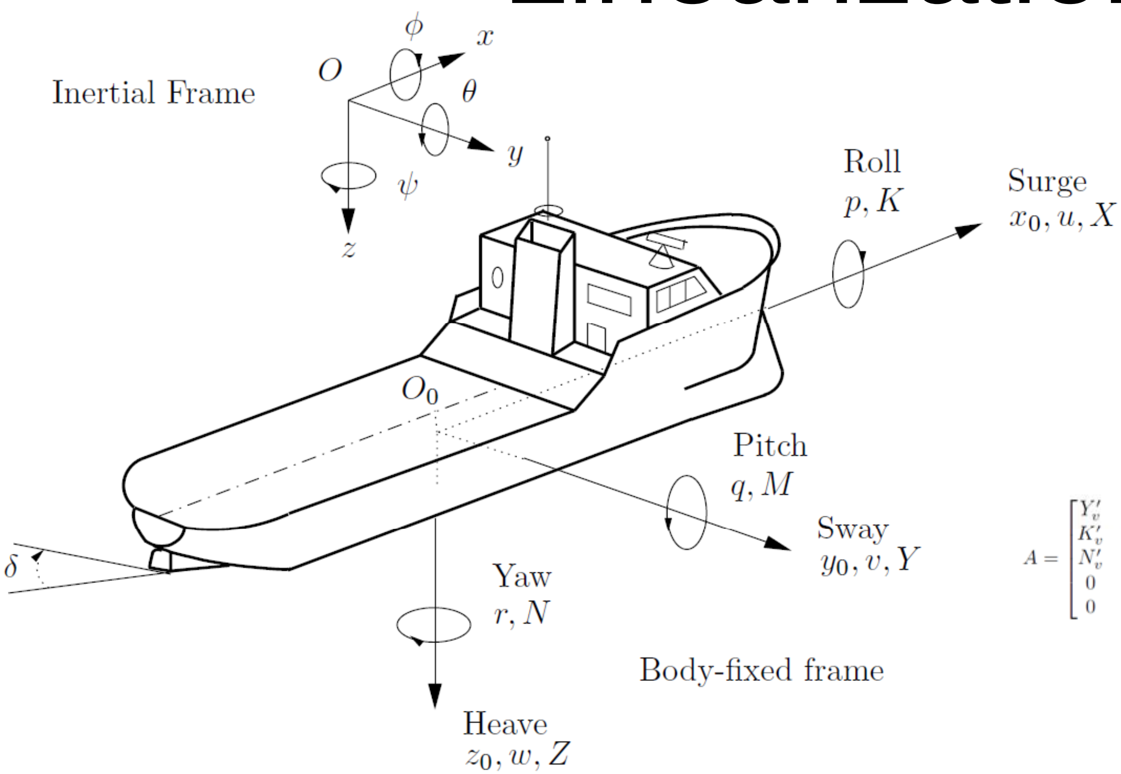
State Space Representation & Linearization



$$A = \left.\frac{\partial f_{hyd}(z,u,\delta)}{\partial z}\right|_{\overline{z},\overline{u},\overline{\delta}} + \left.\frac{\partial f_{rudder}(z,u,V_{av},\delta)}{\partial z}\right|_{\overline{z},\overline{u},\overline{V}_{av},\overline{\delta}}$$

$$B = \left.\frac{\partial f_{rudder}(z,u,V_{av},\delta)}{\partial \delta}\right|_{\overline{x},\overline{u},\overline{V}_{av},\overline{\delta}},$$

$$A = \begin{bmatrix} Y'_v & Y'_p + Y'_{pu}\overline{u}'_a & Y'_r - m'\overline{u}' & Y'_\phi & 0 \\ K'_v & K'_p + K'_{pu}\overline{u}'_a & K'_r + m'z'_G\overline{u}' & -(\rho g \nabla GM)' & 0 \\ N'_v & N'_p + N'_{pu}\overline{u}'_a & N'_r + m'z'_G\overline{u}' & N'_\phi & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} Y'_\delta \\ K'_\delta \\ N'_\delta \\ 0 \\ 0 \end{bmatrix}.$$

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Mathematical Ship Modeling for Control Applications

by

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State-Space Representation Concept

- Method for Combining Multiple ODEs of Arbitrary Order Into System of First-Order ODEs
- Define System Input (Generally Vector But We'll Work with Scalar)
- Define System Output (Generally Vector But We'll Work with Scalar)
- Define "State Variables" & Assemble into State Vector
  - One State for Each Independent Variable in the ODE
  - One State for Each Derivative of Each Independent Variable (Except the Highest-Order Derivative)
- Final Equations = State Equation & Output Equation

$$\underline{\dot{x}} = \underline{f}\left(\underline{x}, u\right) \qquad y = h(\underline{x}, u)$$

Linearization = One-Term Taylor Series Approx.

$$\dot{\underline{x}} = \underline{f}\left(\underline{x}, u\right)$$

Complete Nonlinear Dynamics

$$\underline{f}\left(\underline{x}_o, u_o\right) = \underline{0}$$

"Fixed Point" = Steady Condition (Called "Trim" in Airplane World)

$$\underline{f}\left(\underline{x}, u\right) = \underbrace{\underline{f}\left(\underline{x}_o, u_o\right)}_{=0} + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_o \left(\underline{x} - \underline{x}_o\right) + \left.\frac{\partial \underline{f}}{\partial u}\right|_o \left(u - u_o\right) + \ldots$$

$$h\left(\underline{x}, u\right) = \underbrace{h\left(\underline{x}_o, u_o\right)}_{y_o} + \left.\frac{\partial h}{\partial \underline{x}}\right|_o \left(\underline{x} - \underline{x}_o\right) + \left.\frac{\partial h}{\partial u}\right|_o \left(u - u_o\right) + \ldots$$

$$\Delta \underline{x}(t) \triangleq \underline{x}(t) - \underline{x}_o$$

$$\Delta u(t) \triangleq u(t) - u_o$$

$$\Delta y(t) \triangleq y(t) - y_o$$

$$\left.\begin{array}{l} \boxed{\Delta \dot{\underline{x}} \approx A \Delta \underline{x} + B \Delta u} \\ \boxed{\Delta y \approx C \Delta \underline{x} + D \Delta u} \end{array}\right\}$$

Linearized System

Linearization : Explicit Forms of Matrices

$$A \triangleq \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{(\underline{x}_o, u_o)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(\underline{x}_o, u_o)}$$

State Dynamic Matrix

$$B \triangleq \left.\frac{\partial \underline{f}}{\partial u}\right|_{(\underline{x}_o, u_o)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}\right|_{(\underline{x}_o, u_o)}$$

Control Dynamic Matrix

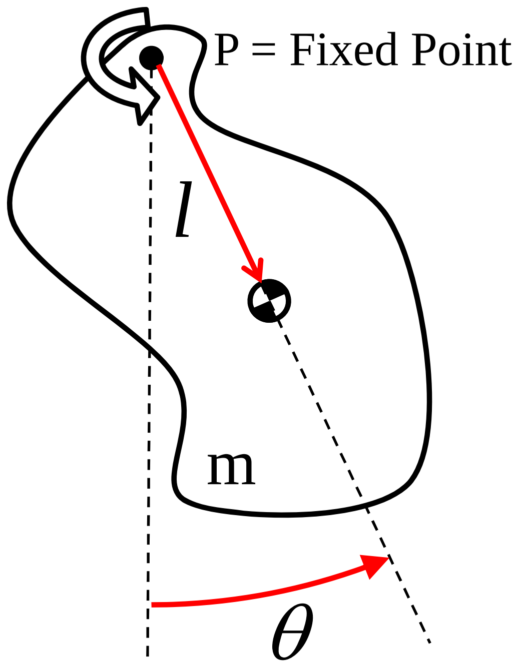
$$C \triangleq \left.\frac{\partial h}{\partial \underline{x}}\right|_{(\underline{x}_o, u_o)} = \left.\begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}\right|_{(\underline{x}_o, u_o)}$$

Output State Matrix

$$D \triangleq \left.\begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix}\right|_{(\underline{x}_o, u_o)}$$

Control Output Matrix

Recall DC Motor Attached to Pendulum



Kinematics of Shaft Rotation

$$\boxed{\frac{d\theta}{dt} = \Omega}$$

Sum Moments on Shaft & Pendulum

$$\boxed{\left(I_P + J\right)\frac{d\Omega}{dt} = Ki - mgl\sin\theta}$$

KVL Applied to Motor

$$\boxed{L\frac{di}{dt} = e - K\Omega - Ri}$$

State Space Representation of Pendulum + Motor

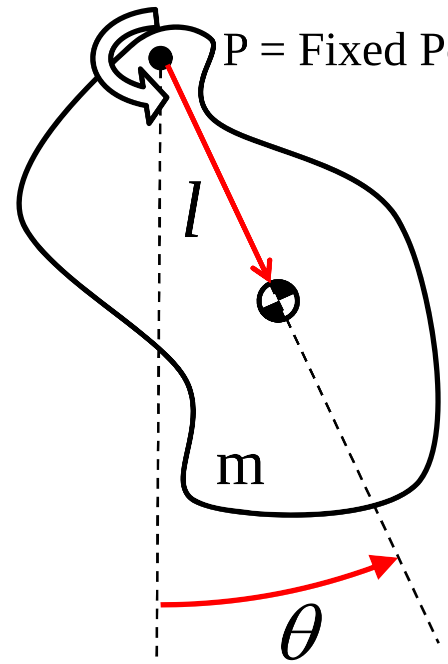
$$x_1 = \theta \qquad x_2 = \Omega \qquad x_3 = i$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Assume We Measure Angle

$$u = e$$

$$y = \theta = h(\underline{x}, u)$$



$$\dot{\underline{x}} = \frac{d}{dt}\begin{bmatrix} \theta \\ \Omega \\ i \end{bmatrix} = \begin{bmatrix} \Omega \\ \frac{1}{J + I_P}\left(-mgl\sin\theta + Ki\right) \\ \frac{1}{L}\left(-Ri - K\Omega + e\right) \end{bmatrix} = \underline{f}\left(\underline{x}, u\right)$$

More State Space Representation

$$\dot{\underline{x}} = \frac{d}{dt}\begin{bmatrix}\theta \\ \Omega \\ i\end{bmatrix} = \begin{bmatrix}\Omega \\ \frac{1}{J+I_P}\left(-mgl\sin\theta + Ki\right) \\ \frac{1}{L}\left(-Ri - K\Omega + e\right)\end{bmatrix} = \underline{f}\left(\underline{x},u\right) \qquad y = \theta \Rightarrow h(\underline{x},u)$$

$$\dot{x}_1 = f_1\left(x_1,x_2,x_3,u\right) = x_2 \qquad h(x_1,x_2,u) = x_1$$

$$\dot{x}_2 = f_2\left(x_1,x_2,x_3,u\right) = \frac{1}{J+I_P}\left(-mgl\sin x_1 + Kx_3\right)$$

$$\dot{x}_3 = f_3\left(x_1,x_2,x_3,u\right) = \frac{1}{L}\left(-Rx_3 - Kx_3 + u\right)$$

- We Often Don’t Bother to Rename the States Using “x” but Keep Physically Significant Variables
- This is All Just Mathematical Bookkeeping!

Trim = Find Equilibrium State ("Fixed Point")

$$\underline{\dot{x}} = \underline{0} \;\Rightarrow\; \underline{f}\left(\underline{x}_o, u_o\right) = \underline{0} \;\Rightarrow\; \begin{cases} x_{2_O} = 0 \\ Kx_{3_O} = mgl \sin x_{1_O} \\ u_o = Rx_{3_O} \end{cases} \;\Rightarrow\; \begin{cases} \Omega_o = 0 \\ Ki_o = mgl \sin \theta_o \\ e_o = Ri_o \end{cases}$$

$$e_o = Ri_o = \frac{R}{K} mgl \sin \theta_o$$

- Understanding Steady Operating Conditions is Critical First Step in Control System Design
- Steady Voltage → Steady Current & Depends On...
  - Steady Angle
  - First Mass Moment of Pendulum (mgl)
  - Motor Resistance (Higher e for Higher R to Get Needed Current)
  - Motor K (More Coils & Stronger Magnets → Lower Voltage Required)

Linearization

$$\frac{d}{dt}\begin{bmatrix}\Delta\theta\\ \Delta\Omega\\ \Delta i\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -\dfrac{mgl\cos\theta_o}{J+I_P} & 0 & \dfrac{K}{J+I_P}\\ 0 & -\dfrac{K}{L} & -\dfrac{R}{L}\end{bmatrix}\begin{bmatrix}\Delta\theta\\ \Delta\Omega\\ \Delta i\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \dfrac{1}{L}\end{bmatrix}\Delta e$$

$$\Delta\theta=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\theta\\ \Delta\Omega\\ \Delta i\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}\Delta e$$