarrow f_1(x_1, x_2, u) = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{ml^2}Q - \frac{g}{l}\sin\theta \Rightarrow f_2(x_1, x_2, u) = \frac{1}{ml^2}u - \frac{g}{l}\sin x_1$$

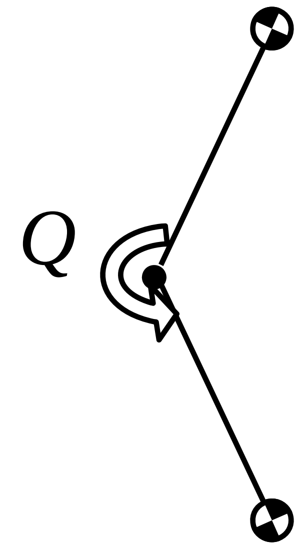
$$y = \theta \Rightarrow h(x_1, x_2, u) = x_1$$

# Linearization : Example Trim Condition

$$\underline{f}\left(\underline{x},u\right)=\begin{pmatrix} x_2 \\ -\dfrac{g}{l}\sin x_1+\dfrac{1}{ml^2}u \end{pmatrix}$$

$$\underline{f}\left(\underline{x}_o,u_o\right)=\underline{0}=\begin{pmatrix} x_{2_o} \\ -\dfrac{g}{l}\sin x_{1_o}+\dfrac{1}{ml^2}u_o \end{pmatrix}$$

$$\sin x_{1_o}=\frac{1}{mgl}u_o \qquad x_{2_o}=0$$




Two Possible Trim Conditions for Given Value of Torque
( -mgl < Q < mgl )

# Linearization : Example Matrices

$$f_2 = \frac{1}{ml^2}u - \frac{g}{l}\sin x_1 \qquad f_1 = x_2 \qquad h(\underline{x},u) = x_1$$

$$A \triangleq \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{(\underline{x}_o,u_o)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_{1_o} & 0 \end{bmatrix} \qquad B \triangleq \left.\frac{\partial \underline{f}}{\partial u}\right|_{(\underline{x}_o,u_o)} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

$$C \triangleq \left.\frac{\partial h}{\partial \underline{x}}\right|_{(\underline{x}_o,u_o)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D \triangleq \left.\left[\frac{\partial h}{\partial u}\right]\right|_{(\underline{x}_o,u_o)} = 0$$

# Example : Laplace Transform

$$\left(sI - A\right)^{-1} = \frac{1}{s^2 + \frac{g}{l}\cos x_{1_o}} \begin{bmatrix} s & 1 \\ -\frac{g}{l}\cos x_{1_o} & s \end{bmatrix}$$

$$C\left(sI - A\right)^{-1} \underline{x}(0) = \frac{s x_1(0) + x_2(0)}{s^2 + \frac{g}{l}\cos x_{1_o}}$$

$$C\left(sI - A\right)^{-1} B = \frac{\frac{1}{mgl}}{s^2 + \frac{g}{l}\cos x_{1_o}}$$

# Initial Condition Response with Zero Torque

$$\sin x_{1_o} = 0 \Rightarrow x_{1_o} = 0, \pi$$

$$\left.\begin{aligned} x_1(0) &= M \\ x_2(0) &= 0 \end{aligned}\right\}$$

Initial Displacement From Trim with Zero Initial Velocity

$$x_{1_o} = 0$$

$$Y(s) = \frac{s x_1(0) + x_2(0)}{s^2 + \dfrac{g}{l}}$$

$$y(t) = M \cos\left(\sqrt{\frac{g}{l}} t\right)$$

Normal Pendulum Oscillates

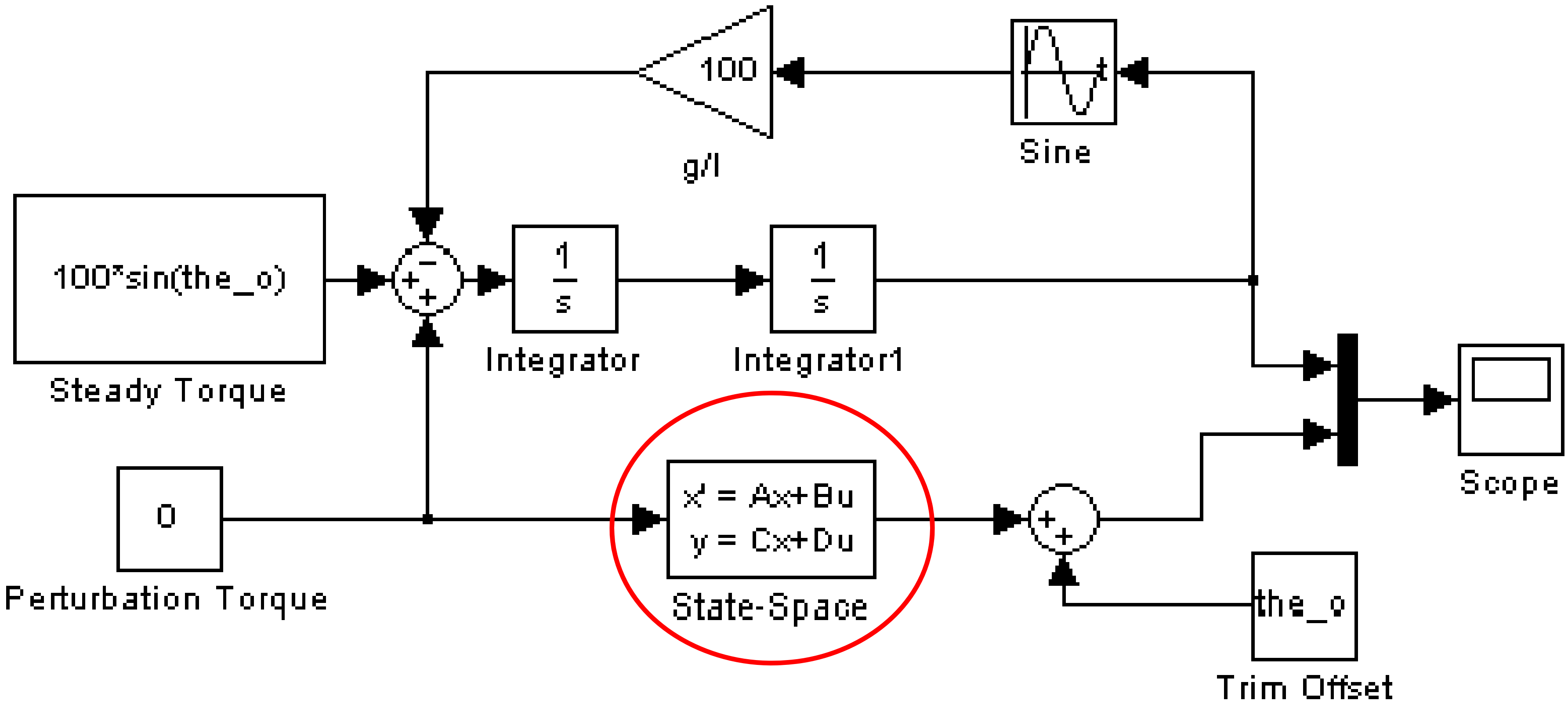
$$x_{1_o} = \pi$$

$$Y(s) = \frac{s x_1(0) + x_2(0)}{s^2 - \dfrac{g}{l}}$$

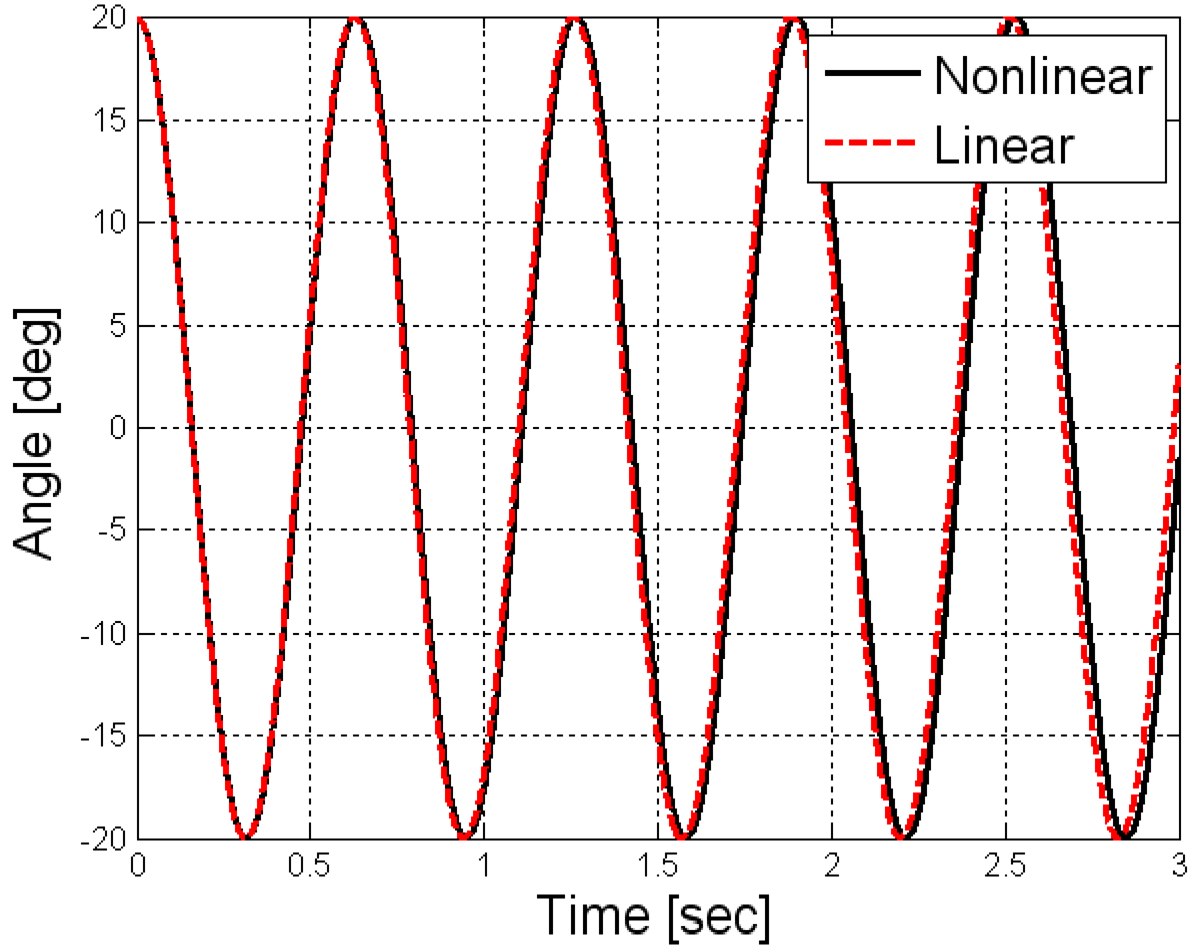
$$y(t) = \frac{M}{2} e^{\sqrt{\frac{g}{l}} t} + \frac{M}{2} e^{-\sqrt{\frac{g}{l}} t}$$

Inverted Pendulum Diverges

# Simulink Model : Linear vs. Nonlinear

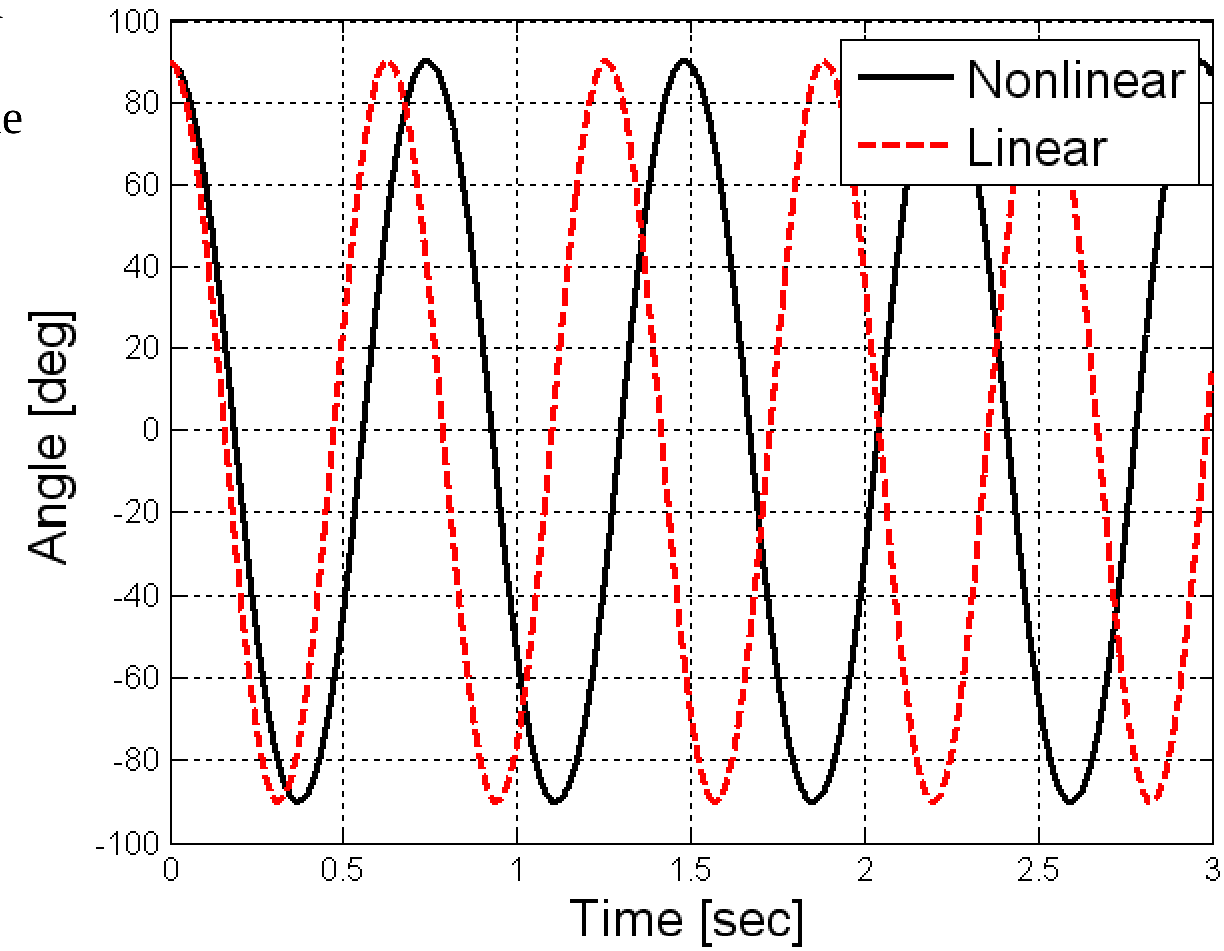


# Simulation Results (Zero-Torque Stable Trim)



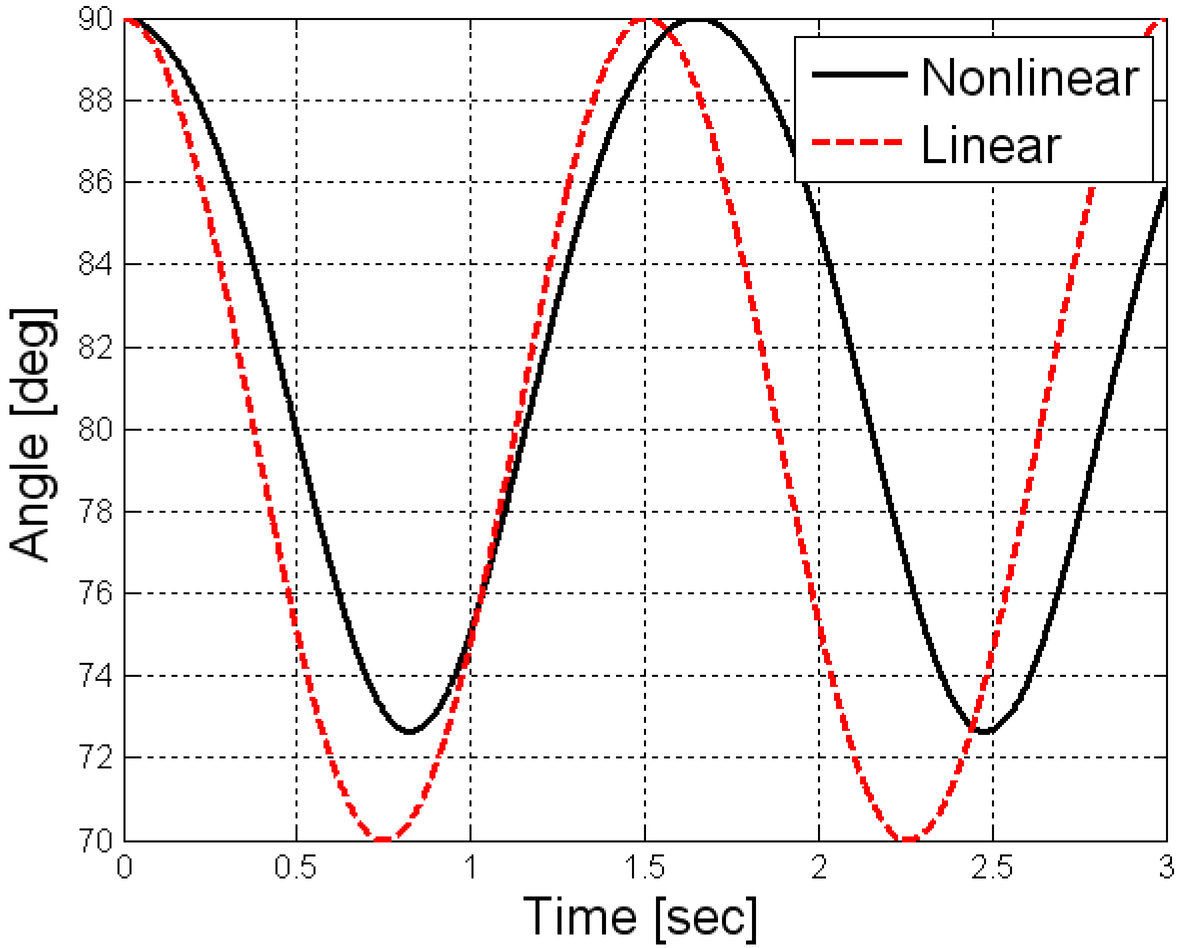
# Larger Amplitude Zero-Torque Stable Trim

Linear Approximation Overestimates Restoring Torque For Large Amplitudes



# Non-Zero Torque Stable Trim (@ 80°)

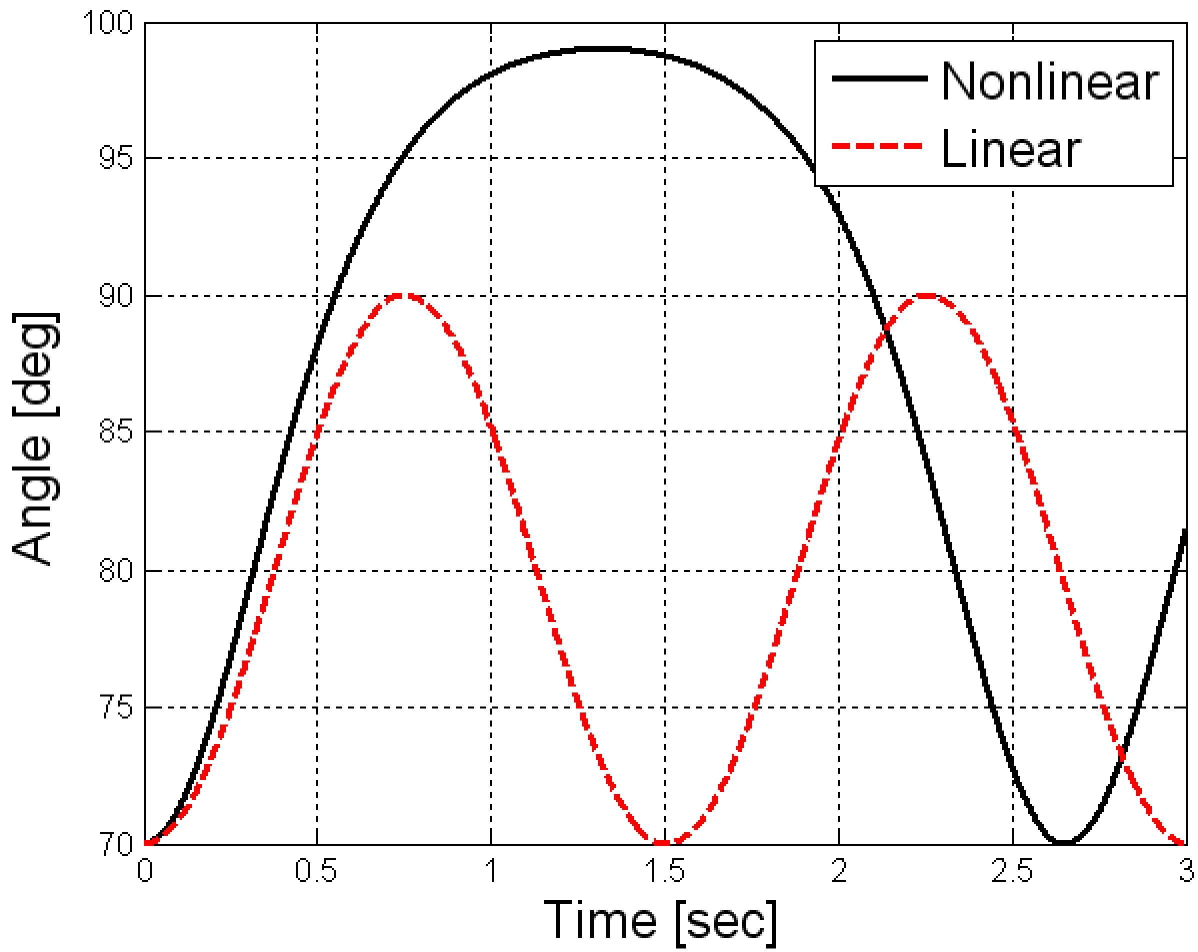
Linear
Approximation
Has Restoring
Torque Too
Large for (+)
Perturbations
and Torque Too
Small for (-)
Perturbations



# Repeat with Negative Initial Perturbation!

Nonlinear Systems Don't Behave Nearly as Nicely as Linear Systems!

Characterizing Behavior of Nonlinear Systems Much More Difficult!



# Zero-Torque Unstable Trim (@ 180°)

Linear Approximation Captures Something Essentially "True" Near Equilibrium, But Badly Misses Global Behavior!