Algebra e matematica discreta, a.a. 2020/2021,

Scuola di Scienze - Corso di laurea: Informatica

ESERCIZIO TIPO 12

Si trovi una base ortonormale del sottospazio di $\mathbb{C}^4$

$$V = \left\langle \begin{pmatrix}1\\0\\i\\0\end{pmatrix}; \begin{pmatrix}i\\1\\0\\-1\end{pmatrix}; \begin{pmatrix}0\\-1\\-1\\1\end{pmatrix}; \begin{pmatrix}0\\0\\0\\i\end{pmatrix} \right\rangle.$$

$\boxed{1^0\textbf{MODO}}$

$\boxed{1}$ Troviamo una base $\mathcal{B}_1$ di V.

Poniamo

$$\mathbf{w_1} = \begin{pmatrix}1\\0\\i\\0\end{pmatrix}, \quad \mathbf{w_2} = \begin{pmatrix}i\\1\\0\\-1\end{pmatrix}, \quad \mathbf{w_3} = \begin{pmatrix}0\\-1\\-1\\1\end{pmatrix}, \quad \mathbf{w_4} = \begin{pmatrix}0\\0\\0\\i\end{pmatrix}$$

e costruiamo la matrice $\mathbf{A} = \begin{pmatrix}\mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} & \mathbf{w_4}\end{pmatrix}$, ossia una matrice tale che $C(\mathbf{A}) = V$.

$$\mathbf{A} = \begin{pmatrix}1 & i & 0 & 0\\0 & 1 & -1 & 0\\i & 0 & -1 & 0\\0 & -1 & 1 & i\end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix}1 & i & 0 & 0\\0 & 1 & -1 & 0\\0 & 1 & -1 & 0\\0 & -1 & 1 & i\end{pmatrix} \xrightarrow{E_{42}(1)E_{32}(-1)}$$

$$\to \begin{pmatrix}1 & i & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & i\end{pmatrix} \xrightarrow{E_{34}} \begin{pmatrix}1 & i & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & i\\0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{E_3(-i)} \begin{pmatrix}1 & i & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix} = \mathbf{U}$$

Dunque $\mathcal{B}_1 = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_4}\}$ è una base di $C(\mathbf{A}) = V$.

$\boxed{2}$ Troviamo una base ortogonale $\mathcal{B}_2$ di V: poniamo $\mathbf{v_1} = \mathbf{w_1}, \mathbf{v_2} = \mathbf{w_2}$ e $\mathbf{v_3} = \mathbf{w_4}$, e applichiamo l'algoritmo di Gram-Schmidt a $\{\mathbf{v_1}; \mathbf{v_2}; \mathbf{v_3}\}$.

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1}, \qquad \mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{12} = \frac{(\mathbf{u_1}|\mathbf{v_2})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_2}) = \mathbf{u_1}^H\mathbf{v_2} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} = i$$

$$(\mathbf{u_1}|\mathbf{u_1}) = \mathbf{u_1}^H\mathbf{u_1} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = 2$$

$$\Longrightarrow \quad \alpha_{12} = i/2$$

$$\begin{aligned} \mathbf{u_2} &= \mathbf{v_2} - \alpha_{12}\mathbf{u_1} = \\ &= \mathbf{v_2} - \tfrac{i}{2}\mathbf{u_1} = \end{aligned}$$

$$= \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} - \tfrac{i}{2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2},$$

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{13} = \frac{(\mathbf{u_1}|\mathbf{v_3})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_3}) = \mathbf{u_1}^H\mathbf{v_3} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = 0$$

$$\Longrightarrow \quad \alpha_{13} = 0$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{23} = \frac{(\mathbf{u_2}|\mathbf{v_3})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_3}) = \mathbf{u_2}^H\mathbf{v_3} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = -i$$

$$(\mathbf{u_2}|\mathbf{u_2}) = \mathbf{u_2}^H\mathbf{u_2} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\Longrightarrow \quad \alpha_{23} = -\frac{2}{5}i$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2} =$$
$$= \mathbf{v_3} + \tfrac{2i}{5}\mathbf{u_2} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + \tfrac{2i}{5} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}$$

$\mathcal{B}_2 = \{\mathbf{u_1}; \mathbf{u_2}; \mathbf{u_3}\}$, dove

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{u_2} = \frac{1}{2} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{u_3} = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortogonale di V.

$\boxed{3}$ Troviamo una base ortonormale $\mathcal{B}$ di V, normalizzando gli elementi di $\mathcal{B}_2$.

$$\|\mathbf{u_1}\|_2 = \sqrt{(\mathbf{u_1}|\mathbf{u_1})} = \sqrt{2}$$

$$\|\mathbf{u_2}\|_2 = \sqrt{(\mathbf{u_2}|\mathbf{u_2})} = \sqrt{5/2}$$

$$\|\mathbf{u_3}\|_2 = \sqrt{(\mathbf{u_3}|\mathbf{u_3})} = \sqrt{\mathbf{u_3}^H\mathbf{u_3}} = \sqrt{\tfrac{1}{5}\begin{pmatrix} -1 & -2i & -i & -3i \end{pmatrix} \tfrac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}} = \tfrac{\sqrt{15}}{5}$$

Concludendo: $\mathcal{B} = \{\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|_2}; \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|_2}; \frac{\mathbf{u_3}}{\|\mathbf{u_3}\|_2}\}$, dove

$$\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{u_3}}{\|\mathbf{u_3}\|_2} = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortonormale di V.

$\boxed{2^0\textbf{MODO}}$

$\boxed{1}$ Prima costruiamo **un insieme di generatori ortogonale di V**: poniamo

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$$

e applichiamo l'algoritmo di Gram-Schimdt a $\{\mathbf{v_1};\mathbf{v_2};\mathbf{v_3};\mathbf{v_4}\}$. Otterremo 4 vettori, $\mathbf{u_1},\mathbf{u_2},\mathbf{u_3},\mathbf{u_4}$, e l'insieme $\{\mathbf{u_1};\mathbf{u_2};\mathbf{u_3};\mathbf{u_4}\}$ sarà un insieme di generatori ortogonale di V.

Per sapere se alcuni degli $\mathbf{u_i}$ saranno nulli, e in tal caso quali, troviamo innanzitutto una forma ridotta di Gauss $\mathbf{U}$ della matrice $\mathbf{A}$ che ha come colnne $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_4}$: le eventuali colonne libere di U corrisponderanno agli $\mathbf{u_i}$ nulli.

$$\mathbf{A} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4}\end{pmatrix} = \begin{pmatrix}1 & i & 0 & 0\\ 0 & 1 & -1 & 0\\ i & 0 & -1 & 0\\ 0 & -1 & 1 & i\end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix}1 & i & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & i\end{pmatrix} \rightarrow$$

$$\xrightarrow{E_{42}(1)E_{32}(-1)} \begin{pmatrix}1 & i & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & i\end{pmatrix} \xrightarrow{E_{34}} \begin{pmatrix}1 & i & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{E_3(-i)}$$

$$\longrightarrow \begin{pmatrix}1 & i & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} = \mathbf{U}$$

Poichè $\mathbf{U}$ ha come unica colonna libera la 3^a, allora applicando l'algoritmo di Gram-Schimdt a $\{\mathbf{v_1};\mathbf{v_2};\mathbf{v_3};\mathbf{v_4}\}$ otterremo $\mathbf{u_3}=\mathbf{0}$.

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix}1\\0\\i\\0\end{pmatrix}$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1}, \qquad \mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{12} = \frac{(\mathbf{u_1}|\mathbf{v_2})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_2}) = \mathbf{u_1}^H\mathbf{v_2} = \begin{pmatrix}1 & 0 & -i & 0\end{pmatrix}\begin{pmatrix}i\\1\\0\\-1\end{pmatrix} = i$$

$$(\mathbf{u_1}|\mathbf{u_1}) = \mathbf{u_1}^H\mathbf{u_1} = \begin{pmatrix}1 & 0 & -i & 0\end{pmatrix}\begin{pmatrix}1\\0\\i\\0\end{pmatrix} = 2$$

$$\Longrightarrow \quad \alpha_{12} = i/2$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1} =$$
$$= \mathbf{v_2} - \tfrac{i}{2}\mathbf{u_1} =$$

$$= \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} - \tfrac{i}{2}\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2},$$

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{13} = \frac{(\mathbf{u_1}|\mathbf{v_3})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_3}) = \mathbf{u_1}^H\mathbf{v_3} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} = i$$

$$(\mathbf{u_1}|\mathbf{u_1}) = 2$$

$$\Longrightarrow \quad \alpha_{13} = \frac{i}{2}$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{23} = \frac{(\mathbf{u_2}|\mathbf{v_3})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_3}) = \mathbf{u_2}^H\mathbf{v_3} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= -1 - \frac{1}{2} - 1 = -\frac{5}{2}$$

$$(\mathbf{u_2}|\mathbf{u_2}) = \mathbf{u_2}^H\mathbf{u_2} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\Longrightarrow \quad \alpha_{23} = -1$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2} =$$
$$= \mathbf{v_3} - \tfrac{i}{2}\mathbf{u_1} + \mathbf{u_2} =$$

$$= \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \tfrac{i}{2}\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\mathbf{u_4} = \mathbf{v_4} - \alpha_{14}\mathbf{u_1} - \alpha_{24}\mathbf{u_2} - \alpha_{34}\mathbf{u_3}$

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{14} = \frac{(\mathbf{u_1}|\mathbf{v_4})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_4}) = \mathbf{u_1}^H\mathbf{v_4} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = 0$$

$$\Longrightarrow \quad \alpha_{14} = 0$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{24} = \frac{(\mathbf{u_2}|\mathbf{v_4})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_4}) = \mathbf{u_2}^H\mathbf{v_4} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = -i$$

$$(\mathbf{u_2}|\mathbf{u_2}) = \mathbf{u_2}^H\mathbf{u_2} = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\Longrightarrow \quad \alpha_{24} = -\frac{2}{5}i$$

$$\mathbf{u_3} = \mathbf{0} \quad \Longrightarrow \quad \alpha_{34} = 0 \quad \text{per def.}$$

$$\begin{aligned} \mathbf{u_4} &= \mathbf{v_4} - \alpha_{24}\mathbf{u_2} = \\ &= \mathbf{v_4} + \tfrac{2i}{5}\mathbf{u_2} = \end{aligned}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + \tfrac{2i}{5} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}$$

Dunque

$$\left\{ \mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{u_2} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}; \mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u_4} = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix} \right\}$$

è un insieme di generatori ortogonale di V.

$\boxed{2}$ Costruiamo **una base ortogonale di V** togliendo dall'insieme di generatori ortogonale di V trovato al punto $\boxed{1}$ gli eventuali $\mathbf{u}_{\boldsymbol{i}}$ nulli. In questo caso poniamo:

$$\mathbf{w_1} = \mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{w_2} = \mathbf{u_2} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}, \quad \mathbf{w_3} = \mathbf{u_4} = \frac{1}{5}\begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}.$$

L'insieme

$$\left\{ \mathbf{w_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{w_2} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}; \mathbf{w_3} = \frac{1}{5}\begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix} \right\}$$

è una base ortogonale di V.

$\boxed{3}$ Costruiamo **base ortonormale di V** normalizzando la base ortogonale trovata al punto $\boxed{2}$, ossia dividendo ciascun elemento della base ortogonale trovata in $\boxed{2}$ per la propria norma euclidea.

Cominciamo con il calcolare la norma euclidea di $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$:

$$\|\mathbf{w_1}\|_2 = \sqrt{(\mathbf{u_1}|\mathbf{u_1})} = \sqrt{2}$$

$$\|\mathbf{w_2}\|_2 = \sqrt{(\mathbf{u_2}|\mathbf{u_2})} = \sqrt{5/2}$$

$$\|\mathbf{w_3}\|_2 = \sqrt{(\mathbf{u_4}|\mathbf{u_4})} = \sqrt{\mathbf{u_4}^H \mathbf{u_4}} = \sqrt{\frac{1}{5}\begin{pmatrix} -1 & -2i & -i & -3i \end{pmatrix} \frac{1}{5}\begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}} = \frac{\sqrt{15}}{5}$$

Allora

$$\mathcal{B} = \left\{ \frac{\mathbf{w_1}}{\|\mathbf{w_1}\|_2}; \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|_2}; \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|_2} \right\},$$

dove

$$\frac{\mathbf{w_1}}{\|\mathbf{w_1}\|_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|_2} = \frac{1}{\sqrt{10}}\begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|_2} = \frac{1}{\sqrt{15}}\begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortonormale di V.