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# SIGNAL PROCESSING FUNDAMENTALS Through NCERT

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# Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.



# Chapter 1

## Harmonics

1.0.1 Suppose that the electric field amplitude of an electromagnetic wave is  $E_0 = 120\text{N/C}$  and that its frequency is  $f = 50.0\text{ MHz}$ .

- (a) Determine,  $B_0, \omega, k$  and  $\lambda$
- (b) Find expressions for  $\mathbf{E}$  and  $\mathbf{B}$

**Solution:**

Table 1.1: Input Parameters

Symbol	Description	value
$f$	frequency of source	50.0 MHz
$E_0$	Electric field amplitude	120 N/C
$c$	speed of light	$3 \times 10^8$ m/s
$\mathbf{e}_2, \mathbf{e}_3$	Standard Basis vectors	N/A

Table 1.2: Formulae and Output

Symbol	Description	Formula	Value
<b>E</b>	Electric field vector	$E_0 \sin(kx - 2\pi ft)\mathbf{e}_2$	$120 \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e}_2$
<b>B</b>	Magnetic field vector	$B_0 \sin(kx - 2\pi ft)\mathbf{e}_3$	$(4 \times 10^{-7}) \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e}_3$
$B_0$	Magnetic field strength	$\frac{E_0}{c}$	400nT
$\omega$	Angular frequency	$2\pi f$	$3.14 \times 10^8 \text{m/s}$
$k$	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
$\lambda$	Wavelength	$\frac{c}{f}$	6.0m

1.0.2 A charged particle oscillates about its mean equilibrium position with a frequency of  $10^9 \text{Hz}$ . What is the frequency of the electromagnetic waves produced by the oscillator?

**Solution:**

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
$f_c$	$10^9$	Frequency of electromagnetic wave
$t$	seconds	Time

Table 1.0.2: Variable description





Figure 1.0.1: Graphs of  $\mathbf{E}$  and  $\mathbf{B}$

1.0.3 Given below are some functions of  $x$  and  $t$  to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a)  $y = 2 \cos(3x) \sin(10t)$

(b)  $y = 2\sqrt{x - vt}$

(c)  $y = 3 \sin(5x - 0.5t) + 4 \cos(5x - 0.5t)$



Figure 1.0.2:  $y(t) = \cos(2\pi \times 10^9 t)$

(d)  $y = \cos x \sin t + \cos 2x \sin 2t$

**Solution:**

Let us assume an equation:

$$y = A(x) \cos(\omega t + \phi(x)) \quad (1.1)$$

Fig. 1.0.3 and Fig. 1.0.3 are self explanatory for stationary and travelling waves. Fig. 1.0.3 and Fig. 1.0.3 are neither stationary nor travelling waves.

TRAVELLING WAVE	STATIONARY WAVE
$y(x, t) = A \sin(kx \pm \omega t)$	$y(x, t) = A \sin kx \cos \omega t$
PARAMETERS	DEFINITION
$A$	Amplitude
$\omega$	Angular Velocity
$x$	Position
$k$	Wavenumber

Table 1.0.3: Travelling wave *vs* Stationary wave

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position $x$ , and it can be expressed as $A(x) = A_0 \cos(\omega t + \alpha)$ where $A_0$ is a constant, $k$ is the wavenumber, $x$ is the position and $\alpha$ is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where $A_0$ is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where $c$ is a constant.	(2) $\phi(x)$ represents a linear expression in $x$ , and it can be expressed as $\phi(x) = kx + \theta$ where $k$ is the wavenumber and $\theta$ is the phase constant.

Table 1.0.3: Travelling wave *vs* Stationary wave



Figure 1.0.3: DIPLACEMENT *vs* TIME-graph1

1.0.4 For the travelling harmonic wave  $y(x, t) = 2.0 \cos 2\pi (10t - 0.0080x + 0.35)$  where  $x$  and  $y$  are in  $cm$  and  $t$  in  $s$ . Calculate the phase difference between oscillatory motion of two points separated by a distance of

- (a)  $4m$
- (b)  $0.5m$
- (c)  $\lambda/2$
- (d)  $3\lambda/4$



Figure 1.0.3: DIPLACEMENT *vs* TIME-graph2

**Solution:**

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \quad (1.2)$$

$$= k(x_2 - x_1) \quad (1.3)$$

Parameter	Description	Value
$y(x_i, t)$	equation of harmonic wave	$A \cos(2\pi ft - kx_i + \phi)$
$k$	angular wave number	$2\pi$ (0.008)
$\lambda = \frac{2\pi}{k}$	wavelength	125 cm
$f$	frequency	10
$A$	amplitude	2.0
$\phi$	phase constant	$2\pi$ (0.35)
$\theta_i$	phase of $i^{th}$ harmonic wave	$(2\pi ft - kx + \phi)$
$x_i$	position of $i^{th}$ harmonic wave	
$t$	time	
$x_2 - x_1$	path difference	400 cm
		50 cm
		$\frac{\lambda}{2}$
		$\frac{3\lambda}{4}$

Table 1.0.4: Given parameters list

Parameter	Description	subquestion	Value
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	$6.4\pi$ radians
		(b)	$0.8\pi$ radians
		(c)	$\pi$ radians
		(d)	$\frac{3\pi}{2}$ radians

Table 1.0.4: Phase differences

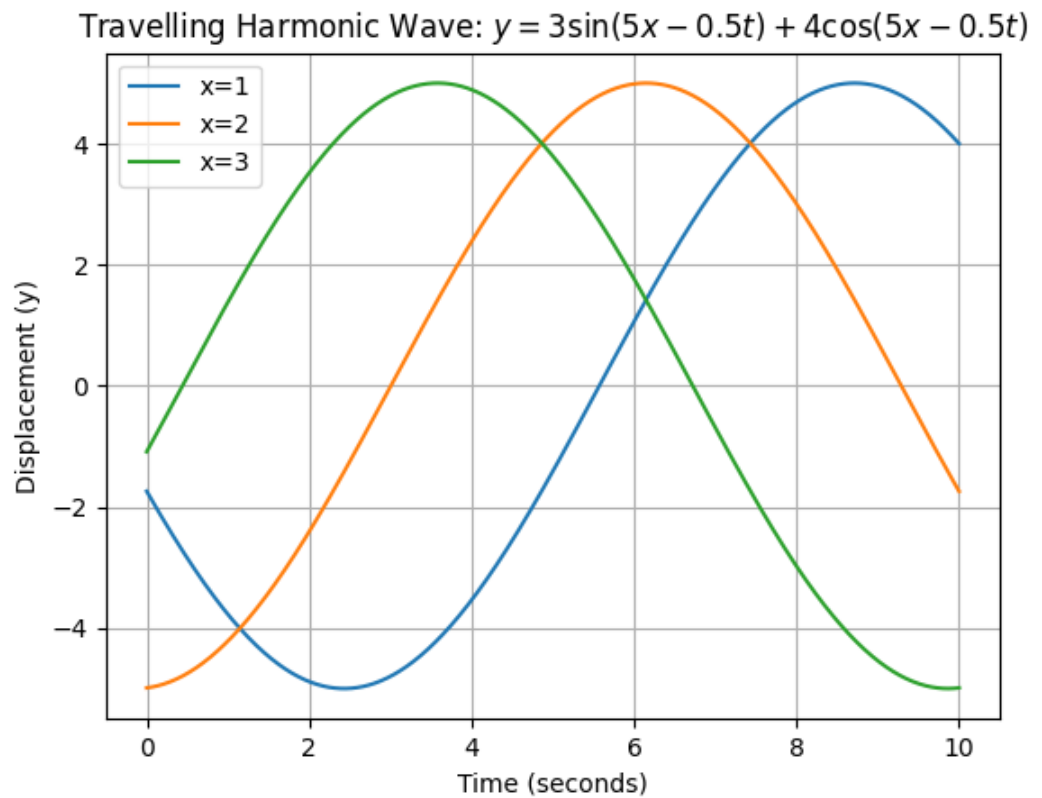


Figure 1.0.3: DIPLACEMENT *vs* TIME-graph3

1.0.5 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

**Solution:**



Figure 1.0.3: DIPLACEMENT *vs* TIME-graph4





Figure 1.0.4:

(a)

$$V_{\text{rms}}^2 = \frac{1}{T} \int_0^T [V(t)]^2 dt \quad (1.4)$$

$$= f \int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi ft + \phi) dt \quad (1.5)$$

$$= \frac{1}{2} V_0^2 \left( 1 - \frac{1}{f} \int_0^{\frac{1}{f}} \cos(4\pi ft + 2\phi) dt \right) \quad (1.6)$$

$$= \frac{1}{2} V_0^2 \left( 1 - \frac{1}{f} \left[ \frac{\sin(4\pi ft + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \quad (1.7)$$

$$= \frac{1}{2} V_0^2 \left( 1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \quad (1.8)$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \quad (1.9)$$



Figure 1.0.4:

To find the RMS voltage ( $V_{\text{rms}}$ ) when the peak voltage ( $V_0$ ) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \quad (1.10)$$



Figure 1.0.4:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T} \int_0^T [I(t)]^2 dt \quad (1.11)$$

$$= f \int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi ft + \phi) dt \quad (1.12)$$

$$= \frac{1}{2} I_0^2 \left( 1 - \frac{1}{f} \left[ \frac{\sin(4\pi ft + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \quad (1.13)$$

$$= \frac{1}{2} I_0^2 \left( 1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \quad (1.14)$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \quad (1.15)$$



Figure 1.0.4:

To find the peak current ( $I_0$ ) when the RMS current ( $I_{\text{rms}}$ ) is given, you can use equation (1.15)

$$I_0 \approx 10 \text{ A} \times 1.414 \approx 14.14 \text{ A} \quad (1.16)$$

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi ft + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi ft + \phi)$	current in terms of time
$V_0$	300 V	peak voltage
$V_{\text{rms}}$	$\sqrt{\frac{1}{T} \int_0^T [V(t)]^2 dt}$	rms value of Voltage
$I_{\text{rms}}$	10 A	rms value of current
$I_0$	$\sqrt{2} \times I_{\text{rms}}$	peak current
$f$	50 Hz	frequency of the sinusoidal wave
$T$	0.02 s	time period of sinusoidal wave

Table 1.0.5: Input Parameter Table



1.0.6 In Young's double-slit experiment using monochromatic light of wavelength  $\lambda$ , the intensity of light at a point on the screen where path difference is  $\lambda$ , is  $K$  units. What is the intensity of light at a point where path difference is  $\lambda/3$ ?

**Solution:**

Parameter	Description	Value
$y_i(t)$	Equation of light from $S_{i^{\text{th}}}$	$A \sin(\omega t - kx_i)$
$k$	Wave number	$\frac{2\pi}{\lambda}$
$I$	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	$\lambda$
		$\frac{\lambda}{3}$
$K$	Intensity of light at $\Delta x = \lambda$	
$A$	Amplitude of wave from source	
$r$	constant	$r \geq 0$

Table 1.0.6: Parameters

From Table 1.0.6:

$$y(t) = A \sin(2\pi ft - kx_1) + A \sin(2\pi ft - kx_2) \quad (1.17)$$

$$y(t) = 2A \cos\left(\frac{k\Delta x}{2}\right) \sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \quad (1.18)$$

From Table 1.0.6 and equation (1.18):

$$\therefore I \propto 4A^2 \cos^2\left(\frac{k\Delta x}{2}\right) \quad (1.19)$$

From Table 1.0.6 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2 \cos^2\left(\frac{2\pi}{2}\right)}{4A^2 \cos^2\left(\frac{\pi}{3}\right)} \implies I_r = \frac{K}{4} \quad (1.20)$$

Hence, the Intensity of light at a point where path difference is  $\frac{\lambda}{3}$  is  $\frac{K}{4}$  units.

Parameter	Description	Value
$I_r$	Net Intensity of light at $\Delta x = \frac{\lambda}{3}$	$\frac{K}{4}$

Table 1.0.6:

Assuming  $\Delta x = r\lambda$ ,

From equation (1.19):



Figure 1.0.6:



1.0.7 In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of  $2.0 \times 10^{10}$  Hz and amplitude  $48 \text{ Vm}^{-1}$ .

(a) What is the wavelength of the wave?

(b) What is the amplitude of the oscillating magnetic field?

(c) Show that the average energy density of the **E** field equals the average energy density of the **B** field. [ $c = 3 \times 10^8 \text{ ms}^{-1}$ ]

1.0.8 (a) For the wave on the string  $y(x, t) = 0.06 \sin(\frac{2\pi x}{3}) \cos(120\pi t)$ , do all the points on the string oscillate with the same (a)frequency, (b)phase, (c)amplitude? Explain your answers.

(b) What is the amplitude of a point 0.375m away from one end?

**Solution:**



## Chapter 2

### Filters

2.0.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with  $L = 3.0\text{ H}$ ,  $C = 27\text{ }\mu\text{F}$ , and  $R = 7.4\text{ }\Omega$ . It is desired to improve the sharpness of the resonance of the circuit by reducing its ‘full width at half maximum’ by a factor of 2. Suggest a suitable way.

**Solution:** Given parameters are:

Symbol	Value	Description
$L$	3.0 H	Inductance
$C$	27 $\mu\text{F}$	Capacitance
R	7.4 $\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
$\omega_0$	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 2.1: Given Parameters



Figure 2.1: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \quad (2.1)$$

Using reactances from Fig. 2.2,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \quad (2.2)$$

$$= I(s) \left( R + Ls + \frac{1}{sC} \right) \quad (2.3)$$

$$\Rightarrow I(s) = \frac{V(s)}{\left( R + Ls + \frac{1}{sC} \right)} \quad (2.4)$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

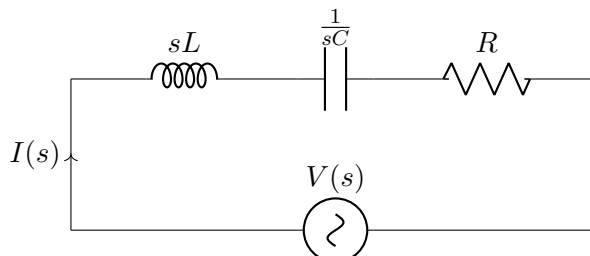


Figure 2.2: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \quad (2.5)$$

$$\implies s = j \frac{1}{\sqrt{LC}} \quad (2.6)$$

$s$  can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \quad (2.7)$$

Comparing (2.6) and (2.7), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (2.8)$$

## (b) Quality Factor

i. Using voltage across inductor,

$$Q = \left( \frac{V_L}{V_R} \right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \quad (2.9)$$

$$= \frac{1}{\sqrt{LC}} \frac{L}{R} \quad (2.10)$$

$$= \frac{1}{R} \sqrt{\frac{L}{C}} \quad (2.11)$$

ii. Using voltage across capacitor,

$$Q = \left( \frac{V_C}{V_R} \right)_{\omega_0} = \frac{\left| \frac{I(s)}{sC} \right|}{|RI(s)|} \quad (2.12)$$

$$= \frac{\sqrt{LC}}{RC} \quad (2.13)$$

$$= \frac{1}{R} \sqrt{\frac{L}{C}} \quad (2.14)$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \quad (2.15)$$

Using (2.4),

$$H(s) = R + sL + \frac{1}{sC} \quad (2.16)$$

$$\Rightarrow H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \quad (2.17)$$

$$\Rightarrow |H(j\omega)| = \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} \quad (2.18)$$

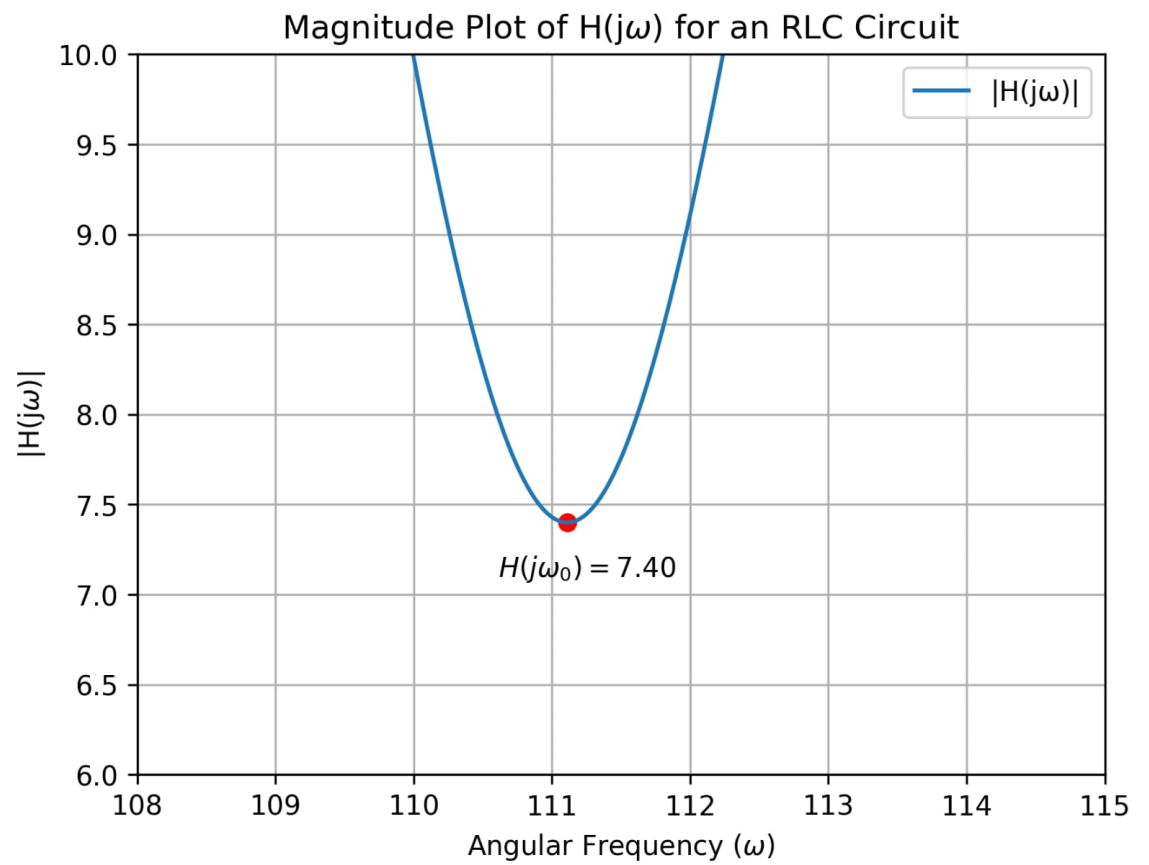


Figure 2.3: Impedance vs  $\omega$  (using values in Table 2.1)





# Chapter 3

## Z-transform

3.0.1 The 4<sup>th</sup> term of a G.P. is square of its second term, and the first term is -3. Determine its 7<sup>th</sup> term.

**Solution:**

Variable	Description	value
$x(0)$	first term of G.P.	-3
$r$	Common ratio of G.P.	?
$x(n)$	general term of the G.P.	$x(0) r^n$
$x(3)$	fourth term	$[x(1)]^2$
$u(n)$	unit step function	-

Table 3.1: A Table with input parameters

from Table 3.1

$$x(0) r^3 = \left(x(0) r^1\right)^2 \tag{3.1}$$

$$= x(0)^2 r^2 \tag{3.2}$$

$$\implies r = x(0) \tag{3.3}$$

$$= -3 \tag{3.4}$$

general term

$$x(n) = x(0) r^n u(n) \quad (3.5)$$

$$= (-3)^{n+1} u(n) \quad (3.6)$$

The 7<sup>th</sup> term of the sequence will be:

$$x(6) = (-3)(-3)^6 \quad (3.7)$$

$$= -2187 \quad (3.8)$$

Z transform of the given G.P is:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} = \frac{-3}{1 + 3z^{-1}} \quad |z| > 3 \quad (3.9)$$



Figure 3.1: Graph showing first 8 terms of the GP

3.0.2 Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

**Solution:**

Parameter	Description	Value
$n$	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$(n^3 + 5n^2 + 8n + 4) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$(n^3 + 4n^2 + 5n + 2) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} :  z  > 1\}$
ROC	Region of convergence	$\{z :  \sum_{n=-\infty}^{\infty} x(n)z^{-n}  < \infty\}$

Table 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \quad (3.10)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 5n^2 + 8n + 4) u(n) z^{-n} \quad (3.11)$$

Using results of equations (B.3.2) to (B.3.5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \quad (3.12)$$

From (A.3.2)

$$y_1(n) = x_1(n) * u(n) \quad (3.13)$$

$$Y_1(z) = X_1(z) U(z) \quad (3.14)$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \quad (3.15)$$

Using partial fractions:

$$\begin{aligned} Y_1(z) &= \frac{22z^{-1}}{(1 - z^{-1})} + \frac{48z^{-2}}{(1 - z^{-1})^2} + \frac{52z^{-3}}{(1 - z^{-1})^3}, \\ &+ \frac{28z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 4, |z| > 1 \end{aligned} \quad (3.16)$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (3.16):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \quad (3.17)$$

$$= \frac{(3n + 8)(n + 1)(n + 2)(n + 3)}{12} u(n) \quad (3.18)$$

## 2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \quad (3.19)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 4n^2 + 5n + 2) u(n) z^{-n} \quad (3.20)$$

Using results of equation (B.3.2) to (B.3.5) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \quad (3.21)$$

From (A.3.2)

$$y_2(n) = x_2(n) * u(n) \quad (3.22)$$

$$Y_2(z) = X_2(z) U(z) \quad (3.23)$$

$$= \frac{2 + 4z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \quad (3.24)$$

Using partial fractions:

$$\begin{aligned} Y_2(z) &= \frac{14z^{-1}}{(1 - z^{-1})} + \frac{36z^{-2}}{(1 - z^{-1})^2} + \frac{44z^{-3}}{(1 - z^{-1})^3} \\ &+ \frac{26z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 2, |z| > 1 \end{aligned} \quad (3.25)$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (3.25):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \quad (3.26)$$

$$= \frac{(3n + 4)(n + 1)(n + 2)(n + 3)}{12} u(n) \quad (3.27)$$

As the sequence start from  $n = 0$ , in RHS of question  $n$  should be replaced by  $n + 1$ :

$$\frac{y_1(n)}{y_2(n)} = \frac{3n + 8}{3n + 4} \quad (3.28)$$

Hence Proved.



Figure 3.2: Stem Plot of  $x_1(n)$



Figure 3.3: Stem Plot of  $x_2(n)$

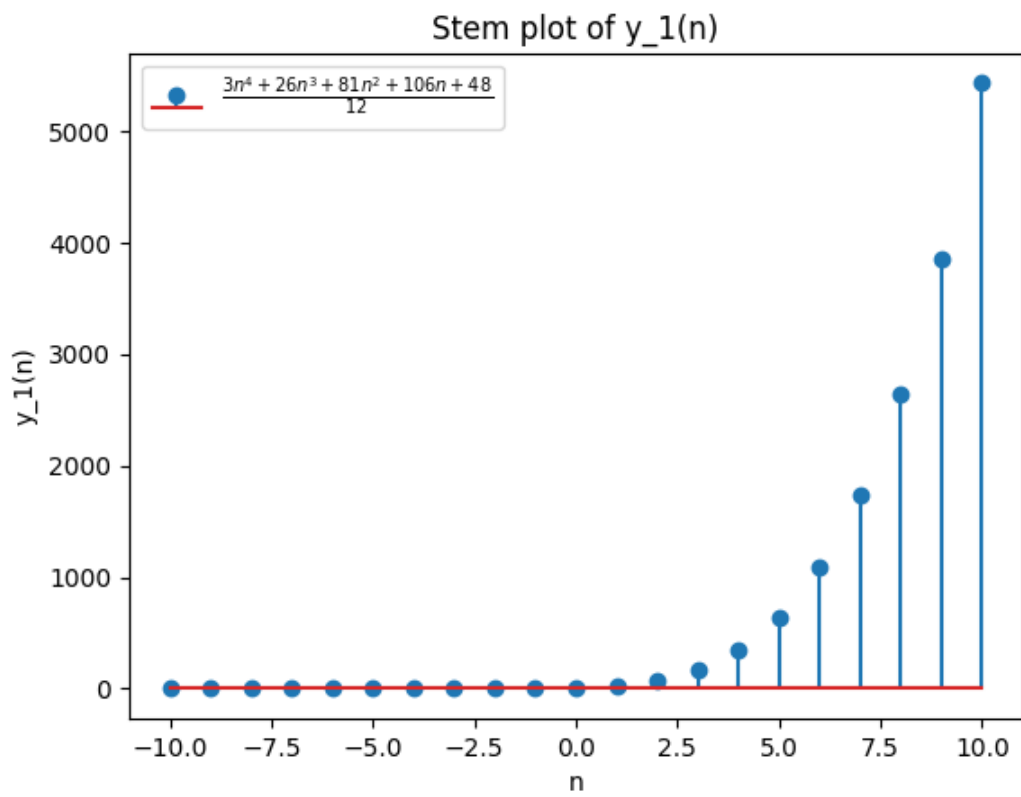


Figure 3.4: Stem Plot of  $y_1(n)$





Figure 3.5: Stem Plot of  $y_2(n)$

3.0.3 Write the five terms at  $n = 1, 2, 3, 4, 5$  of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \quad n = 0 \quad (3.29)$$

$$= \frac{x(n-1)}{n}, \quad n > 0 \quad (3.30)$$

$$= 0, \quad n < 0 \quad (3.31)$$

**Solution:**

$$x(1) = \frac{x(0)}{1} = -1 \quad (3.32)$$

$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \quad (3.33)$$

$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \quad (3.34)$$

$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \quad (3.35)$$

$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \quad (3.36)$$

$$x(n) = \frac{-1}{n!} (u(n)) \quad (3.37)$$

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \quad (3.38)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (3.39)$$

using (3.37),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \quad (3.40)$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \quad (3.41)$$

$$= -e^{z^{-1}} \quad \{z \in \mathbb{C} : z \neq 0\} \quad (3.42)$$

Symbol	Value	Description
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 3.3: Parameters



Figure 3.6: Plot of  $x(n)$  vs  $n$

3.0.4 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

**Solution:**

Parameter	Value	Description
$x(0)$	5000	Initial Income
$d$	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	$n^{th}$ term of the AP

Table 3.4: Input Parameters

From the values given in Table 3.4:

$$7000 = 5000 + 200n \quad (3.43)$$

$$\Rightarrow 2000 = 200n \quad (3.44)$$

$$\therefore n = 10 \quad (3.45)$$

Let Z-transform of  $x(n)$  be  $X(z)$ .

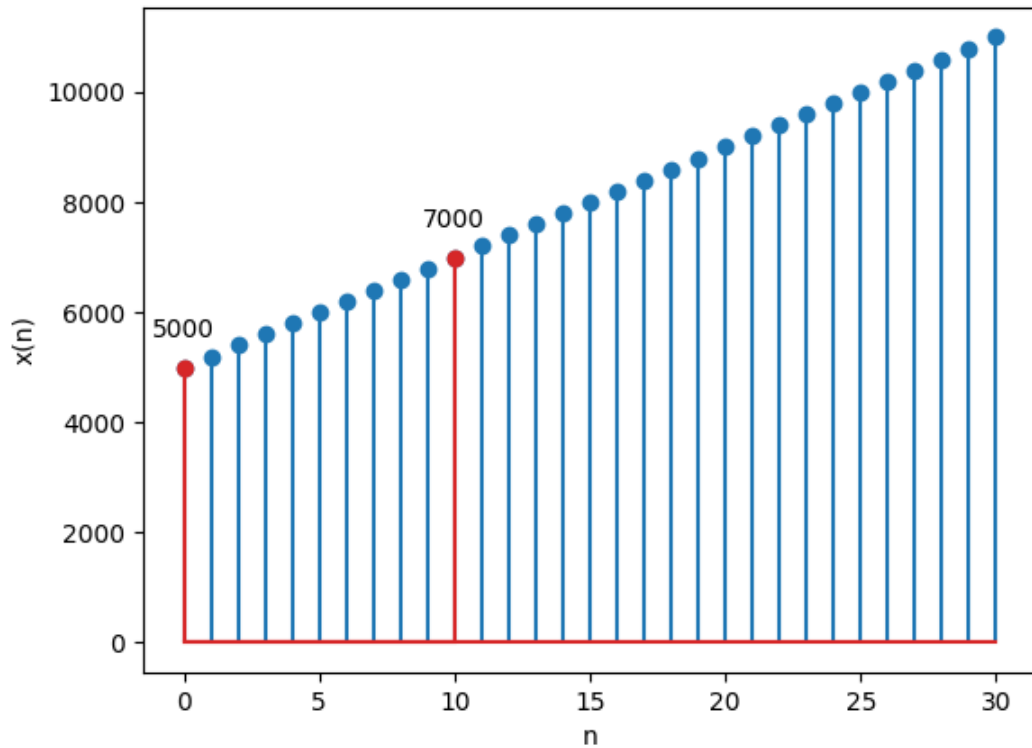


Figure 3.7: Plot of  $x(n)$  vs  $n$ . See Table 3.4 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (3.46)$$

Using the values from Table 3.4:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (3.47)$$

3.0.5 Consider the sequence whose  $n^{\text{th}}$  term is given by  $2^n$ . Find the first 6 terms of this sequence.

**Solution:**

Variable	Description	Value
$x(n)$	general term of sequence	$2^n u(n)$

Table 3.5: input parameters

$$X(Z) = \frac{1}{1 - 2z^{-1}} \quad |z| > |2| \quad (3.48)$$



Figure 3.8: Six terms of given sequence

3.0.6 If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first  $n$  terms.

**Solution:**

Variable	Description
$x(0)$	First term of the AP
$d$	Common difference of the AP
$y(n)$	Sum of $n + 1$ terms of the AP
$x(n)$	General term

Table 3.6: Variables Used

$$y(n) = \frac{n+1}{2} (2x(0) + nd) \quad u(n) \quad (3.49)$$

$$y(6) = 49 \quad (3.50)$$

$$y(16) = 289 \quad (3.51)$$

Then,

$$x(0) + 3d = 7 \quad (3.52)$$

$$x(0) + 8d = 17 \quad (3.53)$$

From equations 3.52 and 3.53, the augmented matrix is:

$$\begin{pmatrix} 1 & 3 & 7 \\ 1 & 8 & 17 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \quad (3.54)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 10 \end{pmatrix} \quad (3.55)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad (3.56)$$

$$\Rightarrow \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.57)$$

$$x(n) = (1 + 2n) u(n) \quad (3.58)$$

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \quad \{z \in \mathbb{C} : |z| > 1\} \quad (3.59)$$

$$y(n) = x(n) * u(n) \quad (3.60)$$

$$Y(z) = X(z) U(z) \quad (3.61)$$

$$\Rightarrow Y(z) = \left( \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \right) \left( \frac{1}{1 - z^{-1}} \right) \quad (3.62)$$

$$= \frac{1}{(1 - z^{-1})^2} + \frac{2z^{-1}}{(1 - z^{-1})^3} \quad (3.63)$$

$$(n + 1) u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})^2} \{z \in \mathbb{C} : |z| > 1\} \quad (3.64)$$

$$n((n + 1) u(n)) \xleftrightarrow{\mathcal{Z}} \frac{2z^{-1}}{(1 - z^{-1})^3} \{z \in \mathbb{C} : |z| > 1\} \quad (3.65)$$



From equations (B.11.1) and (B.11.2), taking the inverse Z Transform,

$$y(n) = (n+1)u(n) + n((n+1)u(n)) \quad (3.66)$$

$$\Rightarrow y(n) = (n+1)^2 u(n) \quad (3.67)$$

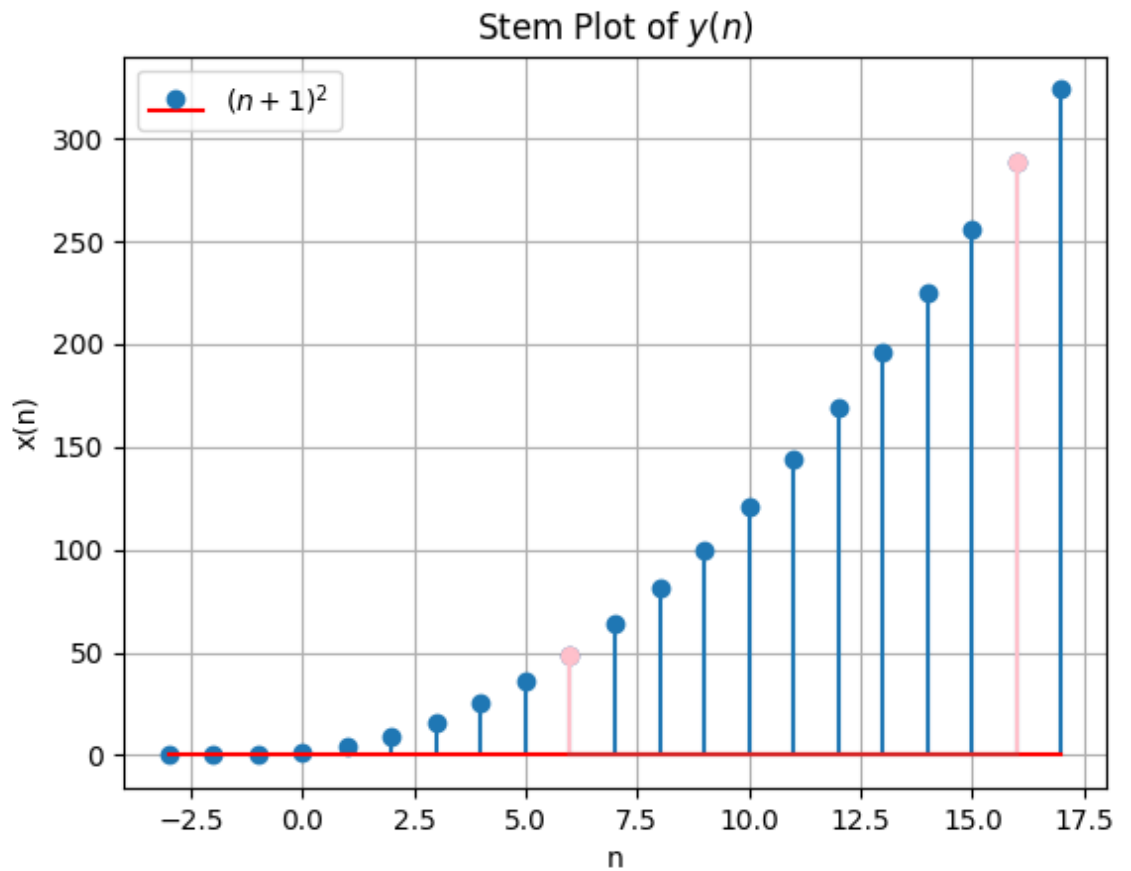


Figure 3.9: Stem Plot of  $y(n)$

3.0.7 Write the first five terms of the sequence and obtain the corresponding series:

$$a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$$

**Solution:**

Parameter	Description	Value
$x(0)$	First term	2
$x(1)$	Second term	2
ROC	Region of convergence	$\{z :  \sum_{n=-\infty}^{\infty} x(n)z^{-n}  < \infty\}$
$x(n)$	General term	$x(n) = \begin{cases} ? & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$

Table 1: Parameter Table

$$x(n) - x(n-1) = 2u(n) - 2u(n-1) - u(n-2) \quad (3.68)$$

$$X(z) - z^{-1}X(z) = \frac{2}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})} - \frac{2z^{-1}}{(1-z^{-1})} \quad (3.69)$$

$$X(z) = \frac{2 - 2z^{-1} - z^{-2}}{(1-z^{-1})^2}, |z| > 1 \quad (3.70)$$

Using partial fractions

$$X(z) = \frac{2z^{-1}}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})^2} + 2 \quad (3.71)$$

Taking inverse Z-transform by result of equation (B.4.6) in equation (3.71):

$$x(n) = 2u(n) + (1-n)u(n-1) \quad (3.72)$$

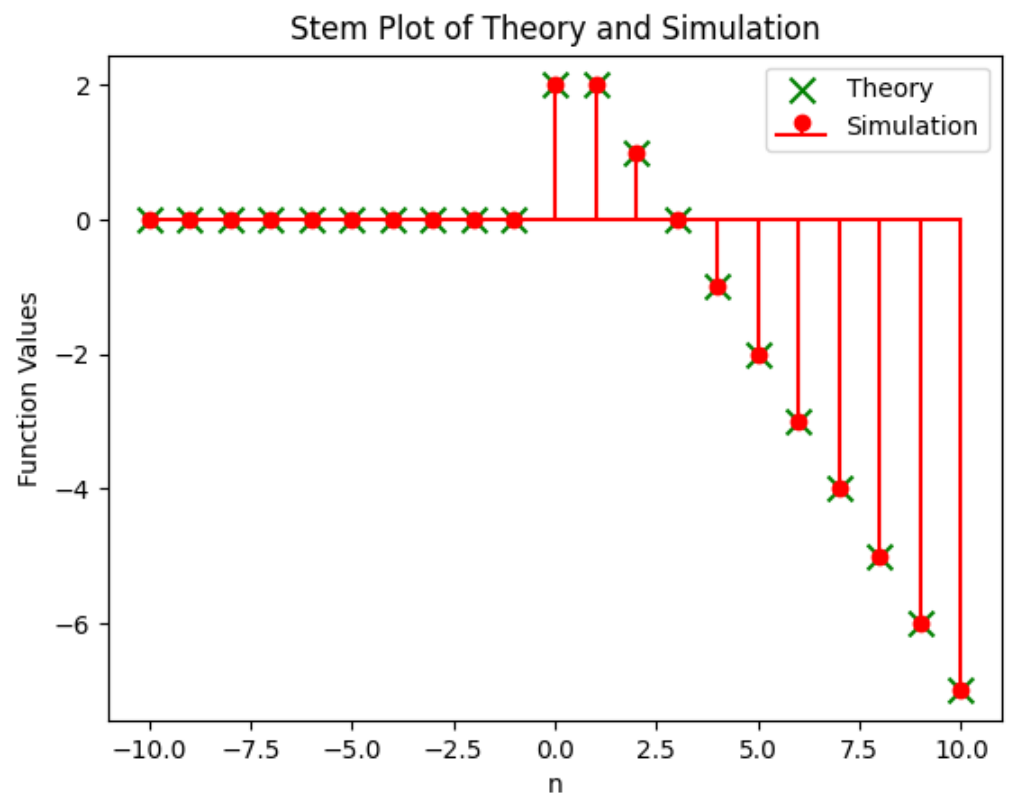


Figure 3.10: Comparison of Theory and Simulated Values

From the figureFig. 3.10 we can see that the theoretical and simulated values overlap.

3.0.8 Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

**Solution:**

Parameter	Description	Value
$x(0)$	First term of G.P.	3
$x(3)$	Fourth term of G.P.	81
$r$	common ratio of G.P.	r

Table 3.8: input values

(a)

$$x(n) = x(0) r^n \quad (3.73)$$

from the values in Table 3.8

$$\frac{x(0) r^3}{x(0)} = 27 \quad (3.74)$$

$$r = 3 \quad (3.75)$$

$\therefore$  Required numbers are 9 and 27.

(b)

$$x(n) = 3^{n+1} u(n) \quad (3.76)$$

$$X(z) = \frac{3}{1 - 3z^{-1}} \quad |z| > 3 \quad (3.77)$$

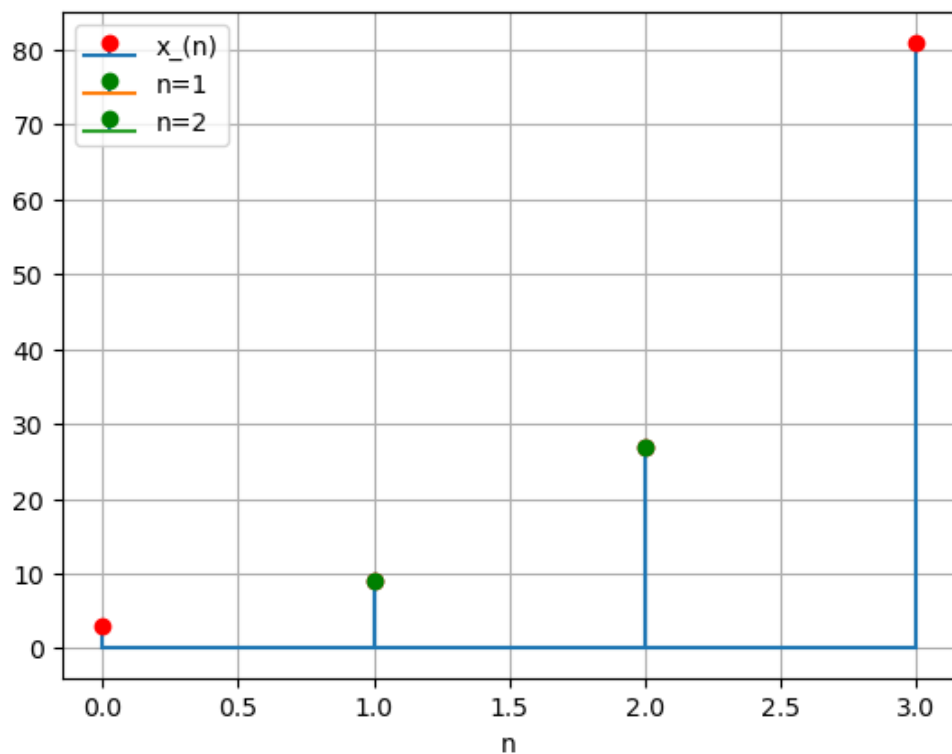


Figure 3.11: Graph of  $x(n)$

3.0.9 What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

**Solution:**

The Z-transform of a sequence  $x(n)$  is given by:

$$x(n) = 500(1.1)^n u(n) \quad (3.78)$$

$$X(Z) = \frac{500}{1 - (1.1)z^{-1}}; |z| > 1.1 \quad (3.79)$$

Parameter	Value	Description
$x(0)$	500	Principal amount before first year
$r$	1.1	Common ratio of GP
$n$	10	Number of years
$x(10)$	$500(1.1)^{10} = 1296.87$	Amount after 10 years

Table 3.9: Parameter Table

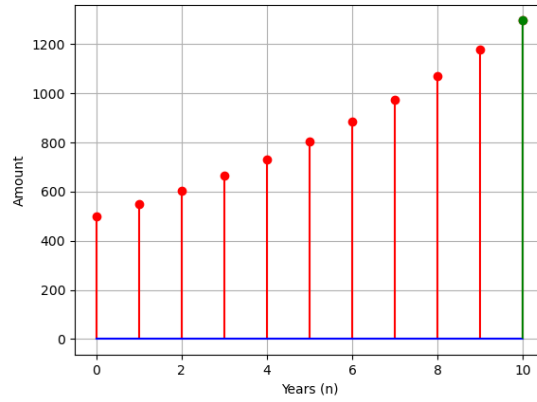


Figure 3.12: Plot of  $x(n) = 500(1.1)^n$

3.0.10 Find the 20<sup>th</sup> term from the last term of the AP: 3, 8, 13.....253.

**Solution:** As the 20<sup>th</sup> term is considered from last,

Parameter	Description	Value
$x(0)$	first term	253
$d$	common difference	$3 - 8 = -5$
$x(n)$	$(n + 1)^{th}$ term	$(x(0) + nd)u(n)$
$u(n)$	unit step function	
$x(n)$	20 <sup>th</sup> term	158

Table 3.10: Input table

From equation (B.1.1) and (B.3.2):

$$X(z) = \frac{253}{1 - z^{-1}} + \frac{-5z^{-1}}{(1 - z^{-1})^2}; \{|z| > 1\} \quad (3.80)$$



Figure 3.13:

3.0.11 Find the sum to  $n$  terms of series , whose  $n^{th}$  term is :  $n(n+1)(n+4)$ .

**Solution:**

Parameter	Description	Value
$x(n)$	$n^{th}$ term of series	$n(n+1)(n+4)u(n)$
$y(n)$	sum of $n$ terms of series	

Table 3.11: Given parameters

$$nu(n) \xleftrightarrow{z} \frac{z^{-1}}{(1-z^{-1})^2} \{|z| > 1\} \quad (3.81)$$

$$n^2u(n) \xleftrightarrow{z} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \{|z| > 1\} \quad (3.82)$$

$$n^3u(n) \xleftrightarrow{z} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \{|z| > 1\} \quad (3.83)$$

$$n^4u(n) \xleftrightarrow{z} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \{|z| > 1\} \quad (3.84)$$

From equation (B.3.2) to (B.3.4),

$$X(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} + \frac{5z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} + \frac{4z^{-1}}{(1-z^{-1})^2} \{|z| > 1\} \quad (3.85)$$

$$Y(z) = X(z)U(z) \quad (3.86)$$

$$= \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^5} + \frac{5z^{-1}(z^{-1}+1)}{(1-z^{-1})^4} + \frac{4z^{-1}}{(1-z^{-1})^3} \quad (3.87)$$



$$\begin{aligned}
&= \frac{1}{4} \left[ \frac{z^{-1} (1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5} \right] \\
&\quad + \frac{13}{6} \left[ \frac{z^{-1} (1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \right] + \frac{19}{4} \left[ \frac{z^{-1} (1 + z^{-1})}{(1 - z^{-1})^3} \right] \\
&\quad + \frac{17}{6} \left[ \frac{z^{-1}}{(1 - z^{-1})^2} \right] \{ |z| > 1 \} \quad (3.88)
\end{aligned}$$

Taking reverse z transform, using equations (3.81) to (3.84)

$$y(n) = \left( \frac{n^4}{4} + \frac{13n^3}{6} + \frac{19n^2}{4} + \frac{17n}{6} \right) u(n) \quad (3.89)$$

$$= \left( \frac{n^4}{4} + \frac{2n^3}{4} + \frac{10n^3}{6} + \frac{n^2}{4} + \frac{15n^2}{6} + \frac{4n^2}{2} + \frac{5n}{6} + \frac{4n}{2} \right) u(n) \quad (3.90)$$

$$\begin{aligned}
&= \left( \frac{n^4 + 2n^3 + n^4}{4} \right) u(n) + \left( \frac{10n^3 + 15n^2 + 5n}{6} \right) u(n) \\
&\quad + \left( \frac{4n^2 + 4n}{2} \right) u(n) \quad (3.91)
\end{aligned}$$

$$= \left( \frac{n^2 (n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \right) u(n) \quad (3.92)$$



Figure 3.14: Sum of n terms of series

3.0.12 Find the indicated terms in the sequence whose  $n$ th terms is  $a(n) = 4n - 3$ . Find  $a(17)$  and  $a(24)$ .

**Solution:** In the question, following information is provided:

$$x(n) = (4n + 1)(u(n)) \quad (3.93)$$

Symbol	Value	Description
$x(n)$	$(4n + 1)u(n)$	The nth term of the sequence
$x(16)$	?	17th term
$x(23)$	?	24th term

Table 3.12: Parameters

$$x(16) = 4 \times 16 + 1 = 65 \quad (3.94)$$

$$x(23) = 4 \times 23 + 1 = 93 \quad (3.95)$$

Using Z-Transform,

$$X(z) = 4 \frac{z^{-1}}{(1 - z^{-1})^2} + \frac{1}{1 - z^{-1}} \quad |z| > 1 \quad (3.96)$$



Figure 3.15:  $x(n)$  vs  $n$

3.0.13 The difference between any two cosecutive interior angles of a polygon is  $5^\circ$ . If the smallest angle is  $120^\circ$ , find the number of sides of polygon.

**Solution:**

Variable	Description	Value
$x(0)$	first term of AP	120
d	common difference of AP	5
$x(n)$	general term of AP	none

Table 3.13: input parameters

Sum of interior angles of a polygon with  $n + 1$  sides is given by

$$S = (n - 1)180 \quad (3.97)$$

Sum of  $n$  terms of AP is given by

$$y(n) = x(n) * u(n) \quad (3.98)$$

where  $x(n) = 120 + 5n$

$$x(n) * u(n) = (n - 1)180 \quad (3.99)$$

$$Y(z) = X(z)U(z) \quad (3.100)$$

$$= \left( \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \right) \frac{1}{1 - z^{-1}} \quad |z| > 1 \quad (3.101)$$

$$= \frac{120}{(1 - z^{-1})^2} + \frac{5z^{-1}}{(1 - z^{-1})^3} \quad |z| > 1 \quad (3.102)$$

$$(n + 1) u(n) \xleftrightarrow{Z} \left( \frac{1}{(1 - z^{-1})^2} \right) \quad |z| > 1 \quad (3.103)$$

$$\frac{(n)(n - 1)}{2} u(n - 1) \xleftrightarrow{Z} \left( \frac{z^{-1}}{(1 - z^{-1})^3} \right) \quad |z| > 1 \quad (3.104)$$

applying inverse Z-transform for each term and solving we get,

$$y(n) = \frac{n + 1}{2} (240 + 5n) u(n) \quad (3.105)$$

now from (3.99)

$$y(n) = (n - 1)180 \quad (3.106)$$

$$\frac{n+1}{2} (240 + 5n) u(n) = (n - 1)180 \quad (3.107)$$

now replace  $n$  by  $n - 1$ :

$$n(235 + 5n) = (n - 2)360 \quad (3.108)$$

$$5n^2 - 125n + 720 = 0 \quad (3.109)$$

$$n = 16, 9 \quad (3.110)$$



Figure 3.16: Plot of the sum of n terms taken from Python

3.0.14 The 5th,8th and 11th terms of a GP are p,q and s respectively .show that  $q^2 = ps$

**Solution:**

Symbol	Value	Description
$x(5)$	$p$	$x(0)r^5$
$x(8)$	$q$	$x(0)r^8$
$x(11)$	$s$	$x(0)r^{11}$
$x(n)$		$x(0)r^nu(n)$
$r$	$\left(\frac{s}{p}\right)^{\frac{1}{6}}$	common ratio

Table 3.14: input parameters

From Table 3.14:

$$q^2 = x(0) r^8 x(0) r^8 \quad (3.111)$$

$$= x(0)^2 r^{16} \quad (3.112)$$

$$ps = x(0) r^5 x(0) r^{11} \quad (3.113)$$

$$= x(0)^2 r^{16} \quad (3.114)$$

$$\Rightarrow q^2 = ps \quad (3.115)$$

now we will find r and x(0):

$$\frac{s}{p} = \frac{x(0) r^{11}}{x(0) r^5} \quad (3.116)$$

$$r = \left(\frac{s}{p}\right)^{\frac{1}{6}} \quad (3.117)$$

$$p = x(0) \left(\frac{s}{p}\right)^{\frac{5}{6}} \quad (3.118)$$

$$x(0) = \frac{p^{\frac{11}{6}}}{s^{\frac{5}{6}}} \quad (3.119)$$

Applying z-Transform:

$$X(z) = \frac{x(0)}{1 - r z^{-1}}, |z| > |r| \quad (3.120)$$

$$\Rightarrow X(z) = \frac{p^3}{p^{\frac{7}{6}} s^{\frac{5}{6}} - q^2 z^{-1}}, |z| > \left| \left(\frac{q}{p}\right)^{\frac{1}{3}} \right| \quad (3.121)$$





Figure 3.17: plot  $x(n)$  vs  $n$   $p = 486$ ,  $q = 13122$ ,  $s = 354294$ ,  $r = 3$

3.0.15 The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112.

If its first term is 11, then find the number of terms.

**Solution:**

$$y(n) = \left[ \frac{(n+1)}{2} (2x(0) + nd) \right] u(n) \quad (3.122)$$

$$\Rightarrow y(3) = \frac{4}{2} (2x(0) + 3d) \quad (3.123)$$

$$(3.124)$$

Symbol	Value	Description
$x(0)$	11	First term of AP
$y(3)$	56	Sum of the first four terms of AP
$y(n) - y(n-4)$	112	Sum of the last four terms of AP

Table 3.15: Input Parameters

From Table 3.15:

$$\frac{4}{2} (2x(0) + 3d) = 56 \quad (3.125)$$

$$2x(0) + 3d = 28 \quad (3.126)$$

$$\implies d = 2 \quad (3.127)$$

$$y(n) - y(n-4) = \frac{4}{2} (2x(n) + 3(-d)) \quad (3.128)$$

From Table 3.15:

$$\frac{4}{2} (2x(n) + 3(-d)) = 112 \quad (3.129)$$

$$2x(n) - 3d = 56 \quad (3.130)$$

$$\implies x(n) = 31 \quad (3.131)$$

$$x(0) + (n)2 = 31 \quad (3.132)$$

$$\implies n = 10 \quad (3.133)$$

$$x(n) = (x(0) + 2n)u(n) \quad (3.134)$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + 2 \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (3.135)$$



Figure 3.18: Plot  $y(n)$  vs  $n$

3.0.16 How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636? **Solution:**

We know the formula

$$S_n = \frac{(n+1)}{2} [2x(0) + d(n)] \quad (3.136)$$

Putting in values from the table

Parameter	Description	Value
$x(0)$	First Term	9
$d$	Common Difference	8
$S_n$	Sum of n terms	636

Table 3.16: Parameter Table

$$636 = \frac{(n+1)}{2} [18 + 8n] \quad (3.137)$$

$$636 = (n+1) [4n + 9] \quad (3.138)$$

$$4n^2 + 13n - 627 = 0 \quad (3.139)$$

On solving this quadratic equation, we get roots

$$n = -12.5 \text{ and } n = 11$$

Since we are looking for positive terms of n, we remove the negative root

$$\implies n = 11 \quad (3.140)$$

$\therefore$  the total number of terms are 12

The Z-Transform of the above question is

$$X(z) = \frac{9}{1 - z^{-1}} + \frac{8z^{-1}}{(1 - z^{-1})^2} \quad (3.141)$$

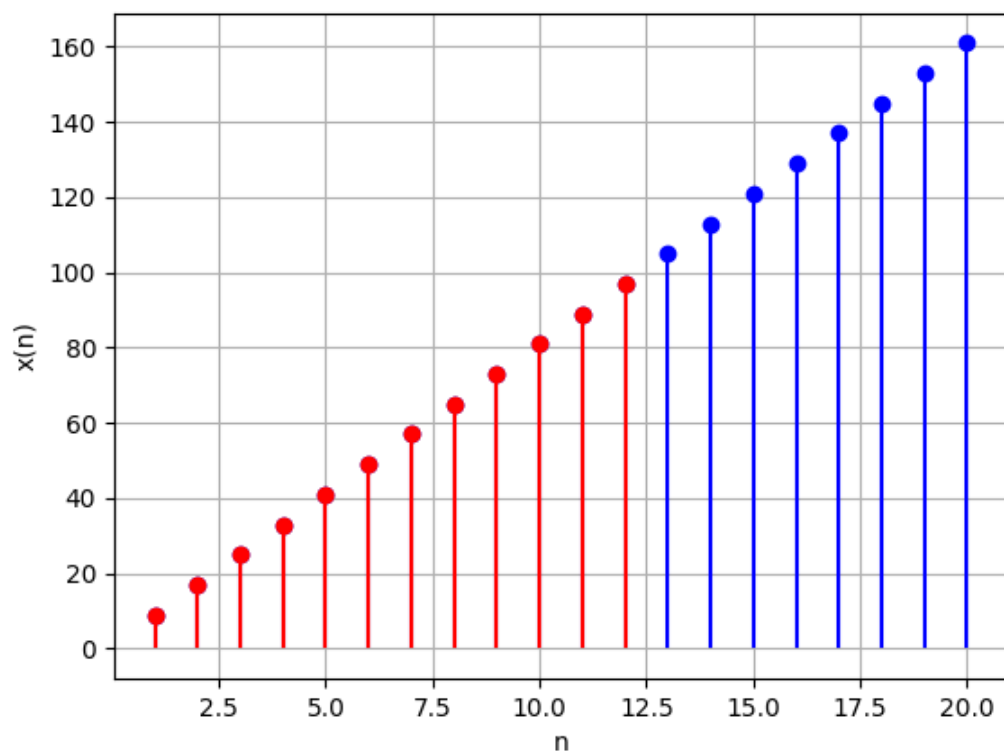


Figure 3.19: Plot of  $x(n)$  vs  $n$

3.0.17 Find the sum to  $n$  terms of the series whose  $n^{th}$  term is given by  $(2n - 1)^2$  ? **Solution:**

3.0.18  $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$  **Solution:**

# Chapter 4

## Sequences

4.0.1 Find the number of terms in each of the following APs.

- (a) 7, 13, 19, ... 205
- (b) 18,  $15\frac{1}{2}$ , 13, ... -47

**Solution:** The number of terms in the AP  $x(n)$  is given by:

Parameter	Used to denote	Values
$x_i(n)$	$n^{th}$ term of $i^{th}$ series ( $i = (1, 2)$ )	$(x_i(0) + nd_i) u(n)$
$x_i(0)$	First term of $i^{th}$ AP	$x_1(0) = 7$ $x_2(0) = 18$
$d_i$	Common difference of $i^{th}$ AP	$d_1 = 6$ $d_2 = -2.5$

Table 4.1: Parameter Table

$$\frac{x(n) - x(0)}{d} + 1 \tag{4.1}$$

$$X_i(z) = \frac{x_i(0)}{1 - z^{-1}} + d_i \frac{z^{-1}}{(1 - z^{-1})^2}, \text{ for } i=1,2 \quad (4.2)$$

$$\text{ROC} : |z| > 1 \text{ as it is an AP} \quad (4.3)$$

(a)

$$x_1(n) = (7 + (n)6)u(n) \quad (4.4)$$

Using the values in Table 4.1 and equation (4.1),

$$k_1 = \frac{205 - 7}{6} + 1 = 34 \quad (4.5)$$

Using the values in Table 4.1 and equation (4.2) :

$$X_1(z) = \frac{7 - z^{-1}}{(1 - z^{-1})^2} \quad (4.6)$$

ROC is  $|z| > 1$

(b)

$$x_2(n) = (18 + n(-2.5))u(n) \quad (4.7)$$

Using the values in Table 4.1 and equation (4.1),

$$k_2 = \frac{-47 - 18}{-2.5} + 1 = 27 \quad (4.8)$$



Using the values in Table 4.1 and equation (4.2) :

$$X_2(z) = \frac{18 - (20.5)z^{-1}}{(1 - z^{-1})^2} \quad (4.9)$$

ROC is  $|z| > 1$ .



Figure 4.1: Plot of  $x_1(n)$

4.0.2 For what value of  $n$ , are the  $n$ th terms of two A.Ps: 63, 65, 67, ... and 3, 10, 17, ... equal?

**Solution:**



Figure 4.2: Plot of  $x_2(n)$

$$x_i(n) = x(0)u(n) + dnu(n) \quad (4.10)$$

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.11)$$

(a)

$$x_1(n) = 63u(n) + 2nu(n) \quad (4.12)$$

$$X_1(z) = \frac{63}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.13)$$

Parameter	Sub-question	Description	Value
$x_i(0)$	$x_1(0)$	1 <sup>st</sup> term of 1 <sup>st</sup> A.P.	63
	$x_2(0)$	1 <sup>st</sup> term of 2 <sup>nd</sup> A.P.	3
$d_i$	$d_1$	Common difference of 1 <sup>st</sup> A.P.	2
	$d_2$	Common difference of 2 <sup>nd</sup> A.P.	7

Table 4.2: input values

(b)

$$x_2(n) = 3u(n) + 7nu(n) \quad (4.14)$$

$$X_2(z) = \frac{3}{1-z^{-1}} + \frac{7z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \quad (4.15)$$

(c) given,

$$x_1(n) = x_2(n) \quad (4.16)$$

$$\therefore 63 + 2n = 7n + 3 \quad (4.17)$$

$$\implies n = 12 \quad (4.18)$$

4.0.3 Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?



Figure 4.3: Graphs of  $x_1(n)$  and  $x_2(n)$  and both are equal at  $n = 12$

**Solution:**

$$x(n) = \{x(0) + nd\}u(n) \quad (4.19)$$

$$x(99) - y(99) = 100 \quad (4.20)$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \quad (4.21)$$

$$\implies x(0) - y(0) = 100 \quad (4.22)$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \quad (4.23)$$

$$= x(0) - y(0) \quad (4.24)$$

$$= 100 \quad (4.25)$$

$$\implies x(999) - y(999) = 100 \quad (4.26)$$

Variable	Description	Value
$x(n)$	$n^{th}$ term of X	none
$y(n)$	$n^{th}$ term of Y	none
$d$	common difference between the terms of AP	none
$x(99) - y(99)$	difference of $99^{th}$ terms of X and Y	100

Table 4.3: input parameters

Let

$$x(n) = \{101, 106, 111, \dots\} \quad (4.27)$$

$$y(n) = \{1, 6, 11, \dots\} \quad (4.28)$$

4.0.4 Check whether -150 is a term of the AP: 11,8,5,2,....

**Solution:**

$$x(n) = x(0) + nd \quad (4.29)$$

$$n = \frac{x(n) - x(0)}{d} \quad (4.30)$$



Figure 4.4:

$$x(n) - x(0) \equiv 0 \pmod{d} \quad (4.31)$$

On substituting values

$$-161 \equiv 2 \pmod{-3} \quad (4.32)$$

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \quad (4.33)$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.34)$$

Variable	Description	Value
$x(0)$	First term of AP	11
$d$	Common difference	-3
$x(n)$	General term of given AP	None

Table 4.4: Input parameters



Figure 4.5: Representation of  $x(n)$

4.0.5 Write the first five terms of the sequence  $a_n = \frac{n(n^2+5)}{4}$ .

**Solution:**

$$x(n) = \left( \frac{n^3 + 3n^2 + 8n + 6}{4} \right) u(n) \quad (4.35)$$

$$n^k u(n) \xleftrightarrow{\mathcal{Z}} (-1)^k z^k \frac{d^k}{dz^k} U(z) \quad (4.36)$$



$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.37)$$

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + z^{-1})}{(1 - z^{-1})^3} \quad |z| > 1 \quad (4.38)$$

$$n^3u(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \quad |z| > 1 \quad (4.39)$$

Referencing the equations from (4.37), (4.38), and (4.39).

$$x(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{4(1 - z^{-1})^4} + \frac{3(z^{-1})(1 + z^{-1})}{4(1 - z^{-1})^3} + \frac{2z^{-1}}{(1 - z^{-1})^2} + \frac{3}{2(1 - z^{-1})} \quad |z| > 1 \quad (4.40)$$

$$x(n) \xleftrightarrow{\mathcal{Z}} \frac{3}{2(1 - z^{-1})^3} + \frac{3z^{-2}}{2(1 - z^{-1})^4} \quad |z| > 1 \quad (4.41)$$

4.0.6 (a) 30th term of the AP: 10, 7, 4, ... is

(b) 11th term of the AP:  $-3, -\frac{1}{2}, 2, \dots$  is

**Solution:**

Parameter	value	Description
$x_i(0)$	10	First term
	-3	
$d_i$	-3	Common difference
	$\frac{5}{2}$	
$x_1(29)$	?	30th term
$x_2(10)$	?	11th term

Table 4.5: Input Parameters

$$x_i(n) = [x_i(0) + nd_i] u(n) \quad (4.42)$$



Figure 4.6: Plot of equation(4.35)

(a) From (4.42) Table 4.5 :

$$x_1(n) = [10 - 3n] u(n) \quad (4.43)$$

$$x_1(29) = -77 \quad (4.44)$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.45)$$

(b) From (4.42) and Table 4.5 :

$$x_2(n) = \left[ -3 + \frac{5}{2}n \right] u(n) \quad (4.46)$$

$$x_2(10) = 22 \quad (4.47)$$

$$X_2(z) = \frac{5.5z^{-1} - 3}{(1 - z^{-1})^2} \quad |z| > 1 \quad (4.48)$$



Figure 4.7: stem plots of  $x_1(n)$  and  $x_2(n)$

4.0.7 Write the first five terms of the sequence whose  $n$ th term is  $\frac{2n-3}{6}$  and obtain the Z transform of the series **Solution:**

$$x(n) = \frac{2n-1}{6} (u(n)) \quad (4.49)$$



Figure 4.8: Plot of  $x(n)$  vs  $n$

$$X(z) = \frac{3z^{-1} - 1}{6(1 - z^{-1})^2} \quad |z| > 1 \quad (4.50)$$

4.0.8 For what values of  $x$ , the numbers  $-\frac{2}{7}, x, -\frac{7}{2}$  are in G.P ?

**Solution:** Let  $r$  be the common ratio

Variable	Description	Value
$x(0)$	First term of the GP	$-\left(\frac{2}{7}\right)$
$x(1)$	Second term of the GP	$x$
$x(2)$	Third term of the GP	$-\left(\frac{7}{2}\right)$
$r$	Common ratio of the GP	
$x(n)$	General term	$x(0) r^n u(n)$

Table 4.6: Variables Used

From Table 4.6:

$$\Rightarrow \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \quad (4.51)$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \quad (4.52)$$

$$x = \pm 1 \quad (4.53)$$

$$\Rightarrow r = \pm \frac{7}{2} \quad (4.54)$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right) \left(\pm \frac{7}{2}\right)^n u(n) \quad (4.55)$$

Applying z-Transform :

$$\Rightarrow X_1(z) = \left(\frac{1}{7}\right) \left(\frac{4}{7z^{-1} + 2}\right) \quad |z| > \frac{7}{2} \quad (4.56)$$

$$\Rightarrow X_2(z) = \left(\frac{1}{7}\right) \left(\frac{4}{7z^{-1} - 2}\right) \quad |z| > \frac{7}{2} \quad (4.57)$$

4.0.9 Find the 20<sup>th</sup> and  $n^{\text{th}}$  terms of the G.P  $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$



Figure 4.9: Stem Plot of  $x_1(n)$

**Solution:**

From Table 4.7:  $Z$ -Transform of  $x(n)$ :

$$\Rightarrow X(z) = \frac{5}{2} \left( \frac{1}{1 - \frac{z^{-1}}{2}} \right); \left\{ z \in \mathbb{C} : |z| > \frac{1}{2} \right\} \quad (4.58)$$

4.0.10 Which term of the following sequences:

(a)  $2, 2\sqrt{2}, 4, \dots$  is 128    (b)  $\sqrt{3}, 3, 3\sqrt{3}, \dots$  is 729

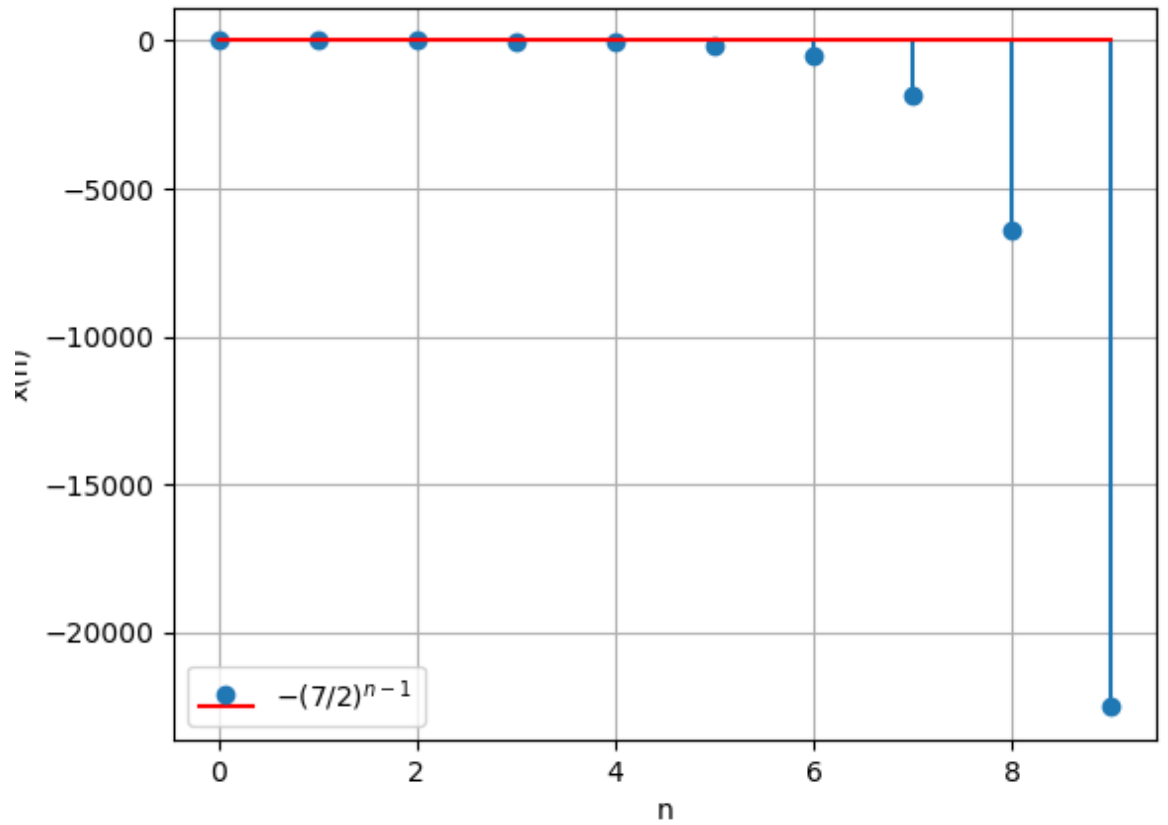


Figure 4.10: Stem Plot of  $x_2(n)$

(c)  $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$  is  $\frac{1}{19683}$

**Solution:** For a general GP series and  $k > 0$ ,

$$x(k) = x(0) r^k \quad (4.59)$$

$$\therefore k = \log_r \frac{x(k)}{x(0)} \quad (4.60)$$

Parameter	Description	Value
$x(0)$	First Term	$\frac{5}{2}$
$r = \frac{x(1)}{x(0)}$	Common Ratio	$\frac{1}{2}$
$x(n)$	$n^{th}$ Term	$\frac{5}{2} \left(\frac{1}{2}\right)^n \cdot u(n)$
$x(19)$	$20^{th}$ Term	$\frac{5}{2} \left(\frac{1}{2}\right)^{19}$
$u(n)$	Unit step function	

Table 4.7: Parameters

And the Z-transform  $X(z)$ :

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \quad (4.61)$$

(a) By Table 4.8, (4.60) and Table 4.8:

$$x_1(n) = x_1(0) r_1^n u(n) \quad (4.62)$$

$$k_1 = \log_{r_1} \frac{128}{x_1(0)} \quad (4.63)$$

$$\therefore k_1 = 12 \quad (4.64)$$

$$X_1(z) = \frac{2}{1 - \sqrt{2}z^{-1}} \quad |z| > \sqrt{2} \quad (4.65)$$





Figure 4.11:

(b) By (4.60), (4.61) and Table 4.8:

$$x_2(n) = x_2(0) r_2^n u(n) \quad (4.66)$$

$$k_2 = \log_{r_2} \frac{729}{x_2(0)} \quad (4.67)$$

$$\therefore k_2 = 11 \quad (4.68)$$

$$X_2(z) = \frac{\sqrt{3}}{1 - \sqrt{3}z^{-1}} \quad |z| > \sqrt{3} \quad (4.69)$$



Figure 1: Plot of  $x_1(n)$  vs  $n$ . See Table 4.8

(c) By (4.60), (4.61) and Table 4.8:

$$x_3(n) = x_3(0) r_3^n u(n) \quad (4.70)$$

$$k_3 = \log_{r_3} \frac{1}{19683x_3(0)} \quad (4.71)$$

$$\therefore k_3 = 8 \quad (4.72)$$

$$X_3(z) = \frac{1}{3 - z^{-1}} \quad |z| > \frac{1}{3} \quad (4.73)$$

Find the 20<sup>th</sup> and  $n^{\text{th}}$  terms of the G.P  $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$



Figure 2: Plot of  $x_2(n)$  vs  $n$ . See Table 4.8

Parameter	Description	Value
$r_i$	Common ratio of G.P (a),(b),(c)	$\sqrt{2}, \sqrt{3}, \frac{1}{3}$
$x_i(0)$	Initial Values	$2, \sqrt{3}, \frac{1}{3}$
$x_i(k_i)$	Given Values	$128, 729, \frac{1}{19683}$
$k_i$	Desired index	12, 11, 8
$x_i(n)$	Series	$x_i(0) r_i^n u(n)$
$X_i(z)$	Z-Transform of $x_i(n)$	$\frac{x_i(0)}{1-rz^{-1}}$

Table 4.8: Table of parameters



Figure 3: Plot of  $x_3(n)$  vs  $n$ . See Table 4.8

4.0.11 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of  $2^{nd}$  hour,  $4^{th}$  hour and  $n^{th}$  hour?

**Solution:** From Table 4.9:

Parameter	Value	Description
$x(0)$	30	Initial no. of bacteria
$r$	2	Ratio of no. of bacteria at end of hour to start of hour (Common Ratio)
$x(n)$	$r^n x(0)u(n)$	$n^{th}$ term of the GP

Table 4.9: Input Parameters

$$x(2) = 120 \quad (4.74)$$

$$x(4) = 480 \quad (4.75)$$

$$x(n) = 30(2^n)u(n) \quad (4.76)$$

$$X(z) = \frac{30z^{-1}}{1 - 2z^{-1}} \quad |z| > 2 \quad (4.77)$$

4.0.12 Ramkali saved Rs 5 in the first week of a year and then increased her weekly savings by Rs 1.75. If in the  $n$ th week, her weekly savings become Rs 20.75, find  $n$ .

**Solution:**

Parameter	Value	Description
$x(0)$	5	First term of AP
$d$	1.75	Common difference of AP
$x(n)$	20.75	$n^{th}$ term of AP

Table 4.10: Parameter List



Figure 4.15: Plot of  $x(n)$  vs  $n$ . See Table 4.9 for details.

$$x(n) = x(0) + (n)(d) \quad (4.78)$$

$$20.75 = 5 + (n)(1.75) \quad (4.79)$$

$$\Rightarrow 15.75 = (n)(1.75) \quad (4.80)$$

$$\Rightarrow n = \frac{15.75}{1.75} \quad (4.81)$$

$$\Rightarrow n = 9 \quad (4.82)$$

$$x(n) = 5u(n) + 1.75nu(n) \quad (4.83)$$

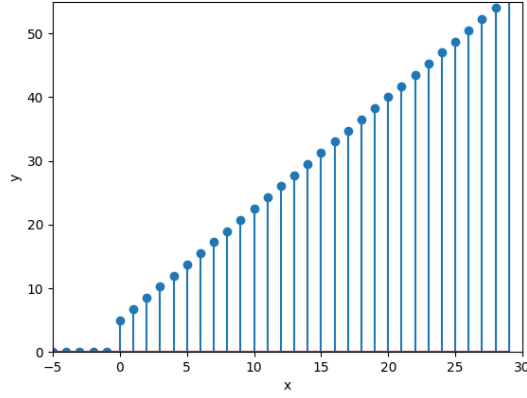


Figure 4.16: Plot of  $x(n) = 5 + 1.75n$

The Z-transform of a sequence  $x(n)$  is given by:

$$X(z) = \frac{5z^{-1}}{1 - z^{-1}} + \frac{1.75z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \quad (4.84)$$

4.0.13 Show that the sum of  $(m + n)^{th}$  and  $(m - n)^{th}$  terms of an *A.P.*, is equal to twice the  $m^{th}$  terms.

**Solution:**

For an *AP*,

$$x(n) = [x(0) + nd]u(n) \quad (4.85)$$

$$\implies x(m + n) + x(m - n) = [x(0) + (m + n)d] + [x(0) + (m - n)d] \quad (4.86)$$

$$= 2[x(0) + md] \quad (4.87)$$

$$\therefore x(m + n) + x(m - n) = 2x(m) \quad (4.88)$$

PARAMETER	VALUE	DESCRIPTION
$x(0)$	$x(0)$	First term
$d$	$d$	common difference
$x(n)$	$[x(0) + nd]u(n)$	General term of the series

Table 4.11: Parameter Table1

4.0.14 The sum of the first three terms of a G.P is  $39/10$  and their product is 1. Find the common ratio and the terms.

**Solution:**

$$y(n) = x(0) \left( \frac{r^{n+1} - 1}{r - 1} \right) u(n) \quad (4.89)$$

From Table 4.13 and (4.89) :

$$y(2) = x(0) \left( \frac{r^3 - 1}{r - 1} \right) \quad (4.90)$$

$$\frac{39}{10} = x(0) (r^2 + r + 1) \quad (4.91)$$

$$\implies \frac{39r}{10} = r^2 + r + 1 \quad (\because x(0)r = 1) \quad (4.92)$$

$$\implies (2r - 5)(5r - 2) = 0 \quad (4.93)$$

$$\implies r = \frac{2}{5} \quad \text{or} \quad \frac{5}{2} \quad (4.94)$$

(a) If  $r = \frac{2}{5}$ , then terms are  $\frac{5}{2}, 1, \frac{2}{5}$ .



(b) If  $r = \frac{5}{2}$ , then terms are  $\frac{2}{5}, 1, \frac{5}{2}$ .



Figure 4.17: stem plots of GP if  $r = \frac{2}{5}$

4.0.15 The ratio of the A.M and G.M of two positive numbers  $a$  and  $b$  is  $m : n$ . Show that

$$a : b = \left( m + \sqrt{m^2 - n^2} \right) : \left( m - \sqrt{m^2 - n^2} \right).$$

**Solution:** Expressing A.M and G.M in terms of  $a$  and  $b$ :

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n} \quad (4.95)$$



Figure 4.18: stem plots of GP if  $r = \frac{5}{2}$

Let's assume that  $x = \sqrt{\frac{a}{b}}$ . Then, we have:

$$\frac{a}{b} = x^2 \quad (4.96)$$

Substituting the value of  $x$  in equation (4.95):

$$\frac{1+x^2}{2x} = \frac{m}{n} \quad (4.97)$$

$$\frac{1}{x} + x = \frac{2m}{n} \quad (4.98)$$

$$x^2 - \frac{2m}{n}x + 1 = 0 \quad (4.99)$$

$$\implies x = \frac{m}{n} \pm \frac{\sqrt{m^2 - n^2}}{n} \quad (4.100)$$

Since  $x = \sqrt{\frac{a}{b}}$ ,  $x$  must be positive.

$$x = \frac{m + \sqrt{m^2 - n^2}}{n} \quad (4.101)$$

Referencing the value of  $x$  from equation(4.96).

$$\frac{a}{b} = \left( \frac{m + \sqrt{m^2 - n^2}}{n} \right)^2 \quad (4.102)$$

Multiplying both the numerator and denominator with  $(m - \sqrt{m^2 - n^2})$ :

$$\frac{a}{b} = \frac{1}{n^2} \frac{(m + \sqrt{m^2 - n^2})^2 (m - \sqrt{m^2 - n^2})}{(m - \sqrt{m^2 - n^2})} \quad (4.103)$$

$$\implies a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2}) \quad (4.104)$$

nth term of the AP :

$$y(n) = [a + n(b - a)] u(n) \quad (4.105)$$

$$n^k u(n) \xleftrightarrow{\mathcal{Z}} (-1)^k z^k \frac{d^k}{dz^k} U(z) \quad (4.106)$$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})} \quad |z| > |1| \quad (4.107)$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \quad (4.108)$$

Referencing the equations from (4.107),(4.108).

$$y(n) \xleftrightarrow{\mathcal{Z}} \frac{a}{(1 - z^{-1})} + \frac{(b - a) z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \quad (4.109)$$

nth term of the GP :

$$y(n) = a \left( \frac{b}{a} \right)^n u(n) \quad (4.110)$$

$$r^n u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - rz^{-1})} \quad |z| > |r| \quad (4.111)$$

Referencing the equation from (4.111).

$$y(n) \xleftrightarrow{\mathcal{Z}} \frac{a^2 z^{-1}}{(a - bz^{-1})} \quad |z| > \left| \frac{b}{a} \right| \quad (4.112)$$

4.0.16 The sum of three numbers in an arithmetic progression (AP) is 24 and the product of those three numbers is 440, find the values of the three numbers.

**Solution:**

4.0.17 The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2 , respectively. Find the last term and the number of terms.

**Solution:**

$$x(n) = x(0)r^n u(n) \quad (4.113)$$

From (B.5.4)

$$X(z) = \frac{5}{1 - 2z^{-1}} \quad |z| > |2| \quad (4.114)$$

By contour integration:

$$y(n) = x(0) \left( \frac{r^{n+1} - 1}{r - 1} \right) u(n) \quad (4.115)$$

$$315 = 5 (2^{n+1} - 1) \quad (4.116)$$

$$\implies n = 5 \quad (4.117)$$

The number of terms is  $n + 1 = 6$

From (4.113):

$$x(5) = 5 (2^5) \quad (4.118)$$

$$= 160 \quad (4.119)$$

4.0.18 Find the sum of n terms of the A.P. whose kth term is  $5k + 1$ .

**Solution:**

4.0.19 How many 3 digit numbers are divisible by 7?

**Solution:**

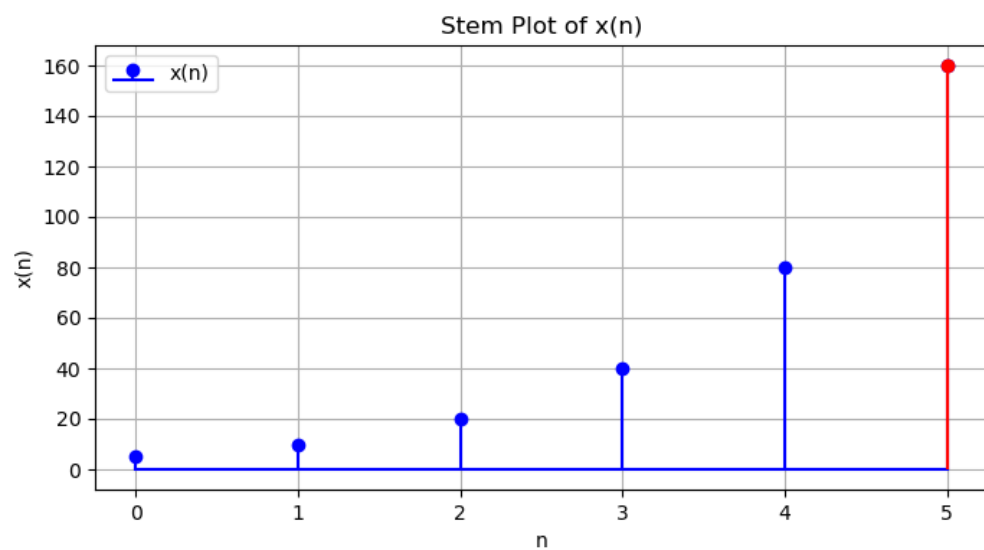


Figure 4.19: Stem plot of  $x(n)$

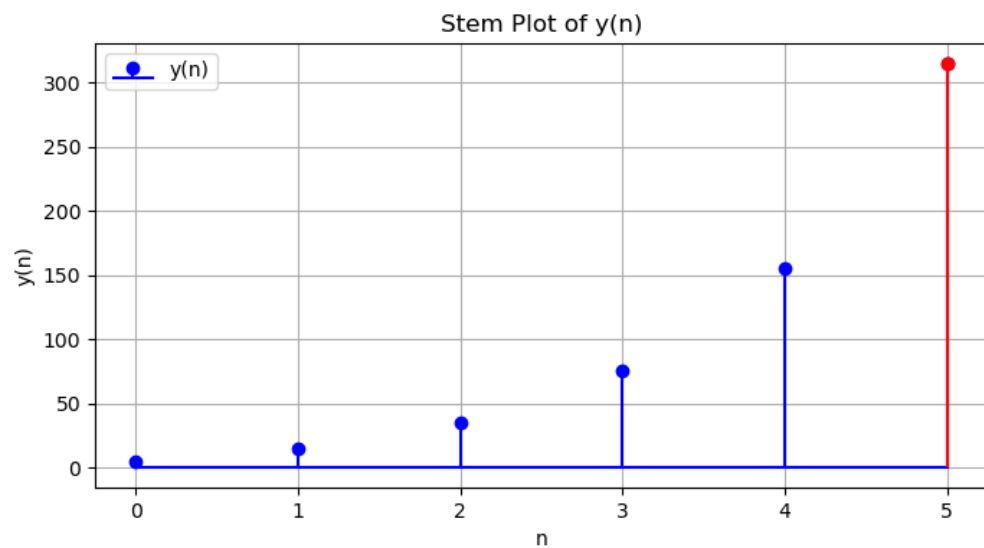


Figure 4.20: Stem plot of  $y(n)$

$x(0)$	3
$d$	2
$m$	6
$n$	2
$x(m+n)$	19
$x(m-n)$	11
$x(m)$	15

Table 4.12: Verified Values

Parameter	Value	Description
$x(0)$		First term
$r$		Common ratio
$x(0)^3 r^3$	1	Product of terms
$x(0) + x(0)r + x(0)r^2$	$\frac{39}{10}$	Sum of terms

Table 4.13: Input Parameters

Parameter	Value	Description
$x(0)$	5	First term
$r$	2	Common ratio
$y(n)$	315	Sum of $n + 1$ terms
$x(n)$	?	Last term

Table 4.14: Input Parameters



# Chapter 5

## Contour Integration

5.1 Find the sum of the first 15 multiples of 8.

**Solution:**

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
$d$	8	common difference
$x(n)$	$[8 + 8n]u(n)$	General term of the series

Table 5.1: Parameter Table1

For an  $AP$ ,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad (5.1)$$

$$\Rightarrow X(z) = \frac{8}{1 - z^{-1}} + \frac{8z^{-1}}{(1 - z^{-1})^2} \quad (5.2)$$

$$= \frac{8}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (5.3)$$

$$y(n) = x(n) * u(n) \quad (5.4)$$

$$\Rightarrow Y(z) = X(z)U(z) \quad (5.5)$$

$$Y(z) = \left( \frac{8}{(1 - z^{-1})^2} \right) \left( \frac{1}{1 - z^{-1}} \right) \quad (5.6)$$

$$= \frac{8}{(1 - z^{-1})^3}, \quad |z| > 1 \quad (5.7)$$

Using Contour Integration to find the inverse  $Z$ -transform,

$$y(14) = \frac{1}{2\pi j} \oint_C Y(z) z^{13} dz \quad (5.8)$$

$$= \frac{1}{2\pi j} \oint_C \frac{8z^{13}}{(1 - z^{-1})^3} dz \quad (5.9)$$

We can observe that the pole is repeated 3 times and thus  $m = 3$ ,

$$R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.10)$$

$$= \frac{1}{(2)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{8z^{16}}{(z-1)^3} \right) \quad (5.11)$$

$$= 4 \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (z^{16}) \quad (5.12)$$

$$= 960 \quad (5.13)$$

$$\therefore \boxed{y(14) = 960} \quad (5.14)$$



Figure 5.1: Plot of  $x(n)$  vs  $n$

5.2 If the sum of  $n$  terms of an A.P. is  $3n^2 + 5n$  and its  $m^{th}$  term is 164, find the value of  $m$ .

**Solution:**

$$Y(z) = \sum_{n=0}^{\infty} y(n) z^{-n} \quad (5.15)$$

$$= \frac{2(4 - z^{-1})}{(1 - z^{-1})^3}, \quad |z| > 1 \quad (5.16)$$

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (5.17)$$

$$X(z) = \frac{Y(z)}{U(z)} \quad (5.18)$$

$$= 2 \left( \frac{1}{1 - z^{-1}} \right) + 6 \left( \frac{1}{(1 - z^{-1})^2} \right) \quad (5.19)$$

$$= \frac{8z^2 - 2z}{(z - 1)^2} \quad (5.20)$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (5.21)$$

$$= \frac{1}{2\pi j} \oint_C \frac{(8z^{n+1} - 2z^n) dz}{(z - 1)^2} \quad (5.22)$$

$$= \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z - a)^m f(z)) \quad (5.23)$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z - 1)^2 \frac{8z^{n+1} - 2z^n}{(z - 1)^2} \right) \quad (5.24)$$

$$= \lim_{z \rightarrow 1} (8(n+1)z^n - 2nz^{n-1}) \quad (5.25)$$

$$\implies x(n) = (6n + 8)(u(n)) \quad (5.26)$$

$$164 = (6m + 8)(u(m)) \quad (5.27)$$

$$\implies m = 26 \quad (5.28)$$

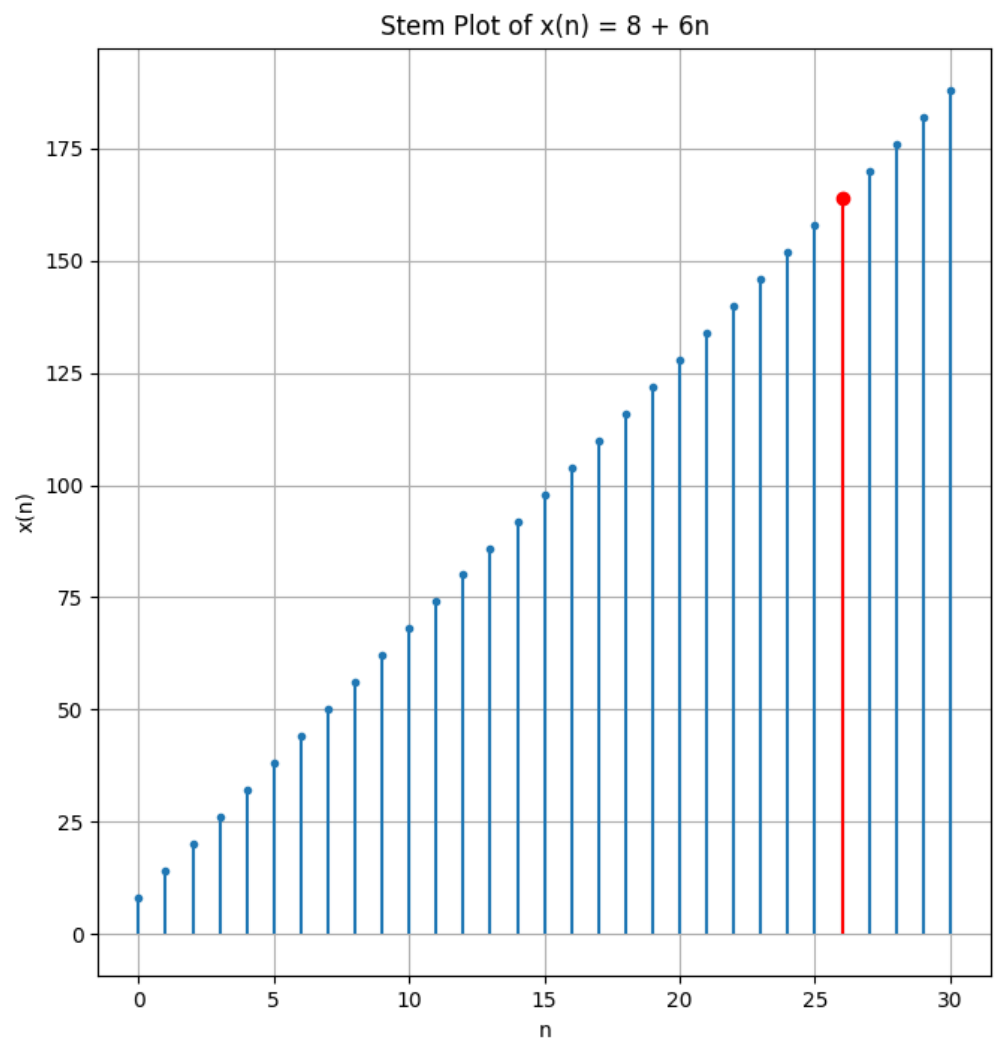


Figure 5.2: Plot of  $x(n)$  vs  $n$

Symbol	Remarks
$y(n) = (3n^2 + 11n + 8)(u(n))$	Sum of $n$ terms
$x(m-1)$	164
$y(n)$	$x(n) * u(n)$

Table 5.2: Parameters

5.3 Show that  $a_0, a_1, a_2, \dots, a_n, \dots$  form an AP where an is defined as below :

(a)  $a_n = (3 + 4n)$

(b)  $a_n = (9 - 5n)$

Also find the sum of the first 15 terms in each case. **Solution:**

Parameter	Description	Value
$x_i(n)$	$i^{th}$ Discrete signal	$(3 + 4n)u(n)$
		$(9 - 5n)u(n)$
$x_i(0)$	First term of $i^{th}$ AP	3
		9
$d_i$	common difference of $i^{th}$ AP	4
		-5

Table 5.3: Given parameters

(a) From equation (B.10.6)

$$X(z) = \frac{3}{1 - z^{-1}} + \frac{4.z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \quad (5.29)$$

$$\therefore y(n) = x(n) * u(n) \quad (5.30)$$

$$Y(z) = X(z)U(z) \quad (5.31)$$

$$= \left[ \frac{3}{(1 - z^{-1})^2} + \frac{4z^{-1}}{(1 - z^{-1})^3} \right] \quad (5.32)$$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j} \int Y(z)z^{13}dz \quad (5.33)$$

$$= \frac{1}{2\pi j} \int \frac{3.z^{15}}{(z - 1)^2}dz + \frac{1}{2\pi j} \int \frac{4.z^{15}}{(z - 1)^3}dz \quad (5.34)$$

$$\therefore R = \frac{1}{(m - 1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z - a)^m f(z)) \quad (5.35)$$

$$R_1 = \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z - 1)^2 \cdot \frac{3.z^{15}}{(z - 1)^2} \right) \quad (5.36)$$

$$= 45 \quad (5.37)$$

$$R_2 = \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z - 1)^3 \cdot \frac{4.z^{15}}{(z - 1)^3} \right) \quad (5.38)$$

$$= 420 \quad (5.39)$$

$$\implies y(14) = R_1 + R_2 \quad (5.40)$$

$$= 465 \quad (5.41)$$

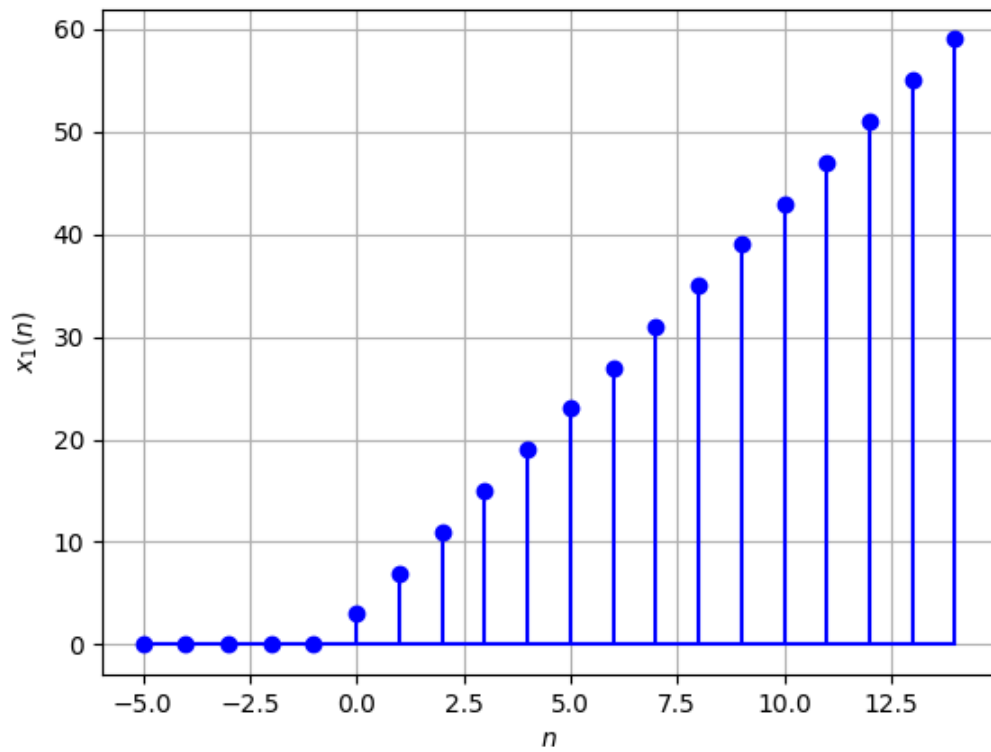


Figure 5.3:  $x_1(n) = (3 + 4n)u(n)$

(b) From equation (B.10.6)

$$X(z) = \frac{9}{1 - z^{-1}} - \frac{5z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \quad (5.42)$$

$$\therefore y(n) = x(n) * u(n) \quad (5.43)$$

$$Y(z) = X(z)U(z) \quad (5.44)$$

$$= \left[ \frac{9}{(1 - z^{-1})^2} - \frac{5z^{-1}}{(1 - z^{-1})^3} \right] \quad (5.45)$$



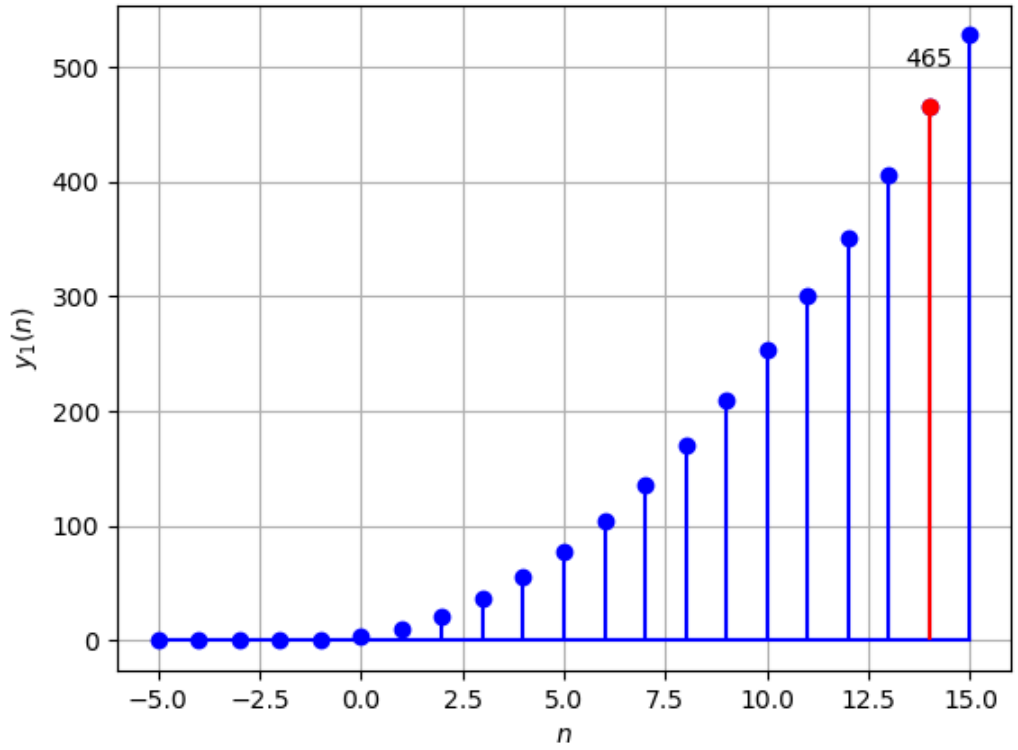


Figure 5.4:  $x_1(n) = (2n^2 + 5n + 3)u(n)$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j} \int Y(z)z^{13}dz \quad (5.46)$$

$$= \frac{1}{2\pi j} \int \frac{9.z^{15}}{(z-1)^2}dz - \frac{1}{2\pi j} \int \frac{5.z^{15}}{(z-1)^3}dz \quad (5.47)$$

$$\therefore R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.48)$$

$$R_1 = \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z-1)^2 \cdot \frac{9.z^{15}}{(z-1)^2} \right) \quad (5.49)$$

$$= 135 \quad (5.50)$$

$$R_2 = \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \cdot \frac{5.z^{15}}{(z-1)^3} \right) \quad (5.51)$$

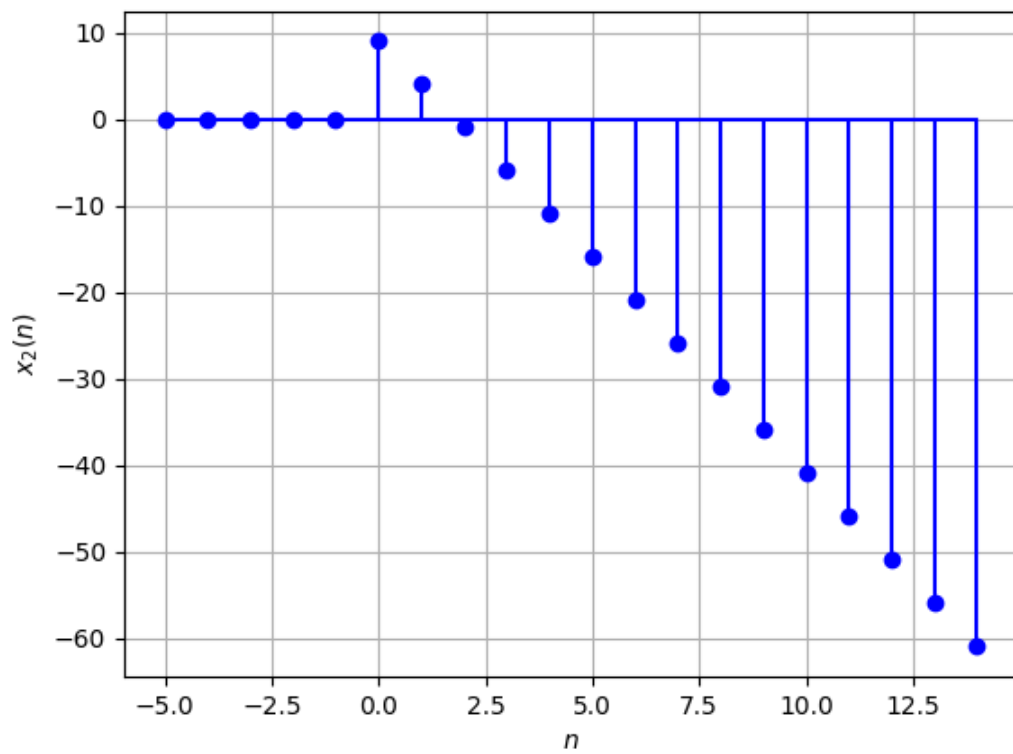


Figure 5.5:  $x_2(n) = (9 - 5n)u(n)$

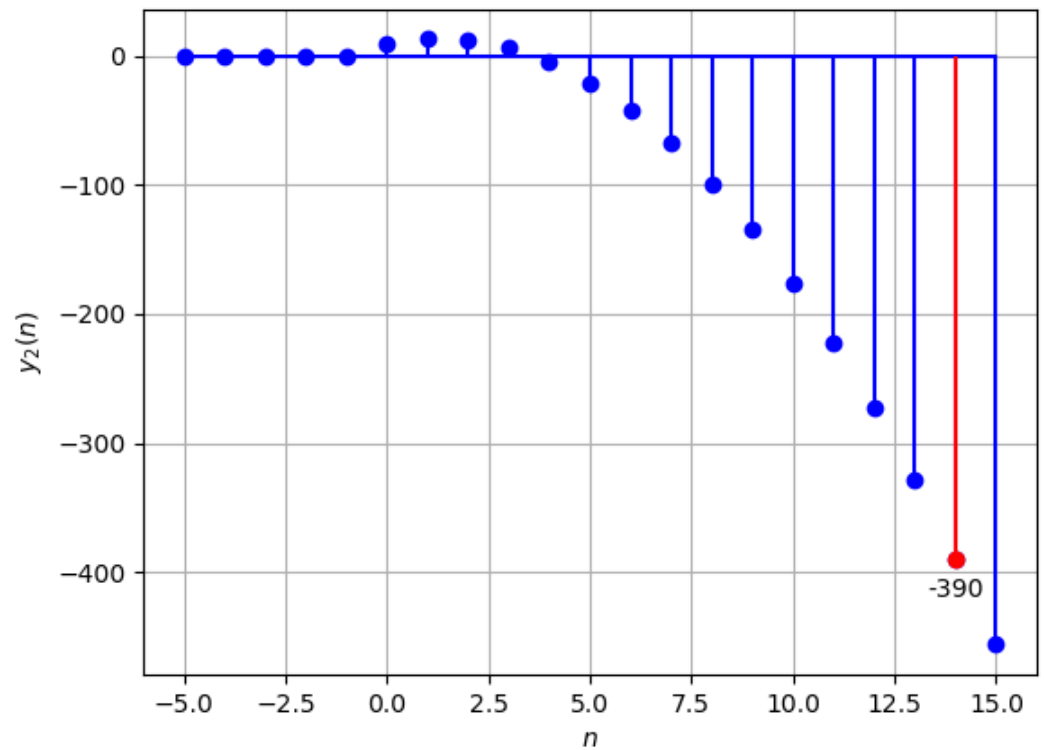


Figure 5.6:  $x_2(n) = (-5n^2 + 13n + 18)u(n)$

5.4 If the sum of  $n$  terms of an AP is  $(pn + qn^2)$ , where  $p$  and  $q$  are constants, find the common difference. **Solution:**

Symbol	Value	Description
$y(n)$	$(pn + qn^2)$	Sum of $n$ terms
$x(n)$		$n^{th}$ term of AP
$d$	$x(n + 1) - x(n)$	Common Difference

Table 5.4: Given Parameters

Sum of  $n$  terms, as a discrete signal:

$$y(n) = (pn + qn^2)u(n) \quad (5.55)$$

Taking the Z-Transform,

$$(a) \mathcal{Z}\{u(n)\}$$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - z^{-1}} \{|z| > 1\} \quad (5.56)$$

$$(b) \mathcal{Z}\{nu(n)\}$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \{|z| > 1\} \quad (5.57)$$

$$(c) \mathcal{Z}\{n^2u(n)\}$$

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \{|z| > 1\} \quad (5.58)$$

Taking the Z-Transform of (5.55) using (5.57) and (5.58)

$$Y(z) = p \left( \frac{z^{-1}}{(1 - z^{-1})^2} \right) + q \left( \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \right) \quad (5.59)$$

Now,

$$y(n) = x(n) * u(n) \quad (5.60)$$

$$\implies Y(z) = X(z)U(z) \quad (5.61)$$

$$\implies X(z) = \frac{Y(z)}{U(z)} \quad (5.62)$$

Using (5.56) in (5.62),

$$X(z) = p \left( \frac{z^{-1}}{(1 - z^{-1})} \right) + q \left( \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \quad (5.63)$$

Using contour integration for inverse Z-Transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (5.64)$$

$$= \frac{1}{2\pi j} \oint_C \left[ p \left( \frac{z^{-1}}{(1 - z^{-1})} \right) + q \left( \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \right] z^{n-1} dz \quad (5.65)$$

Calculating the residues  $R_1$  and  $R_2$  at pole  $z = 1$ :

$$R_1 = \frac{1}{0!} \lim_{z \rightarrow 1} (z - 1) \left( p \left( \frac{z^{-1}}{1 - z^{-1}} \right) \right) z^{n-1} \quad (5.66)$$

$$= p \quad (5.67)$$

$$R_2 = \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z - 1)^2 q \left( \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \right) z^{n-1} \quad (5.68)$$

$$= q \lim_{z \rightarrow 1} \frac{d}{dz} (z^n + z^{n-1}) \quad (5.69)$$

$$= q(2n - 1) \quad (5.70)$$

$$\Rightarrow x(n) = R_1 + R_2 \quad (5.71)$$

$$= p + q(2n - 1) \quad (5.72)$$

Writing  $x(n)$  as a discrete signal we get:

$$x(n) = (p - q)u(n) + 2qnu(n) \quad (5.73)$$

To simplify, use  $n = 0$ :

$$y(0) = x(0) \tag{5.74}$$

$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{5.75}$$

$$\implies p = q \tag{5.76}$$

$\therefore$  (5.73) can be written as:

$$x(n) = 2qnu(n) \tag{5.77}$$

Common difference is given by:

$$d = x(n + 1) - x(n) \tag{5.78}$$

$$= 2q \tag{5.79}$$

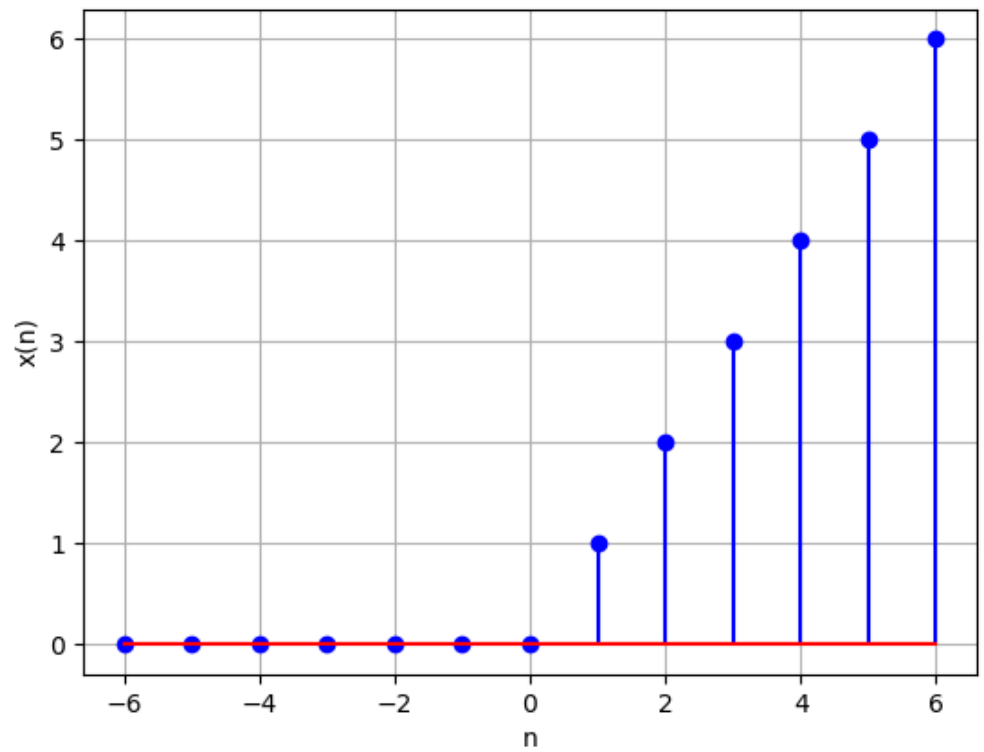


Figure 5.7: Plot of  $x(n)$  vs  $n$  for  $p = q = 0.5$

5.5 Find the sum of the first 40 positive integers divisible by 6

**Solution:**

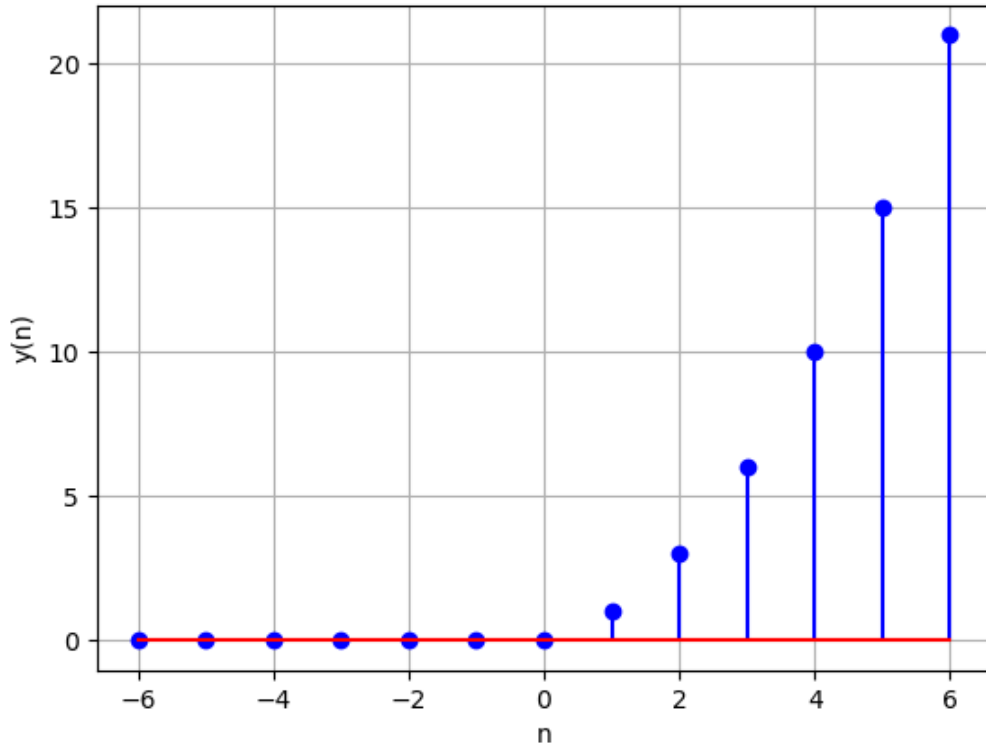


Figure 5.8: Plot of  $y(n)$  vs  $n$  for  $p = q = 0.5$

$$x(n) = (6 + 6n)(u(n)) \quad (5.80)$$

$$\Rightarrow X(z) = \frac{6}{1 - z^{-1}} + \frac{6z^{-1}}{(1 - z^{-1})^2} \quad (\text{B.10.6}) \quad (5.81)$$

$$\Rightarrow X(z) = \frac{6}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (5.82)$$

$$y(n) = x(n) * u(n) \quad (5.83)$$

$$\Rightarrow Y(z) = X(z)U(z) \quad (5.84)$$

$$= \frac{6}{(1 - z^{-1})^3}, \quad |z| > 1 \quad (5.85)$$



Parameter	Description	Value
x(0)	First Term	6
d	Common Difference	6

Table 5.5: Parameter Table 10.5.3.12

Using contour integration to find the inverse Z-transform:

$$\Rightarrow y(39) = \frac{1}{2\pi j} \oint_C Y(z) z^{38} dz \quad (5.86)$$

$$= \frac{1}{2\pi j} \oint_C \frac{6z^{41}}{(z-1)^3} dz \quad (5.87)$$

We can observe that there is only a three times repeated pole at  $z=1$ ,

$$\Rightarrow R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.88)$$

$$= \frac{1}{(2)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{6z^{41}}{(z-1)^3} \right) \quad (5.89)$$

$$= 3 \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (z^{41}) \quad (5.90)$$

$$= 4920 \quad (5.91)$$

$$\therefore \boxed{y(39) = 4920} \quad (5.92)$$

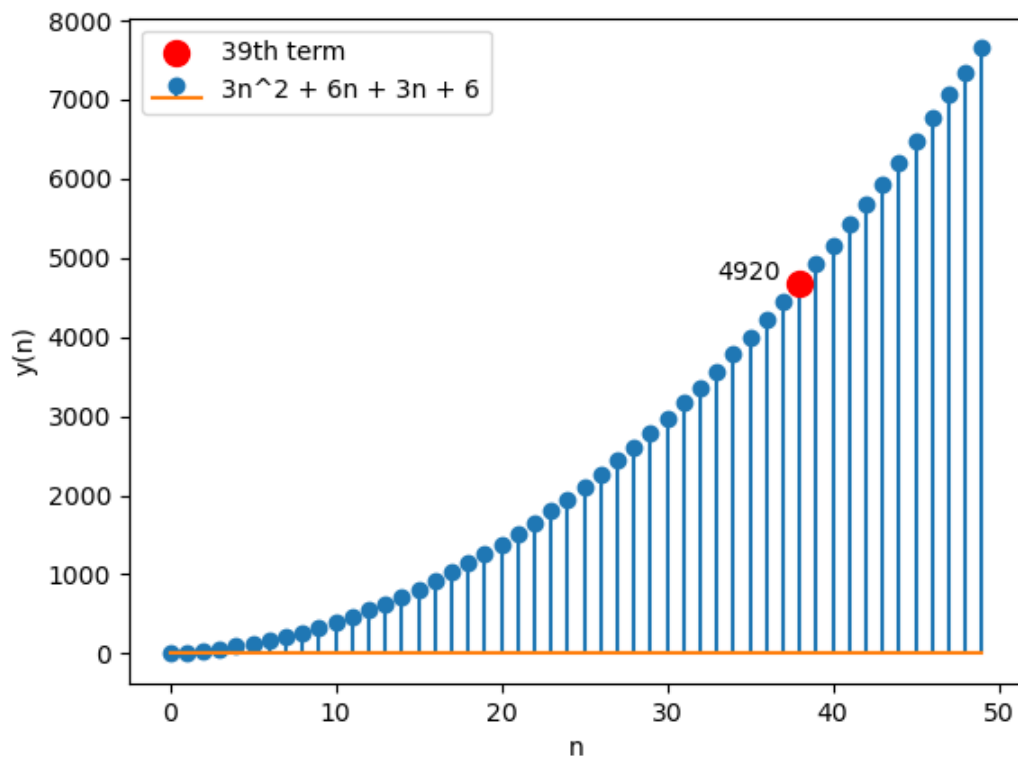


Figure 5.9: Plot of  $y(n)$  vs  $n$

5.6 If the sum of certain number of terms in a AP 25,22,19,... is 116. Find the last term.

**Solution:**

$$x(n) = (25 - 3n)u(n) \quad (5.93)$$

Symbol	Value	Description
$x(0)$	25	first term of AP
$d$	-3	common difference
$x(n)$	$(25 - 3n)u(n)$	$n$ -th term of AP
$y(n)$	116	sum of terms

Table 5.6: Input Parameters

Applying Z transform:

$$x(z) = \frac{25}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad (5.94)$$

$$= \frac{25 - 28z^{-1}}{(1 - z^{-1})^2} \quad (5.95)$$

Region of Convergence or R.O.C :

$$|z| > 1 \quad (5.96)$$

For AP, the sum of first  $n+1$  terms can be written as :

$$y(n) = x(n) * u(n) \quad (5.97)$$

Applying Z transform on both sides

$$Y(z) = X(z)U(z) \quad (5.98)$$

$$= \frac{25}{(1 - z^{-1})^2} - \frac{3z^{-1}}{(1 - z^{-1})^3} \quad (5.99)$$

Using contour integration to find inverse Z transform:

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z) z^{n-1} dz \quad (5.100)$$

$$= \frac{1}{2\pi j} \oint_C \left[ \frac{25}{(1-z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})^3} \right] z^{n-1} dz \quad (5.101)$$

The sum of the terms of the sequence is computed using the residue theorem, expressed as  $R_i$ , which represents the residue of the Z-transform at  $z = 1$  for the expression  $Y(z)$ .

$$R_i = R_1 + R_2 \quad (5.102)$$

$R_1$  and  $R_2$  are residues calculated at the poles of the Z-transform.

$$R_1 = \frac{1}{(2-1)!} \left. \frac{d(25z^{n+1})}{dz} \right|_{z=1} \quad (5.103)$$

$$= 25(n+1) \quad (5.104)$$

$$R_2 = \frac{1}{(3-1)!} \left. \frac{d^2(-3z^{n+1})}{dz^2} \right|_{z=1} \quad (5.105)$$

$$= \frac{-3}{2} (n+1)(n) \quad (5.106)$$

The sum of terms is given by  $R_i$ :

$$25(n+1) + \frac{-3}{2} n(n+1) = 116 \quad (5.107)$$

Solving the equation gives :

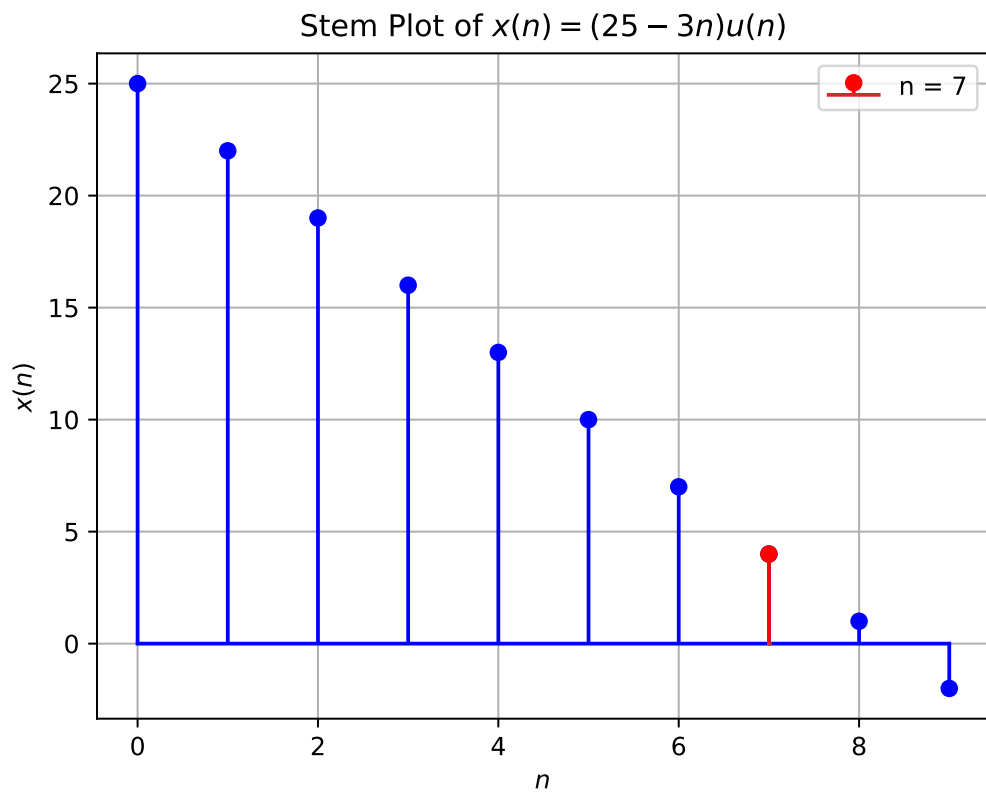
$$n = 7 \quad (5.108)$$

$$n = 8.667 \quad (5.109)$$

Since  $n$  can take only integer values,  $n = 8.667$  is rejected. Upon substituting the value of  $n$  in equation (5.93):

$$x(7) = 4 \quad (5.110)$$

Hence the last term of the given AP is 4.



5.7 The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

**Solution:**

$$x(n) = (x(0) + nd)u(n) \quad (5.111)$$

$$x(l) = (17 + 9l)u(l) \quad (5.112)$$

Thus,

$$l = 37 \quad (5.113)$$

Parameters in expression		
Symbol	Description	Value
$x(n)$	$n^{th}$ term of series	
$x(l)$	Last( $l^{th}$ ) term of series	350
$x(0)$	Starting ( $0^{th}$ ) term of series	17
d	Common difference of AP	9

Table 5.7: Parameters

Using (B.10.6),

$$X(z) = \frac{(17 - 8z^{-1})}{(1 - z^{-1})^2}, \quad |z| > |1| \quad (5.114)$$

$$y(n) = x(n) * u(n) \quad (5.115)$$

$$\implies Y(z) = X(z)U(z) \quad (5.116)$$

$$= \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} \quad (5.117)$$

Using contour integral to find Z transform, we get

$$y(37) = \frac{1}{2\pi j} \oint_C Y(z) z^{36} dz \quad (5.118)$$

$$= \frac{1}{2\pi j} \oint_C \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} z^{36} dz \quad (5.119)$$

Now, using Cauchy's residual theorem and observing the fact that 3 repeated poles

exist at  $z = 1$ ,

$$R = \frac{1}{(k-1)!} \lim_{z \rightarrow c} \frac{d^{k-1}}{dz^{k-1}} ((z-c)^k f(z)) \quad (5.120)$$

$$= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^{k-1}}{dz^{k-1}} ((z-1)^3 \frac{(17-8z^{-1})}{(1-z^{-1})^3} z^{36}) \quad (5.121)$$

$$= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (17z^{39} - 8z^{38}) \quad (5.122)$$

$$= 6973 \quad (5.123)$$



Figure 5.10: Stem Plot of  $x(n)$  v/s  $n$



5.8 A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete. Each step has a rise of 1/4 m and a tread of 1/2 m. Calculate the total volume of concrete required to build the terrace. [Hint: Volume of concrete required to build the first step=

$$V = \frac{1}{4} \cdot \frac{1}{2} \cdot 50 \quad (5.124)$$

**Solution:** here

parameter	description	value
$x(0)$	first term	6.25
$d$	common difference	6.25
$n$	no of terms -1	14
$x(n)$	volume of $(n+1)$ th step	$(6.25 + 6.25n) u(n)$

Table 5.8: formula parameters

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \quad (5.125)$$

$$\Rightarrow X(z) = \left( \frac{6.25}{(1 - z^{-1})^2} \right) \quad |z| > |1| \quad (5.126)$$



Figure 5.11: plot  $y(n)$  vs  $n$

$$y(n) = x(n) * u(n) \quad (5.127)$$

$$\Rightarrow Y(z) = X(z)U(z) \quad (5.128)$$

$$U(z) = \frac{1}{1 - z^{-1}} \quad |z| > |1| \quad (5.129)$$

$$Y(z) = \left( \frac{6.25}{1 - z^{-1}} + \frac{6.25z^{-1}}{(1 - z^{-1})^2} \right) \left( \frac{1}{1 - z^{-1}} \right) \quad |z| > |1| \quad (5.130)$$

$$Y(z) = \frac{6.25z^3}{(z - 1)^3} \quad |z| > |1| \quad (5.131)$$

contour integration to find inverse z transform

$$y(14) = \frac{1}{2\pi j} \oint_c Y(z) z^{13} dz \quad (5.132)$$

$$= \frac{1}{2\pi j} \oint_c \frac{6.25z^{16}}{(z-1)^3} \quad (5.133)$$

pole at 1 repeated 3 times

$$\Rightarrow m = 3 \quad (5.134)$$

$$R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.135)$$

$$\Rightarrow y(14) = \frac{1}{(2!)} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{6.25z^{16}}{(z-1)^3} \right) \quad (5.136)$$

$$= 3.125 \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (z^{16}) \quad (5.137)$$

$$y(14) = 750 \quad (5.138)$$

5.9 Given a GP with a = 729 and 7<sup>th</sup> term 64, find  $S_7$ .

**Solution:**

Parameter	Description	Value
$x(0)$	First Term	729
$r$	Common Ratio	
$x(n)$	$(n + 1)^{th}$ Term	$x(0) r^n$
$x(6)$	7 <sup>th</sup> Term	64
$y(k)$	Sum of first $(k + 1)$ terms	

Table 5.9: Parameter Table

from Table 5.9 :

$$x(6) = x(0) r^6 \quad (5.139)$$

$$\implies 64 = 729r^6 \quad (5.140)$$

$$\therefore r = \frac{2}{3} \quad (5.141)$$

using Table 5.9 and equation (5.141)

$$X(z) = \frac{729}{1 - \frac{2}{3}z^{-1}}, |z| > \frac{2}{3} \quad (5.142)$$

using Table 5.9 and equation (5.142)

$$Y(z) = \frac{729}{\left(1 - \frac{2}{3}z^{-1}\right)(1 - z^{-1})} \quad (5.143)$$

$$= 2187 \left( \frac{1}{1 - z^{-1}} - \frac{\frac{2}{3}}{1 - \frac{2}{3}z^{-1}} \right), |z| > 1 \quad (5.144)$$

Using contour integration for inverse z transform,

$$y(6) = \frac{1}{2\pi j} \oint Y(z) z^5 dz \quad (5.145)$$

Using equation (5.144) in (5.145) :

$$y(6) = \frac{1}{2\pi j} \left( \oint \frac{2187z^6}{z-1} dz - \oint \frac{1458z^6}{z-\frac{2}{3}} dz \right) \quad (5.146)$$

$$\frac{1}{2\pi j} \left( \oint \frac{2187z^6}{z-1} dz \right) = 2187 \quad (5.147)$$

$$\frac{1}{2\pi j} \left( \oint \frac{1458z^6}{z-\frac{2}{3}} dz \right) = 128 \quad (5.148)$$

using equations (5.147) and (5.148) in (5.146):

$$y(6) = 2187 - 128 \quad (5.149)$$

$$= 2059 \quad (5.150)$$



Figure 5.12: Plot of  $y(n]$

5.10 Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

**Solution:**

$$x(n) = (105 + 5n)(u(n)) \quad (5.151)$$



Figure 5.13: Plot of  $x(n)$

Parameter	Description	Value
$x(0)$	First Term	105
$d$	Common Difference	5
$n$	Total terms	179
$x(178)$	Last Term	995
$m$	No of poles	3

Table 5.10: Given Parameters

On taking Z transform

$$X(z) = \frac{x(0)}{(1 - z^{-1})} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad (5.152)$$

$$= \frac{105}{1 - z^{-1}} + \frac{5z^{-1}}{(1 - z^{-1})^2} \quad (5.153)$$

$$\Rightarrow X(z) = \frac{105 - 100z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (5.154)$$

$$y(n) = x(n) * u(n) \quad (5.155)$$

$$\Rightarrow Y(z) = X(z)U(z) \quad (5.156)$$

$$= \frac{105 - 100z^{-1}}{(1 - z^{-1})^2} \frac{1}{(1 - z^{-1})} \quad (5.157)$$

$$= \frac{105 - 100z^{-1}}{(1 - z^{-1})^3} \quad |z| > 1 \quad (5.158)$$

Using contour integration to find the inverse Z-transform:

$$\Rightarrow y(178) = \frac{1}{2\pi j} \oint_C Y(z) z^{177} dz \quad (5.159)$$

$$= \frac{1}{2\pi j} \oint_C \frac{(105 - 100z^{-1}) z^{177}}{(1 - z^{-1})^3} dz \quad (5.160)$$

We can observe that there is only a 3 times repeated pole at  $z = 1$ ,

$$\Rightarrow R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.161)$$



$$= \frac{1}{(2)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{(105 - 100z^{-1}) z^{180}}{(z-1)^3} \right) \quad (5.162)$$

$$= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (105z^{180} - 100z^{179}) \quad (5.163)$$

$$= 98450 \quad (5.164)$$

$$\therefore y(178) = 98450 \quad (5.165)$$



Figure 5.14: Plot of  $x(n)$  vs  $n$



## Chapter 6

# Laplace Transform

6.0.1 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of

- (a) The spring constant  $K$
- (b) The damping constant  $b$  for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

**Solution:** The parameters are :

Parameter	Value(SI)	Description
$x_0$	0.15	Initial spring compression
$m$	750	Mass
$g$	9.8	Gravitational acc
$k$	$mg/x_0$	Spring constant
$b$		Damping constant

Table 6.1: Input Parameters

Parameter	Value(SI)	Description
$x$		Spring Extension
$F_1$	$kx$	Spring Force
$F_2$	$b\frac{dx}{dt}$	Damping Force
$s$		Complex Frequency
$s_1, s_2$		Values of s

Table 6.2: Intermediate Parameters

Initially the automobile is in rest, so we can use,

$$mg = kx_0 \quad (6.1)$$

$$\implies k = \frac{mg}{x_0} \quad (6.2)$$

Now, as the oscillation begins, from the FBD, we have net force on the mass as,

$$F = F_1 + F_2 + mg u(t) \quad (6.3)$$

$$\implies -m \frac{d^2 x(t)}{dt^2} = kx(t) + b \frac{dx(t)}{dt} + mg u(t) \quad (6.4)$$

$$\implies \frac{d^2 x(t)}{dt^2} + \left(\frac{b}{m}\right) \frac{dx(t)}{dt} + \left(\frac{k}{m}\right) x(t) = -gu(t) \quad (6.5)$$

Now, taking the Laplace transform on both sides,

$$s^2 X(s) + \frac{b}{m} s X(s) + \frac{k}{m} X(s) = -\frac{g}{s} \quad (6.6)$$

$$\implies X(s) = -\frac{g}{s \left( s^2 + \frac{b}{m} s + \frac{k}{m} \right)} \quad (6.7)$$

$$\implies X(s) = -\frac{g}{s(s - s_1)(s - s_2)} \quad (6.8)$$



Figure 6.1: FBD of the damped oscillation system

Where

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \quad (6.9)$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \quad (6.10)$$

From (6.8) we get,

$$\begin{aligned} \Rightarrow X(s) &= \frac{g}{(s_1 - s_2)} \left[ \frac{1}{s_2(s - s_2)} - \frac{1}{s_1(s - s_1)} \right] \\ &+ \frac{g}{s_1 s_2} \left( \frac{1}{s} \right) \end{aligned} \quad (6.11)$$

Now again taking the inverse Laplace transform we have,

$$x(t) = \frac{g}{s_1 s_2} u(t) + \frac{g}{(s_1 - s_2)} \left[ \frac{1}{s_2} e^{s_2 t} - \frac{1}{s_1} e^{s_1 t} \right] u(t) \quad (6.12)$$

$$\begin{aligned} \Rightarrow x(t) &= \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} e^{-bt/2m} u(t) \\ &\sin \left( \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} t + \tan^{-1} \left( \frac{2mg\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}{gb} \right) \right) \\ &+ \frac{mg}{k} u(t) \end{aligned} \quad (6.13)$$

(Substituting the values of  $s_1$  and  $s_2$  from (6.9) and (6.10))

From (6.13) we have the amplitude after one time period  $T$ ,

$$\begin{aligned} \frac{1}{2} \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} &= \\ \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} e^{-bT/2m} \end{aligned} \quad (6.14)$$

$$\Rightarrow e^{\pi b/\sqrt{mk}} = 2 \quad (6.15)$$

$$\Rightarrow b = \frac{\sqrt{mk} \ln 2}{\pi} \quad (6.16)$$



Figure 6.2: Displacement Vs. Time Graph





# Appendix A

## Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad (\text{A.1.1})$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.2.1})$$

A.3 If

$$x(n) = 0, \quad n < 0, \quad (\text{A.3.1})$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^n x(k) \quad (\text{A.3.2})$$



## Appendix B

# Z-transform

B.1 The Z-transform of  $p(n)$  is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n)z^{-n} \quad (\text{B.1.1})$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \quad (\text{B.2.1})$$

$$P(z) = P_1(z)P_2(z) \quad (\text{B.2.2})$$

B.3

$$nx(n) \xleftrightarrow{\mathcal{Z}} -zX'(z) \quad (\text{B.3.1})$$

From (B.3.1)

$$\implies nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \quad (\text{B.3.2})$$

$$\implies n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, |z| > 1 \quad (\text{B.3.3})$$

$$\implies n^3u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, |z| > 1 \quad (\text{B.3.4})$$

$$\implies n^4u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \quad (\text{B.3.5})$$

where  $|z| > 1$

B.4

$$x(n-k) \xleftrightarrow{\mathcal{Z}} z^{-k}X(z) \quad (\text{B.4.1})$$

Using (B.4.1):

$$nu(n-1) \xleftrightarrow{\mathcal{Z}} z \frac{2z^{-2}}{(1-z^{-1})^2} \quad (\text{B.4.2})$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \xleftrightarrow{\mathcal{Z}} \frac{z^{-2}}{(1-z^{-1})^2} \quad (\text{B.4.3})$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \xleftrightarrow{\mathcal{Z}} \frac{z^{-3}}{(1-z^{-1})^3} \quad (\text{B.4.4})$$

$\vdots$

$$\frac{(n-1)(n-2)\dots(n-k+1)}{(k-1)!}u(n-k) \xleftrightarrow{\mathcal{Z}} \frac{z^{-k}}{(1-z^{-1})^k} \quad (\text{B.4.5})$$

$$\Rightarrow Z^{-1} \left[ \frac{z^{-2}}{(1 - z^{-1})^2} \right] = (n - 1) u(n - 1) \quad (\text{B.4.6})$$

$$\Rightarrow Z^{-1} \left[ \frac{z^{-3}}{(1 - z^{-1})^3} \right] = \frac{(n - 1)(n - 2)}{2} u(n - 1) \quad (\text{B.4.7})$$

$$\Rightarrow Z^{-1} \left[ \frac{z^{-4}}{(1 - z^{-1})^4} \right] = \frac{(n - 1)(n - 2)(n - 3)}{6} u(n - 1) \quad (\text{B.4.8})$$

$$\Rightarrow Z^{-1} \left[ \frac{z^{-5}}{(1 - z^{-1})^5} \right] = \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24} \quad (\text{B.4.9})$$

$$u(n - 1)$$

B.5 For a Geometric progression

$$x(n) = x(0) r^n u(n), \quad (\text{B.5.1})$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(0) r^n z^{-n} \quad (\text{B.5.2})$$

$$= \sum_{n=0}^{\infty} x(0) (rz^{-1})^n \quad (\text{B.5.3})$$

$$= \frac{x(0)}{1 - rz^{-1}}, \quad |z| > |r| \quad (\text{B.5.4})$$

B.6 Substituting  $r = 1$  in (B.5.4),

$$u(n) \xleftrightarrow{Z} U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (\text{B.6.1})$$

B.7 From (B.3.1) and (B.6.1),

$$nu(n) \xleftrightarrow{Z} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (\text{B.7.1})$$

B.8 For an AP,

$$x(n) = [x(0) + nd] u(n) = x(0)u(n) + dnu(n) \quad (\text{B.8.1})$$

$$\Rightarrow X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (\text{B.8.2})$$

upon substituting from (B.6.1) and (B.7.1).

B.9 From (A.3.2), the sum to  $n$  terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \quad (\text{B.9.1})$$

where  $x(n)$  is defined in (B.5.1). From (B.2.2), (B.5.4) and (B.6.1),

$$Y(z) = X(z) U(z) \quad (\text{B.9.2})$$

$$= \left( \frac{x(0)}{1 - rz^{-1}} \right) \left( \frac{1}{1 - z^{-1}} \right) \quad |z| > |r| \cap |z| > |1| \quad (\text{B.9.3})$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \quad (\text{B.9.4})$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1} \left( \frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}} \right) \quad (\text{B.9.5})$$

using partial fractions. Again, from (B.5.4) and (B.6.1), the inverse of the above can be expressed as

$$y(n) = x(0) \left( \frac{r^{n+1} - 1}{r - 1} \right) u(n) \quad (\text{B.9.6})$$

B.10 For the AP  $x(n)$ , the sum of first  $n + 1$  terms can be expressed as

$$y(n) = \sum_{k=0}^n x(k) \quad (\text{B.10.1})$$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \quad (\text{B.10.2})$$

$$= x(n) * u(n) \quad (\text{B.10.3})$$

Taking the Z-transform on both sides, and substituting (B.8.2) and (B.6.1),

$$Y(z) = X(z)U(z) \quad (\text{B.10.4})$$

$$\Rightarrow Y(z) = \left( \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \right) \frac{1}{1-z^{-1}} \quad |z| > 1 \quad (\text{B.10.5})$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \quad (\text{B.10.6})$$

B.11 From (B.4.1) and (B.7.1),

$$(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \quad (\text{B.11.1})$$

From (B.11.1) and (B.3.1),

$$n(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \quad (\text{B.11.2})$$

B.12 Taking the inverse Z-transform of (B.10.6),

$$y(n) = x(0) [(n+1)u(n)] + \frac{d}{2} [n(n+1)u(n)] \quad (\text{B.12.1})$$

$$= \frac{n+1}{2} \{2x(0) + nd\} u(n) \quad (\text{B.12.2})$$





## Appendix C

# Contour Integration

C.1

$$x(n) \xrightarrow{Z} X(z) \quad (C.1.1)$$

$$\implies X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \quad (C.1.2)$$

Multiplying both side with  $z^{k-1}$  and integrating on a contour integral enclosing the region of convergence. Where  $C$  is a counter-clockwise closed contour in region of convergence.

$$\frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz = \frac{1}{2\pi j} \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{-n+k-1} dz \quad (C.1.3)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \frac{1}{2\pi j} \oint_C z^{-n+k-1} dz \quad (C.1.4)$$

From cauchy's integral theorem

$$\frac{1}{2\pi j} \oint_C z^{-k} dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \quad (C.1.5)$$

$$= \delta(1 - k) \quad (C.1.6)$$

So eq (C.1.4) becomes

$$\frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz = \sum_{k=-\infty}^{\infty} x(k) \delta(k-n) \quad (C.1.7)$$

$$\Rightarrow x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (C.1.8)$$

Contour integrals like (C.1.8) can be evaluated using Cauchy's residue theorem.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (C.1.9)$$

$$= \sum [\text{Residue of } X(z) z^{n-1} \text{ at poles inside } C] \quad (C.1.10)$$

C.2 Question: Find the sum of n terms of an AP where common difference = d using Contour Integration.

**Solution:**

By performing inverse Z transform on S(z) using contour integration

$$s(n) = \frac{1}{2\pi j} \oint_C S(z) z^{n-1} dz \quad (C.2.1)$$

$$s(n) = \frac{1}{2\pi j} \oint_C \left( \frac{x(0) z^{n-1}}{(1-z^{-1})^2} + \frac{dz^{n-2}}{(1-z^{-1})^3} \right) dz \quad (C.2.2)$$

For  $R_1$  we can observe that the pole has been repeated twice.

$$R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (C.2.3)$$

$$R_1 = \frac{1}{(1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z-1)^2 \frac{x(0) z^{n+1}}{(z-1)^2} \right) \quad (C.2.4)$$

$$= x(0) (n+1) \lim_{z \rightarrow 1} (z^n) \quad (C.2.5)$$

$$= x(0) (n+1) \quad (C.2.6)$$

For  $R_2$  we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{dz^{n+1}}{(z-1)^3} \right) \quad (\text{C.2.7})$$

$$= \frac{d(n+1)}{2} \lim_{z \rightarrow 1} \frac{d}{dz} (z^n) \quad (\text{C.2.8})$$

$$= \frac{d(n+1)(n)}{2} \lim_{z \rightarrow 1} (z^{n-1}) \quad (\text{C.2.9})$$

$$= \frac{d(n)(n+1)}{2} \quad (\text{C.2.10})$$

$$\implies R = R_1 + R_2 \quad (\text{C.2.11})$$

Using (C.2.6) and (C.2.10) in (C.2.11)

$$R = x(0)(n+1) + \frac{d(n)(n+1)}{2} \quad (\text{C.2.12})$$

Finally,

$$s(n) = x(0)(n+1)u(n) + d \left( \frac{n(n+1)}{2} \right) u(n) \quad (\text{C.2.13})$$

$$= \frac{n+1}{2} (2x(0) + nd) u(n) \quad (\text{C.2.14})$$

C.3 Question: Find the sum of n terms of GP where common ratio is  $r$  using Contour Integration.

**Solution:**

Symbol	Value	Description
$x(n)$	$x(0)r^n u(n)$	$n^{th}$ $n^{th}$ term of gp G.P
$x(0)$	$x(0)$	$1^{st}$ term of the G.P
$d$	$r$	Common ratio
$s(n)$	$\sum_{k=0}^n x(k)$	Sum of n terms of GP

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{C.3.1}$$

$$= \sum_{n=-\infty}^{\infty} x(0)r^n u(n) z^{-n} \tag{C.3.2}$$

$$= \sum_{n=0}^{\infty} x(0)r^n z^{-n} \tag{C.3.3}$$

$$= \frac{x(0)}{1 - rz^{-1}} \tag{C.3.4}$$

$$U(z) = \frac{1}{1 - z^{-1}} \text{ if } |z| > 1 \tag{C.3.5}$$

Now we will perform inverse Z transform on  $S(z)$  using contour integration to find  $s(n)$

$$s(n) = \frac{1}{2\pi j} \oint_C S(z) z^{n-1} dz \quad (\text{C.3.10})$$

$$= \frac{1}{2\pi j} \oint_C \frac{x(0)z^{n-1}}{(1-rz^{-1})(1-z^{-1})} dz \quad (\text{C.3.11})$$

$$= \frac{1}{2\pi j} \oint_C \frac{x(0)z^{n+1}}{(z-r)(z-1)} dz \quad (\text{C.3.12})$$

$$= \frac{x(0)}{r-1} \left( \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z-r} dz - \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z-1} dz \right) \quad (\text{C.3.13})$$

we already know;

$$R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (\text{C.3.14})$$

Now for first contour integral,

$$R_1 = \frac{1}{(1-1)!} \lim_{z \rightarrow a} ((z-a)f(z)) \quad (\text{C.3.15})$$

$$= \lim_{z \rightarrow r} \left( (z-r) \frac{z^{n+1}}{z-r} \right) \quad (\text{C.3.16})$$

$$= \lim_{z \rightarrow r} (z^{n+1}) \quad (\text{C.3.17})$$

$$= r^{n+1} \quad (\text{C.3.18})$$

for second contour integral,

$$R_2 = \frac{1}{(1-1)!} \lim_{z \rightarrow a} ((z-a)f(z)) \quad (\text{C.3.19})$$

$$= \lim_{z \rightarrow 1} \left( (z-1) \frac{z^{n+1}}{z-1} \right) \quad (\text{C.3.20})$$

$$= \lim_{z \rightarrow 1} (z^{n+1}) \quad (\text{C.3.21})$$

$$= 1 \quad (\text{C.3.22})$$

So finally the sum of n terms of the GP is given by:

$$s(n) = \frac{x(0)}{r-1} (R_1 - R_2) \quad (\text{C.3.23})$$

$$= \frac{x(0)}{r-1} (r^{n+1} - 1) \quad (\text{C.3.24})$$