LEZIONE 37

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V sp. vett. euclideo, $(\cdot|\cdot)$ il prodotto interno di V

DATO $\mathcal{S}$ INSIEME DI GENERATORI DI V: $\mathcal{S} = \{\underline{v}_1, \dots, \underline{v}_n\}$

TROVO $\mathcal{S}_0$ INS. DI GENERAT. ORTOGONALE DI V: $\mathcal{S}_0 = \{\underline{u}_1, \dots, \underline{u}_n\}$

$\underline{v}_1, \underline{v}_2, \ldots\ldots, v_n$

$\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$

$$\underline{u}_1 = \underline{v}_1$$

$$\underline{u}_2 = \underline{v}_2 - \alpha_{12}\underline{u}_1$$

$$\alpha_{12} = \begin{cases} \dfrac{(\underline{u}_1|\underline{v}_2)}{(\underline{u}_1|\underline{u}_1)} & \text{se } \underline{u}_1 \neq \underline{0} \\ 0 & \text{se } \underline{u}_1 = \underline{0} \end{cases}$$

$$\underline{u}_3 = \underline{v}_3 - \alpha_{13}\underline{u}_1 - \alpha_{23}\underline{u}_2$$

$$\alpha_{13} = \begin{cases} \dfrac{(\underline{u}_1|\underline{v}_3)}{(\underline{u}_1|\underline{u}_1)} & \text{se } \underline{u}_1 \neq \underline{0} \\ 0 & \text{se } \underline{u}_1 = \underline{0} \end{cases} \longrightarrow \alpha_{23} = \begin{cases} \dfrac{(\underline{u}_2|\underline{v}_3)}{(\underline{u}_2|\underline{u}_2)} & \text{se } \underline{u}_2 \neq \underline{0} \\ 0 & \text{se } \underline{u}_2 = \underline{0} \end{cases}$$

$$\underline{u}_4 = \underline{v}_4 - \alpha_{14}\underline{u}_1 - \alpha_{24}\underline{u}_2 - \alpha_{34}\underline{u}_3$$

.....

$$\underline{u}_j = \underline{v}_j - \alpha_{1j}\underline{u}_1 - \alpha_{2j}\underline{u}_2 - \cdots - \alpha_{j-1\,j}\underline{u}_{j-1}$$

$$= \underline{v}_j - \sum_{i=1}^{j-1} \alpha_{ij}\underline{u}_i$$

$$\alpha_{ij} = \begin{cases} 0 & \text{se } \underline{u}_i = \underline{0} \\ \dfrac{(\underline{u}_i \mid \underline{v}_j)}{(\underline{u}_i \mid \underline{u}_i)} & \text{se } \underline{u}_i \neq \underline{0} \end{cases}$$

CALCOLO DI DETERMINANTI

Sia A una matrice QUADRATA di ordine n.
Il determinante di A è un numero associato ad A. Si indica $\mathrm{Det}\, A$ o $\det A$. Incominciamo a calcolarlo con $n=1,2,3\ldots$

$n=1$ $\qquad A=[a_{11}] \qquad \mathrm{Det}\, A = a_{11}$

$n=2$ $\qquad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathrm{Det}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$$\mathrm{Det}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = 2\cdot 5 - 3\cdot 4 = 10 - 12 = -2$$

$$\mathrm{Det}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} + a_{12}(-a_{21})$$

$$= a_{11}\overbrace{\boxed{(-1)^{1+1}\,\mathrm{Det}[a_{22}]}}^{A_{11}} + a_{12}\overbrace{\boxed{(-1)^{1+2}\,\mathrm{Det}[a_{21}]}}^{A_{12}}$$

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & \boxed{a_{22}} \end{bmatrix} \qquad [a_{22}] = C_{11} \qquad A_{11} = (-1)^{1+1}\,\mathrm{Det}\,C_{11}$$

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix} \qquad [a_{21}] = C_{12} \qquad A_{12} = (-1)^{1+2}\,\mathrm{Det}\,C_{12}$$

$A_{n\times n}$ $\quad (i,j)$

$(n-1)\times(n-1)$

C_{ij} = la matrice che si ottiene da A cancellando la i-esima riga di A e la j-esima colonna di A (MATRICE COMPLEMENTARE DI POSTO (i,j) DI A)

$$A_{ij} = (-1)^{i+j}\,\mathrm{Det}\,C_{ij}$$ = IL COFATTORE DI POSTO (i,j) IN A

$$\operatorname{Det}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix}$$

$$n=3 \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\operatorname{Det} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$A_{11} = (-1)^{1+1}\operatorname{Det}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = (-1)^{1+2}\operatorname{Det}\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad A_{13} = (-1)^{1+3}\operatorname{Det}\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\mathrm{Det}\begin{pmatrix} 1 & -5 & 0 & 3 \\ 6 & 2 & 0 & 4 \\ -2 & 0 & 0 & 2 \\ -1 & 7 & 5 & 1 \end{pmatrix} =$$

$$= [1 \quad -5 \quad 0 \quad 3] \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{bmatrix} =$$

$$= \overset{-20}{\boxed{A_{11}}} - 5 \overset{100}{\boxed{A_{12}}} + \cancel{0 A_{13}} + 3 \overset{-20}{\boxed{A_{14}}} = -20 - 500 - 60 = -580$$

FORMULA DEL DETERMINANTE SVILUPPATO RISPETTO ALLA 1A RIGA

$A = (a_{ij}) \quad m \times m$

$$\mathrm{Det}\, A = [a_{11} \; a_{12} \cdots a_{1n}] \begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1n} \end{bmatrix} =$$

$$= a_{11} A_{11} + a_{12} A_{12} + \ldots + a_{1n} A_{1n}$$

SVILUPPO DI LAPLACE

$$A_{11} = \cancel{(-1)^{1+1}}\, \mathrm{Det}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 2 \\ 7 & 5 & 1 \end{bmatrix} = 2 B_{11} + \cancel{0 B_{12}} + 4 B_{13} =$$

$$= 2\, \cancel{(-1)^{1+1}}\, \mathrm{Det}\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix} + 4\, \cancel{(-1)^{1+3}}\, \mathrm{Det}\begin{bmatrix} 0 & 0 \\ 7 & 5 \end{bmatrix} = 2(0 \cdot 1 - 2 \cdot 5) + 4(0 \cdot 5 - 0 \cdot 7)$$

$$= 2(-10) = -20$$

$$A = \begin{pmatrix} 1 & -5 & 0 & 3 \\ 6 & 2 & 0 & 4 \\ -2 & 0 & 0 & 2 \\ -1 & 7 & 5 & 1 \end{pmatrix}$$

$$A_{12} = (-1)^{1+2} \text{ Det} \begin{pmatrix} 6 & 0 & 4 \\ -2 & 0 & 2 \\ -1 & 5 & 1 \end{pmatrix} = -\left[6 B_{11} + \cancel{0 B_{12}} + 4 B_{13} \right] =$$

$$= -\left[6 \cancel{(-1)^{1+1}} \text{ Det} \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} + 4 \cancel{(-1)^{1+3}} \text{ Det} \begin{pmatrix} -2 & 0 \\ -1 & 5 \end{pmatrix} \right] =$$

$$= -\left[6 (0 \cdot 1 - 2 \cdot 5) + 4 ((-2) \cdot 5 - 0 \cdot (-1)) \right]$$

$$= -\left[6 (-10) + 4 (-10) \right] = -(-100) = 100$$

$$A = \begin{pmatrix} 1 & -5 & 0 & 3 \\ 6 & 2 & 0 & 4 \\ -2 & 0 & 0 & 2 \\ -1 & 7 & 5 & 1 \end{pmatrix}$$

$$A_{14} = (-1)^{1+4} \det \begin{pmatrix} 6 & 2 & 0 \\ -2 & 0 & 0 \\ -1 & 7 & 5 \end{pmatrix} = -\left[6B_{11} + 2B_{12} + \cancel{0B_{13}} \right] =$$

$$= -\left[6 \cdot \cancel{(-1)^{1+1}} \det \begin{pmatrix} 0 & 0 \\ 7 & 5 \end{pmatrix} + 2(-1)^{1+2} \det \begin{pmatrix} -2 & 0 \\ -1 & 5 \end{pmatrix} \right] =$$

$$= -\left[\cancel{6 \cdot 0} + 2 \cdot (-1) \cdot (-10) \right]$$

$$= -2 \cdot (-1) \cdot (-10) = 2 \cdot (-10) = -20$$

SVILUPPO DI LAPLACE DEL DETERMINANTE DI A RISPETTO ALLA i-ESIMA RIGA

$$\operatorname{Det} A = \operatorname{Det}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} =$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} A_{i1} \\ A_{i2} \\ \vdots \\ A_{in} \end{bmatrix} =$$

$$= a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}$$

$$A = \begin{pmatrix} 1 & -5 & 0 & 3 \\ 6 & 2 & 0 & 4 \\ \boxed{-2 \quad 0 \quad 0 \quad 2} \\ -1 & 7 & 5 & 1 \end{pmatrix}$$

$$\det A = \begin{bmatrix} \underset{3,1}{-2} & \underset{3,2}{0} & \underset{3,3}{0} & \underset{3,4}{2} \end{bmatrix} \begin{bmatrix} A_{31} \\ A_{32} \\ A_{33} \\ A_{34} \end{bmatrix} = -2\overset{130}{\boxed{A_{31}}} + \cancel{0A_{32}} + \cancel{0A_{33}} + 2\overset{-160}{A_{34}}$$

$$= -260 + 2 \cdot (-160)$$

$$= -580$$

$$A_{31} = \cancel{(-1)^{3+1}} \det \begin{pmatrix} -5 & 0 & 3 \\ \boxed{\underset{2,1}{2} \quad \underset{2,2}{0} \quad \underset{2,3}{4}} \\ 7 & 5 & 1 \end{pmatrix} = 2B_{21} + \cancel{0B_{22}} + 4B_{23}$$

$$= 2(-1)^{2+1} \det \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix} + 4(-1)^{2+3} \det \begin{pmatrix} -5 & 0 \\ 7 & 5 \end{pmatrix} = \quad 130$$

$$= 2 \cdot (-1)(0-15) + 4(-1)(-25) = 2 \cdot (-1)(-15) + 4 \cdot 25 = 30 + 100$$

$$A = \begin{pmatrix} 1 & -5 & 0 & 3 \\ 6 & 2 & 0 & 4 \\ \boxed{-2 \quad 0 \quad 0 \quad 2} \\ -1 & 7 & 5 & 1 \end{pmatrix}$$

$$A_{34} = (-1)^{3+4} \det \begin{pmatrix} 1 & -5 & 0 \\ \underset{2,1}{6} & \underset{2,2}{2} & \underset{2,3}{0} \\ -1 & 7 & 5 \end{pmatrix} = -\left[6B_{21} + 2B_{22} + 0\cancel{B_{23}} \right]$$

$$= -\left[6(-1)^{2+1} \det \begin{pmatrix} -5 & 0 \\ 7 & 5 \end{pmatrix} + 2 \cancel{(-1)^{2+2}} \det \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} \right] =$$

$$= -\left[6 \cdot (-1)(-25) + 2 \cdot 5 \right] = -\left[6 \cdot 25 + 10 \right] = -(150+10) = -160$$

$$\boxed{\text{SVILUPPO RISPETTO ALLA } j\text{-ESIMA COLONNA}}$$

$$A = \begin{pmatrix} a_{11} \dots & a_{1j} \dots & a_{1n} \\ a_{21} \dots & a_{2j} \dots & a_{2n} \\ a_{n1} \dots & a_{nj} \dots & a_{nn} \end{pmatrix}$$

$$\det A = [a_{1j} \; a_{2j} \dots a_{nj}] \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{bmatrix} = a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$$

1,3 2,3

$$\det\begin{pmatrix}1&-5&0&3\\6&2&0&4\\-2&0&0&2\\-1&7&5&1\end{pmatrix}=(0\;\;0\;\;0\;\;5)\begin{bmatrix}A_{13}\\A_{23}\\A_{33}\\A_{43}\end{bmatrix}=$$

3,3 4,3

$$=\cancel{0\cdot A_{13}}+\cancel{0\,A_{23}}+\cancel{0\,A_{33}}+5\,A_{43}=$$

1,2

$$=5\cdot(-1)^{4+3}\det\begin{pmatrix}1&-5&3\\6&2&4\\-2&0&2\end{pmatrix}=-5\left[-5\,B_{12}+2\,B_{22}+\cancel{0\,B_{32}}\right]$$

2,2 3,2

$$=-5\left[-5\,(-1)^{1+2}\det\begin{pmatrix}6&4\\-2&2\end{pmatrix}+2\,\cancel{(-1)^{2+2}}\det\begin{pmatrix}1&3\\-2&2\end{pmatrix}\right]=-5\left[5(12+8)+2(2+6)\right]=$$

$$=-5[100+16]=-5\cdot 116=-580$$

PROPRIETÀ DEL DETERMINANTE

A $m \times n$

$\boxed{1}$ $\mathrm{Det}\,\overline{A} = \overline{\mathrm{Det}\,A}$

$\mathrm{Det}(A^T) = \mathrm{Det}\,A$

$\mathrm{Det}(A^H) = \overline{\mathrm{Det}\,A}$

$\boxed{2}$ B $m \times n$ $\qquad \mathrm{Det}(AB) = \mathrm{Det}(A) \cdot \mathrm{Det}(B)$

$\boxed{3}$ A è non singolare $\iff \mathrm{Det}\,A \neq 0$

e in tal caso $\quad \mathrm{Det}(A^{-1}) = \dfrac{1}{\mathrm{Det}\,A}$