ESE 406/505 & MEAM 513 - SPRING 2013 - HOMEWORK #3
More Linearization & Laplace Transforms
DUE 30-Jan-2013 (Late Pass 4-Feb-2013)

1. Write the following ordinary differential equation in state-space form and then solve using Laplace transforms.

$$\frac{d^2y}{dt^2}+10\frac{dy}{dt}+169y=0\,,\ y(0)=2\,,\ \frac{dy}{dt}(0)=-10$$

Hint: See HW#1.

2. Solve the following equation using Laplace transforms. What is the system transfer function?

$$\frac{dx}{dt}=-\omega x+\omega u$$
$$y=-x+u$$
$$\text{with } x(0)=0 \ \ \&\ \ u=3+2\sin\left(\omega t\right)$$

Answer: $y=3e^{-\omega t}+\sin\left(\omega t\right)+\cos\left(\omega t\right)-e^{-\omega t}=2e^{-\omega t}+\sin\left(\omega t+\frac{\pi}{4}\right)$

$$H(s)=\frac{s}{s+\omega}$$

3. In this problem, we will play with Laplace transforms and derive a few important results. We will often make use of the "unit step" function, which can be written as follows:

$$\hat{u}(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$$

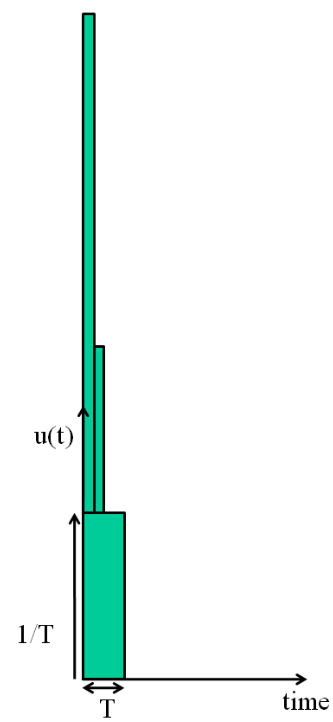
In class, we showed that the Laplace transform of this function is $\hat{U}(s)=\frac{1}{s}$.

Now consider the following function:

$$\delta_T(t)=\frac{1}{T}\left[\hat{u}(t)-\hat{u}(t-T)\right]$$

This function is equal to $\frac{1}{T}$ for $0<t<T$ zero otherwise. Let's take the Laplace transform of this function. We will need to use the important result that delay in time corresponds to multiplication by e^{-Ts} in the s-domain:

$$L\left\{f(t-T)\right\}=e^{-Ts}F(s)$$



from which you should be able to show that $\delta_T(s) = L\{\delta_T(t)\} = \frac{1}{Ts}\left[1-e^{-Ts}\right]$. Now we want to take the limit as $T \to 0$. You might have to remind yourself how to use L'Hôpital's Rule to show:

$$\delta(s) = L\{\delta(t)\} = \lim_{T\to 0}\delta_T(s) = 1$$

What does this mean? Notice that $\int_{-\infty}^{\infty}\delta_T(t)dt = 1$, regardless of the value of *T*. As $T \to 0$, we get a function whose amplitude gets infinitely large for zero time, but whose integral value is still unity. This function can be thought of as the derivative of $\hat{u}(t)$, which is easy to see in the Laplace transforms, since $\delta(s) = s\hat{U}(s)$. We call $\delta(t)$ the "unit impulse function".

What good is a function that has infinite amplitude for zero time? There are many physical processes in which something with very large amplitude occurs over a very short time interval. An example from elementary mechanics is firing a bullet from a gun. The force is very large, but lasts for a very short time. When we analyze this process (freshman physics), we typically write $mV = \int F(t)dt = I(t)$, where *I(t)* is the "impulse" delivered by the gun powder. An example from basic electronics is the charging of a capacitor connected directly to a power supply. When we turn the power supply on, the resistance in the wires is very low, so a very large current flows, until the capacitor has accumulated enough charge for its voltage to be equal to the voltage of the power supply: $Q = CV = \int i(t)dt$. (Obviously, "*V*" is velocity in mechanics and voltage in electronics.)

In situations such as these, we often don't care about the details of the very fast process. By pretending, with math, that they happen infinitely fast, we save ourselves from having to invent arbitrary details and we capture only the fundamental characteristics that actually matter[1].

A couple other useful observations about the unit impulse. First, if we apply a unit impulse as the input to a linear system, the output will simply be the inverse Laplace transform of the transfer function:

$$Y(s) = H(s)\delta(s) \Rightarrow y(t) = L^{-1}\{H(s)\}$$

Put another way, ***the transfer function is the Laplace transform of the unit impulse response***.

Finally, there is a very useful mathematical property of the unit impulse that is sometimes handy:

$$\int_{t=\tau^-}^{t=\tau^+}\delta(t-\tau)f(t)dt = f(\tau)$$

where $t=\tau^-$ and $t=\tau^+$ are arbitrary times before and after $t=\tau$, respectively.

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[1] We sometimes approximate a unit impulse with a large-amplitude input that lasts only a short time. But we have to be careful thinking about the response of any real system to a unit step or unit impulse input. Most actuators have "rate limits"--they can only move so fast. Thus, we can typically never achieve a true unit step input on a physical system, let alone a true unit impulse. Nevertheless, studying the response of systems to unit steps and unit impulses is an invaluable tool for helping us understand what is going on in real systems. We just have to remember that the real system response might be significantly affected by limitations that are not present in our simple linear models. Nonlinear simulation (using a tool such as SIMULINK) plays a very important complementary role to linear analysis in the engineering design process for control systems.

IS THERE A QUESTION HERE? All you have to submit is a graph of the unit impulse response (with zero initial conditions) of the following system. Briefly explain what is going on.

$$\frac{d^2 y}{dt^2} + 10\frac{dy}{dt} + 169y = u(t)$$

4. [ESE 505 & MEAM 513 Only] 3D rotation of a rigid body is governed by the following equation:

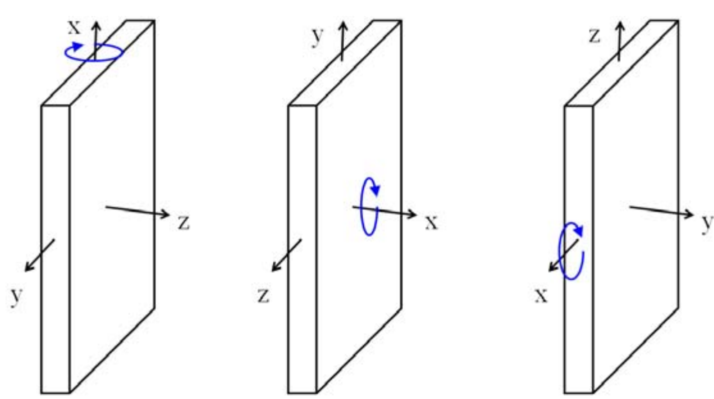
$$\frac{d\vec{H}}{dt} = \vec{\vec{I}}\,\dot{\vec{\omega}} + \vec{\omega} \times \vec{\vec{I}}\,\vec{\omega} = \vec{M}$$

where $\vec{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is the angular velocity, $\vec{\vec{I}} = \begin{bmatrix} I_X & 0 & 0 \\ 0 & I_Y & 0 \\ 0 & 0 & I_Z \end{bmatrix}$ is the inertia tensor, and $\vec{M}$ is the applied moment. Expanding the above expression in the case of no applied moment ($\vec{M} = \vec{0}$) yields the following nonlinear state-space equations

$$\frac{dp}{dt} = \frac{1}{I_X}\left(I_Y - I_Z\right)qr$$
$$\frac{dq}{dt} = \frac{1}{I_Y}\left(I_Z - I_X\right)rp$$
$$\frac{dr}{dt} = \frac{1}{I_Z}\left(I_X - I_Y\right)pq$$

Linearize these equations about a trim condition of spinning steadily about the x-axis: $\vec{\omega}_o = \begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix}$.

Consider the response to a small perturbation in the form of a non-zero angular velocity about the y-axis: $\Delta\vec{\omega}(0) = \begin{pmatrix} 0 \\ \varepsilon\Omega \\ 0 \end{pmatrix}$. You don't have to solve for the response, but discuss the stability of response. Note that there are 3 distinct situations to consider, depending on the relative inertias of the X, Y, and Z axes, as shown in the figure at right.



$I_X < I_Y < I_Z$ $\quad$ $I_Y < I_Z < I_X$ $\quad$ $I_Z < I_X < I_Y$

<u>BONUS</u>: Use SIMULINK to solve the nonlinear equations and plot representative time histories of angular velocities for each of the 3 cases (use inertias of 1, 2, and 4 and an initial angular velocity perturbation of ε=0.1). Do the results make sense in light of your linear analysis?