SIGNAL PROCESSING FUNDAMENTALS Through NCERT

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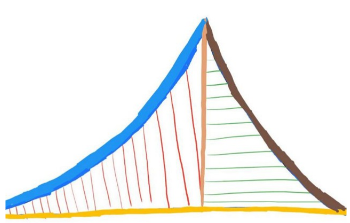


Contents

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Harmonics

1.0.1 A circular disk of mass 10kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5s. The radius of the disc is 15cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation J=-$\alpha\theta$, where J is the restoring couple and θ is the angle of twist).

Solution:

Symbols	Description	Values
T	Time period of a body following the law $x_{(t)} = a\sin\omega t$	$\frac{2\pi}{\omega}$
α	Torsional constant	(?)
m	Mass of pendulum	$9kg$
r	Radius of disc	$15cm$
I	Moment of inertia of the disc	$0.1125kgm^{-2}$
R	Residual formula for m=1	$\lim_{s\to a}\left((s-a)\theta(t)e^{st}\right)$

Table 1.1: Parameters, Descriptions, and Values

A torsional pendulum is governed by the following law:

$$J = -\alpha\theta(t) \tag{1.1}$$

From $J = I\alpha$ and $\alpha = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}$

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{\alpha\theta(t)}{I} = 0 \tag{1.2}$$

For the Laplace transform of the differential : Assuming $\theta(0) = 0$,

$$0 = s^2\mathscr{L}\left(\theta(t)\right) - s\theta(0) - \theta'(0) + \frac{\alpha\mathscr{L}\left(\theta(t)\right)}{I} \tag{1.3}$$

$$\mathscr{L}\left(\theta(t)\right) = \frac{\theta'(0)}{s^2 + \frac{\alpha}{I}} \tag{1.4}$$

For the Inverse Laplace transform of $\mathscr{L}\left(\theta(t)\right)$: Using Bromwich integral,

$$\theta(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \mathscr{L}\left(\theta(t)\right) e^{st}\,dt, c > 0 \tag{1.5}$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{\theta'(0)e^{st}}{s^2 + \frac{\alpha}{I}}\,dt \tag{1.6}$$

Since the poles $s = j\sqrt{\frac{\alpha}{I}}$ and $s = -j\sqrt{\frac{\alpha}{I}}$ (both non repeated, i.e. $m = 1$) lie inside a semicircle for some $c > 0$. Using Jordans lemma and method of residues from 1.1 :

$$R_1 = \lim_{s \to j\sqrt{\frac{\alpha}{I}}} \left(\left(s - j\sqrt{\frac{\alpha}{I}} \right) \left(\frac{\theta'(0)}{s^2 + \frac{\alpha}{I}} \right) e^{st} \right) \tag{1.7}$$

$$= \left(\frac{\theta'(0) I e^{j\sqrt{\frac{\alpha}{I}}t}}{2j\alpha} \right) \tag{1.8}$$

$$R_2 = \lim_{s \to -j\sqrt{\frac{\alpha}{I}}} \left(\left(s + j\sqrt{\frac{\alpha}{I}} \right) \left(\frac{\theta'(0)}{s^2 + \frac{\alpha}{I}} \right) e^{st} \right) \tag{1.9}$$

$$= \left(\frac{-\theta'(0) I e^{-j\sqrt{\frac{\alpha}{I}}t}}{2j\alpha} \right) \tag{1.10}$$

$$\theta(t) = R_1 + R_2 \tag{1.11}$$

$$\theta(t) = \left(\frac{\theta'(0) I \left(e^{j\sqrt{\frac{\alpha}{I}}t} - e^{-j\sqrt{\frac{\alpha}{I}}t} \right)}{2j\alpha} \right) \tag{1.12}$$

$$\implies \theta(t) = \theta'(0) \sin\left(t\sqrt{\frac{\alpha}{I}} \right) \tag{1.13}$$

For calculating the torsional constant : From table 1.1.

$$T = 2\pi\sqrt{\frac{I}{\alpha}} \tag{1.14}$$

$$\implies \alpha = 1.972Nms^{-1} \tag{1.15}$$

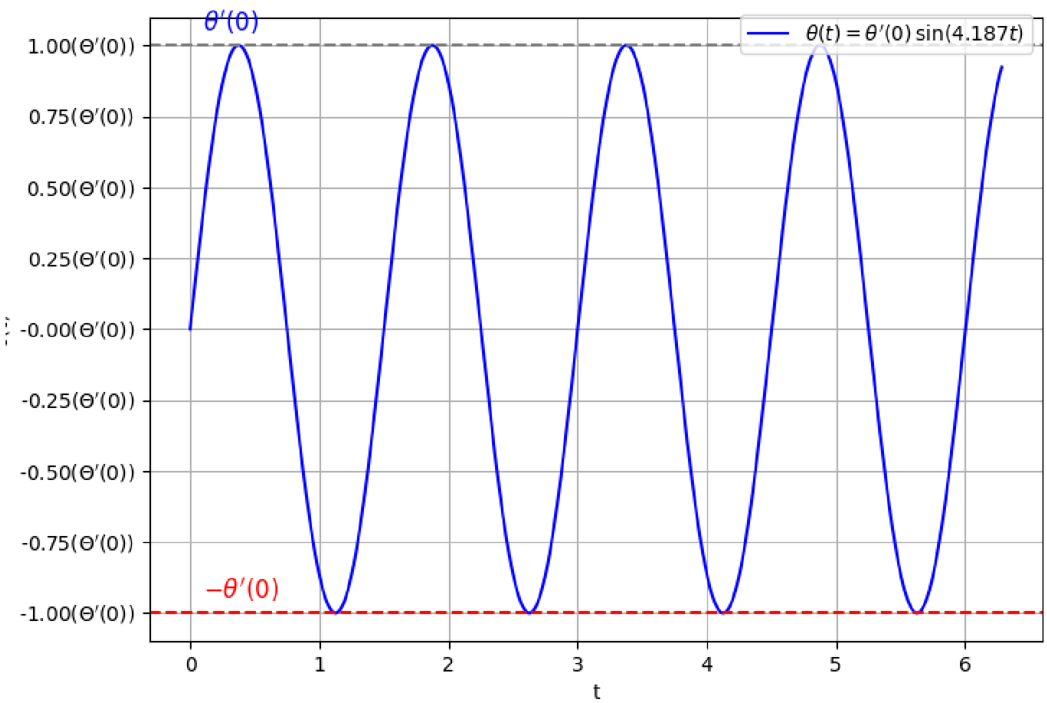


Figure 1.1: $\theta(t)$ vs t

1.0.2 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0$ MHz.

(a) Determine, B_0, ω, k and λ

(b) Find expressions for **E** and **B**

Solution:

Table 1.2: Input Parameters

Symbol	Description	value
f	frequency of source	50.0 MHz
E_0	Electric field amplitude	120 N/C
c	speed of light	3 x 10^8 m/s
$\mathbf{e_2}, \mathbf{e_3}$	Standard Basis vectors	N/A

1.0.3 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Table 1.3: Formulae and Output

Symbol	Description	Formula	Value
$\mathbf{E}$	Electric field vector	$E_0 \sin(kx - 2\pi ft)\mathbf{e_2}$	$120 \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_2}$
$\mathbf{B}$	Magnetic field vector	$B_0 \sin(kx - 2\pi ft)\mathbf{e_3}$	$(4x10^{-7}) \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_3}$
B_0	Magnetic field strength	$\frac{E_0}{c}$	400nT
ω	Angular frequency	$2\pi f$	3.14 x 10^8m/s
k	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
λ	Wavelength	$\frac{c}{f}$	6.0m

1.0.4 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

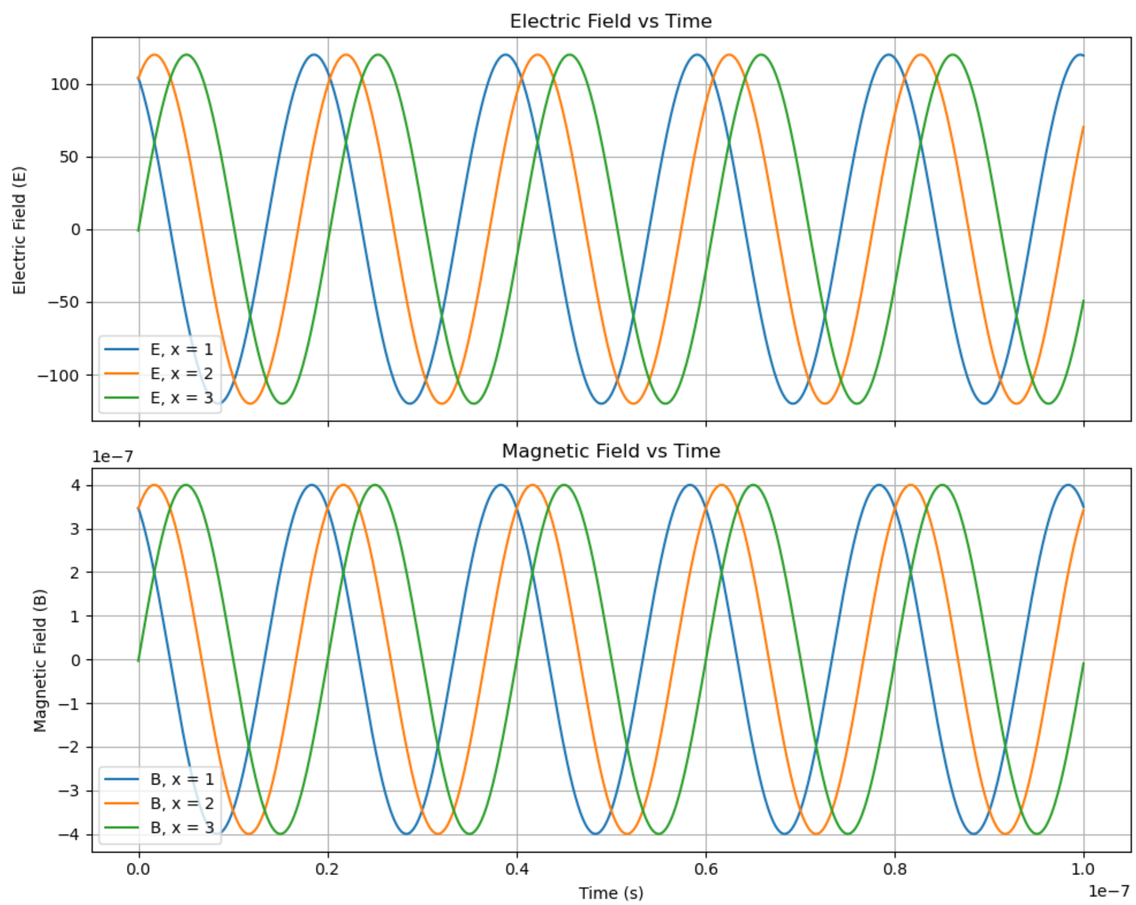


Figure 1.0.2: Graphs of **E** and **B**

Let us assume an equation:

$$y = A(x)\cos\left(\omega t + \phi\left(x\right)\right) \tag{1.16}$$

Fig. 1.0.4 and Fig. 1.0.4 are self explanatory for stationary and travelling waves.Fig. 1.0.4 and Fig. 1.0.4 are neither stationary nor travelling waves.

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.0.3: Variable description

TRAVELLING WAVE	STATIONARY WAVE
$y(x,t) = A\sin(kx \pm \omega t)$	$y(x,t) = A\sin kx \cos \omega t$

PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.0.4: Travelling wave *vs* Stationary wave

1.0.5 For the travelling harmonic wave $y(x,t) = 2.0\cos 2\pi(10t - 0.0080x + 0.35)$ where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi f t - kx_1 + \phi) - (2\pi f t - kx_2 + \phi) \tag{1.17}$$

$$= k(x_2 - x_1) \tag{1.18}$$

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t+\alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx+\theta$ where k is the wavenumber and θ is the phaseconstant.

Table 1.0.4: Travelling wave *vs* Stationary wave

Parameter	**Description**	**Value**
$y(x_i, t)$	equation of harmonic wave	$A\cos(2\pi ft - kx_i + \phi)$
k	angular wave number	$2\pi(0.008)$
$\lambda = \frac{2\pi}{k}$	wavelength	$125\,cm$
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	$2\pi(0.35)$
θ_i	phase of i^{th} harmonic wave	$(2\pi ft - kx + \phi)$
x_i	position of i^{th} harmonic wave	
t	time	
$x_2 - x_1$	path difference	$400\,cm$
		$50\,cm$
		$\frac{\lambda}{2}$
		$\frac{3\lambda}{4}$

Table 1.0.5: Given parameters list

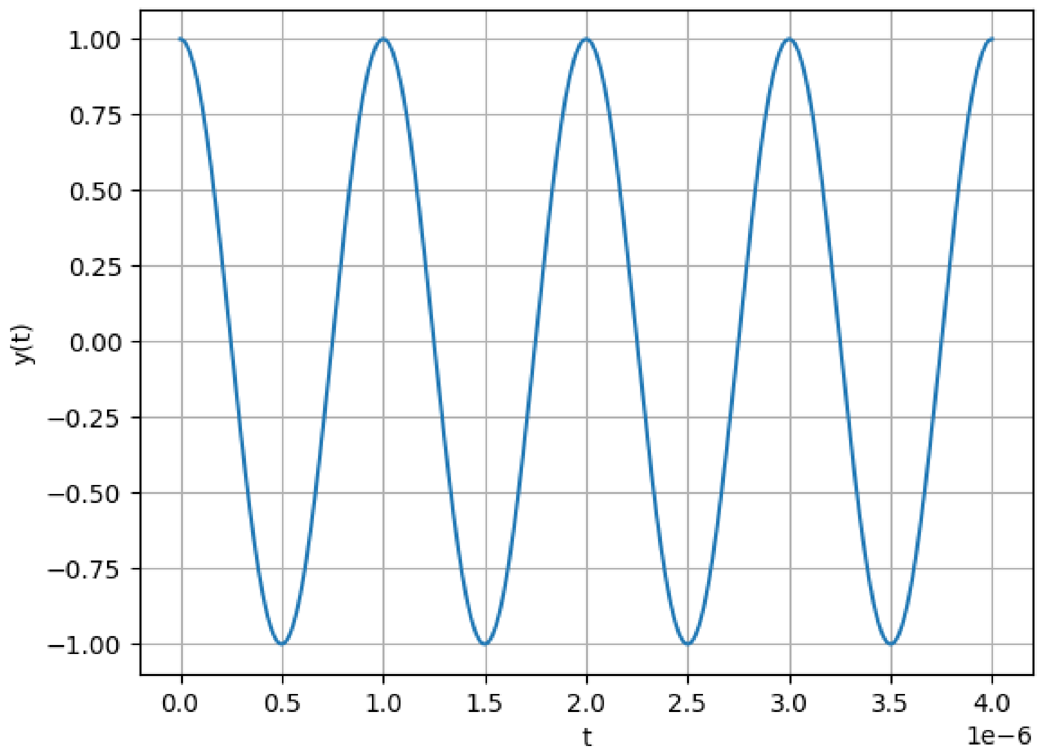


Figure 1.0.3: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

1.0.6 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

Solution:

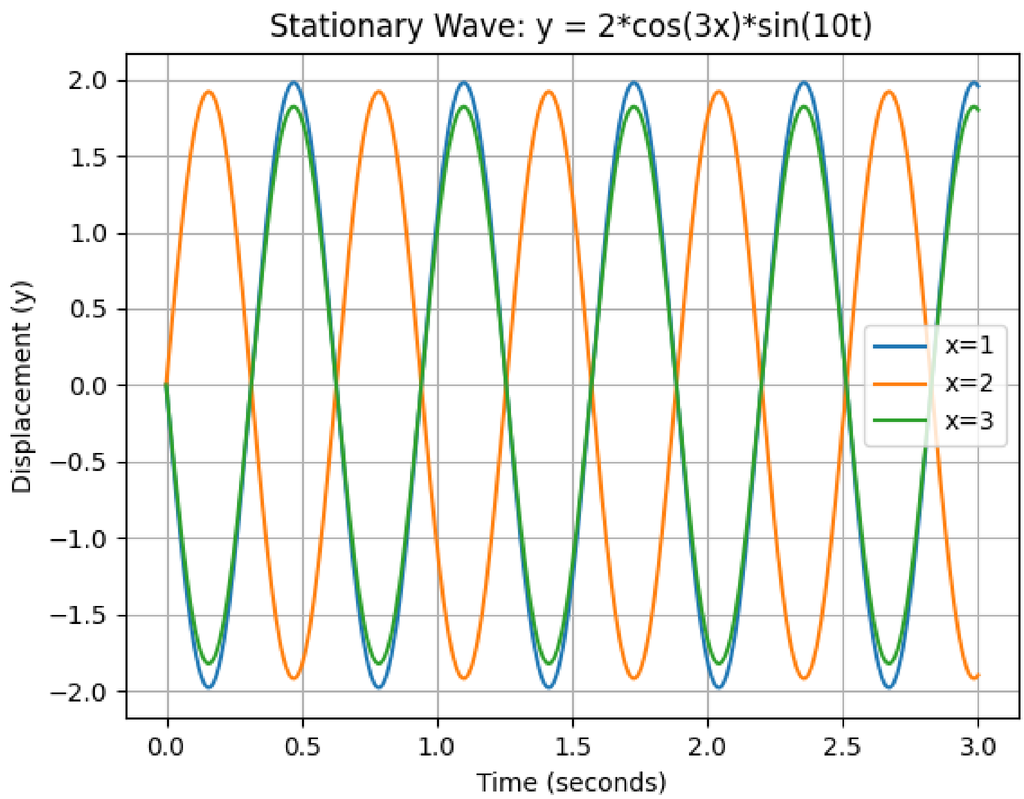


Figure 1.0.4: DIPLACEMENT vs TIME-graph1

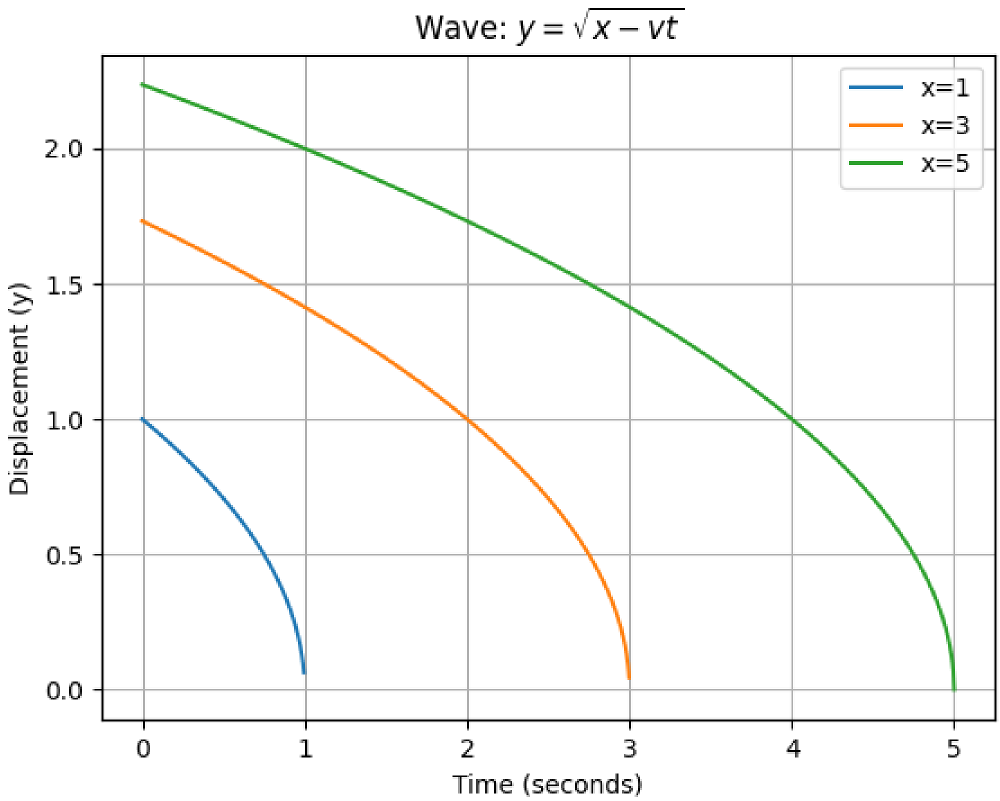


Figure 1.0.4: DIPLACEMENT vs TIME-graph2

(a)

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T [V(t)]^2\,dt \tag{1.19}$$

$$= f\int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi f t + \phi)\,dt \tag{1.20}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\int_0^{\frac{1}{f}} \cos(4\pi f t + 2\phi)\,dt\right) \tag{1.21}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.22}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin\left(4\pi + 2\phi\right) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.23}$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \tag{1.24}$$

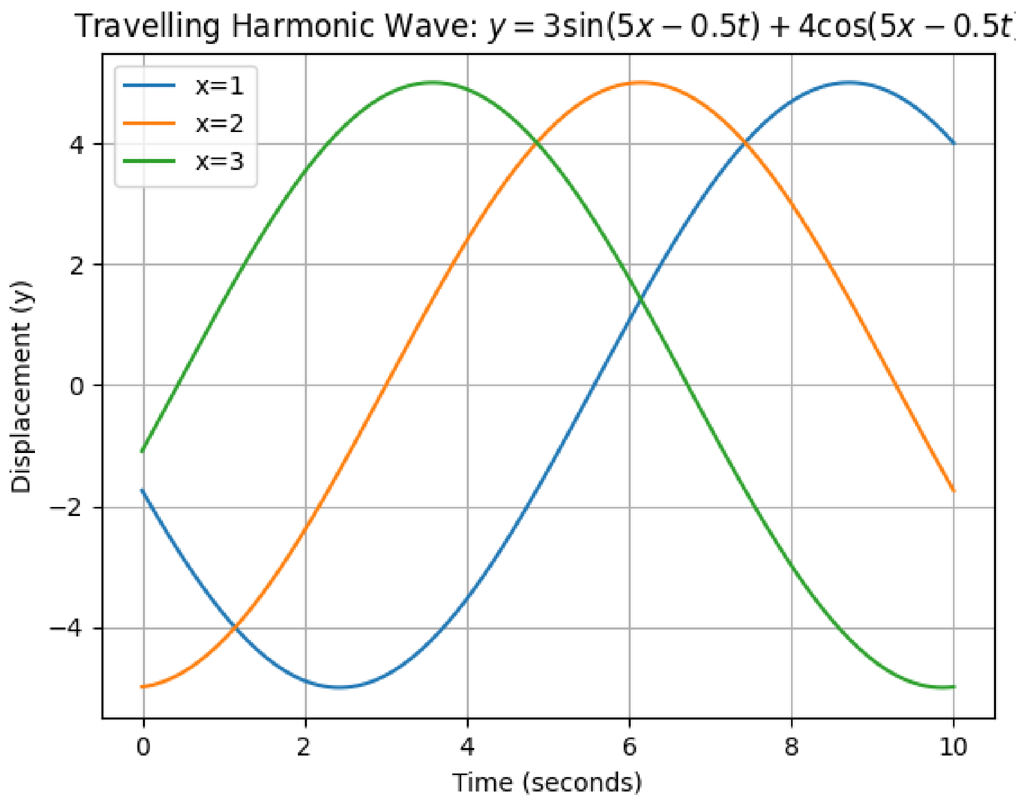


Figure 1.0.4: DIPLACEMENT vs TIME-graph3

To find the RMS voltage (V_{rms}) when the peak voltage (V_0) is 300V, you can use equation (1.24)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \tag{1.25}$$

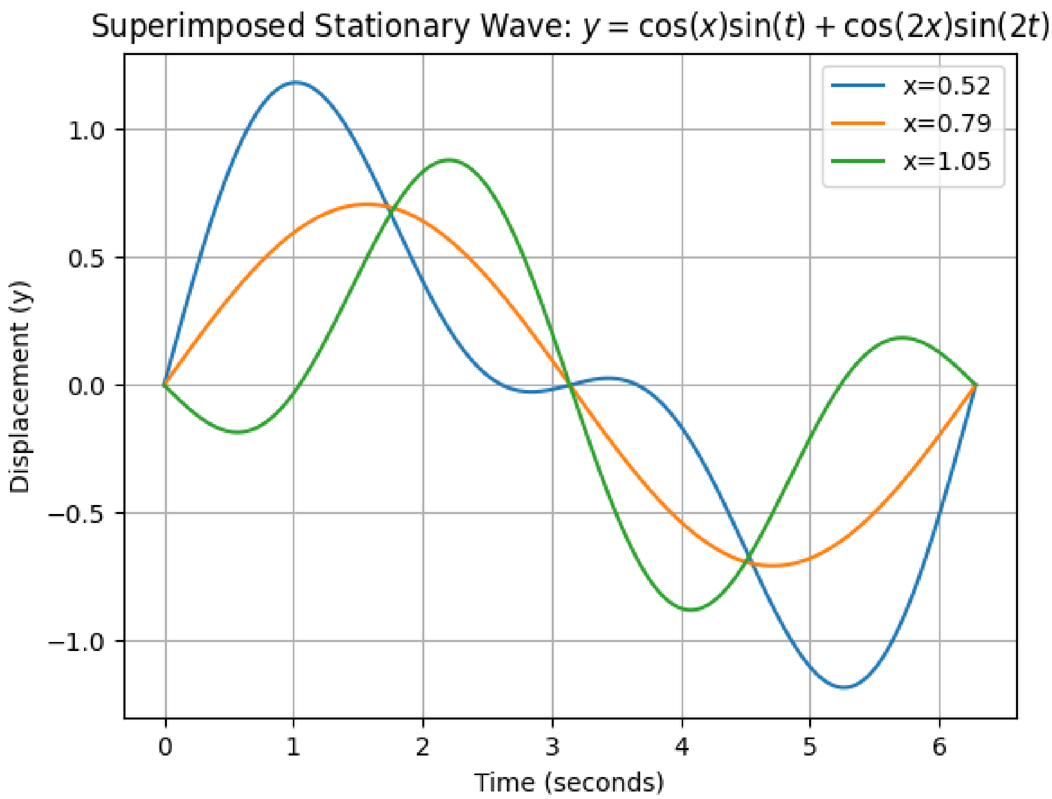


Figure 1.0.4: DIPLACEMENT *vs* TIME-graph4

Parameter	**Description**	**subquestion**	**Value**
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\frac{3\pi}{2}$ radians

Table 1.0.5: Phase differences

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi f t + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi f t + \phi)$	current in terms of time
V_0	$300\,\mathrm{V}$	peak voltage
V_{rms}	$\sqrt{\frac{1}{T}\int_0^T [V(t)]^2\,dt}$	rms value of Voltage
I_{rms}	$10\,\mathrm{A}$	rms value of current
I_0	$\sqrt{2} \times I_{\text{rms}}$	peak current
f	$50\,\mathrm{Hz}$	frequency of the sinusoidal wave
T	$0.02\,\mathrm{s}$	time period of sinusoidal wave

Table 1.0.6: Input Parameter Table

(b)

$$I_{\text{rms}}^2 = \frac{1}{T}\int_0^T [I(t)]^2\,dt \tag{1.26}$$

$$= f\int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi f t + \phi)\,dt \tag{1.27}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.28}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.29}$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \tag{1.30}$$

To find the peak current (I_0) when the RMS current (I_{rms}) is given, you can use equation (1.30)

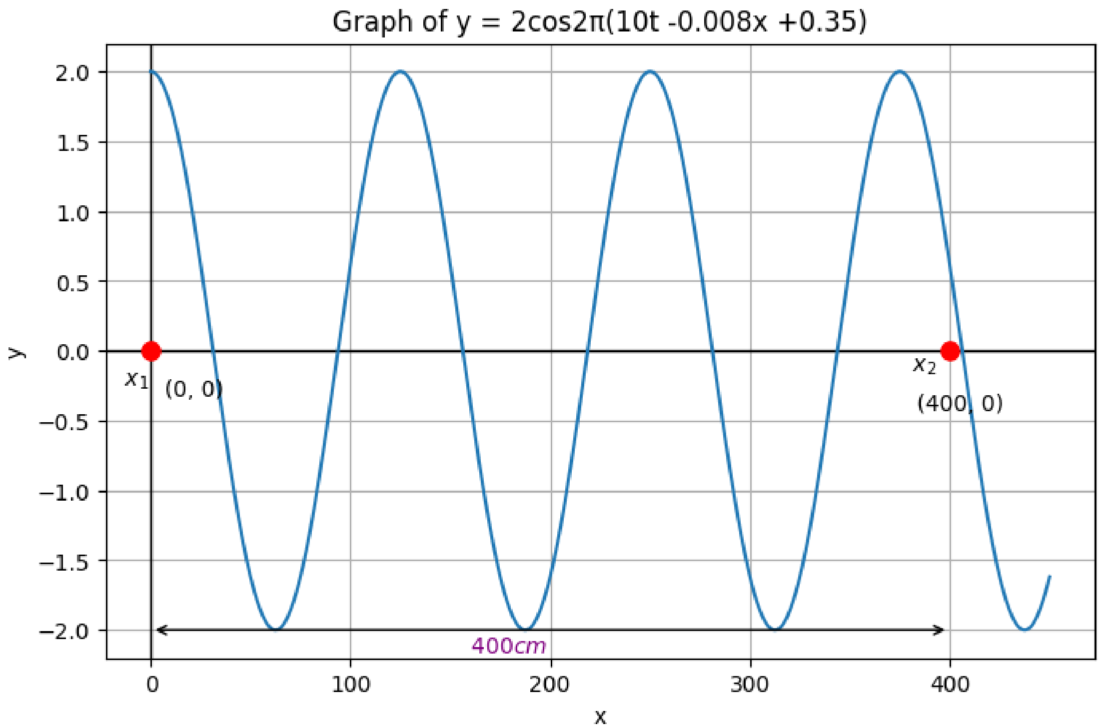
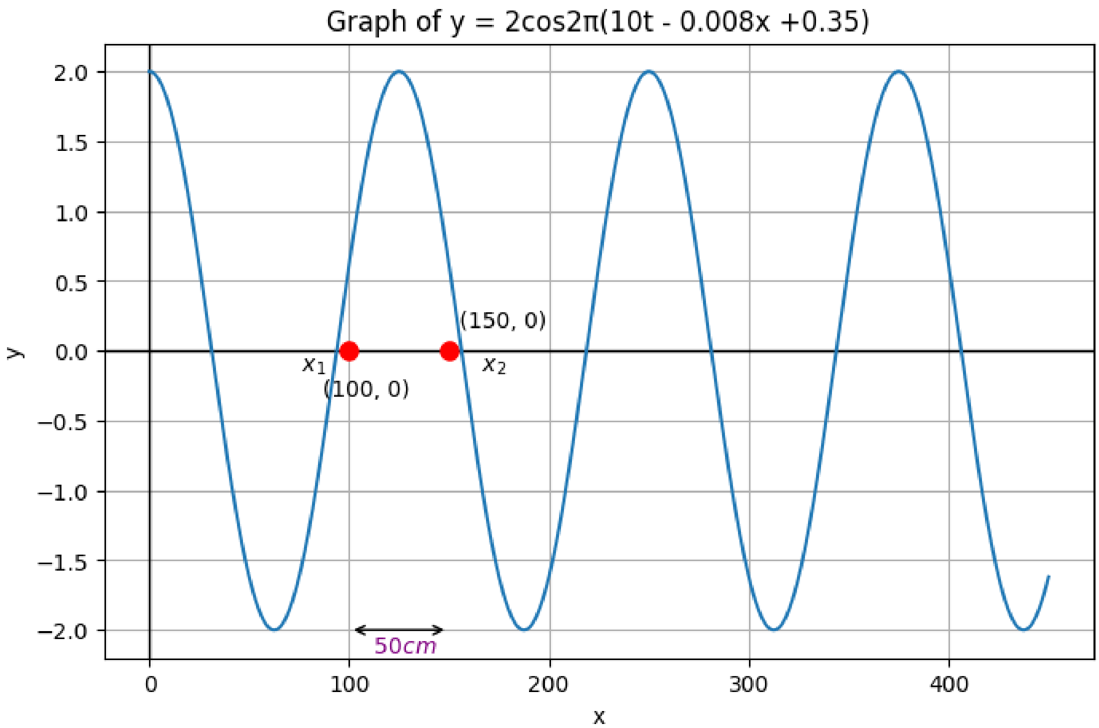


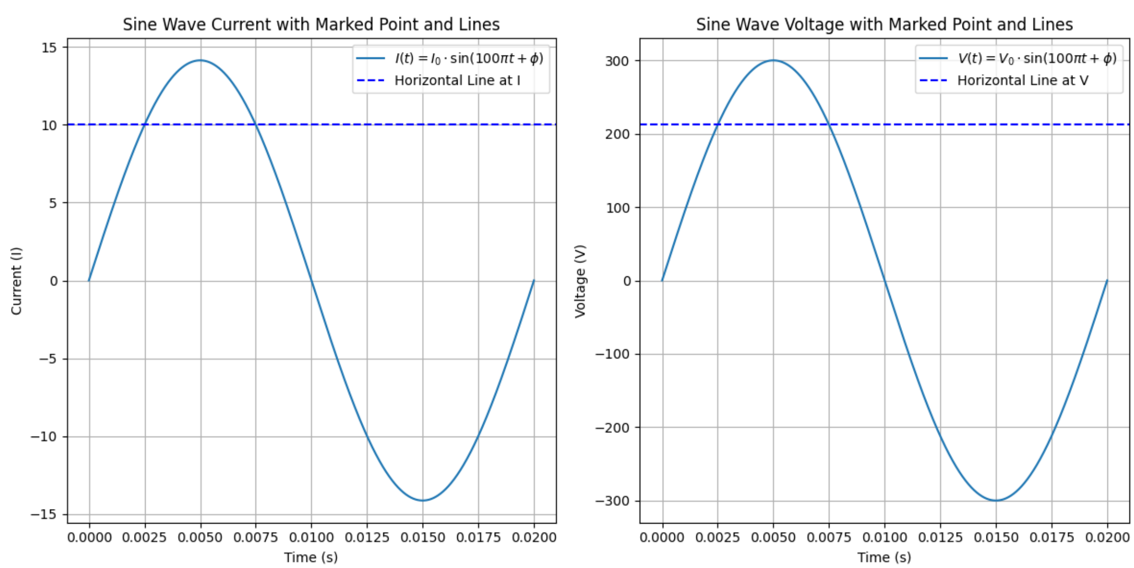
Figure 1.0.5:

$$I_0 \approx 10\,\mathrm{A} \times 1.414 \approx 14.14\,\mathrm{A} \tag{1.31}$$



Graph of y = 2cos2π(10t - 0.008x +0.35)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y
(150, 0)
x_1
x_2
(100, 0)
$50cm$
0
100
200
300
400
x

Figure 1.0.5:



Sine Wave Current with Marked Point and Lines
$I(t) = I_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at I
Current (I)
Time (s)
Sine Wave Voltage with Marked Point and Lines
$V(t) = V_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at V
Voltage (V)
Time (s)

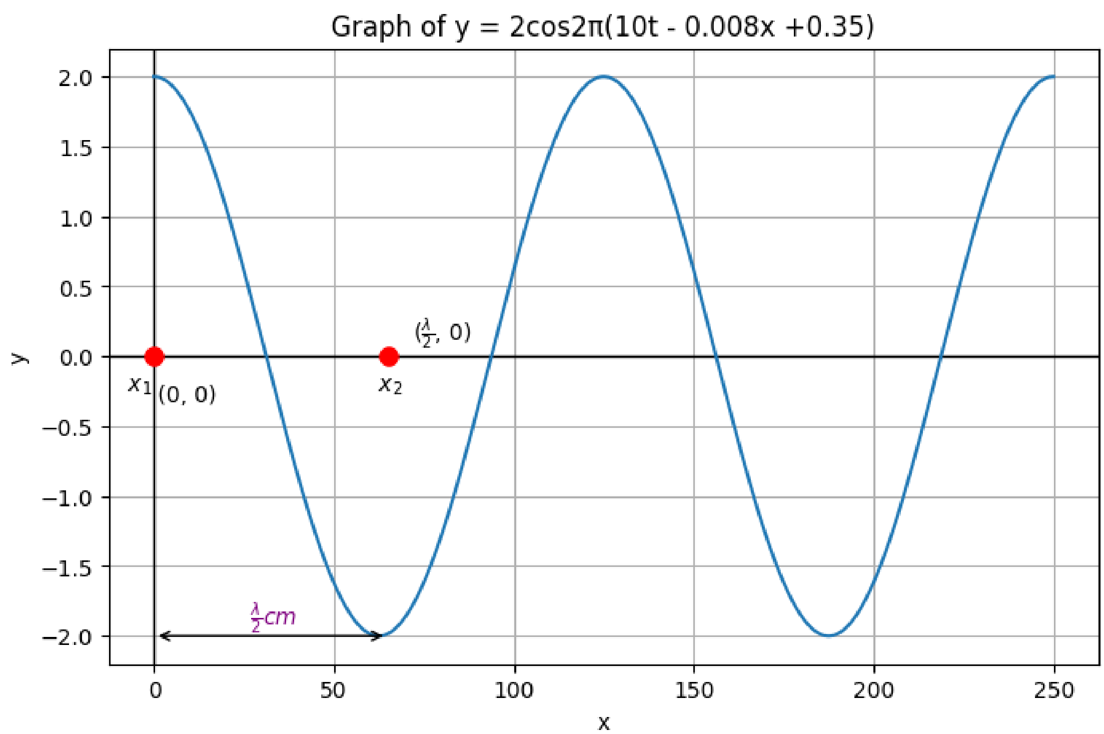


Figure 1.0.5:

1.0.7 In Young's double-slit experiment using monochromatic light of wavelength λ, the intensity of light at a point on the screen where path difference is λ, is K units. What is the intensity of light at a point where path difference is $\lambda/3$?

Solution:

From Table 1.0.7:

$$y(t) = A\sin(2\pi ft - kx_1) + A\sin(2\pi ft - kx_2) \tag{1.32}$$

$$y(t) = 2A\cos\left(\frac{k\Delta x}{2}\right)\sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \tag{1.33}$$

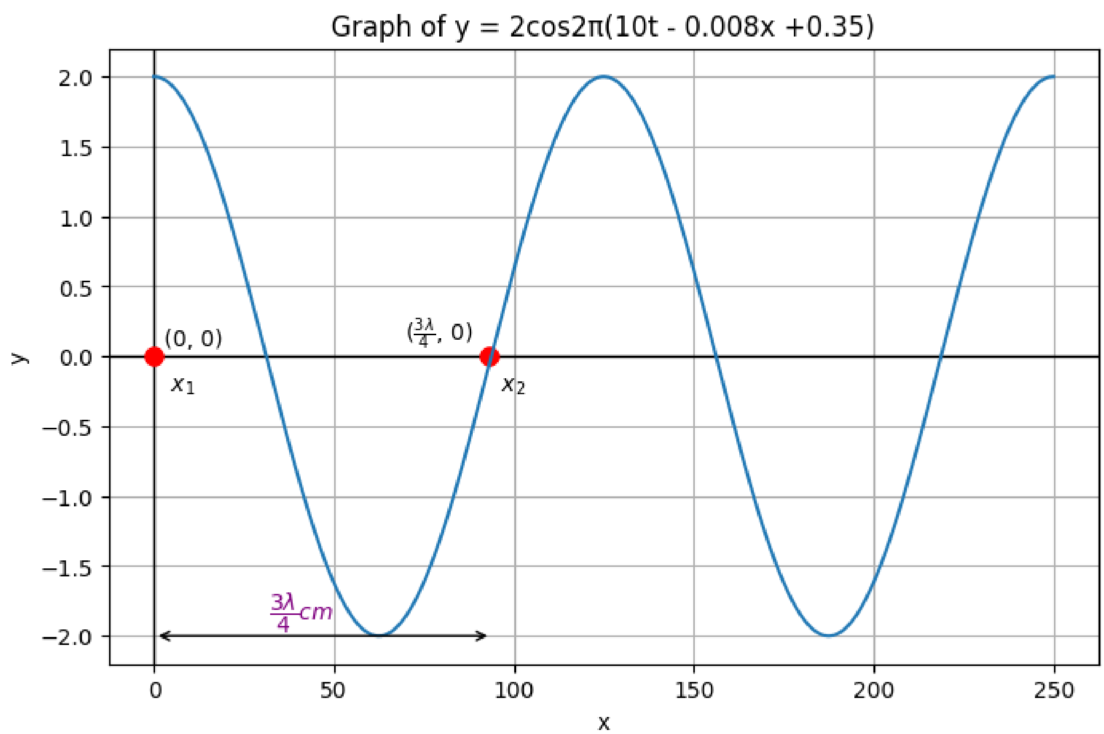


Figure 1.0.5:

From Table 1.0.7 and equation (1.33):

$$\therefore I \propto 4A^2 \cos^2 \left(\frac{k\Delta x}{2} \right) \tag{1.34}$$

From Table 1.0.7 and equation (1.34):

$$\frac{K}{I_r} = \frac{4A^2 \cos^2 \left(\frac{2\pi}{2} \right)}{4A^2 \cos^2 \left(\frac{\pi}{3} \right)} \implies I_r = \frac{K}{4} \tag{1.35}$$

Hence, the Intensity of light at a point where path difference is $\frac{\lambda}{3}$ is $\frac{K}{4}$ units.

Parameter	**Description**	**Value**
$y_i\left(t\right)$	Equation of light from $S_{i^{\text{th}}}$	$A\sin(\omega t - kx_i)$
k	Wave number	$\frac{2\pi}{\lambda}$
I	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	λ
		$\frac{\lambda}{3}$
K	Intensity of light at $\Delta x = \lambda$	
A	Amplitude of wave from source	
r	constant	$r \geq 0$

Table 1.0.7: Parameters

Parameter	**Description**	**Value**
I_r	Net Intensity of light at $\Delta x = \frac{\lambda}{3}$	$\frac{K}{4}$

Table 1.0.7:

Assuming $\Delta x = r\lambda$,

From equation (1.34):

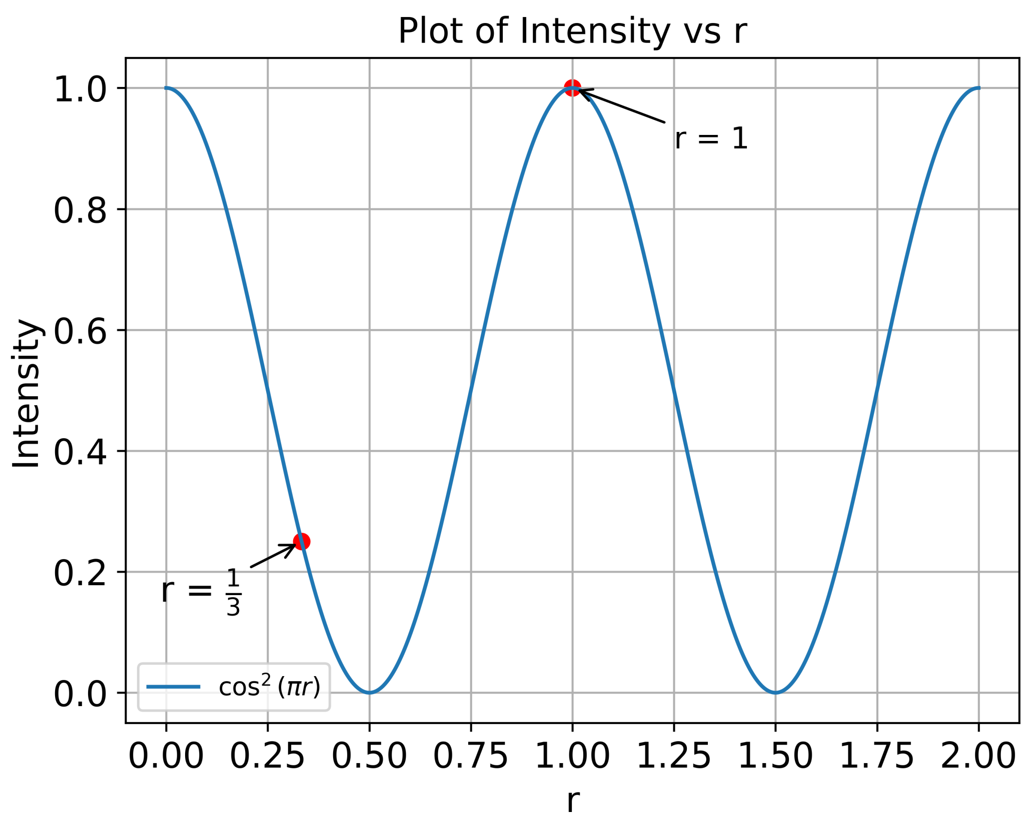


Figure 1.0.7:

1.0.8 In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of 2.0 x 10^{10} Hz and amplitude 48 Vm^{-1}.

(a) What is the wavelength of the wave?

(b) What is the amplitude of the oscillating magnetic field?

(c) Show that the average energy density of the **E** field equals the average energy density of the **B** field. [$c = 3$ x $10^8 ms^{-1}$]

1.0.9 (a) For the wave on the string $y(x,t) = 0.06\sin(\frac{2\pi x}{3})\cos(120\pi t)$, do all the points on the string oscillate with the same (a)frequency , (b)phase , (c)amplitude ? Explain your answers.

(b) What is the amplitude of a point 0.375m away from one end?

Solution:

1.0.10 A transverse harmonic wave on a string is described by

$$y(x,t) = 3.0 \sin\left(36t + 0.018x + \frac{\pi}{4}\right) \tag{1.36}$$

where x and y are in cm and t in s. The positive direction of x is from left to right.

(a) Is this a travelling wave or a stationary wave? If it is travelling, what are the speed and direction of its propogation?

(b) What are its amplitude and frequency?

(c) What is the initial phase at the origin?

(d) What is the least distance between two succesive crests in the wave?

Solution:

1.0.11 In deriving the single slit diffraction pattern, it was stated that the intensity is zero at angles of $\frac{n\lambda}{a}$. Justify this by suitably dividing the slit to bring out the cancellation.

Solution:

1.0.12 A 60 μ F capacitor is connected to a 110 V, 60 Hz ac supply. Determine the rms value of the current in the circuit.

Solution:

1.0.13 A charged $30\mu F$ capacitor is connected to a $27mH$ inductor. What is the angular frequency of free oscillations of the circuit?

Solution:

Symbol	**Description**	**Value**
C	Capacitance	$30\mu F$
L	Inductance	$27mH$
ω_0	Angular Frequency	??

Table 1.0.13: Input Parameters

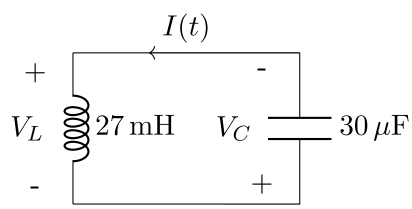


Figure 1.0.13: LC Circuit Diagram

at $t = 0^-$ $\quad V_C = -V_0, \quad I(0) = 0, \quad V_L = V_0$

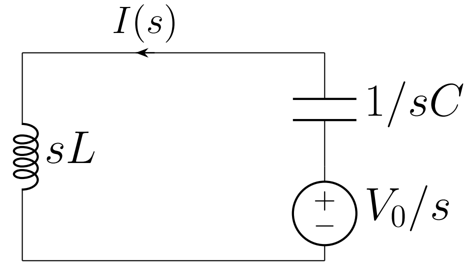


Figure 1.0.13: LC Circuit Diagram in S domain

$$sLI(s) + \frac{1}{sC}I(s) = \frac{V_0}{s} \tag{1.37}$$

$$I(s) = \frac{V_0 C}{s^2 LC + 1} \tag{1.38}$$

$$\mathcal{L}^{-1}\{I(s)\} = I(t) \tag{1.39}$$

$$I(t) = \frac{V_0 C}{\sqrt{LC}} \sin\left(t\frac{1}{\sqrt{LC}}\right) \tag{1.40}$$

$$I(t) = V_0\sqrt{\frac{C}{L}} \sin\left(t\frac{1}{\sqrt{LC}}\right) \tag{1.41}$$

Net impedence of LC circuit

$$Z = R_L + R_C \tag{1.42}$$

$$= Ls + \frac{1}{sC} \tag{1.43}$$

At resonance, the resistance of capacitor and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{1.44}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{1.45}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{1.46}$$

on comparing (1.45) and (1.46)

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.47}$$

$$\omega_0 = \frac{1}{\sqrt{(30 \times 10^{-6}) \times (27 \times 10^{-3})}} \tag{1.48}$$

$$= \frac{1}{\sqrt{8.1 \times 10^{-7}}} \tag{1.49}$$

$$\approx 1.11 \times 10^{3}\,\text{rad/s} \tag{1.50}$$

Assuming $V_0 = 1$volt

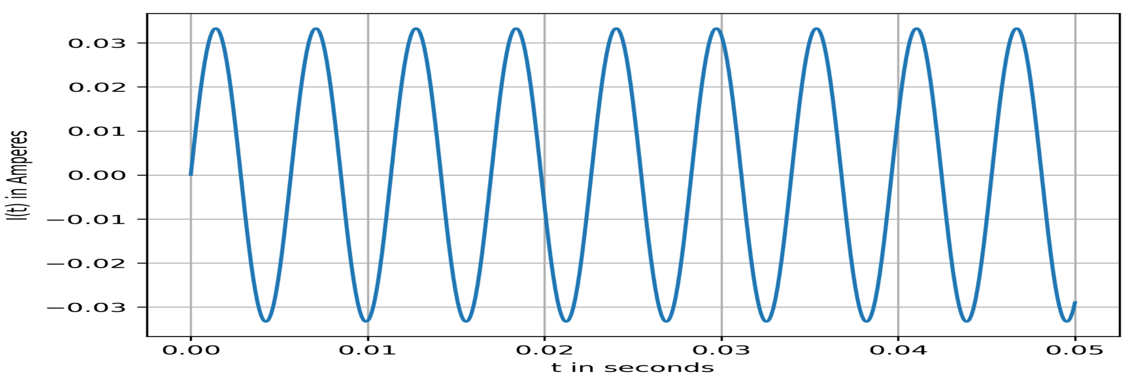


Figure 1.0.13: I(t) vs t

1.0.14 Obtain the resonance frequency of a series LCR circuit with $L = 2.0\,H$, $C = 32\,\mu F$, and $R = 10\,\Omega$. What is the Q-value of the circuit.

Solution:

1.0.15 A charged 30 μF capacitor is connected to a 27 mH inductor. Suppose the initial charge on the capacitor is 6mC.What is the total energy stored in the circuit initially? What is the total energy at later time?

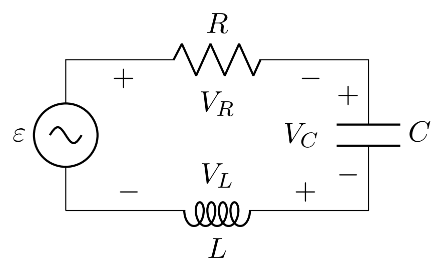
Solution:

1.0.16 A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5×10^{-2} kg, and its linear mass density is 4.0×10^{-2} kg/m. The length of the wire is 0.875 m. Determine the speed of a transverse wave on the string and the tension in the string.

Solution:

1.0.17 The given figure shows a series LCR circuit connected to a variable frequency 230 V source.

L = 5.0 H, C = 80 μF, R = 40 Ω.



(a) Determine the source frequency which drives the circuit in resonance.

(b) Obtain the impedance of the circuit and the amplitude of current at the resonating frequency.

(c) Determine the rms potential drops across the three elements of the circuit. Show that the potential drop across the LC combination is zero at the resonating frequency.

Solution:

1.0.18 Q23) A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium.

(a) Does the pulse have a definite (i) frequency, (ii) wavelength, (iii) speed of propagation?

(b) If the pulse rate is 1 after every 20 s, (that is the whistle is blown for a split of second after every 20 s), Is the frequency of note produced by whistle equal to 1/20 or 0.05 Hz ?

Solution:

1.0.19 Suppose that the electric field part of an electromagnetic wave in vacuum given as

$\mathbf{E}$ =$\{$(3.1N/C)cos[(1.8 rad/m)y+(5.4×10^6rad/s)t]$\}\mathbf{e_1}$

(a) What is the direction of propagation ?

(b) What is the wavelength ?

(c) What is the frequency ?

(d) What is the amplitude of the magnetic field part of the wave?

(e) Write an expression for the magnetic field part of the wave.

Solution:

Symbol	Values	Description
λ	$\frac{2\pi}{k}$	Wave length of E.M wave.
f	$0.859 \times 10^6 Hz$	Frequency of E.M wave.
c	$3 \times 10^6 m/s$	Velocity of propagation of E.M wave.
ω	$2\pi f$	Angular frequency of E.M wave.
k	$1.8 rad/m$	Wave number of E.M wave
B_o	$\frac{E_o}{c}$	Amplitude of magnetic part of E.M wave
E_o	$3.1 N/C$	Amplitude of electric part of E.M wave.
$\mathbf{e_1}$	-	Base vector in direction of electric field.
$\mathbf{e_2}$	-	Base vector in direction of propagation.
$\mathbf{e_3}$	-	Base vector in direction of magnetic field.

Table 1.0.19: Input Parameters

(a)

As the wave is in form of $cos(ky + wt)$ the wave is propagating along $-y$ axis, represented by $\mathbf{e_2}$

(b)

$$k = \frac{2\pi}{\lambda} \tag{1.51}$$

$$\Rightarrow \lambda = \frac{2\pi}{1.8} \tag{1.52}$$

$$\approx 3.5m \tag{1.53}$$

(c)

$$\omega = 2\pi.f \tag{1.54}$$

$$5.4x10^6 = 2.\pi.f \tag{1.55}$$

$$\Rightarrow f = 0.859 \times 10^6 Hz \tag{1.56}$$

(d)

$$B_o = \frac{E_o}{c} \tag{1.57}$$

where c is velocity of propagation of wave which is given by

$$c = \frac{\omega}{k} \tag{1.58}$$

$$= \frac{5.4 \times 10^6}{1.8} \tag{1.59}$$

$$= 3 \times 10^6 m/s. \tag{1.60}$$

$$B_o = \frac{3.1}{3 \times 10^6} \tag{1.61}$$

$$= 1.03 \times 10^{-6} T \tag{1.62}$$

(e) Direction of magnetic field is $\mathbf{e_3}$

where,

$$\mathbf{e_3} = \mathbf{e_2} \times \mathbf{e_1} \tag{1.63}$$

$$\mathbf{B} = B_o cos(ky + wt)\mathbf{e_3} \tag{1.64}$$

From (1.62),(1.64)

$$\mathbf{B} = 1.03 \times 10^{-6} T\{cos[(1.8rad/m)y + (5.4x10^6 rad/s)t]\}\mathbf{e_3} \tag{1.65}$$

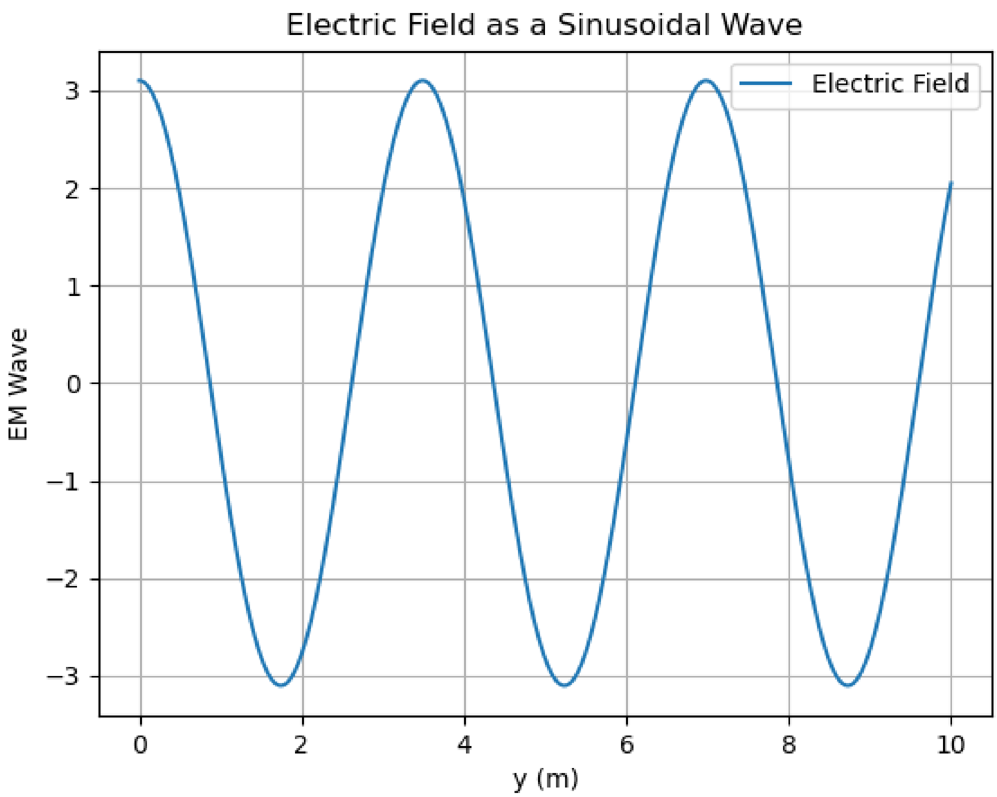


Figure 1.0.19: Electric field part

1.0.20 A 44 mH inductor is connected to 220 V, 50 Hz ac supply. Determine the rms value of the current in the circuit.

Solution:

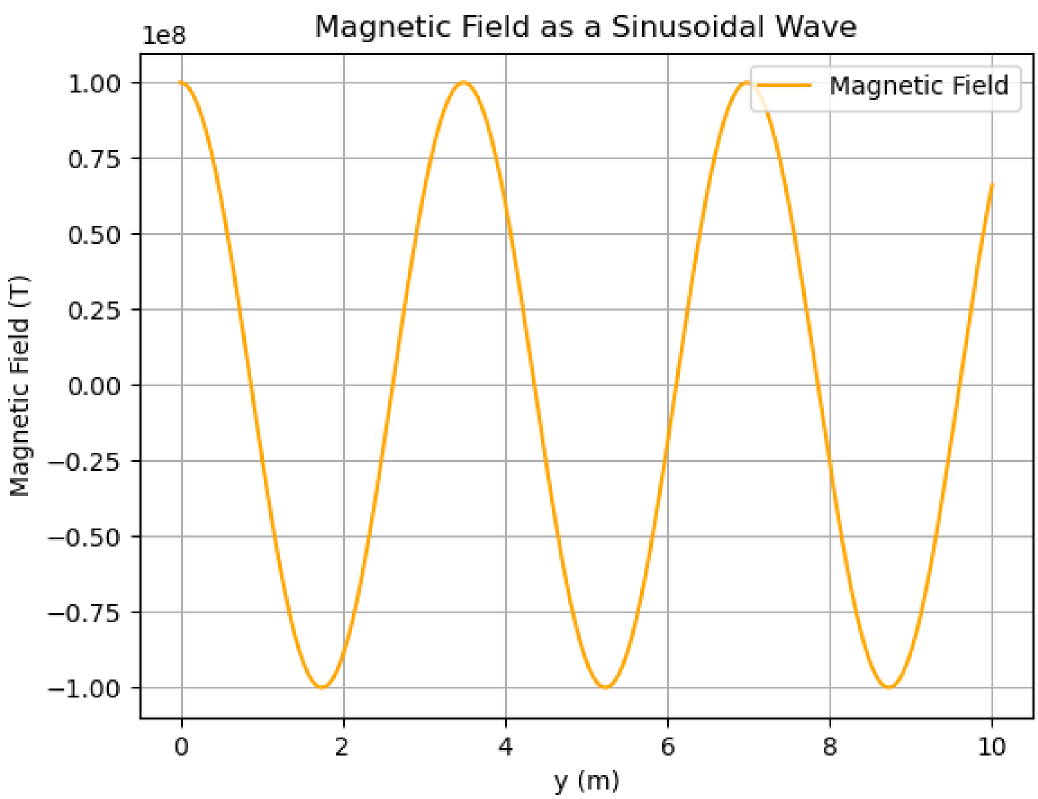


Figure 1.0.19: Magnetic field part

$$V = I\left(j\omega L\right) \tag{1.66}$$

$$I = \frac{V}{j\omega L}A \tag{1.67}$$

$$I = \frac{220\sqrt{2}}{j\left(314\right)\left(44\text{x}10^{-3}\right)}A \tag{1.68}$$

$$I = \frac{22.52}{j}A \tag{1.69}$$

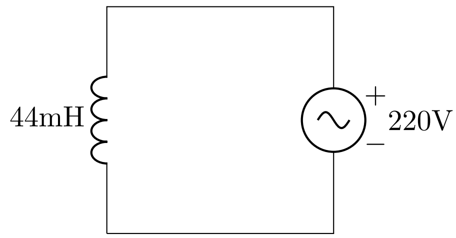


Figure 1.0.20: Circuit-1

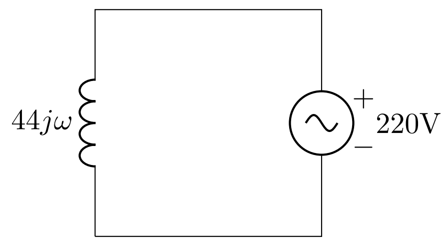


Figure 1.0.20: Circuit-2

$$I_{rms} = \frac{I}{\sqrt{2}}A \tag{1.70}$$

$$I_{rms} = \frac{15.92}{j}A \tag{1.71}$$

$$|I_{rms}| = 15.92A \tag{1.72}$$

Table 1.0.20: Input Parameters

Symbol	Description	value
L	Inductor	44mH
V_{rms}	RMS Voltage	220V
f	Frequency	50Hz

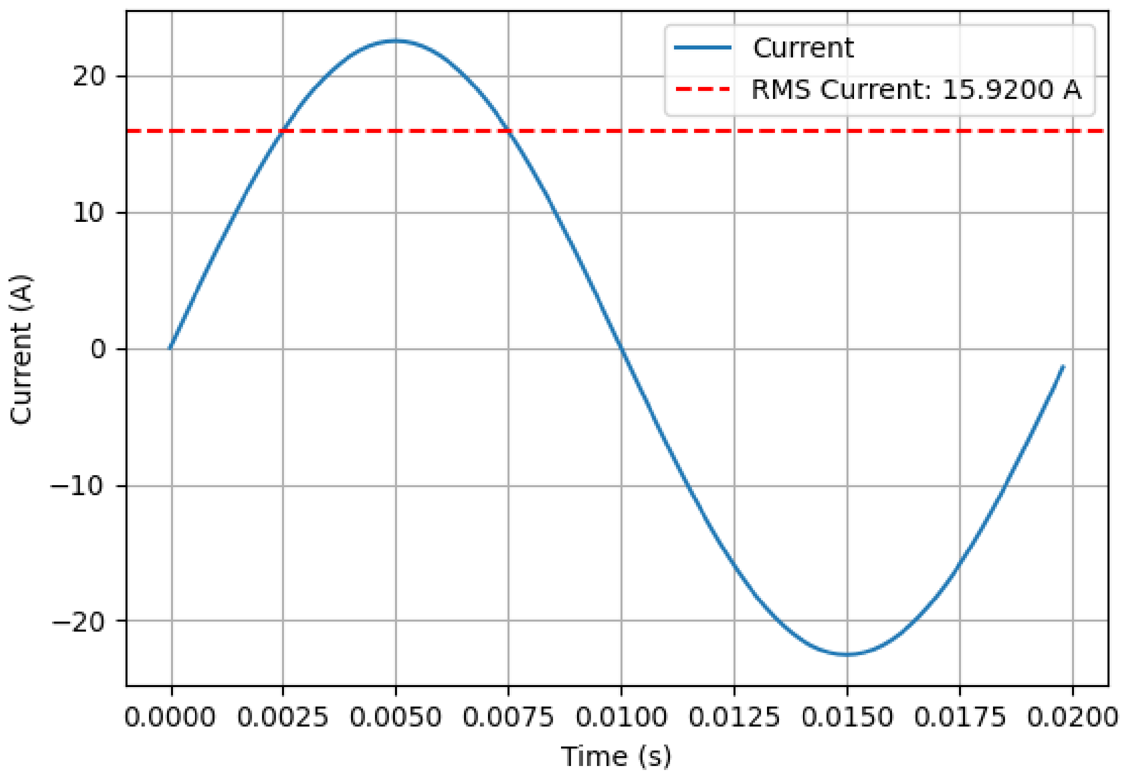


Figure 1.0.20: Plot of I vs time

1.0.21 The 6563 Å Hα line emitted by hydrogen in a star is found to be redshifted by 15 Å. Estimate the speed with which the star is receding from the Earth. **Solution:**

Table 1.0.20: Formulae and output

Symbol	Description	Formulae	Value
X_L	Inductive Reactance	2 $\pi f L$	13.816 Ω
ω	Angular Frequency	$2\pi f$	314 rad/sec
I_{rms}	Rms current	$\frac{V}{X_L}$	15.92A

1.0.22 The amplitude of the magnetic part of a harmonic elctromagnetic wave is $B_0 =$ 510nT.What is the amplitude of the electric part of the electromagnetic wave.

Solution:

$$\frac{E_0}{B_0} = c \tag{1.73}$$

$$E_0 = c * B_0 \tag{1.74}$$

$$E_0 = 153V - m \tag{1.75}$$

Parameter	Description	Value
B_0	Amplitude of the Electric Field	510nT
c	Speed of Electro Magnetic Wave	$3 \times 10^8 ms^{-1}$
E_0	amplitude of the Electric Field	153V-m

Table 1.0.22: Parameter Table

Consider the general equation of Electric Field and Magnetic field

$$E = E_0 \sin(\omega t - kx) \tag{1.76}$$

$$B = B_0 \sin(\omega t - kx) \tag{1.77}$$

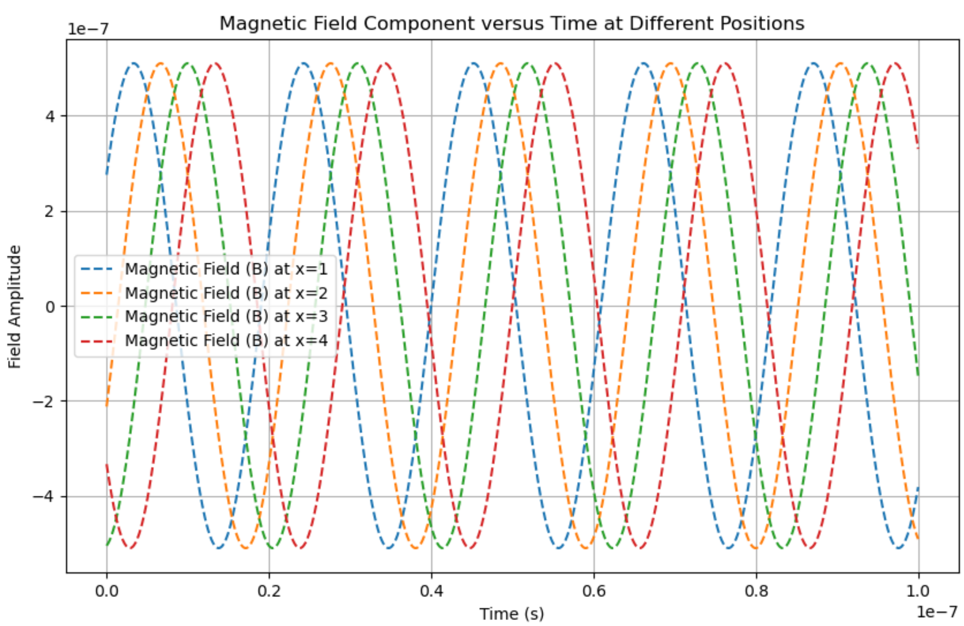


Figure 1.0.22: Graph of Magnetic Field vs Time

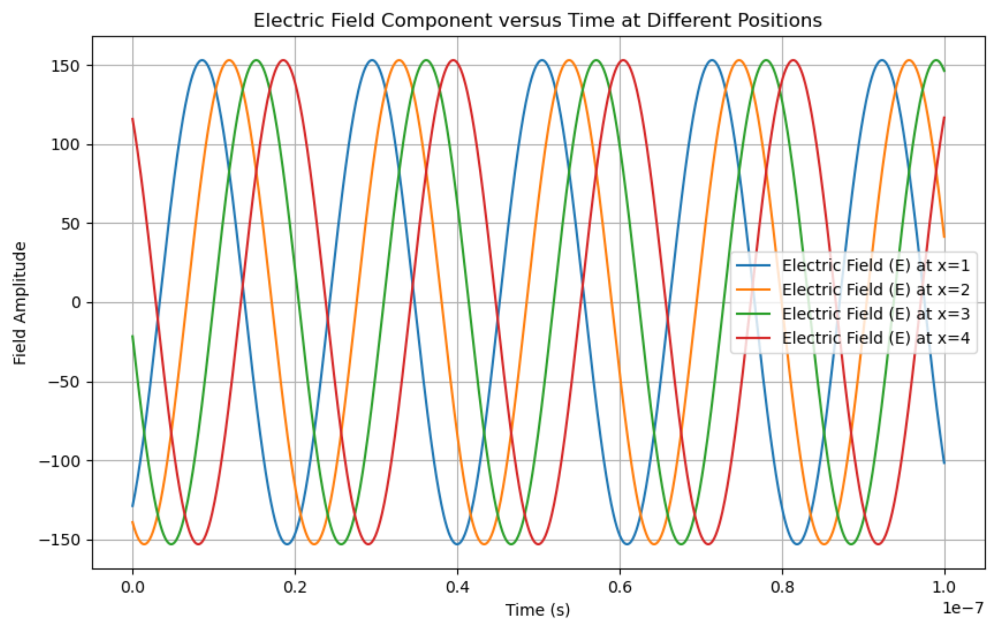


Figure 1.0.22: Graph of Electric Field vs Time

1.0.23 A 100μF capacitor in series with a 40Ω resistance is connected to a 110V, 60Hz supply.

(a) What is the maximum current in the circuit?

(b) What is the time lag between the current maximum and the voltage maximum?

Solution:

Symbol	Value	Description
V	110 V	Voltage Supplied
ν	$60Hz$	Frequency
R	$40\,\Omega$	Resistance
C	$100\,\mu$F	Capacitance
ω	$2\pi\nu$	Angular Frequency
ϕ	$tan^{-1}\frac{1}{\omega CR}$	Phase Angle
I_0	$\frac{V_0}{Z}$	Max Current
V_0	V $\sqrt{2}$	Peak Voltage
Z	$\sqrt{R^2+\frac{1}{\omega^2C^2}}$	Impedance
$H(s)$	$\frac{V(s)}{I(s)}$	Transfer Function

Table 1.0.23: Given Parameters

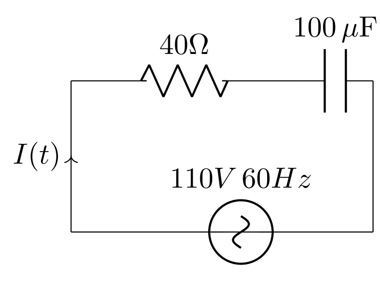


Figure 1.0.23: RC Circuit

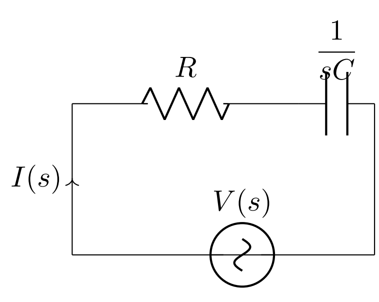


Figure 1.0.23: RC Circuit

(a) V_{out} across capacitor,

$$V_{out} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{in} \tag{1.78}$$

$$\frac{V_{out}}{V_{in}} = H(s) \tag{1.79}$$

$$\implies H(s) = \frac{1}{1 + sRC} \tag{1.80}$$

$$= \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{-tan^{-1}\frac{1}{(\omega RC)}} \tag{1.81}$$

On taking fourier transform of $H(s)$,

$$V_{out} = \frac{110}{\sqrt{1 + (\omega RC)^2}} cos\left(\omega t - tan^{-1}\frac{1}{(\omega RC)}\right) \tag{1.82}$$

For current across circuit,

$$\implies I = \frac{V_{out}}{Z} \tag{1.83}$$

$$= \frac{110 j\omega C}{\sqrt{1 + (\omega RC)^2}} cos\left(\omega t - tan^{-1}\frac{1}{(\omega RC)}\right) \tag{1.84}$$

$$= -\frac{110\omega C}{\sqrt{1 + (\omega RC)^2}} sin\left(\omega t - tan^{-1}\frac{1}{(\omega RC)}\right) \tag{1.85}$$

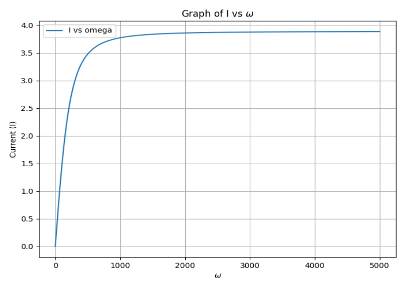


Figure 1.0.23: Current vs ω

Maximum current in the circuit ,

$$\implies I_0 = \frac{110\omega C}{\sqrt{1 + (\omega RC)^2}} \tag{1.86}$$

$$= 3.24\ A \tag{1.87}$$

(b) In a capacitor circuit,the voltage lags behind the current by a phase angle of ϕ .

$$\implies \phi = tan^{-1}\frac{1}{(\omega RC)} \tag{1.88}$$

$$= \frac{33.56\pi}{180 \times 120\pi} \tag{1.89}$$

$$\phi = \frac{33.56\pi}{180} rad \tag{1.90}$$

$$\implies Time\ lag = \frac{\phi}{\omega} \tag{1.91}$$

$$= \frac{33.56\pi}{180 \times 120\pi} \tag{1.92}$$

$$= 1.55ms \tag{1.93}$$

(c) Plot of Impedance vs Angular Frequency

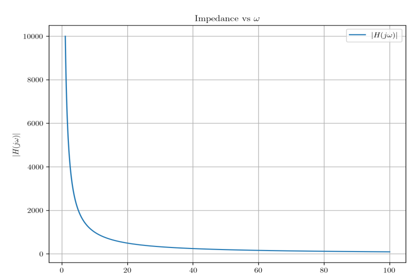


Figure 1.0.23: Impedance vs ω

1.0.24 A 100 Ω resistor is connected to $220V$, $50Hz$ AC supply.

(1) What is the rms value of current in the circuit?

(2) What is the net power consumed over a full cycle?

Solution:

Symbol	Value	Description
V_{rms}	220V	rms value of voltage
I_{rms}	$\frac{V_{rms}}{R}$	rms value of current
P_{avg}	$V_{rms}.I_{rms}$	Average power consumed per cycle
R	100Ω	Resistance

Table 1.0.24: Variable description

1)

$$\begin{aligned} I_{rms} &= \frac{V_{rms}}{R} \\ &= \frac{220V}{100\Omega} \\ &= 2.2A \end{aligned}$$

2)

$$\begin{aligned} \text{Net power consumed} &= \frac{V^2}{R} \\ &= \frac{220^2}{100} \\ &= 484W \end{aligned}$$

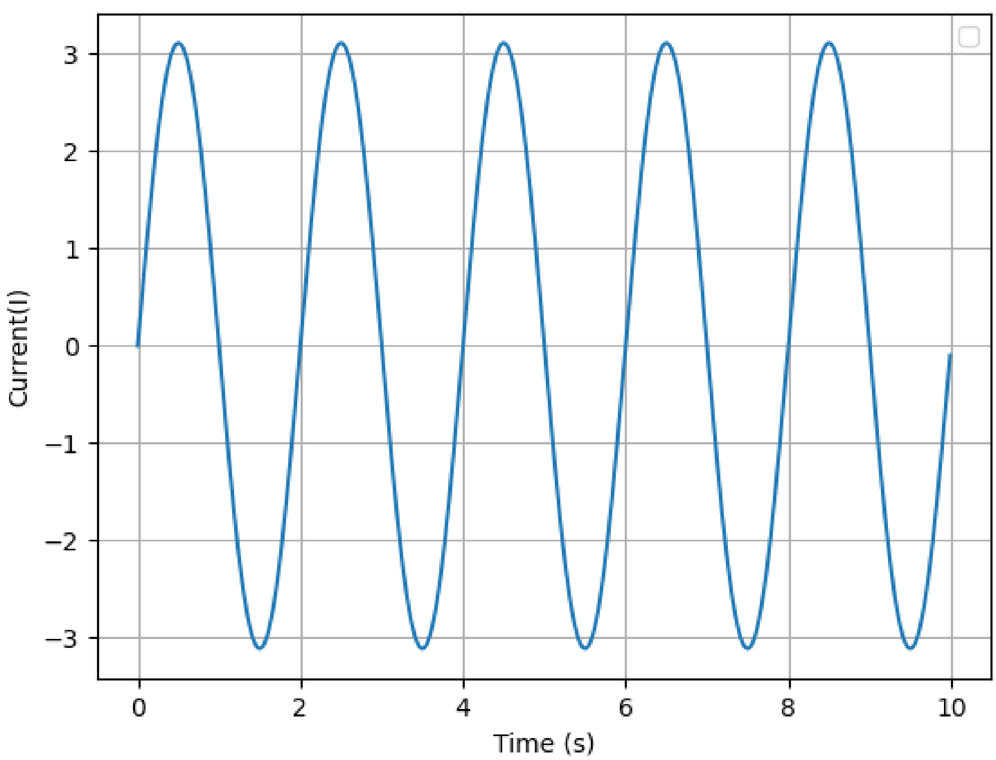


Figure 1.0.24: Current v/s time

1.0.25 Two towers on top of two hills are 40 km apart.This line joining them passes 50 m above a hill halfway between the towers.What is the longest wavelength of radio waves,which can be sent between the towers without appereciable diffraction effects?

Solution:

1.0.26 A circuit containing a 80mH inductor and a 60μF capacitor in series is connected to a 230V, 50Hz supply. The resistance of the circuit is negligible.

(a) Obtain the current amplitude and rms value.

(b) Obtain the rms value of potential drops across each element.

(c) What is the average power transferred to the inductor ?

(d) What is the average power transferred to the capacitor ?

(e) What is the total average power absorbed by the circuit ? ('Average' implies 'averaged ov

Solution:

1.0.27 A coil of inductance 0.50 H and resistance 100 Ω is connected to a 240 V, 50 Hz ac supply.

(a) What is the maximum current in the coil?

(b) What is the time lag between the voltage maximum and the current maximum?

Solution:

1.0.28 A plane electromagnetic wave travels in vacuum along the z-direction. What can you say about the directions of its electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) field vectors? If the frequency of the wave is 30 MHz, what can you say about its wavelength?

Solution:

1.0.29 Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and longitudinal (P) sound waves. Typically the speed of S wave is about $4.0km/s$, and that of P wave is $8.0km/s$. A seismograph records P and S waves from an earthquake. The first P wave arrives $4min$ before the first S wave. Assuming the waves travel in straight line, at what distance does the earthquake occur ? **Solution:**

1.0.30 A hospital uses an ultrasonic scanner to locate tumors in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is 1.7 km/s? The operating frequency of the scanner is 4.2 MHz. **Solution:**

1.0.31 In double-slit experiment using light of wavelength $600nm$, the angular width of a fringe formed on a distant screen is $0.1°$. What is the spacing between the two slits?

Solution:

1.0.32 A spring having with a spring constant 1200 Nm^{-1} is mounted on a horizontal table as shown in Fig.A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass. **Solution:**

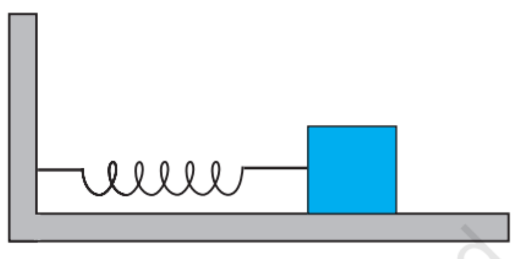


Figure 1.0.32:

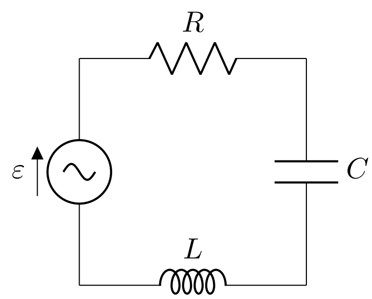
1.0.33 A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does he disturbance take to reach the other end ? **Solution:**

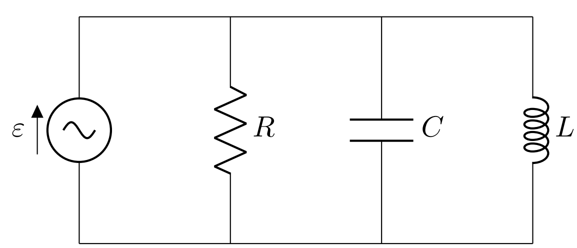
1.0.34 In a Young's double-slit experiment, the slits ar e separated by 0.28 mm and the screen is placed $1.4m$ away. The distance between the central bright fringe and the fourth bright fringe is measured to be $1.2cm$. Determine the wavelength of light used in the experiment.

Solution:

1.0.35 Keeping the source frequency equal to the resonating frequency of the series LCR circuit, if the three elements, L, C, and R are arranged in parallel, show that the total current in the parallel LCR circuit is minimum at this frequency. Obtain the current rms value in each branch of the circuit for the elements and source specified in Exercise 7.11 for this frequency

$\varepsilon = 230V$, $L = 5.0H$, $C = 80\mu$F, $R = 40\Omega$





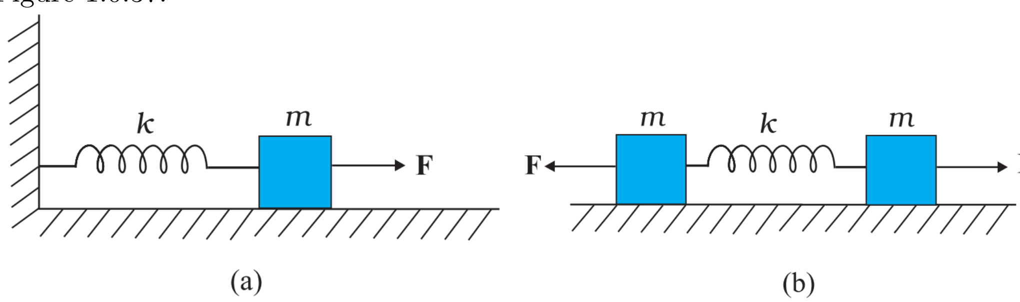
Solution:

1.0.36 A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of 360 km/hr. What is the frequency of sound reflected by the submarine? Take the speed of sound in water to be 1450 m/s.

Solution:

1.0.37 Figure 1.0.35 (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure 1.0.35 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 1.0.35(b) is stretched by the same force F.

Figure 1.0.37:



(a) What is the maximum extension of the spring in the two cases ?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case ?

Solution:

1.0.38 A bat is flitting about in a cave, navigating via ultrasonic beeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall ? NCERT Analog 11.15.27

Solution:

1.0.39 One end of a long string of linear mass density $8.0 \times 10^{-3}\,\mathrm{kg\,m^{-1}}$ is connected to an electrically driven tuning fork of frequency $256\,\mathrm{Hz}$. The other end passes over a pulley and is tied to a pan containing a mass of $90\,\mathrm{kg}$. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At $t = 0$, the left end (fork end) of the string $x = 0$ has zero transverse displacement ($y = 0$) and is moving along positive y-direction. The amplitude of the wave is $5.0\,\mathrm{cm}$. Write down the transverse displacement y as a function of x and t that describes the wave on the string. NCERT Analog 11.15.24

Solution:

1.0.40 A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at 20 $^{\circ}C = 343ms^{-1}$ NCERT Analog 11.15.3

Solution:

1.0.41 What is the Brewster angle for air to glass transition?(Refractive index of glass =1.5)

Solution:

1.0.42 A parallel beam of light with a wavelength of 500 nm falls on a narrow slit, and the resulting diffraction pattern is observed on a screen 1 m away. The distance to the first minimum from the center of the screen is 2.5 mm.

Find the width of the slit given that $y = 0.0025$ m, $L = 1$ m, and $\lambda = 5 \times 10^{-7}$ m.

Solution:

1.0.43 Two sitar strings A and B playing the note 'Ga' are slightly out of tune and produce beats of frequency 6Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3Hz. If the original frequency of A is 324Hz,what is the frequency of B?

Solution:

1.0.44 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):

(a) $\sin(\omega t) - \cos(\omega t)$

(b) $\sin^3(\omega t)$

(c) $3\cos\left(\frac{\pi}{4} - 2\omega t\right)$

(d) $\cos(\omega t) + \cos(3\omega t) + \cos(5\omega t)$

(e) $\exp\left(-\omega^2 t^2\right)$

(f) $1 + \omega t + \omega^2 t^2$

Solution:

(a) Periodic function:

$$x(t+T) = x(t) \quad \forall x \in \mathbb{R} \tag{1.94}$$

where min T s.t $T > 0$ is time period

(b) SHM:

For a function to be in shm it must satisfy

$$\frac{d^2x(t)}{dt^2} = -(2\pi f_0)^2 x(t) \tag{1.95}$$

Variable	**Description**	formula
$x(t)$	Displacemen wrt mean position	none
ω	Angular frequncy	$2\pi f$
T	Time period	$\frac{1}{f}$
ϕ	phase angle	none

Table 1.0.44: input parameters

(a) $\sin(2\pi ft) - \cos(2\pi ft)$

The function can be rewritten as:

$$= \sin(2\pi ft) - \sin\left(\frac{\pi}{2} - 2\pi ft\right) \tag{1.96}$$

$$= 2\cos\left(\frac{\pi}{4}\right)\sin\left(2\pi ft - \frac{\pi}{4}\right) \tag{1.97}$$

$$= \sqrt{2}\sin\left(2\pi ft - \frac{\pi}{4}\right) \tag{1.98}$$

$$\frac{d^2(\sin(2\pi ft) - \cos(2\pi ft))}{dt^2} = -(2\pi f)^2(\sin(2\pi ft) - \cos(2\pi ft)) \tag{1.99}$$

$$\frac{d^2x(t)}{dt^2} = -(2\pi f)^2 x(t) \tag{1.100}$$

$$\tag{1.101}$$

$\therefore$ SHM, T is $\frac{1}{f}$and ϕ is $\left(\frac{-\pi}{4}\right)$ or $\left(\frac{7\pi}{4}\right)$

$$\sin\left(2\pi f\left(t+\frac{1}{f}\right)\right)-\cos\left(2\pi f\left(t+\frac{1}{f}\right)\right)=\sin(2\pi ft)-\cos(2\pi ft) \tag{1.102}$$

Graph of function is shown in (Fig. 1.0.44)

(3) $3\cos\left(\frac{\pi}{4}-4\pi ft\right)$

This function can be rewritten as

$$\begin{aligned}
&= 3\cos\left(4\pi ft-\frac{\pi}{4}\right) && (1.103)\\
\frac{d^2\left(3\cos\left(\frac{\pi}{4}-4\pi ft\right)\right)}{dt^2} &= -3(4\pi f)^2\left(\cos\frac{\pi}{4}-4\pi ft\right) && (1.104)\\
\frac{d^2x(t)}{dt^2} &= -(4\pi f)^2 x(t) && (1.105)
\end{aligned}$$

$\therefore$ SHM, T is $\frac{1}{2f}$ and ϕ is $\left(\frac{-\pi}{4}\right)$ *or* $\left(\frac{7\pi}{4}\right)$

$$3\cos\left(\frac{\pi}{4}-4\pi f\left(t+\frac{1}{2f}\right)\right)=3\cos\left(\frac{\pi}{4}-4\pi ft\right) \tag{1.106}$$

$$\tag{1.107}$$

Graph of function is shown in (Fig. 1.0.44)

(4) $\cos(2\pi ft)+\cos(6\pi ft)+\cos(10\pi ft)$

This function can be rewritten as

$$= \cos(2\pi ft) + \cos(10\pi ft) + \cos(6\pi ft) \tag{1.108}$$

$$= 2\cos\left(\frac{2\pi ft + 10\pi ft}{2}\right)\cos\left(\frac{10\pi ft - 2\pi ft}{2}\right) + \tag{1.109}$$

$$= 2\cos(6\pi ft)\cos(2\pi ft) + \cos(6\pi ft) \tag{1.110}$$

$$= \cos(6\pi ft)(1 + 2\cos(2\pi ft)) \tag{1.111}$$

$$\frac{d^2\cos(2\pi ft) + \cos(6\pi ft) + \cos(10\pi ft)}{dt^2} = (2\pi f)^2\cos(2\pi ft) + (6\pi f)^2\cos(6\pi ft) + (10\pi f)^2 \tag{1.112}$$

$$\frac{d^2x(t)}{dt^2} \neq -(2\pi f)^2 x(t) \tag{1.113}$$

Period of $\cos(6\pi ft)$ is $\frac{1}{3f}$

Period of $1 + 2\cos(2\pi ft)$ is $\frac{1}{f}$

Lcm is $\frac{1}{f}$

$\therefore$ SHM,T is $\frac{1}{f}$

$$\cos\left(2\pi f\left(t + \frac{1}{f}\right)\right) + \cos\left(6\pi f\left(t + \frac{1}{f}\right)\right) + \cos\left(10\pi f\left(t + \frac{1}{f}\right)\right) = \cos(2\pi ft) + \cos(6 \tag{1.114}$$

Graph of function is shown in (Fig. 1.0.44)

(5) $\exp\left(-(2\pi f)^2t^2\right)$

$$\text{As } T \to \infty \tag{1.115}$$

$$\exp\left(-(2\pi f)^2t^2\right) \to \infty \tag{1.116}$$

$$\frac{d^2(\exp\left(-(2\pi f)^2t^2\right))}{dt^2} = 2(2\pi ft)^2\exp\left(-(2\pi f)^2t^2\right) + 2(2\pi ft)^4\exp\left(-(2\pi f)^2t^2\right) \tag{1.117}$$

$$\frac{d^2x(t)}{dt^2} \neq -(2\pi f)^2x(t) \tag{1.118}$$

$\therefore$ This never repeats and non periodic

Graph of function is shown in (Fig. 1.0.44)

(6) $1 + 2\pi ft + (2\pi f)^2t^2$

$$\text{As } T \to \infty \tag{1.119}$$

$$1 + 2\pi ft + (2\pi f)^2t^2 \to \infty \tag{1.120}$$

$$\frac{d^2(1 + 2\pi ft + (2\pi f)^2t^2)}{dt^2} = 2(2\pi f)^2 \tag{1.121}$$

$$\frac{d^2x(t)}{dt^2} \neq -(2\pi f)^2x(t) \tag{1.122}$$

$\therefore$ This never repeats and non periodic

Graph of function is shown in (Fig. 1.0.44)

:

Table 1.0.44: Summary

	Function	**Periodic**	**Simple harmonic motion**	**Non Per**
(a)	$\sin(2\pi ft) - \cos(2\pi ft)$	Yes	Yes	No
(b)	$\sin^3(2\pi ft)$	Yes	No	No
(c)	$3\cos\left(\frac{\pi}{4} - 4\pi ft\right)$	Yes	Yes	No
(d)	$\cos(2\pi ft) + \cos(6\pi ft) + \cos(10\pi ft)$	Yes	No	No
(e)	$\exp\left(-(2\pi ft)^2\right)$	No	No	Yes
(f)	$1 + (2\pi f)t + (2\pi f)^2 t^2$	No	No	Yes

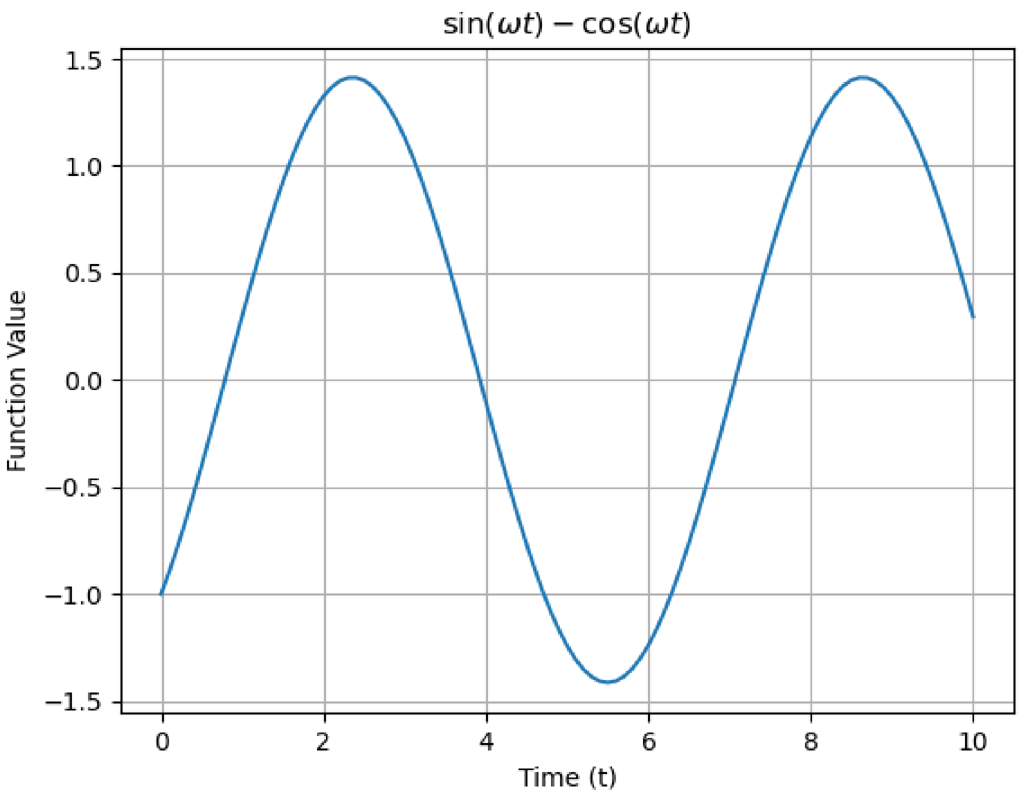


Figure 1.0.44: $\sin(2\pi ft) - \cos(2\pi ft)$

1.0.45 A metre-long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.

Solution:

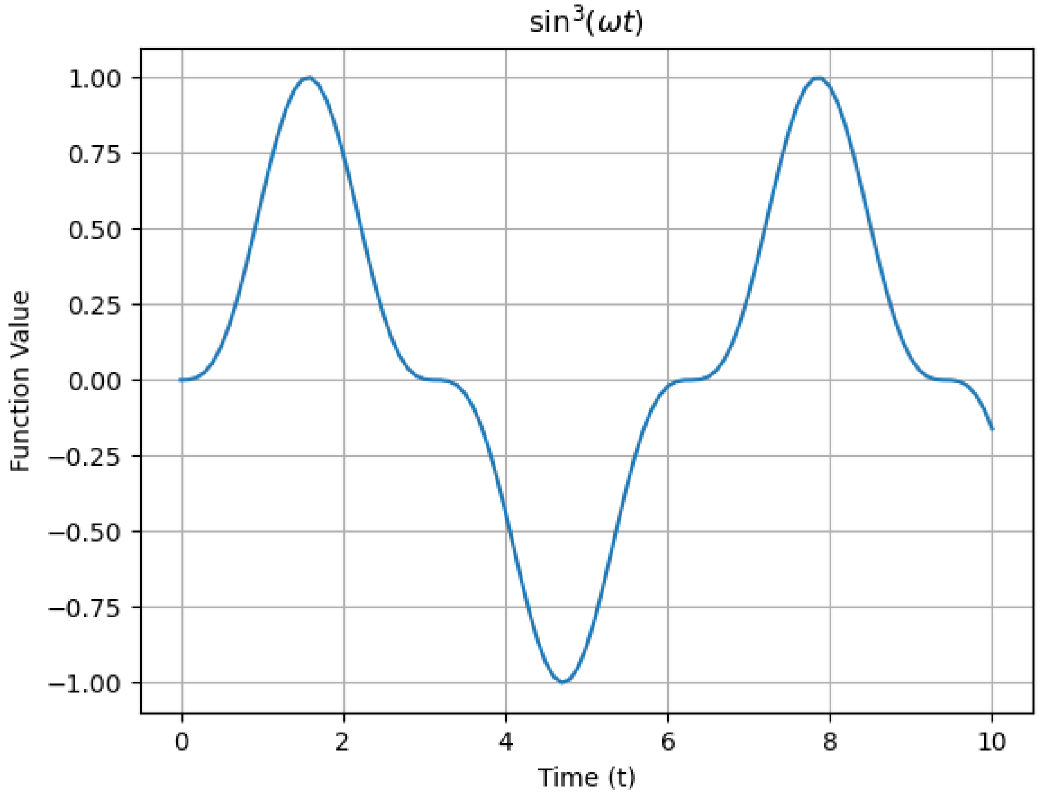


Figure 1.0.44: $\sin^3(2\pi ft)$

1.0.46 In a double-slit experiment the angular width of a fringe is found to be 0.2° on a screen placed 1 m away. The wavelength of light used is 600 nm. What will be the angular width of the fringe if the entire experimental apparatus is immersed in water? Take refractive index of water to be $\frac{4}{3}$ **Solution:**

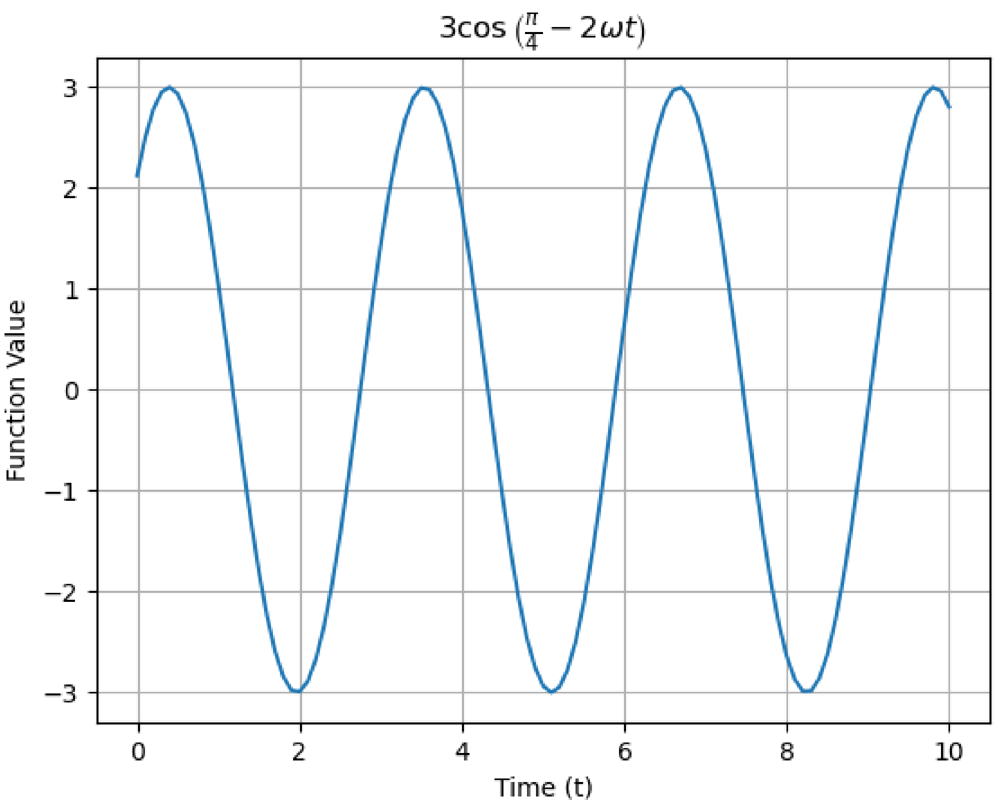


Figure 1.0.44: $3\cos\left(\frac{\pi}{4} - 4\pi f t\right)$

1.0.47 An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up down without any friction. Show that when the ball is pressed down a little and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal.

Solution:

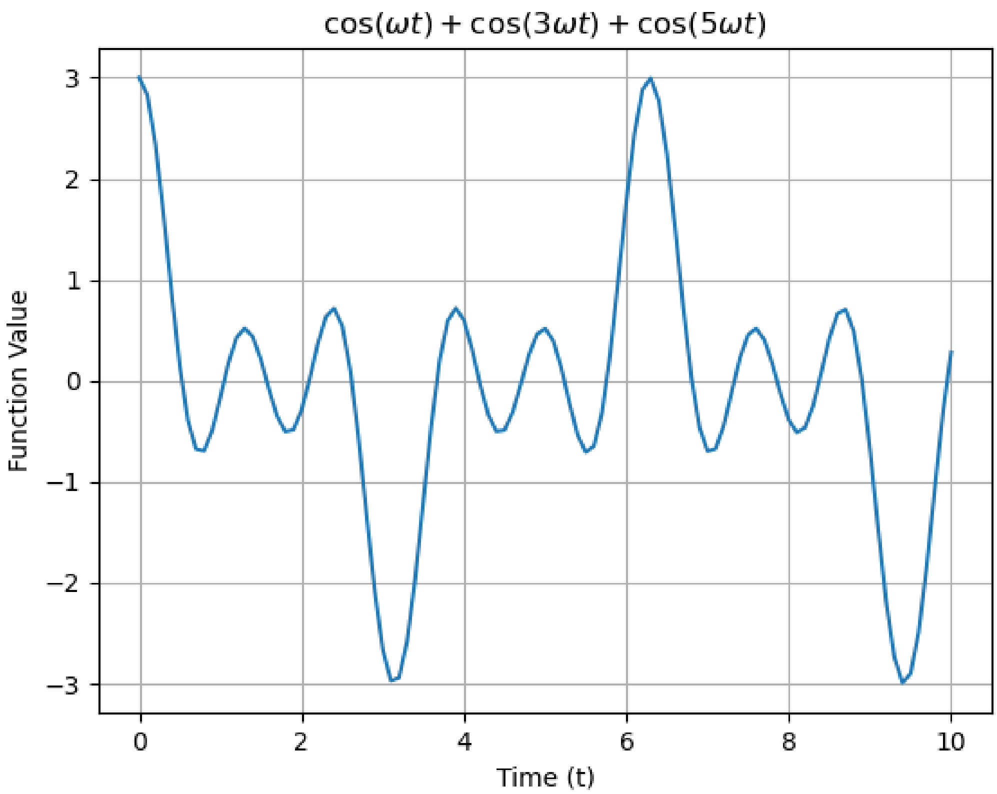


Figure 1.0.44: $\cos(2\pi ft) + \cos(6\pi ft) + \cos(10\pi ft)$

1.0.48 (a) The refractive index of glass is 1.5. What is the speed of light in glass?(Speed of light in vacuum is $3.0 \times 10^8 ms^{-1}$)

(b) Is the speed of light in glass independent of the colour of light? If not, which of the two colours red and violet travels slower in a glass prism?

Solution:

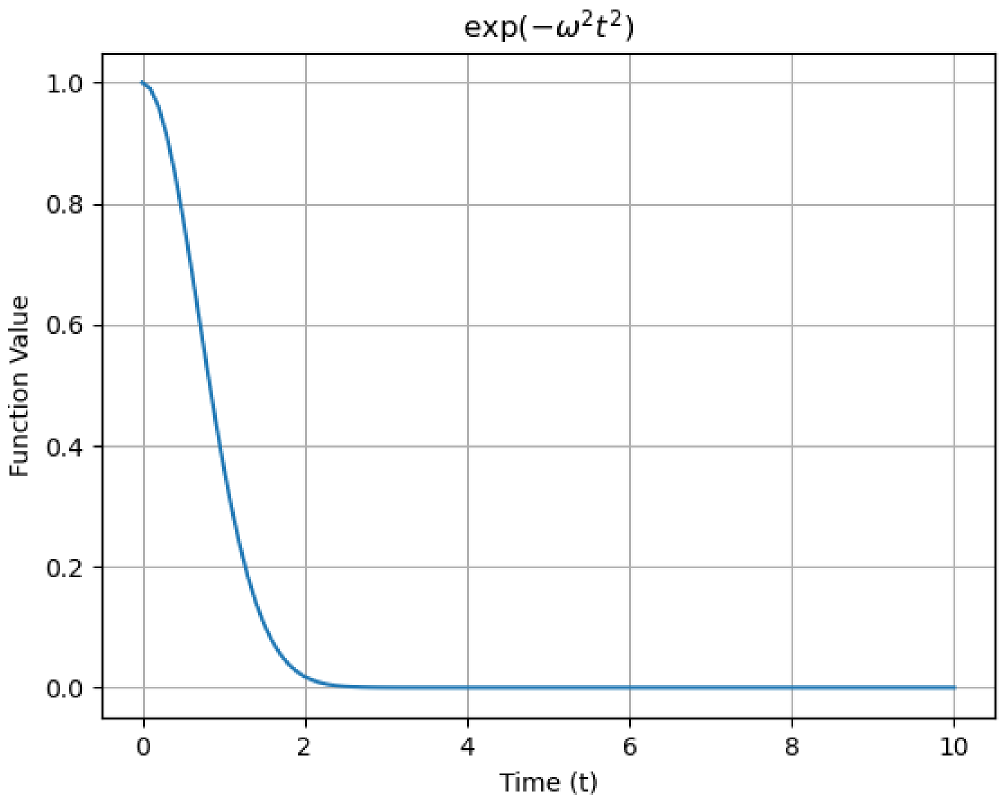


Figure 1.0.44: $exp^{\left(-(2\pi f t)^2\right)}$

1.0.49 The motion of a particle executing simple harmonic motion is described by the displacement function, $x(t) = A\ cos\ (\omega t + \phi)$. If the initial $(t = 0)$ position of the particle is $1cm$ and its initial velocity is $\omega \quad cm/s$, what are its amplitude and initial phase angle ? The angular frequency of the particle is $\pi \quad s^{-1}$. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B\ sin\ (\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

Solution:

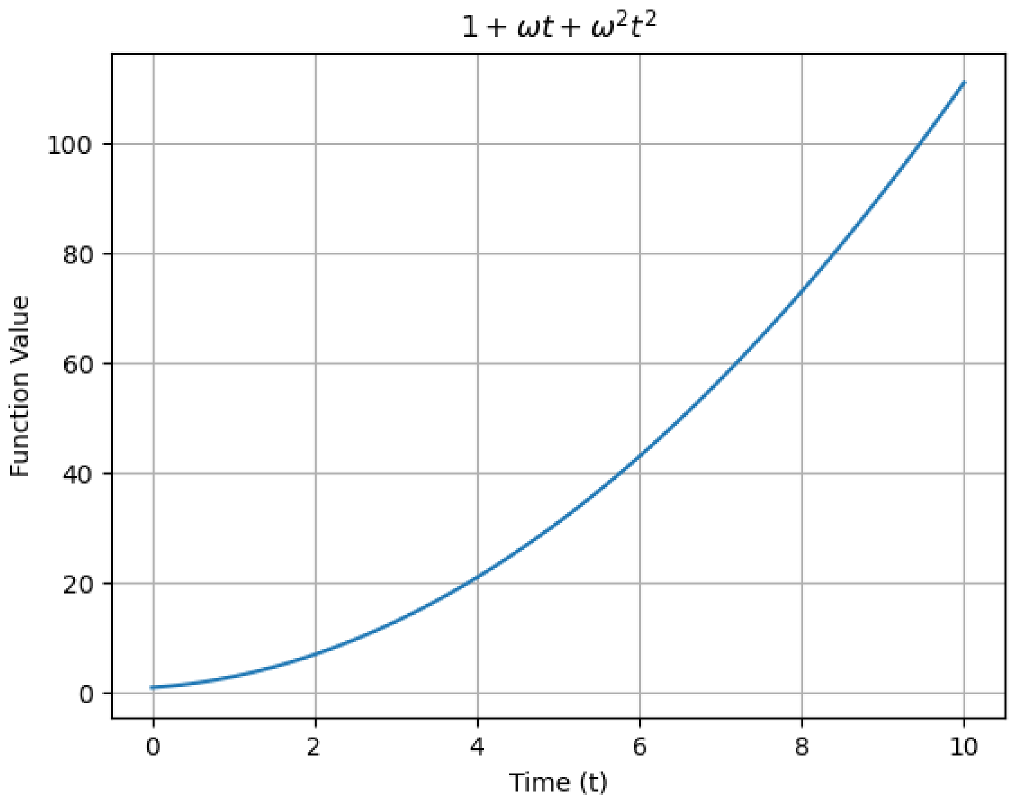


Figure 1.0.44: $1 + 2\pi f t + (2\pi f t)^2$

1.0.50 A stone dropped from the top of a tower of height 300 m splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of the sound in air is 340 m s^{-1}.(g=9.8 m s^{-2})

Solution:

1.0.51 Monochromatic light of wavelength 589nm is incident from air on a water surface.

What are the wavelength, frequency and speed of

(a) Reflected light?

(b) refracted light? Refractive index of water is 1.33.

Solution:

1.0.52 A bat emits ultrasonic sound of frequency $1000kHz$ in air. If the sound meets a water surface, what is the wavelength of

(a) the reflected sound

(b) the transmitted sound?

Speed of sound in air is $340ms^{-1}$ and in water is $1486ms^{-1}$.

Solution:

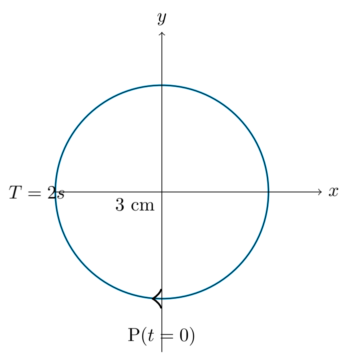
1.0.53 A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is

resonantly excited by a 430 Hz source ? Will the same source be in resonance with

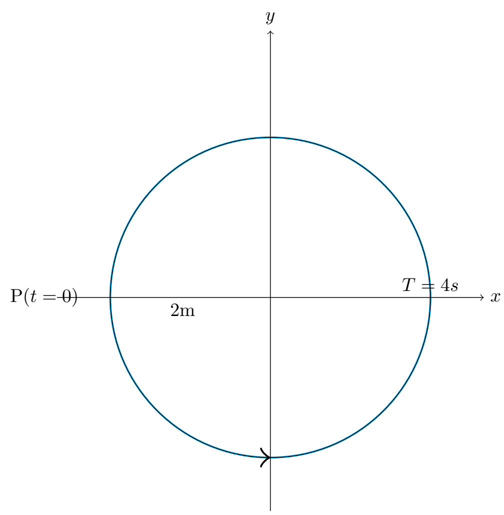
the pipe if both ends are open? (speed of sound in air is 340 m s^{-1}).

Solution:

1.0.54 Figures correspond to two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution(i.e. clockwise or anti-clockwise) are indicated on each figure. Obtain the corresponding simple harmonic motions of the x-projections of the radius vector of resolving particle P in each case.



(a)



(b)

1

Solution:

Parameter	**Value(a)**	**Value(b)**	**Description**
Radius(r)	3cm	2m	Radius of each circle
Time Period(T)	2s	4s	Time period
Sense	clockwise	anti-clockwise	Indicated by arrow
Initial Phase(ϕ)	$\frac{\pi}{2}$	π	Initial angle with x-axis

Table 1.0.54: Input parameters table

Given $\begin{pmatrix} r \end{pmatrix}$ as radius vector making angle θ with positive x-axis, its x-projection = $\begin{pmatrix} r \end{pmatrix} \cos\theta$

a. At $t = 0$, the radius vector makes an angle $\frac{\pi}{2}$ with the positive x-axis, $\phi = \frac{\pi}{2}$,

From Table 1.0.54, equation of x-projection of radius:

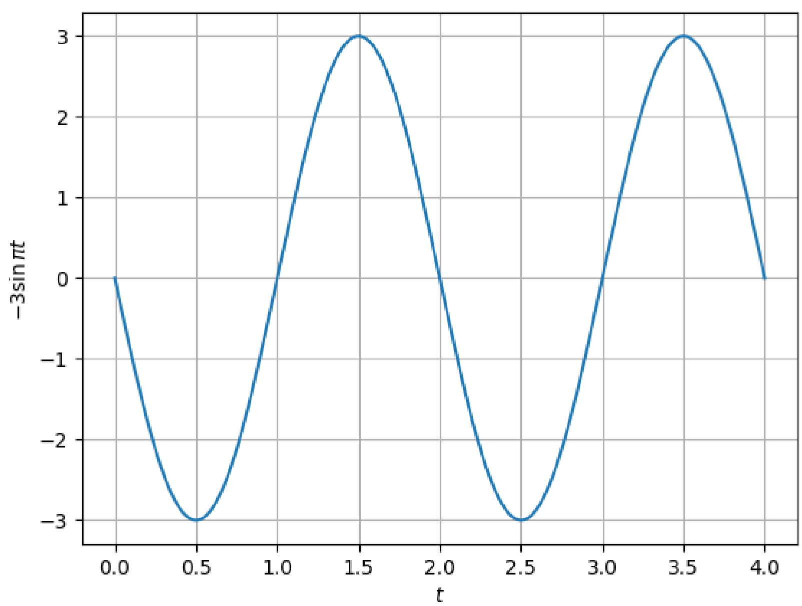
$$x(t) = r\cos\left(\frac{2\pi}{T}t + \phi\right) \tag{1.123}$$

$$= 3\cos\left(\frac{2\pi}{2}t + \frac{\pi}{2}\right) \tag{1.124}$$

$$= -3\sin\left(\pi t\right) \text{cm} \tag{1.125}$$

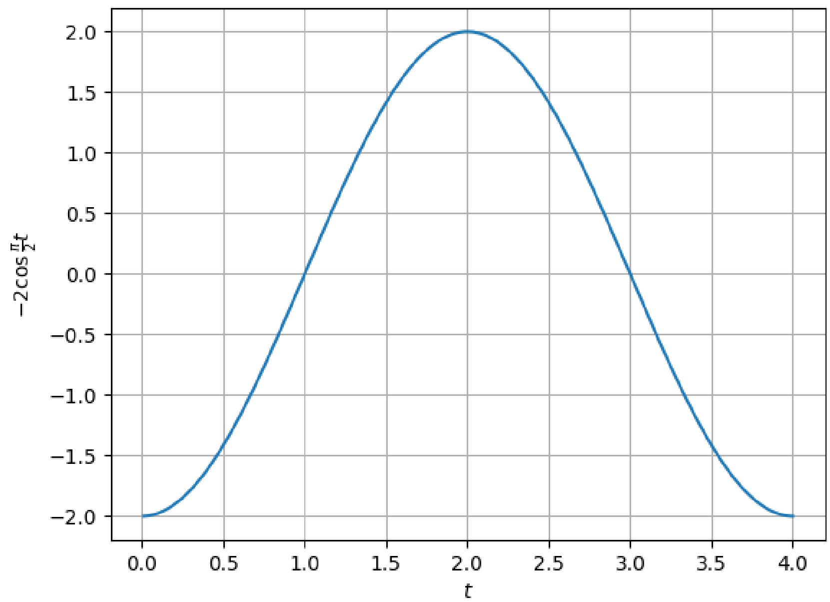
b. Similarly,

At $t = 0$, radius vector makes an angle π with x-axis in anti-clockwise direction,



$\phi = \pi$,

$$x(t) = r\cos\left(\frac{2\pi}{T}t + \phi\right) \tag{1.126}$$
$$= 2\cos\left(\frac{2\pi}{4}t + \pi\right) \tag{1.127}$$
$$= -2\cos\left(\frac{\pi}{2}t\right)\,\mathrm{m} \tag{1.128}$$



1.0.55 A steel rod 100cm long is clamped at its middle. The fundamental frequency of the longitudinal vibrations of the rod are given to be 2.53kHz. What is the speed of sound in steel?

Solution:

1.0.56 A train, standing in a station yard, blows a whistle of frequency 400 Hz in still air. The wind starts blowing in the direction from the yard to the station with a speed of 10 m/s. What are the frequency, wavelength, and the speed of sound for an observer standing on the station's platform? Is the situation exactly identical to the case when the air is still and the observer runs towards the yard at a speed of 10 m/s? The speed of sound in still air can be taken as 340 m/s.

Solution:

1.0.57 A radio can tune in to any station in the 7.5 MHz to 12 MHz band. What is the corresponding wavelength band?

Solution:

1.0.58 A travelling harmonic wave on a string is described by

$$y(x,t) = 7.5\sin(0.0050x + 12t + \frac{\pi}{4})$$

(a) What are the displacement and velocity of oscillation of a point at $x = 1\,\text{cm}$ and $t = 1\,\text{s}$? Is this velocity equal to the velocity of wave propagation?

(b) Locate the points on the string which have the same transverse displacements and velocity as the point at $x = 1\,\text{cm}$ at $t = 2\,\text{s}$, $t = 5\,\text{s}$, and $t = 11\,\text{s}$.

Solution:

1.0.59 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ, The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period $T = 2\pi\sqrt{\dfrac{h\rho}{\rho_1 g}}$

Solution:

1.0.60 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t = 0) position of the particle, the radius of the circle,and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s)

a) $x = -2\sin(3t + \frac{\pi}{3})$

b) $x = \cos(\frac{\pi}{6} - t)$

c) $x = 3\sin(2\pi t + \frac{\pi}{4})$

d) $x = 2\cos(\pi t)$

Solution:

S.No	f**(in Hz)**	A**(in cm)**	ϕ
1.	$\frac{3}{2\pi}$	2	$\frac{5\pi}{6}$
2.	$\frac{1}{2\pi}$	1	$\frac{-\pi}{6}$
3.	1	3	$\frac{3\pi}{4}$
4.	$\frac{1}{2}$	1	0

Table 1.0.60: Input table

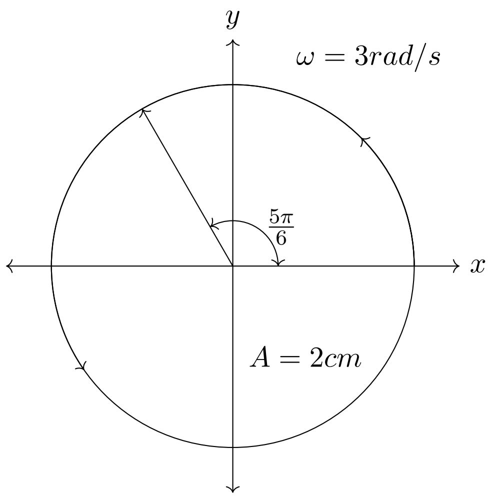
i)

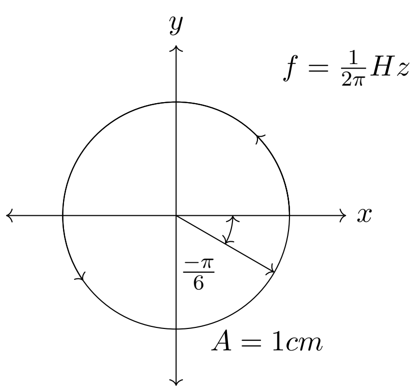
$$x = -2\sin\left(3t + \frac{\pi}{3}\right) \tag{1.129}$$

$$x = 2\cos\left(3t + \frac{5\pi}{6}\right) \tag{1.130}$$

ii)

$$x = \cos\left(\frac{\pi}{6} - t\right) \tag{1.131}$$

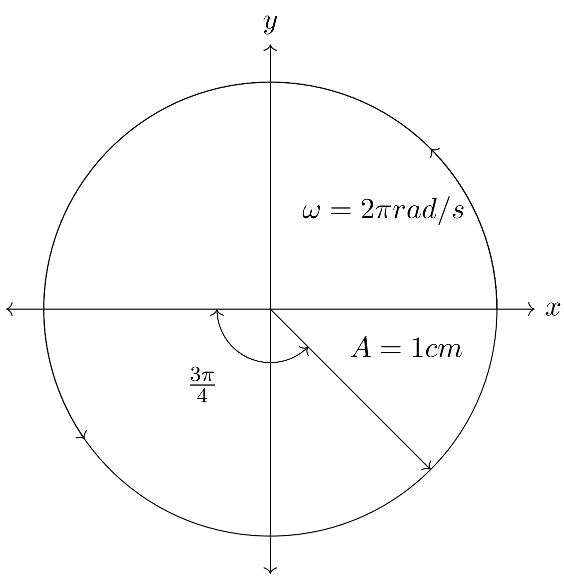




iii)

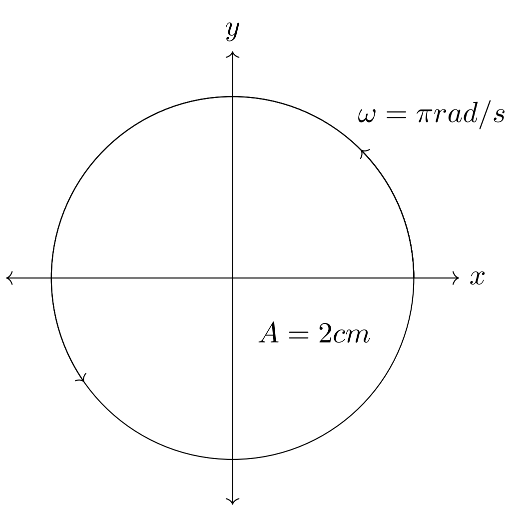
$$x = 3\sin\left(2\pi t + \frac{\pi}{4}\right) \tag{1.132}$$

$$x = -3\cos\left(2\pi t + \frac{3\pi}{4}\right) \tag{1.133}$$



iv)

$$x = 2\cos(\pi t) \tag{1.134}$$



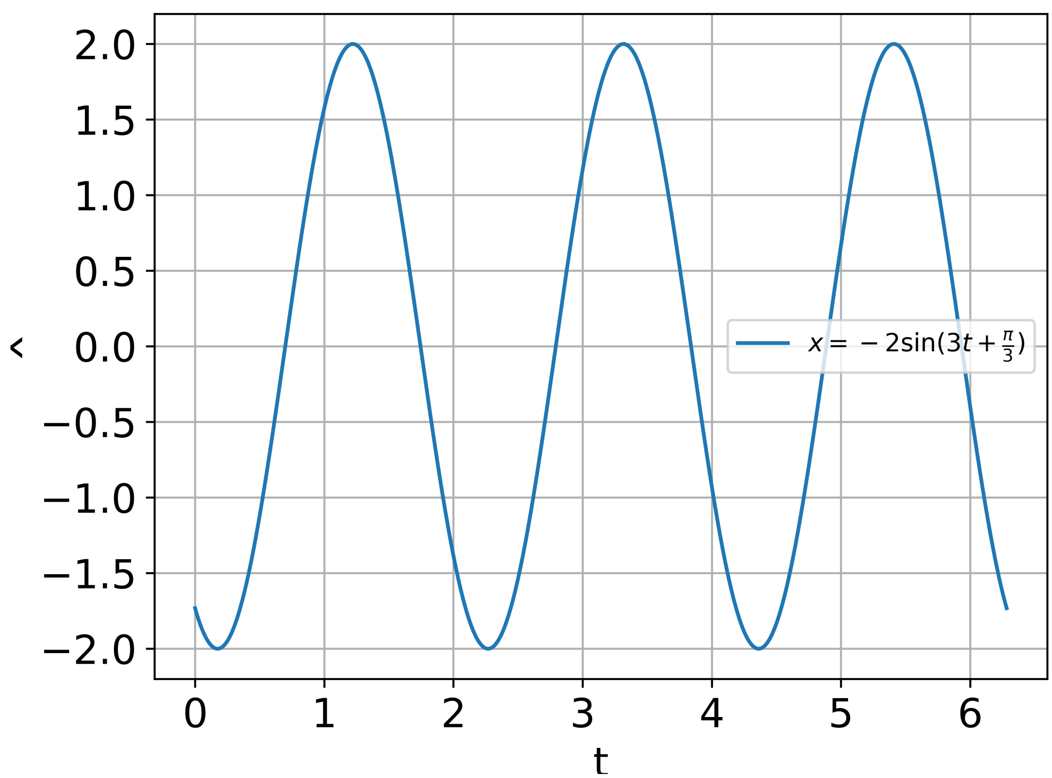


Figure 1.0.60:

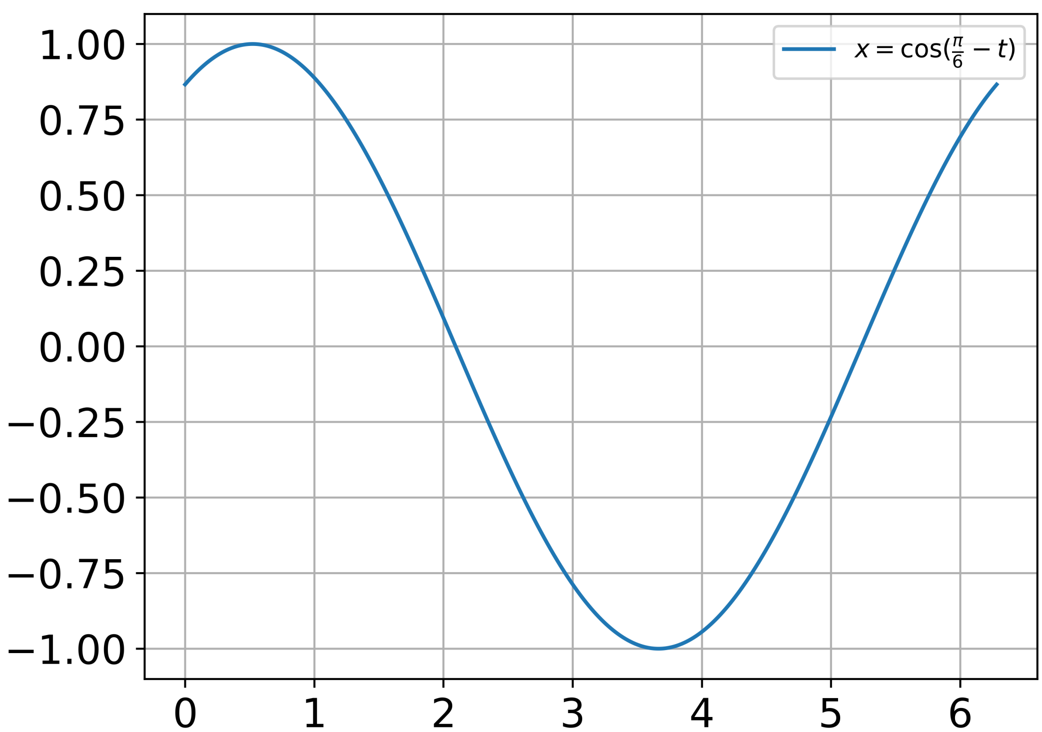


Figure 1.0.60:

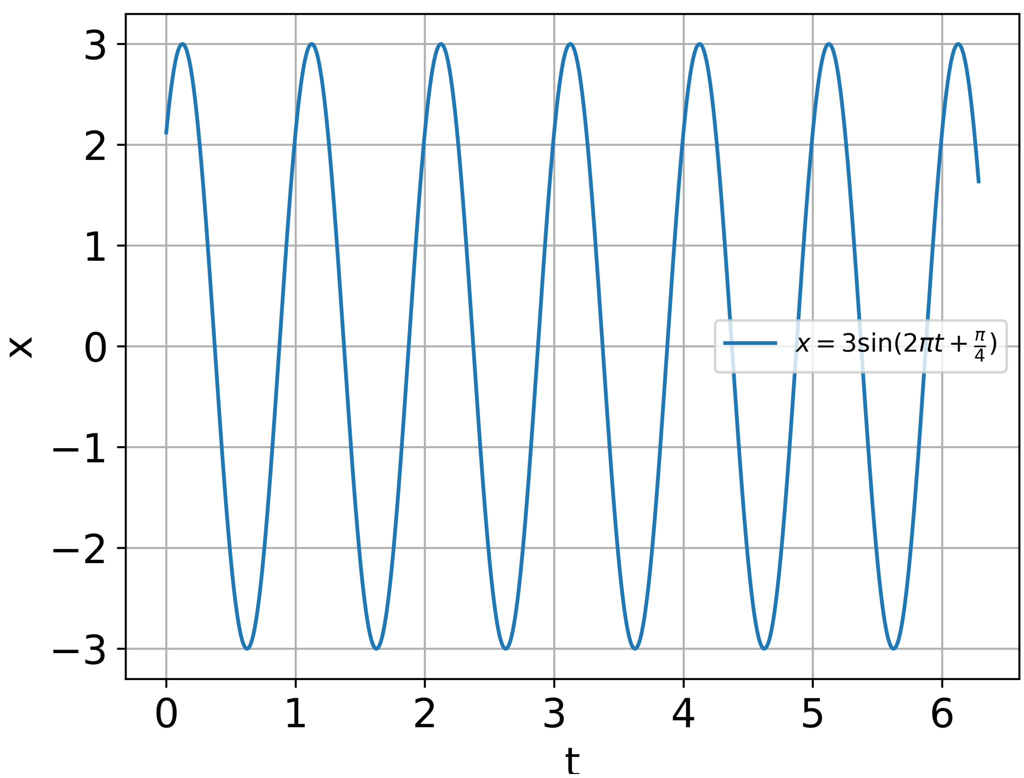


Figure 1.0.60:

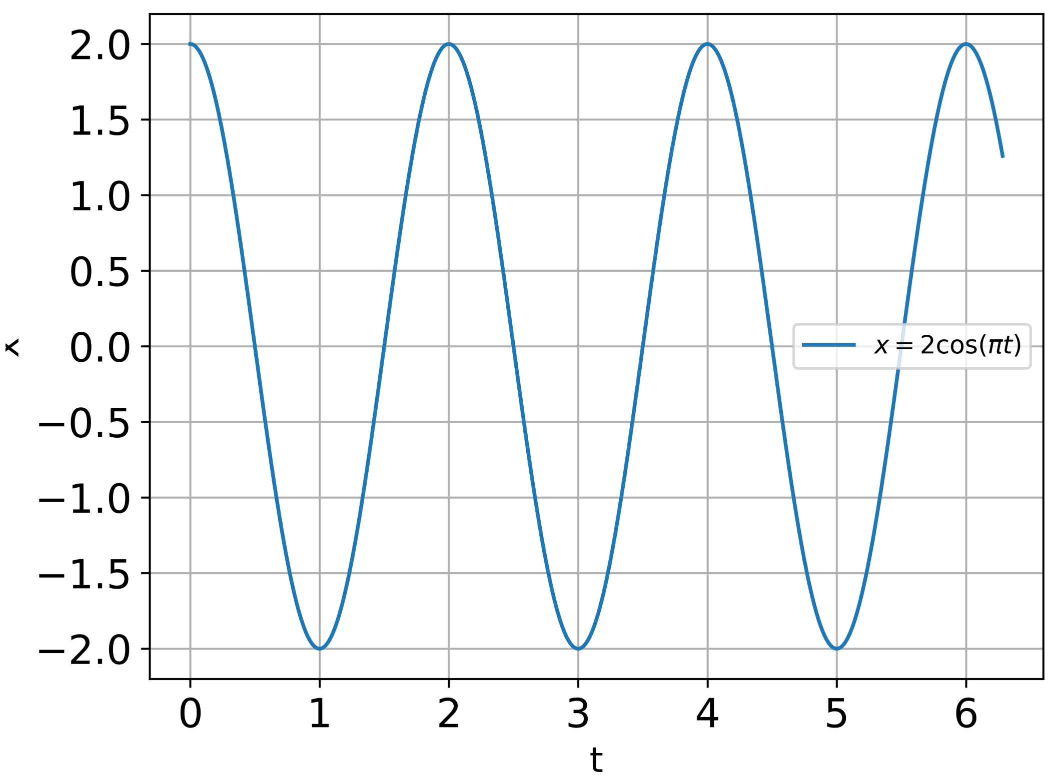


Figure 1.0.60:

Chapter 2

Filters

2.0.1 An LC circuit contains a $50\mu H$ inductor and a $50\mu F$ capacitor with an initial charge of $10mC$. The resistance of the circuit is negligible. Let the instant the circuit is closed by $t = 0$.

a) What is the total energy stored initially? Is it conserved during LC oscillations?

b) What is the natural frequency of the circuit?

c) At what time is the energy stored **(i)** completely electrical (i.e., stored in the capacitor)? **(ii)** completely magnetic (i.e., stored in the inductor)?

d) At what times is the total energy shared equally between the inductor and the capacitor?

e) If a resistor is inserted in the circuit, how much energy is eventually dissipated as heat?

(NCERT-Physics 12.7 12Q)

Solution:

Parameter	**Description**	**Value**
L	Inductance	$50\mu H$
C	Capacitance	$50\mu F$
$E(0)$	Initial Energy of Capacitor	?
$q(0)$	Initial Charge on Capacitor	$10mC$
$v(0^-)$	Initial Voltage on Capacitor	$200V$
ω_o	Angular Resonant frequency	?

Table 1: Parameter Table

(a) Initial energy stored :

$$E(0) = \frac{1}{2}C\left(v\left(0^-\right)\right)^2 \tag{2.1}$$

$$= 1J \tag{2.2}$$

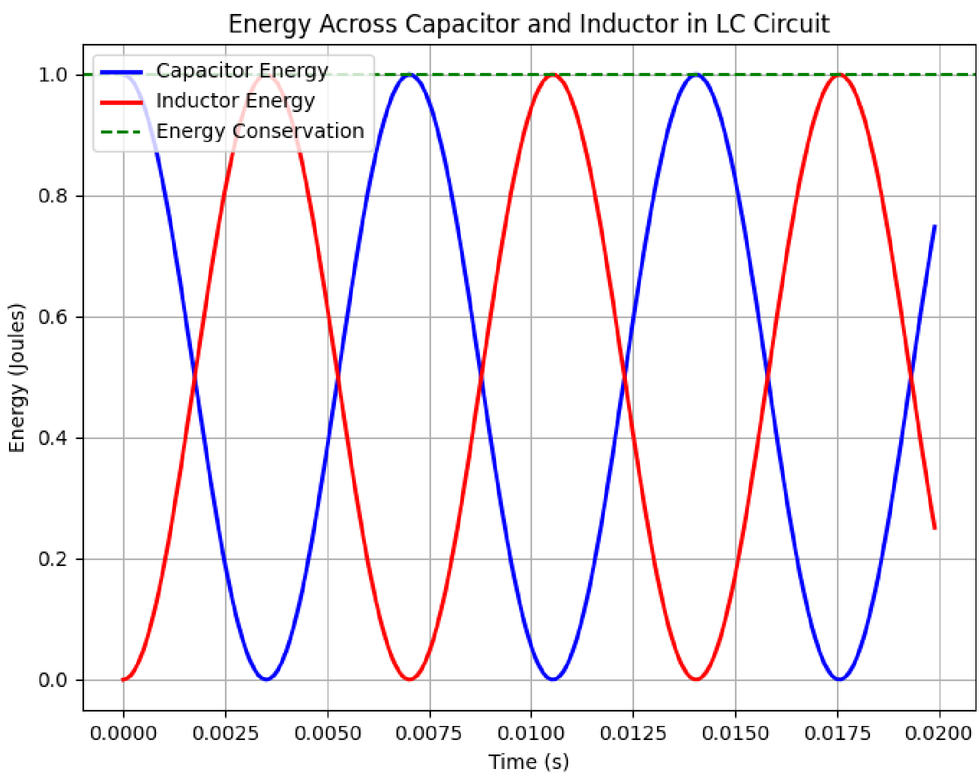


Figure 2.1: Energy is Conserved During Oscillations total energy being limited to the initial energy

(b) The Laplace domain circuit :

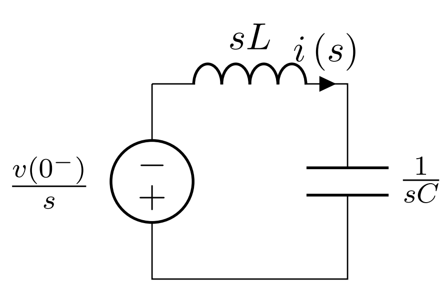


Figure 2.2: LC Circuit in lapalace domain

Writing KVL in Fig. 2.2

$$\frac{-200}{s} - I(s) Ls - I(s) \frac{1}{sC} = 0 \tag{2.3}$$

$$I(s) = \frac{-10^8}{25s^2 + 10^{10}} \tag{2.4}$$

Simplifying ,

$$I(s) = -200 \left(\frac{2 \times 10^4}{s^2 + (2 \times 10^4)^2} \right) \tag{2.5}$$

Now,

$$\sin(at) u(t) \xleftrightarrow{\mathcal{L}} \frac{a}{s^2 + a^2} \tag{2.6}$$

$$\sin\left(2 \times 10^4 t\right) u(t) \xleftrightarrow{\mathcal{L}} \frac{2 \times 10^4}{s^2 + (2 \times 10^4)^2} \tag{2.7}$$

Uisng (2.7) on (2.4)and taking inverse-Laplace transform:

$$i(t) = -200 \sin(2 \times 10^4 t) u(t) \tag{2.8}$$

From Fig. 2.2 voltage across capacitor

$$V(s) = -\left(I(s) \frac{1}{sC} + \frac{V(0^-)}{s} \right) \tag{2.9}$$

$$= \frac{5 \times 10^3 s}{25s^2 + 10^{10}} \tag{2.10}$$

Simplifying,

$$V(s) = 200 \left(\frac{s}{s^2 + (2 \times 10^4)^2} \right) \tag{2.11}$$

$$\cos(at)\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+a^2} \tag{2.12}$$

$$\cos\left(2\times10^4 t\right)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\left(2\times10^4\right)^2} \tag{2.13}$$

Using (2.13) on (2.10)and taking inverse-Laplace transform:

$$v(t) = 200\cos(2\times10^4 t)u(t) \tag{2.14}$$

The impedance of the circuit :

$$Z = \frac{V(s)}{I(s)} = sL + \frac{1}{sC} \tag{2.15}$$

s can be expressed in angular frequency as :

$$s = j\omega \tag{2.16}$$

$$Z = \left(\omega L - \frac{1}{\omega C}\right)j \tag{2.17}$$

$$|Z| = \omega L - \frac{1}{\omega C} \tag{2.18}$$

For resonant angular frequency imaginary part of impedance is zero :

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{2.19}$$

$$= 20,000\text{rad/sec} \tag{2.20}$$

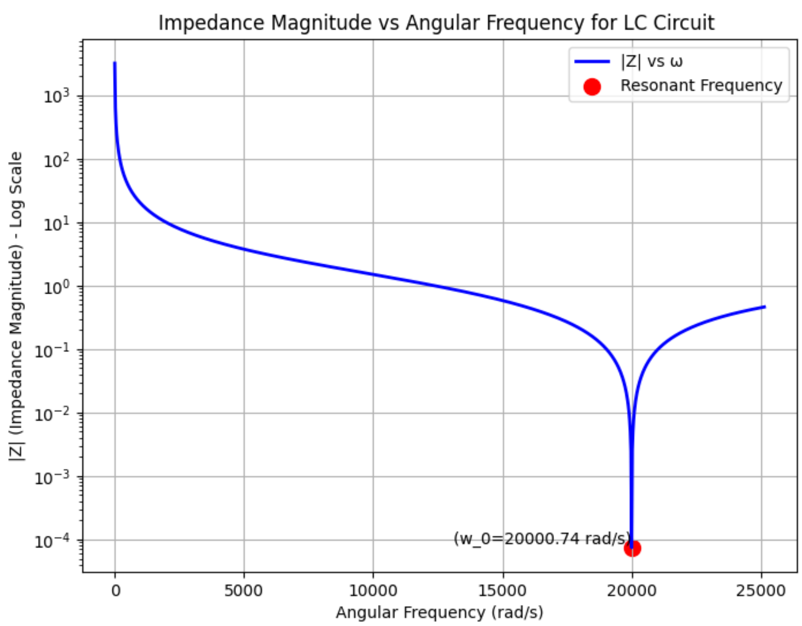


Figure 2.3: The frequency at which impedance is minimum is resonant frequency which is at 20000 rad/sec

(c) The energy stored is completely electrical when $i(t) = 0$.

The energy stored is completely magnetic when $v(t) = 0$

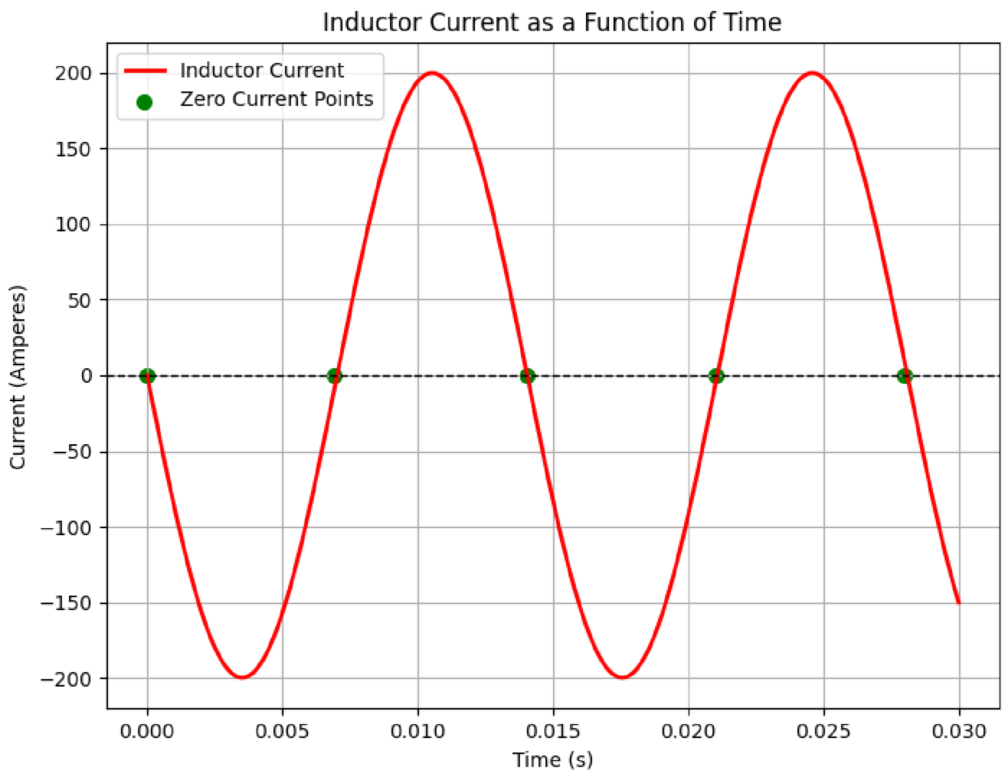


Figure 2.4: Energy is completely electrical at the marked points.

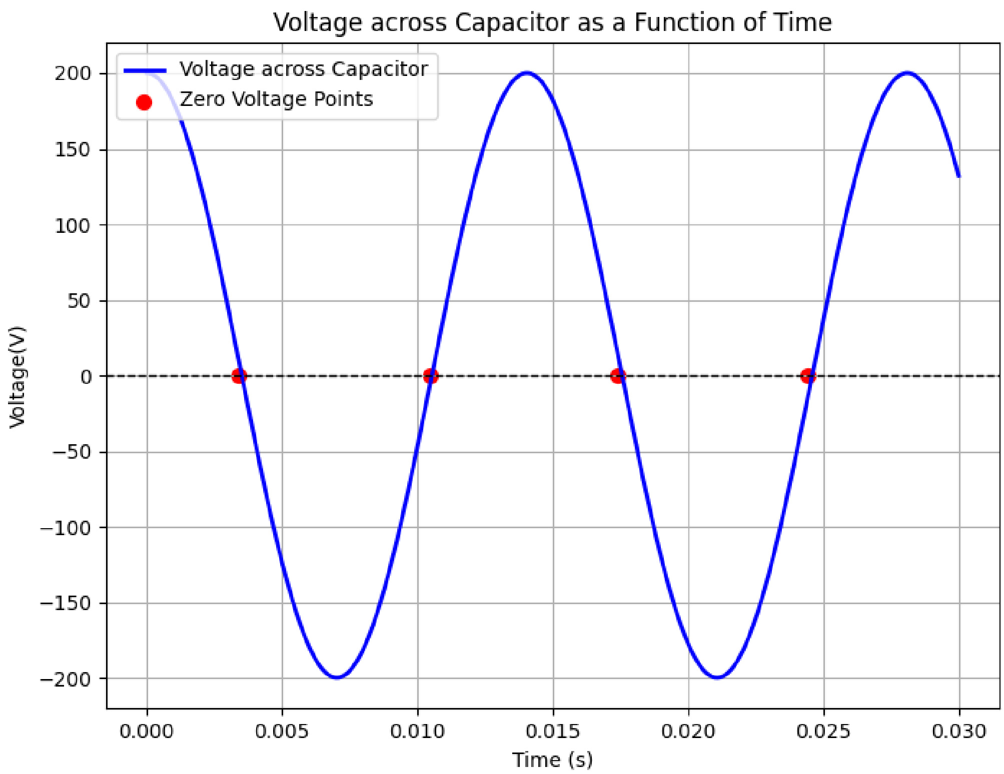


Figure 2.5: Energy is completely magnetic at the marked points.

(d) When energy is equally shared then the capacitor has half of the maximum energy.

$$\frac{1}{2}C\left(v\left(t\right)\right)^2 = \frac{1}{2} \tag{2.21}$$

$$\cos\omega_o t = \frac{1}{\sqrt{2}} \tag{2.22}$$

$$t = \frac{(2n+1)T}{8}, T = \frac{2\pi}{\omega_o} \tag{2.23}$$

Hence, the total energy is equally shared between the inductor and capacitor at the time, $t = \frac{T}{8}, \frac{3T}{8}, \frac{5T}{8} \ldots$

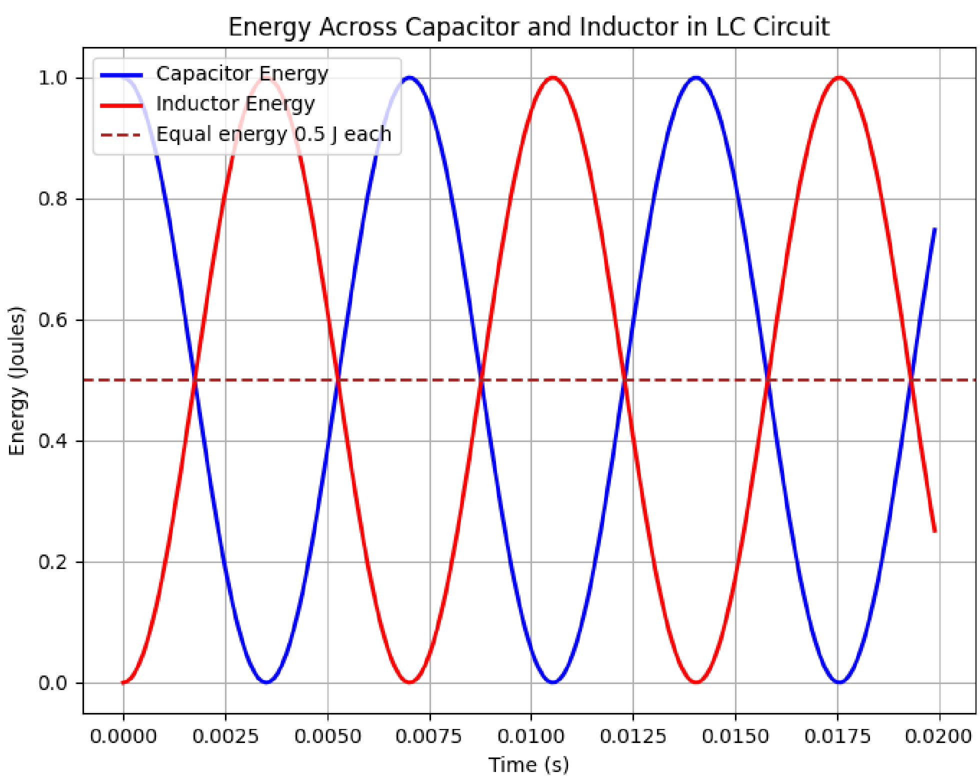


Figure 2.6: At the intersection points with 0.5 J horizontal line both capacitor and inductor have equal energy.

(e) Once the resistor is added to the LC circuit, it starts dissipating energy in the form of heat.

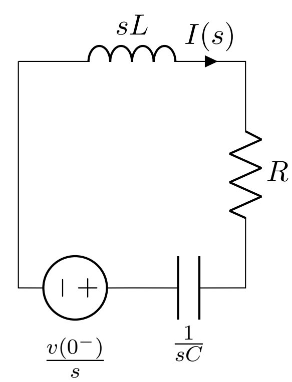


Figure 2.7: Laplace domain LCR Circuit with $R = 1\Omega$

Applying KVL in Fig. 2.7:

$$\left(\frac{1}{sC} + sL + R\right) I\left(s\right) = \frac{v\left(0^-\right)}{s} \tag{2.24}$$

$$I\left(s\right) = \frac{10^{10}}{2500s^2 + 50 \times 10^6 s + 10^{12}} \tag{2.25}$$

$$= -\frac{400}{\sqrt{3}}\left(\frac{\sqrt{3} \times 10^4}{\left(s + 10^4\right)^2 + \left(\sqrt{3} \times 10^4\right)^2}\right) \tag{2.26}$$

By frequency-shifting property:

$$e^{-\alpha t} x\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} X\left(s + \alpha\right) \tag{2.27}$$

Applying (2.27) on (2.7)

$$e^{-10^4 t} \sin(10^4\sqrt{3}t) u\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} \left(\frac{\sqrt{3} \times 10^4}{\left(s + 10^4\right)^2 + \left(\sqrt{3} \times 10^4\right)^2}\right) \tag{2.28}$$

Using (2.28) and Taking inverse Laplace transform:

$$i\left(t\right) = -\frac{400}{\sqrt{3}} e^{-10^4 t} \sin(10^4\sqrt{3}t) \tag{2.29}$$

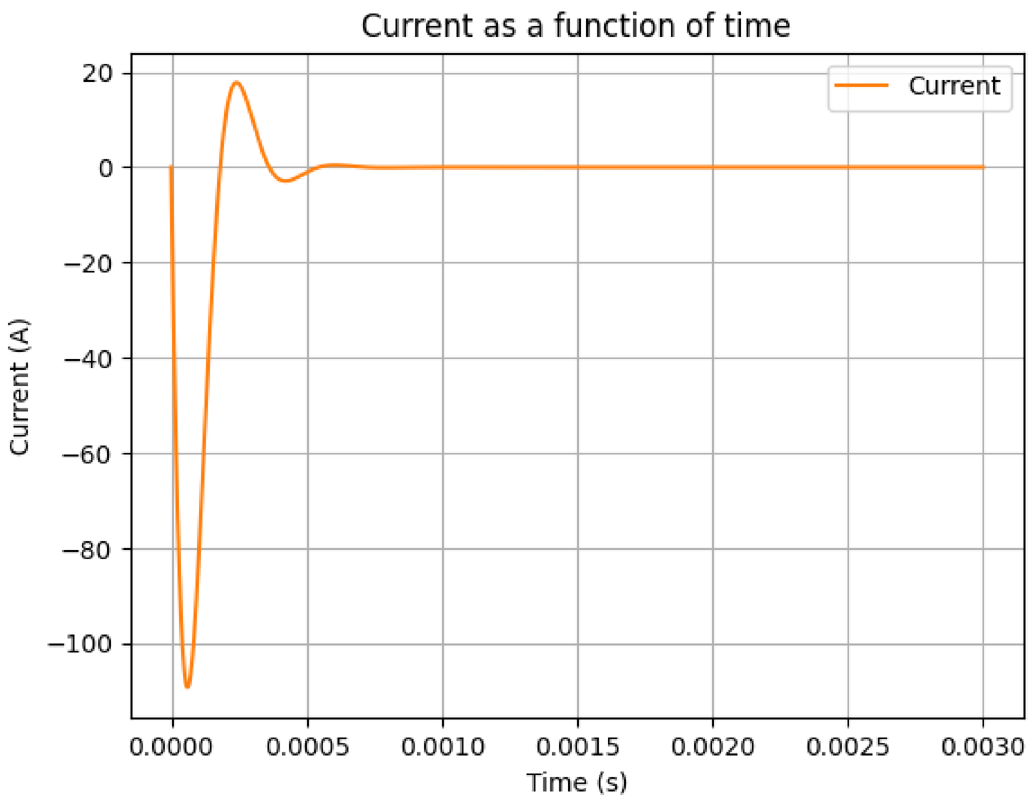


Figure 2.8: Graph of current in the circuit. Becomes zero after all the energy is dissipated, $R = 1\Omega$

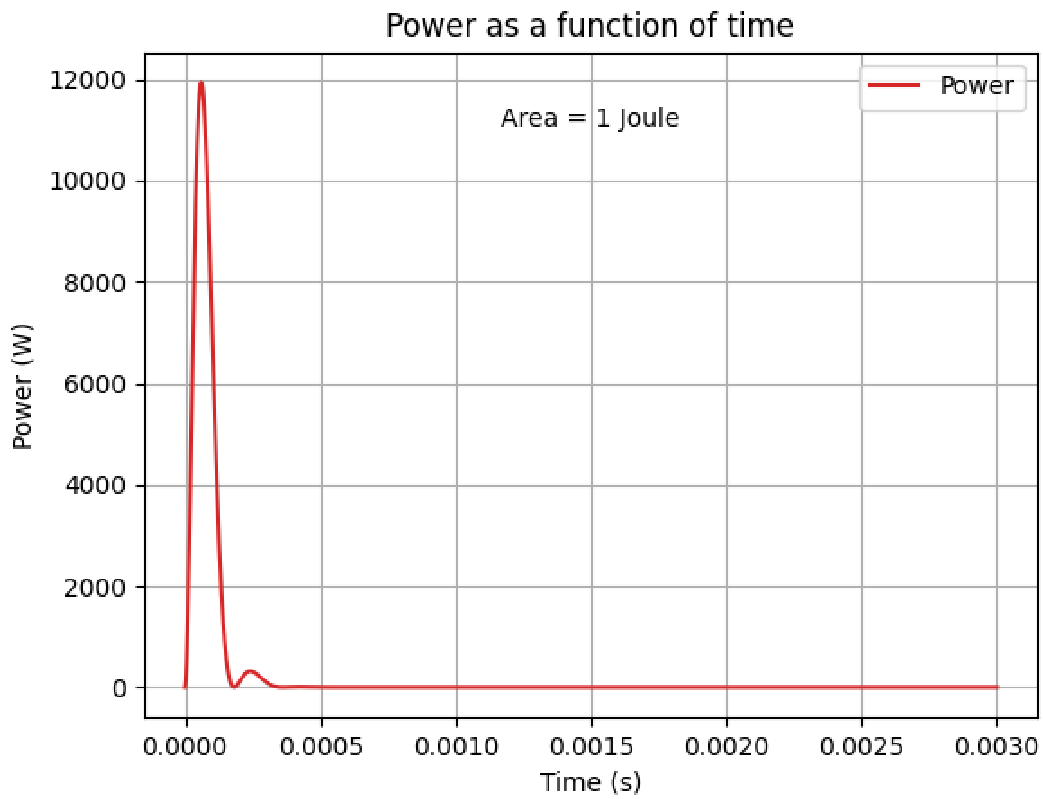


Figure 2.9: Power Dissipated across resistor, $R = 1\Omega$. The area under the curve is 1J.

2.0.2 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Symbol	**Value**	**Description**
L	$3.0\,\mathrm{H}$	Inductance
C	$27\,\mu\mathrm{F}$	Capacitance
R	$7.4\,\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 2.2: Given Parameters

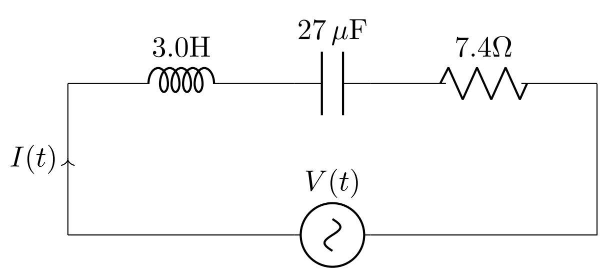


Figure 2.10: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{2.30}$$

Using reactances from Fig. 2.11,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{2.31}$$

$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{2.32}$$

$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{2.33}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

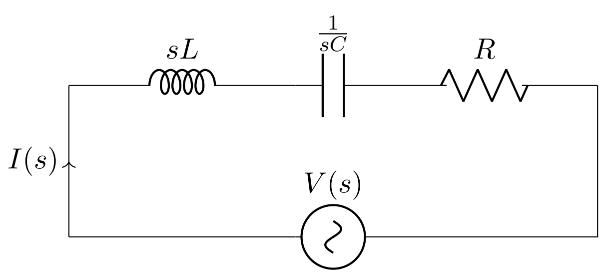


Figure 2.11: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{2.34}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{2.35}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{2.36}$$

Comparing (2.35) and (2.36), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.37}$$

(b) Quality Factor

i. Using voltage across inductor,

$$\begin{aligned}
Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} &= \frac{|sLI(s)|}{|RI(s)|} && (2.38)\\
&= \frac{1}{\sqrt{LC}}\frac{L}{R} && (2.39)\\
&= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.40)
\end{aligned}$$

ii. Using voltage across capacitor,

$$\begin{aligned}
Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} &= \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} && (2.41)\\
&= \frac{\sqrt{LC}}{RC} && (2.42)\\
&= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.43)
\end{aligned}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{2.44}$$

Using (2.33),

$$H(s) = R + sL + \frac{1}{sC} \tag{2.45}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{2.46}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{2.47}$$

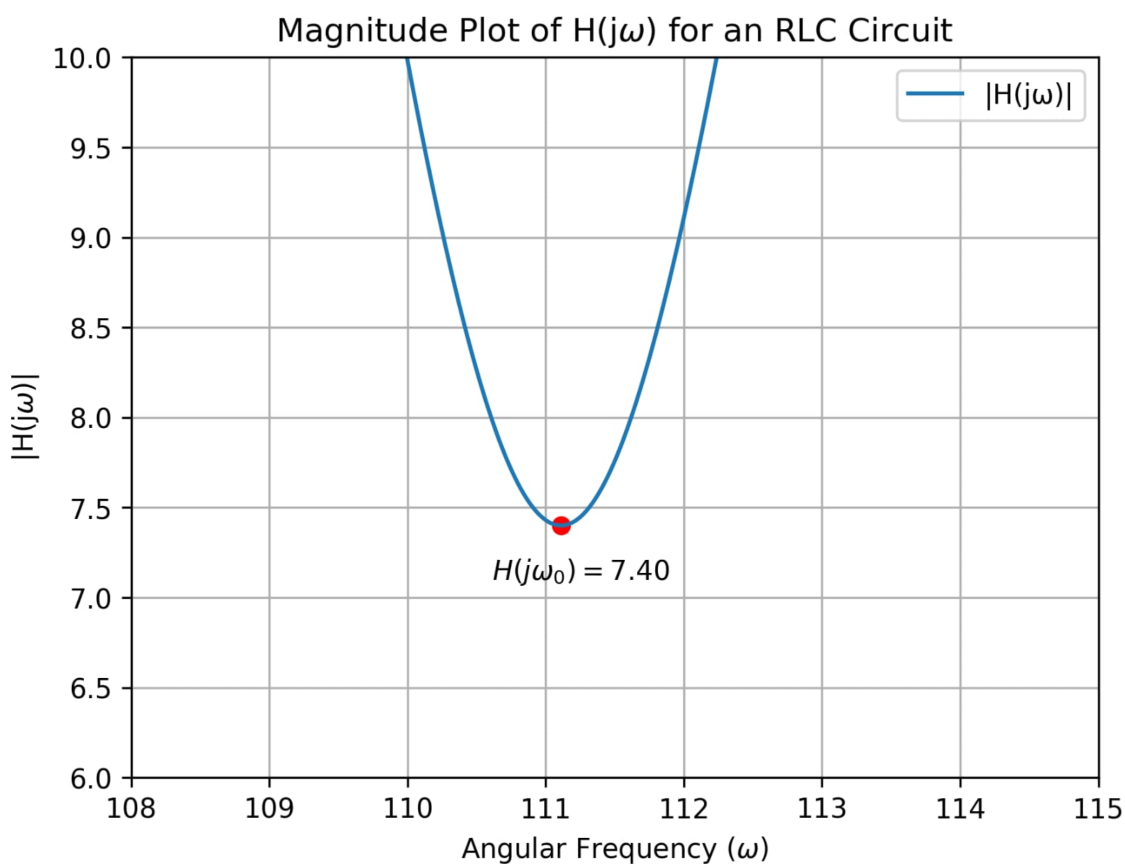


Figure 2.12: Impedance vs ω (using values in Table 2.2)

2.0.3 A circuit containing a $80mH$ inductor and a $60\mu F$ capacitor in series is connected to a $230V$, $50Hz$ supply. A resistance of 15Ω is connected in series. Obtain the average power transferred to each element of the circuit, and the total power absorbed.

Solution:

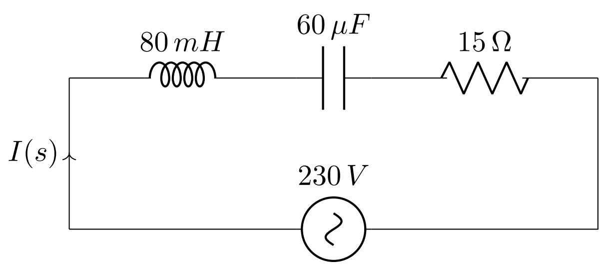


Figure 2.13: LCR Circuit

In Fig. 2.13 the following information is provided:

Symbol	Value	Description
L	$80mH$	Inductance
C	$60\,\mu F$	Capacitance
R	$15\,\Omega$	Resistance
V_{rms}	$230\,V$	Voltage
f	$50\,Hz$	Frequency
ω	$2\pi f = 100\pi$	Angular Frequency
ϕ	$-$	Phase difference between current and voltage
I_{rms}	$-$	rms value of current
V_m	$-$	Maximum voltage
I_m	$-$	Maximum current
P_m	$-$	Maximum Power

Table 2.3: Given Parameters

Applying Kirchoff's Voltage Law in the Fig. 2.14

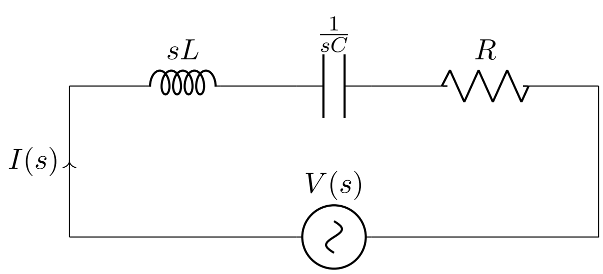


Figure 2.14: s domain circuit

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{2.48}$$

$$= I(s)\left(R + sL + \frac{1}{sC}\right) \tag{2.49}$$

$$I(s) = \frac{V(s)}{\left(R + sL + \frac{1}{sC}\right)} \tag{2.50}$$

$$H(s) = \frac{V(s)}{I(s)} \tag{2.51}$$

$$H(s) = R + sL + \frac{1}{sC} \tag{2.52}$$

Substituting s with $j\omega$

$$H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{2.53}$$

$$\Rightarrow |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{2.54}$$

Let the input voltage be:

$$V = V_m sin\,(\omega t) \tag{2.55}$$

Let the current at a given instant be:

$$I = I_m sin\,(\omega t - \phi) \tag{2.56}$$

Instantaneous power is given by:

$$P = VI \tag{2.57}$$

$$P = V_m sin\,(\omega t) \times I_m sin\,(\omega t - \phi) \tag{2.58}$$

Average power is given by:

$$P_{av} = \frac{W}{T} \tag{2.59}$$

$$dW = Pdt \tag{2.60}$$

Integrating on both sides

$$\begin{aligned}
W &= V_m I_m \int_0^T \sin(\omega t)\sin(\omega t - \phi)\; dt && (2.61)\\
&= V_m I_m \int_0^T \sin(\omega t)\left(\sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)\right)\; dt && (2.62)\\
&= V_m I_m \int_0^T (\sin(\omega t))^2 \cos(\phi)\, dt - V_m I_m \int_0^T \sin(\omega t)\cos(\omega t)\sin(\phi)\, dt && (2.63)\\
&= V_m I_m \int_0^T \frac{1-\cos(2\omega t)}{2}\cos(\phi)\, dt - V_m I_m \int_0^T \sin(2\omega t)\sin(\phi)\, dt && (2.64)
\end{aligned}$$

After solving the integral we get,

$$W = \frac{1}{2} V_m I_m T cos\phi \tag{2.65}$$

Relation between V_{rms} and V_m:

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{2.66}$$

Relation between I_{rms} and I_m:

$$I_{\text{rms}} = \frac{I_m}{\sqrt{2}} \tag{2.67}$$

a) The average power dissipated in a RLC circuit is given by :

$$P = V_{rms} I_{rms} cos(\phi) \tag{2.68}$$

The phase difference is given by:

$$tan\,(\phi) = \frac{\frac{1}{\omega C} - \omega L}{R} \tag{2.69}$$

After substituting the values from Table 2.3:

$$tan\,(\phi) = 1.86 \tag{2.70}$$

Rms value of current I_{rms} is given by :

$$I_{rms} = \frac{V_{rms}}{R} = \frac{230}{15} = 15.33A \tag{2.71}$$

Now, susbtituting the value of ϕ, I_{rms} and values from Table 2.3 in (2.68) we obtain the total power :

$$P_{av} = 789.62W \tag{2.72}$$

b) Average power transferred to the capacitor, P_C:

For a capacitor the phase angle is:

$$\phi = \frac{\pi}{2} \tag{2.73}$$

$$cos(\phi) = 0 \tag{2.74}$$

$$P_C = 0 \tag{2.75}$$

c) Average power transferred to the inductor, P_L:

For an inductor the phase angle is:

$$\phi = -\frac{\pi}{2} \tag{2.76}$$

$$cos(\phi) = 0 \tag{2.77}$$

$$P_L = 0 \tag{2.78}$$

d) Average Power transferred to the resistor, P_R:

$$P_{avg} = P_R + P_C + P_L \tag{2.79}$$

$$P_R = P_{avg} - P_C - P_L \tag{2.80}$$

$$P_R = 789.62 - 0 - 0 \tag{2.81}$$

$$P_R = 789.62W \tag{2.82}$$

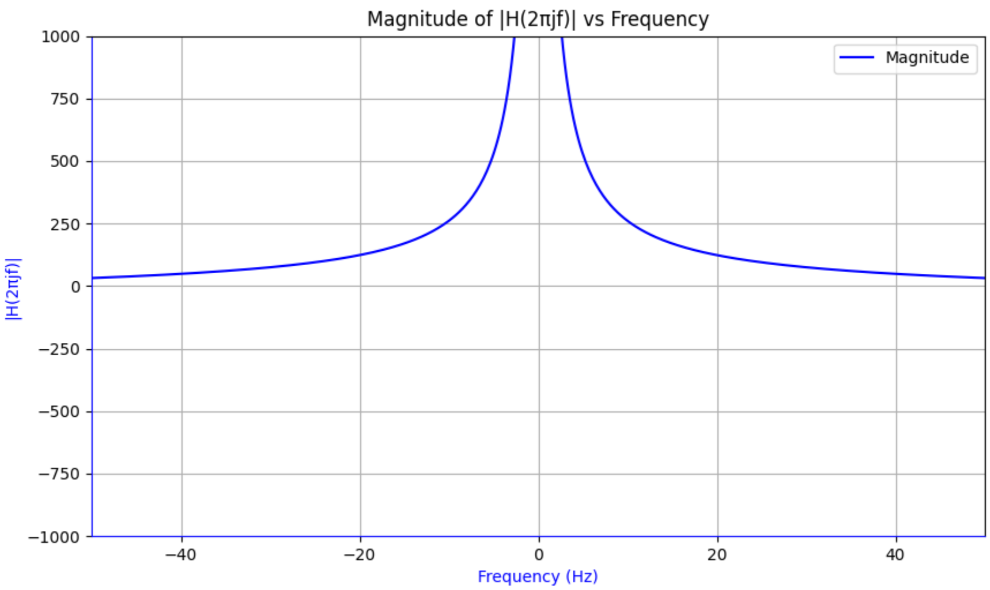


Figure 2.15: $|H(j/omega)|$ vs ω

Bandwidth is defined as the range of frequencies, where power ranges from its maximium value to half of its maximum value.

$$I_{rms} = \frac{V_{rms}}{|H(j\omega)|} \tag{2.83}$$

At maximum power, $|H(j\omega)|$ will be minimum,

$$|H(j\omega)| = R \tag{2.84}$$

$$I_m = \frac{V_{rms}}{R} \tag{2.85}$$

when, power is half of the maximum value of power

$$P = \frac{P_m}{2} \tag{2.86}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}} \tag{2.87}$$

$$|H(j\omega)| = \sqrt{2}R \tag{2.88}$$

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}R \tag{2.89}$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2 \tag{2.90}$$

This equation has 2 roots, ω_1 and ω_2:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4R} + \frac{1}{LC}} \tag{2.91}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4R} + \frac{1}{LC}} \tag{2.92}$$

Thus Bandwidth of circuit is :

$$\omega_2 - \omega_1 = \frac{R}{L} = 187.5 \tag{2.93}$$

$$f = \frac{\omega_2 - \omega_1}{2\pi} = 29.85 \tag{2.94}$$

2.0.4 A series LCR circuit with $L = 0.12H$ $C = 480nF$ $R = 23\Omega$ is connected to a $230V$ variable frequency supply.

(a) What is the source frequency for which current amplitude is maximum? Obtain this maximum value.

(b) What is the source frequency for which the average power absorbed by the circuit is maximum? Obtain the value of this maximum power.

(c) For which frequencies of the source is the power transferred to the circuit half the power at resonant frequency? What is the current amplitude at these frequencies?

(d) What is the Q-factor of the given circuit?

Solution:

Given parameters are:

Table 2.1: Given Data

Symbol	**Value**	**Parameter**
L	0.12 H	Inductance
C	480 nF	Capacitance
R	23 Ω	Resistance
V	230 V	Supply voltage

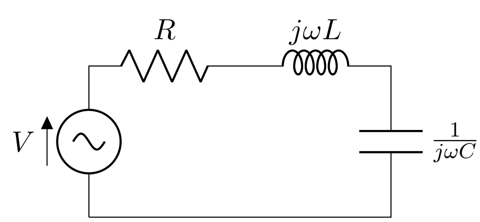


Figure 2.16: Circuit diagram with sinusoidal voltage source, resistor, inductor, and capacitor.

The impedance of the above circuit is given as:

$$H(s) = \frac{V(s)}{I(s)} \tag{2.95}$$

$$H(s) = R + sL + \frac{1}{sC} \tag{2.96}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{2.97}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{2.98}$$

(a) **Part (a):**

At resonance, the circuit becomes purely resistive. The reactances of capacitor and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{2.99}$$

$$\implies s = j\frac{1}{\sqrt{LC}} = j\omega \tag{2.100}$$

The current (I) is given by Ohm's Law as:

$$I = \frac{V}{Z} = \frac{V}{R + j(\omega L - \frac{1}{\omega C})} \tag{2.101}$$

Substitute the expression for Z into the current equation:

$$I = \frac{V}{R + j(\omega L - \frac{1}{\omega C})} \tag{2.102}$$

$$\tag{2.103}$$

$$|I| = \frac{V}{\sqrt{(\omega L - \frac{1}{\omega C})^2 + (R)^2}} \tag{2.104}$$

The source frequency for maximum current amplitude is given by:

$$\omega_{\text{max}} = \frac{1}{\sqrt{LC}} \tag{2.105}$$

Substitute the values and calculate:

$$\omega_{\text{max}} \approx 4166.67\,\text{rad/s} \tag{2.106}$$

(b) **Part (b):**

The source frequency for which the average power absorbed by the circuit is maximum is the same as the resonance frequency.

$$I_{\max} = \frac{V}{Z_{\text{total}}} = \frac{V}{R} \tag{2.107}$$

At resonance, $Z_{\text{total}} = R$, so $I_{\max} = \frac{V}{R}$.

$$P_{\text{avg}} = \frac{1}{2} I_{\max}^2 R \tag{2.108}$$

Substitute $I_{\max} = \frac{V}{R}$ into the expression for P_{avg}:

$$P_{\text{avg}} = \frac{1}{2} \left(\frac{V}{R}\right)^2 R \tag{2.109}$$

$$P_{\text{avg}} = \frac{1}{2} \frac{V^2}{R} \tag{2.110}$$

Substitute the given values and calculate:

$$P_{\text{avg}} = 1150\,\text{W} \tag{2.111}$$

(c) **Part (c):**

The power in the circuit is given by $P_{\max} = i_{\max}^2 R$. At half power frequencies, $P = \frac{P_{\max}}{2}$, and the current is $\frac{i_{\max}}{\sqrt{2}}$. Then, $V = \frac{i_{\max}}{\sqrt{2}} Z$.

$$Z^2 = R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2 \tag{2.112}$$

$$2R^2 = R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2 \tag{2.113}$$

$$R^2 = \left(2\pi fL - \frac{1}{2\pi fC}\right)^2 \tag{2.114}$$

$$R = \pm\left(2\pi fL - \frac{1}{2\pi fC}\right) \tag{2.115}$$

This leads to two equations:

$$R = 2\pi f_1 L - \frac{1}{2\pi f_1 C} \tag{2.116}$$

$$R = \frac{1}{2\pi f_2 C} - 2\pi f_2 L \tag{2.117}$$

Solving these equations gives the half power frequencies f_1 and f_2.

Additionally, the bandwidth $\Delta\omega$ is related to R and L by $\Delta\omega = \frac{R}{2L}$. In terms of angular frequency ω, we have $\omega_1 - \omega_2 = \frac{R}{L}$.

$$\omega' = \omega_R \pm \Delta\omega \tag{2.118}$$

$$\Delta\omega = \frac{R}{2L} \tag{2.119}$$

Substitute the given values and calculate:

$$\Delta\omega = 95.83\,\text{rad/s} \tag{2.120}$$

Finally,

$$\omega_1' = \omega_{\text{max}} + \Delta\omega = 4262.3\,\text{rad/s} \tag{2.121}$$

$$\omega_2' = \omega_{\text{max}} - \Delta\omega = 4070.87\,\text{rad/s} \tag{2.122}$$

The amplitude of current at these frequencies is the RMS value, which is 10 A.

(d) **Part (d):**

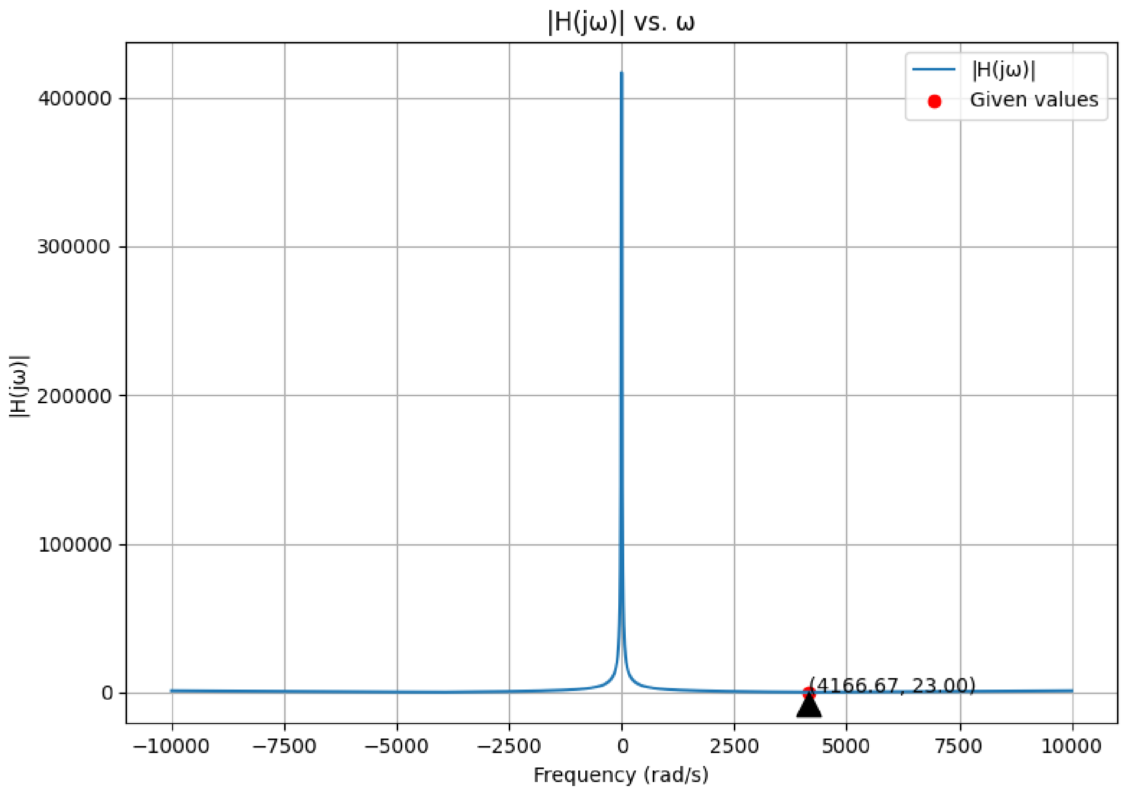
The Q-factor (Q) of a series RLC circuit is given by the formula:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Substitute the given values and calculate:

$$Q \approx \frac{1}{23}\sqrt{\frac{0.12}{480 \times 10^{-9}}} \tag{2.123}$$

$$Q \approx 39.6826 \tag{2.124}$$



2.0.5 A radio can tune over the frequency range of a portion of the MW broadcast band: (800 kHz to 1200 kHz). If its LC circuit has an effective inductance (L) and a variable capacitor with capacitance (C), what must be the range of C?

Solution:

Chapter 3

Z-transform

3.0.1 Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

Solution:

Symbols	Description	Values
r	Common ratio of the GP	-2
$x(n)$	$(n+1)^{th}$ term of the Sequence	$x(0)r^n u(n)$
$x(0)$	First term of the GP	3
$x(2)-x(0)$	First constraint	9
$x(1)-x(3)$	Second constraint	18

Table 3.1: Parameters, Descriptions, and Values

From the constraints given in 3.1:

$$x\left(0\right)r^2 - 9 = x\left(0\right) \tag{3.1}$$

$$x\left(0\right)r + 18 = x\left(0\right)r^3 \tag{3.2}$$

$$\implies x(0)(r^2-1) = 9 \tag{3.3}$$

$$\implies x(0)r(r^2-1) = 18 \tag{3.4}$$

By dividing (3.3) and (3.4) and solving ,we get:

$$\implies x(0) = 3 \tag{3.5}$$

$$\implies r = -2 \tag{3.6}$$

Z-Transform for x(n) : Using (**??**) :

$$X(z) = \frac{1}{1+2z^{-1}}, \quad |z| > |2| \tag{3.7}$$

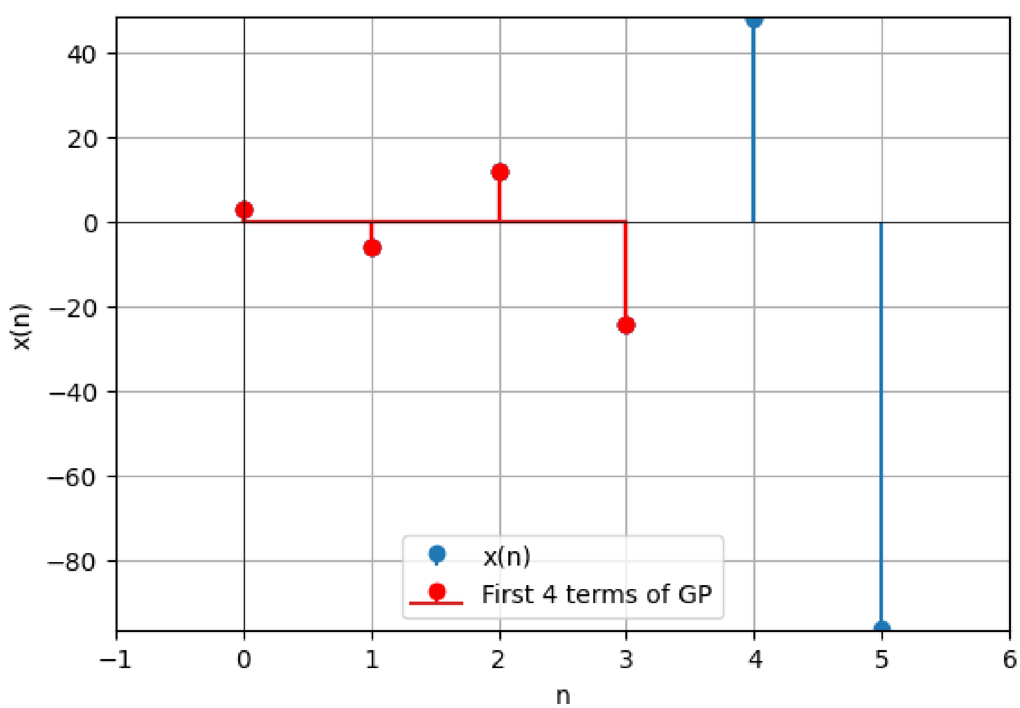


Figure 3.1: $x(n)$ vs n

3.0.2 The 4^{th} term of a G.P. is square of its second term, and the first term is -3. Determine its 7^{th} term.

Solution:

Variable	**Description**	**value**
$x(0)$	first term of G.P.	-3
r	Common ratio of G.P.	?
$x(n)$	general term of the G.P.	$x(0)\, r^n$
$x(3)$	fourth term	$[x(1)]^2$
$u(n)$	unit step function	-

Table 3.2: A Table with input parameters

from Table 3.2

$$x(0)\, r^3 = \left(x(0)\, r^1\right)^2 \tag{3.8}$$

$$= x(0)^2\, r^2 \tag{3.9}$$

$$\implies r = x(0) \tag{3.10}$$

$$= -3 \tag{3.11}$$

general term

$$x(n) = x(0)\, r^n u(n) \tag{3.12}$$

$$= (-3)^{n+1}\, u(n) \tag{3.13}$$

The 7^{th} term of the sequence will be:

$$x(6) = (-3)(-3)^6 \tag{3.14}$$

$$= -2187 \tag{3.15}$$

Z transform of the given G.P is:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} = \frac{-3}{1 + 3z^{-1}}. \quad |z| > 3 \tag{3.16}$$

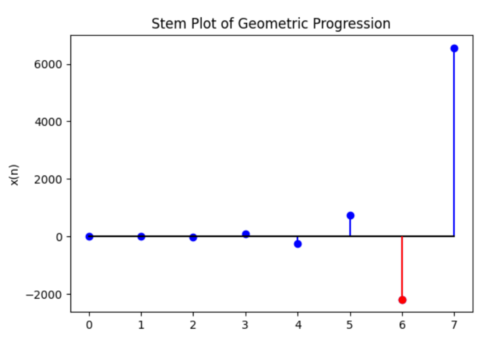


Figure 3.2: Graph showing first 8 terms of the GP

3.0.3 Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

Parameter	**Description**	**Value**
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3+5n^2+8n+4\right)\cdot u\left(n\right)$
$x_2(n)$	General Term of Denominator	$\left(n^3+4n^2+5n+2\right)\cdot u\left(n\right)$
$y_1\left(n\right)$	Sum of terms of numerator	?
$y_2\left(n\right)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}},\{z\in\mathbb{C}:\lvert z\rvert>1\}$
ROC	Region of convergence	$\left\{z:\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\rvert<\infty\right\}$

Table 1: Parameter Table

1. <u>Analysis of Numerator:</u>

$$X_1\left(z\right)=\sum_{n=-\infty}^{\infty}x_1\left(n\right)z^{-n} \tag{3.17}$$

$$=\sum_{n=-\infty}^{\infty}\left(n^3+5n^2+8n+4\right)u\left(n\right)z^{-n} \tag{3.18}$$

Using results of equations (**??**) to (**??**) we get:

$$\therefore X_1\left(z\right)=\frac{4+2z^{-1}}{\left(1-z^{-1}\right)^4},\lvert z\rvert>1 \tag{3.19}$$

From (**??**)

$$y_1(n) = x_1(n) * u(n) \tag{3.20}$$

$$Y_1(z) = X_1(z)\,U(z) \tag{3.21}$$

$$= \frac{4+2z^{-1}}{(1-z^{-1})^5}, |z| > 1 \tag{3.22}$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{(1-z^{-1})} + \frac{48z^{-2}}{(1-z^{-1})^2} + \frac{52z^{-3}}{(1-z^{-3})^3}, \tag{3.23}$$
$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4, |z| > 1$$

Substituting results of equation (**??**) to (**??**) in equation (3.23):

$$y_1(n) = \frac{3n^4+26n^3+81n^2+106n+48}{12}u(n) \tag{3.24}$$

$$= \frac{(3n+8)(n+1)(n+2)(n+3)}{12}u(n) \tag{3.25}$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\,z^{-n} \tag{3.26}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3+4n^2+5n+2\right)u(n)\,z^{-n} \tag{3.27}$$

Using results of equation (**??**) to (**??**) we get:

$$\therefore X_2(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, |z| > 1 \tag{3.28}$$

From (**??**)

$$y_2(n) = x_2(n) * u(n) \tag{3.29}$$

$$Y_2(z) = X_2(z)\,U(z) \tag{3.30}$$

$$= \frac{2+4z^{-1}}{\left(1-z^{-1}\right)^5}, |z| > 1 \tag{3.31}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{\left(1-z^{-1}\right)} + \frac{36z^{-2}}{\left(1-z^{-1}\right)^2} + \frac{44z^{-3}}{\left(1-z^{-3}\right)^3} + \frac{26z^{-4}}{\left(1-z^{-1}\right)^4} + \frac{6z^{-5}}{\left(1-z^{-1}\right)^5} + 2, |z| > 1 \tag{3.32}$$

Substituting results of equation (**??**) to (**??**) in equation (3.32):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{3.33}$$

$$= \frac{(3n+4)(n+1)(n+2)(n+3)}{12} u(n) \tag{3.34}$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n+1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n+8}{3n+4} \tag{3.35}$$

Hence Prooved.

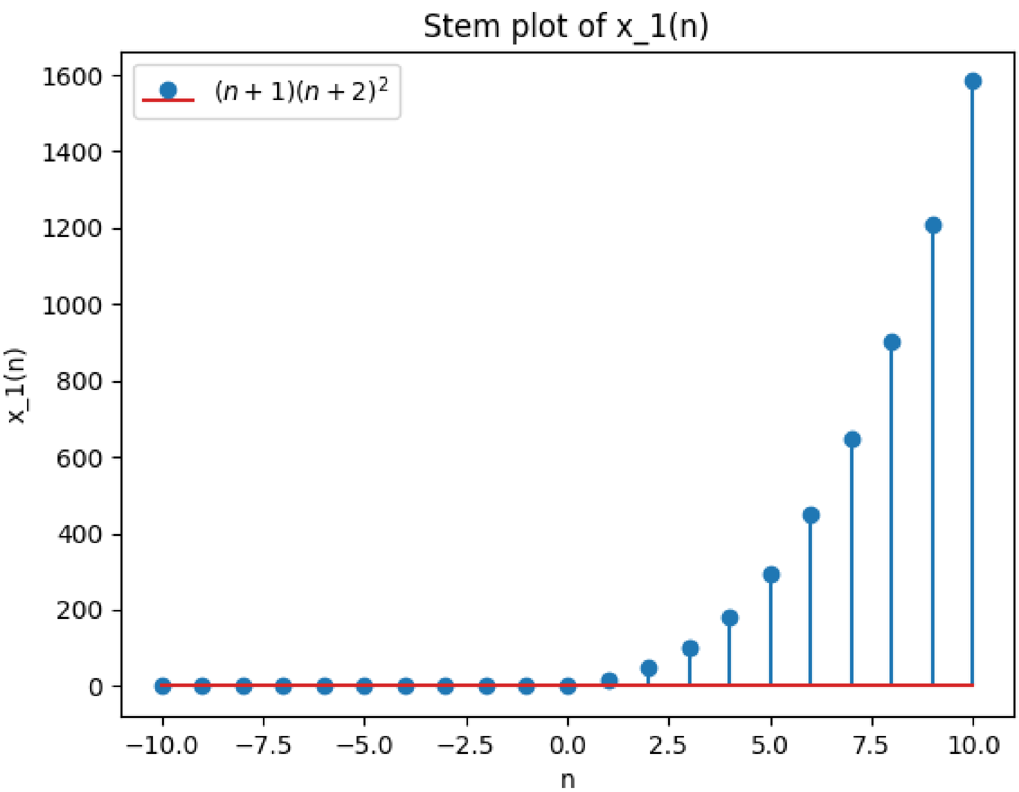


Figure 3.3: Stem Plot of $x_1(n)$

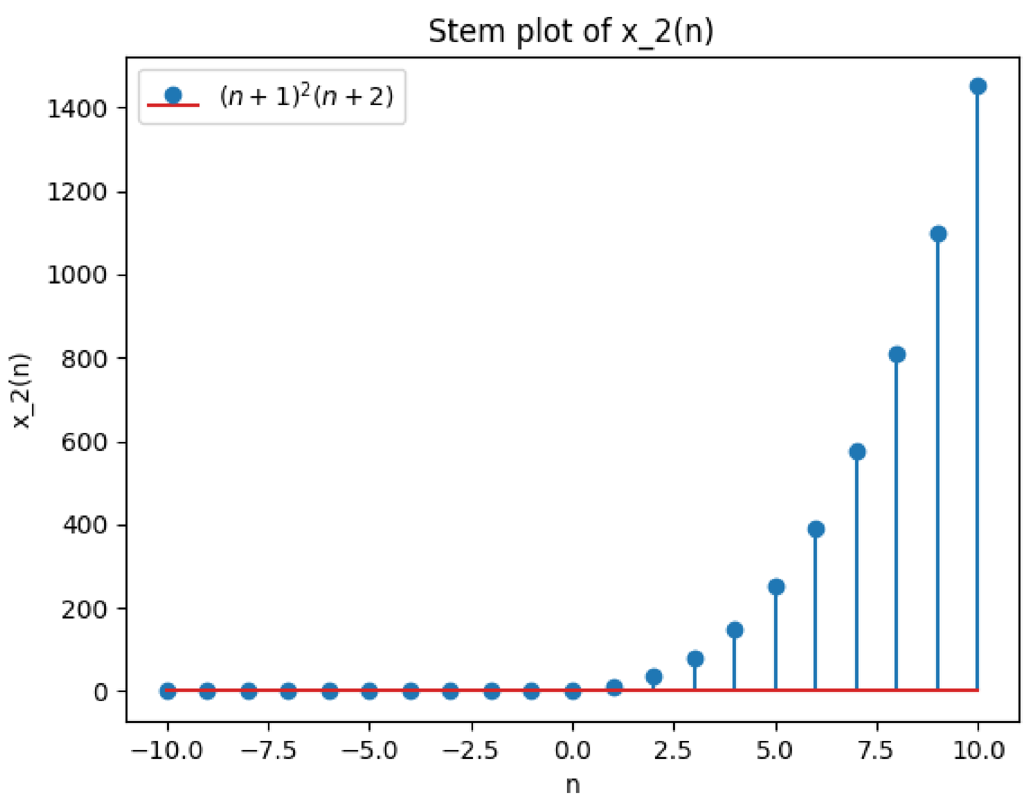


Figure 3.4: Stem Plot of $x_2\,(n)$

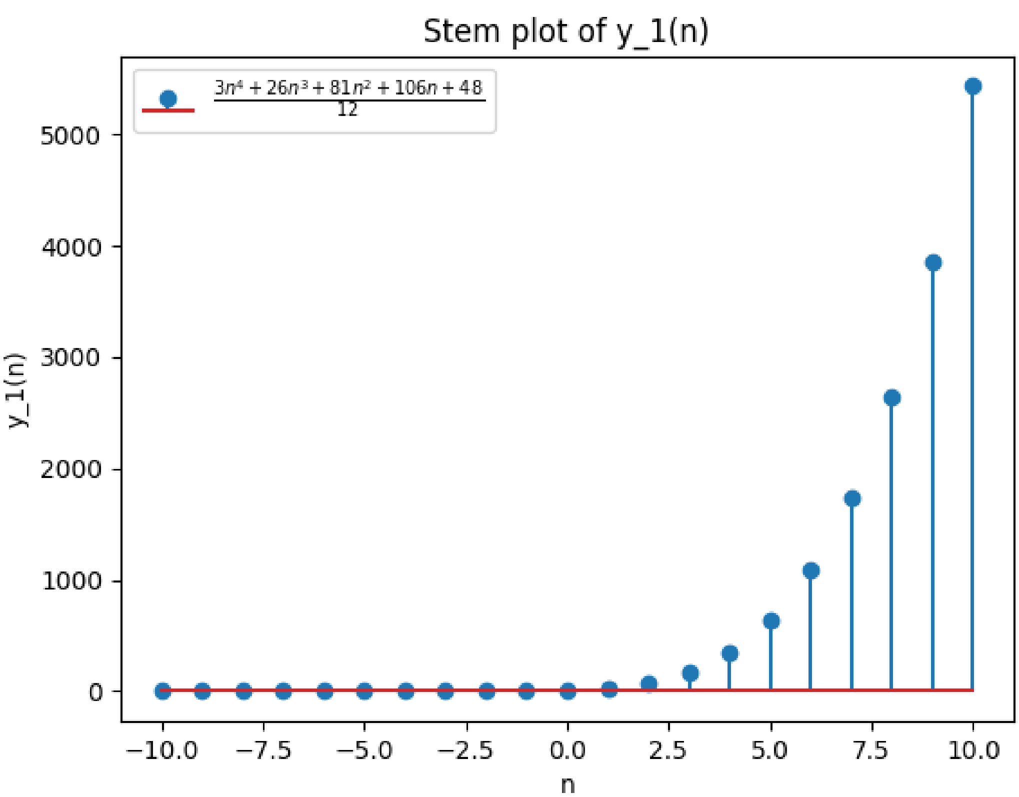


Figure 3.5: Stem Plot of $y_1(n)$

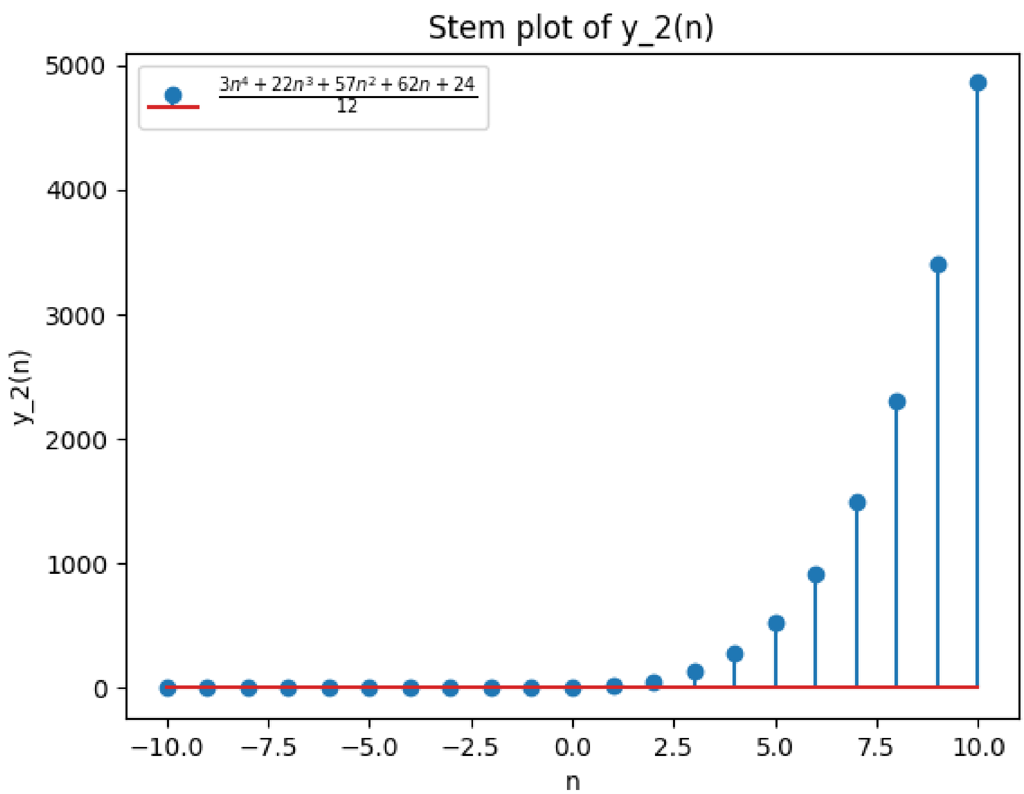


Figure 3.6: Stem Plot of $y_2(n)$

3.0.4 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \quad n = 0 \tag{3.36}$$

$$= \frac{x(n-1)}{n}, \quad n > 0 \tag{3.37}$$

$$= 0, \quad n < 0 \tag{3.38}$$

Solution:

$$x(1) = \frac{x(0)}{1} = -1 \tag{3.39}$$

$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{3.40}$$

$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{3.41}$$

$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{3.42}$$

$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{3.43}$$

$$x(n) = \frac{-1}{n!}(u(n)) \tag{3.44}$$

$$x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{3.45}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.46}$$

using (3.44),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \tag{3.47}$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{3.48}$$

$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{3.49}$$

Symbol	**Value**	**Description**
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 3.4: Parameters

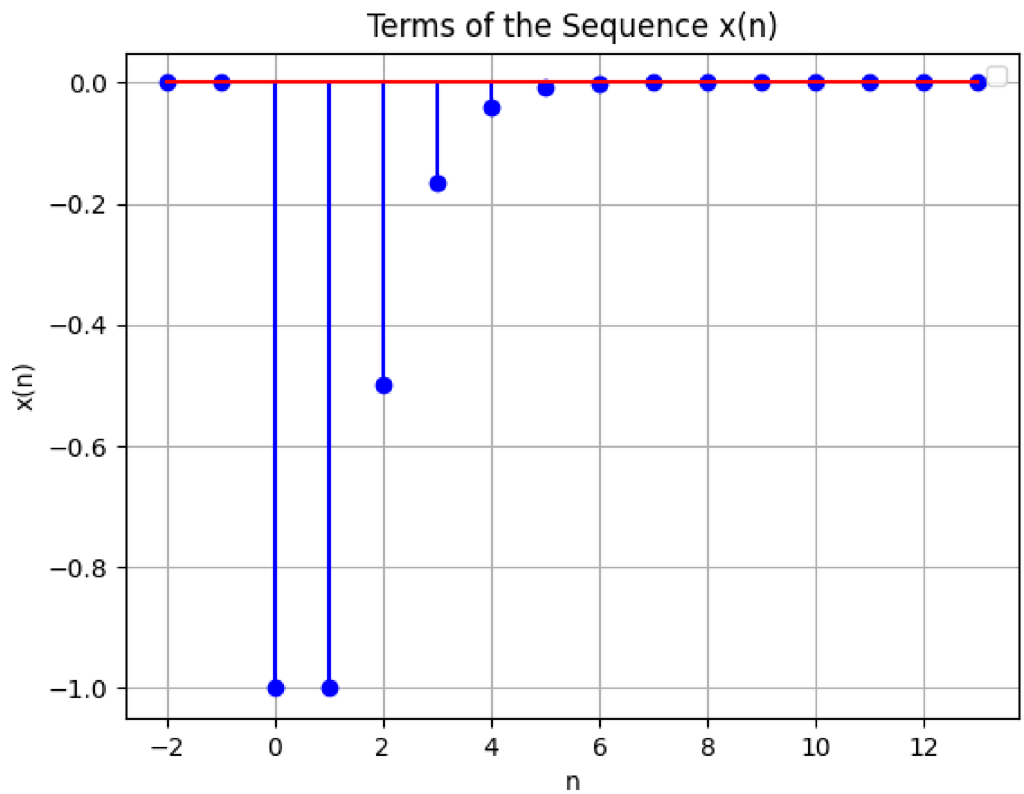


Figure 3.7: Plot of x(n) vs n

3.0.5 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of the AP

Table 3.5: Input Parameters

From the values given in Table 3.5:

$$7000 = 5000 + 200n \tag{3.50}$$

$$\implies 2000 = 200n \tag{3.51}$$

$$\therefore n = 10 \tag{3.52}$$

Let Z-transform of $x(n)$ be $X(z)$.

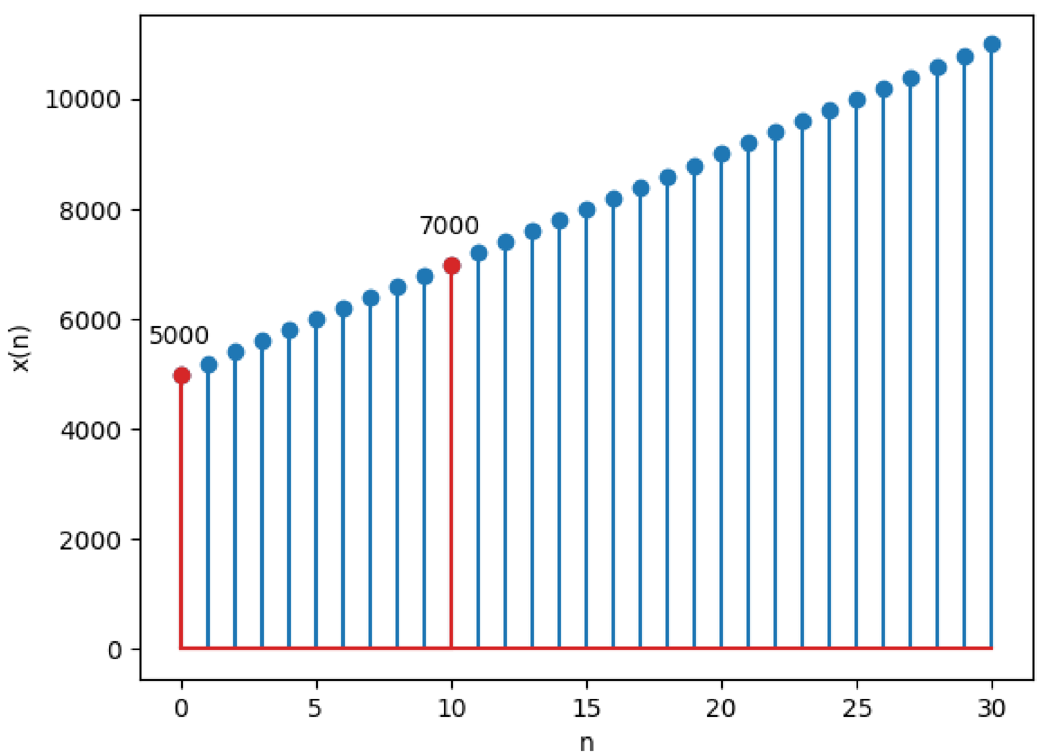


Figure 3.8: Plot of $x(n)$ vs n. See Table 3.5 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.53}$$

Using the values from Table 3.5:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.54}$$

3.0.6 Consider the sequence whose n^{th} term is given by 2^n. Find the first 6 terms of this sequence.

Solution:

Variable	**Description**	**Value**
$x(n)$	general term of sequence	$2^n u(n)$

Table 3.6: input parameters

$$X(Z) = \frac{1}{1 - 2z^{-1}} \quad |z| > |2| \tag{3.55}$$

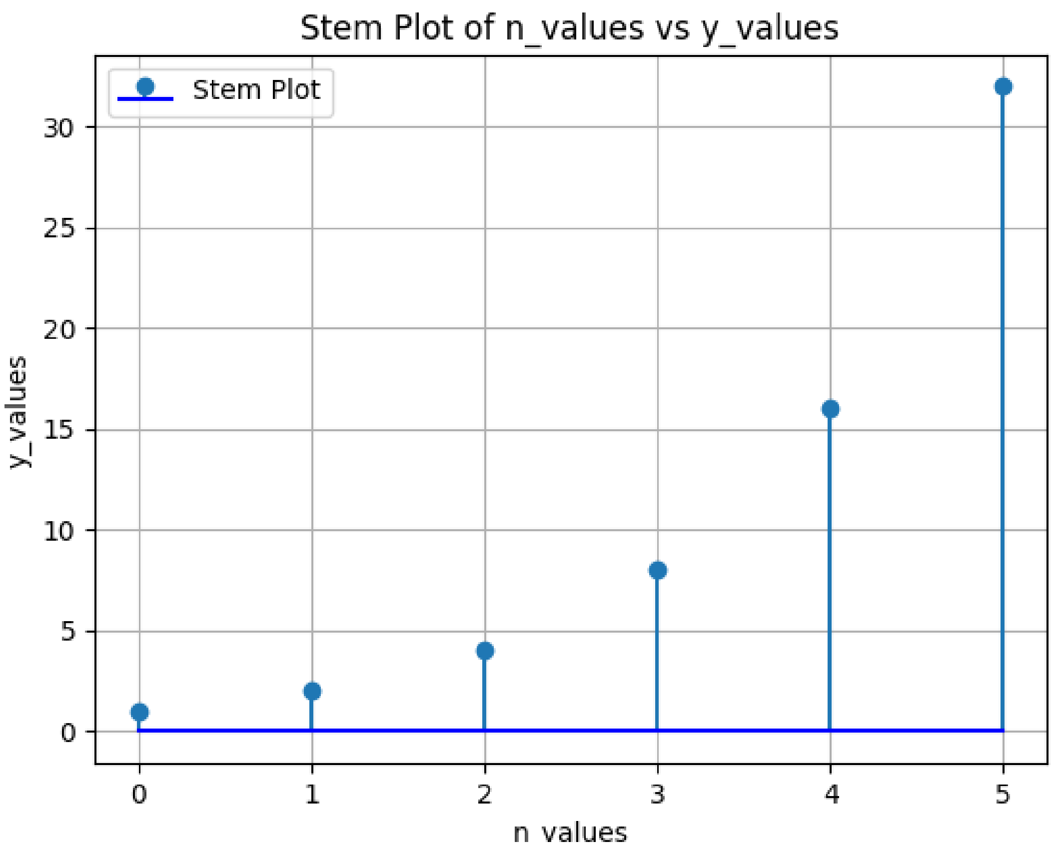


Figure 3.9: Six terms of given sequence

3.0.7 If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

Solution:

Variable	Description
$x(0)$	First term of the AP
d	Common difference of the AP
$y(n)$	Sum of $n+1$ terms of the AP
$x(n)$	General term

Table 3.7: Variables Used

$$y(n) = \frac{n+1}{2}\left(2x(0) + nd\right) u(n) \tag{3.56}$$

$$y(6) = 49 \tag{3.57}$$

$$y(16) = 289 \tag{3.58}$$

Then,

$$x(0) + 3d = 7 \tag{3.59}$$

$$x(0) + 8d = 17 \tag{3.60}$$

From equations 3.59 and 3.60, the augmented matrix is:

$$\begin{pmatrix} 1 & 3 & 7 \\ 1 & 8 & 17 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \tag{3.61}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{5} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 10 \end{pmatrix} \tag{3.62}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{3.63}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.64}$$

$$x(n) = (1 + 2n) u(n) \tag{3.65}$$

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \quad \{z \in \mathbb{C} : |z| > 1\} \tag{3.66}$$

$$y(n) = x(n) * u(n) \tag{3.67}$$

$$Y(z) = X(z)\, U(z) \tag{3.68}$$

$$\implies Y(z) = \left(\frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \right) \left(\frac{1}{1 - z^{-1}} \right) \tag{3.69}$$

$$= \frac{1}{(1 - z^{-1})^2} + \frac{2z^{-1}}{(1 - z^{-1})^3} \tag{3.70}$$

$$(n+1) u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})^2} \{z \in \mathbb{C} : |z| > 1\} \tag{3.71}$$

$$n((n+1) u(n)) \xleftrightarrow{\mathcal{Z}} \frac{2z^{-1}}{(1 - z^{-1})^3} \{z \in \mathbb{C} : |z| > 1\} \tag{3.72}$$

From equations (**??**) and (**??**), taking the inverse Z Transform,

$$y(n) = (n+1)\,u(n) + n\,((n+1)\,u(n)) \tag{3.73}$$

$$\implies y(n) = (n+1)^2\, u(n) \tag{3.74}$$

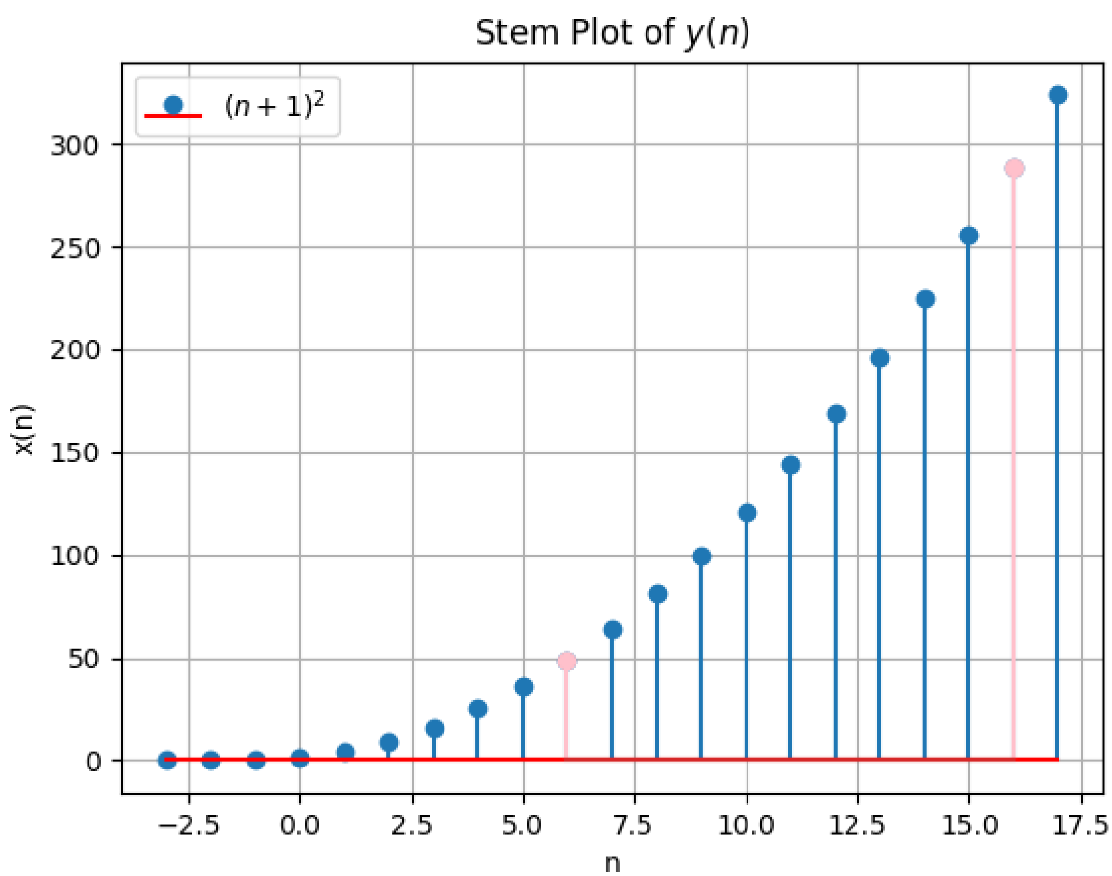


Figure 3.10: Stem Plot of y(n)

3.0.8 Write the first five terms of the sequence and obtain the corresponding series:

$a_1 = a_2 = 2,\, a_n = a_{n-1} - 1,\, n > 2$

Solution:

Parameter	Description	Value
$x(0)$	First term	2
$x(1)$	Second term	2
ROC	Region of convergence	$\{z : \lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\rvert < \infty\}$
$x(n)$	General term	$x(n) = \begin{cases} ? & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$

Table 1: Parameter Table

$$x(n) - x(n-1) = 2u(n) - 2u(n-1) - u(n-2) \tag{3.75}$$

$$X(z) - z^{-1}X(z) = \frac{2}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})} - \frac{2z^{-1}}{(1-z^{-1})} \tag{3.76}$$

$$X(z) = \frac{2 - 2z^{-1} - z^{-2}}{(1-z^{-1})^2}, |z| > 1 \tag{3.77}$$

Using partial fractions

$$X(z) = \frac{2z^{-1}}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})^2} + 2 \tag{3.78}$$

Taking inverse Z-transform by result of equation (**??**) in equation (3.78):

$$x(n) = 2u(n) + (1-n)u(n-1) \tag{3.79}$$

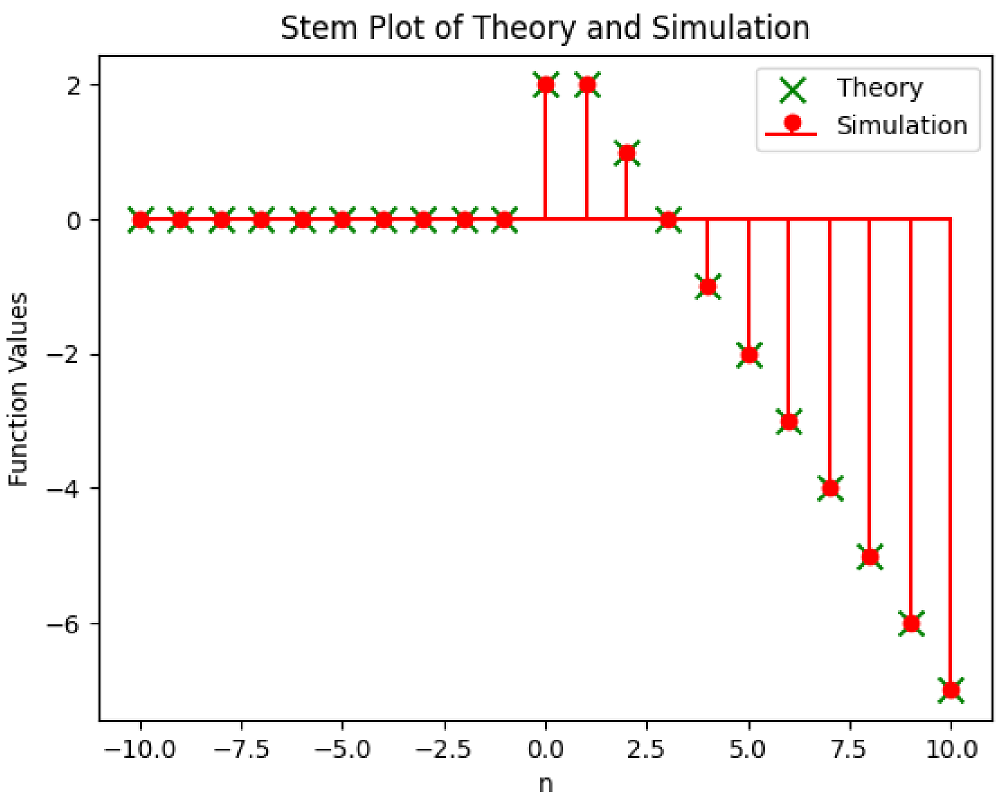


Figure 3.11: Comparison of Theory and Simulated Values

From the figureFig. 3.11 we can see that the theoretical and simulated values overlap.

3.0.9 Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution:

Parameter	**Description**	**Value**
$x(0)$	First term of G.P.	3
$x(3)$	Fourth term of G.P.	81
r	common ratio of G.P.	r

Table 3.9: input values

(a)

$$x(n) = x(0)\, r^n \tag{3.80}$$

from the values in Table 3.9

$$\frac{x(0)\, r^3}{x(0)} = 27 \tag{3.81}$$

$$r = 3 \tag{3.82}$$

$\therefore$ Required numbers are 9 and 27.

(b)

$$x(n) = 3^{n+1} u(n) \tag{3.83}$$

$$X(z) = \frac{3}{1 - 3z^{-1}} \quad |z| > 3 \tag{3.84}$$

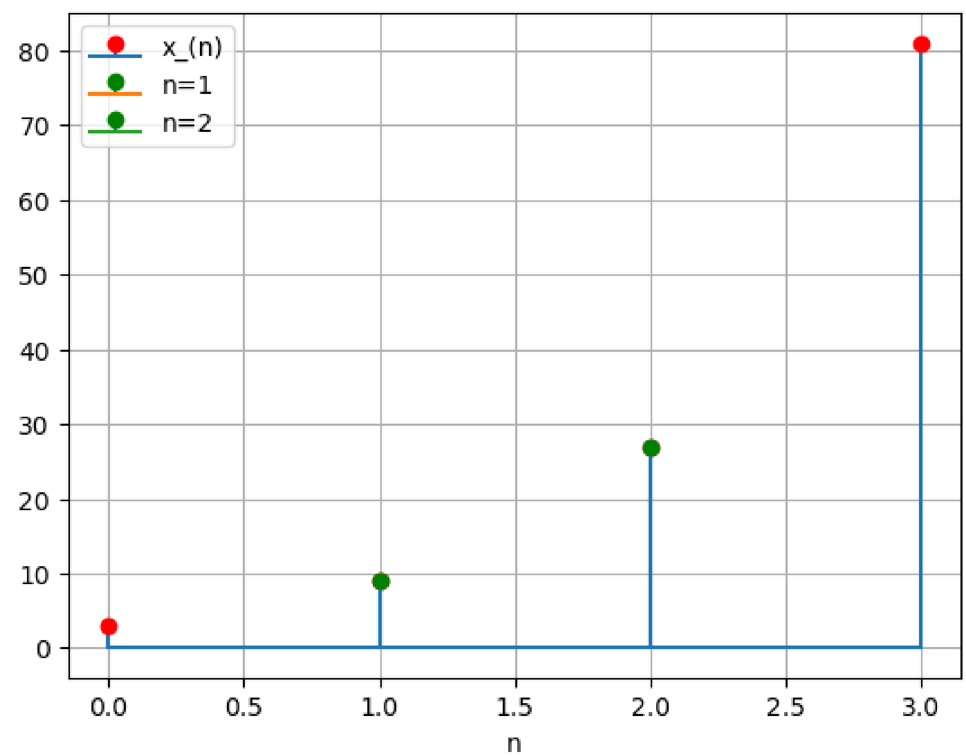


Figure 3.12: Graph of $x\,(n)$

3.0.10 What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Solution:

The Z-transform of a sequence $x(n)$ is given by:

$$x(n) = 500(1.1)^n u(n) \tag{3.85}$$

$$X(Z) = \frac{500}{1-(1.1)z^{-1}}; |z| > 1.1 \tag{3.86}$$

Parameter	Value	Description
$x(0)$	500	Principal amount before first year
r	1.1	Common ratio of GP
n	10	Number of years
$x(10)$	$500(1.1)^{10} = 1296.87$	Amount after 10 years

Table 3.10: Parameter Table

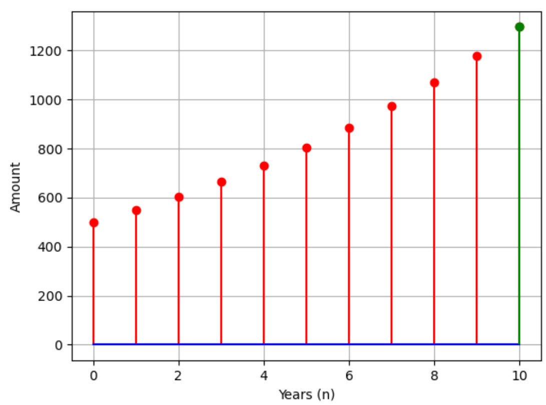


Figure 3.13: Plot of $x(n) = 500(1.1)^n$

3.0.11 Find the 20^{th} term from the last term of the AP: $3, 8, 13.....253$.

Solution: As the 20^{th} term is considered from last,

Parameter	Description	Value
$x(0)$	first term	253
d	common difference	$3 - 8 = -5$
$x(n)$	$(n+1)^{th}$ term	$(x(0) + nd)u(n)$
$u(n)$	unit step function	
$x(n)$	20^{th} term	158

Table 3.11: Input table

From equation (??) and (??):

$$X(z) = \frac{253}{1 - z^{-1}} + \frac{-5z^{-1}}{\left(1 - z^{-1}\right)^2}; \{|z| > 1\} \tag{3.87}$$

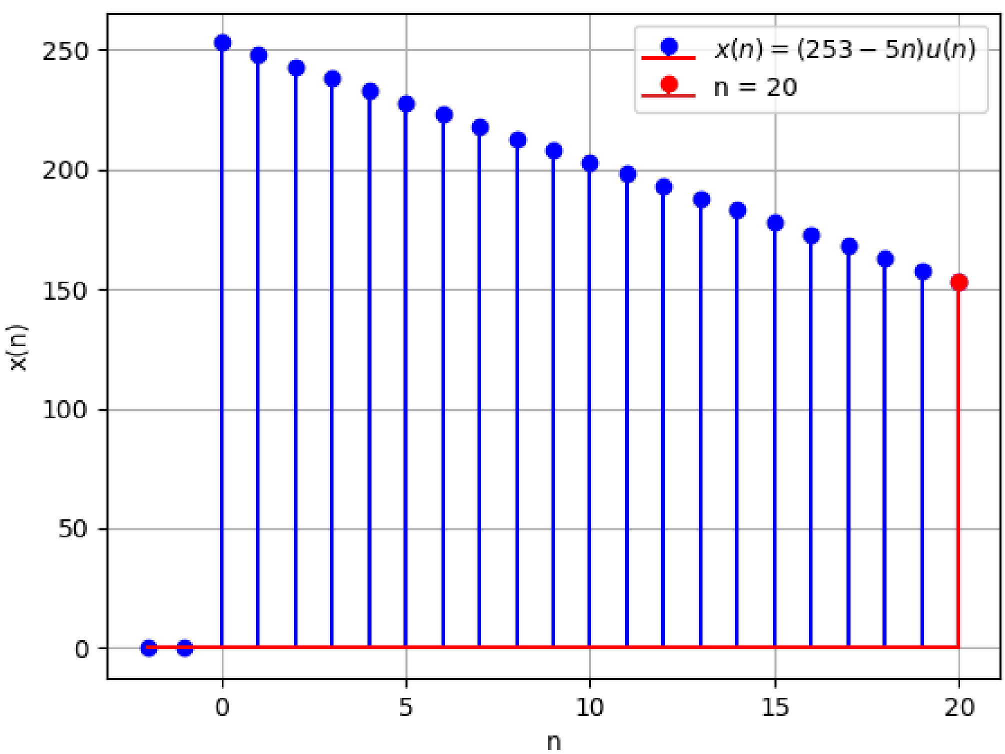


Figure 3.14:

3.0.12 Find the sum to n terms of series , whose n^{th} term is : $n(n+1)(n+4)$.

Solution:

Parameter	**Description**	**Value**
$x(n)$	n^{th} term of series	$n(n+1)(n+4)u(n)$
$y(n)$	sum of n terms of series	

Table 3.12: Given parameters

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \{|z|>1\} \tag{3.88}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3} \{|z|>1\} \tag{3.89}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4} \{|z|>1\} \tag{3.90}$$

$$n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5} \{|z|>1\} \tag{3.91}$$

From equation (**??**) to (**??**),

$$X(z) = \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4} + \frac{5z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3} + \frac{4z^{-1}}{\left(1-z^{-1}\right)^2} \{|z|>1\} \tag{3.92}$$

$$Y(z) = X(z)U(z) \tag{3.93}$$

$$= \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^5} + \frac{5z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^4} + \frac{4z^{-1}}{\left(1-z^{-1}\right)^3} \tag{3.94}$$

$$= \frac{1}{4}\left[\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5}\right] + \frac{13}{6}\left[\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}\right] + \frac{19}{4}\left[\frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3}\right] + \frac{17}{6}\left[\frac{z^{-1}}{\left(1-z^{-1}\right)^2}\right]\{|z|>1\} \quad (3.95)$$

Taking reverse z transform, using equations (3.88) to (3.91)

$$y(n) = \left(\frac{n^4}{4} + \frac{13n^3}{6} + \frac{19n^2}{4} + \frac{17n}{6}\right)u\,(n) \quad (3.96)$$

$$= \left(\frac{n^4}{4} + \frac{2n^3}{4} + \frac{10n^3}{6} + \frac{n^2}{4} + \frac{15n^2}{6} + \frac{4n^2}{2} + \frac{5n}{6} + \frac{4n}{2}\right)u\,(n) \quad (3.97)$$

$$= \left(\frac{n^4+2n^3+n^4}{4}\right)u\,(n) + \left(\frac{10n^3+15n^2+5n}{6}\right)u\,(n) + \left(\frac{4n^2+4n}{2}\right)u\,(n) \quad (3.98)$$

$$= \left(\frac{n^2\,(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}\right)u\,(n) \quad (3.99)$$

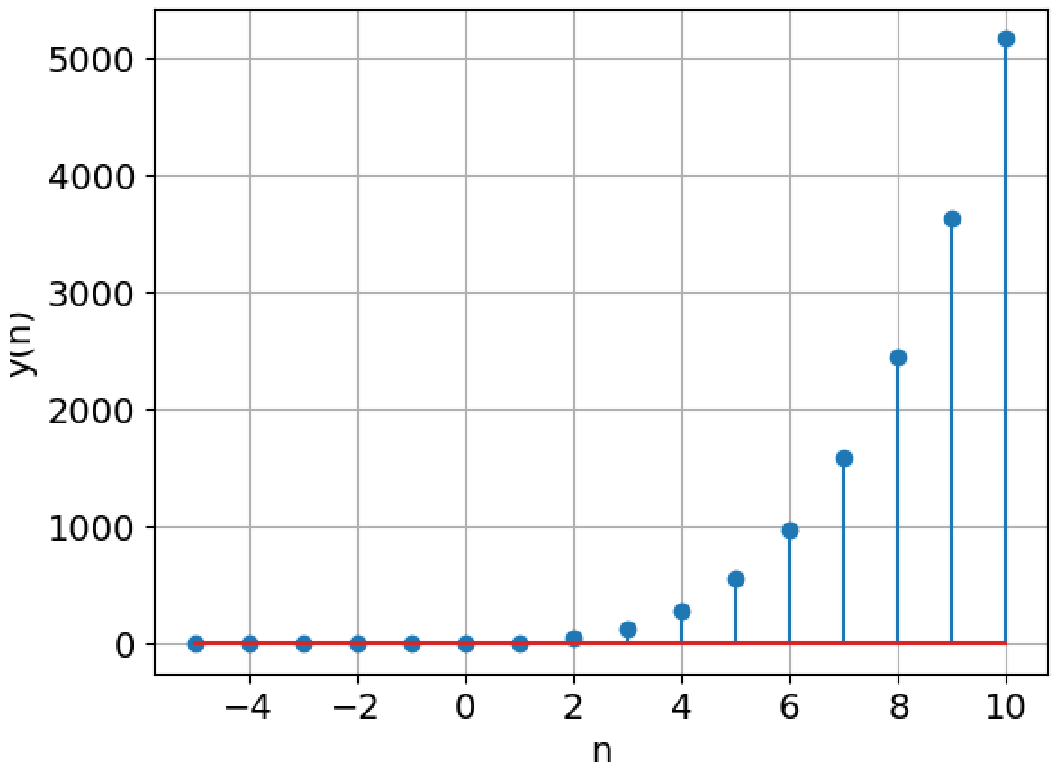


Figure 3.15: Sum of n terms of series

3.0.13 Find the indicated terms in the sequence whose nth terms is $a(n) = 4n - 3$. Find $a(17)$ and $a(24)$.

Solution: In the question, following information is provided:

$$x(n) = (4n + 1)(u(n)) \tag{3.100}$$

Symbol	Value	Description
$x(n)$	$(4n+1)u(n)$	The nth term of the sequence
$x(16)$	?	17th term
$x(23)$	?	24th term

Table 3.13: Parameters

$$x(16) = 4 \times 16 + 1 = 65 \tag{3.101}$$

$$x(23) = 4 \times 23 + 1 = 93 \tag{3.102}$$

Using Z-Transform,

$$X(z) = 4\frac{z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{3.103}$$

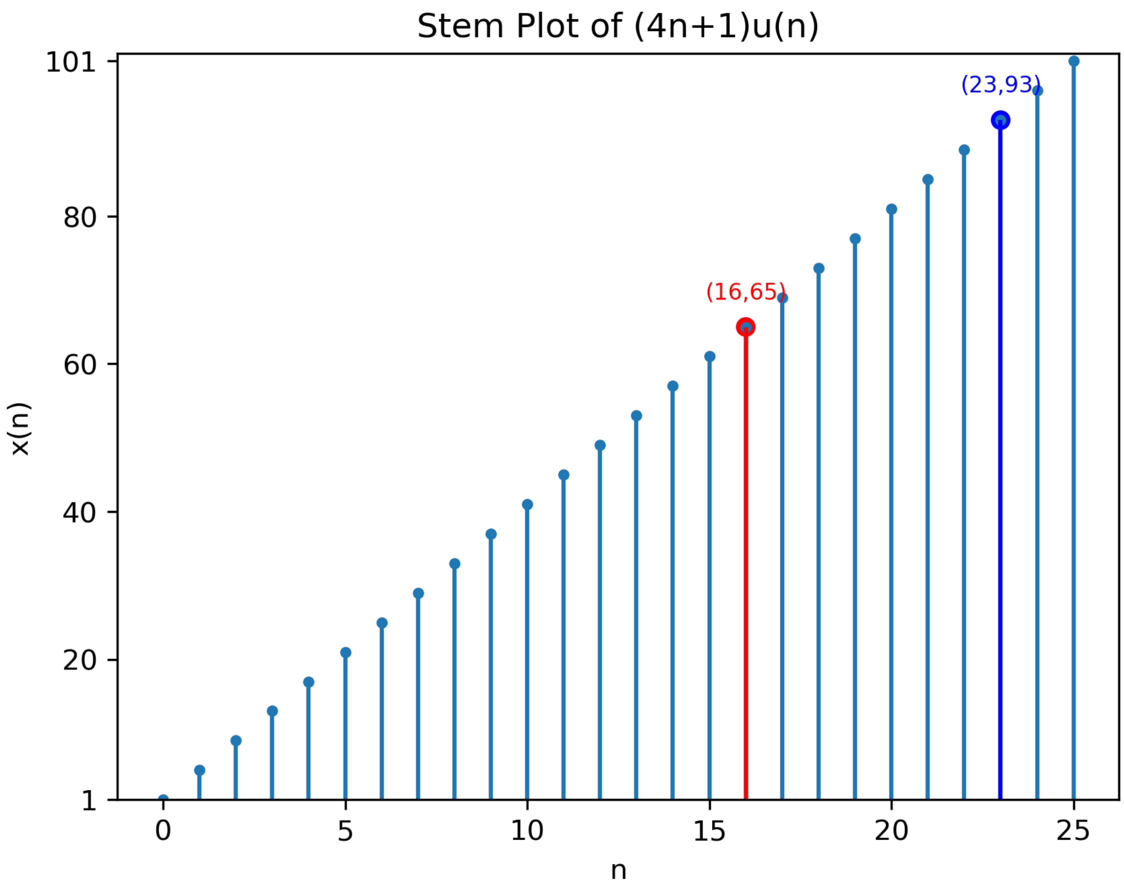


Figure 3.16: $x(n)$ vs n

3.0.14 The difference between any two cosecutive interior angles of a polygon is 5°.If the smallest angle is 120°,find the number of sides of polygon.

Solution:

Variable	**Description**	**Value**
$x(0)$	first term of AP	120
d	common difference of AP	5
$x(n)$	general term of AP	none

Table 3.14: input parameters

Sum of interior angles of a polygon with $n+1$ sides is given by

$$S = (n-1)180 \tag{3.104}$$

Sum of n terms of AP is given by

$$y(n) = x(n) * u(n) \tag{3.105}$$

where $x(n) = 120 + 5n$

$$x(n) * u(n) = (n-1)180 \tag{3.106}$$

$$Y(z) = X(z)U(z) \tag{3.107}$$

$$= \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \right) \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{3.108}$$

$$= \frac{120}{(1-z^{-1})^2} + \frac{5z^{-1}}{(1-z^{-1})^3} \quad |z| > 1 \tag{3.109}$$

$$(n+1)\,u\,(n) \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{(1-z^{-1})^2} \right) \quad |z| > 1 \tag{3.110}$$

$$\frac{(n)\,(n-1)}{2} u\,(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{z^{-1}}{(1-z^{-1})^3} \right) \quad |z| > 1 \tag{3.111}$$

applying inverse Z-transform for each term and solving we get,

$$y\,(n) = \frac{n+1}{2}\,(240+5n)\,u(n) \tag{3.112}$$

now from (3.106)

$$y(n) = (n-1)180 \tag{3.113}$$

$$\frac{n+1}{2}(240+5n)\,u(n) = (n-1)180 \tag{3.114}$$

now replace n by $n-1$:

$$n(235+5n) = (n-2)360 \tag{3.115}$$

$$5n^2 - 125n + 720 = 0 \tag{3.116}$$

$$n = 16, 9 \tag{3.117}$$

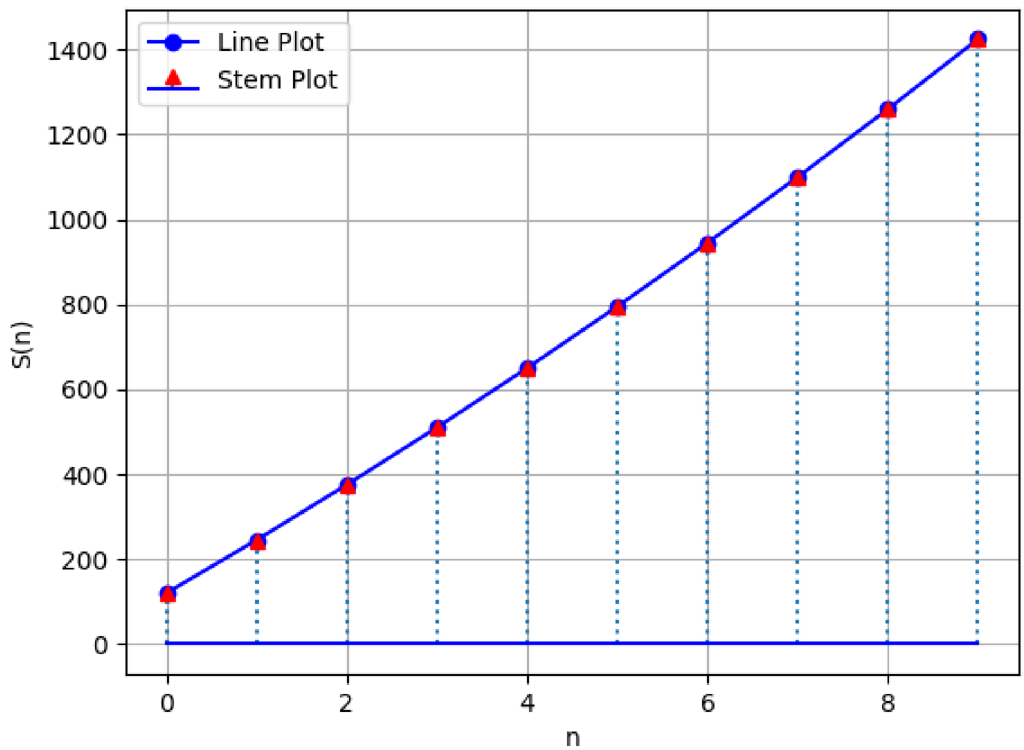


Figure 3.17: Plot of the sum of n terms taken from Python

3.0.15 The 5th,8th and 11th terms of a GP are p,q and s respectively .show that $q^2 = ps$

Solution:

Symbol	Value	Description
$x(5)$	p	$x(0)\, r^5$
$x(8)$	q	$x(0)\, r^8$
$x(11)$	s	$x(0)\, r^{11}$
$x(n)$		$x(0)\, r^n u(n)$
r	$\left(\frac{s}{p}\right)^{\frac{1}{6}}$	common ratio

Table 3.15: input parameters

From Table 3.15:

$$q^2 = x\,(0)\ r^8\,x\,(0)\ r^8 \tag{3.118}$$

$$= x\,(0)^2\ r^{16} \tag{3.119}$$

$$ps = x\,(0)\ r^5\,x(0)\,r^{11} \tag{3.120}$$

$$= x\,(0)^2\ r^{16} \tag{3.121}$$

$$\Longrightarrow\ q^2 = ps \tag{3.122}$$

now we will find r and x(0):

$$\frac{s}{p} = \frac{x\,(0)\,r^{11}}{x\,(0)\,r^5} \tag{3.123}$$

$$r = \left(\frac{s}{p}\right)^{\frac{1}{6}} \tag{3.124}$$

$$p = x\,(0)\left(\frac{s}{p}\right)^{\frac{5}{6}} \tag{3.125}$$

$$x\,(0) = \frac{p^{\frac{11}{6}}}{s^{\frac{5}{6}}} \tag{3.126}$$

Applying z-Transform:

$$X(z) = \frac{x\,(0)}{1 - r\,z^{-1}}\ \ , |z| > |r| \tag{3.127}$$

$$\Longrightarrow\ X(z) = \frac{p^3}{p^{\frac{7}{6}}\,s^{\frac{5}{6}} - q^2 z^{-1}}\ , |z| > \left|\left(\frac{q}{p}\right)^{\frac{1}{3}}\right| \tag{3.128}$$

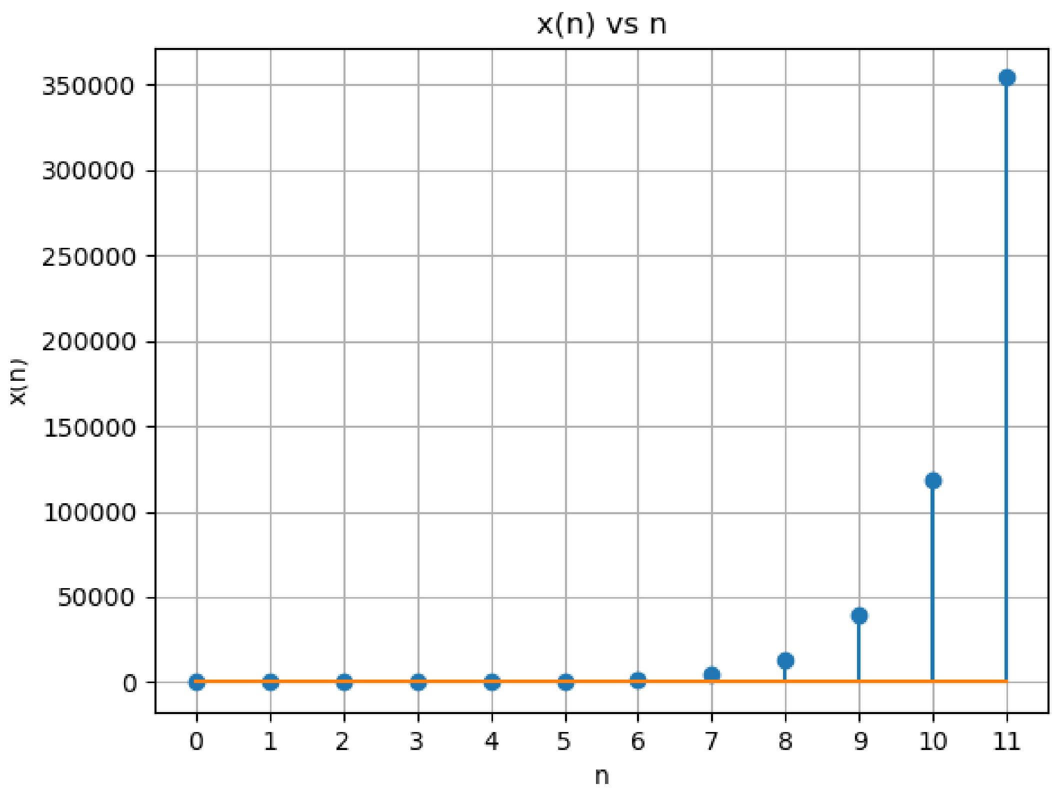


Figure 3.18: plot x(n)vs n $p = 486, \ \ q = 13122, \ \ s = 354294, \ \ r = 3$

3.0.16 The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution:

$$y(n) = \left[\frac{(n+1)}{2}(2x(0) + nd)\right] u(n) \tag{3.129}$$

$$\implies y(3) = \frac{4}{2}(2x(0) + 3d) \tag{3.130}$$

$$\tag{3.131}$$

Symbol	Value	Description
$x(0)$	11	First term of AP
$y(3)$	56	Sum of the first four terms of AP
$y(n) - y(n-4)$	112	Sum of the last four terms of AP

Table 3.16: Input Parameters

From Table 3.16:

$$\frac{4}{2}\left(2x\left(0\right)+3d\right)=56 \tag{3.132}$$

$$2x\left(0\right)+3d=28 \tag{3.133}$$

$$\implies d=2 \tag{3.134}$$

$$y\left(n\right)-y\left(n-4\right)=\frac{4}{2}\left(2x\left(n\right)+3\left(-d\right)\right) \tag{3.135}$$

From Table 3.16:

$$\frac{4}{2}\left(2x\left(n\right)+3\left(-d\right)\right)=112 \tag{3.136}$$

$$2x\left(n\right)-3d=56 \tag{3.137}$$

$$\implies x\left(n\right)=31 \tag{3.138}$$

$$x\left(0\right)+\left(n\right)2=31 \tag{3.139}$$

$$\implies n=10 \tag{3.140}$$

$$x\left(n\right)=\left(x\left(0\right)+2n\right)u\left(n\right) \tag{3.141}$$

$$\implies X\left(z\right)=\frac{x\left(0\right)}{1-z^{-1}}+2\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}. \quad |z|>1 \tag{3.142}$$

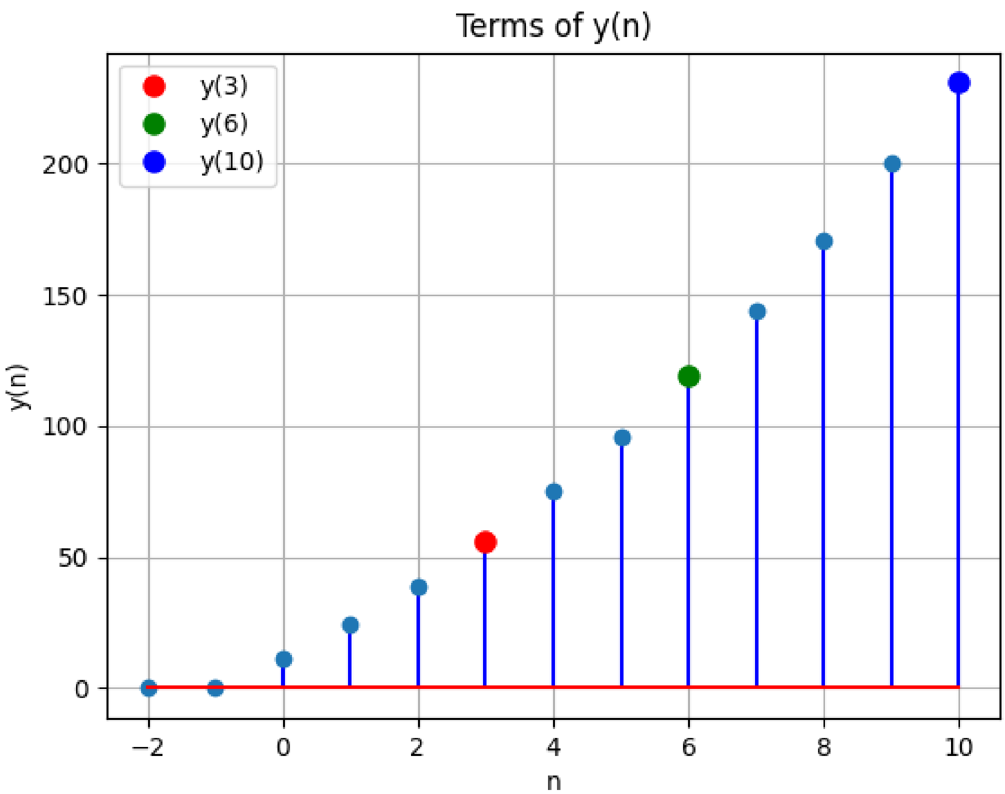


Figure 3.19: Plot y(n) vs n

3.0.17 Find the sum to n terms of the series whose n^{th} term is given by $(2n-1)^2$? **Solution:** Sum

Variable	**Description**	**Value**
$x(n)$	n^{th} term of sequence	$(2n+1)^2 u(n)$

Table 3.17: input parameters

of n terms of AP is given by

$$y(n) = x(n) * u(n) \tag{3.143}$$

$$x(n) = (2n+1)^2 u(n) \tag{3.144}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})} \quad |z|>1 \tag{3.145}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \tag{3.146}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \quad |z|>1 \tag{3.147}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4} \quad |z|>1 \tag{3.148}$$

$$\implies X(z) = \frac{4z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{1}{(1-z^{-1})} + \frac{4z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \tag{3.149}$$

$$Y(z) = X(z)U(z) \tag{3.150}$$

$$= \left(\frac{4z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} + \frac{1}{(1-z^{-1})} + \frac{4z^{-1}}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{3.151}$$

$$= \frac{4z^{-1}(z^{-1}+1)}{(1-z^{-1})^4} + \frac{1}{(1-z^{-1})^2} + \frac{4z^{-1}}{(1-z^{-1})^3} \tag{3.152}$$

$$\implies Y(Z) = \frac{1}{(1-z^{-1})} + \frac{9z^{-1}}{(1-z^{-1})} + \frac{25z^{-2}}{(1-z^{-1})^2} + \frac{24z^{-3}}{(1-z^{-1})^3} + \frac{8z^{-4}}{(1-z^{-1})^4} \quad |z|>1 \tag{3.153}$$

Now from (3.145), (3.146), (3.147), (3.148), (3.153) By using inverse Z-transform pairs,

$$y\,(n) = u(n) + 9u(n-1) + 25(n-1)u(n-2) + 24\frac{(n-1)(n-2)}{2}u(n-3) + 8\frac{(n-1)(n-2)(n-3)}{6}u(n-4) \tag{3.154}$$

$$\implies y\,(n) = \left(\frac{4n^3+12n^2+11n+3}{3}\right)u(n) \tag{3.155}$$

$\therefore$ Sum of n terms of the series whose n^{th} term is given by $(2n+1)^2$ is $\frac{4n^3+12n^2+11n+3}{3}$.

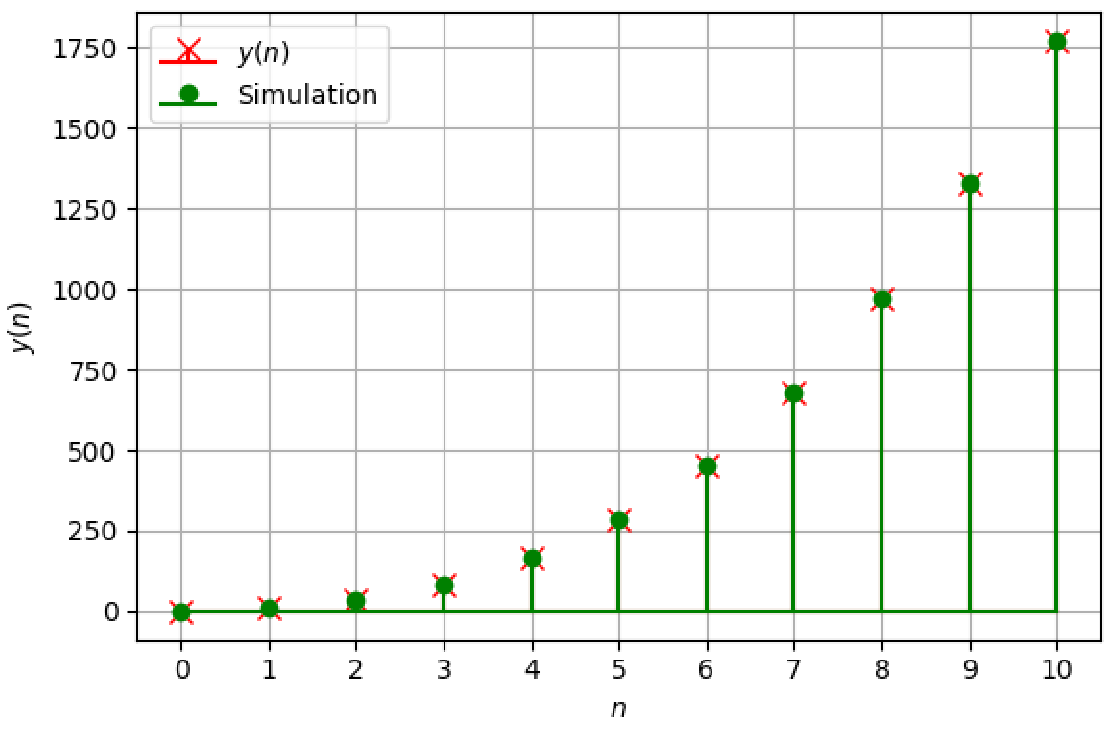


Figure 3.20: Theory vs Simulation

3.0.18 If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y, and z, respectively. Prove that $x,\ y,\ z$ are in G.P.

Solution:

Symbol	**Value**	**Description**
x	$x(0)\,r^4$	$x(4)$
y	$x(0)\,r^{10}$	$x(10)$
z	$x(0)\,r^{16}$	$x(16)$
r	?	$\frac{x(n)}{x(n-1)}$
$x(0)$	?	First term
$x(n)$	$x(0)\,r^n u(n)$	General Term

Table 3.18: **Given Information**

(a) From Table 3.0.18,

$$x = x(3) = x(0)\,r^3 \tag{3.156}$$

$$y = x(9) = x(0)\,r^9 \tag{3.157}$$

$$z = x(15) = x(0)\,r^{15} \tag{3.158}$$

Consider $\frac{x(9)}{x(3)}$ and $\frac{x(15)}{x(9)}$;

$$\frac{x(9)}{x(3)} = \frac{x(0)\,r^9}{x(0)\,r^3} = r^6 = \frac{x(15)}{x(9)} = \frac{x(0)\,r^{15}}{x(0)\,r^9} \tag{3.159}$$

From (3.159), $x(3)$, $x(9)$, $x(15)$ are in G.P.

$\therefore\ x,\ y,\ z$ are in G.P.

(b) $x(0)$ and r can be expressed in terms of x, y, and z in the following manner.

$$\frac{y}{x} = r^6 \tag{3.160}$$

$$\Longrightarrow r = \sqrt[6]{\frac{y}{x}} = \left(\frac{y}{x}\right)^{\frac{1}{6}} \tag{3.161}$$

$$\Longrightarrow x(0) = \frac{x}{r^3} = x\left(\frac{x}{y}\right)^{\frac{3}{6}} \tag{3.162}$$

$$\therefore\ x(0) = x^{\frac{5}{3}} y^{-\frac{2}{3}} \text{ and } r = \left(\frac{y}{x}\right)^{\frac{1}{6}} = y^{\frac{1}{6}} x^{-\frac{1}{6}} \tag{3.163}$$

(c) From (**??**) Z-transform of a G.P. is

$$X(z) = \frac{x(0)}{1 - rz^{-1}}; |z| > |r| \tag{3.164}$$

Substituting r and $x(0)$ from (3.163),

$$X(z) = \frac{x^{\frac{5}{3}} y^{-\frac{2}{3}}}{1 - \left(\frac{y}{x}\right)^{\frac{1}{6}} z^{-1}} \tag{3.165}$$

(d) Example Let $x(0) = 1$ and $r = 1.2$

$$x = x(3) = (1.2)^3 \tag{3.166}$$

$$y = x(9) = = (1.2)^9 \tag{3.167}$$

$$z = x(15) = (1.2)^{15} \tag{3.168}$$

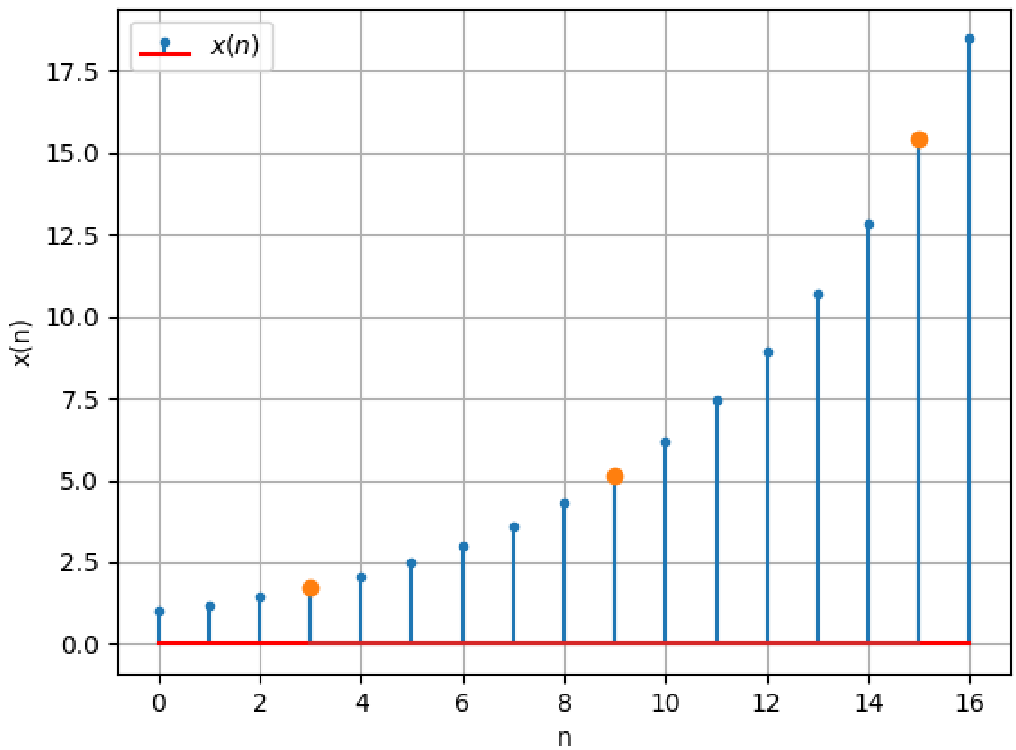


Figure 3.21: Stem Plot of $x(n)$ vs n

3.0.19 Show that the ratio of the sum of the first n terms of a geometric progression (G.P.) to the sum of terms from $(n+1)$th to $(2n)$th term is $\frac{1}{r^n}$. **Solution:**

$$x(n) = x(0)r^n u(n) \tag{3.169}$$

where

$$u(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

Table 3.19: Input Parameters

Variable	**Description**	**Value**
$x(0)$	First term of G.P	
r	Common ratio of G.P	
$x(n)$	General Term	$x(0) \cdot r^n \cdot u(n)$

$$y(n) = \sum_{k=-\infty}^{n} x(k) \tag{3.170}$$

$$y(n) = x(n) * u(n) \tag{3.171}$$

Taking z transform

$$Y(z) = X(z)U(z) \tag{3.172}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > 1 \tag{3.173}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{3.174}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1}\left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{3.175}$$

Using partial fractions, again the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^n - 1}{r - 1}\right)u(n) \tag{3.176}$$

$$y(2n) = x(0)\left(\frac{r^{2n-1}-1}{r-1}\right)u(2n) \tag{3.177}$$

Now we have to find $\frac{y(n)}{y(2n)-y(n)}$

$$\frac{y(n)}{y(2n)-y(n)} = \frac{x(0)\left(\frac{r^n-1}{r-1}\right)u(n)}{x(0)\left(\frac{r^{2n}-1}{r-1}\right)u(2n) - x(0)\left(\frac{r^n-1}{r-1}\right)u(n)} \tag{3.178}$$

$$= \frac{\left(\frac{r^n-1}{r-1}\right)}{\left(\frac{r^{2n}-1}{r-1}\right) - \left(\frac{r^n-1}{r-1}\right)} \tag{3.179}$$

$$= \frac{r^n-1}{(r^{2n}-1)-(r^n-1)} \tag{3.180}$$

$$= \frac{r^n}{(r^{2n})-(r^n)} \tag{3.181}$$

$$= \frac{1}{r^n} \tag{3.182}$$

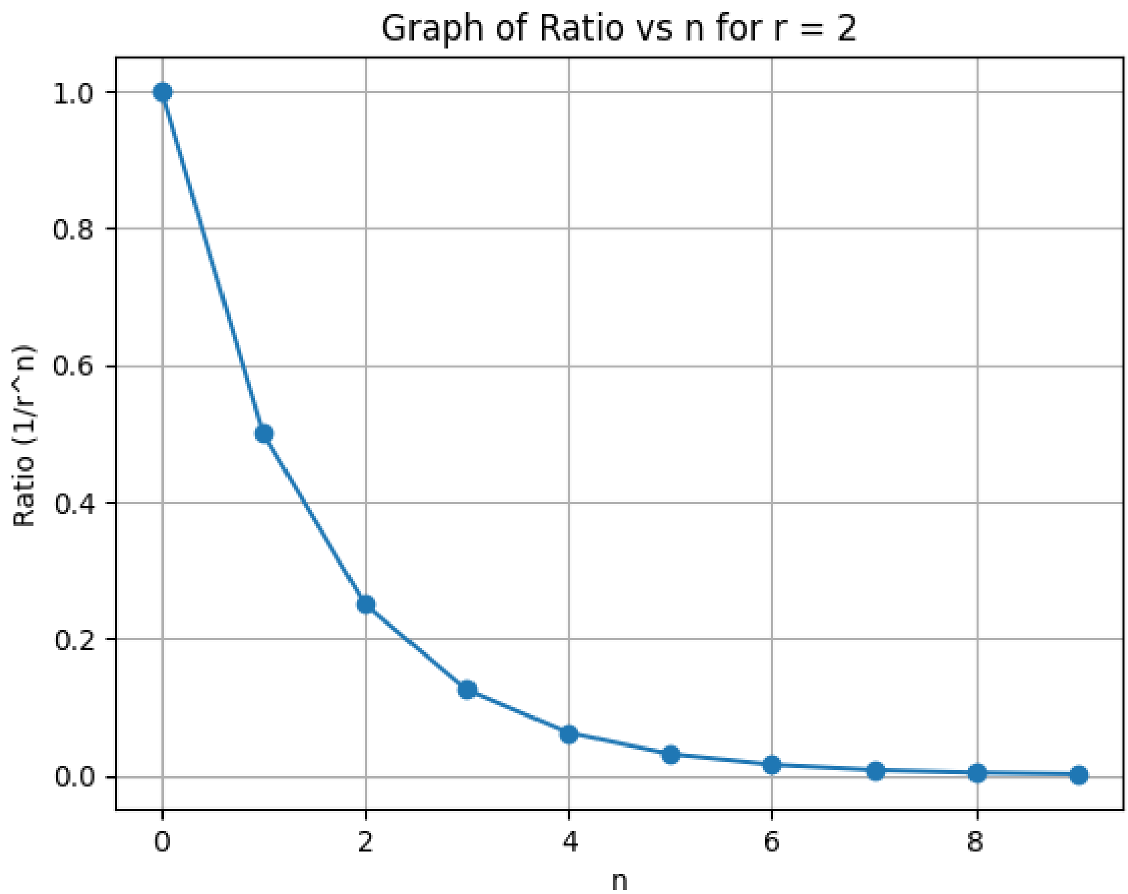


Figure 3.22: Plot of ratio vs $1/r^n$ for r = 2

3.0.20 A G.P consists of an even number of terms. If the sum of all terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution: Solving the Question in time domain:

$$x(n) = x(0)r^n \tag{3.183}$$

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{3.184}$$

Parameter	Description	condition
N	Number of terms in the G.P	-
M	number of odd place terms	N=2M
$x(0)$	first term in the G.P	-
r	common ratio in the G.P	-
$x(n)$	$n+1$ th term in the G.P	$x(n) = x(0)r^n$
$y(n)$	sum of G.P series	$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n)$
$x_o(n)$	$n+1$ th term of G.P of odd places	$x_o(n) = x(0)r^{2n}$
$y_o(n)$	sum of terms in odd places	$y_o(n) = x(0)\left(\frac{r^{n+1}-1}{r^2-1}\right)u(n)$

Table 3.20: Input Parameters

The sum of terms in odd places:

$$x_o(n) = x(0)r^{2n} \tag{3.185}$$

$$y_o(n) = x(0)\left(\frac{r^{n+1}-1}{r^2-1}\right)u(n) \tag{3.186}$$

Then from (3.184) and (3.186)

$$x(0)\left(\frac{r^N-1}{r-1}\right)u(n) = 5\left(x(0)\left(\frac{r^{2M}-1}{r^2-1}\right)u(n)\right) \tag{3.187}$$

$$\frac{r^2-1}{r-1} = 5 \tag{3.188}$$

$$\text{as } r \neq 1, \quad \text{hence } r = 4 \tag{3.189}$$

$$\tag{3.190}$$

X,Y,Xo,Yo are frequency counterparts of the above GP

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \tag{3.191}$$

$$X_o(z) = \frac{x(0)}{1 - r^2 z^{-1}} \tag{3.192}$$

$$Y(z) = \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \tag{3.193}$$

$$Y_o(z) = \frac{x(0)}{\left(1 - rz^{\frac{-1}{2}}\right)(1 - z^{-1})} \tag{3.194}$$

3.0.21 Find the sum to indicated number of terms in the geometric progression $x^3, x^5, x^7, ...n$ terms (if $x \neq \pm 1$).

Solution:

Input Parameters	Values	Description
$x(0)$	x^3	Initial term
r	x^2	Common ratio
$x(n)$	$x^{2n+3}u(n)$	General term

Table 3.21: Given inputs

From Table 3.21,

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \tag{3.195}$$

$$= \frac{x^3}{1 - x^2z^{-1}} \quad |z| > x^2 \tag{3.196}$$

$$y(n) = x(n) * u(n) \tag{3.197}$$

$$Y(z) = X(z)U(z) \tag{3.198}$$

$$= \frac{x^3}{(1 - x^2z^{-1})(1 - z^{-1})} \quad |z| > x^2 \cap |z| > 1 \tag{3.199}$$

$$= \frac{x^3}{x^2 - 1}\left(\frac{x^2}{1 - x^2z^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{3.200}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{3.201}$$

$$x^{2n+2}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{x^2}{1 - x^2z^{-1}} \quad |z| > x^2 \tag{3.202}$$

Taking inverse Z transform of $Y(z)$,

$$y(n) = x^3\left(\frac{x^{2n+2} - 1}{x^2 - 1}\right)u(n) \tag{3.203}$$

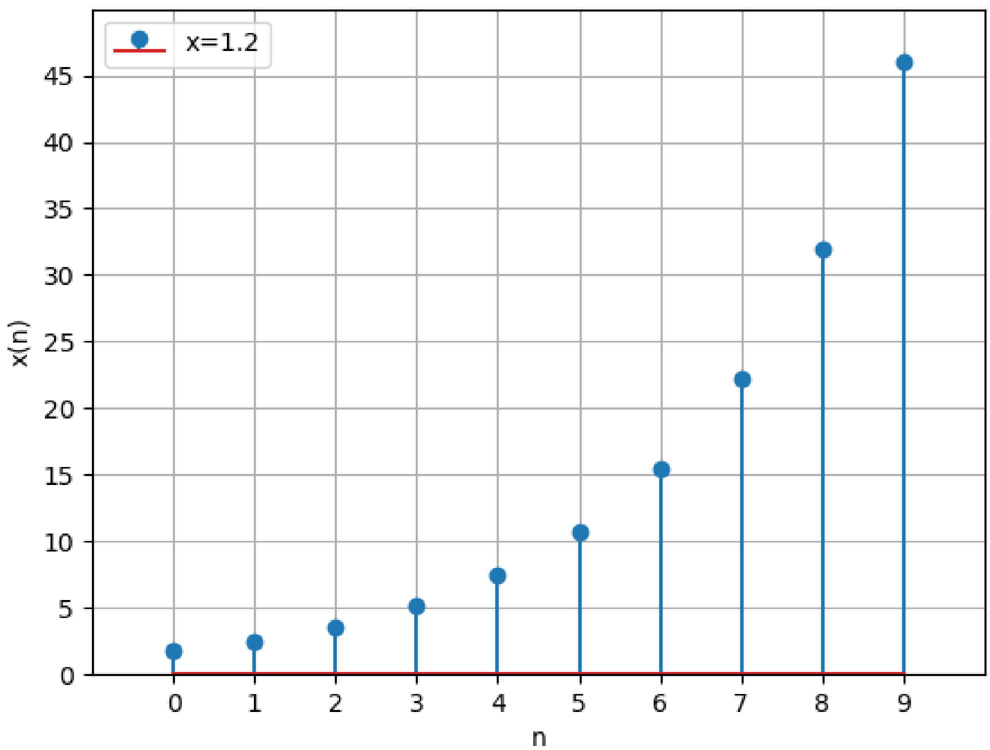


Figure 3.23: Plot of $x(n)$ for $x = 1.2$

3.0.22 Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

Solution:

Parameter	**Value**	**Description**
$x(6) - x(4)$	12	7th term exceeds 5th by 12
$x(2)$	16	Third term
d	?	Common difference
$x(0)$	?	First term of AP
$x(n)$	$(x(0) + nd)u(n)$	General term

Table 3.22: Input parameters table

From Table 3.22

$$x(0) + 6d - x(0) - 4d = 12 \tag{3.204}$$

$$\implies 2d = 12 \tag{3.205}$$

$$\implies d = 6 \tag{3.206}$$

Also,

$$x(0) + 2d = 16 \tag{3.207}$$

$$\implies x(0) + 2(6) = 16 \tag{3.208}$$

$$\implies x(0) = 4 \tag{3.209}$$

$$\therefore x(n) = 6n + 4 \tag{3.210}$$

From Table 3.22

$$X(z) = x(0)\frac{1}{1 - z^{-1}} + d\frac{z^{-1}}{(1 - z^{-1})^2} \tag{3.211}$$

$$= 4\frac{1}{1 - z^{-1}} + 6\frac{z^{-1}}{(1 - z^{-1})^2} \tag{3.212}$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.213}$$

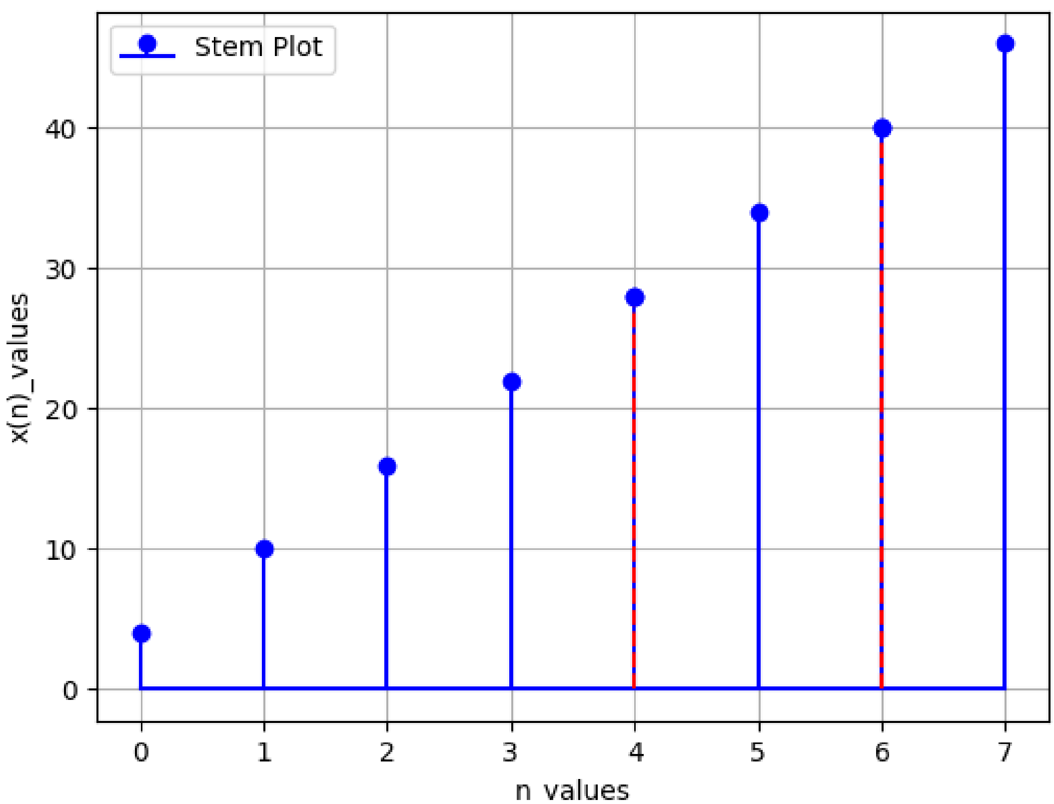


Figure 3.24: Given AP

3.0.23 Find the seventh term of the sequence where the nth term is given by $a_n = \frac{n^2}{2^n}$

Solution:

$$x(n) = \frac{(n+1)^2}{2^{(n+1)}} u(n) \tag{3.214}$$

Parameter	**Value**
$x(n)$	$\frac{(n+1)^2}{2^(n+1)} u(n)$
$x(6)$	?

Table 3.23: Input Parameters

$$x(6) = \frac{(6+1)^2}{2^{(6+1)}} \tag{3.215}$$

$$= \frac{49}{128} \tag{3.216}$$

(a) Scaling property:

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-az^{-1})}, \quad |z| > |a| \tag{3.217}$$

(b) Differentiation property:

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)\frac{dY(z)}{dz} \tag{3.218}$$

$$\implies nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{3.219}$$

$$\implies n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, \quad |z| > 1 \tag{3.220}$$

(c) Time shifting property:

$$y(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}Y(z) \tag{3.221}$$

The Z transform of $x(n)$ is given by:

from(4)

$$\frac{u(n)}{2^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-(2z)^{-1})}, \quad |z| > \frac{1}{2} \tag{3.222}$$

from(5)

$$\frac{n}{2^n}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(2z)^{-1}}{(1-(2z)^{-1})^2}, \quad |z| > \frac{1}{2} \tag{3.223}$$

$$\frac{n^2}{2^n}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(2z)^{-1}(1+(2z)^{-1})}{(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.224}$$

from(8)

$$\frac{(n+1)^2}{2^(n+1)}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (z)\frac{(2z)^{-1}(1+(2z)^{-1})}{(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.225}$$

$$X(z) = \frac{1+(2z)^{-1}}{2(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.226}$$

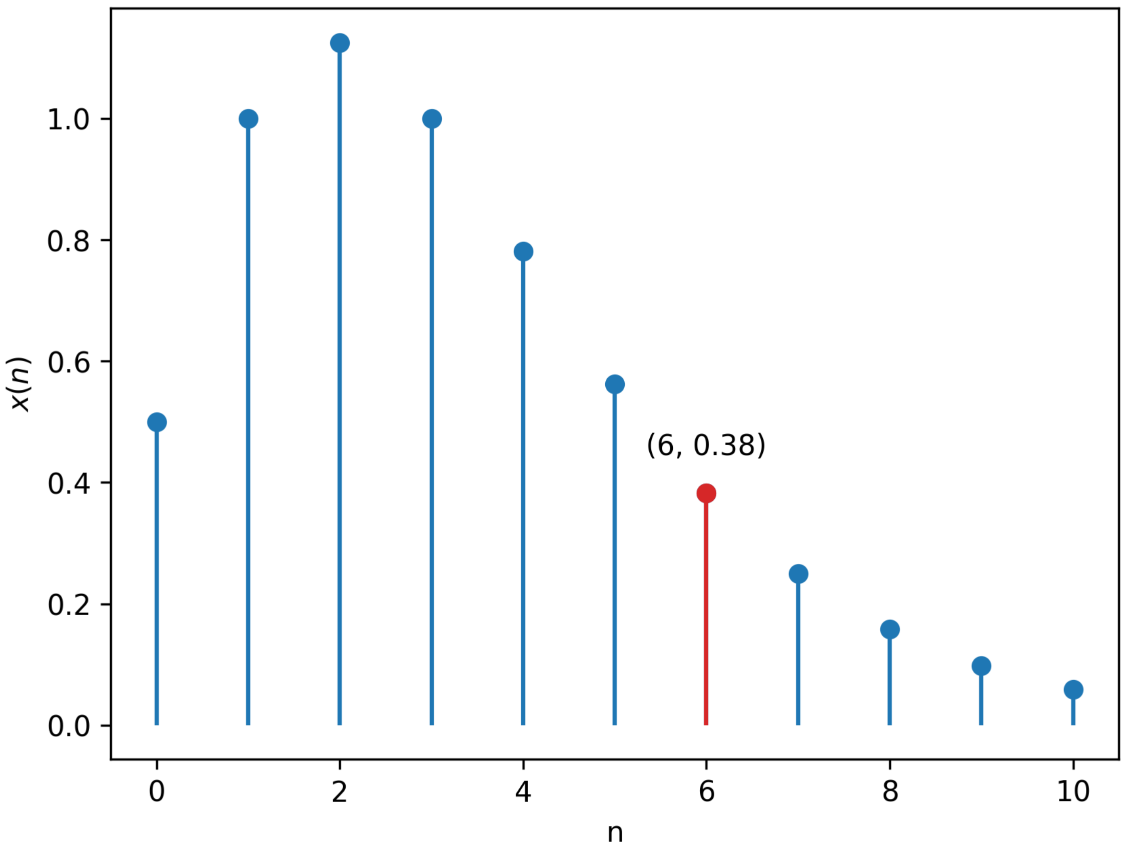


Figure 3.25: Stem plot of $x(n)$

3.0.24 Find sum to n terms of the following series:

$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots$ (NCERT 11.9.4.4)

Solution:

$$x(n) = \frac{1}{(n+1)(n+2)} u(n) \tag{3.227}$$

$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) u(n) \tag{3.228}$$

Symbol	Description	Value
$x(n)$	n^{th} term of series	$\frac{1}{(n+1)(n+2)}u(n)$
$y(n)$	Sum of n terms of series	?

Table 3.24: Parameters

Using (**??**) and (**??**), we get,

$$X(z) = -zlog\left(1 - z^{-1}\right) + z^2 \log\left(1 - z^{-1}\right) + z \tag{3.229}$$

$$= z(z-1)\log\left(1 - z^{-1}\right) + z \tag{3.230}$$

$$Y(z) = X(z)U(z) \tag{3.231}$$

$$= z^2 \log\left(1 - z^{-1}\right) + \frac{z}{1 - z^{-1}} \tag{3.232}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{3.233}$$

$$u(n+k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^k}{1 - z^{-1}}, \forall k \in \mathbb{R}, \quad |z| > 1 \tag{3.234}$$

Using (3.234) and (**??**) ,

$$y(n) = u(n+1) - \frac{1}{n+2}u(n), \ n \geq 0 \tag{3.235}$$

Since $y(n)$ is only defined for $n \geq 0$, the above expression can be equivalently written as

$$y(n) = \left(1 - \frac{1}{n+2}\right)u(n) \tag{3.236}$$

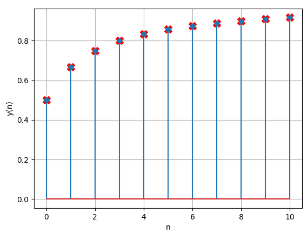


Figure 3.26: Stem Plot of $y(n)$ v/s n

3.0.25 1x2x3 + 2x3x4 + 3x4x5 +.....

Solution:

3.0.26 Find the sum of the following series up to n terms:

(a) $5 + 55 + 555 + \ldots$

(b) $.6 + .66 + .666 + \ldots$

Solution:

Symbol	**Value**	**Z-Transform**
$x_1(n)$	$\{5, 55, 555, \ldots\}$	$X_1(z)$
$x_2(n)$	$\{0.6, 0.66, 0.666, \ldots\}$	$X_2(z)$
$s_1(n)$	$\{5 + 55 + 555 + \ldots$ up to n terms$\}$	$S_1(z)$
$s_2(n)$	$\{0.6 + 0.66 + 0.666 + \ldots$ up to n terms$\}$	$S_2(z)$

Table 3.25: **Input Parameters**

(a) For $x_1(n)$:

$$x_1(n) = 5\left(\frac{10^{n+1} - 1}{10 - 1}\right)u(n) \tag{3.237}$$

$$x_1(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \tag{3.238}$$

$$X_1(z) = \frac{50}{9}\left(\frac{1}{1 - 10z^{-1}}\right) - \frac{5}{9}\left(\frac{1}{1 - z^{-1}}\right), |z| > 10 \tag{3.239}$$

$$s_1(n) = 5\sum_{i=0}^{n-1} \frac{(10^{i+1} - 1)}{10 - 1} \tag{3.240}$$

$$s_1(n) = \left(5\frac{(10^{n+1} - 1)}{10 - 1}\right) * u(n) \tag{3.241}$$

$$s_1(n) \overset{\mathcal{Z}}{\longleftrightarrow} S_1(z) \tag{3.242}$$

$$S_1(z) = \left(\frac{50}{9}\frac{1}{(1 - 10z^{-1})} - \frac{5}{9}\frac{1}{(1 - z^{-1})}\right)\left(\frac{1}{1 - z^{-1}}\right), |z| > 10 \tag{3.243}$$

$$= \frac{50}{81}\left(\frac{10}{1 - 10z^{-1}} - \frac{1}{1 - z^{-1}}\right) - \frac{5}{9}\left(\frac{1}{(1 - z^{-1})^2}\right), |z| > 10 \tag{3.244}$$

$$s_1(n) = \frac{50}{81}(10^{n+1} - 1)u(n) - \frac{5}{9}(n + 1)u(n) \tag{3.245}$$

$$= \frac{5}{81}(10^{n+2} - 9n - 19)u(n) \qquad \{n \geq 0\} \tag{3.246}$$

(b) For $x_2(n)$:

$$x_2(n) = 0.6\left(\frac{1-10^{-(n+1)}}{1-0.1}\right)u(n) \tag{3.247}$$

$$x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) \tag{3.248}$$

$$X_2(z) = \frac{2}{3}\left(\frac{1}{1-z^{-1}}\right) - \frac{1}{15}\left(\frac{1}{1-(10z)^{-1}}\right), |z| > 1 \tag{3.249}$$

$$s_2(n) = 0.6\sum_{i=0}^{n-1}\frac{(1-10^{-(i+1)})}{1-10^{-1}} \tag{3.250}$$

$$s_2(n) = \left(0.6\frac{(1-10^{-(n+1)})}{1-0.1}\right) * u(n) \tag{3.251}$$

$$s_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} S_2(z) \tag{3.252}$$

$$S_2(z) = \left(\frac{2}{3}\frac{1}{(1-z^{-1})} - \frac{1}{15}\frac{1}{(1-(10z)^{-1})}\right)\left(\frac{1}{1-z^{-1}}\right), |z| > 1 \tag{3.253}$$

$$\begin{aligned} &= \frac{2}{3}\left(\frac{1}{(1-z^{-1})^2}\right) \\ &- \frac{2}{27}\left(\frac{1}{1-z^{-1}} - \frac{10^{-1}}{1-(10z)^{-1}}\right), |z| > 1 \end{aligned} \tag{3.254}$$

$$s_2(n) = \frac{2}{3}(n+1)u(n) - \frac{2}{27}(1-10^{-(n+1)})u(n) \tag{3.255}$$

$$= \frac{2}{27}(10^{-(n+1)} + 9n + 8)u(n) \qquad \{n \geq 0\} \tag{3.256}$$

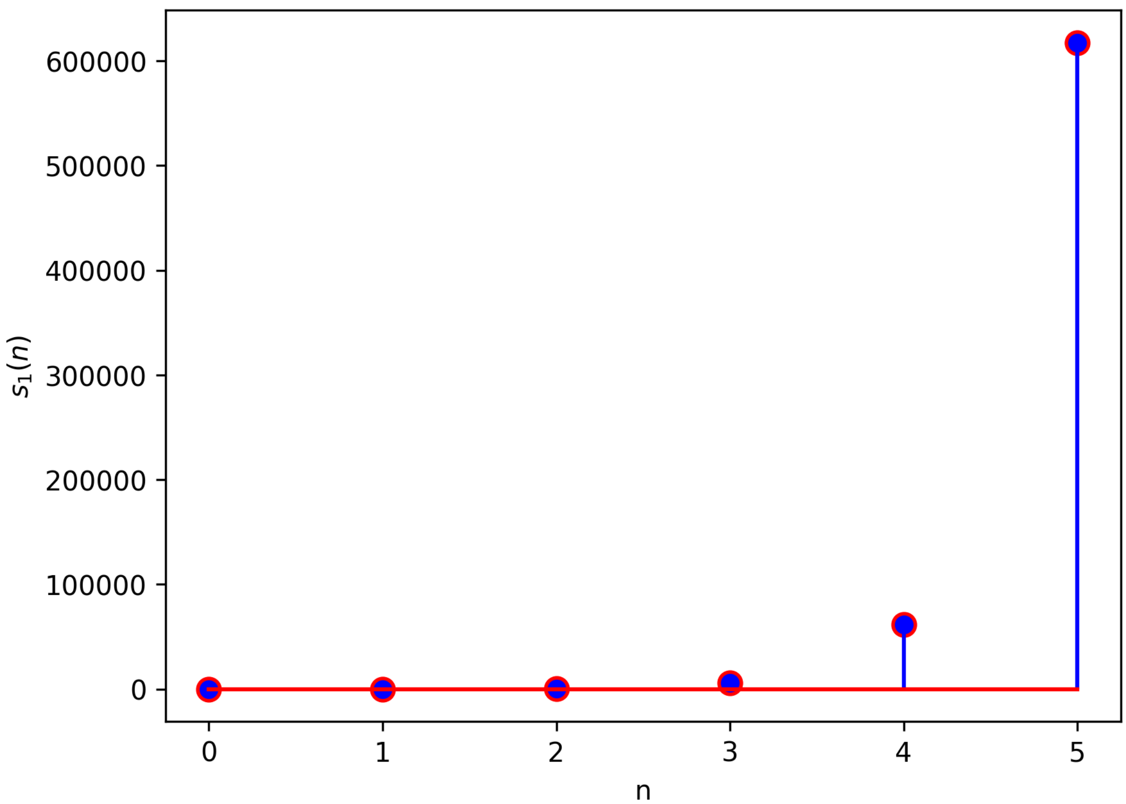


Figure 3.27: Stem plot of $s_1(n)$

3.0.27 Find a_9 in the sequence $a_n = (-1)^{n-1} n^3$

Solution:

Symbol	Value	Description
x(0)	1	First term of the sequence
x(n)	$(-1)^n (n+1)^3 u(n)$	$(n+1)^{th}$ term of the sequence

Table 3.26: Table of parameters

To obtain 9^{th} term of the sequence put n=8 in x(n)

$$x(8) = 729 \tag{3.257}$$

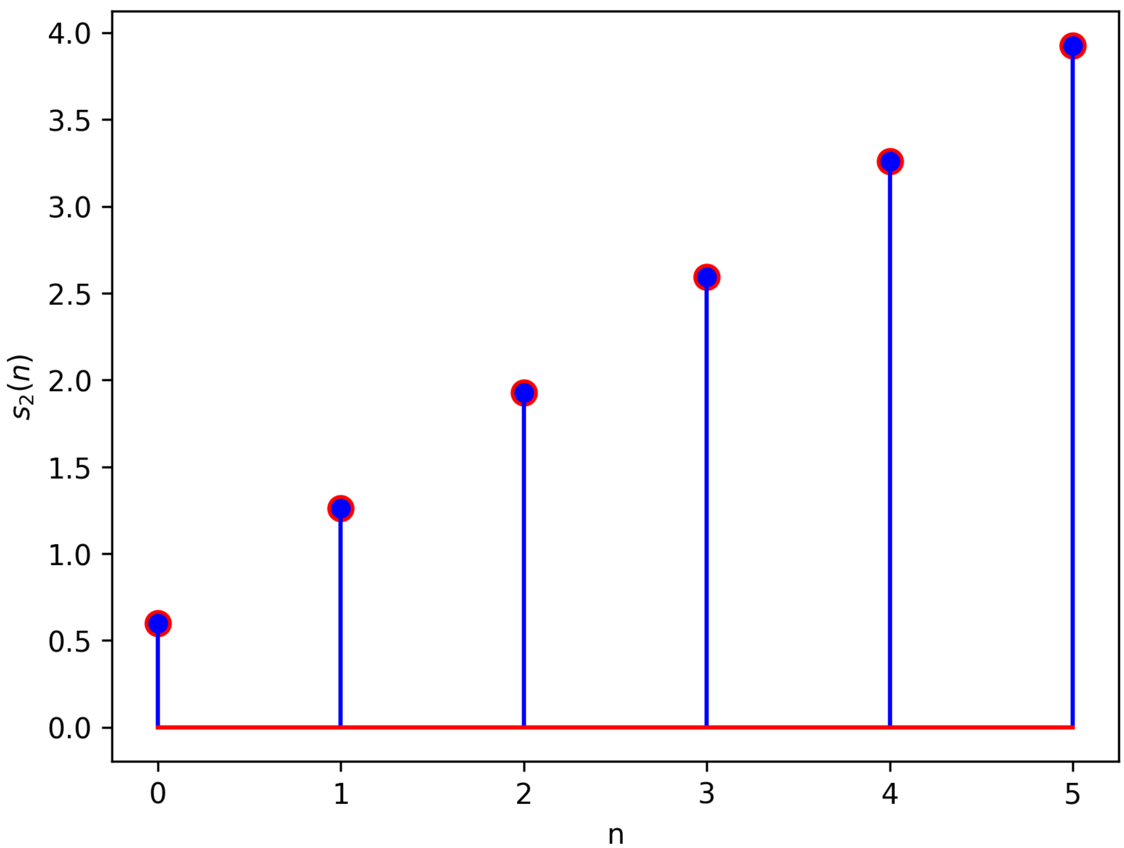


Figure 3.28: Stem plot of $s_2(n)$

Using Z transform,

$$X(z) = \sum_{n=-\infty}^{\infty} (-1)^n (n+1)^3 u(n) z^{-n} \tag{3.258}$$

$$= \sum_{n=-\infty}^{\infty} (n+1)^3 u(n) (-z)^{-n} \tag{3.259}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 3n^2 + 3n + 1\right) u(n) (-z)^{-n} \tag{3.260}$$

Replace z by $-z$ in (**??**),(**??**),(**??**),(**??**)

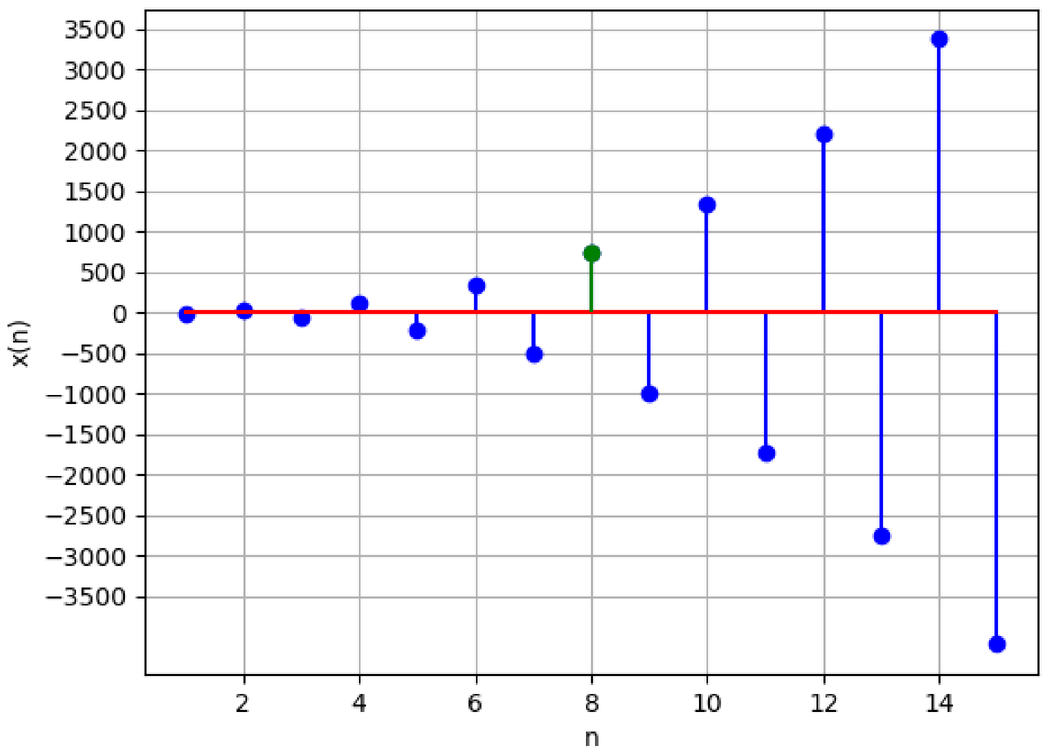
$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+z^{-1}}, |z| > 1 \tag{3.261}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{-z^{-1}}{\left(1+z^{-1}\right)^2}, |z| > 1 \tag{3.262}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}-1\right)}{\left(1+z^{-1}\right)^3}, |z| > 1 \tag{3.263}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{-z^{-1}\left(1-4z^{-1}+z^{-2}\right)}{\left(1+z^{-1}\right)^4}, |z| > 1 \tag{3.264}$$

$$X(z) = \frac{z^{-2}-z^{-1}+1}{\left(1+z^{-1}\right)^4}, |z| > 1 \tag{3.265}$$



3.0.28 Q10) The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Solution: Table of Parameters

(n+1)$^{\text{th}}$ term of GP x(n) is given by:

$$x(n) = x(0)\, r^n \; u(n) \tag{3.266}$$

Then from given table of parameters,

$$x(0) + x(1) + x(2) = 56 \tag{3.267}$$

Input Variable	Condition
$x(0)$, $x(n)$	first term and general term of a GP
r	common ratio of a GP
$x(0), x(1), x(2)$	three terms in GP
$x_i(n)$	general term of i^{th} GP sequence
$x_i(0)$	first term of i^{th} GP sequence
r_i	common ratio of i^{th} GP sequence

$$x(0) \implies \frac{56}{(1+r+r^2)} \tag{3.268}$$

and from given another case following are in AP,

$$x(0)-1, x(1)-7, x(2)-21$$

$$2(x(1)-7) = x(0)-1+x(2)-21 \tag{3.269}$$

$$x(0)\left(r^2-2r+1\right) = 8 \tag{3.270}$$

and from (3.268)

$$\frac{56.\left(r^2-2r+1\right)}{(1+r+r^2)} = 8 \tag{3.271}$$

$$r_1 = 2\ ,\ r_2 = \frac{1}{2} \tag{3.272}$$

so from (3.268),

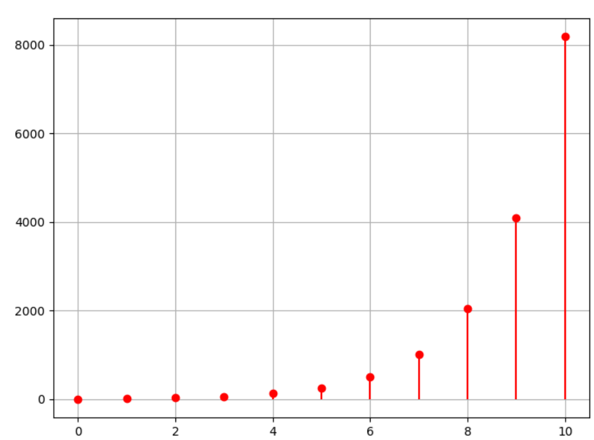
$$x_1(0) = 8 \ , \ x_2(0) = 32 \tag{3.273}$$

Then from (3.266)

$$x_1(n) = 8.2^n = 2^{n+3} \ u(n) \tag{3.274}$$

$$x_2(n) = 32. \left(\frac{1}{2}\right)^n u(n) = 2^{5-n} \ u(n) \tag{3.275}$$

$x_1(0)$, $x_1(1)$ and $x_1(2)$ are 8, 16, 32 (or) $x_2(0)$, $x_2(1)$ and $x_2(2)$ are 32, 16, 8 respectively



Graph of $x_1(n)$

z-transform of $x_1(n)$ is given by:

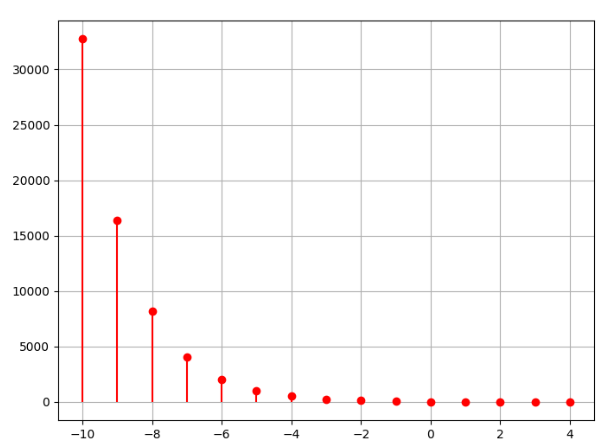
$$X_1(z) = \sum_{k=-\infty}^{\infty} x_1(k) \,.z^{-k} \tag{3.276}$$

from (3.274),

$$X_1(z) = \sum_{k=0}^{\infty} 2^{k+3}\ z^{-k} \tag{3.277}$$

Hence,

$$X_1(z) = \frac{8}{1-2z^{-1}}, \quad |2z^{-1}| < 1 \tag{3.278}$$



Graph of $x_2\left(n\right)$

and also from (3.275),

$$X_2\left(z\right) = \sum_{k=-\infty}^{\infty} x_2\left(k\right).z^{-k} \tag{3.279}$$

$$X_2\left(z\right) = \sum_{k=0}^{\infty} 2^{5-k}\ z^{-k} \tag{3.280}$$

Hence,

$$X_2\left(z\right) = \frac{32}{1-\left(2z\right)^{-1}}, \quad |\left(2z\right)^{-1}| < 1 \tag{3.281}$$

3.0.29 Find the sum to n terms for the given series: $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$

Solution: Sum of n terms of AP is given by

Variable	**Description**	**Value**
$x(n)$	n^{th} term of sequence	$(3n+3)(3n+8)u(n)$

Table 3.27: input parameters

$$x(n) = (3n+3)(3n+8)u(n) \tag{3.282}$$

$$y(n) = x(n) * u(n) \tag{3.283}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})} \quad |z| > 1 \tag{3.284}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{3.285}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \quad |z| > 1 \tag{3.286}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4} \quad |z| > 1 \tag{3.287}$$

$$\implies X(z) = 9z^{-1}\frac{\left(1+z^{-1}\right)}{(1-z^{-1})^3} + \frac{33\left(z^{-1}\right)}{(1-z^{-1})^2} + \frac{24}{(1-z^{-1})}\,|z| > 1 \tag{3.288}$$

$$Y(z) = X(z)U(z) \tag{3.289}$$

$$\implies Y(z) = 9z^{-1}\frac{\left(1+z^{-1}\right)}{(1-z^{-1})^4} + \frac{33\left(z^{-1}\right)}{(1-z^{-1})^3} + \frac{24}{(1-z^{-1})^2}\,|z| > 1 \tag{3.290}$$

Now from (3.284), (3.285), (3.286), (3.287), (3.290) By using inverse Z-transform pairs,

$$y(n) = \left(\frac{9n(n+1)(2n+1)}{6} + \frac{33n(n+1)}{2} + 24(n+1)\right)u(n) \tag{3.291}$$

$\therefore$ Sum of n terms of the series whose n^{th} term is given by $(3n+3)(3n+8)\,u(n)$ is $(\frac{9n(n+1)(2n+1)}{6} + \frac{33n(n+1)}{2} + 24(n+1))\mathrm{u}(n)$

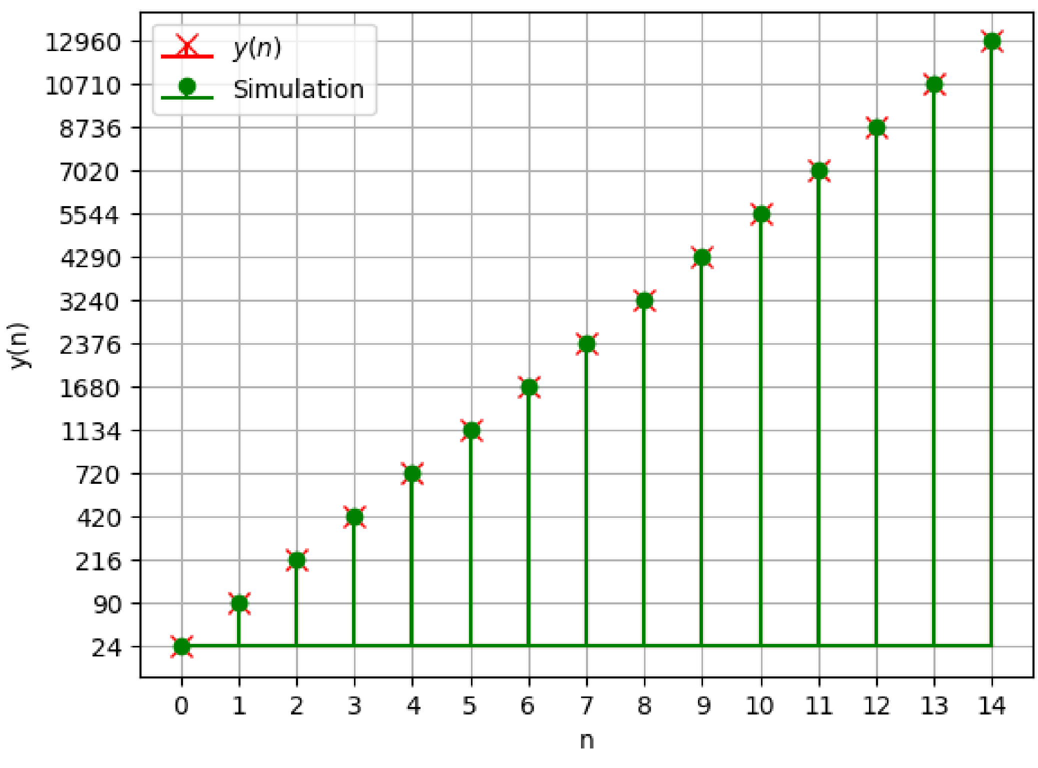


Figure 3.29: Theory vs Simulation

3.0.30 Find the sum to n terms of the series:

$1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + ...$ (NCERT 11.9.4.7)

Solution:

Variable	Description	Value
$y(n)$	Sum of $n+1$ terms of the series	?
$x(n)$	General term	$(n+1)^2 u(n)$

Table 3.28: Variables Used

$$y(n) = 1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + ... \tag{3.292}$$

Let,

$$x(n) = (n+1)^2 u(n) \tag{3.293}$$

$$\implies y(n) = x(n) * u(n) * u(n) \tag{3.294}$$

$$Y(z) = X(z)\,(U(z))^2 \tag{3.295}$$

From (**??**),

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \quad \{|z| > 1\} \tag{3.296}$$

Using (**??**),

$$(n+1)^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1 + z^{-1}}{\left(1 - z^{-1}\right)^3} \quad \{|z| > 1\} \tag{3.297}$$

From (3.297),

$$Y(z) = \left(\frac{1+z^{-1}}{\left(1-z^{-1}\right)^3}\right)\left(\frac{1}{1-z^{-1}}\right)^2 \tag{3.298}$$

$$= \frac{1+z^{-1}}{\left(1-z^{-1}\right)^5} \tag{3.299}$$

$$= \frac{1}{\left(1-z^{-1}\right)^5} + \frac{z^{-1}}{\left(1-z^{-1}\right)^5} \quad \{|z|>1\} \tag{3.300}$$

From (**??**), using (**??**),

$$\frac{(n+1)(n+2)(n+3)(n+4)}{24}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1-z^{-1}\right)^5} \ \{|z|>1\} \tag{3.301}$$

$$\frac{(n)(n+1)(n+2)(n+3)}{24}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^5} \ \{|z|>1\} \tag{3.302}$$

From (3.301) and (3.302), taking the Inverse Z Transform,

$$y(n) = \left(\frac{(n+1)(n+2)(n+3)(n+4)}{24}u(n)\right) + \left(\frac{(n)(n+1)(n+2)(n+3)}{24}u(n)\right) \tag{3.303}$$

$$\implies y(n) = \frac{(n+1)(n+2)^2(n+3)}{12}u(n) \tag{3.304}$$

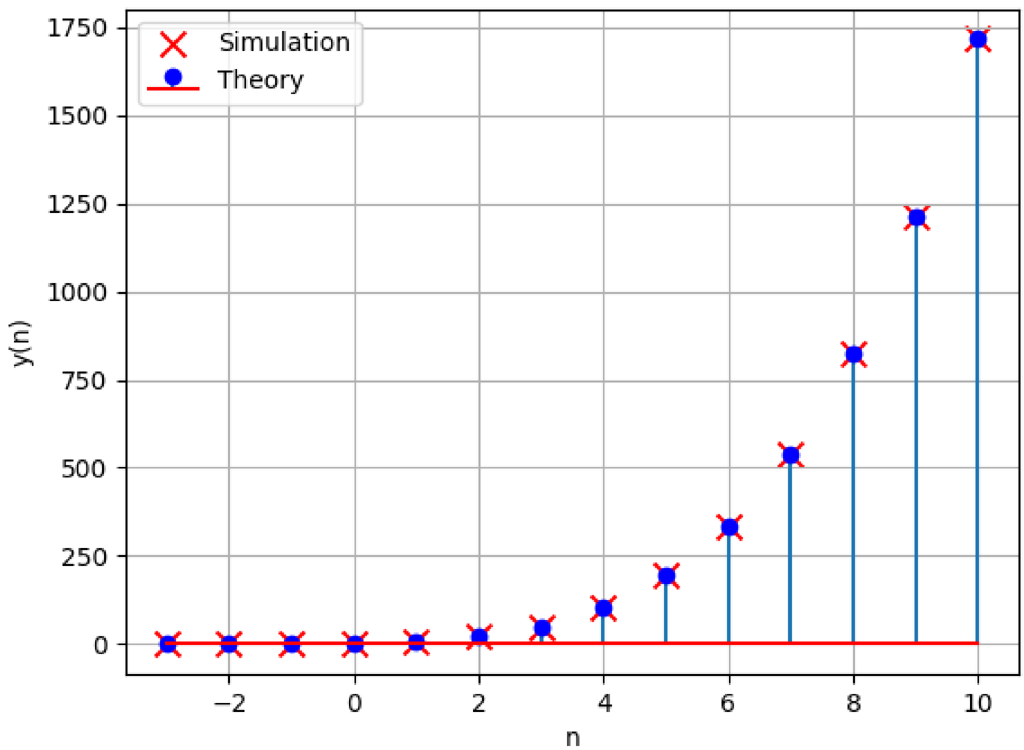


Figure 3.30: Stem Plot of $y\left(n\right)$

3.0.31 Find a GP for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution: From Table 3.29:

$$x(0)r^4 = 4x(0)r^2 \tag{3.305}$$

$$\implies r = \pm 2 \tag{3.306}$$

From Table 3.29 and (3.306) :

Parameter	Description	Value
$x(0)$	First term of AP	–
r	Common ratio	–
$x(n)$	General term of given AP	$x(0)r^n u(n)$
$x(0)+x(1)$	sum of 1st and 2nd term	-4
$\frac{x(4)}{x(2)}$	Ratio of 5th and 3rd term	4

Table 3.29: Input Parameters

$$x(0)r + x(0) = -4 \tag{3.307}$$

$$\implies x(0) = \frac{-4}{r+1} \tag{3.308}$$

$$x(0) = \begin{cases} \frac{-4}{3}, & r = +2 \\ 4, & r = -2 \end{cases} \tag{3.309}$$

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad , |z| > |r| \tag{3.310}$$

$$X(z) = \begin{cases} \frac{4}{3(2z^{-1}-1)}, & r = +2 \\ \frac{4}{1+2z^{-1}}, & r = -2 \end{cases} \tag{3.311}$$

$|z| > 2$

3.0.32 Find the sum of the following series up to n terms and obtain the Z-transform:

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots$$

Solution:

Symbol	**Description**
$s(n)$	sum of n terms
$S(z)$	Z-transform of s(n)
$s_1(n)$	$(n+1)^3 * u(n)$
$S_1(z)$	Z-transform of $s_1(n)$
$s_2(n)$	$(2n+1) * u(n)$
$S_2(z)$	Z-transform of $s_2(n)$

Table 3.30: Parameters

$$s(n) = \sum_{r=0}^{n} \left(\frac{\sum_{i=0}^{r} (i+1)^3}{\sum_{i=0}^{r} (2i+1)} \right) \tag{3.312}$$

$$= \frac{(n+1)^3 * u(n)}{(2n+1) * u(n)} * u(n) \tag{3.313}$$

$$= \frac{s_1(n)}{s_2(n)} * u(n) \tag{3.314}$$

$$s_1(n) = (n+1)^3 * u(n) \tag{3.315}$$

$$s_1(n) \overset{\mathcal{Z}}{\longleftrightarrow} S_1(z) \tag{3.316}$$

$$(n+1)^3 \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1+4z^{-1}+z^{-2}}{(1-z^{-1})^4} \tag{3.317}$$

$$S_1(z) = \left(\frac{1+4z^{-1}+z^{-2}}{(1-z^{-1})^4} \right) \left(\frac{1}{1-z^{-1}} \right) \qquad \{|z| > 1\} \tag{3.318}$$

$$= \frac{1}{(1-z^{-1})^5} + \frac{4z^{-1}}{(1-z^{-1})^5} + \frac{z^{-2}}{(1-z^{-1})^5} \tag{3.319}$$

using (**??**) and (**??**)

$$\begin{aligned}
s_1(n) = {} & \frac{(n+4)(n+3)(n+2)(n+1)}{24}u(n+4) \\
& + \frac{(n+3)(n+2)(n+1)n}{6}u(n+3) \\
& + \frac{(n+2)(n+1)n(n-1)}{24}u(n+2) && (3.320) \\
= {} & \frac{(n+2)^2(n+1)^2}{4} \qquad \{n \geq 0\} && (3.321)
\end{aligned}$$

$$\begin{aligned}
s_2(n) &= (2n+1) * u(n) && (3.322) \\
s_2(n) &\overset{z}{\longleftrightarrow} S_2(z) && (3.323) \\
2n+1 &\overset{z}{\longleftrightarrow} \left(\frac{1+z^{-1}}{(1-z^{-1})^2}\right) && (3.324) \\
S_2(z) &= \left(\frac{1+z^{-1}}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \qquad \{|z| > 1\} && (3.325) \\
&= \frac{1}{(1-z^{-1})^3} + \frac{z^{-1}}{(1-z^{-1})^3} && (3.326)
\end{aligned}$$

using (**??**) and (**??**)

$$\begin{aligned}
s_2(n) = {} & \frac{(n+2)(n+1)}{2}u(n+2) \\
& + \frac{(n+1)(n)}{2}u(n+1) && (3.327) \\
= {} & (n+1)^2 \qquad \{n \geq 0\} && (3.328)
\end{aligned}$$

replacing (3.321) and (3.328) in (3.314)

$$s\left(n\right)=\frac{\left(n+2\right)^{2}}{4}*u\left(n\right) \tag{3.329}$$

$$s\left(n\right)\overset{\mathcal{Z}}{\longleftrightarrow}S\left(z\right) \tag{3.330}$$

$$\frac{\left(n+2\right)^{2}}{4}\overset{\mathcal{Z}}{\longleftrightarrow}\left(\frac{4-3z^{-1}+z^{-2}}{4\left(1-z^{-1}\right)^{3}}\right) \tag{3.331}$$

$$S\left(z\right)=\left(\frac{4-3z^{-1}+z^{-2}}{4\left(1-z^{-1}\right)^{3}}\right)\left(\frac{1}{1-z^{-1}}\right)\qquad\{|z|>1\} \tag{3.332}$$

$$\begin{aligned}&=\frac{1}{(1-z^{-1})^{4}}-\frac{3z^{-1}}{4(1-z^{-1})^{4}}\\&+\frac{z^{-2}}{4(1-z^{-1})^{4}}\end{aligned} \tag{3.333}$$

$$\begin{aligned}s\left(n\right)&=\frac{\left(n+3\right)\left(n+2\right)\left(n+1\right)}{6}u\left(n+3\right)\\&-\frac{\left(n+2\right)\left(n+1\right)\left(n\right)}{8}u\left(n+2\right)\\&+\frac{\left(n+1\right)\left(n\right)\left(n-1\right)}{24}u\left(n+1\right)\end{aligned} \tag{3.334}$$

$$=\left(1+\frac{37n}{24}+\frac{5n^{2}}{8}+\frac{n^{3}}{12}\right)u\left(n\right) \tag{3.335}$$

3.0.33 Insert five numbers between 8 and 26 such that the resulting sequence is an A.P. and obtain the Z-transform of the sequence.

Solution: Given,

symbol	**value**	**description**
$x(0)$	8	first term of the series
$x(6)$	26	last term of the series
N	$2+5=7$	number terms in the series

Table 3.31: Parameters

$$d = \frac{x(6) - x(0)}{N-1}, \tag{3.336}$$

$$= 3 \tag{3.337}$$

$$x(n) = [x(0) + nd]u(n) \tag{3.338}$$

the A.P. sequence is:

$$8, 11, 14, 17, 20, 23, 26 \tag{3.339}$$

using eq (**??**),

$$\implies X(z) = \frac{8}{1-z^{-1}} + \frac{3z^{-1}}{(1-z^{-1})^2} \qquad \{ROC : z \neq 1\} \tag{3.340}$$

3.0.34 If S_1, S_2, S_3 are the sum of the first n natural numbers, their squares, and their cubes, respectively, show that

$$9(S_2)^2 = (S_3)(1 + 8(S_1))$$

Solution:

3.0.35 If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P

Solution:

Table 3.32: Input Parameters

Symbol	Remarks
$x(0)$	a
$x(1)$	b
$x(2)$	c
$x(3)$	d
r	ratio of G.P a,b,c....
r_1	ratio of G.P $a^n + b^n, b^n + c^n, ...$
$X(z)$	z transform of G.P a,b,c....
$Y(z)$	z transform of G.P $a^n + b^n, b^n + c^n, ...$

From Table 3.32

$$r = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \tag{3.341}$$

From eq (3.341)

$$\frac{b^n + c^n}{a^n + b^n} = \frac{(ar)^n + \left(ar^2\right)^n}{(a)^n + (ar)^n} \tag{3.342}$$

$$= \frac{a^n r^n \left(1 + r^n\right)}{a^n \left(1 + r^n\right)} \tag{3.343}$$

$$= r^n \tag{3.344}$$

$$\frac{c^n + d^n}{b^n + c^n} = \frac{\left(ar^2\right)^n + \left(ar^3\right)^n}{(ar)^n + (ar^2)^n} \tag{3.345}$$

$$= \frac{a^n r^{2n} \left(1 + r^n\right)}{a^n r^n \left(1 + r^n\right)} \tag{3.346}$$

$$= r^n \tag{3.347}$$

$$\frac{b^n + c^n}{a^n + b^n} = \frac{c^n + d^n}{b^n + c^n} \tag{3.348}$$

Hence proved they are in in G.P

$$x\left(n\right) = a\left(\frac{b}{a}\right)^n u\left(n\right) \tag{3.349}$$

$$X\left(z\right) = \frac{a}{1 - \left(\frac{b}{a}\right) z^{-1}}, \quad |z| > \left|\frac{b}{a}\right| \tag{3.350}$$

$$r_1 = \frac{b^n + c^n}{a^n + b^n} = \frac{c^n + d^n}{b^n + c^n} \tag{3.351}$$

From eq (3.351)

$$y(n) = (a^n + b^n)\left(\frac{b^n + c^n}{a^n + b^n}\right)^n u(n) \tag{3.352}$$

$$Y(z) = \frac{a^n + b^n}{1 - \left(\frac{b^n + c^n}{a^n + b^n}\right) z^{-1}}, \quad |z| > \left|\frac{b^n + c^n}{a^n + b^n}\right| \tag{3.353}$$

3.0.36 Write the first five terms of the sequence whose n^{th} terms $a_n = \frac{n}{n+1}$

Solution:

Term	Value	Description
$x(n)$	$\frac{n+1}{n+2}u(n)$	General term

Table 3.33: Input Parameters

Here, Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n} \tag{3.354}$$

$$= \sum_{n=-\infty}^{\infty} \frac{n+1}{n+2}.u(n).z^{-n} \tag{3.355}$$

$$= \sum_{n=-\infty}^{\infty} u(n).z^{-n} - \frac{1}{n+2}u(n).z^{-n} \tag{3.356}$$

Now,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{3.357}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty} -\frac{1}{n+2}u(n).z^{-n} &= -\frac{1}{2} - \frac{z^{-1}}{3} - \frac{z^{-2}}{4}... \\ &= z^2[-z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3}...] + z \\ &= z + z^2\log(1-z^{-1})\end{aligned}$$

$$\frac{-1}{n+2}.u(n) \overset{Z}{\longleftrightarrow} \frac{1}{z^{-1}} + \frac{\log(1-z^{-1})}{z^{-2}}, \quad |z| > 1 \tag{3.358}$$

$$X(z) = \frac{1}{1-z^{-1}} + \frac{1}{z^{-1}} + \frac{\log(1-z^{-1})}{z^{-2}}, \quad |z| > 1 \tag{3.359}$$

3.0.37 150 workers were engaged to finish a job in a certain number of days, 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution:

3.0.38 The Fibonacci sequence is defined by 1 = a1 = a_2 and $a_n = an - 1 + a_{n-2}$, n > 2

Find $\frac{a_{n+1}}{a_n}$, for n = 1, 2, 3, 4, 5

Solution:

Parameter	Value	Description
n	≥ 0	Non negative Integer
$x(n)$	$x(n) = x(n-1) + x(n-2) + u(-n)$	$(n+1)^{th}$ term
$y(n)$	$\frac{x(n+1)}{x(n)}$	Required function
$x(0)$	1	1^{st} term
$x(1)$	1	2^{nd} term

Table 3.34: Input Table

Here, $a_1 = 1, a_2 = 1$

$$a_n = a_{n-1} + a_{n-2}, n{>}2 \tag{3.360}$$

Applying z transform,

$$X(z) = z^{-1}X(z) + z^{-2}X(z) + z^{-0} \tag{3.361}$$

$$= \frac{1}{1 - z^{-1} - z^{-2}} \tag{3.362}$$

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad , \quad |\ z\ | > |\ \alpha\ | \tag{3.363}$$

Where, $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$

Using partial fractions,

$$X(z) = \frac{\alpha}{(\alpha - \beta)} \frac{1}{(1 - \alpha z^{-1})} - \frac{\beta}{(\alpha - \beta)} \frac{1}{(1 - \beta z^{-1})} \tag{3.364}$$

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \quad |\ z\ | > |\ a\ | \tag{3.365}$$

Substituting this result,

$$x(n) = \frac{\alpha}{(\alpha - \beta)}(\alpha^n u(n)) - \frac{\beta}{(\alpha - \beta)}(\beta^n u(n)) \tag{3.366}$$

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \tag{3.367}$$

$$x(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} u(n) \tag{3.368}$$

$$y(n) = \frac{x(n+1)}{x(n)} \tag{3.369}$$

$$y(n) = \frac{1}{2}\left[\frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}\right] \tag{3.370}$$

3.0.39 A farmer buys a used tractor for Rs 12000. He pays Rs 6000 in cash and agrees to pay the balance in annual installments of Rs 500 plus 12 % interest on the unpaid amount. How much will the tractor cost him? **Solution:**

Parameter	**Value**	**Description**
$n+1$	-	Number of years
a	6000	Amount paid
r	6000	Remaining Amount
$x(n)$	$(1220-60n)\ u(n)$	Amount to be paid at $(n+1)^{th}$ year
$y(n)$	$\left(1220+1190n-30n^2\right)\ u(n)$	Total amount after $(n+1)$ yrs

Table 3.35: Input Table

Number of years taken to pay the remaining amount,

$$n+1 = \frac{r}{500} \tag{3.371}$$

$$n = 11 \tag{3.372}$$

The amount to be paid by the farmer after $(n+1)$ year/s is,

$$x(n) = 500 + 0.12\,(6000 - 500n) \tag{3.373}$$

$$x(n) = (1220 - 60n)\ u(n) \tag{3.374}$$

Some results,

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{3.375}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}} : U(z) \tag{3.376}$$

$$n.u(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\frac{1}{1-z^{-1}} = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{3.377}$$

By taking z transform,

$$X(z) = \frac{1220}{1 - z^{-1}} - \frac{60z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{3.378}$$

$$X(z) = \frac{1220 - 1280z^{-1}}{\left(1 - z^{-1}\right)^2} \quad , \quad |\, z \,| > 1 \tag{3.379}$$

Convolution in time domain is multiplication in z domain :

$$y(n) = x(n) * u(n) \tag{3.380}$$

$$Y(z) = X(z)\,.U(z) \tag{3.381}$$

$$Y(z) = \frac{1220 - 1280z^{-1}}{\left(1 - z^{-1}\right)^2} . \frac{1}{1 - z^{-1}} \tag{3.382}$$

$$Y(z) = \frac{1220 - 1280z^{-1}}{\left(1 - z^{-1}\right)^3} \quad , \quad |\, z \,| > 1 \tag{3.383}$$

Using Partial fractions,

$$Y(z) = \frac{Az}{z - 1} + \frac{Bz}{\left(z - 1\right)^2} + \frac{Cz}{\left(z - 1\right)^3} \tag{3.384}$$

$$A = 1220 \tag{3.385}$$

$$B = 1160 \tag{3.386}$$

$$C = -60 \tag{3.387}$$

$$\frac{n(n-1)(n-2)\ldots(n-k-2)}{k!} u(n) \xleftrightarrow{\quad z \quad} \frac{z}{\left(z - 1\right)^k} \tag{3.388}$$

Using this result,

$$y\left(n\right) = 1220\ u\left(n\right) + 1160\ n\ u\left(n\right) - 60\ \frac{n\left(n-1\right)}{2}\ u\left(n\right) \tag{3.389}$$

$$y\left(n\right) = \left(1220 + 1190n - 30n^2\right)\ u\left(n\right) \tag{3.390}$$

$$y\left(11\right) = 10680 \tag{3.391}$$

The total tractor cost to be paid,

$$= a + y\left(11\right) \tag{3.392}$$

$$= 16,680 \tag{3.393}$$

3.0.40 1) Find the sum to n terms for the given series: $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$ **Solution:**

3.0.41 The sum of the first n terms of two arithmetic progressions (AP) is in the ratio $5n+4 : 9n+6$. Find the ratio of their 18th terms.

Solution:

3.0.42 How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636? **Solution:**

Parameter	Description	Value
x(0)	First Term	9
d	Common Difference	8
S_n	Sum of n terms	636

Table 3.36: Parameter Table

We know the formula

$$S_n = \frac{(n+1)}{2}[2x(0) + d(n)] \tag{3.394}$$

Putting in values from the table

$$636 = \frac{(n+1)}{2}[18 + 8n] \tag{3.395}$$

$$636 = (n+1)[4n+9] \tag{3.396}$$

$$4n^2 + 13n - 627 = 0 \tag{3.397}$$

On solving this quadratic equation, we get roots

$n = -12.5$ and $n = 11$

Since we are looking for positive terms of n, we remove the negative root

$$\implies n = 11 \tag{3.398}$$

$\therefore$ the total number of terms are 12

The Z-Transform of the above question is

$$X(z) = \frac{9}{1-z^{-1}} + \frac{8z^{-1}}{(1-z^{-1})^2} \tag{3.399}$$

3.0.43 How many terms of the A.P. -6,$-\frac{11}{2}$, -5, ... are needed to give the sum -25?

Solution:

Symbol	Value	Description
$x(0)$	-6	first term of AP
d	$\frac{1}{2}$	common difference of AP
$n+1$	?	number of terms
$x(n)$	$x(0)+nd$	nth term of the AP

Table 1: Input data

$$y\left(n\right) = x\left(n\right) * u\left(n\right) \tag{3.400}$$

$$Y\left(z\right) = X\left(z\right)U\left(z\right) \tag{3.401}$$

$$Y\left(z\right) = \frac{x\left(0\right)}{\left(1-z^{-1}\right)^2} + \frac{dz^{-1}}{\left(1-z^{-1}\right)^3} \quad |z| > 1 \tag{3.402}$$

$$Y\left(z\right) = \frac{-6}{\left(1-z^{-1}\right)^2} + \frac{0.5z^{-1}}{\left(1-z^{-1}\right)^3} \quad |z| > 1 \tag{3.403}$$

Some Results:

$$(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{3.404}$$

$$(n^2+n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{2z^{-1}}{\left(1-z^{-1}\right)^3} \quad |z| > 1 \tag{3.405}$$

Using (5) and (6) and taking inverse Z-transform

$$y\left(n\right) = (-6(n+1) + \frac{1}{4}(n^2+n))u(n) \tag{3.406}$$

$$\implies -25 = \frac{1}{4}n^2 - \frac{23}{4}n - 6 \tag{3.407}$$

$$\implies 0 = n^2 - 23n + 76 \tag{3.408}$$

$$n = 19 or 4 \tag{3.409}$$

Hence number of terms required is 5 or 20.

3.0.44 Find the sum to indicated number of terms in the geometric progression:

$1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).

Solution:

3.0.45 Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2 R^n = S^n$.

Solution:

3.0.46 Find the sum to n terms to the series $3(1)^2 + 5(2)^2 + 7(3)^2 + \dots$

Solution: Given series is $3(1)^2 + 5(2)^2 + 7(3)^2 + \dots$

$x(0)$	3	1st term
$x(n)$	?	$(n+1)$th term
$y(n-1)$	?	sum of n terms

Table 3.38: parameters

$$x(n) = (2n+3)(n+1)^2 \tag{3.410}$$

$$y(n) = x(n) * u(n) \tag{3.411}$$

$$Y(z) = X(z)U(z) \tag{3.412}$$

$$X(z) = \frac{3 + 8z^{-1} + z^{-2}}{(1 - z^{-1})^4} \tag{3.413}$$

$$U(z) = \frac{1}{1 - z^{-1}} \tag{3.414}$$

$$\implies Y(z) = \frac{3 + 8z^{-1} + z^{-2}}{(1 - z^{-1})^5} \tag{3.415}$$

$$Y(z) = \frac{23z^{-1}}{1 - z^{-1}} + \frac{63z^{-2}}{(1 - z^{-1})^2} + \frac{81z^{-3}}{(1 - z^{-1})^3} + \frac{50z^{-4}}{(1 - z^{-1})^4} + \frac{12z^{-5}}{(1 - z^{-1})^5} + 3 \tag{3.416}$$

$$\delta(n) \xleftrightarrow{Z} 1 \tag{3.417}$$

$$u\left(n-1\right) \xleftrightarrow{Z} \frac{z^{-1}}{1-z^{-1}} \tag{3.418}$$

$$\left(n-1\right)u\left(n-1\right) \xleftrightarrow{Z} \frac{z^{-2}}{\left(1-z^{-1}\right)^2} \tag{3.419}$$

$$\frac{\left(n-1\right)\left(n-2\right)u\left(n-1\right)}{2} \xleftrightarrow{Z} \frac{z^{-3}}{\left(1-z^{-1}\right)^3} \tag{3.420}$$

$$\frac{\left(n-3\right)\left(n-2\right)\left(n-1\right)u\left(n-1\right)}{6} \xleftrightarrow{Z} \frac{z^{-4}}{\left(1-z^{-1}\right)^4} \tag{3.421}$$

$$\frac{\left(n-4\right)\left(n-3\right)\left(n-2\right)\left(n-1\right)u\left(n-1\right)}{24} \xleftrightarrow{Z} \frac{z^{-5}}{\left(1-z^{-1}\right)^5} \tag{3.422}$$

By using above 6 equations, we get

$$\begin{aligned} y\left(n\right) = {} & 3\delta n + 23u\left(n-1\right) + 63\left(n-1\right)u\left(n-1\right) \\ & + \frac{81\left(n-1\right)\left(n-2\right)u\left(n-1\right)}{2} \\ & + \frac{50\left(n-3\right)\left(n-2\right)\left(n-1\right)u\left(n-1\right)}{6} \\ & + \frac{\left(n-4\right)\left(n-3\right)\left(n-2\right)\left(n-1\right)u\left(n-1\right)}{24} \end{aligned} \tag{3.423}$$

3.0.47 Find the sum to n terms of the sequence $8, 88, 888, 8888\ldots$

Solution:

Parameter	**Value**	**description**
$x(0)$	8	First term
$x(1)$	88	Second term
$x(n)$	$(\sum_{k=0}^{n} 8\left(10^k\right) u(n)$	General term
$S(n)$	$S(n) = \sum_{k=0}^{n-1} x(k)$	Sum of n terms

Table 3.39: Input parameters

From Table 3.39

$$s(n) = x(n) * u(n) \tag{3.424}$$

Z transform of general term

$$X(z) = \sum_{n=-\infty}^{\infty} (\sum_{k=0}^{n} 8(10)^k) u(n) z^{-n} \tag{3.425}$$

$$X(z) = 8 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (10)^k \right) u(n) z^{-n} \tag{3.426}$$

$$X(z) = 8 \left(\sum_{n=0}^{\infty} (10)^n (z^{-n}) \right) \left(\sum_{n=0}^{\infty} z^{-n} \right) \tag{3.427}$$

$$\implies X(z) = \left(\frac{8}{(1 - 10z^{-1})(1 - z^{-1})} \right) \quad |z| > 10 \tag{3.428}$$

From (3.424), we get

$$S(z) = (X(z))(U(z)) \tag{3.429}$$

$$S(z) = \left(\frac{8}{(1-10z^{-1})(1-z^{-1})}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{3.430}$$

$$S(z) = \left(\frac{8}{(1-10z^{-1})(1-z^{-1})^2}\right) \tag{3.431}$$

$$S(z) = \frac{-224z^{-1}}{81(1-z^{-1})} - \frac{8z^{-2}}{9(1-z^{-1})^2} + \frac{8000z^{-1}}{81(1-10z^{-1})} + 8 \tag{3.432}$$

$$\delta(n) \overset{Z}{\longleftrightarrow} 1 \tag{3.433}$$

$$u(n-1) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{1-z^{-1}} \quad |z| > 1 \tag{3.434}$$

$$a^{n-k}u(n-k) \overset{Z}{\longleftrightarrow} \frac{z^{-k}}{1-az^{-1}} \quad |z| > a \tag{3.435}$$

$$(n-k)\,u(n-k) \overset{Z}{\longleftrightarrow} \frac{z^{-k-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{3.436}$$

From the above 4 equations, we get

$$\begin{aligned} s(n) = {} & \frac{-224u(n-1)}{81} - \frac{8(n-1)\,u(n-1)}{9} \\ & + \frac{8000(10^{n-1})u(n-1)}{81} + 8\delta(n) \end{aligned} \tag{3.437}$$

3.0.48 If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms , prove that $P^2 = (ab)^n$

Solution:

Parameter	**Value**	**Description**
$x(0)$	a	First Term
$x(n)$	b	n^{th} term
r	$\left(\frac{b}{a}\right)^{\frac{1}{n}}$	Common Ratio
P	?	Product of n terms

Table 3.40: Parameter Table 11.9.3.22

' The n^{th} term of GP is :-

$$x(n) = x(0)\, r^n u(n) \tag{3.438}$$

$$P = \prod_{k=0}^{n} x(0)\, r^k = (x(0))^n\, r^{\frac{n^2}{2}} = (ab)^{\frac{n}{2}} \tag{3.439}$$

$$\implies P^2 = (ab)^n \tag{3.440}$$

Z-transform of $x(n)$:

$$X(z) = \frac{a}{1 - \left(\frac{b}{a}\right)^{\frac{1}{n}} z^{-1}}, \quad |z| > \left(\frac{b}{a}\right)^{\frac{1}{n}} \tag{3.441}$$

3.0.49 Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7 th and (m - 1) th numbers is 5:9. Find the value of m.

Solution:

Symbol	Value	description
$x(0)$	1	First term of A.P
$x(n)$	31	$(n+1)$ th term
$\frac{x(7)}{x(m-1)}$	$\frac{5}{9}$	ratio of 7 th and $(m-1)$ th numbers
n	$m+2$	number of terms

Table 3.41:

The last term is

$$x(n) = x(0) + (n)\,d \tag{3.442}$$

$$\implies 31 = 1 + (m+1)\,d \tag{3.443}$$

$$\implies 30 = (m+1)\,d \tag{3.444}$$

$$\implies \frac{30}{m+1} = d \tag{3.445}$$

Now 7th and $(m-1)$th terms

$$x\,(7) = x(0) + 7d \tag{3.446}$$

$$x\,(m-1) = x(0) + (m-1)\,d \tag{3.447}$$

From equations (3.446) and (3.447)

$$\frac{x(0) + 7d}{x(0) + (m-1)\,d} = \frac{5}{9} \tag{3.448}$$

Substituting (3.445) in (3.448)

$$\implies \frac{1+7\left(\frac{30}{m+1}\right)}{1+(m-1)\left(\frac{30}{m+1}\right)} = \frac{5}{9} \tag{3.449}$$

$$\implies \frac{m+1+210}{m+1+30m-30} = \frac{5}{9} \tag{3.450}$$

$$\implies \frac{m+181}{31m-29} = \frac{5}{9} \tag{3.451}$$

$$\implies 9m+1899 = 155m-145 \tag{3.452}$$

$$\implies 155m-9m = 1899+145 \tag{3.453}$$

$$\implies 146m = 2044 \tag{3.454}$$

$$\implies m = 14 \tag{3.455}$$

Therefore, $m = 14$.

General term of AP is

$$x(n) = (2n+1)\, u(n) \tag{3.456}$$

$$x(n) = (2n)\, u(n) + u(n) \tag{3.457}$$

The Z-Transform is

$$X(z) = 2\left(\frac{z}{(z-1)^2}\right) + U(z) \tag{3.458}$$

$$= \frac{2z}{(z-1)^2} + \frac{1}{1-z^{-1}} \tag{3.459}$$

$$X(z) = \frac{z^2+z}{(z-1)^2} \quad |z| > 1 \tag{3.460}$$

3.0.50 Find the sum of n terms of this sequence:

$$5^2 + 6^2 + 7^2 ... + 20^2$$

Solution: The standard z transforms,

Parameter	**Description**	**Formulae/Value**
n	Iteration number starting from zero till 15	-
$x(n)$	General term of the sequence from $n = 0$ to $n = 15$	$(n+5)^2$ u(n)
$x(0)$	First term of the sequence	5

Table 3.42: Parameters

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, |z| > 1 \tag{3.461}$$

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \tag{3.462}$$

$$n^2 u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}\left(1 + z^{-1}\right)}{(1 - z^{-1})^3}, |z| > 1 \tag{3.463}$$

As

$$x(n) = \left(n^2 + 10n + 25\right) u(n) \tag{3.464}$$

The z transform of general term can be written as ,

$$X(z) = \frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3} + 10\frac{z^{-1}}{\left(1-z^{-1}\right)^2} + \frac{25}{1-z^{-1}} \tag{3.465}$$

$$X(z) = \frac{16z^{-2} - 39z^{-1} + 25}{\left(1-z^{-1}\right)^3}; |z| > 1 \tag{3.466}$$

On convolution for finding the sum

$$y(n) = x(n) * u(n) \tag{3.467}$$

On z transform,

$$Y(z) = X(z) \cdot U(z) \tag{3.468}$$

$$= \left(\frac{16z^{-2} - 39z^{-1} + 25}{(1-z^{-1}))^3}\right) \cdot \frac{1}{1-z^{-1}} \tag{3.469}$$

$$\implies Y(z) = \frac{16z^{-2} - 39z^{-1} + 25}{\left(1-z^{-1}\right)^4}; \quad |z| > 1 \tag{3.470}$$

Using the contour integration to find the inverse z transform,

$$y(n) = \oint_c Y(z) \cdot z^{n-1} dz \tag{3.471}$$

$$y(21) = \oint_c \left(\frac{16z^{-2} - 39z^{-1} + 25}{\left(1-z^{-1}\right)^4}\right) z^{14} dz \tag{3.472}$$

As there are four poles from observation, so $m = 4$

$$y\left(21\right) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{3.473}$$

$$= \frac{1}{3!} \lim_{z \to 1} \frac{d^3}{dz^3} \left((z-1)^4 \frac{\left(16z^{-2} - 39z^{-1} + 25\right)}{(1-z^{-1})^4} z^{14} \right) \tag{3.474}$$

$$= \frac{1}{6} \lim_{z \to 1} \frac{d^3}{dz^3} \left(\left(16z^{-2} - 39z^{-1} + 25\right) z^{18}\right) \tag{3.475}$$

$$= \frac{1}{6} \lim_{z \to 1} \frac{d^3}{dz^3} \left(16z^{16} - 39z^{17} + 25z^{18}\right) \tag{3.476}$$

$$= \frac{1}{6} \left(16 \times 18 \times 17 \times 16 + 14 \times 17 \times 16 \times 15\right) \tag{3.477}$$

$$\implies y\left(21\right) = 2840 \tag{3.478}$$

Hence the sum of the terms of the sequence is 2840.

3.0.51 In an A.P., if the p-th term is $\frac{1}{q}$ and q-th term is $\frac{1}{p}$, prove that the sum of the first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.

Solution:

$$\frac{1}{q} = x(0) + pd \tag{3.479}$$

$$\frac{1}{p} = x(0) + qd \tag{3.480}$$

From equations (3.479) and (3.480), the argumented matrix is:

$$\begin{pmatrix} 1 & p & \frac{1}{q} \\ 1 & q & \frac{1}{p} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & p & \frac{1}{q} \\ 0 & q-p & \frac{q-p}{pq} \end{pmatrix} \tag{3.481}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{q-p}} \begin{pmatrix} 1 & p & \frac{1}{q} \\ 0 & 1 & \frac{1}{pq} \end{pmatrix} \tag{3.482}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - pR_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{pq} \end{pmatrix} \tag{3.483}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{pq} \end{pmatrix} \tag{3.484}$$

$$x(n) = \left(\frac{n}{pq}\right) u(n) \tag{3.485}$$

$$X(z) = \frac{z^{-1}}{pq\left(1 - z^{-1}\right)^2} \qquad , |z| > 1 \tag{3.486}$$

Finding sum of pq terms,

$$y(n) = \frac{n}{2}\left(2x(0) + (n+1)d\right) \tag{3.487}$$

$$y(pq) = \frac{1}{2}(pq+1) \tag{3.488}$$

Symbols	Values	Description
$x(n)$	$\left(\frac{n}{pq}\right) u(n)$	general term of the series
$y(n)$	$\frac{(n+1)}{2}$	sum of n terms

Table 3.43: Input Parameters

3.0.52 Find the sum of n terms of the series: $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

Solution:

Symbols	Definition
$x(n)$	General term
$y(n)$	Sum of terms till n_{th}term
$Y(z)$	Z-Transformation Of $y(n)$

Table 3.44: Parameter Table

$$x(n) = (n+1)(n+2)u(n) \tag{3.489}$$

By Z-transformation property:

$$Z[nf(n)] = -z\frac{d}{dz}F(z) \tag{3.490}$$

By (3.490),We have the formulas for:

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \tag{3.491}$$

$$n^2u(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} \tag{3.492}$$

Using (3.491),(3.492) for z-transformation of $x(n)$

$$X(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} + \frac{3z^{-1}}{(1-z^{-1})^2} + \frac{2}{1-z^{-1}}, \quad |z| > |1| \tag{3.493}$$

$$Y(z) = X(z) * U(z) \tag{3.494}$$

$$= \left(\frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} + \frac{3z^{-1}}{(1-z^{-1})^2} + \frac{2}{1-z^{-1}}\right)\frac{1}{1-z^{-1}} \tag{3.495}$$

$$= \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^4} + \frac{3z^{-1}}{(1-z^{-1})^3} + \frac{2}{(1-z^{-1})^2} \tag{3.496}$$

By using inverse Z transformation,we have:

$$\frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^4} \xleftrightarrow{Z^{-1}} \frac{2n^3+3n^2+n}{6} \tag{3.497}$$

$$\frac{3z^{-1}}{\left(1-z^{-1}\right)^3} \xleftrightarrow{Z^{-1}} \frac{3n^2+3n}{2} \tag{3.498}$$

$$\frac{2}{\left(1-z^{-1}\right)^2} \xleftrightarrow{Z^{-1}} 2n+2 \tag{3.499}$$

By adding (3.497),(3.498) and (3.499),we get:

$$y\left(n\right) = \frac{2n^3-3n^2+n}{6}+\frac{3n^2+3n}{2}+2n+2 \tag{3.500}$$

$$= \frac{2n^3+12n^2+22n+12}{6} \tag{3.501}$$

$$= \frac{\left(n+1\right)\left(n+2\right)\left(n+3\right)}{3} \tag{3.502}$$

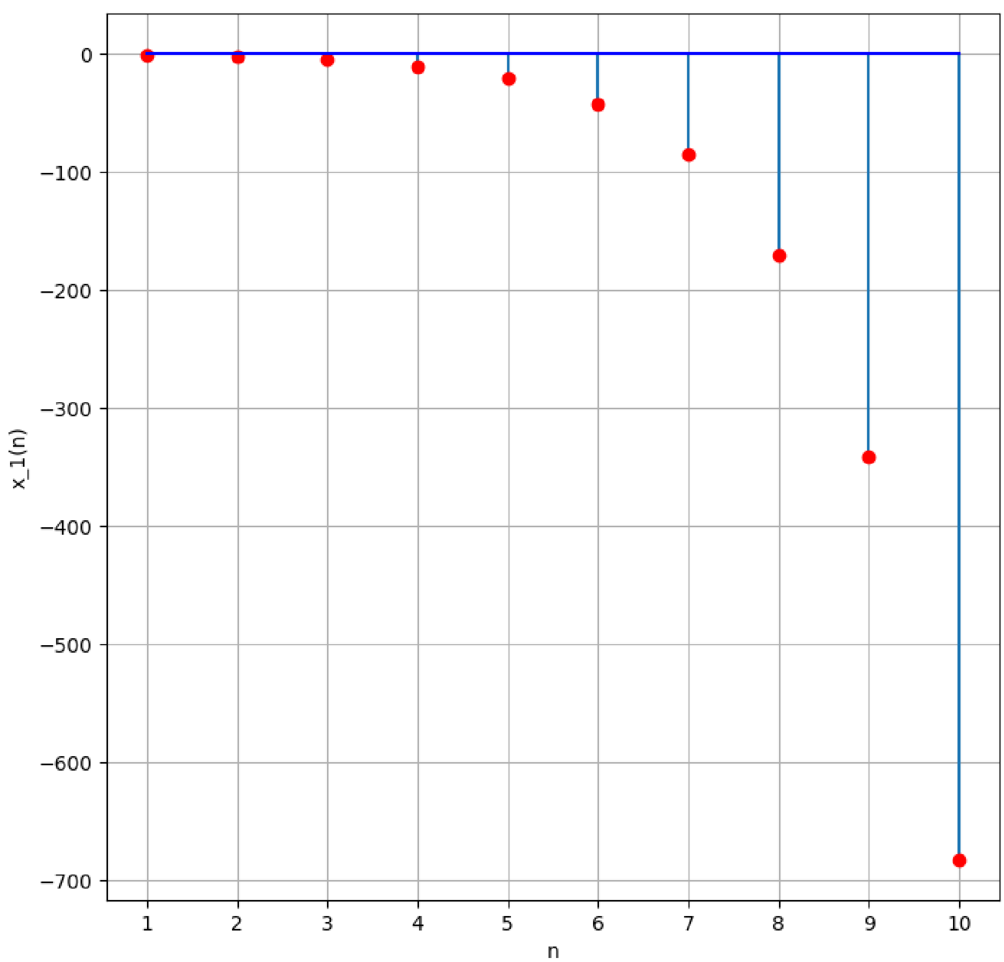


Figure 3.31: Representation of x(n) for $r = 2$

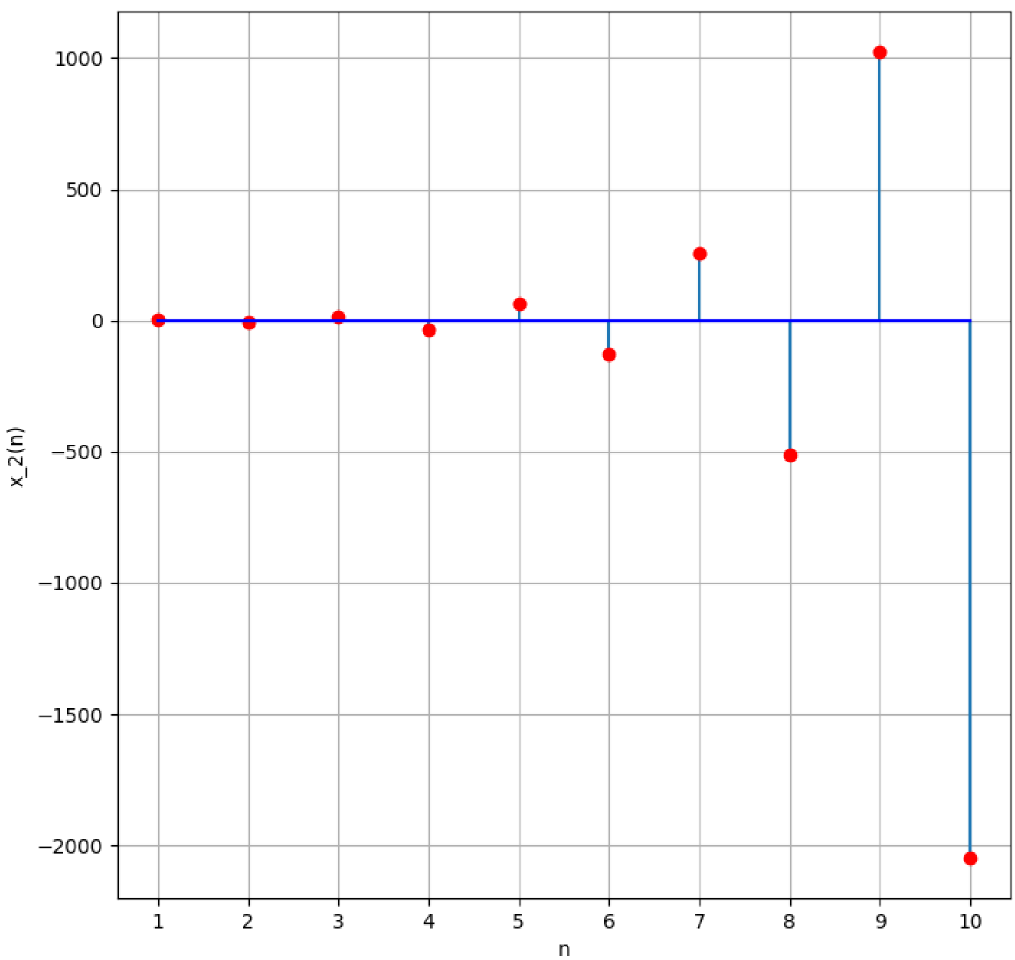


Figure 3.32: Representation of x(n) for $r = -2$

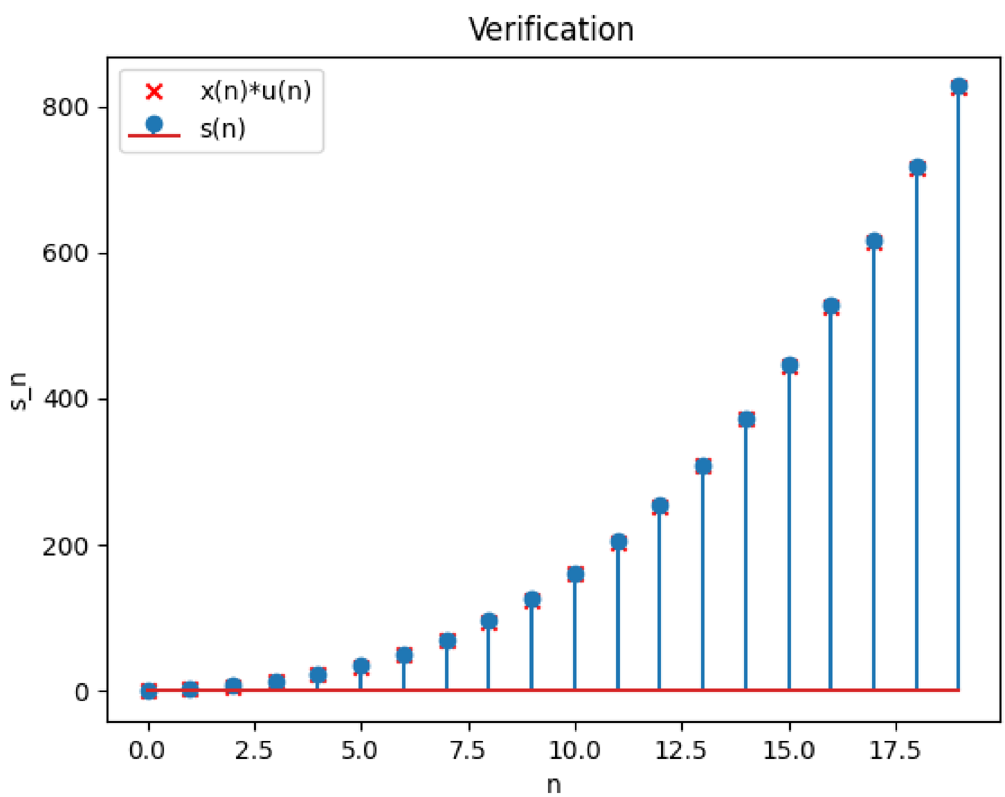


Figure 3.33: Plot of s(n) vs n

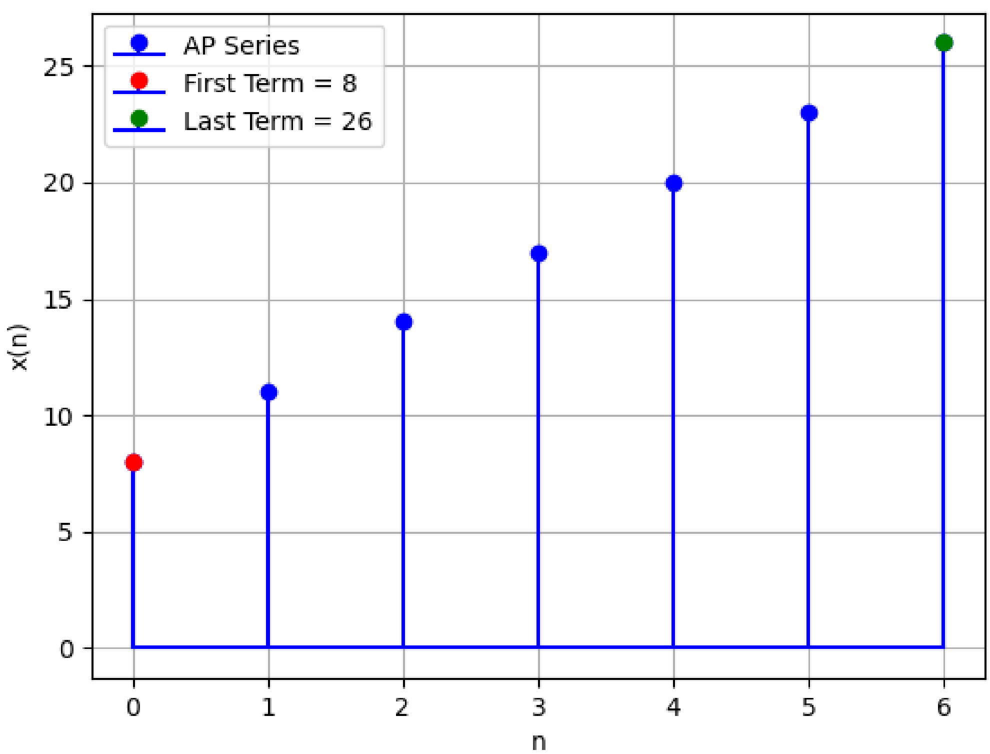


Figure 3.34: Plot of x(n) vs n

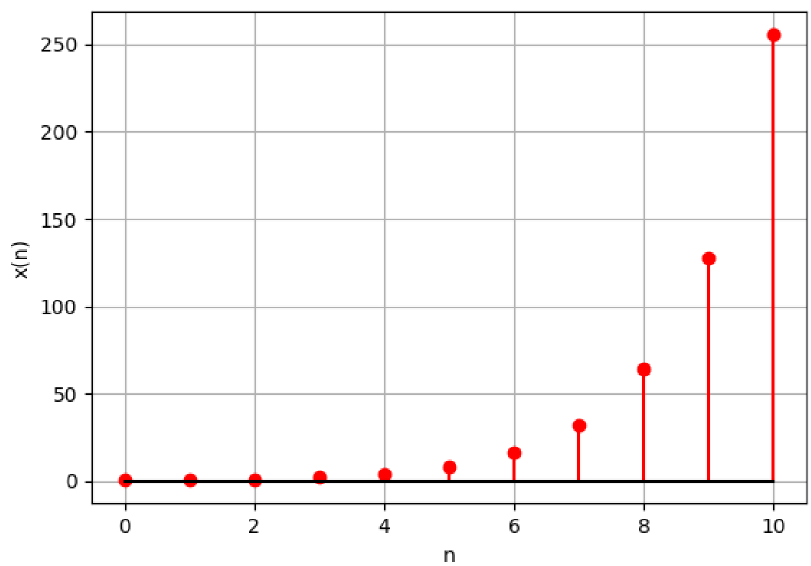


Figure 3.35: Stem Plot of $x(n) = (0.25)2^n u(n)$, $a = 0.25, r = 2$

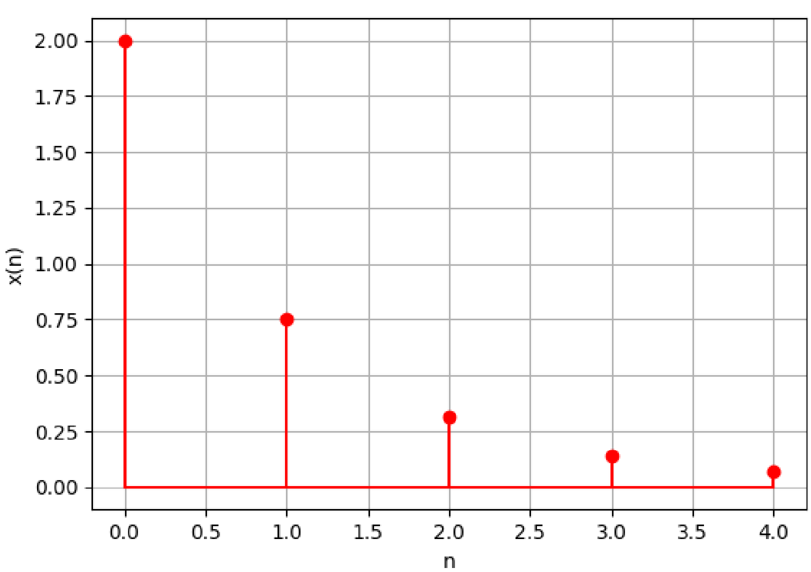


Figure 3.36: Stem Plot of $x(n) = (0.25^n + 0.5^n)\, u(n)$, $a = 0.25, b = 0.5$

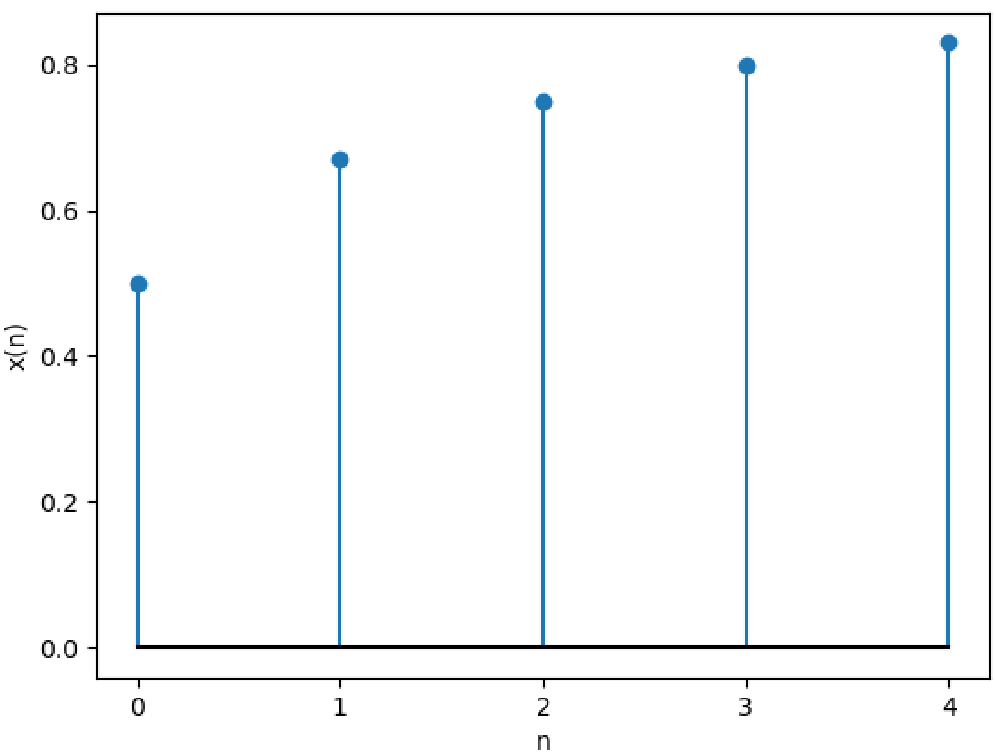


Figure 3.37: Stem plot for x(n)

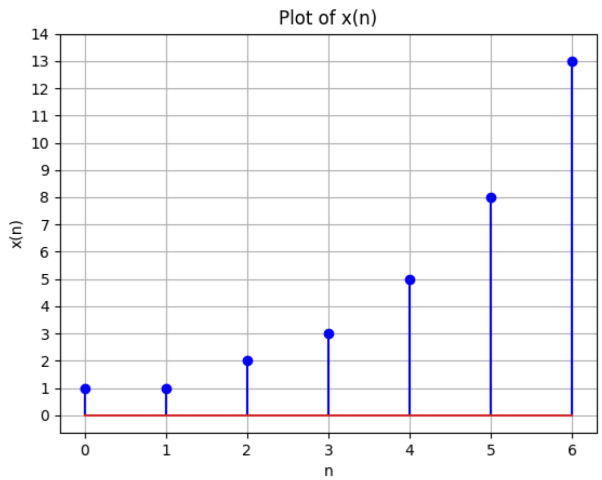


Figure 3.38: *

(a) Plot of $x(n)$ vs n

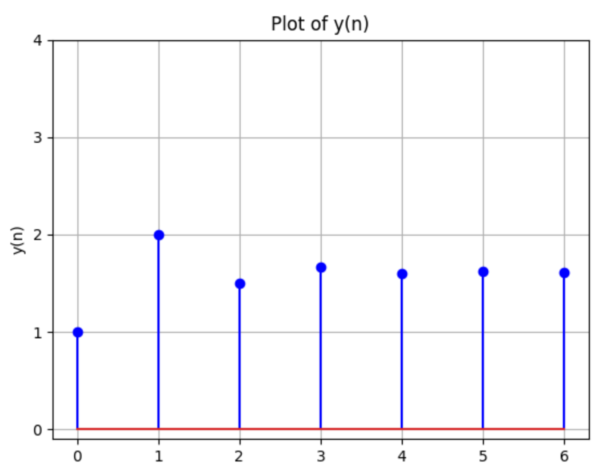


Figure 3.39: *

(b) Plot of $y(n)$ vs n

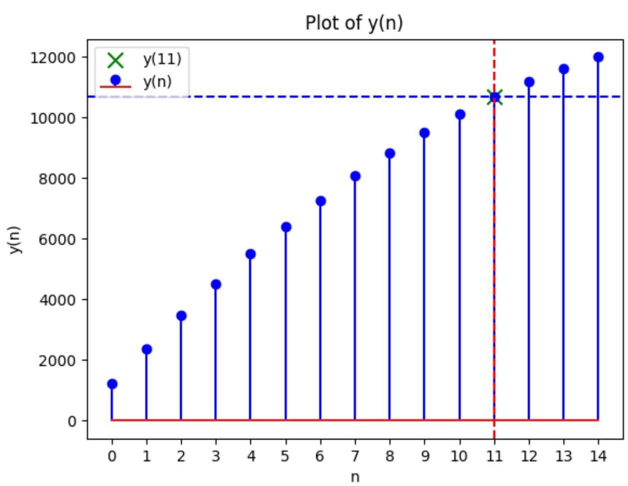


Figure 3.40: (a) Plot of $y\left(n\right)$ vs n

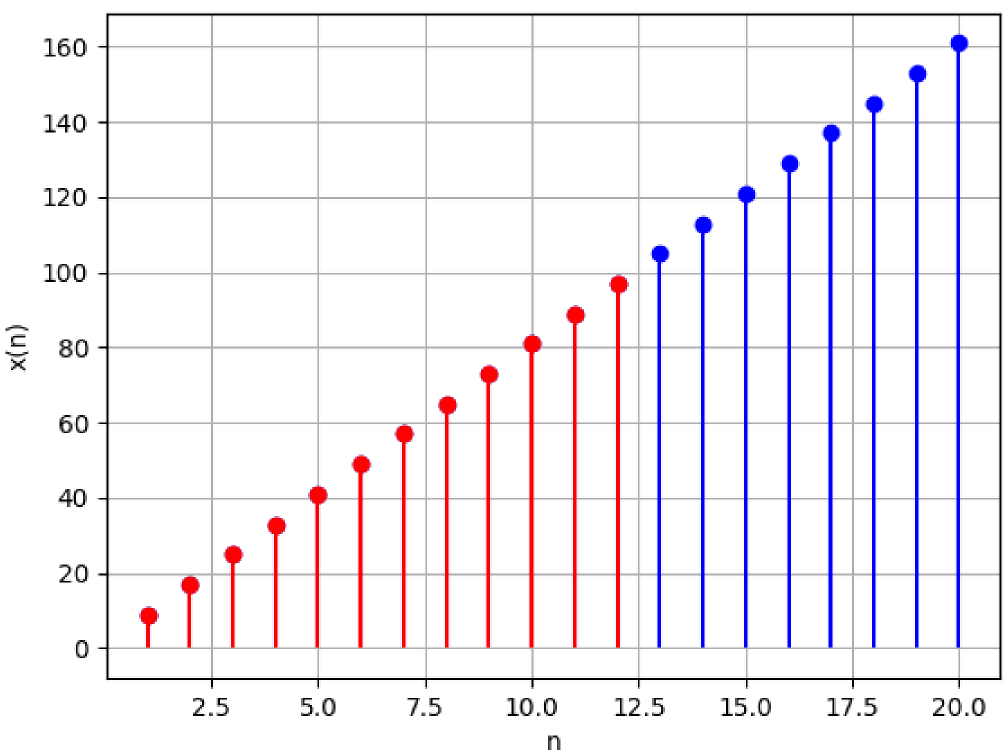


Figure 3.41: Plot of x(n) vs n

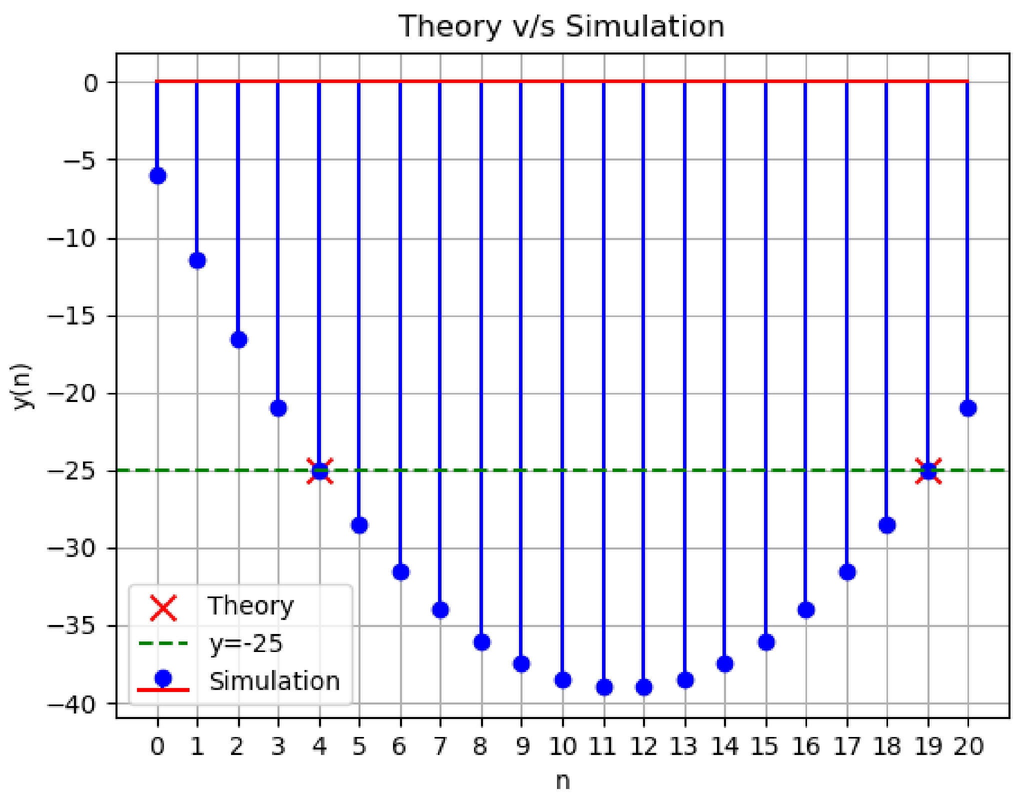


Figure 3.42: Theory matches with simulated values

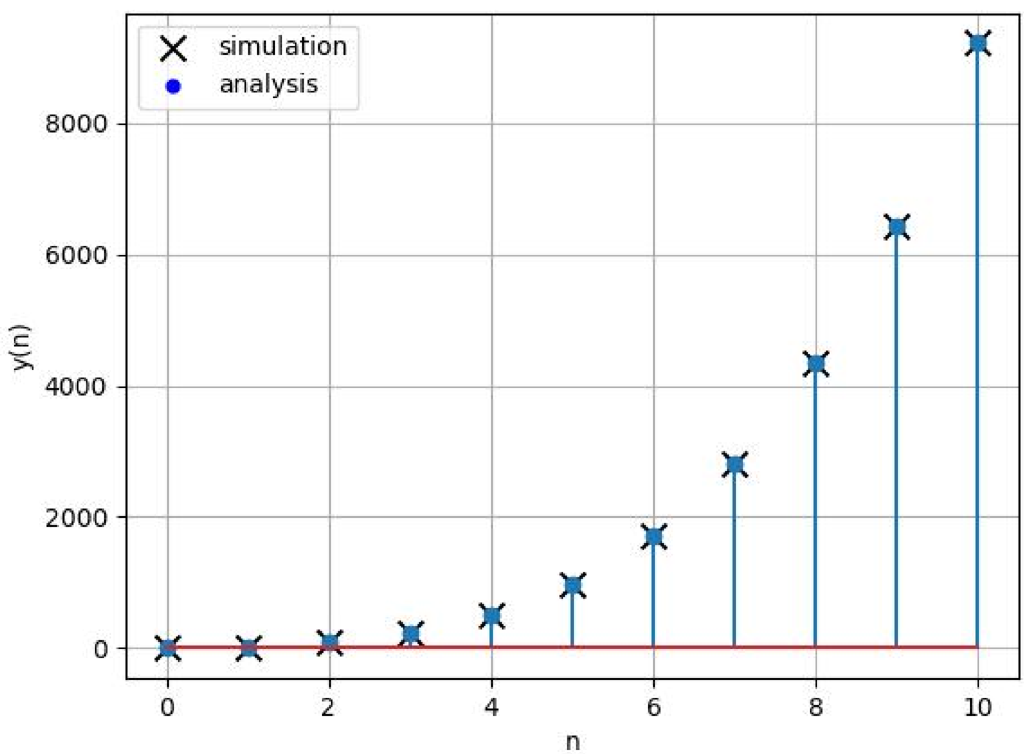


Figure 3.43: simulation vs analysis of y(n)

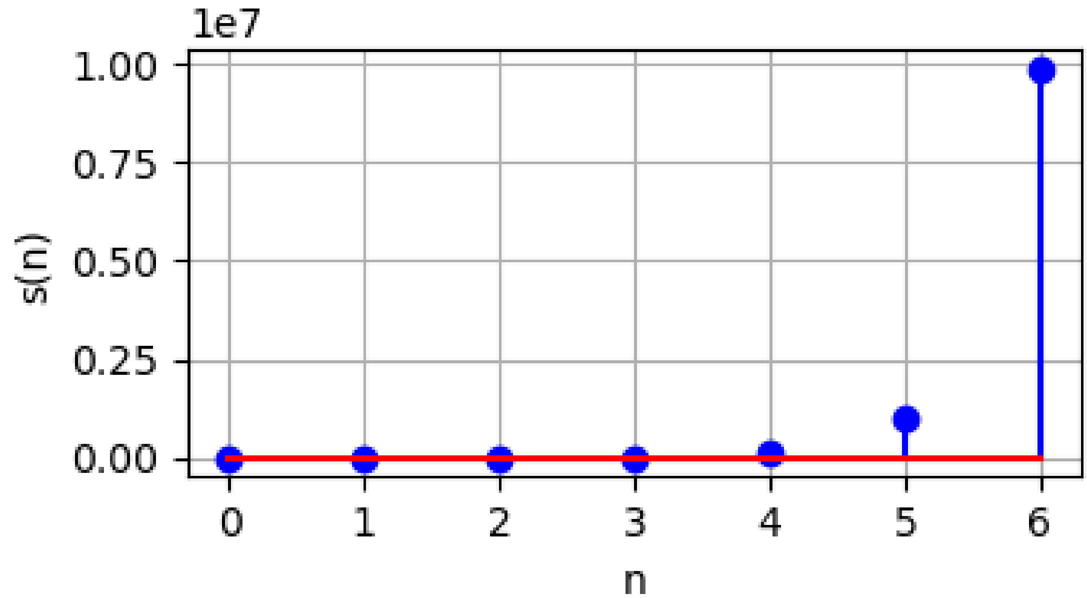


Figure 3.44: graph of sum of n terms

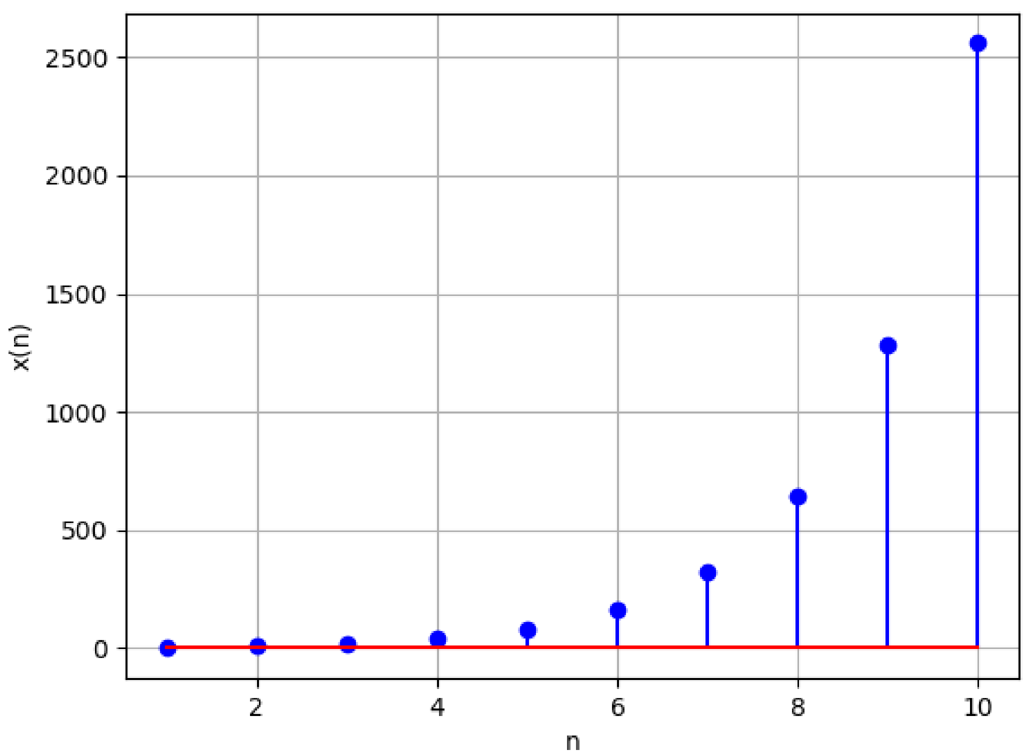


Figure 3.45: (x(0)=5, r=2) Plot of x(n)= (5) (2)^n

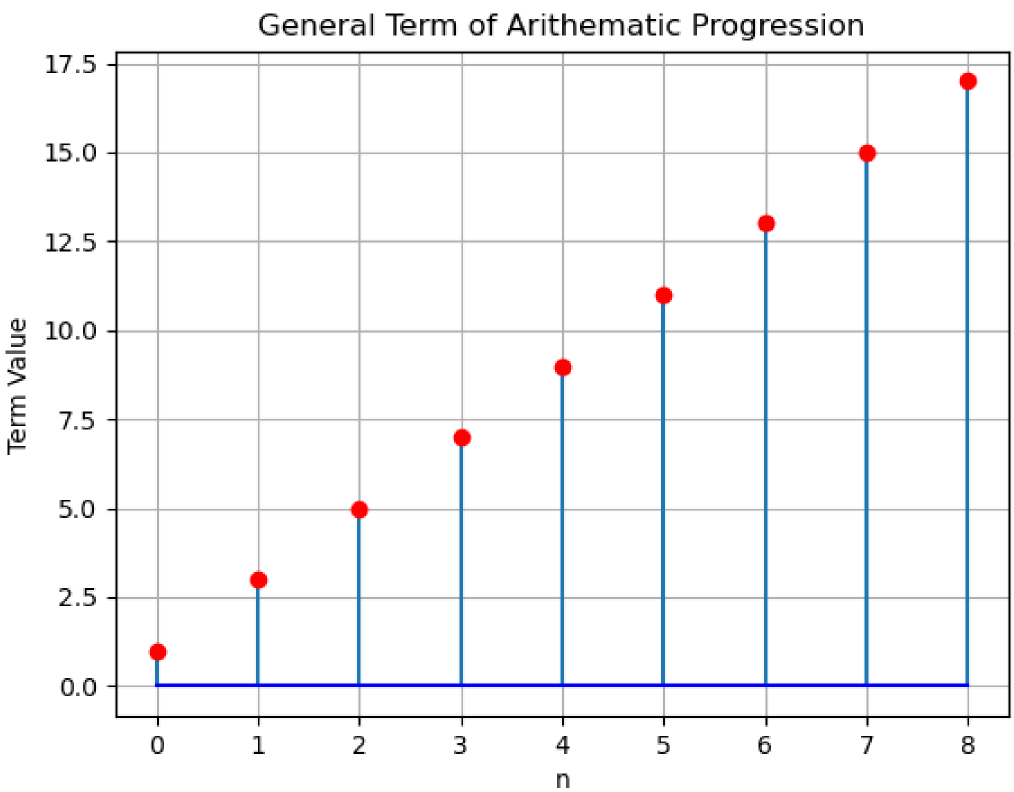


Figure 3.46: Plot of x(n) vs n

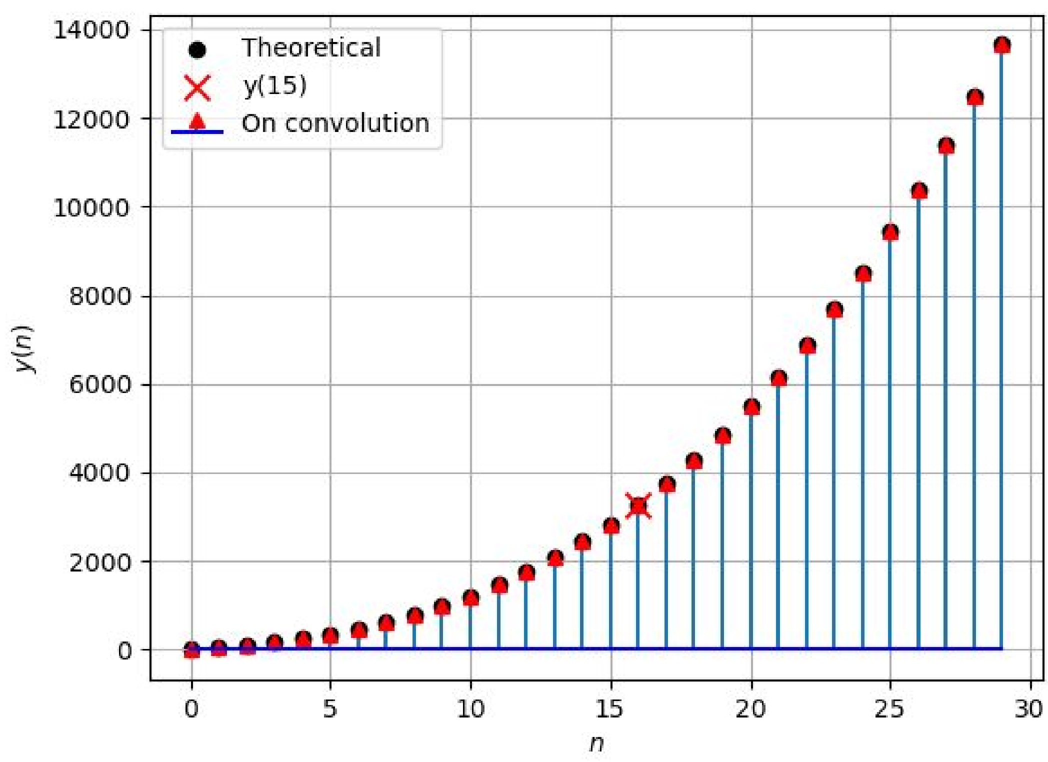


Figure 3.47: Simulation v/s theoretical

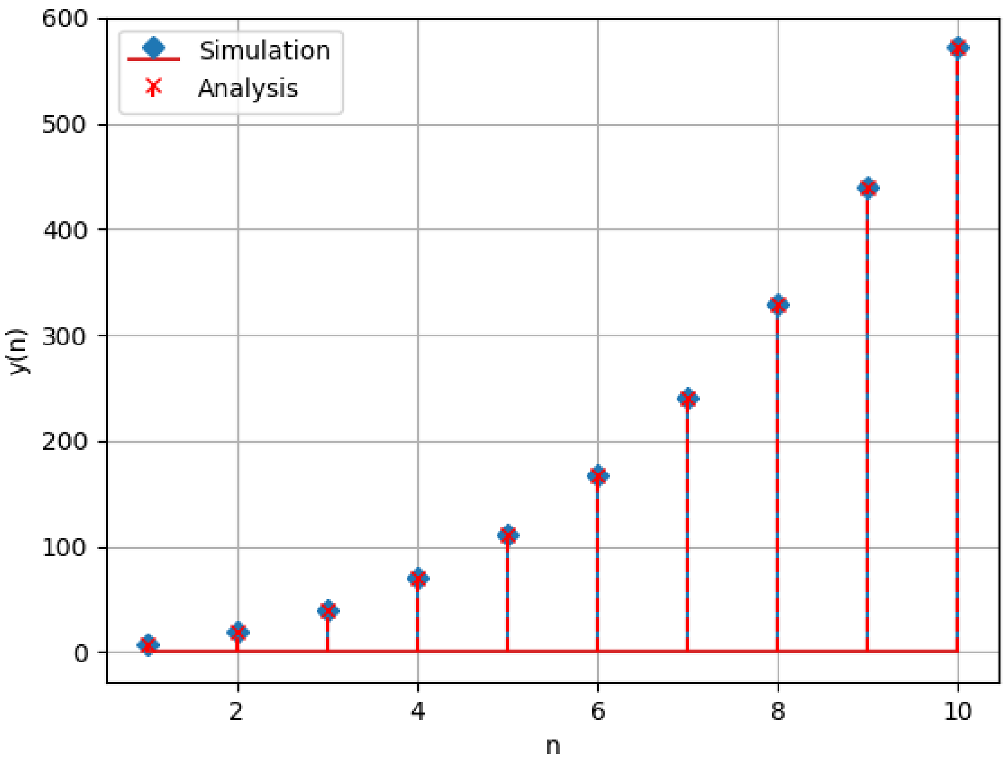


Figure 3.48: Simulation v/s Analysis

Chapter 4

Sequences

4.0.1 Find the number of terms in each of the following APs.

(a) 7, 13, 19, ... 205

(b) 18, $15\frac{1}{2}$, 13, ... -47

Solution: The number of terms in the AP x(n) is given by:

Parameter	**Used to denote**	**Values**
$x_i(n)$	n^{th} term of i^{th} series $(i = (1, 2))$	$(x_i(0) + nd_i)\, u(n)$
$x_i(0)$	First term of i^{th} AP	$x_1(0) = 7$ $x_2(0) = 18$
d_i	Commmon difference of i^{th} AP	$d_1 = 6$ $d_2 = -2.5$

Table 4.1: Parameter Table

$$\frac{x(n) - x(0)}{d} + 1 \tag{4.1}$$

$$X_i(z) = \frac{x_i\left(0\right)}{1-z^{-1}} + d_i \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \text{ , for i=1,2} \tag{4.2}$$

$$\text{ROC} : |z| > 1 \text{ as it is an AP} \tag{4.3}$$

(a)

$$x_1\left(n\right) = \left(7 + \left(n\right)6\right)u\left(n\right) \tag{4.4}$$

Using the values in Table 4.1 and equation (4.1),

$$k_1 = \frac{205-7}{6} + 1 = 34 \tag{4.5}$$

Using the values in Table 4.1 and equation (4.2) :

$$X_1\left(z\right) = \frac{7-z^{-1}}{\left(1-z^{-1}\right)^2} \tag{4.6}$$

ROC is $|z| > 1$

(b)

$$x_2\left(n\right) = \left(18 + n\left(-2.5\right)u\left(n\right)\right) \tag{4.7}$$

Using the values in Table 4.1 and equation (4.1),

$$k_2 = \frac{-47-18}{-2.5} + 1 = 27 \tag{4.8}$$

Using the values in Table 4.1 and equation (4.2) :

$$X_2(z) = \frac{18 - (20.5)\, z^{-1}}{(1 - z^{-1})^2} \tag{4.9}$$

ROC is $|z| > 1$.

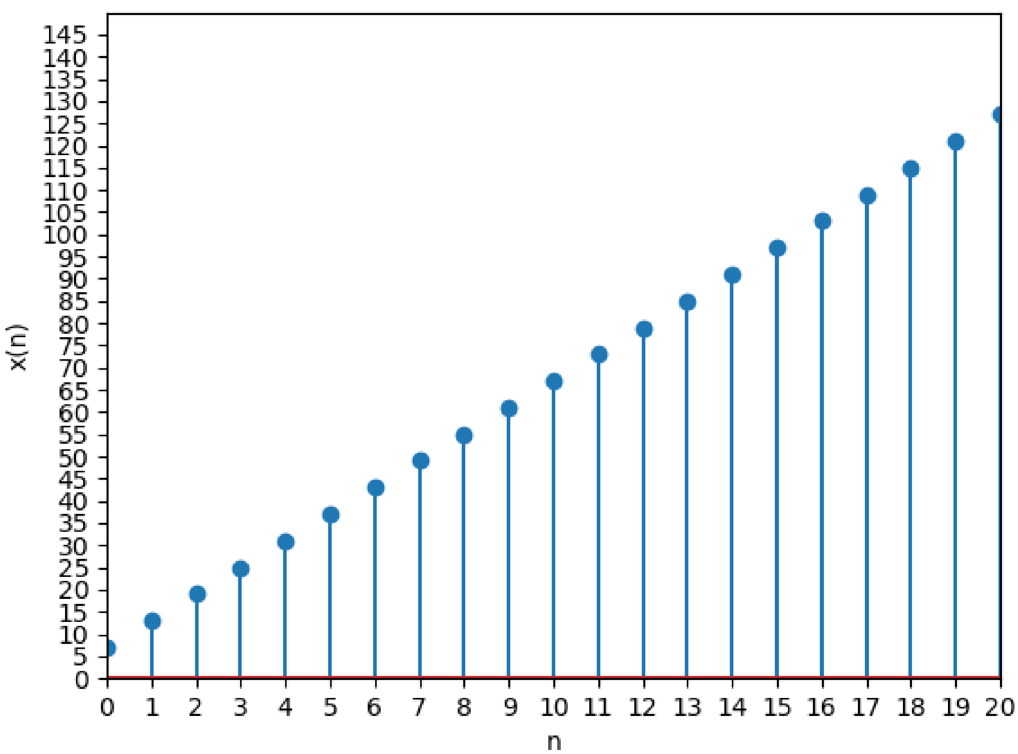


Figure 4.1: Plot of $x_1(n)$

4.0.2 For what value of n, are the nth terms of two A.Ps: 63, 65, 67,. . . and 3, 10, 17,. . . equal?

Solution:

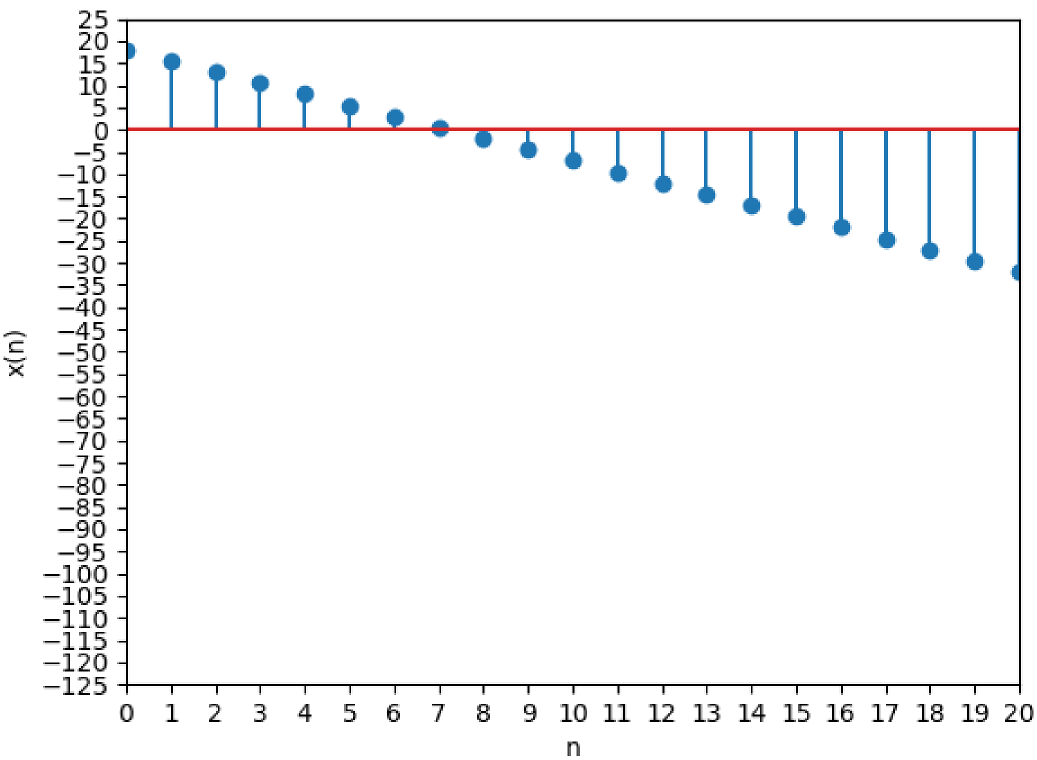


Figure 4.2: Plot of $x_2(n)$

$$x_i(n) = x(0)\,u(n) + dnu(n) \tag{4.10}$$

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{4.11}$$

(a)

$$x_1(n) = 63u(n) + 2nu(n) \tag{4.12}$$

$$X_1(z) = \frac{63}{1 - z^{-1}} + \frac{2z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{4.13}$$

Parameter	Sub-question	Description	Value
$x_i(0)$	$x_1(0)$	1^{st} term of 1^{st} A.P.	63
	$x_2(0)$	1^{st} term of 2^{nd} A.P.	3
d_i	d_1	Common difference of 1^{st} A.P.	2
	d_2	Common difference of 2^{nd} A.P.	7

Table 4.2: input values

(b)

$$x_2(n) = 3u(n) + 7nu(n) \tag{4.14}$$

$$X_2(z) = \frac{3}{1 - z^{-1}} + \frac{7z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{4.15}$$

(c) given,

$$x_1(n) = x_2(n) \tag{4.16}$$

$$\therefore 63 + 2n = 7n + 3 \tag{4.17}$$

$$\implies n = 12 \tag{4.18}$$

4.0.3 Two APs have the same common difference.The difference between their 100th terms is 100,what is the difference between their 1000th terms?

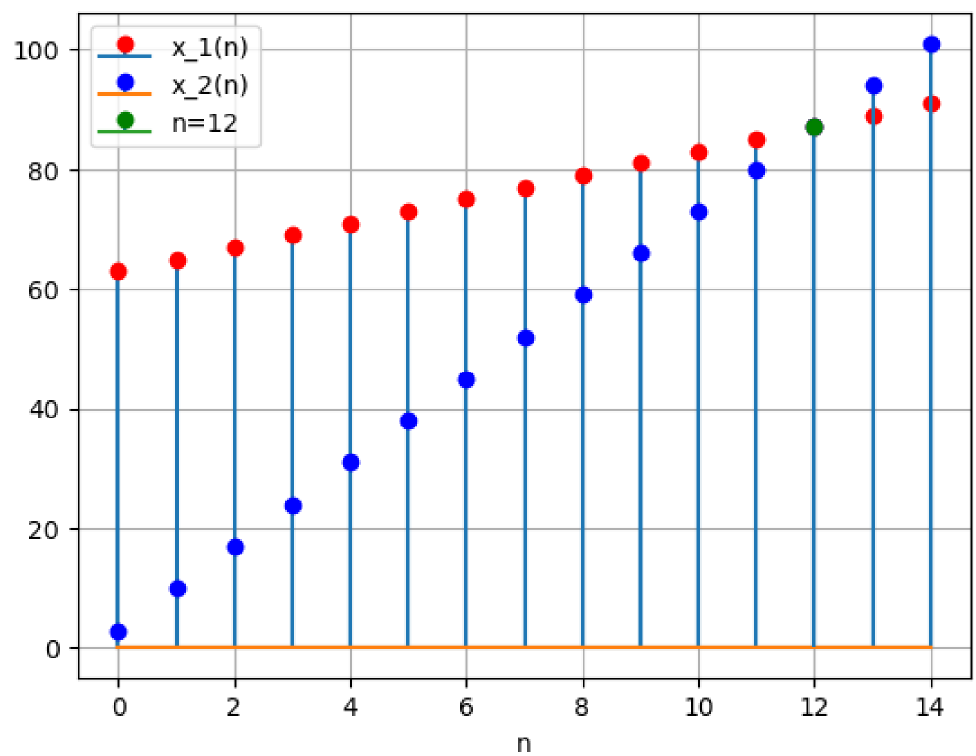


Figure 4.3: Graphs of $x_1\,(n)$ and $x_2\,(n)$ and both are equal at $n = 12$

Solution:

$$x(n) = \{x(0) + nd\}u(n) \tag{4.19}$$

$$x(99) - y(99) = 100 \tag{4.20}$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \tag{4.21}$$

$$\implies x(0) - y(0) = 100 \tag{4.22}$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \tag{4.23}$$

$$= x(0) - y(0) \tag{4.24}$$

$$= 100 \tag{4.25}$$

$$\implies x(999) - y(999) = 100 \tag{4.26}$$

Variable	**Description**	**Value**
$x(n)$	n^{th} term of X	none
$y(n)$	n^{th} term of Y	none
d	common difference between the terms of AP	none
$x(99) - y(99)$	difference of 99^{th} terms of X and Y	100

Table 4.3: input parameters

Let

$$x(n) = \{101, 106, 111, ...\} \tag{4.27}$$

$$y(n) = \{1, 6, 11, ...\} \tag{4.28}$$

4.0.4 Check whether -150 is a term of the AP: 11,8,5,2,....

Solution:

$$x(n) = x(0) + nd \tag{4.29}$$

$$n = \frac{x(n) - x(0)}{d} \tag{4.30}$$

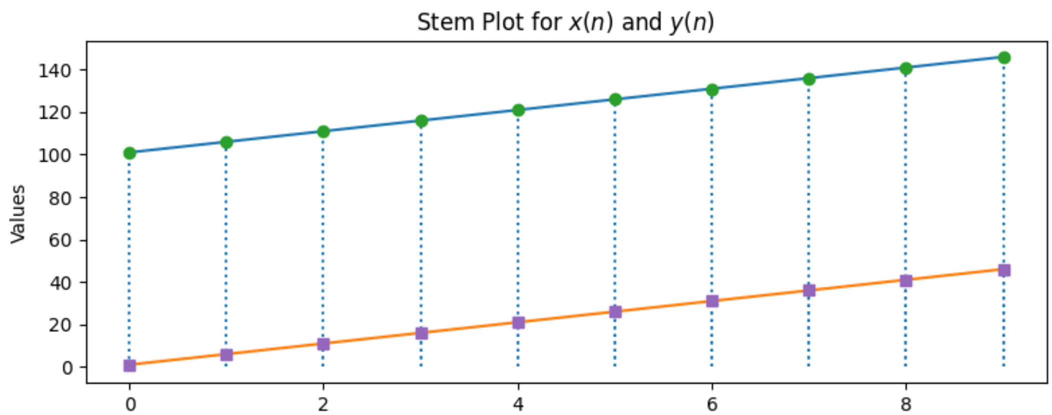


Figure 4.4:

$$x(n) - x(0) \equiv 0 \pmod{d} \tag{4.31}$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \tag{4.32}$$

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \tag{4.33}$$

$$X(z) = \frac{11}{1-z^{-1}} - \frac{3z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{4.34}$$

Variable	Description	Value
$x(0)$	First term of AP	11
d	Common difference	-3
$x(n)$	General term of given AP	None

Table 4.4: Input parameters

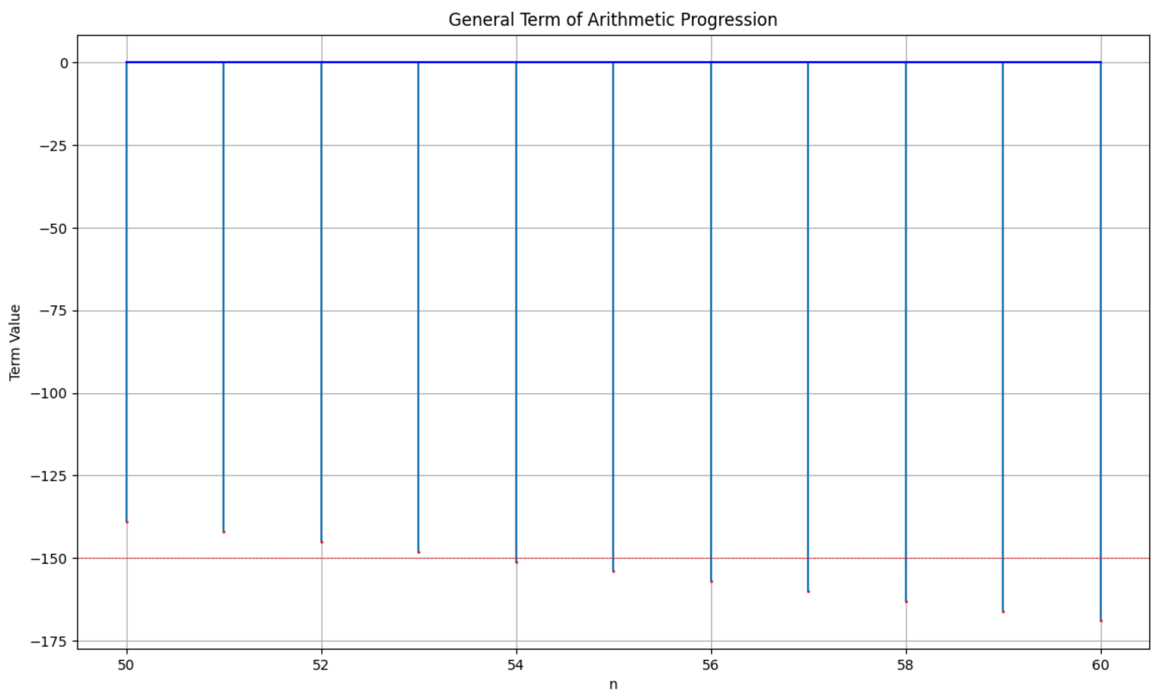


Figure 4.5: Representation of x(n)

4.0.5 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

Solution:

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4}\right) u(n) \tag{4.35}$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{4.36}$$

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{4.37}$$

$$n^2u(n) \overset{z}{\longleftrightarrow} \frac{(z^{-1})(1+z^{-1})}{(1-z^{-1})^3} \quad |z| > 1 \tag{4.38}$$

$$n^3u(n) \overset{z}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \quad |z| > 1 \tag{4.39}$$

Referencing the equations from (4.37), (4.38), and (4.39).

$$x(n) \overset{z}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{4(1-z^{-1})^4} + \frac{3(z^{-1})(1+z^{-1})}{4(1-z^{-1})^3} + \frac{2z^{-1}}{(1-z^{-1})^2} + \frac{3}{2(1-z^{-1})} \quad |z| > 1 \tag{4.40}$$

$$x(n) \overset{z}{\longleftrightarrow} \frac{3}{2(1-z^{-1})^3} + \frac{3z^{-2}}{2(1-z^{-1})^4} \quad |z| > 1 \tag{4.41}$$

4.0.6 (a) 30th term of the AP: 10, 7, 4, . . . is

(b) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

Solution:

Parameter	**value**	**Description**
$x_i(0)$	10	First term
	-3	
d_i	-3	Common difference
	$\frac{5}{2}$	
$x_1(29)$	?	30th term
$x_2(10)$	?	11th term

Table 4.5: Input Parameters

$$x_i(n) = [x_i(0) + nd_i]\, u(n) \tag{4.42}$$

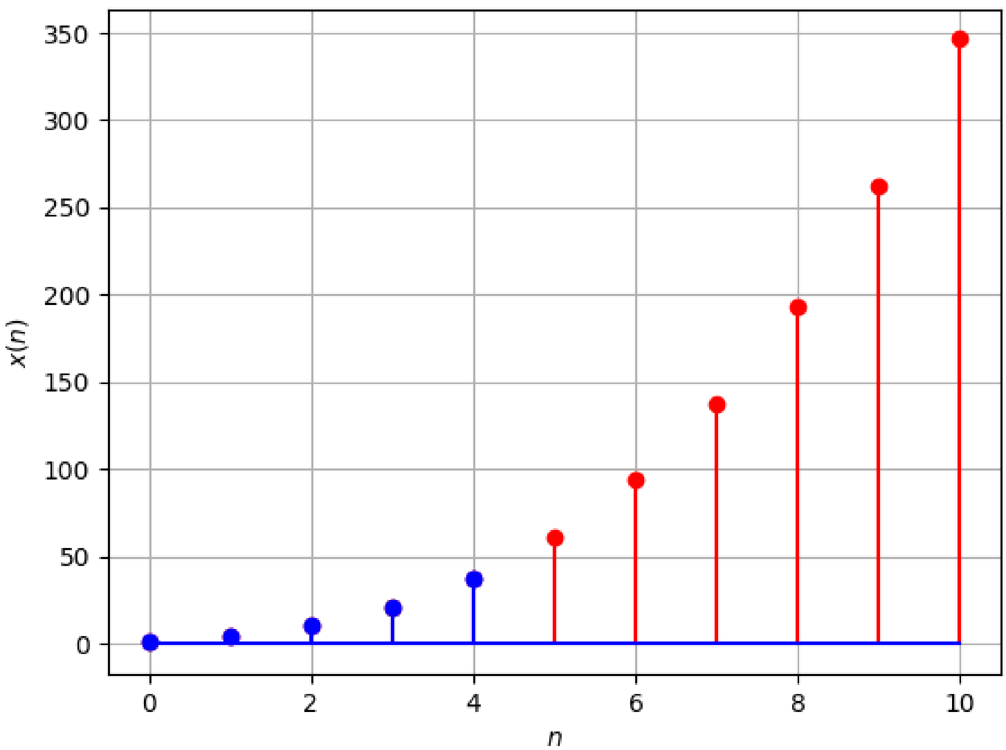


Figure 4.6: Plot of equation(4.35)

(a) From (4.42) Table 4.5 :

$$x_1(n) = [10 - 3n]\, u(n) \tag{4.43}$$

$$x_1(29) = -77 \tag{4.44}$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.45}$$

(b) From (4.42) and Table 4.5 :

$$x_2(n) = \left[-3 + \frac{5}{2}n\right] u(n) \tag{4.46}$$

$$x_2(10) = 22 \tag{4.47}$$

$$X_2(z) = \frac{5.5z^{-1} - 3}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.48}$$

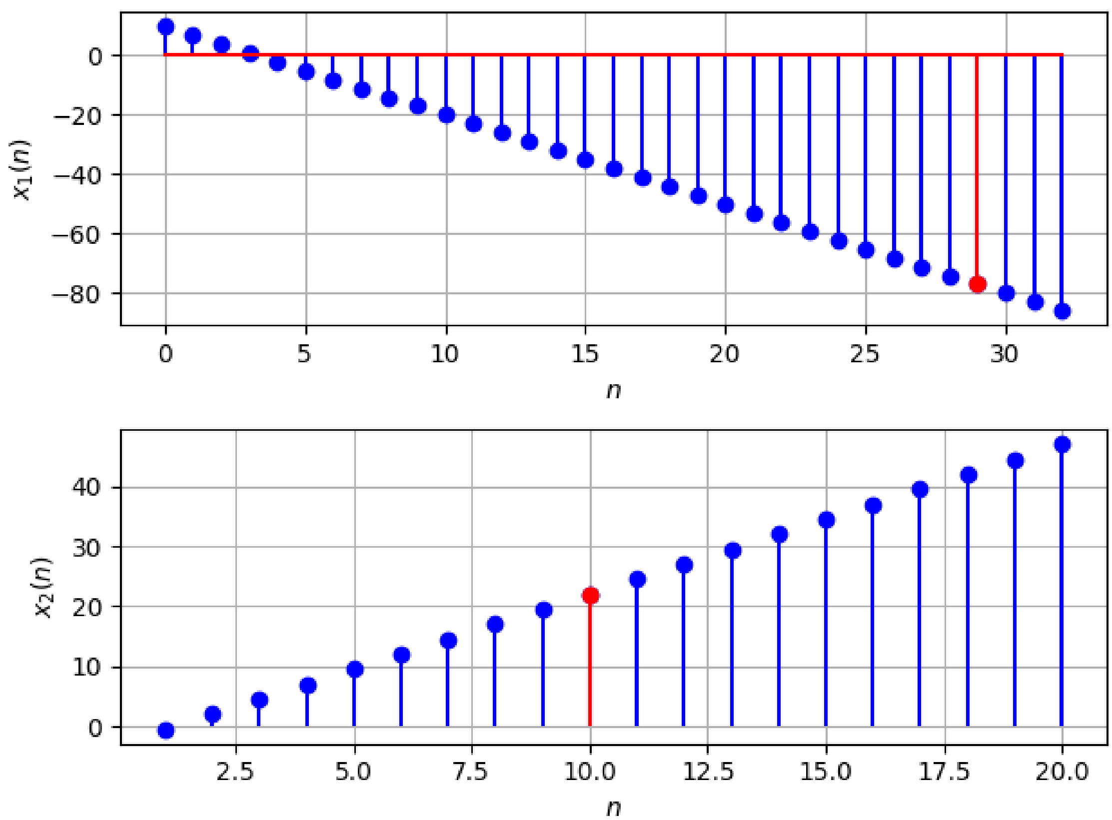


Figure 4.7: stem plots of $x_1(n)$ and $x_2(n)$

4.0.7 Write the first five terms of the sequence whose nth term is $\frac{2n-3}{6}$ and obtain the Z transform of the series **Solution:**

$$x(n) = \frac{2n-1}{6}(u(n)) \tag{4.49}$$

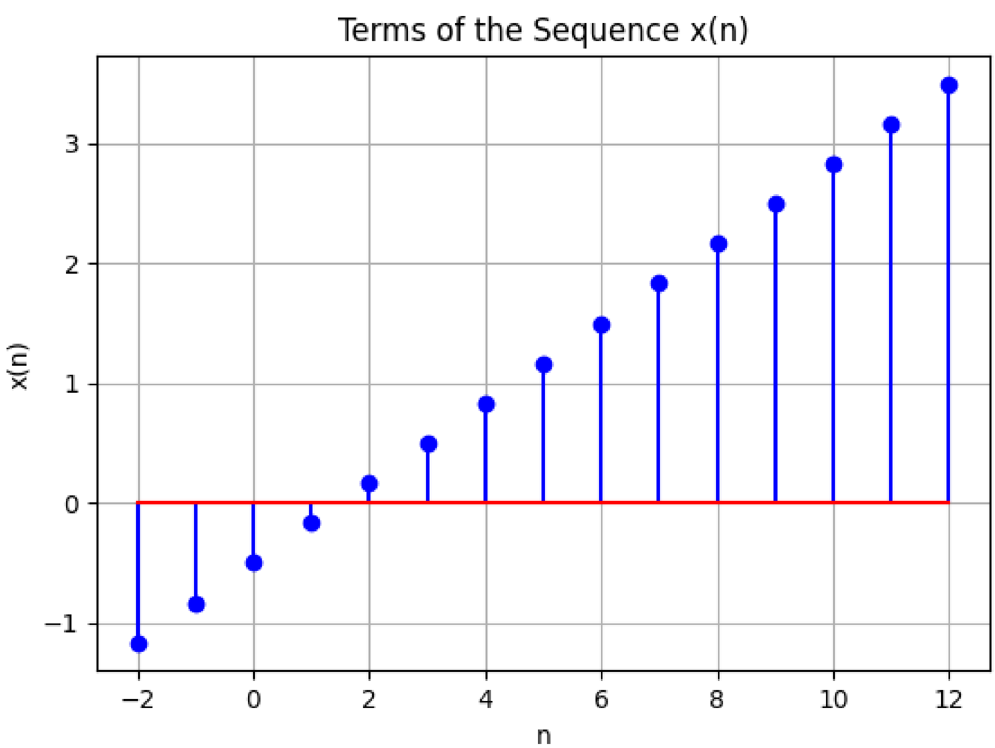


Figure 4.8: Plot of x(n) vs n

$$X(z) = \frac{3z^{-1}-1}{6(1-z^{-1})^2} \quad |z| > 1 \tag{4.50}$$

4.0.8 For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P ?

Solution: Let r be the common ratio

Variable	**Description**	**Value**
$x(0)$	First term of the GP	$-\left(\frac{2}{7}\right)$
$x(1)$	Second term of the GP	x
$x(2)$	Third term of the GP	$-\left(\frac{7}{2}\right)$
r	Common ratio of the GP	
$x(n)$	General term	$x(0)\, r^n\, u(n)$

Table 4.6: Variables Used

From Table 4.6:

$$\implies \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \tag{4.51}$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \tag{4.52}$$

$$x = \pm 1 \tag{4.53}$$

$$\implies r = \pm\frac{7}{2} \tag{4.54}$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right)\left(\pm\frac{7}{2}\right)^n u(n) \tag{4.55}$$

Applying z-Transform :

$$\implies X_1(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}+2}\right) \quad |z| > \frac{7}{2} \tag{4.56}$$

$$\implies X_2(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}-2}\right) \quad |z| > \frac{7}{2} \tag{4.57}$$

4.0.9 Find the 20^{th} and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

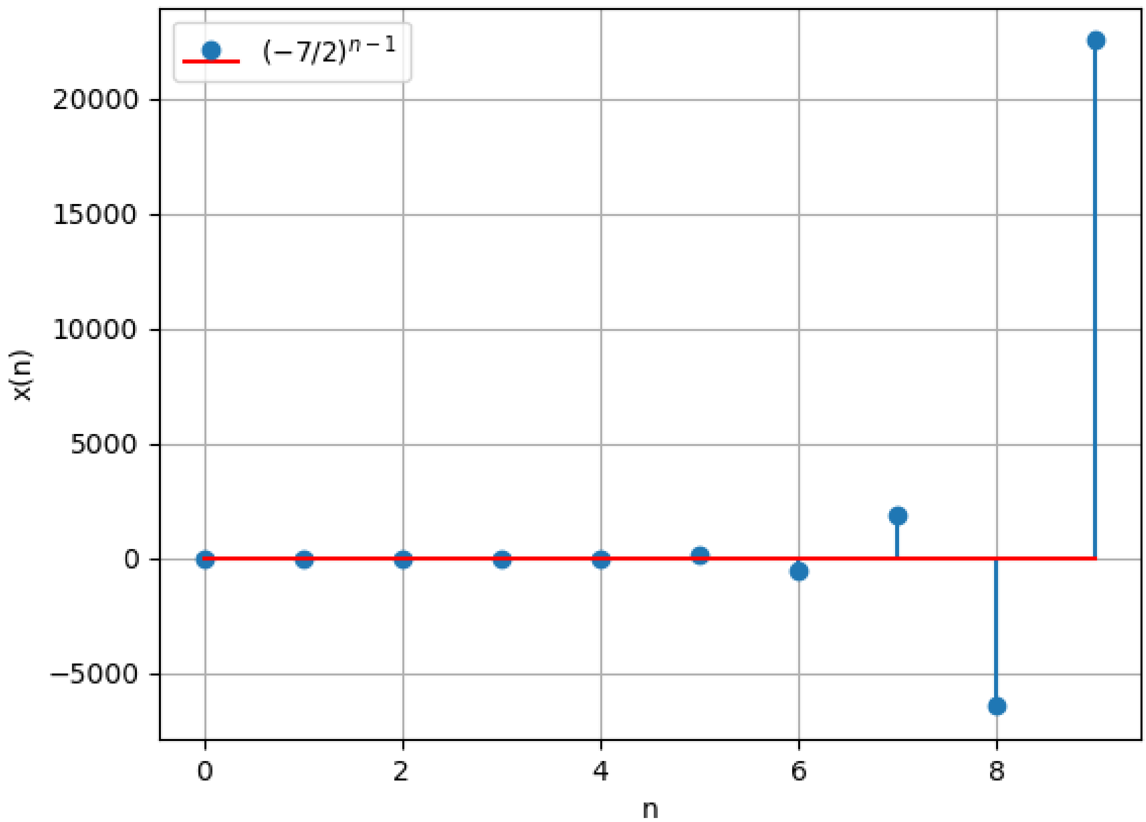


Figure 4.9: Stem Plot of x_1(n)

Solution:

From Table 4.7: Z-Transform of $x(n)$:

$$\implies X(z) = \frac{5}{2}\left(\frac{1}{1-\frac{z^{-1}}{2}}\right); \left\{z \in \mathbb{C} : |z| > \frac{1}{2}\right\} \tag{4.58}$$

4.0.10 Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4 \ldots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3} \ldots$ is 729

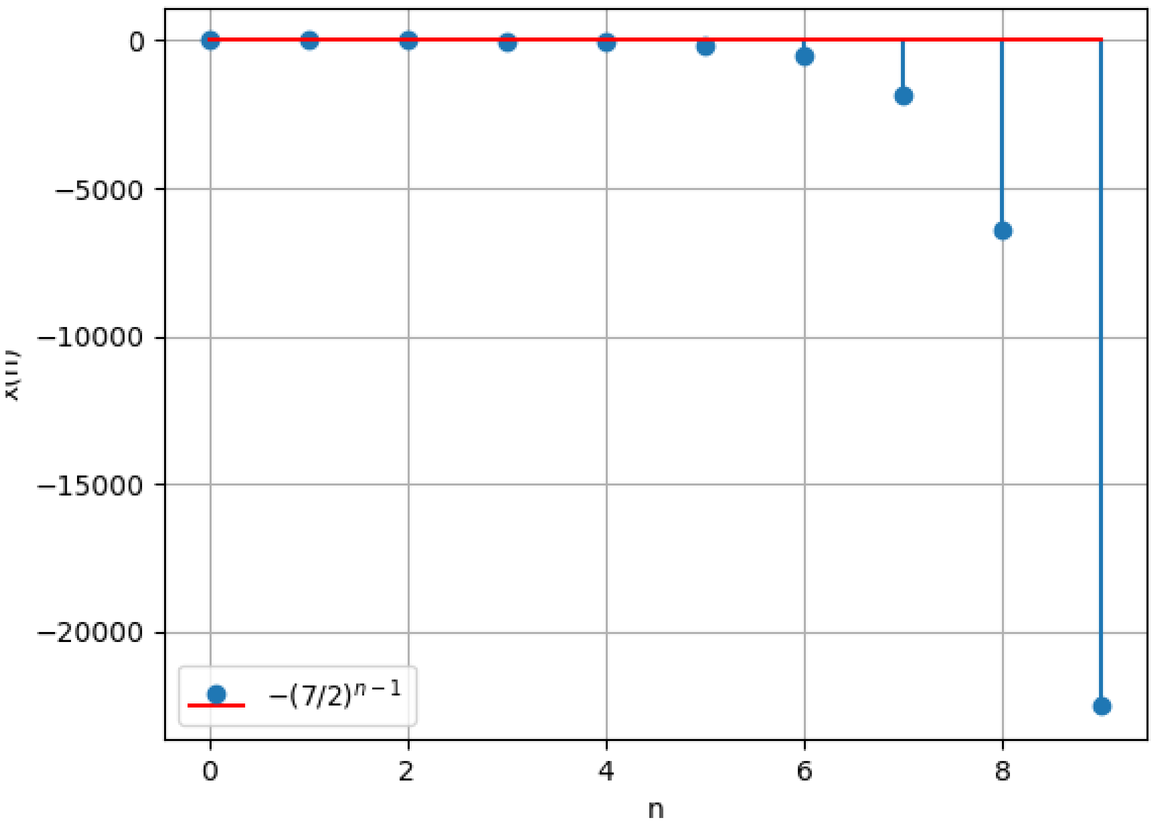


Figure 4.10: Stem Plot of $x_2(n)$

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \dots$ is $\frac{1}{19683}$

Solution: For a general GP series and $k > 0$,

$$x(k) = x(0)\, r^k \tag{4.59}$$

$$\therefore k = \log_r \frac{x(k)}{x(0)} \tag{4.60}$$

Parameter	**Description**	**Value**
$x(0)$	First Term	$\frac{5}{2}$
$r = \frac{x(1)}{x(0)}$	Common Ratio	$\frac{1}{2}$
$x(n)$	n^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{n} \cdot u(n)$
$x(19)$	20^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{19}$
$u(n)$	Unit step function	

Table 4.7: Parameters

And the Z-transform $X(z)$:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \tag{4.61}$$

(a) By Table 4.8, (4.60) and Table 4.8:

$$x_1(n) = x_1(0)\, r_1^n u(n) \tag{4.62}$$

$$k_1 = \log_{r_1} \frac{128}{x_1(0)} \tag{4.63}$$

$$\therefore k_1 = 12 \tag{4.64}$$

$$X_1(z) = \frac{2}{1 - \sqrt{2}z^{-1}} \quad |z| > \sqrt{2} \tag{4.65}$$

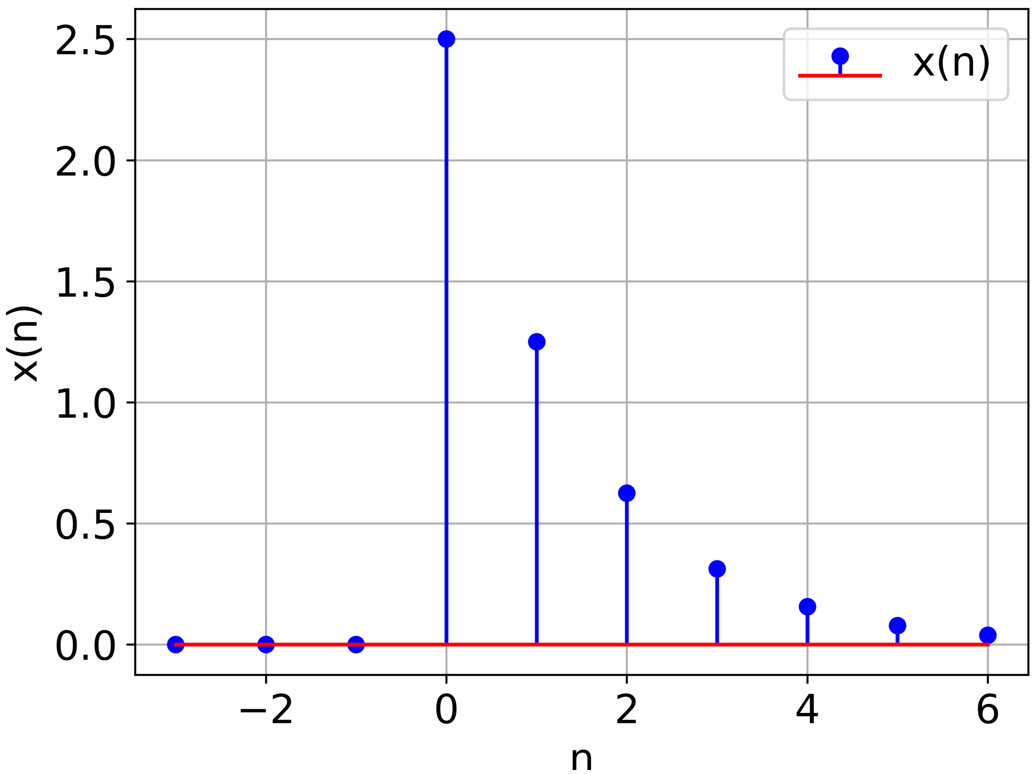


Figure 4.11:

(b) By (4.60), (4.61) and Table 4.8:

$$x_2(n) = x_2(0)\, r_2^n u(n) \tag{4.66}$$

$$k_2 = \log_{r_2} \frac{729}{x_2(0)} \tag{4.67}$$

$$\therefore k_2 = 11 \tag{4.68}$$

$$X_2(z) = \frac{\sqrt{3}}{1 - \sqrt{3} z^{-1}} \quad |z| > \sqrt{3} \tag{4.69}$$

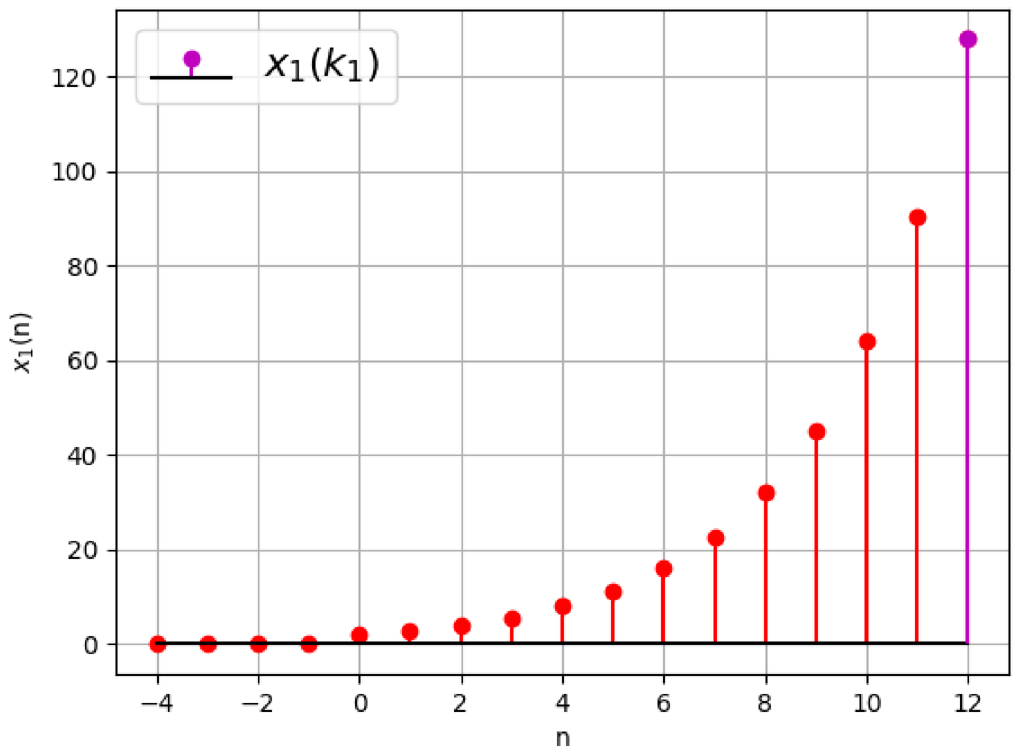


Figure 1: Plot of $x_1(n)$ vs n. See Table 4.8

(c) By (4.60), (4.61) and Table 4.8:

$$x_3(n) = x_3(0)\, r_3^n u(n) \tag{4.70}$$

$$k_3 = \log_{r_3} \frac{1}{19683 x_3(0)} \tag{4.71}$$

$$\therefore k_3 = 8 \tag{4.72}$$

$$X_3(z) = \frac{1}{3 - z^{-1}} \quad |z| > \frac{1}{3} \tag{4.73}$$

Find the 20^{th} and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

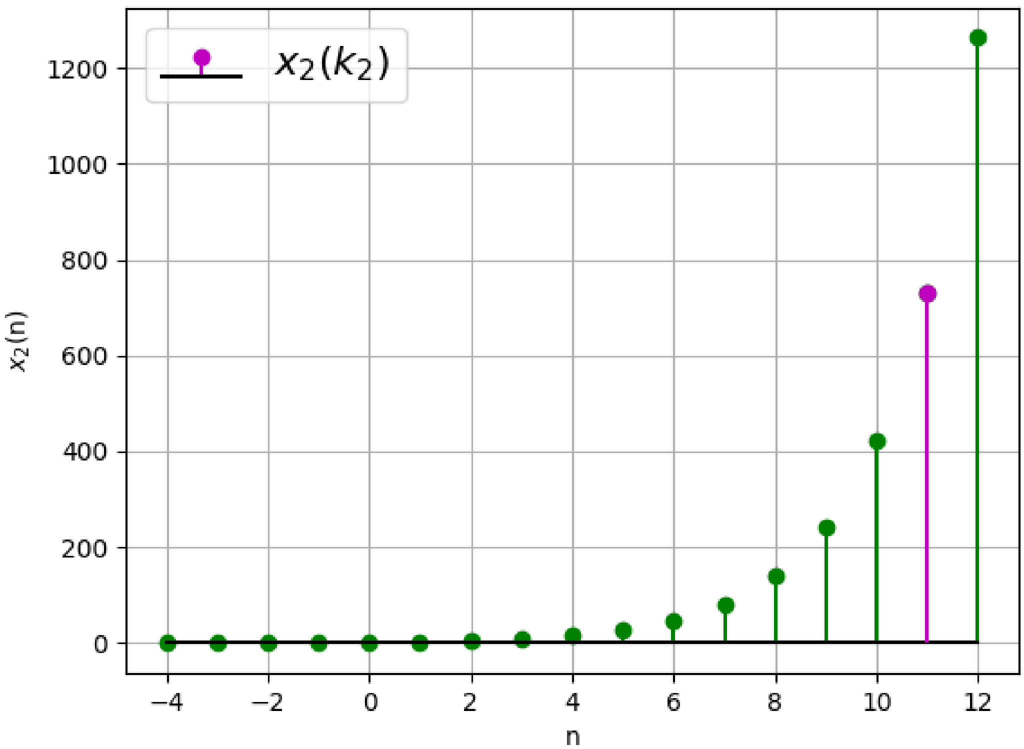


Figure 2: Plot of $x_2(n)$ vs n. See Table 4.8

Parameter	Description	Value
r_i	Common ratio of G.P (a),(b),(c)	$\sqrt{2}, \sqrt{3}, \frac{1}{3}$
$x_i(0)$	Initial Values	$2, \sqrt{3}, \frac{1}{3}$
$x_i(k_i)$	Given Values	$128, 729, \frac{1}{19683}$
k_i	Desired index	$12, 11, 8$
$x_i\left(n\right)$	Series	$x_i\left(0\right) r_i^n u\left(n\right)$
$X_i\left(z\right)$	Z-Transform of $x_i\left(n\right)$	$\frac{x(0)}{1-rz^{-1}}$

Table 4.8: Table of parameters

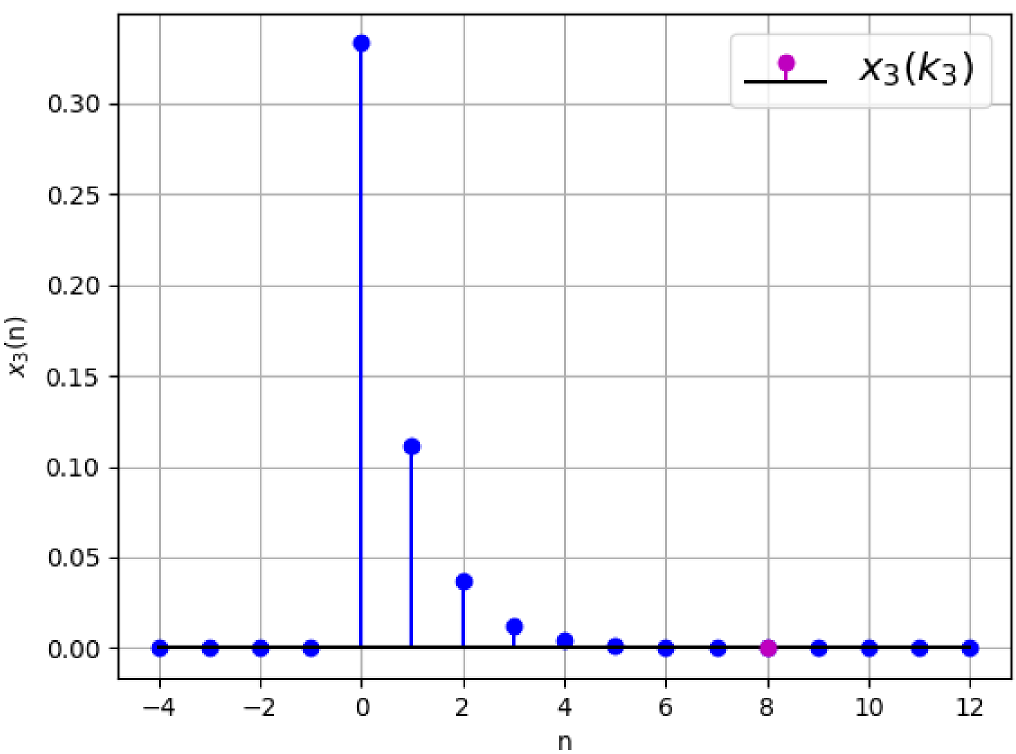


Figure 3: Plot of $x_3(n)$ vs n. See Table 4.8

4.0.11 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour?

Solution: From Table 4.9:

Parameter	Value	Description
$x(0)$	30	Initial no. of bacteria
r	2	Ratio of no. of bacteria at end of hour to start of hour (Common Ratio)
$x(n)$	$r^n x(0)u(n)$	n^{th} term of the GP

Table 4.9: Input Parameters

$$x(2) = 120 \tag{4.74}$$

$$x(4) = 480 \tag{4.75}$$

$$x(n) = 30(2^n)u(n) \tag{4.76}$$

$$X(z) = \frac{30z^{-1}}{1 - 2z^{-1}} \quad |z| > 2 \tag{4.77}$$

4.0.12 Ramkali saved Rs 5 in the first week of a year and then increased her weekly savings by Rs 1.75. If in the nth week, her weekly savings become Rs 20.75, find n.

Solution:

Parameter	Value	Description
$x(0)$	5	First term of AP
d	1.75	Common difference of AP
$x(n)$	20.75	n^{th} term of AP

Table 4.10: Parameter List

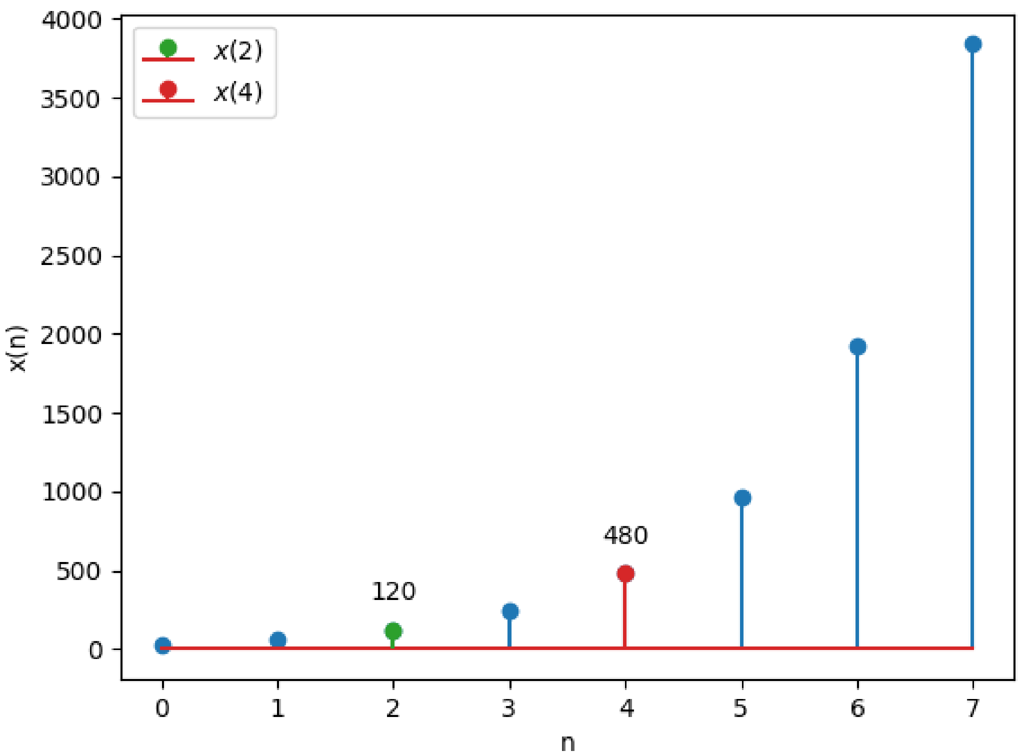


Figure 4.15: Plot of $x(n)$ vs n. See Table 4.9 for details.

$$x(n) = x(0) + (n)(d) \tag{4.78}$$

$$20.75 = 5 + (n)(1.75) \tag{4.79}$$

$$\implies 15.75 = (n)(1.75) \tag{4.80}$$

$$\implies n = \frac{15.75}{1.75} \tag{4.81}$$

$$\implies n = 9 \tag{4.82}$$

$$x(n) = 5u(n) + 1.75nu(n) \tag{4.83}$$

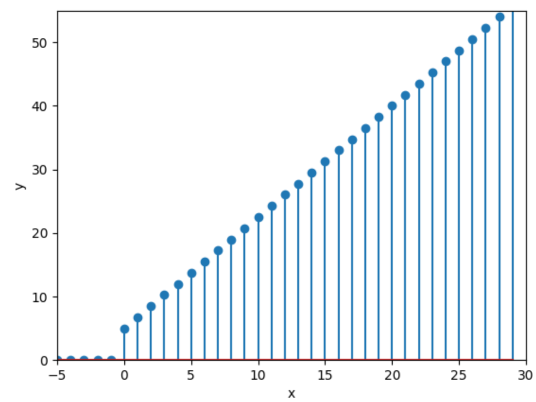


Figure 4.16: Plot of $x(n) = 5 + 1.75n$

The Z-transform of a sequence $x(n)$ is given by:

$$X(z) = \frac{5z^{-1}}{1 - z^{-1}} + \frac{1.75z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{4.84}$$

4.0.13 Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an $A.P.$, is equal to twice the m^{th} terms.

Solution:

For an AP,

$$x(n) = [x(0) + nd]u(n) \tag{4.85}$$

$$\implies x(m+n) + x(m-n) = [x(0) + (m+n)d] + [x(0) + (m-n)d] \tag{4.86}$$

$$= 2[x(0) + md] \tag{4.87}$$

$$\therefore x(m+n) + x(m-n) = 2x(m) \tag{4.88}$$

PARAMETER	VALUE	DESCRIPTION
$x\left(0\right)$	$x\left(0\right)$	First term
d	d	common difference
$x(n)$	$\left[x\left(0\right)+nd\right]u\left(n\right)$	General term of the series

Table 4.11: Parameter Table1

4.0.14 The sum of the first three terms of a G.P is 39/10 and their product is 1. Find the common ratio and the terms.

Solution:

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{4.89}$$

From Table 4.13 and (4.89) :

$$y(2) = x(0)\left(\frac{r^3-1}{r-1}\right) \tag{4.90}$$

$$\frac{39}{10} = x(0)\left(r^2+r+1\right) \tag{4.91}$$

$$\implies \frac{39r}{10} = r^2+r+1 \quad (\because x(0)r = 1) \tag{4.92}$$

$$\implies (2r-5)(5r-2) = 0 \tag{4.93}$$

$$\implies r = \frac{2}{5} \quad or \quad \frac{5}{2} \tag{4.94}$$

(a) If $r = \frac{2}{5}$, then terms are $\frac{5}{2}$, 1, $\frac{2}{5}$.

(b) If $r = \frac{5}{2}$, then terms are $\frac{2}{5}$, 1, $\frac{5}{2}$.

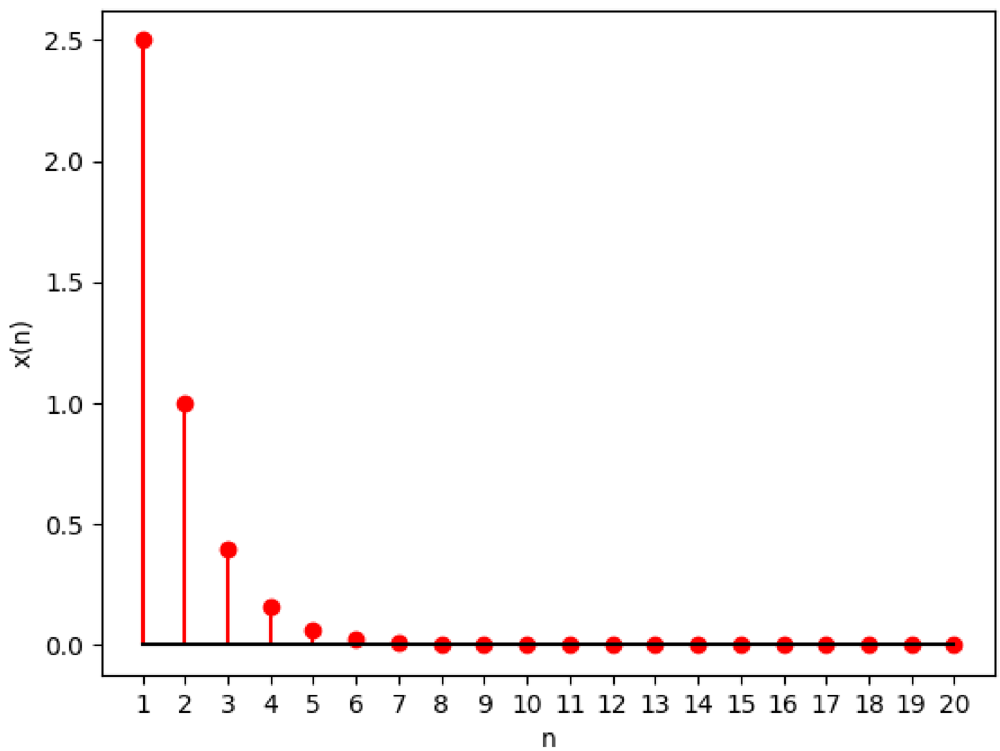


Figure 4.17: stem plots of GP if $r = \frac{2}{5}$

4.0.15 The ratio of the A.M and G.M of two positive numbers a and b is $m : n$. Show that $a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$.

Solution: Expressing A.M and G.M in terms of a and b:

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n} \tag{4.95}$$

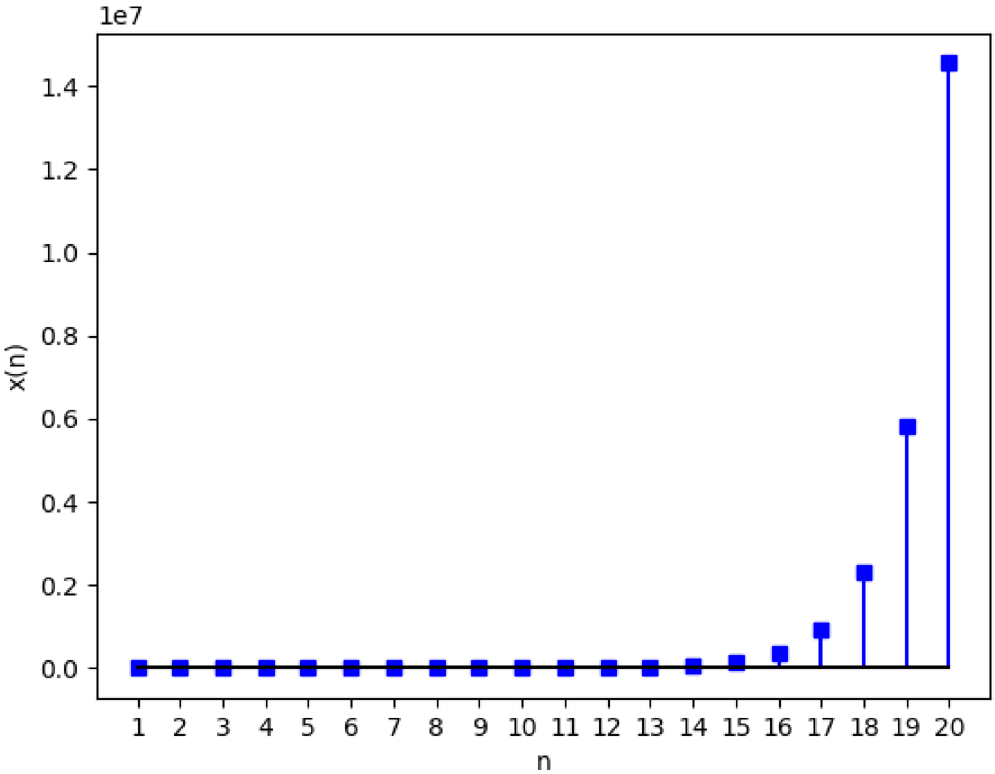


Figure 4.18: stem plots of GP if $r = \frac{5}{2}$

Let's assume that $x = \sqrt{\frac{a}{b}}$. Then, we have:

$$\frac{a}{b} = x^2 \tag{4.96}$$

Substituting the value of x in equation (4.95):

$$\frac{1+x^2}{2x} = \frac{m}{n} \tag{4.97}$$

$$\frac{1}{x} + x = \frac{2m}{n} \tag{4.98}$$

$$x^2 - \frac{2m}{n}x + 1 = 0 \tag{4.99}$$

$$\implies x = \frac{m}{n} \pm \frac{\sqrt{m^2 - n^2}}{n} \tag{4.100}$$

Since $x = \sqrt{\frac{a}{b}}$, x must be positive.

$$x = \frac{m + \sqrt{m^2 - n^2}}{n} \tag{4.101}$$

Referencing the value of x from equation(4.96).

$$\frac{a}{b} = \left(\frac{m + \sqrt{m^2 - n^2}}{n}\right)^2 \tag{4.102}$$

Multiplying both the numerator and denominator with $\left(m - \sqrt{m^2 - n^2}\right)$:

$$\frac{a}{b} = \frac{1}{n^2}\frac{\left(m + \sqrt{m^2 - n^2}\right)^2\left(m - \sqrt{m^2 - n^2}\right)}{\left(m - \sqrt{m^2 - n^2}\right)} \tag{4.103}$$

$$\implies a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right) \tag{4.104}$$

nth term of the AP :

$$y(n) = [a + n(b - a)]\, u(n) \tag{4.105}$$

$$n^k u(n) \xleftrightarrow{\mathcal{Z}} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{4.106}$$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})} \quad |z| > |1| \tag{4.107}$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \tag{4.108}$$

Referencing the equations from (4.107),(4.108).

$$y(n) \xleftrightarrow{\mathcal{Z}} \frac{a}{(1 - z^{-1})} + \frac{(b - a)\, z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \tag{4.109}$$

nth term of the GP :

$$y(n) = a \left(\frac{b}{a}\right)^n u(n) \tag{4.110}$$

$$r^n u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - rz^{-1})} \quad |z| > |r| \tag{4.111}$$

Referencing the equation from (4.111).

$$y(n) \xleftrightarrow{\mathcal{Z}} \frac{a^2 z^{-1}}{(a - bz^{-1})} \quad |z| > \left|\frac{b}{a}\right| \tag{4.112}$$

4.0.16 The sum of three numbers in an arithmetic progression (AP) is 24 and the product of those three numbers is 440, find the values of the three numbers.

Solution: The following information is provided in the question:

Let the three numbers in the arithmetic progression be denoted as $x(0)$, $x(1)$, and $x(2)$.

From Table 4.14

$$x(0) + x(1) + x(2) = 24 \tag{4.113}$$

$$(x(1) - d) + x(1) + (x(1) + d) = 24 \tag{4.114}$$

$$3x(1) = 24 \tag{4.115}$$

$$\implies x(1) = 8 \tag{4.116}$$

$$x(0) \cdot x(1) \cdot x(2) = 440 \tag{4.117}$$

$$(8 - d) \cdot (8) \cdot (8 + d) = 440 \tag{4.118}$$

$$(8 - d) \cdot (8 + d) = 55 \tag{4.119}$$

$$64 - d^2 = 55 \tag{4.120}$$

$$\implies d = 3 \tag{4.121}$$

$$\implies x(0) = 5 \tag{4.122}$$

$$x(n) = (5 + 3n)\, u(n) \tag{4.123}$$

From equation (**??**):

$$X(z) = \frac{5 - 8z^{-1}}{(1 - z^{-1})^2}; \quad |z| > |1| \tag{4.124}$$

Therefore, the required three numbers in AP are 5, 8, and 11.

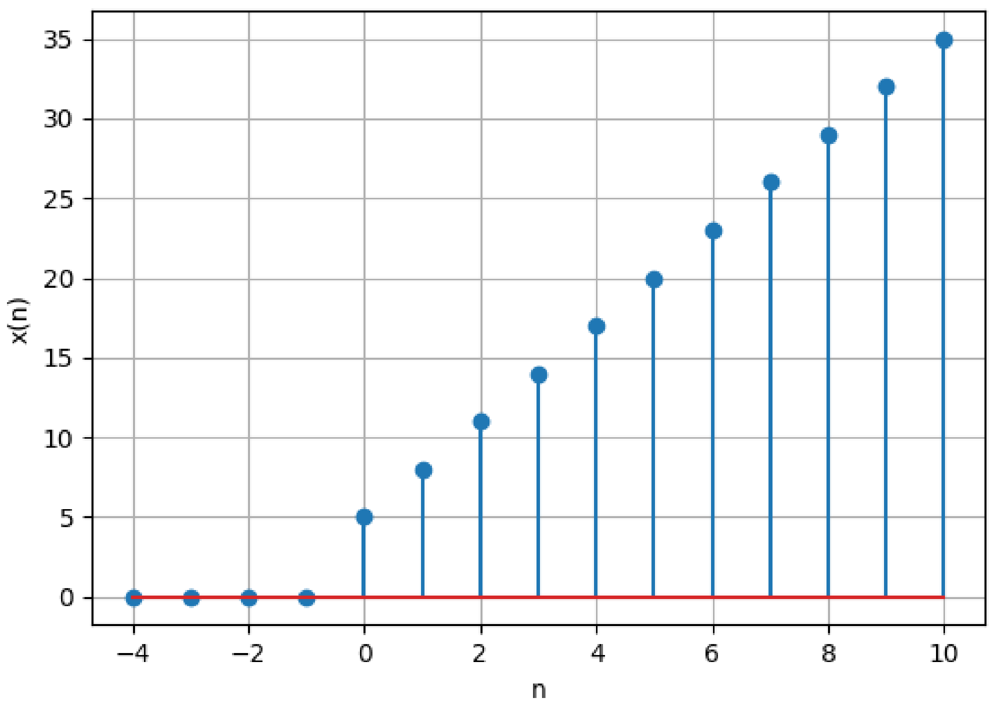


Figure 4.19: stem plots of x(n)

4.0.17 The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2 , respectively. Find the last term and the number of terms.

Solution:

$$x(n) = x(0)r^n u(n) \tag{4.125}$$

From (**??**)

$$X(z) = \frac{5}{1-2z^{-1}} \quad |z| > |2| \tag{4.126}$$

By contour integration:

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{4.127}$$

$$315 = 5\left(2^{n+1}-1\right) \tag{4.128}$$

$$\implies n = 5 \tag{4.129}$$

The number of terms is $n + 1 = 6$

From (4.125):

$$x(5) = 5\left(2^5\right) \tag{4.130}$$

$$= 160 \tag{4.131}$$

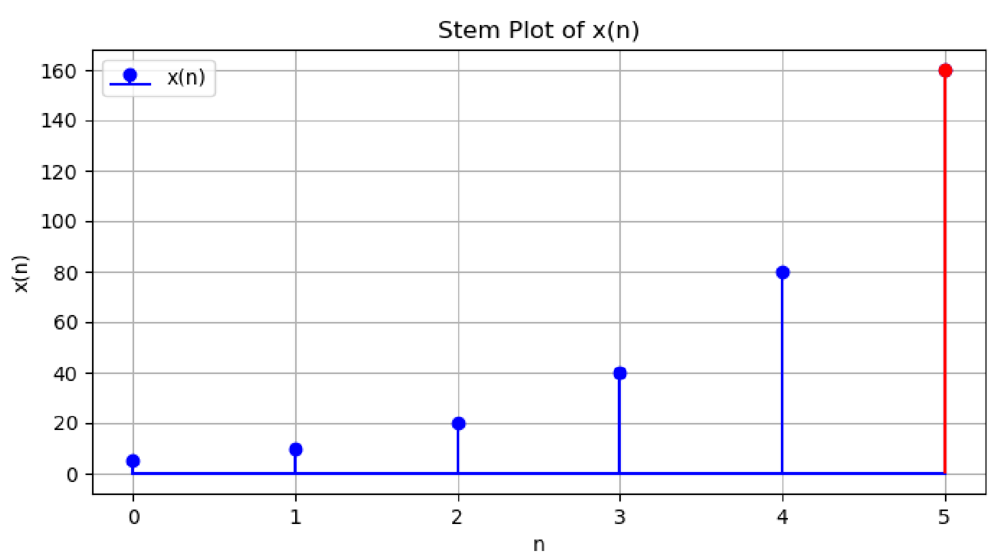


Figure 4.20: Stem plot of x(n)

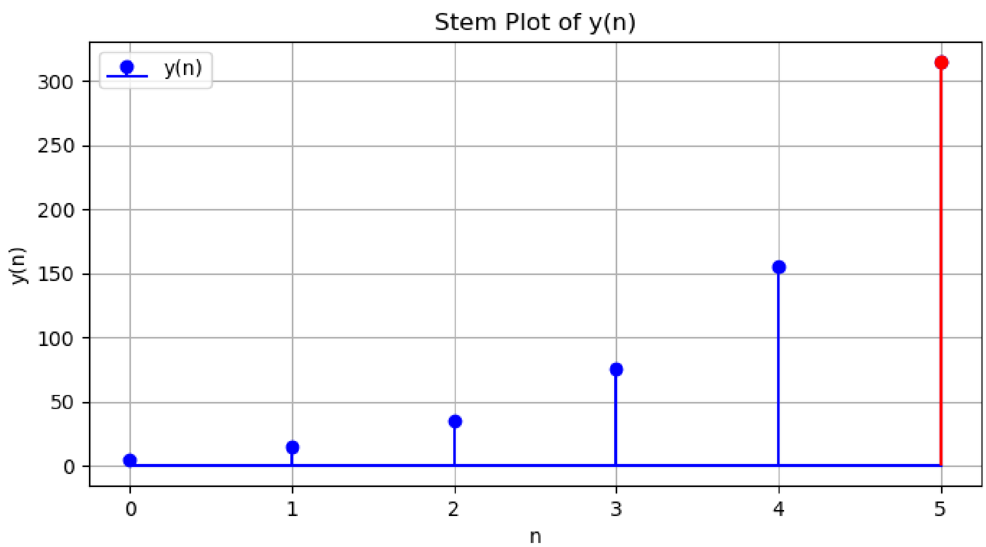


Figure 4.21: Stem plot of y(n)

4.0.18 Find the sum of n terms of the A.P. whose kth term is $5k+1$.

Solution:

Apply the Z-transform to $x(n)$:

$$X(z) = \frac{5z^{-1}}{(1-z^{-1})^2} + \frac{1}{(1-z^{-1})} \quad |z| > 1 \tag{4.132}$$

Sum of First n Terms:

$$y(n) = x(n) * u(n) \tag{4.133}$$

Applying Z transform on both sides:

$$Y(z) = X(z)\,U(z) \tag{4.134}$$

$$= \frac{1}{(1-z^{-1})^2} + \frac{5}{2}\cdot\frac{2z^{-1}}{(1-z^{-1})^3} \tag{4.135}$$

Now we can compare the above pairs as;

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \tag{4.136}$$

$$u(n) \overset{Z}{\longleftrightarrow} \frac{1}{(1-z^{-1})} \tag{4.137}$$

$$n\left(n-1\right)u\left(n\right)\overset{\mathrm{Z}}{\longleftrightarrow}\frac{2z^{-1}}{(1-z^{-1})^{3}} \tag{4.138}$$

On referring the above equations and comparing, we can obtain the Z transform inverse as follows:

$$y\left(n\right)=\left(n+1\right)u\left(n\right)+\frac{5}{2}n\left(n-1\right)u\left(n\right) \tag{4.139}$$

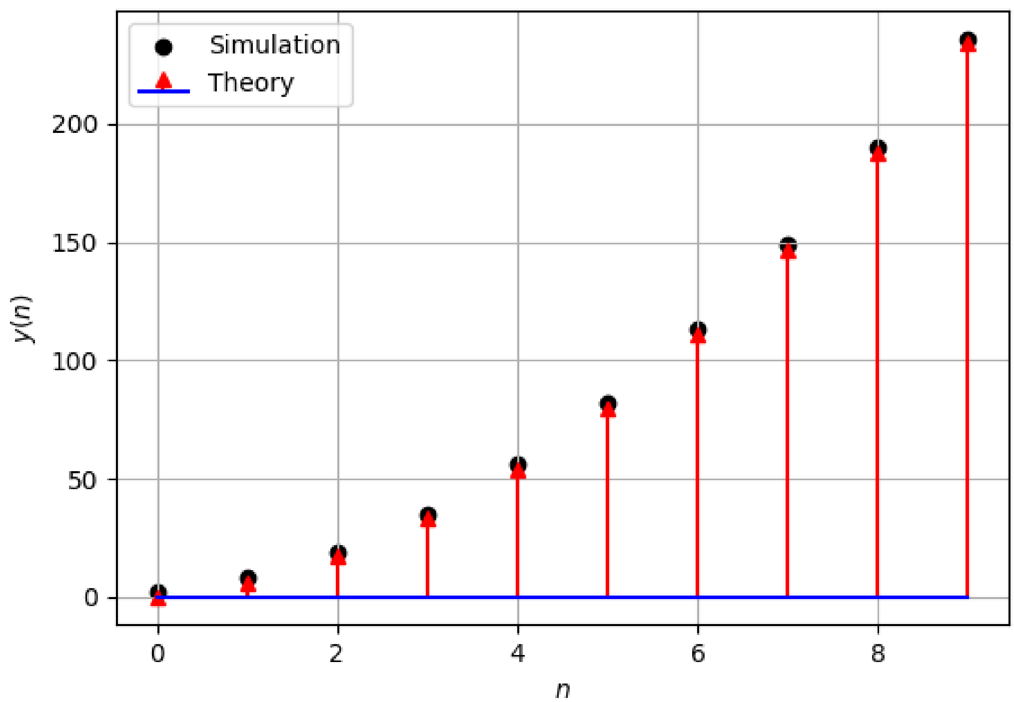
$$=\left(n+1+\frac{5}{2}n\left(n-1\right)\right)u\left(n\right) \tag{4.140}$$

Since we are taking n starting from 0 we replace n with n+1 to make our simulation match with the theory

Therefore, we have got the sum of n terms as:

$$y\left(n\right)=\left(n+2+\frac{5}{2}n\left(n+1\right)\right)u\left(n+1\right) \tag{4.141}$$

The stem plot is given as



4.0.19 How many 3 digit numbers are divisible by 7?

Solution:

We can use modular arithmetic to determine last three digit number divisible by 7 .

$$x(k-1) \equiv 0 \bmod 7 \tag{4.142}$$

So we need to find the largest multiple of 7 less than 1000. We can find this by subtracting the remainder when 1000 is divided by 7 from 1000.

$$1000 - (1000 \bmod 7) = 1000 - 6 \tag{4.143}$$

$$x(k-1) = 994 \tag{4.144}$$

From Table 4.17, the number of terms in the AP, k is:

$$k = \frac{x(k-1) - x(0)}{d} + 1 \tag{4.145}$$

$$\frac{994 - 105}{7} + 1 = 128 \tag{4.146}$$

Taking z transform using **??** :

$$X(z) = \frac{105 - 98z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.147}$$

4.0.20 A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

Solution:

$$x(n) = x(0)\, r^n u(n) \tag{4.148}$$

On taking Z transform

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \tag{4.149}$$

$$= \frac{4}{1 - 4z^{-1}} \tag{4.150}$$

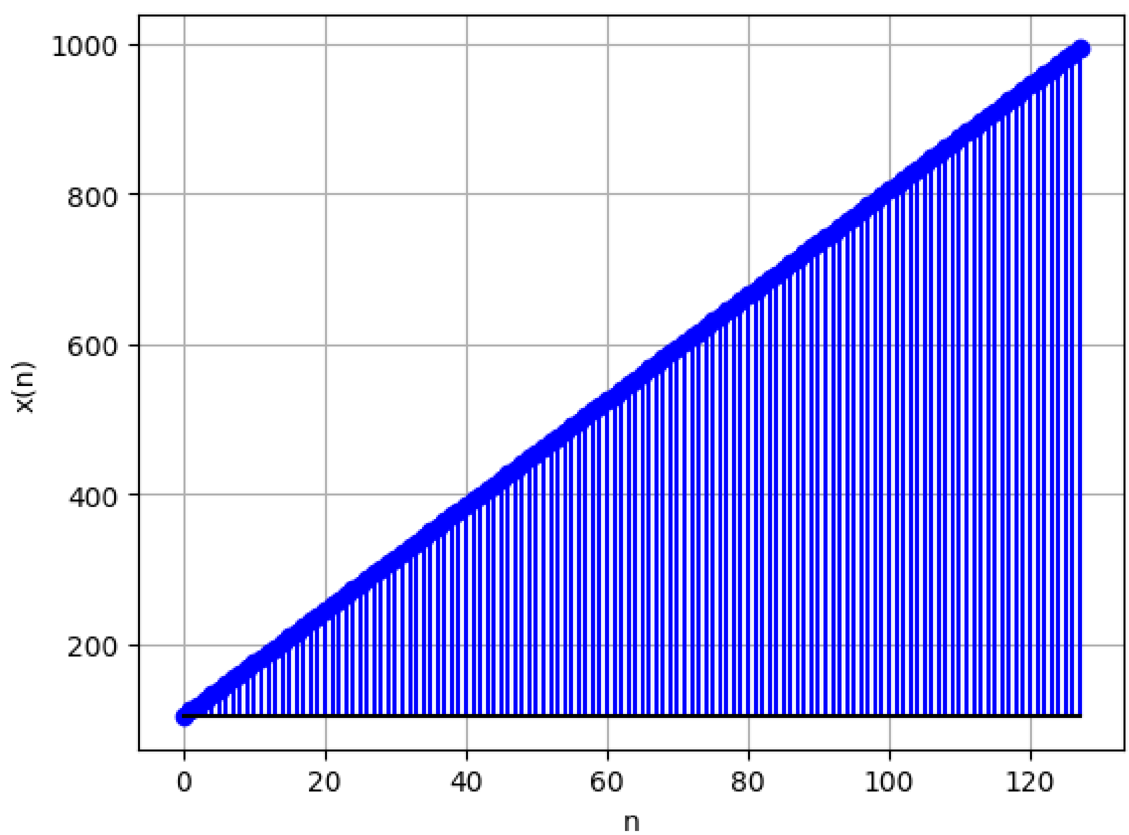


Figure 4.22: Plot of $x\left(n\right)$

$$y\left(n\right) = x\left(n\right) * u\left(n\right) \tag{4.151}$$

$$\implies Y\left(z\right) = X\left(z\right)U\left(z\right) \tag{4.152}$$

$$= \frac{4}{\left(1-4z^{-1}\right)}\frac{1}{\left(1-z^{-1}\right)} \quad |z| > |r| \cap |z| > |1| \tag{4.153}$$

Using contour integration to find the inverse Z-transform:

$$\implies y(7) = \frac{1}{2\pi j}\oint_C Y(z)\ z^6\ dz \tag{4.154}$$

$$= \frac{1}{2\pi j}\oint_C \frac{4z^6}{(1-4z^{-1})(1-z^{-1})}\ dz \tag{4.155}$$

$$= \frac{4}{3}\left(\frac{1}{2\pi j}\oint_C \frac{z^9}{z-4}\ dz - \frac{1}{2\pi j}\oint_C \frac{z^9}{z-1}\right) dz \tag{4.156}$$

We know that

$$\implies R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{4.157}$$

For first contour integral,

$$R_1 = \frac{1}{(1-1)!}\lim_{z\to a}\left((z-a) f(z)\right) \tag{4.158}$$

$$= r^{n+1} \tag{4.159}$$

For second contour integral,

$$R_2 = \frac{1}{(1-1)!}\lim_{z\to a}\left((z-a) f(z)\right) \tag{4.160}$$

$$= 1 \tag{4.161}$$

The sum of n terms of a GP is given by :

$$s(n) = \frac{x(0)}{r-1}(R_1 - R_2) \tag{4.162}$$

$$= 87380 \tag{4.163}$$

$$\tag{4.164}$$

$$\therefore \text{Total amount spent on postage} = 87380 \times 0.5 \tag{4.165}$$

$$= Rs.\ 43690 \tag{4.166}$$

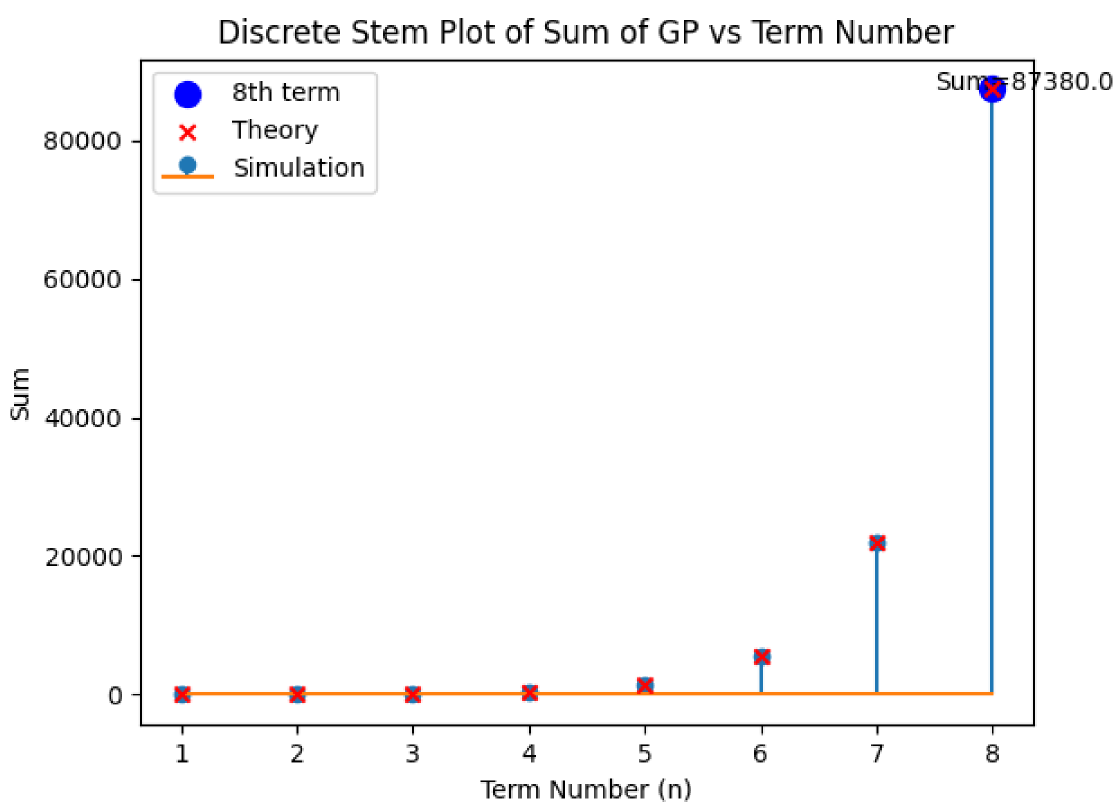


Figure 4.23: Plot of $x(n)\ vs\ n$

4.0.21 If a, b, c are in A.P.; b, c, d are in G.P and $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

Solution:

$$b - a = c - b \tag{4.167}$$

$$2b = a + c \tag{4.168}$$

$$c^2 = b \times d \tag{4.169}$$

$$d = \frac{c^2}{b} \tag{4.170}$$

$$\frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d} \tag{4.171}$$

$$\frac{2}{d} = \frac{1}{c} + \frac{1}{e} \tag{4.172}$$

From (4.170),

$$\frac{2b}{c^2} = \frac{1}{c} + \frac{1}{e} \tag{4.173}$$

From (4.168),

$$\frac{a+c}{c^2} = \frac{1}{c} + \frac{1}{e} \tag{4.174}$$

$$\frac{a}{c^2} + \frac{1}{c} = \frac{1}{c} + \frac{1}{e} \tag{4.175}$$

$$a \times e = c^2 \tag{4.176}$$

So, a,c,e are in G.P

(a) For $y(n)$:

$$y(n) = a\left(\frac{c}{a}\right)^n u(n) \tag{4.177}$$

$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$$

$$Y(z) = \frac{c}{1 - \frac{c}{a}z^{-1}}, \quad |z| > \left|\frac{c}{a}\right| \tag{4.178}$$

(b) For $x_1(n)$:

$$x_1(n) = (b + n(b-a))u(n) \tag{4.179}$$

$$x_1(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)$$

$$X_1(z) = \frac{a}{1 - z^{-1}} + \frac{(b-a)z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{4.180}$$

(c) For $x_2(n)$:

$$x_2(n) = b\left(\frac{c}{b}\right)^n u(n) \tag{4.181}$$

$$x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$$

$$X_2(z) = \frac{c}{1-\frac{c}{b}z^{-1}}, \quad |z| > \left|\frac{c}{b}\right| \tag{4.182}$$

(d) For $x_3(n)$:

$$x_3(n) = \left(\frac{1}{c} + n\left(\frac{1}{c} - \frac{1}{d}\right)\right) u(n) \tag{4.183}$$

$$x_3(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_3(z)$$

$$X_3(z) = \frac{\frac{1}{c}}{1-z^{-1}} + \left(\frac{1}{d} - \frac{1}{c}\right)\frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{4.184}$$

4.0.22 Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

Solution:

From Table 4.20

$$x(0) + 10d = 38 \tag{4.185}$$

$$x(0) + 15d = 73 \tag{4.186}$$

From equations 4.185 and 4.186, the augmented matrix is:

$$\begin{pmatrix} 1 & 10 & 38 \\ 1 & 15 & 73 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 10 & 38 \\ 0 & 5 & 35 \end{pmatrix} \tag{4.187}$$

$$\xleftrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 5 & 35 \end{pmatrix} \tag{4.188}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 1 & 7 \end{pmatrix} \tag{4.189}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} -32 \\ 7 \end{pmatrix} \tag{4.190}$$

The general term and the Z-transform are given by

$$x(n) = (-32 + 7n)\, u(n) \tag{4.191}$$

$$\tag{4.192}$$

The 31st term of this A.P. is

$$x(30) = 178 \tag{4.193}$$

From (??), the Z-Transform of $x(n)$ is given by

$$X(z) = \frac{-32}{1 - z^{-1}} + \frac{7z^{-1}}{(1 - z^{-1})^2} \tag{4.194}$$

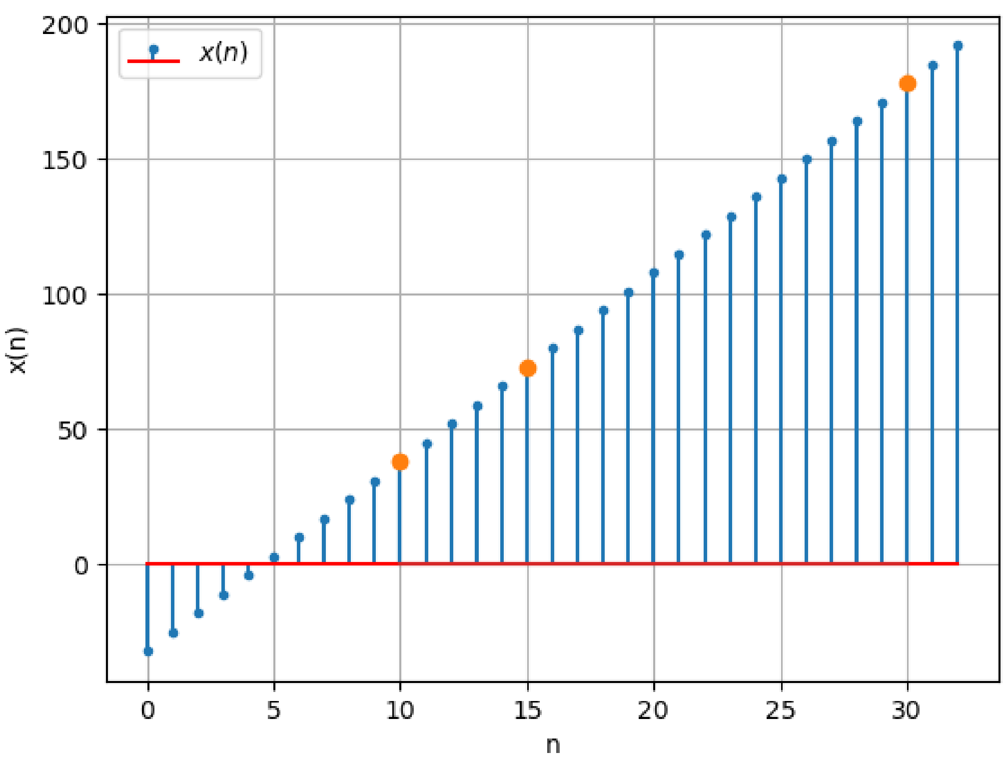


Figure 4.24: Stem plot of $x(0)$ v/s n

4.0.23 If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in arithmetic progression (AP), prove that a, b, c are also in AP.

Solution: Common difference can be written as:

$$b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right) \tag{4.195}$$

$$\implies (b-a)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=(c-b)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{4.196}$$

$$\implies b-a=c-b \tag{4.197}$$

Hence proved that a, b, c are in AP.

From table Table 4.21

$$X(z)=x(0)\left(\frac{1}{1-z^{-1}}\right)+d\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) \tag{4.198}$$

$$=a\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1-z^{-1}}\right)+(b-a)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) \quad |z|>1 \tag{4.199}$$

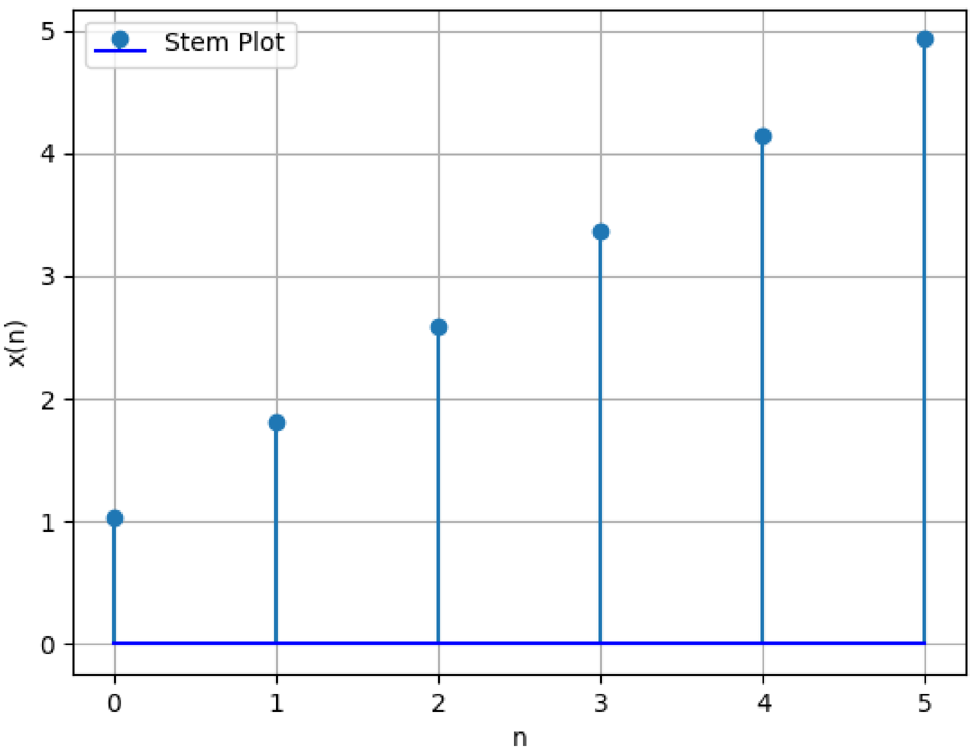


Figure 4.25: graph with value of $a = 3, b = 5, c = 7$

4.0.24 If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is A.M between a and b, then find value of n.

Solution:

A.M of two numbers a,b is $\frac{a+b}{2}$.

$$x(n) = x(0) + n \cdot d \cdot u(n) \tag{4.200}$$

Where,

$$x(1) = \frac{x(0)^n + x(2)^n}{x(0)^{n-1} + x(2)^{n-1}} \tag{4.201}$$

$$= \frac{a+b}{2} \tag{4.202}$$

$$\Rightarrow \frac{x(0)^n + x(2)^n}{x(0)^{n-1} + x(2)^{n-1}} = \frac{x(0) + x(2)}{2} \tag{4.203}$$

$$\Rightarrow x(0)^{n-1}(x(0) - x(2)) = x(2)^{n-1}(x(0) - x(2)) \tag{4.204}$$

From (4.204)

$$\Rightarrow n \begin{cases} = 1 & \text{if } a \neq b \\ \in R & \text{if } a = b \end{cases} \tag{4.205}$$

From (4.200)

$$d = x(1) - x(0) \tag{4.206}$$

$$= \frac{a+b}{2} - a \tag{4.207}$$

$$= \frac{b-a}{2} \tag{4.208}$$

Using Z transform.

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{4.209}$$

$$X(z) = \frac{a}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{4.210}$$

From (4.208)

$$X(z) = \frac{a}{1 - z^{-1}} + \frac{(b - a)z^{-1}}{2(1 - z^{-1})^2} \tag{4.211}$$

4.0.25 The 17th term of ap exceeds its 10th term by 7. FInd its common difference?

Solution:

$$x(n) = \{x(0) + nd\}u(n) \tag{4.212}$$

$$x(17) - x(10) = 7 \tag{4.213}$$

$$\implies x(0) + 17d - x(0) + 10d = 7 \tag{4.214}$$

$$\implies 17d - 10d = 7 \tag{4.215}$$

$$\implies 7d = 7 \tag{4.216}$$

$$\implies d = 1 \tag{4.217}$$

Taking Z-Transform:

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}\{|z| > 1\} \tag{4.218}$$

(b) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}\ \{|z| > 1\} \tag{4.219}$$

0 Taking Z-Transform of (4.212) using (4.218)and (4.219)

$$X(n) = 100\frac{1}{1 - z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2} \tag{4.220}$$

Let

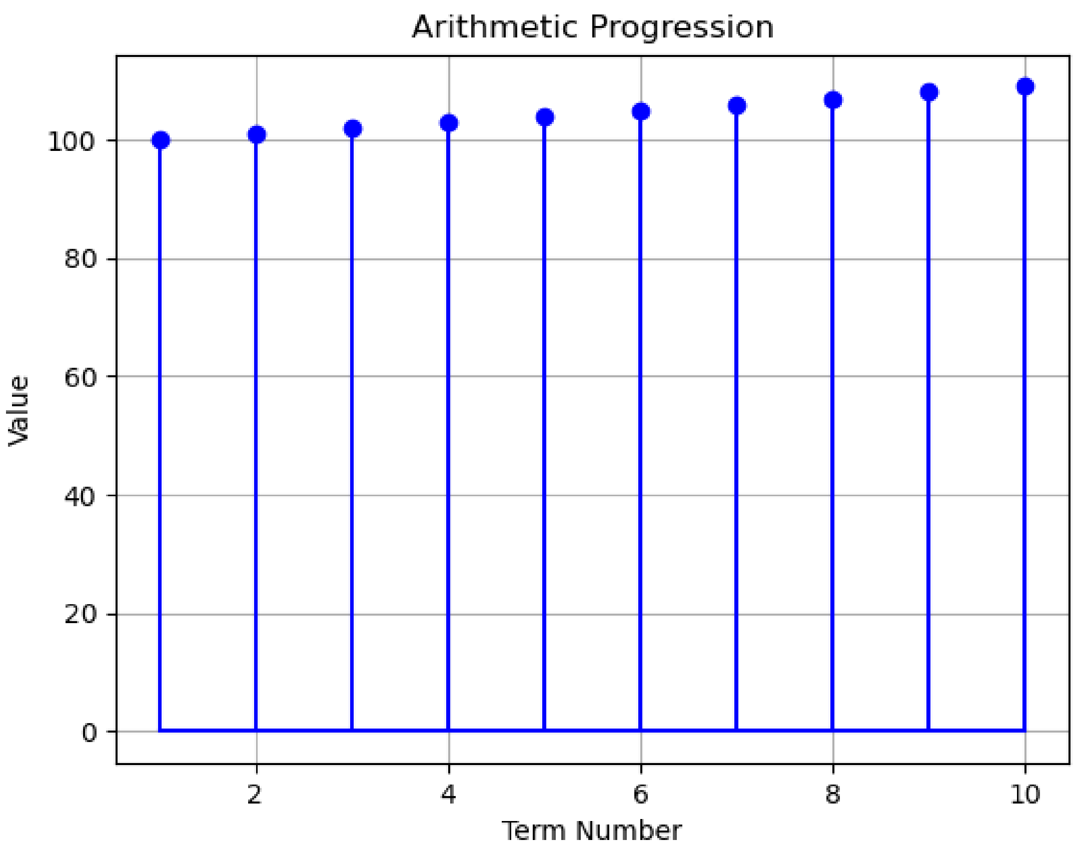


Figure 4.26:

$$x(n) = \{101, 102, 103, ...\} \tag{4.221}$$

4.0.26 If p^{th}, q^{th}, r^{th} term of a GP are a, b and c respectively Prove that

$$a^{q-r}b^{r-p}c^{p-q} = 1$$

Solution:

$$x(n) = (x(0)d^n)u(n) \tag{4.222}$$

$$a = x(p) = (x(0)d^p) \tag{4.223}$$

$$b = x(q) = (x(0)d^q) \tag{4.224}$$

$$c = x(r) = (x(0)d^r) \tag{4.225}$$

$$a^{q-r}b^{r-p}c^{p-q} = x(0)^{q-r}d^{p(q-r)}x(0)^{r-p}d^{q(r-p)}x(0)^{p-q}d^{r(p-q)} \tag{4.226}$$

$$= x(0)^{q-r+r-p+p-q}d^{p(q-r)+q(r-p)+r(p-q)} \tag{4.227}$$

$$= x(0)^0d^0 \tag{4.228}$$

$$a^{q-r}b^{r-p}c^{p-q} = 1 \tag{4.229}$$

Taking Z-Transform:

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}\{|z| > 1\} \tag{4.230}$$

(b) $\mathcal{Z}\{d^nu(n)\}$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-dz^{-1})}\{|z| > |d|\} \tag{4.231}$$

Taking Z-Transform of (4.222) using (4.230)and (4.231)

$$X(z) = \frac{x(0)}{1 - dz^{-1}} \qquad |z| > |d| \tag{4.232}$$

4.0.27 Write the first five terms of the sequence whose n^{th} term is : $x(n) = (-1)^{n-1}5^{n+1}$.

Solution:

$$x(n) = (-1)^n 5^{n+2} u(n) \tag{4.233}$$

$$= 25(-5)^n u(n) \tag{4.234}$$

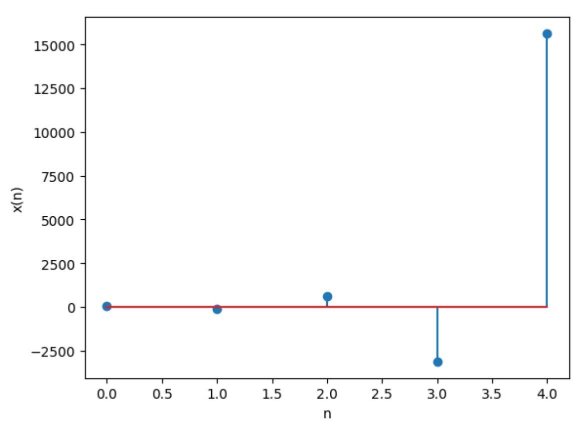
On substituting n = 0, 1, 2, 3 and 4, we get the first five terms.

Hence, the required terms are 25, -125, 625, -3125, 15625 .

$$x(n) \longleftrightarrow X(z)$$

$$a^n u(n) \longleftrightarrow \frac{1}{1 - az^{-1}} \ ; \ |z| > |a| \tag{4.235}$$

$$\therefore \ X(z) = \frac{25}{1 + 5z^{-1}} \ ; \ (|z| > 5) \tag{4.236}$$



4.0.28 The ratio of sums of m and n terms of an A.P. is $m^2 : n^2$.Show that the ratio of m^{th} and n^{th} term is (2m-1):(2n-1).

Solution:

4.0.29 If a and b are the roots of $x^2 - 3x + p = 0$ and c , d are roots of $x^2 - 12x + q = 0$ where a, b, c, d form a G.P. Prove that $(q + p) : (q - p) =$ 17:15 .

Solution:

Given $x_1(0)$ and $x_1(1)$ are the roots of $x^2–3x + p = 0$ So, we have :

$$a + b = 3 \tag{4.237}$$

$$ab = p \tag{4.238}$$

Also, $x_1(2)$ and $x_1(3)$ are the roots of $x^2–12x + q = 0$, so,

$$c + d = 12 \tag{4.239}$$

$$cd = q \tag{4.240}$$

From 4.237 and 4.239 , we get ,

$$a(1 + r) = 3 \tag{4.241}$$

And ,

$$ar^2(1 + r) = 12 \tag{4.242}$$

On dividing eq. 4.241 and eq. 4.242, we get

$$\frac{ar^2(1+r)}{a(1+r)} = \frac{12}{3} \tag{4.243}$$

$$r^2 = 4 \tag{4.244}$$

$$r = \pm 2 \tag{4.245}$$

When r = 2, a = 1

When r = -2, a= -3

Case 1 : When $r = 2$ and $a = 1$

$$p = ab \tag{4.246}$$

$$p = 2 \tag{4.247}$$

$$q = cd \tag{4.248}$$

$$q = 32 \tag{4.249}$$

$$\frac{q+p}{q-p} = \frac{32+2}{32-2} \tag{4.250}$$

$$= \frac{17}{15} \tag{4.251}$$

Case 2 : When $r = -2$ and $a = -3$

$$p = ab \tag{4.252}$$

$$p = -18 \tag{4.253}$$

$$q = cd \tag{4.254}$$

$$q = 288 \tag{4.255}$$

$$\frac{q+p}{q-p} = \frac{288-18}{288+18} \tag{4.256}$$

$$= \frac{135}{153} \tag{4.257}$$

Hence , case 1 satisfies the condition .

$$x_1(n) \longleftrightarrow X(z)$$

$$ar^n u(n) \longleftrightarrow \frac{a}{1-rz^{-1}} \; ; \; |z| > |r| \tag{4.258}$$

$$\therefore \; X(z) = \frac{1}{1-2z^{-1}} \; ; \; (|z| > 2) \tag{4.259}$$

4.0.30 Write the first five terms in the sequence defined recursively as follows:

$$a_0 = 3$$

$$a_n = 3a_{n-1} + 2 \quad \text{for } n > 0$$

Solution:

4.0.31

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx} \tag{4.260}$$

then show that a,b,c,d are in G.P

Solution: let,

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=r \tag{4.261}$$

$$\frac{a+bx}{a-bx}=\frac{a+arx}{a-arx} \tag{4.262}$$

$$=\frac{1+rx}{1-rx} \tag{4.263}$$

$$\frac{b+cx}{b-cx}=\frac{ar+ar^2x}{ar-ar^2x} \tag{4.264}$$

$$=\frac{1+rx}{1-rx} \tag{4.265}$$

$$\frac{c+dx}{c-dx}=\frac{ar^2+ar^3x}{ar^2-ar^3x} \tag{4.266}$$

$$=\frac{1+rx}{1-rx} \tag{4.267}$$

As, equations

$$(4)=(6)=(8) \tag{4.268}$$

so, a,b,c,d are in G.P

Applying z-transform

$$X(z) = \frac{a^2}{a - bz^{-1}} \quad |z| > \left|\frac{b}{a}\right| \tag{4.269}$$

4.0.32 Sum of the first p, q and r terms of an A.P. are a, b and c, respectively. Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$ NCERT-discrete 11.9.2.11

Solution:

$$y(n)=\frac{n+1}{2}(2x(0)+nd)\,u(n) \tag{4.270}$$

Using y(n),

$$a=\frac{p}{2}(2x(0)+(p-1)d) \tag{4.271}$$
$$b=\frac{q}{2}(2x(0)+(q-1)d) \tag{4.272}$$
$$c=\frac{r}{2}(2x(0)+(r-1)d) \tag{4.273}$$

which can be represented as,

$$p.x(0)+\frac{p(p-1)}{2}.d+a.(-1)=0 \tag{4.274}$$
$$q.x(0)+\frac{q(q-1)}{2}.d+b.(-1)=0 \tag{4.275}$$
$$r.x(0)+\frac{r(r-1)}{2}.d+c.(-1)=0 \tag{4.276}$$

resulting in the matrix equation,

$$\begin{pmatrix} p & \frac{p(p-1)}{2} & a \\ q & \frac{q(q-1)}{2} & b \\ r & \frac{r(r-1)}{2} & c \end{pmatrix}\mathbf{x}=0 \tag{4.277}$$

where,

$$\mathbf{x} = \begin{pmatrix} x(0) \\ d \\ -1 \end{pmatrix} \tag{4.278}$$

solving the equations (4.271),(4.272) and (4.273) by row reducing the matrix in (4.277),

$$\begin{pmatrix} p & \frac{p(p-1)}{2} & a \\ q & \frac{q(q-1)}{2} & b \\ r & \frac{r(r-1)}{2} & c \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{R_1}{p}, R_2 \leftarrow \frac{R_2}{q}]{R_3 \leftarrow \frac{R_3}{r}} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 1 & \frac{q-1}{2} & \frac{b}{q} \\ 1 & \frac{r-1}{2} & \frac{c}{r} \end{pmatrix} \tag{4.279}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 0 & \frac{q-p}{2} & \frac{b}{q} - \frac{a}{p} \\ 0 & \frac{r-p}{2} & \frac{c}{r} - \frac{a}{p} \end{pmatrix} \tag{4.280}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{\frac{q-p}{2}}} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right)\frac{2}{q-p} \\ 0 & \frac{r-p}{2} & \frac{c}{r} - \frac{a}{p} \end{pmatrix} \tag{4.281}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{p-1}{2} R_2]{R_3 \leftarrow R_3 - \frac{r-p}{2} R_2} \begin{pmatrix} 1 & 0 & \frac{a}{p} - \frac{\left(\frac{b}{q} - \frac{a}{p}\right)(p-1)}{q-p} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right)\frac{2}{q-p} \\ 0 & 0 & \left(\frac{c}{r} - \frac{a}{p}\right) - \frac{\left(\frac{b}{q} - \frac{a}{p}\right)(r-p)}{q-p} \end{pmatrix} \tag{4.282}$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & \frac{aq(q-1) - bp(p-1)}{pq(q-p)} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right)\frac{2}{q-p} \\ 0 & 0 & \frac{\frac{a}{p}(r-q) + \frac{b}{q}(p-r) + \frac{c}{r}(q-p)}{q-p} \end{pmatrix} \tag{4.283}$$

After row reduction of matrix we get,

$$x(0) = \left(\frac{aq(q-1) - bp(p-1)}{pq(q-p)} \right) \tag{4.284}$$

$$d = \left(\frac{b}{q} - \frac{a}{p} \right) \frac{2}{q-p} \tag{4.285}$$

$$\frac{\frac{a}{p}(r-q) + \frac{b}{q}(p-r) + \frac{c}{r}(q-p)}{q-p} = 0 \tag{4.286}$$

$$\therefore \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0 \tag{4.287}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{4.288}$$

$$X(z) = \frac{aq(q-1) - bp(p-1)}{pq(q-p)(1-z^{-1})} + \frac{2\left(\frac{b}{q} - \frac{a}{p}\right) z^{-1}}{(q-p)(1-z^{-1})^2} \tag{4.289}$$

$$R.O.C\,(|z| > 1) \tag{4.290}$$

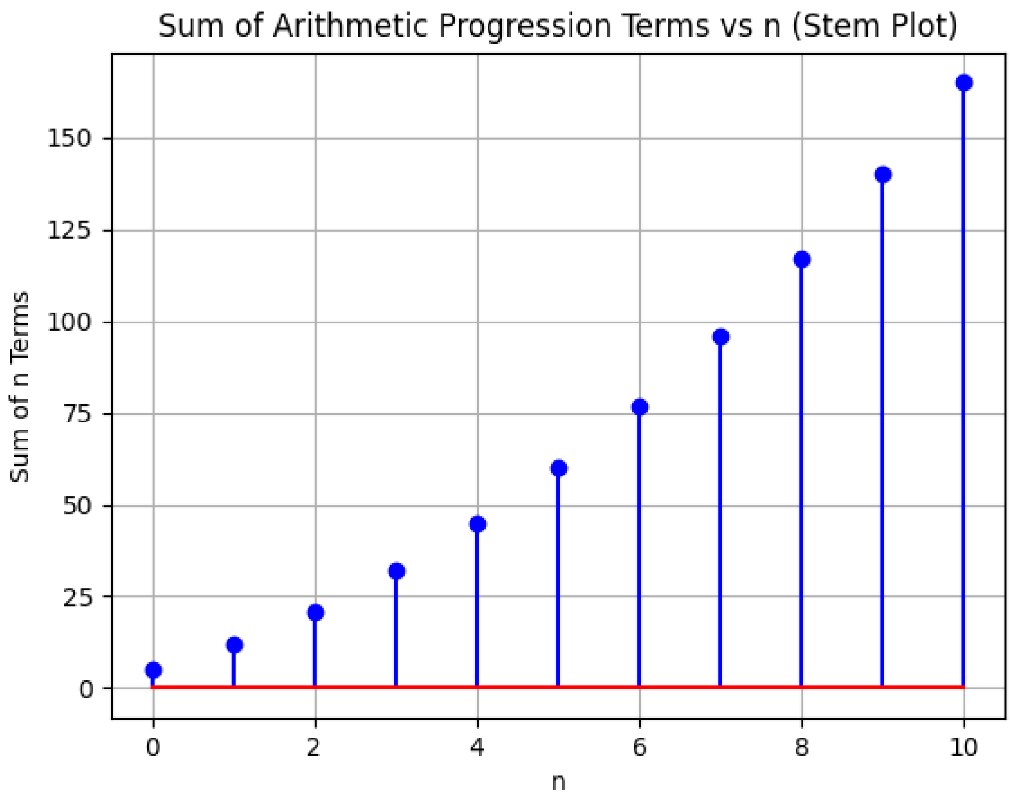


Figure 4.27: Plot of x(n) vs n

4.0.33 The pth, qth and rth terms of an AP are a,b,c respectively. Show that

$$(q-r)a+(r-p)b+(p-q)c=0$$

Solution: The AP has the following parameters

Now,

$$x\,(0) + pd = a \tag{4.291}$$

$$x\,(0) + qd = b \tag{4.292}$$

$$x\,(0) + rd = c \tag{4.293}$$

which can be represented as,

$$x\,(0) + p.d + a.\,(-1) = 0 \tag{4.294}$$

$$x\,(0) + q.d + b.\,(-1) = 0 \tag{4.295}$$

$$x\,(0) + r.d + c.\,(-1) = 0 \tag{4.296}$$

resulting in the matrix equation,

$$\begin{pmatrix} 1 & p & a \\ 1 & q & b \\ 1 & r & c \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{4.297}$$

where,

$$\mathbf{x} = \begin{pmatrix} x\,(0) \\ d \\ -1 \end{pmatrix} \tag{4.298}$$

solving the equations (4.291),(4.292) and (4.293) by row reducing the matrix in

(4.297),

$$\begin{pmatrix} 1 & p & a \\ 1 & q & b \\ 1 & r & c \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & p & a \\ 0 & q-p & b-a \\ 0 & r-p & c-a \end{pmatrix} \tag{4.299}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{q-p}} \begin{pmatrix} 1 & p & a \\ 0 & 1 & \frac{b-a}{q-p} \\ 0 & r-p & c-a \end{pmatrix} \tag{4.300}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - p.R_2} \begin{pmatrix} 1 & 0 & a - p.\frac{b-a}{q-p} \\ 0 & 1 & \frac{b-a}{q-p} \\ 0 & r-p & c-a \end{pmatrix} \tag{4.301}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (r-p).R_2} \begin{pmatrix} 1 & 0 & a - p.\frac{b-a}{q-p} \\ 0 & 1 & \frac{b-a}{q-p} \\ 0 & 0 & (c-a) - \frac{(r-p)(b-a)}{q-p} \end{pmatrix} \tag{4.302}$$

$$\implies \begin{pmatrix} 1 & 0 & \frac{aq-pb}{q-p} \\ 0 & 1 & \frac{b-a}{q-p} \\ 0 & 0 & \frac{a(r-q)+b(p-r)+c(q-p)}{q-p} \end{pmatrix} \tag{4.303}$$

After row reduction of matrix we get,

$$x\,(0) = \frac{aq - pb}{q - p} \tag{4.304}$$

$$d = \frac{b-a}{q-p} \tag{4.305}$$

$$\frac{a\,(r-q) + b\,(p-r) + c\,(q-p)}{q-p} = 0 \tag{4.306}$$

$$\therefore (q-r)\,a + (r-p)\,b + (p-q)\,c = 0 \tag{4.307}$$

$$x(n) \overset{z}{\longleftrightarrow} X(z) \tag{4.308}$$

$$X(z) = \frac{aq - pb}{(q - p)(1 - z^{-1})} + \frac{(b - a)z^{-1}}{(q - p)(1 - z^{-1})^2} \tag{4.309}$$

$$R.O.C\,(|z| > 1) \tag{4.310}$$

Hence proved

4.0.34 Find the sum to indicated number of term in each of the geometric progressions in $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, ...n$ terms

Solution:

$$X(z) = x(0)\left(\frac{1}{1-rz^{-1}}\right), \quad |rz^{-1}| < 1 \tag{4.311}$$

$$y(n) = x(n) * u(n) \tag{4.312}$$

$$Y(z) = X(z)U(z) \tag{4.313}$$

$$= \sqrt{7}\left(\frac{1}{1-\sqrt{3}z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right), \quad |z| > \sqrt{3} \tag{4.314}$$

$$= \left(\frac{\sqrt{7}}{\sqrt{3}-1}\right)\left(\left(\frac{\sqrt{3}}{1-\sqrt{3}z^{-1}}\right) - \left(\frac{1}{1-z^{-1}}\right)\right) \tag{4.315}$$

$$\frac{1}{1-rz^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} r^n u(n), \quad |z| > r \tag{4.316}$$

$$y(n) = \sqrt{7}\left(\frac{\sqrt{3}^{n+1}-1}{\sqrt{3}-1}\right)u(n), \quad |z| > \sqrt{3} \tag{4.317}$$

4.0.35 How many multiples of 4 lie between 10 and 250?

Solution:

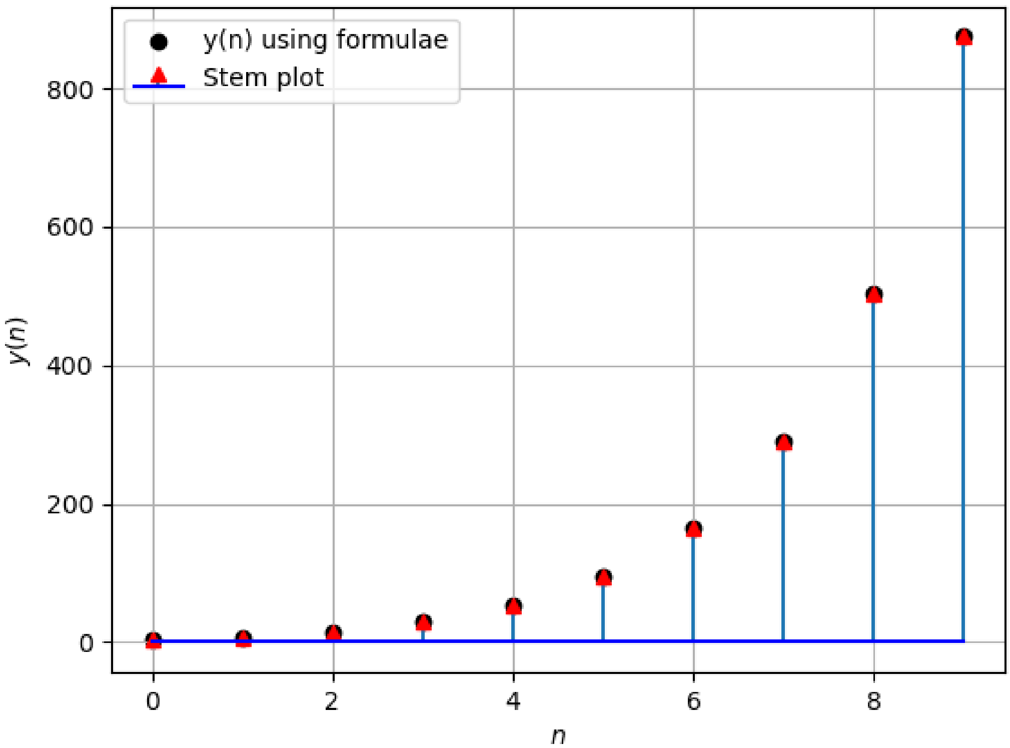


Figure 4.28: STEM PLOT OF $y(n)$

4.0.36 if a, b, c and d are in GP then show that $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

Solution:

4.0.37 In an A.P. the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20^{th} term is -112. (NCERT MATHS 11.9.2.3)

Solution:

General term can be written as

$$x(n) = (x(0) + nd)\, u(n) \tag{4.318}$$

By referreing (**??**)

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{4.319}$$

Taking the inverse Z-transform by contour integration by refering (**??**),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{4.320}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\} u(n) \tag{4.321}$$

Therefore,

$$y(4) = 5x(0) + 10d \tag{4.322}$$

$$y(9) = 10x(0) + 45d \tag{4.323}$$

Given,

$$\sum_{n=0}^{4} x(n) = \frac{1}{4} \sum_{n=5}^{9} x(n) \tag{4.324}$$

Simplifying:

$$y(4) = \frac{1}{4}\left(y(9) - y(4)\right) \tag{4.325}$$

$$\implies 5x(0) + 10d = \frac{1}{4}\left(5x(0) + 35d\right) \tag{4.326}$$

$$x(0) = \frac{-d}{3} \tag{4.327}$$

$$\implies d = -6 \tag{4.328}$$

From (4.328) and Table 1:

$$x(n) = (2 - 6n)\, u(n) \tag{4.329}$$

From (4.329):

$$x(19) = x(0) + 19d \tag{4.330}$$

$$= -112 \tag{4.331}$$

From (4.329) and (4.319):

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{6z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{4.332}$$

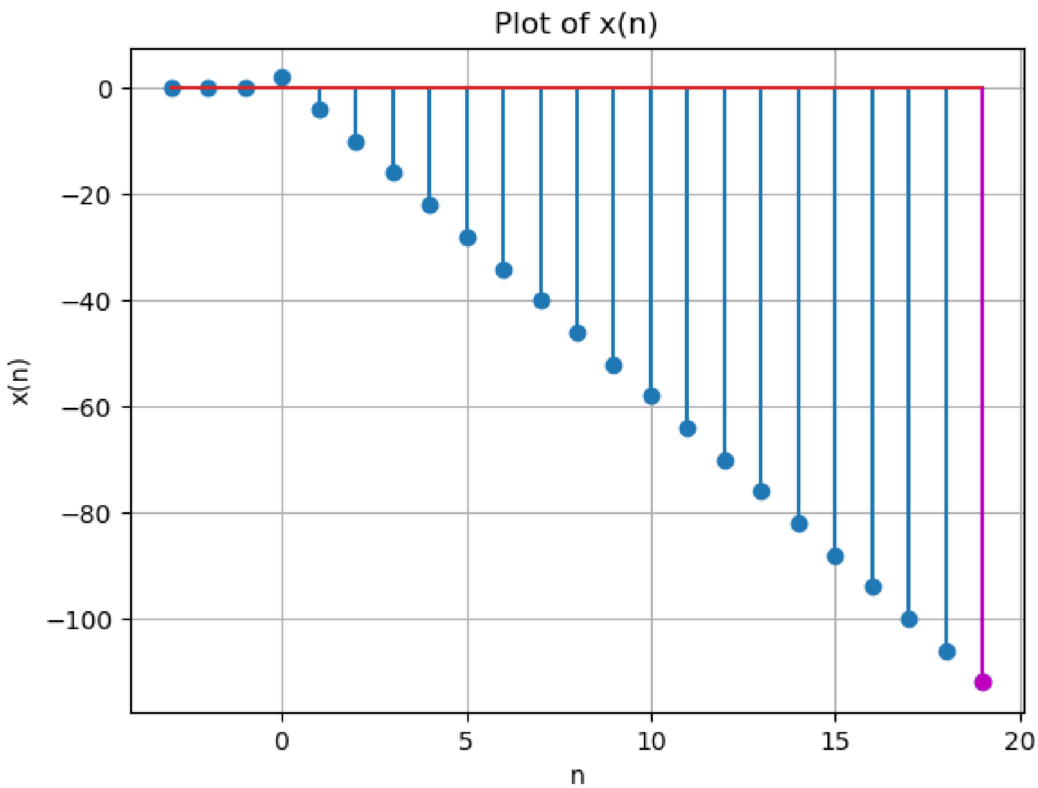


Figure 1: graph of $x(n) = 2 - 6n$

4.0.38 If the 3rd and the 9th terms of an AP are 4 and -8, respectively, which term of this AP is zero?

Solution:

4.0.39 Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$. **Solution:**

4.0.40 Let the sum of $n, 2n, 3n$ terms of an AP be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$

Solution:

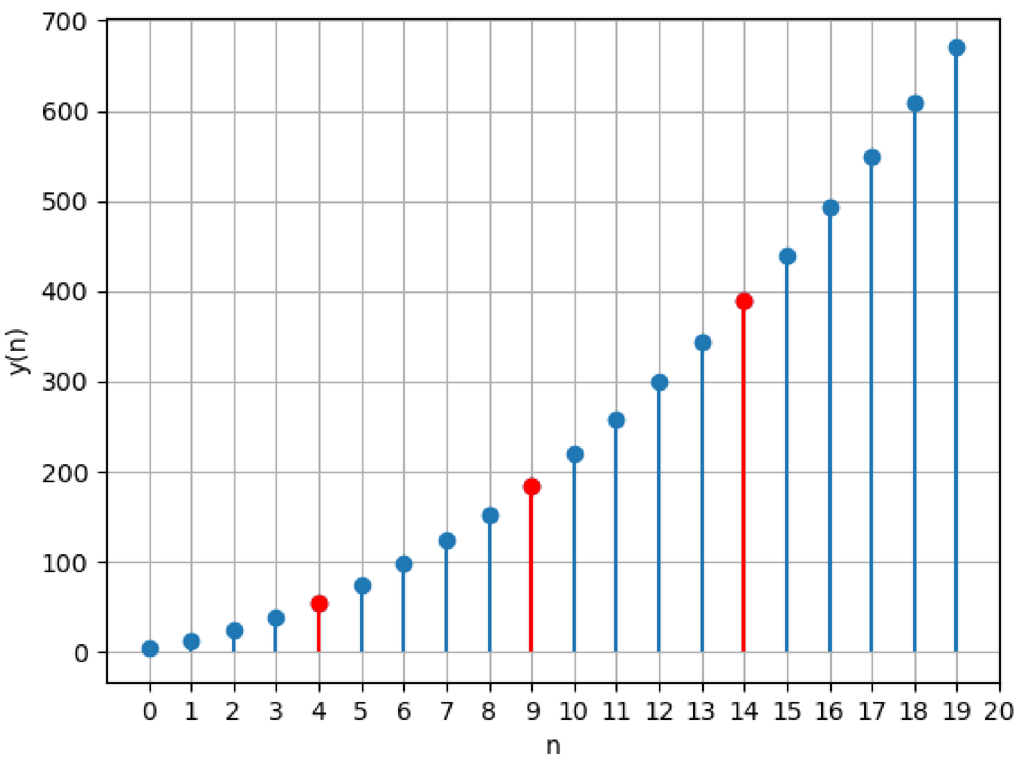


Figure 4.30: Verification plot for the AP[y(n)=$\frac{n+1}{2}(2(5) + n(3))u(n)$]

By equation(**??**)

$$y(n) = \frac{n+1}{2}(2x(0) + nd)u(n) \tag{4.333}$$

$$y(2n) = \frac{2n+1}{2}(2x(0) + 2nd)u(n) \tag{4.334}$$

$$y(3n) = \frac{3n+1}{2}(2x(0) + 3nd)u(n) \tag{4.335}$$

$$3(y(2n) - y(n)) = \frac{3n+1}{2}(2x(0) + 3nd)u(n) \tag{4.336}$$

4.0.41 Show that the products of the corresponding terms of the sequences $a, ar, ar2, \ldots ar^{n-1}$ and $A, AR, AR2, \ldots AR^{n-1}$ form a G.P, and find the common ratio. **Solution:** General term of the n^{th} term of the 1^{st} G.P.,

$$x_1(n) = ar^n u(n) \tag{4.337}$$

Now the sequence in the z domain would be,

$$X_1(z) = \sum_{n=-\infty}^{\infty} ar^n u(n) z^{-n} \tag{4.338}$$

$$= \frac{a}{1 - rz^{-1}}, \quad |z| > |r| \tag{4.339}$$

General for the 2^{nd} G.P. is given as,

$$x_2(n) = AR^n u(n) \tag{4.340}$$

And the z-Transform,

$$X_2(z) = \sum_{n=-\infty}^{\infty} AR^n u(n) z^{-n} \tag{4.341}$$

$$= \frac{A}{1 - Rz^{-1}}, \quad |z| > |R| \tag{4.342}$$

Now taking the product will result in a sequence as,

$$y(n) = x_1(n) x_2(n) \tag{4.343}$$

$$= aA(rR)^n u(n) \tag{4.344}$$

z-Transform of the resulting sequence,

$$Y(z) = \sum_{n=-\infty}^{\infty} aA (rR)^n u(n) z^{-n} \tag{4.345}$$

$$= \frac{aA}{1 - rRz^{-1}}, \qquad |z| > |rR| \tag{4.346}$$

So, form 4.344, taking the ratio of two consecutive terms,

$$\frac{y(n)}{y(n-1)} = \frac{aA (rR)^n u(n)}{aA (rR)^{n-1} u(n-1)} \tag{4.347}$$

$$= rR \tag{4.348}$$

As we can see the ratio of any two consecutive terms, rR, is a constant. Which means the product of the corresponding terms of the two G.P.s results in another G.P. And the common ratio is rR.

4.0.42 The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

Solution:

From Table 4.35

$$x(0) + 3d + x(0) + 7d = 24 \tag{4.349}$$

$$x(0) + 5d + x(0) + 9d = 44 \tag{4.350}$$

Subtracting (4.349) from (4.350)

$$4d = 20 \tag{4.351}$$

$$\implies d = 5 \tag{4.352}$$

Putting (4.352) in (4.349)

$$2x(0) + 10d = 24 \tag{4.353}$$

$$2x(0) + 10(5) = 24 \tag{4.354}$$

$$\implies x(0) = -13 \tag{4.355}$$

Now , general term becomes

$$x(n) = (-13 + 5n) u(n) \tag{4.356}$$

Taking Z-transform of $x(n)$

$$X(z) = \frac{-13}{1 - z^{-1}} + \frac{5z^{-1}}{(1 - z^{-1})^2} \tag{4.357}$$

$$X(z) \implies \frac{18z^{-1} - 13}{z^{-2} - 2z^{-1} + 1} \quad \text{, ROC: } |z| > 1 \tag{4.358}$$

Plotting $x(n)$ v n :

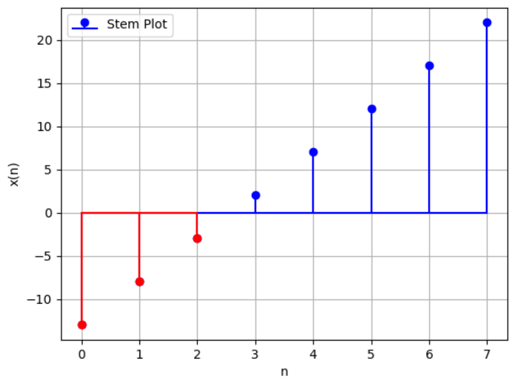


Figure 4.31: Given AP

From Fig. 4.31 first three terms will be

$$\{x(0), x(1), x(2)\} = \{-13, -8, -3\} \tag{4.359}$$

4.0.43 If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$

Solution:

4.0.44 A man starts repaying a loan as first instalment of Rs.100. If he increases the instalment by Rs 5 every month, what amount he will pay in the 30^{th} instalment?

Solution:

$$x(n) = 100 + 5n \tag{4.360}$$

$$x(29) = x(0) + 29d \tag{4.361}$$

$$x(29) = 100 + 145 \tag{4.362}$$

$$x(29) = 245 \tag{4.363}$$

Z transform of $x(n) = 100 + 5n$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.364}$$

$$= \sum_{n=-\infty}^{\infty} (100 + 5n) u(n) z^{-n} \tag{4.365}$$

$$= \sum_{n=-\infty}^{\infty} (100) u(n) z^{-n} + \sum_{n=-\infty}^{\infty} (5n) u(n) z^{-n} \tag{4.366}$$

$$\implies X(z) = \frac{100}{1 - z^{-1}} + \frac{5z^{-2}}{(1 - z^{-1})^2} \tag{4.367}$$

$$ROC : |z| > 1$$

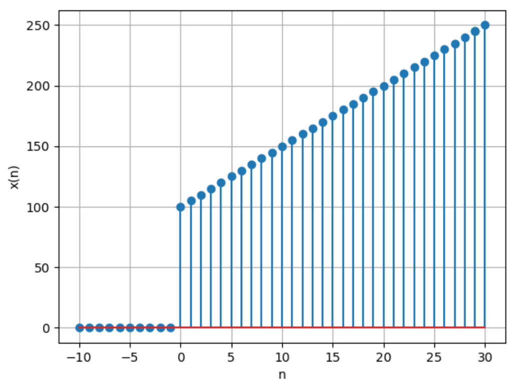


Figure 4.32: Stem Plot of $x\left(n\right)$ v/s n

4.0.45 Write the first five terms of the sequence $a_n = n(n+2)$.

Solution:

from table 4.37

$$X(z) = \sum_{n=-\infty}^{\infty} (n+1)(n+3)\, u(n)\, z^{-n} \tag{4.368}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^2 u(n) + 4n\, u(n) + 3u(n)\right) z^{-n} \tag{4.369}$$

Using eq (**??**) and eq (**??**)

$$X(z) = \frac{3 - z^{-1}}{\left(1 - z^{-1}\right)^3}\ , \qquad |z| > 1 \tag{4.370}$$

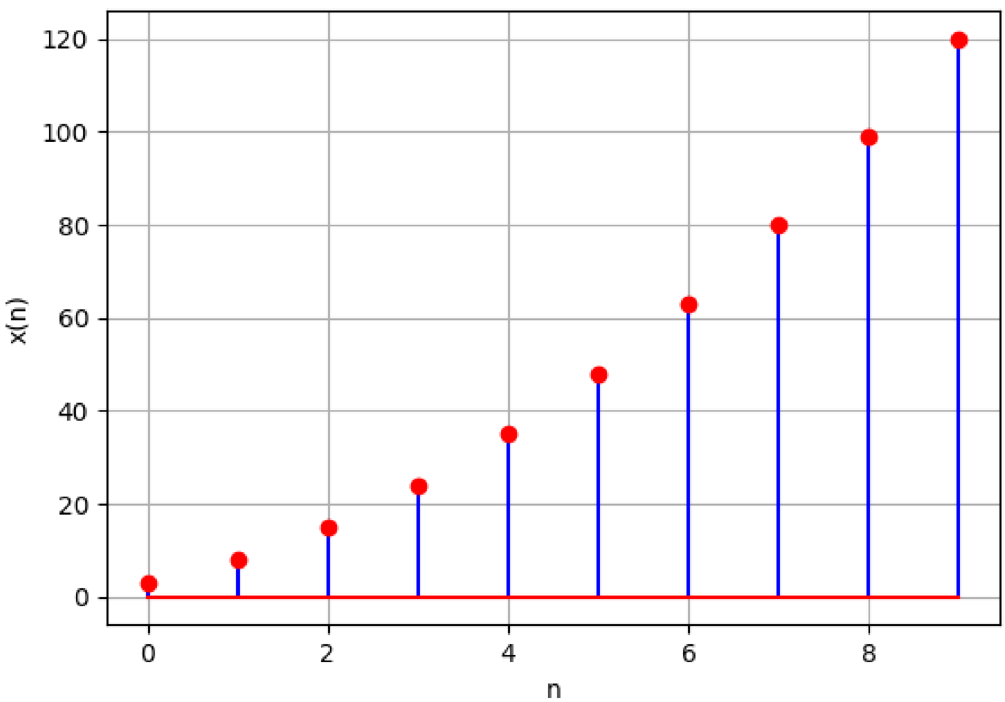


Figure 4.33: Plot of x(n) vs n

4.0.46 If A.M. and G.M. of roots of a quadratic equation are 8 and 5,respectively,then obtain the quadratic equation. **Solution:**

$$x_1 \cdot x_2 = 25 \tag{4.371}$$

$$x_1 + x_2 = 16 \tag{4.372}$$

$$\implies x^2 - 16x + 25 = 0 \tag{4.373}$$

$$\implies x_1 = 8 + \sqrt{39} \tag{4.374}$$

$$\implies x_2 = 8 - \sqrt{39} \tag{4.375}$$

For AP,

$$x\left(0\right) = 8+\sqrt{39} \tag{4.376}$$

$$d = -2\sqrt{39} \tag{4.377}$$

$$x\left(n\right) = \left(8+\sqrt{39}+n\left(-2\sqrt{39}\right)\right)u\left(n\right) \tag{4.378}$$

$$X\left(z\right) = \frac{8+\sqrt{39}}{1-z^{-1}} + \frac{\left(-2\sqrt{39}\right)\cdot z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > |1| \tag{4.379}$$

$$\implies X\left(z\right) = \frac{8+\sqrt{39}-\left(8+3\sqrt{39}\right)\cdot z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > |1| \tag{4.380}$$

For GP,

$$x\left(0\right) = 8+\sqrt{39} \tag{4.381}$$

$$r = \frac{8-\sqrt{39}}{8+\sqrt{39}} \tag{4.382}$$

$$x\left(n\right) = \left(\left(8+\sqrt{39}\right)\cdot\left(\frac{8-\sqrt{39}}{8+\sqrt{39}}\right)^n\right)u\left(n\right) \tag{4.383}$$

$$X\left(z\right) = \frac{8+\sqrt{39}}{1-\frac{\left(8-\sqrt{39}\right)z^{-1}}{8+\sqrt{39}}} \quad |z| > \frac{103-16\sqrt{39}}{25} \tag{4.384}$$

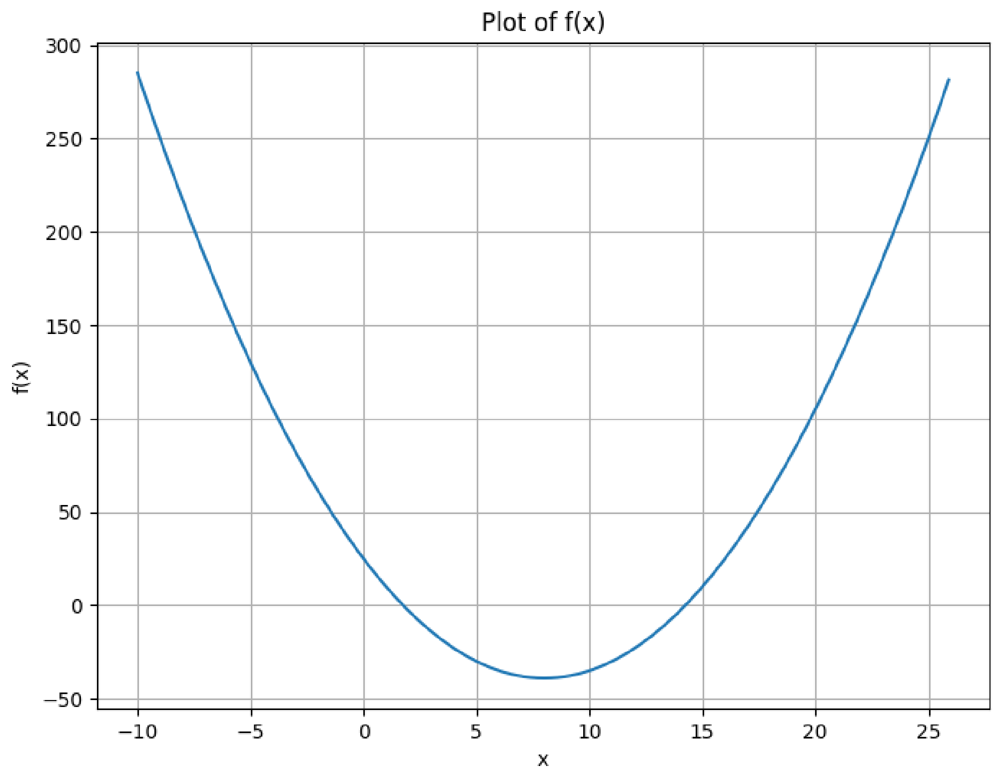


Figure 4.34: Plot of $f(x) = x^2 - 16x + 25 = 0$

4.0.47 An AP consists of 50 terms of which 3^{rd} term is 12 and the last term is 106. Find the 29^{th} term.

Solution:

Parameter	**Value**	**description**
$x(2)$	12	Third term
$x(49)$	106	Last term
$x(0)$		First term
d		Common difference
$x(n)$	$(x(0)+nd)u(n)$	general term

Table 4.39: Input parameters

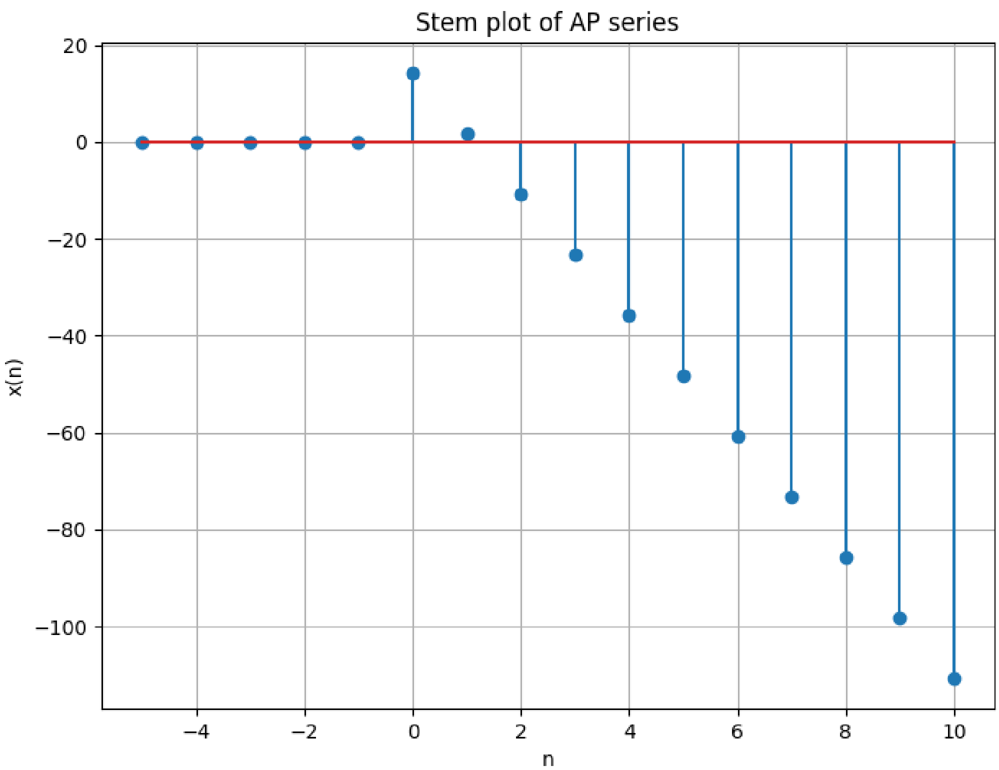


Figure 4.35: Plot of $x\left(n\right)=\left(8+\sqrt{39}+n\left(-2\sqrt{39}\right)\right)u\left(n\right)$

$$\begin{pmatrix} x(2) \\ x(49) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 49 \end{pmatrix} \begin{pmatrix} x(0) \\ d \end{pmatrix} \tag{4.385}$$

$$\begin{pmatrix} 12 \\ 106 \end{pmatrix} = \begin{pmatrix} x(0)+2d \\ x(0)+49d \end{pmatrix} \tag{4.386}$$

$$\text{converting to augmented matrix} \tag{4.387}$$

$$= \begin{pmatrix} x(0)+2d & |12 \\ x(0)+49d & |106 \end{pmatrix} \tag{4.388}$$

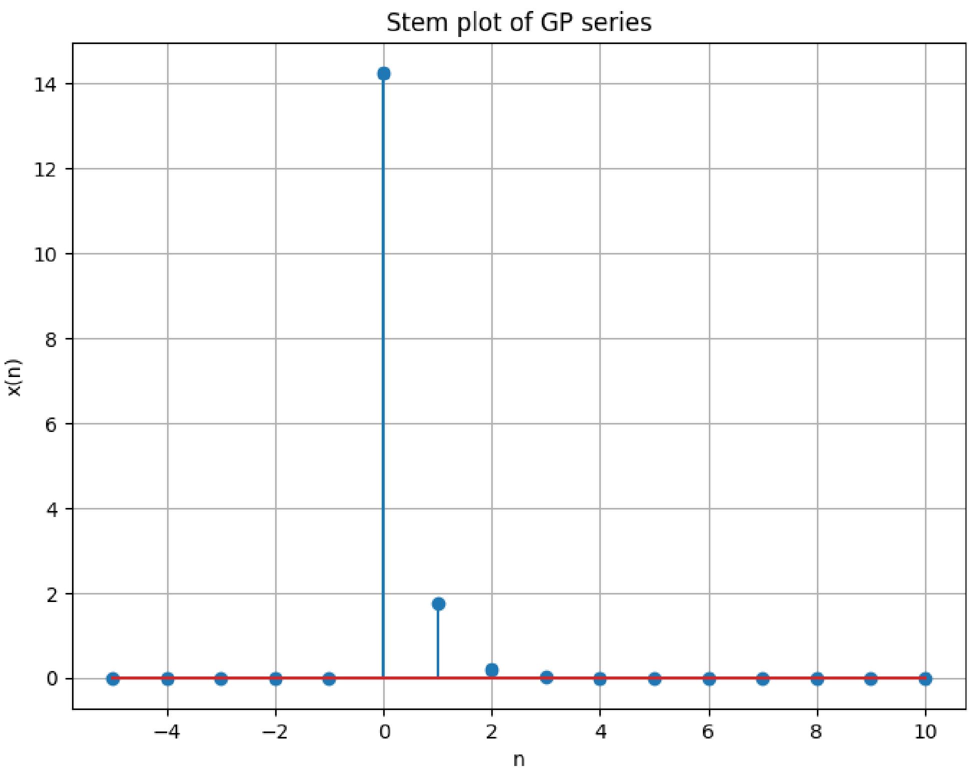


Figure 4.36: Plot of $x\left(n\right)=\left(\left(8+\sqrt{39}\right)\cdot\left(\frac{8-\sqrt{39}}{8+\sqrt{39}}\right)^{n}\right)u\left(n\right)$

From (4.390), we get

$$\implies x(0) = 8 \tag{4.391}$$

$$\implies d = 2 \tag{4.392}$$

From the Table 4.39 :

$$\implies \ x(n) = (8+2n)u(n) \tag{4.393}$$

Finding $x(28)$:

$$x(28) = x(0) + 28(2) \tag{4.394}$$

$$\implies x(28) = 64 \tag{4.395}$$

Z-transform :

$$\implies X(z) = \frac{8 - 6z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.396}$$

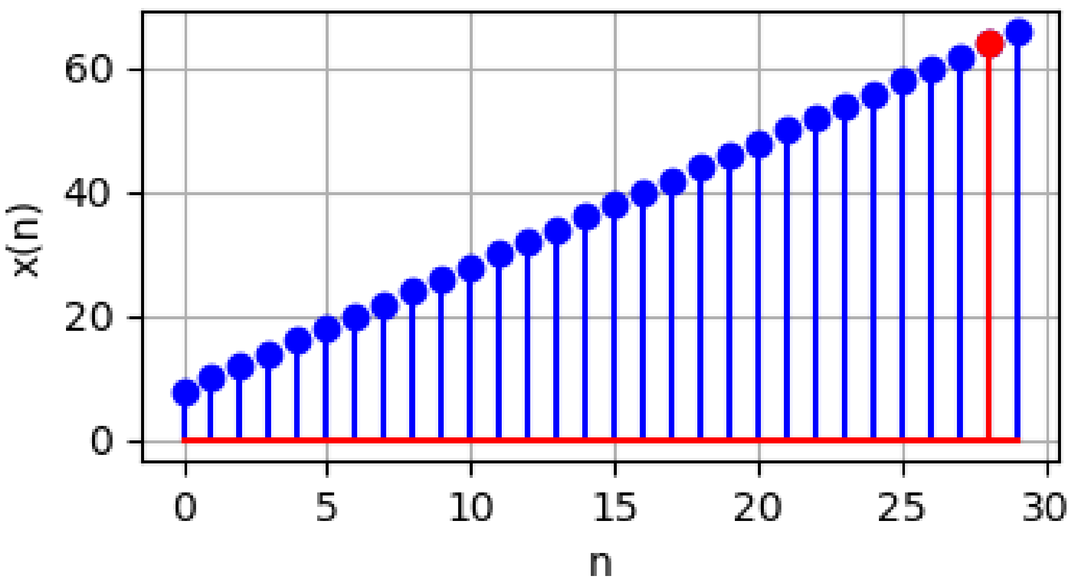


Figure 4.37: graph of the given AP

4.0.48 The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

Solution: Given AP is 5, . . . , 45.

$$x(n) = x(0) + nd \tag{4.397}$$

$$40 = nd \tag{4.398}$$

$$y(n) = \frac{n+1}{2}\left[2x(0) + nd\right] \tag{4.399}$$

$$\implies n = 15 \tag{4.400}$$

$$\implies d = \frac{8}{3} \tag{4.401}$$

by substituting equation (4.400) in the equation (4.398), we get the equation (4.401). z transform of $x(n), y(n)$ are $X(z), Y(z)$

$$X(z) = \frac{7}{3(1-z^{-1})} + \frac{8}{3(1-z^{-1})^2} \tag{4.402}$$

$$Y(z) = \frac{7}{3(1-z^{-1})^2} + \frac{8}{3(1-z^{-1})^3} \tag{4.403}$$

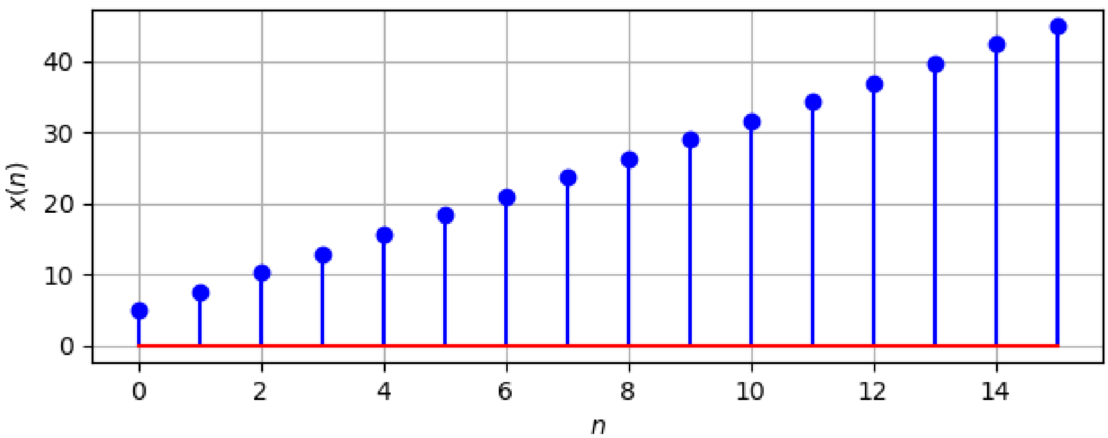


Figure 4.38: analysis of x(n)

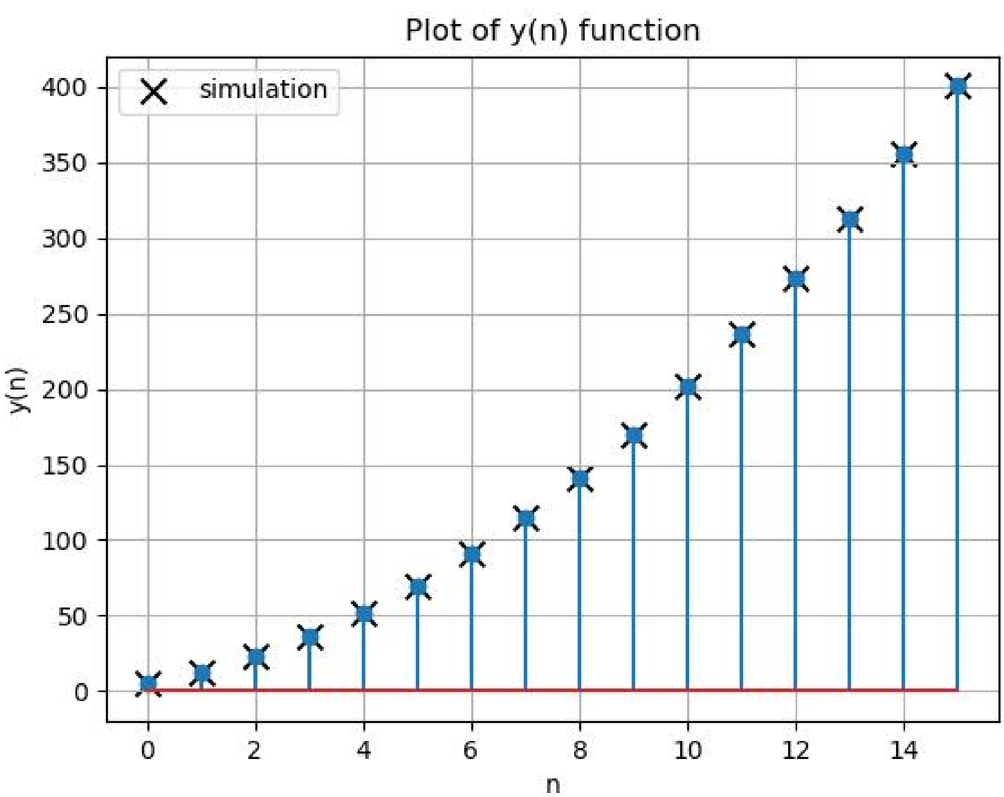


Figure 4.39: simulation vs analysis of y(n)

4.0.49 Which term of the arithmetic progression (AP): $3, 8, 13, 18, \ldots$ is 78?

Solution:

$$x(n) = (3 + (n)5)\, u(n) \tag{4.404}$$

$$78 = 3 + (k)5 \tag{4.405}$$

$$k = 15 \tag{4.406}$$

So, the term of the arithmetic progression that is equal to 78 is the 16th term.

$$X(z) = \frac{3 + 2z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.407}$$

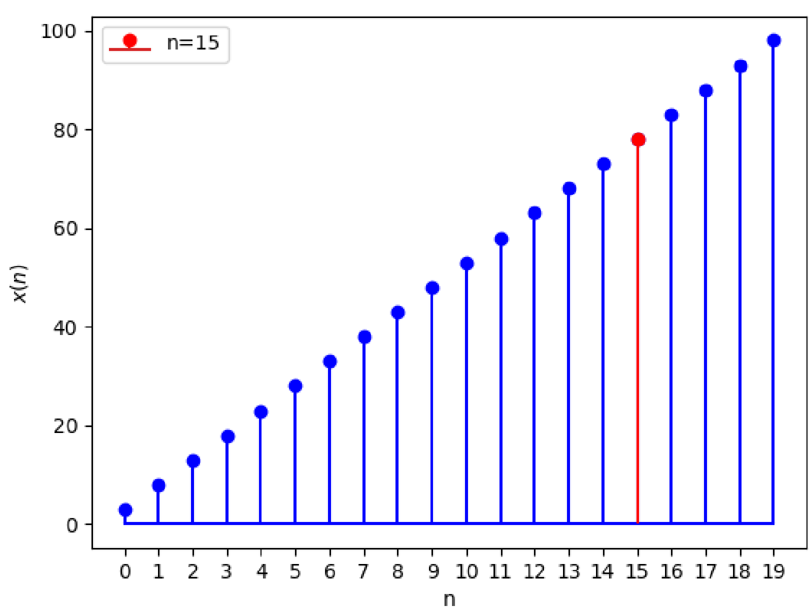


Figure 4.40: Arithmetic Progression Plot

4.0.50 The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\dfrac{(3+2\sqrt{2})}{(3-2\sqrt{2})}$. **Solution:** Let the two numbers be $x(0)$ and $x(2)$ such that $x(2) \geq x(0)$

From Table 4.42:

$$x(0) + x(2) = 6x(1) \tag{4.408}$$

$$\implies x(0) + x(0)r^2 = 6x(0)r \tag{4.409}$$

$$\implies r^2 - 6r + 1 = 0 \tag{4.410}$$

$$\implies r = 3 \pm 2\sqrt{2} \tag{4.411}$$

$$\therefore \frac{x(2)}{x(0)} = (3+2\sqrt{2})^2 \tag{4.412}$$

$$= \frac{(3+2\sqrt{2})}{(3-2\sqrt{2})} \tag{4.413}$$

$$x(n) = (x(0)(3+2\sqrt{2})^n)u(n) \tag{4.414}$$

Taking z - Transform of $x(n)$:

$$X(z) = \frac{x(0)}{1-(3+2\sqrt{2})z^{-1}}; |z| > (3+2\sqrt{2}) \tag{4.415}$$

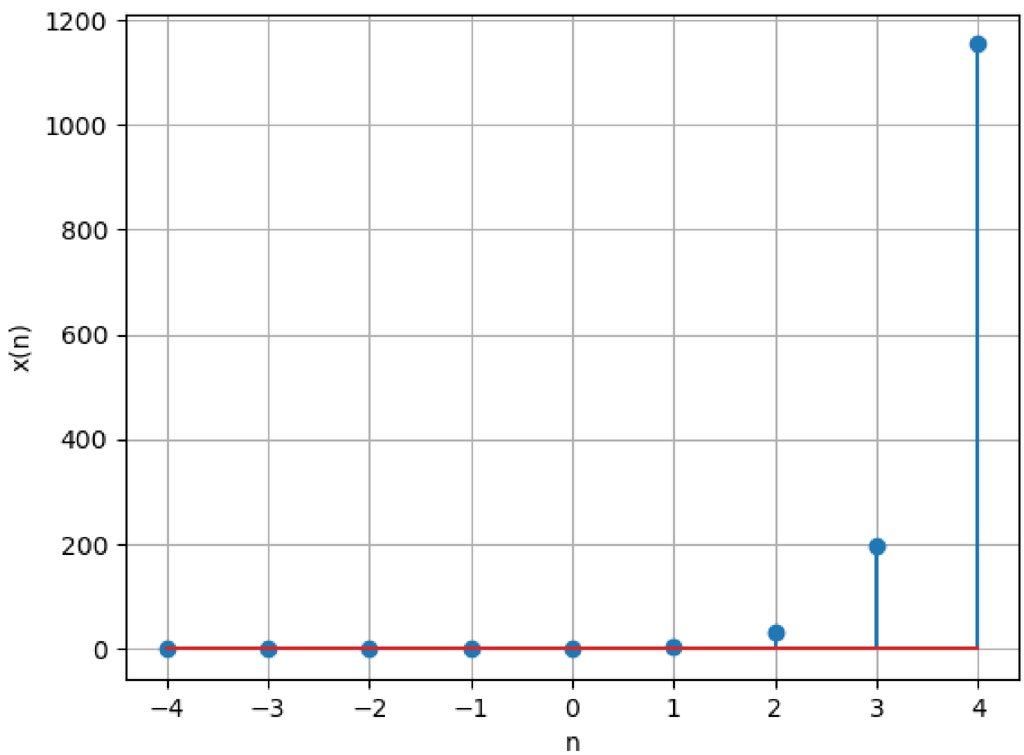


Figure 4.41:

4.0.51 Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution:

Consider a GP as in Table 4.43,

$$\therefore \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = x(1) \tag{4.416}$$

$$\implies a^{n+1}+b^{n+1} = a^{n+\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{n+\frac{1}{2}} \tag{4.417}$$

$$\implies a^{n+\frac{1}{2}}(a^{\frac{1}{2}} - b^{\frac{1}{2}}) = b^{n+\frac{1}{2}}(a^{\frac{1}{2}} - b^{\frac{1}{2}}) \tag{4.418}$$

$$\implies (\frac{a}{b})^{n+\frac{1}{2}} = (\frac{a}{b})^0 \tag{4.419}$$

$$\implies n = -\frac{1}{2} \tag{4.420}$$

From Table 4.43,

$$X(z) = \frac{a}{1-(\sqrt{\frac{b}{a}})z^{-1}} \quad |z| > \left|\sqrt{\frac{b}{a}}\right| \tag{4.421}$$

4.0.52 Which term of the AP : 121, 117, 113, . . . , is its first negative term?

Solution:

4.0.53 The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution:

4.0.54 If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first $(p+q)$ terms.

Solution:

4.0.55 How many terms of G.P.3,3^2,3^3, $\ldots$ are needed to give the sum 120 ?

Solution:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \qquad |z| > |r| \tag{4.422}$$

$$= \frac{3}{1 - 3z^{-1}} \tag{4.423}$$

$$U(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1 \tag{4.424}$$

$$s(n) = x(n) * u(n) \tag{4.425}$$

$$S(z) = X(z)U(z) \tag{4.426}$$

$$= \left(\frac{3}{1 - 3z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > 3 \tag{4.427}$$

by using sum to n terms in G.P

$$s(n) = a(\frac{r^n - 1}{r - 1}) \tag{4.428}$$

$$120 = \frac{3^{n+1} - 3}{2} \tag{4.429}$$

$$n = 4 \tag{4.430}$$

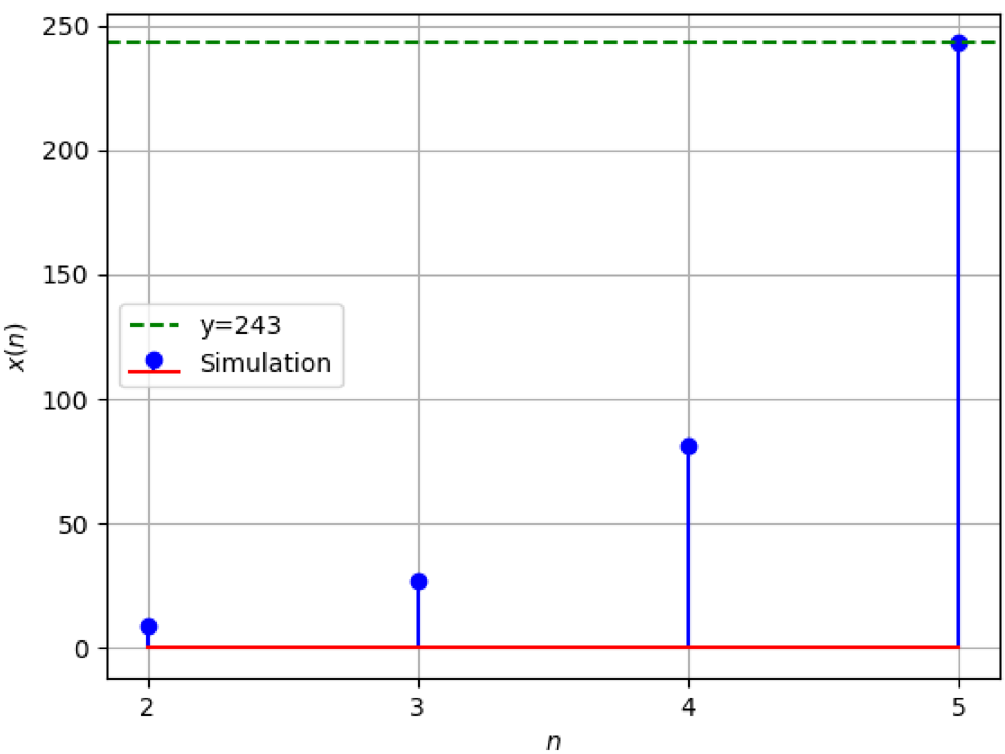


Figure 4.42: Stem plot of x(n)

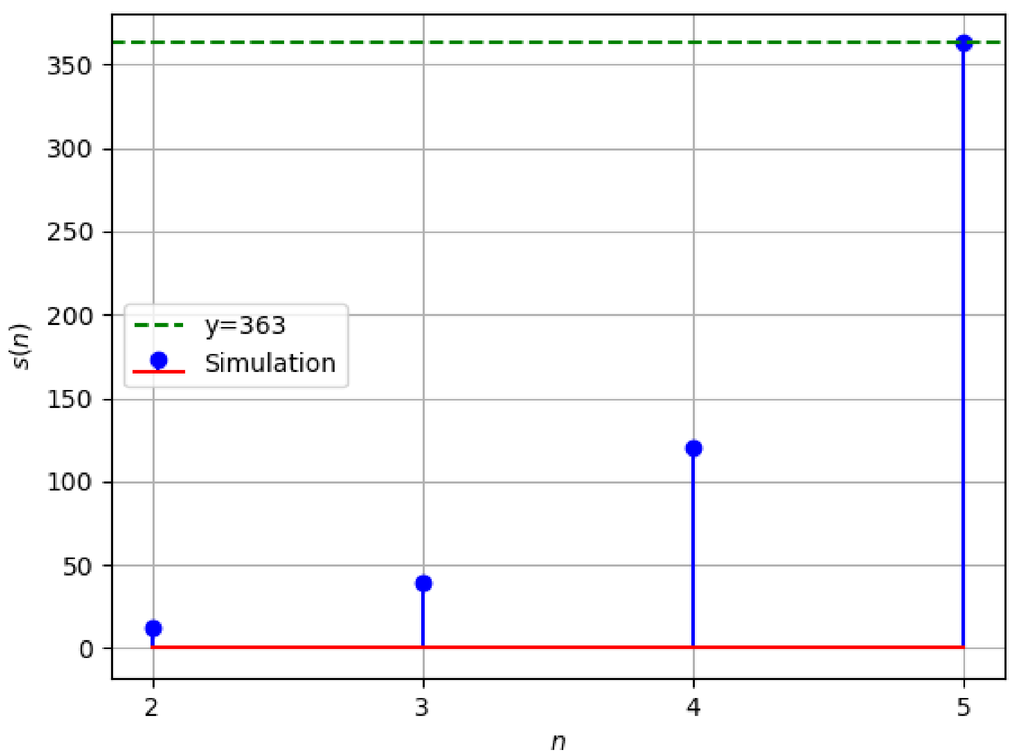


Figure 4.43: Stem plot of s(n)

4.0.56 Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

Solution:

For an AP,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{4.431}$$

$$\implies X(z) = \frac{10}{(1 - z^{-1})} + \frac{4z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \tag{4.432}$$

$$y(n) = x(n) * u(n) \tag{4.433}$$

$$Y(z) = X(z) U(z) \tag{4.434}$$

$$Y(z) = \frac{10}{(1 - z^{-1})^2} + \frac{4z^{-1}}{(1 - z^{-1})^3} \tag{4.435}$$

$$\implies Y(z) = \frac{(-6z^{-1} + 10)}{(1 - z^{-1})^3}, |z| > 1 \tag{4.436}$$

Using Contour Integration to find the inverse Z-transform,

$$y(50) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{49}\, dz \tag{4.437}$$

$$= \frac{1}{2\pi j} \oint_C \frac{(-6z^{-1} + 10)z^{49}}{(1 - z^{-1})^3}\, dz \tag{4.438}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{4.439}$$

$$\implies R = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{(-6z^{-1} + 10)z^{52}}{(z-1)^3} \right) \tag{4.440}$$

$$\implies R = \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} (10z^{52} - 6z^{51}) \tag{4.441}$$

$$\implies R = 5610 \tag{4.442}$$

$$\therefore y(50) = 5610 \tag{4.443}$$

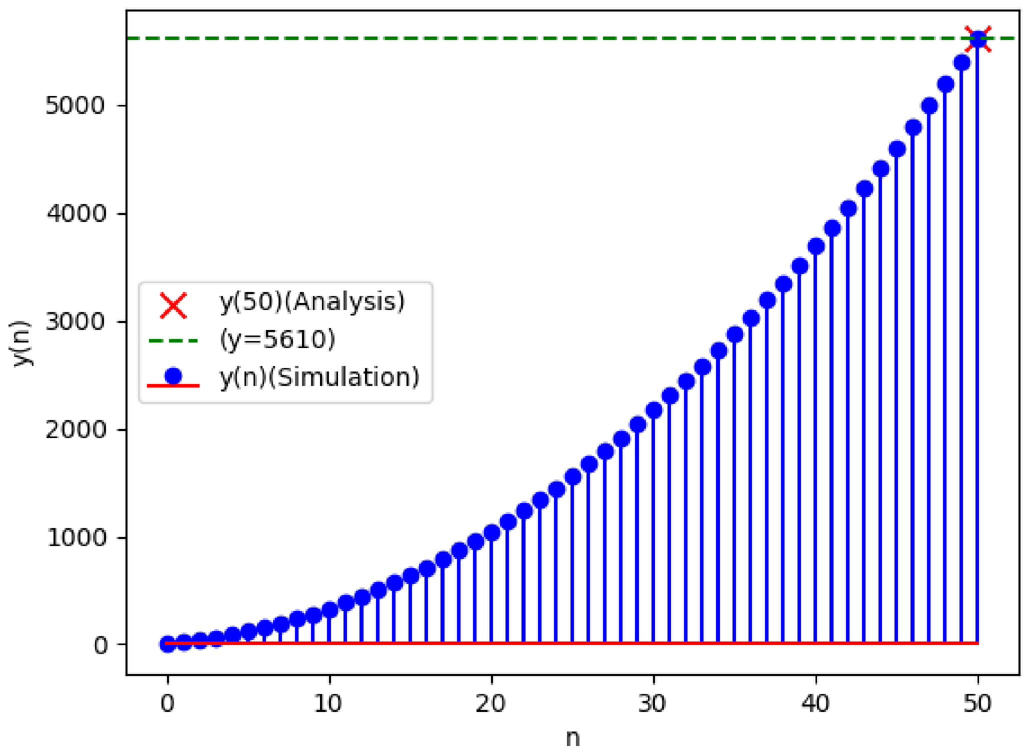


Figure 4.44: Analysis vs Simulation

4.0.57 Fill in the blanks in the following table given that a is the first term, d is the common difference, and a_n is the nth term of the AP.

Solution: for A.P,

$$x_i(n) = [x_i(0) + nd_i]\, u(n) \tag{4.444}$$

(a) From (4.444) :

$$x_1(n) = [7 + 3n]\, u(n) \tag{4.445}$$

$$x_1(7) = 28 \tag{4.446}$$

$$X_1(z) = \frac{7 - 4z^{-1}}{(1 - z^{-1})^2} \quad |z| \neq 1 \tag{4.447}$$

(b) From (4.444):

$$x_2(n) = [-18 + d_2 n]\, u(n) \tag{4.448}$$

$$x_2(9) = 0 \tag{4.449}$$

$$d_2 = 2 \tag{4.450}$$

$$X_2(z) = \frac{-18 + 20z^{-1}}{(1 - z^{-1})^2} \quad |z| \neq 1 \tag{4.451}$$

(c) From (4.444) :

$$x_3(n) = [x_3(0) - 3n]\, u(n) \tag{4.452}$$

$$x_3(17) = -5 \tag{4.453}$$

$$x_3(0) = 49 \tag{4.454}$$

$$X_3(z) = \frac{49 - 52z^{-1}}{(1 - z^{-1})^2} \quad |z| \neq 1 \tag{4.455}$$

(d) From (4.444) :

$$x_4(n) = [18.9 + 2.5n]\, u(n) \tag{4.456}$$

$$x_4(n) = 3.6 \tag{4.457}$$

$$n_4 = 9 \tag{4.458}$$

$$X_4(z) = \frac{18.9 - 16.4z^{-1}}{(1 - z^{-1})^2} \quad |z| \neq 1 \tag{4.459}$$

(e) From (4.444) :

$$x_5(n) = [3.5]\, u(n) \tag{4.460}$$

$$x_5(104) = 3.5 \tag{4.461}$$

$$X_5(z) = \frac{3.5}{1 - z^{-1}} \quad |z| \neq 1 \tag{4.462}$$

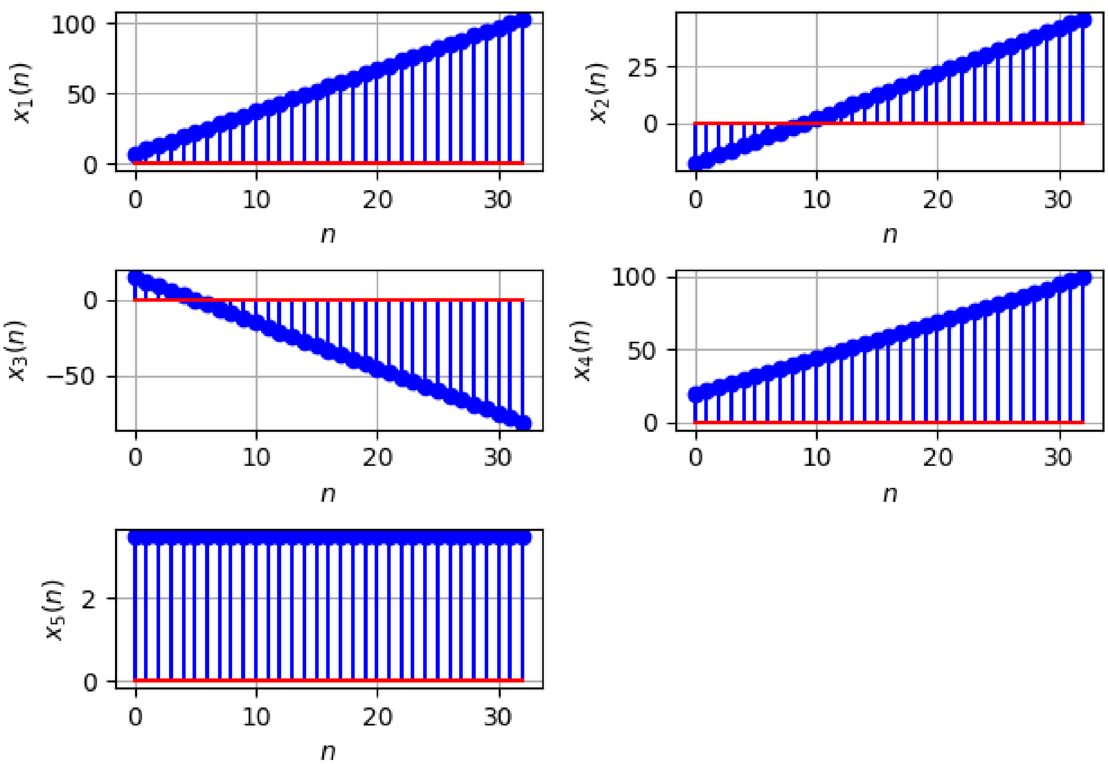


Figure 4.45: stem plots

4.0.58 If a function Satisfying f($x+y$) = f(x) f(y) for all $x, y\ inN$ such that f(1) =3 and $\sum_{x=1}^{n} f(x) = 120$, find the value of n .

Solution: $x = 1$ and $y = 1$, we get

$$f\,(2) = f\,(1+1) \tag{4.463}$$

$$= f\,(1)f\,(1) \tag{4.464}$$

$$= [f\,(1)]^2 \tag{4.465}$$

$$f\,(3) = f\,(2+1) \tag{4.466}$$

$$= f\,(2)f\,(1) \tag{4.467}$$

$$= [f\,(1)]^3 \tag{4.468}$$

Using induction , we get ;

$$f\,(x) = [f\,(1)]^x \tag{4.469}$$

$$r = f\,(1) \tag{4.470}$$

$$= 3 \tag{4.471}$$

Refering from table ;

$$X(z) = \frac{3}{1-3z^{-1}} \quad |z| > |3| \tag{4.472}$$

$$Y(z) = \frac{x(0)}{(1-rz^{-1})\,(1-z^{-1})} \quad |z| > |r| \tag{4.473}$$

Using partial differentiation

$$Y(z) = \frac{x\,(0)}{r-1}\left(\frac{r}{1-rz^{-1}} - \frac{1}{1-z^{-1}}\right) \quad |z| > |r| \tag{4.474}$$

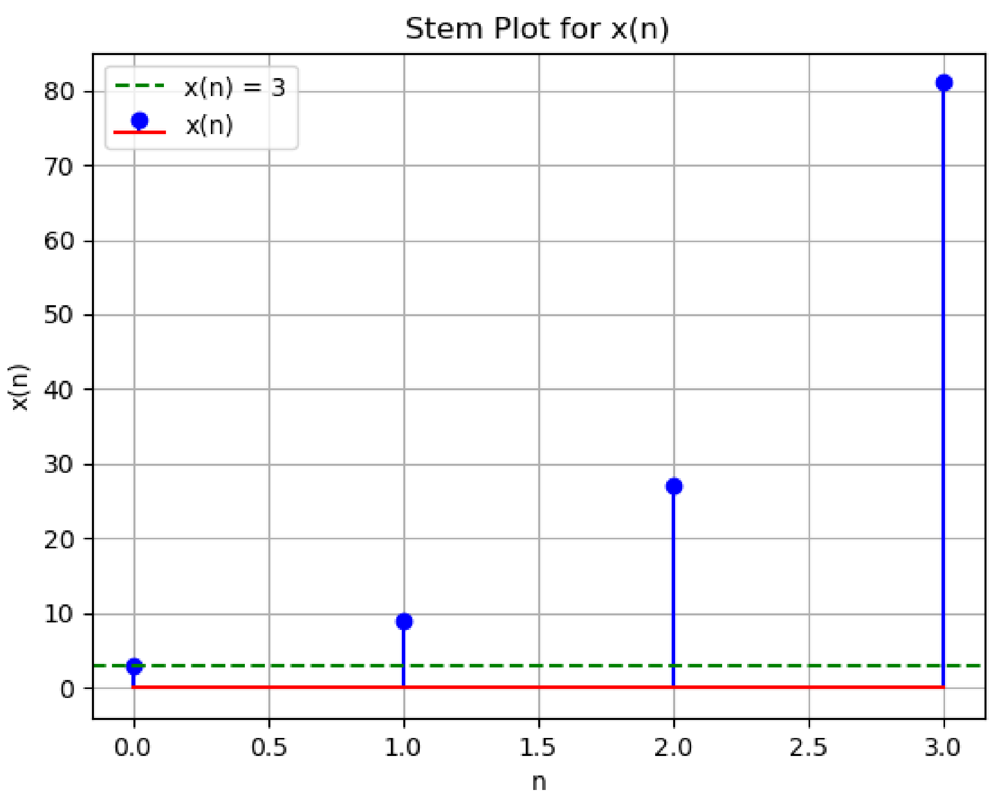
applying inverse z transform ;

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{4.475}$$

$$\implies 120 = 3\left(\frac{3^{n+1}-1}{3-1}\right) \tag{4.476}$$

$$n = 3 \tag{4.477}$$

Ans . n take values from $n = 0$ to $n = 3$, so there are total four terms .



Stem Plot for x(n)
x(n) = 3
x(n)
80
70
60
50
40
30
20
10
0
x(n)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
n

$x\left(0\right)$	3
d	2
m	6
n	2
$x\left(m+n\right)$	19
$x\left(m-n\right)$	11
$x\left(m\right)$	15

Table 4.12: Verified Values

Parameter	**Value**	**Description**
$x(0)$		First term
r		Common ratio
$x(0)^3r^3$	1	Product of terms
$x(0)+x(0)r+x(0)r^2$	$\frac{39}{10}$	Sum of terms

Table 4.13: Input Parameters

Parameter	Value	Description
$x(n)$	$(x(0) + n \cdot d)\, u(n)$	$(n + 1)$th term
d	3	common difference
$x(0) + x(1) + x(2)$	24	sum of the terms
$x(0) \cdot x(1) \cdot x(2)$	440	product of terms

Table 4.14: Parameters

Parameter	**Value**	**Description**
$x(0)$	5	First term
r	2	Common ratio
$y(n)$	315	Sum of $n + 1$ terms
$x(n)$	?	Last term

Table 4.15: Input Parameters

Symbol	**Value**	**Parameter**
$x(0)$	1	First Term
$x(n)$	$(5n + 1)u(n)$	kth Term
d	5	Common Difference

Table 4.16: Given Parameters

Table 4.17: Input Parameters

Parameter	**Used to denote**	**Values**
$x(0)$	First three digit number divisible by 7	105
$x(k-1)$	Last three digit number divisible by 7	?
d	Common difference of A.P	7
k	Number of 3 digit terms divisible by 7	?

Parameter	Description	Value
$x(0)$	First Term	4
r	Common Ratio	4
n	Total terms	8
$s(n)$	Sum of n terms of GP	$r^n x(0)u(n)$
m	No of poles	2

Table 4.18: Given Parameters

Symbol	Value	Description	Z-Transform
$x_1(n)$	$\{a, b, c...\}$	A.P Sequence	$X_1(z)$
$x_2(n)$	$\{b, c, d...\}$	G.P Sequence	$X_2(z)$
$x_3(n)$	$\{\frac{1}{c}, \frac{1}{d}, \frac{1}{e}...\}$	A.P Sequence	$X_3(z)$
$y(n)$	$\{a, c, e...\}$	Sequence	$Y(z)$

Table 4.19: Parameters

Symbol	Value	Description
$x\,(0)$	-32	First term
$x\,(10)$	38	11th term
$x\,(15)$	73	16th term
d	7	Common Difference
$x\,(n)$	$x(0) + nd$	$(n + 1)$th term

Table 4.20: Given Values

parameter	value	description
$x(0)$	$a\left(\frac{1}{b} + \frac{1}{c}\right)$	First Term of given AP
d	(b - a)$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$	Common Difference of given AP
$x(n)$	$(x(0) + nd)u(n)$	General Term of given AP

Table 4.21: Input Parameter Table

Symbol	Values	Description
$x(0)$	a	First term of A.P
$x(1)$	$\frac{a+b}{2}$	A.M of first and third terms of A.P
$x(2)$	b	Third term of A.P

Table 4.22: parameters

Variable	Description	Value
$x(n)$	n^{th} term of AP	none
d	common difference between the terms of AP	none
$x(17) - x(10)$	difference of 17^{th} and 10^{th} term of X	7

Table 4.23: input parameters

Variable	**Description**	**Value**
$x(n)$	n^{th} term of GP	none
$x(0)$	First term of GP	none
d	common ratio between the terms of GP	none
$x(p)$	a	$x(0)d^p$
$x(q)$	b	$x(0)d^q$
$x(r)$	c	$x(0)d^r$

Table 4.24: input parameters

Parameter	**Value**	**Description**
$x(n)$	$(-1)^n 5^{n+2}$	General Term
$x(0)$	25	First term of G.P.
r	-5	Common ratio of G.P.
$X(z)$	-	Z-Transform

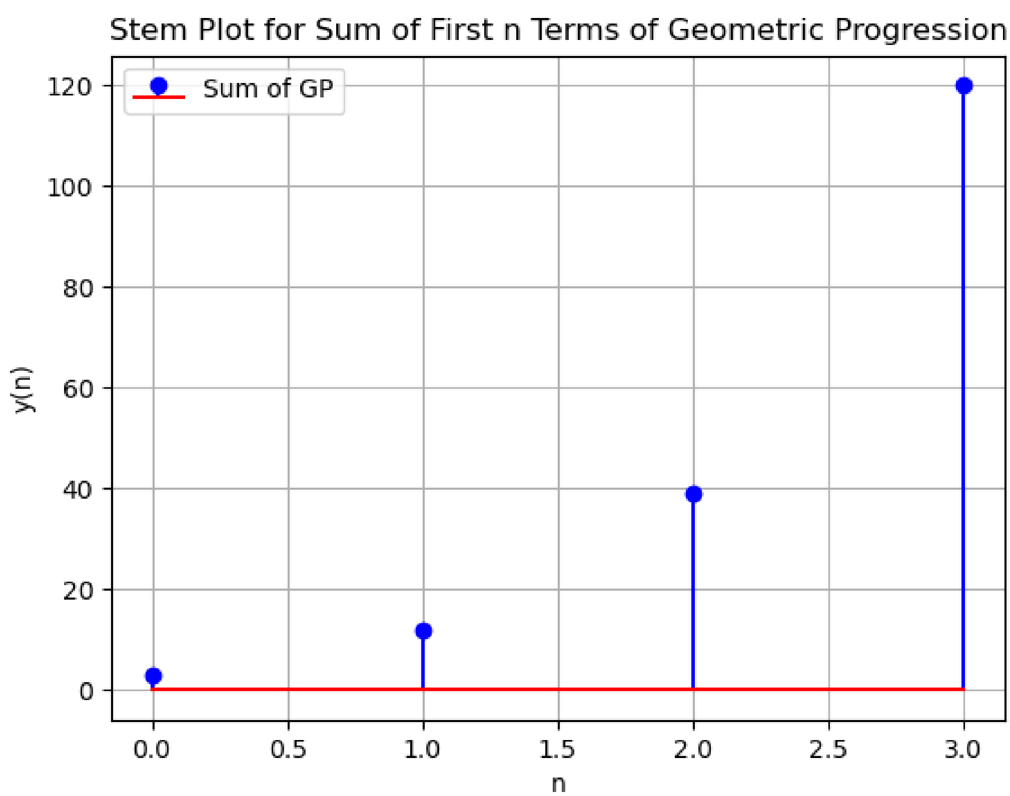
Table 4.25: Given Parameters

Parameter	**Value**	**Description**
$x_1(n)$	-	G.P. Sequence
$x_1(0)$	a	First term of G.P.
$x_1(1)$	b	Second term of G.P.
$x_1(2)$	c	Third term of G.P.
$x_1(3)$	d	Fourth term of G.P.
r	$\frac{b}{a}$	Common ratio

Table 4.26: Given Parameters

parameter	description	value
$x\left(0\right)$	first term	a
$x\left(1\right)$	second term	b
$x\left(2\right)$	third term	c
$x\left(3\right)$	fourth term	d
r	common ratio	$\frac{b}{a}$
n	no of terms	4
$x\left(n\right)$	$n/^{th}$ term	$x\left(0\right)r^{n}$

Table 4.27: input parameters



Symbol	Value	Description
$x(n)$	$(x(0)+nd)u(n)$	n^{th} term of an A.P
$x(0)$	$x(0)$	1^{st} term of the A.P
d	d	Common difference
$y(n)$	$x(n)*u(n)$	Sum of n terms of an AP
a	$y(p-1)$	Sum of first p terms of the AP
b	$y(q-1)$	Sum of first q terms of the AP
c	$y(r-1)$	Sum of first r terms of the AP

Table 4.28: Variable description

$x(0)$	5
d	2
p	8
q	10
r	4
a	96
b	140
c	32

Table 4.29: Verified Values

Term	**Value**	**Description**
$x\left(0\right)$	-	First term
d	-	Common Difference
$x\left(n\right)$	$\left(x\left(0\right)+nd\right)u\left(n\right)$	General term
$x\left(p\right)$	a	pth term
$x\left(q\right)$	b	qth term
$x\left(r\right)$	c	rth term

Table 4.30: Input Parameters

variable	value	description
$x(0)$	$\sqrt{7}$	first term of the geometric progession
r	$\sqrt{3}$	common ratio of the geometeric progression
$x(n)$	$\sqrt{7(3^n)}u\left(n\right)$	n^{th} term of the geometric progession
$y(n)$	$\frac{x(0)(r^{n+1}-1)}{r-1}u\left(n\right)$	Sum of the n term of the geometric progression

Table 4.31: Input parameters

Parameter	**Description**	**Value**
$x(0)$	First term	2
$x(19)$	20^{th} term	-112
$y(n)$	sum upto n^{th} term	

Table 1: Input data

Variable	**Description**
x(0)	First term of AP
d	common difference in the AP
n	number of terms in AP
y(n)	sum of n terms of the AP

Table 4.33: Variables Used

Input Parameters	Values	Description
a		First term of 1^{st} G.P.
r		Common ratio of 1^{st} G.P.
$x_1(n)$	$x_1(n) = ar^n u(n)$	General term of 1^{st} G.P.
$X_1(z)$		z-Transform of 1^{st} G.P.
A		First term of 2^{nd} G.P.
R		Common ratio of 2^{nd} G.P.
$x_2(n)$	$x_1(n) = AR^n u(n)$	General term of 2^{nd} G.P.
$X_2(z)$		z-Transform of 2^{nd} G.P.

Table 4.34: Parameters

Parameter	Value	Description
$x(0)$	?	First term of AP
d	?	Common difference
$x(3) + x(7)$	24	Sum of 4th , 8th term
$x(5) + x(9)$	44	Sum of 6th , 10th term
$x(n)$	$(x(0) + nd)\, u(n)$	General term

Table 4.35: Input parameters table

Parameter	Value	Description
$x(n)$	$(x(0) + nd)u(n)$	general term
$x(0)$	100	first term
d	5	Common difference
$x(29)$	$(x(0) + 29d)\, u(n)$	30^{th}term

Table 4.36: Parameters

Symbol	Parameters	value
$u(n)$	unit step function	1, if n$\geq$ 0; 0 otherwise
$x(n)$	general term of the series	$(n+1)(n+3)\, u(n)$
$X(z)$	Z-transform of $x(n)$	?

Table 4.37: Input Parameters

Parameter	Description	Value
x_1, x_2	Roots of a quadratic equation	?
$\frac{x_1+x_2}{2}$	A.M. of roots	8
$\sqrt{x_1 \cdot x_2}$	G.M. of roots	5

Table 4.38: Input Parameters

$x(0)$	5	1st term
$x(n)$	45	$(n+1)$th term
$y(n)$	400	sum of $(n+1)$ terms
n+1	?	no.of terms
d	?	common difference

Table 4.40: parameters

Parameters	Value	Description
$x(0)$	3	Initial Term
d	5	Common Difference
$x(k)$	78	Target Term
k	?	Target Term Number
$x(n)$	$x(0) + (n)d$	General term

Table 4.41: Parameters for the Arithmetic Progression

Parameter	**Description**	**Value**
$x(0)$	first number	
r	common ratio	
$x(2)$	second number	$x(0)r^2$
$x(1)$	G.M	$x(0)r$
$x(n)$	$(n+1)^{th}$term	$(x(0)r^n)u(n)$

Table 4.42: Input table

Parameter	**Value**	**Description**
$x(0)$	a	First term
$x(2)$	b	Third term
$x(1)$	$\sqrt{ab} = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$	Second term
r	$\sqrt{\frac{b}{a}}$	Common ratio
n	-	Given variable
$x(k)$	$ar^k u(k)$	General term

Table 4.43: Input parameters table

Parameter	Description	Value
n	No. of terms in the G.P	4
$x(0)$	first term in the G.P	3
r	common ratio in the G.P	3
$x(n)$	n^{th} term in G.P	$x(0)r^n u(n)$

Table 4.44: variables

Variable	**Description**	**Value**
$x(1)$	Second term of AP	14
$x(2)$	Third term of AP	18
$x(0)$	First term of AP	$2x(1) - x(2) = 10$
d	Common difference of AP $(x(2) - x(1))$	4
$x(n)$	n^{th} term of sequence	$(4n + 10)u(n)$

Table 4.45: input parameters

i	$x_i(0)$	d_i	n_i	$a_i(n)$
1	7	3	7 (8^{th} term)	...
2	-18	...	9 (10^{th} term)	0
3	...	-3	17 (18^{th} term)	-5
4	-18.9	2.5	...	3.6
5	3.5	0	104 (105^{th} term)	...

Symbol	Value	Description
$x(0)$	3	first term
r	3	common ratio
$y\left(n\right)$	120	sum of all n terms
$x(n)$	$x\left(0\right)r^{n}u\left(n\right)$	$n+1^{th}$ term

Table 4.46:

Chapter 5

Contour Integration

5.1 In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato,and the other potatoes are placed 3 m apart in a straight line. There are ten potatoes in the line. A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?

(NCERT-Maths 10.5.3.20Q)

Solution:

Parameter	**Description**	**Value**
$x(0)$	First term	10
d	Common Difference	6
$y(9)$	Total distance covered	?

Table 1: Parameter Table

From (**??**) :

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.1}$$

$$= \frac{10}{1-z^{-1}} + \frac{6z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.2}$$

$$= \frac{10-4z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{5.3}$$

From (**??**)

$$y(n) = \sum_{k=0}^{n} x(k) = x(n) * u(n) \tag{5.4}$$

Taking z transform :

$$Y(z) = X(z)U(z) \tag{5.5}$$

$$\implies Y(z) = \frac{10-4z^{-1}}{\left(1-z^{-1}\right)^3} \quad |z| > 1 \tag{5.6}$$

Taking inverse z transform :

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{n-1} dz \tag{5.7}$$

$$y(9) = \frac{1}{2\pi j} \oint_C Y(z)\ z^8\ dz \tag{5.8}$$

$$= \frac{1}{2\pi j} \oint_C \frac{10z^{11} - 4z^{10}}{(z-1)^3}\ dz \tag{5.9}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left(f(z) \right) \tag{5.10}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(10z^{11} - 4z^{10} \right) \tag{5.11}$$

$$= 370 \tag{5.12}$$

$$\therefore y(9) = 370 \tag{5.13}$$

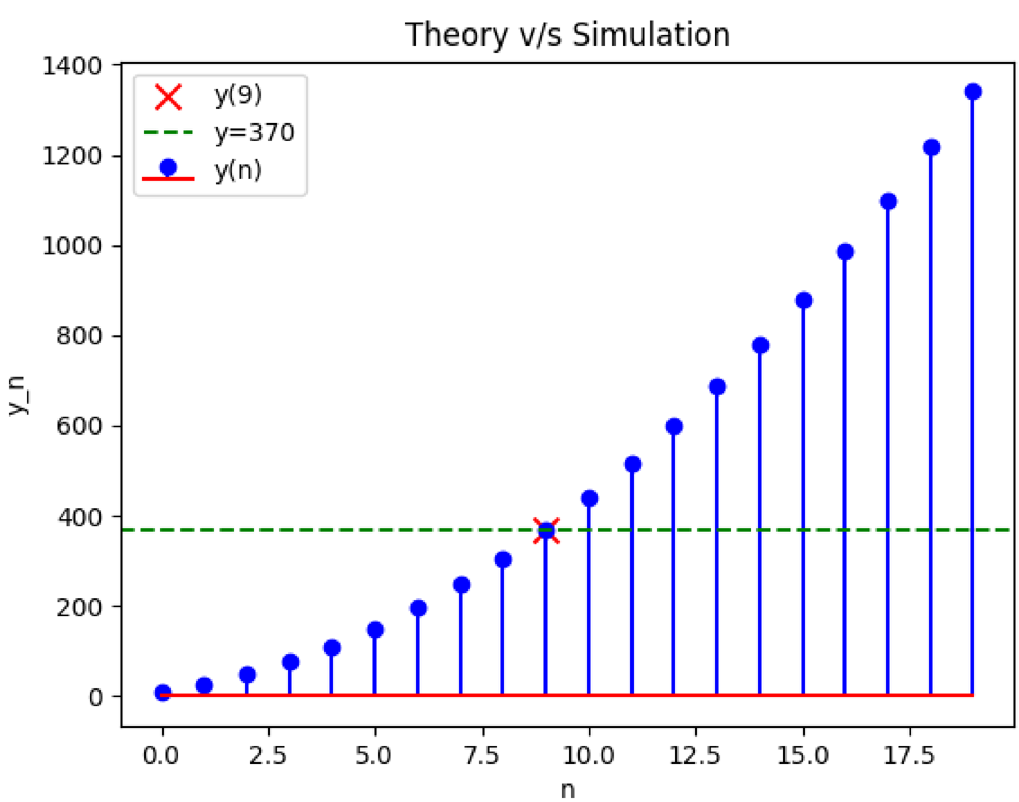


Figure 5.1: Theory matches with the simulated values

5.2 Find the sum of the first 15 multiples of 8.

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
d	8	common difference
$x(n)$	$[8+8n]u(n)$	General term of the series

Table 5.2: Parameter Table1

For an AP,

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{5.14}$$

$$\implies X(z) = \frac{8}{1-z^{-1}} + \frac{8z^{-1}}{(1-z^{-1})^2} \tag{5.15}$$

$$= \frac{8}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.16}$$

$$y(n) = x(n) * u(n) \tag{5.17}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.18}$$

$$Y(z) = \left(\frac{8}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{5.19}$$

$$= \frac{8}{(1-z^{-1})^3}, \quad |z| > 1 \tag{5.20}$$

Using Contour Integration to find the inverse Z-transform,

$$y(14) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{13}\, dz \tag{5.21}$$

$$= \frac{1}{2\pi j}\oint_C \frac{8z^{13}}{(1-z^{-1})^3}\, dz \tag{5.22}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.23}$$

$$= \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{8z^{16}}{(z-1)^3}\right) \tag{5.24}$$

$$= 4\lim_{z\to 1}\frac{d^2}{dz^2}(z^{16}) \tag{5.25}$$

$$= 960 \tag{5.26}$$

$$\therefore \boxed{y(14) = 960} \tag{5.27}$$

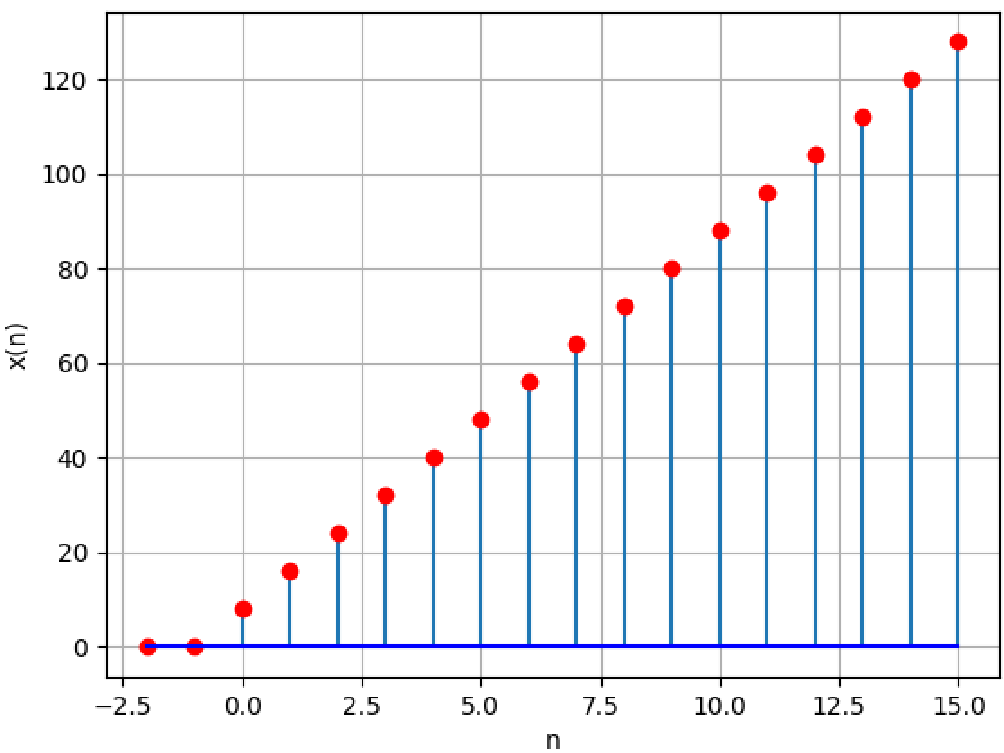


Figure 5.2: Plot of x(n) *vs* n

ss

5.3 If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of m.

Solution:

$$Y(z) = \sum_{n=0}^{\infty} y(n) z^{-n} \tag{5.28}$$

$$= \frac{2\left(4 - z^{-1}\right)}{\left(1 - z^{-1}\right)^3}, \qquad |z| > 1 \tag{5.29}$$

$$U(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1 \tag{5.30}$$

$$X(z) = \frac{Y(z)}{U(z)} \tag{5.31}$$

$$= 2\left(\frac{1}{1 - z^{-1}}\right) + 6\left(\frac{1}{\left(1 - z^{-1}\right)^2}\right) \tag{5.32}$$

$$= \frac{8z^2 - 2z}{(z-1)^2} \tag{5.33}$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{5.34}$$

$$= \frac{1}{2\pi j} \oint_C \frac{\left(8z^{n+1} - 2z^n\right) dz}{(z-1)^2} \tag{5.35}$$

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.36}$$

$$= \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{8z^{n+1} - 2z^n}{(z-1)^2}\right) \tag{5.37}$$

$$= \lim_{z \to 1} \left(8(n+1) z^n - 2nz^{n-1}\right) \tag{5.38}$$

$$\implies x(n) = (6n + 8)(u(n)) \tag{5.39}$$

$$164 = (6m + 8)(u(m)) \tag{5.40}$$

$$\implies m = 26 \tag{5.41}$$

Symbol	Remarks
$y(n) = \left(3n^2 + 11n + 8\right)(u(n))$	Sum of n terms
$x(m-1)$	164
$y(n)$	$x(n) * u(n)$

Table 5.3: Parameters

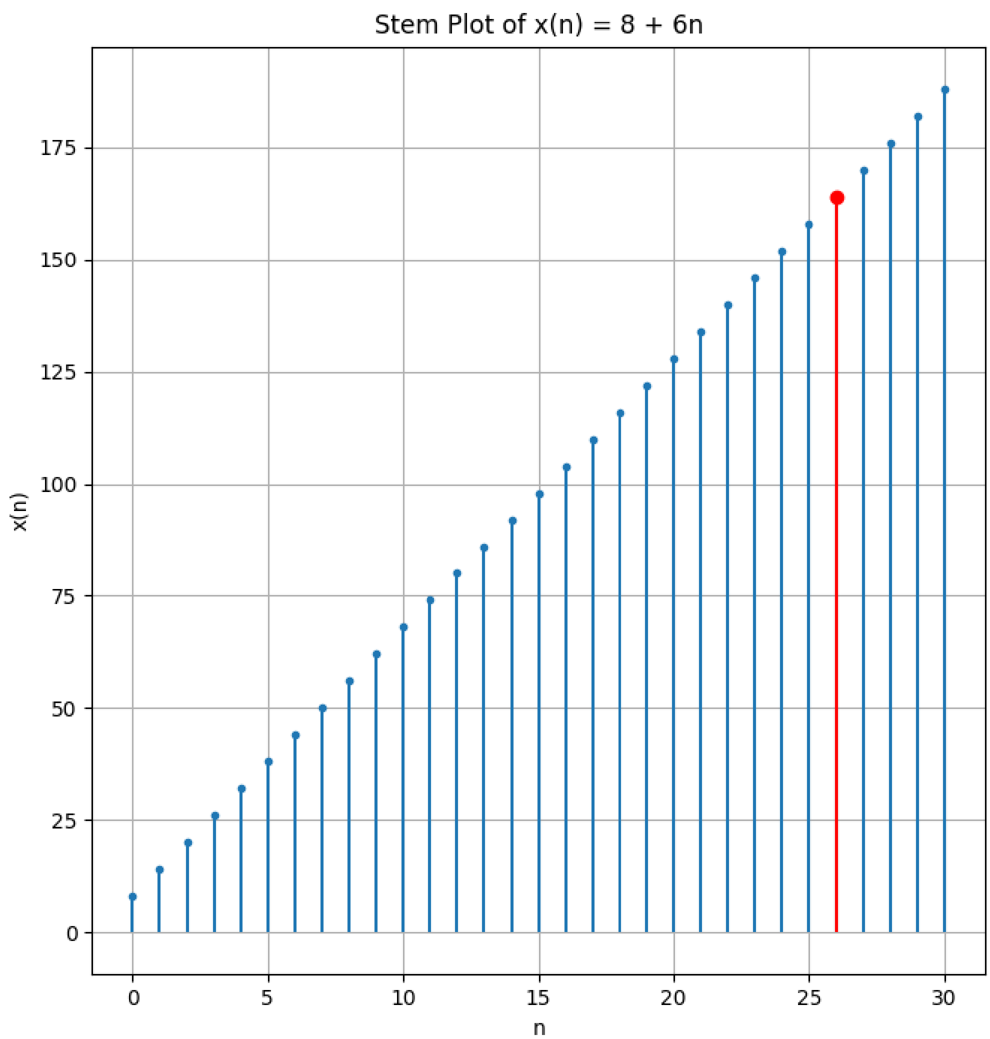


Figure 5.3: Plot of x(n) vs n

5.4 Find the sums given below:

(a) $7 + \frac{21}{2} + 14... + 84$

(b) $34 + 32 + 30... + 10$

(c) $-5 + -8 + -11... - 230$

Solution:

Symbols	Description	Values
d_i	Common Difference for i^{th} AP	3.5
		-2
		-3
$x_i(n)$	n^{th} term for i^{th} Sequence	$(7 + \frac{7n}{2})u_{(n)}$
		$(34$ - $2n)u_{(n)}$
		$(-5 + -3n)u_{(n)}$
$x_i(0)$	First term for i^{th} AP	7
		34
		-5

Table 5.4: Parameters , Descriptions And Values

(a) $7 + \frac{21}{2} + 14... + 84$

$$x_1(n) = (x_1(0) + nd_1)\, u_{(n)} \tag{5.42}$$

$$\implies 84 = 7 + \frac{7n}{2} \tag{5.43}$$

$$\implies n = 22 \tag{5.44}$$

i. z-Transform of $x_1(n)$: Using (**??**)

$$X_1(z) = \frac{7z}{z-1} + \frac{7z}{2(z-1)^2}, \quad |z| > |1| \tag{5.45}$$

ii. Z-Transform of $y_1(n)$:

$$y_1(n) = x_1(n) * h(n) \tag{5.46}$$

$$h(n) = u(n) \tag{5.47}$$

$$H(z) = \frac{z}{z-1} \tag{5.48}$$

$$Y_1(z) = X_1(z) * H(z) \tag{5.49}$$

$$= \left(\frac{7z}{z-1} + \frac{7z}{2(z-1)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.50}$$

iii. Inversion of $Y_1(z)$: Using Contour Integration :

$$y_1(22) = \frac{1}{2\pi j}\oint_C Y(z)\ z^{21}\ dz \tag{5.51}$$

$$\implies = \frac{1}{2\pi j}\oint_C \left(\frac{7z^{23}}{(z-1)^2} + \frac{7z^{23}}{2(z-1)^3}\right)\ dz \tag{5.52}$$

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.53}$$

For R_1 , $m = 2$, where m corresponds to number of repeated poles .

$$R_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{7z^{23}}{(z-1)^2}\right) \tag{5.54}$$

$$= 7\lim_{z\to 1}\frac{d}{dz}(z^{23}) \tag{5.55}$$

$$= 161 \tag{5.56}$$

For R_2 , $m = 3$

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{(7z^{13})}{2(z-1)^3} \right) \tag{5.57}$$

$$= \left(\frac{7}{4} \right) \lim_{z \to 1} \frac{d^2}{dz^2} (z^{23}) \tag{5.58}$$

$$= \frac{1771}{2} \tag{5.59}$$

$$R_1 + R_2 = \frac{2093}{2} \tag{5.60}$$

$$\implies y_1(22) = \frac{2093}{2} \tag{5.61}$$

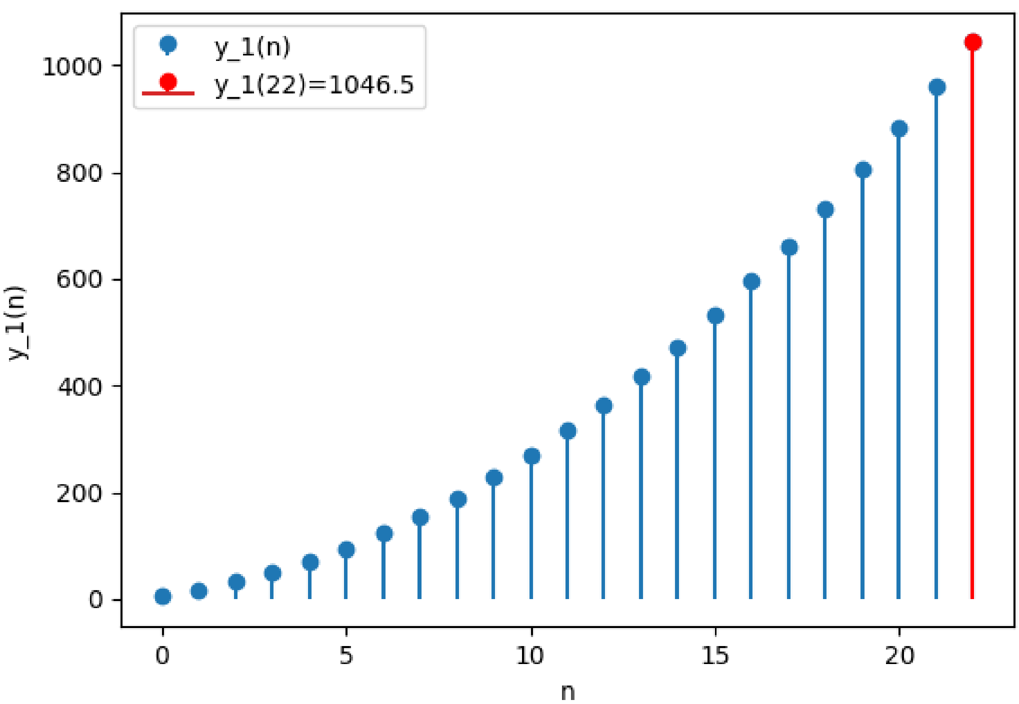


Figure 5.4: $y_1(n)$ vs n

(b) 34 + 32 + 30 ... + 10

$$x_2\left(n\right) = \left(x_2\left(0\right) + nd_2\right)u_{(n)} \tag{5.62}$$

$$\implies 10 = 34 - 2n \tag{5.63}$$

$$\implies n = 12 \tag{5.64}$$

i. Z-Transform of $x_2\left(n\right)$: Using (**??**)

$$X_2(z) = \frac{34z}{z-1} - \frac{2z}{(z-1)^2}, \quad |z| > |1| \tag{5.65}$$

ii. Z-Transform of $y_2\left(n\right)$:

$$y_2\left(n\right) = x_2\left(n\right) * h\left(n\right) \tag{5.66}$$

$$h\left(n\right) = u\left(n\right) \tag{5.67}$$

$$Y_2\left(z\right) = X_2\left(z\right) * H\left(z\right) \tag{5.68}$$

$$= \left(\frac{34z}{(z-1)^1} - \frac{2z}{(z-1)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.69}$$

iii. Inversion of $Y_2\left(z\right)$: Using Contour Integration :

$$y_2(12) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{11}\, dz \tag{5.70}$$

$$\implies = \frac{1}{2\pi j}\oint_C \left(\frac{34z^{13}}{(z-1)^2} - \frac{2z^{13}}{(z-1)^3}\right) dz \tag{5.71}$$

Using (**??**) For R_1 , $m = 2$:

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{34z^{13}}{(z-1)^2} \right) \tag{5.72}$$

$$= 34 \lim_{z \to 1} \frac{d}{dz} (z^{13}) \tag{5.73}$$

$$= 442 \tag{5.74}$$

For R_2 , $m = 3$:

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{(-2z^{13})}{(z-1)^3} \right) \tag{5.75}$$

$$= - \lim_{z \to 1} \frac{d^2}{dz^2} (z^{13}) \tag{5.76}$$

$$= -156 \tag{5.77}$$

$$R_1 + R_2 = 286 \tag{5.78}$$

$$\implies y_2(12) = 286 \tag{5.79}$$

(c) -5 + -8 + -11 ... -230

$$x_3(n) = (x_3(0) - 3n) u_{(n)} \tag{5.80}$$

$$\implies -230 = -5 - 3n \tag{5.81}$$

$$\implies n = 75 \tag{5.82}$$

i. Z-Transform of $x_3(n)$: Using (**??**)

$$X_3(z) = \frac{-5z}{(z-1)^1} - \frac{3z}{(z-1)^2}, \quad |z| > |1| \tag{5.83}$$

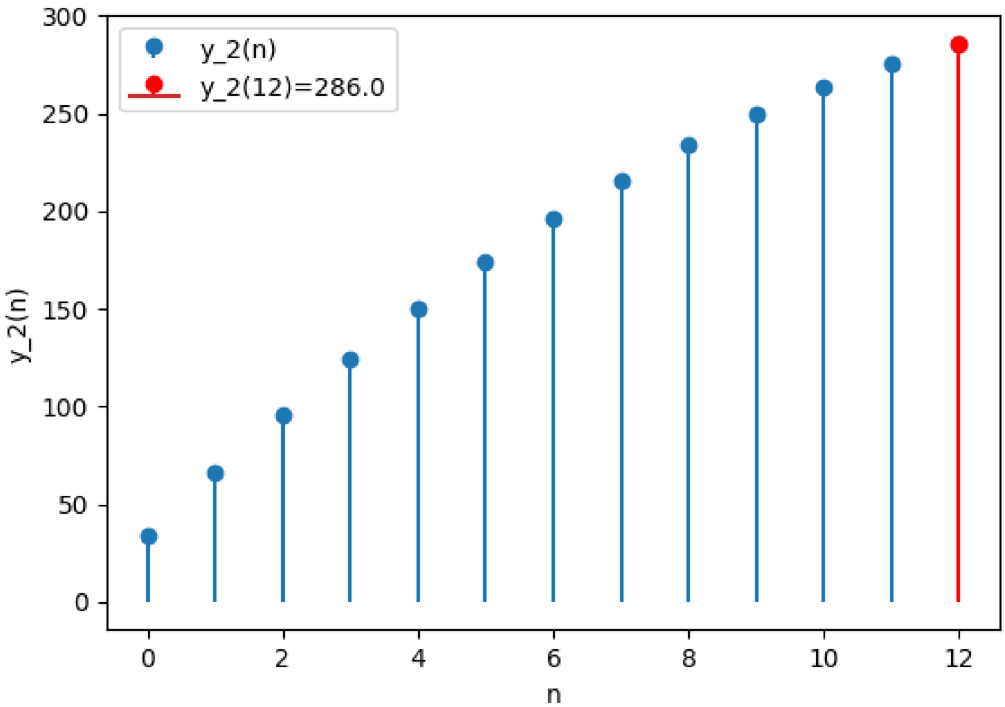


Figure 5.5: $y_2(n)$ vs n

ii. Z-Transform of $y_3(n)$:

$$y_3(n) = x_3(n) * h(n) \tag{5.84}$$

$$h(n) = u(n) \tag{5.85}$$

$$Y_3(z) = X_3(z) * H(z) \tag{5.86}$$

$$= \left(\frac{-5z}{(z-1)^1} - \frac{3z}{(z-1)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.87}$$

iii. Inversion of $Y_3\,(z)$: Using Contour Integration :

$$y_1(75) = \frac{1}{2\pi j} \oint_C Y(z)\; z^{74}\; dz \tag{5.88}$$

$$\implies \; = \frac{1}{2\pi j} \oint_C \left(\frac{-5z^{76}}{(z-1)^2} - \frac{3z^{76}}{(z-1)^3} \right)\; dz \tag{5.89}$$

Using (**??**) For R_1 , $m = 2$:

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{-5z^{76}}{(z-1)^2} \right) \tag{5.90}$$

$$= -5 \lim_{z \to 1} \frac{d}{dz}(z^{76}) \tag{5.91}$$

$$= -380 \tag{5.92}$$

For R_2 , $m = 3$:

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{3z^{76}}{(z-1)^3} \right) \tag{5.93}$$

$$= 1.5 \lim_{z \to 1} \frac{d^2}{dz^2}(z^{76}) \tag{5.94}$$

$$= -8550 \tag{5.95}$$

$$R_1 + R_2 = -8930 \tag{5.96}$$

$$\implies y_3(75) = -8930 \tag{5.97}$$

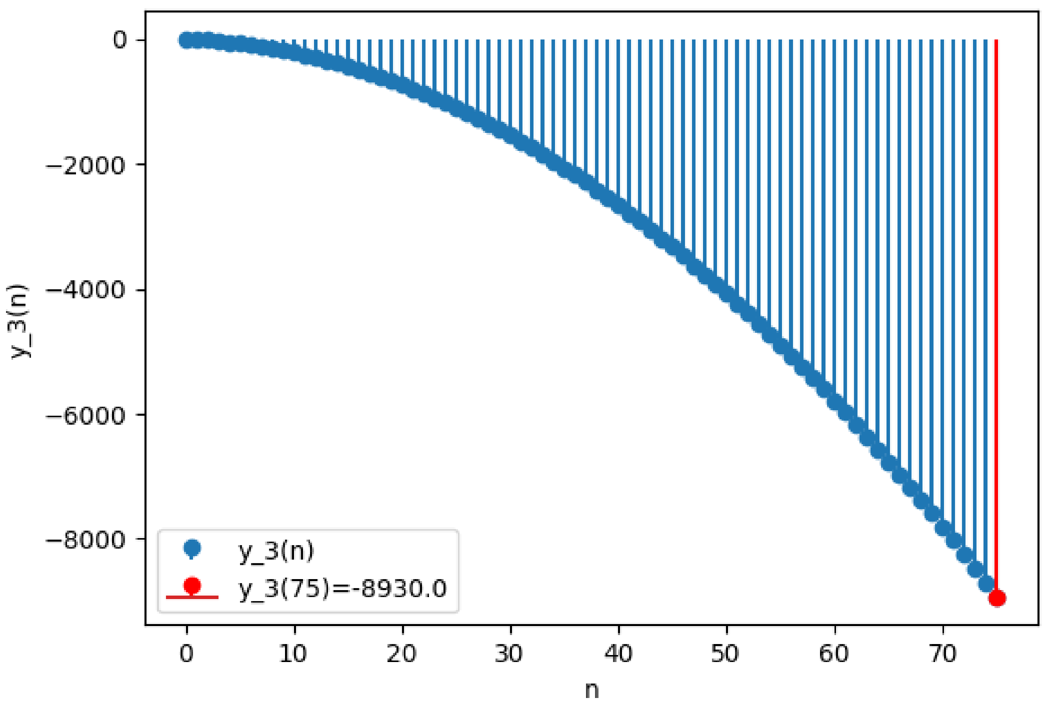


Figure 5.6: $y_3\left(n\right)$ vs n

5.5 Show that a_0 , a_1 , a_2 , . . ., a_n , . . . form an AP where an is defined as below :

(a) $a_n = (3 + 4n)$

(b) $a_n = (9 - 5n)$

Also find the sum of the first 15 terms in each case. **Solution:**

Parameter	Description	Value
$x_i(n)$	i^{th}Discrete signal	$(3+4n)u(n)$
		$(9-5n)u(n)$
$x_i(0)$	First term of i^{th}AP	3
		9
d_i	common difference of i^{th}AP	4
		-5

Table 5.5: Given parameters

(a) From equation (**??**)

$$X(z) = \frac{3}{1-z^{-1}} + \frac{4.z^{-1}}{(1-z^{-1})^2}; |z| > 1 \tag{5.98}$$

$$\because y(n) = x(n) * u(n) \tag{5.99}$$

$$Y(z) = X(z)U(z) \tag{5.100}$$

$$= \left[\frac{3}{(1-z^{-1})^2} + \frac{4z^{-1}}{(1-z^{-1})^3}\right] \tag{5.101}$$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{5.102}$$

$$= \frac{1}{2\pi j}\int \frac{3.z^{15}}{(z-1)^2}dz + \frac{1}{2\pi j}\int \frac{4.z^{15}}{(z-1)^3}dz \tag{5.103}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.104}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{3.z^{15}}{(z-1)^2}\right) \tag{5.105}$$

$$= 45 \tag{5.106}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{4.z^{15}}{(z-1)^3}\right) \tag{5.107}$$

$$= 420 \tag{5.108}$$

$$\implies y(14) = R_1 + R_2 \tag{5.109}$$

$$= 465 \tag{5.110}$$

(b) From equation (**??**)

$$X(z) = \frac{9}{1-z^{-1}} - \frac{5.z^{-1}}{(1-z^{-1})^2}; |z| > 1 \tag{5.111}$$

$$\because y(n) = x(n) * u(n) \tag{5.112}$$

$$Y(z) = X(z)U(z) \tag{5.113}$$

$$= \left[\frac{9}{(1-z^{-1})^2} - \frac{5z^{-1}}{(1-z^{-1})^3}\right] \tag{5.114}$$

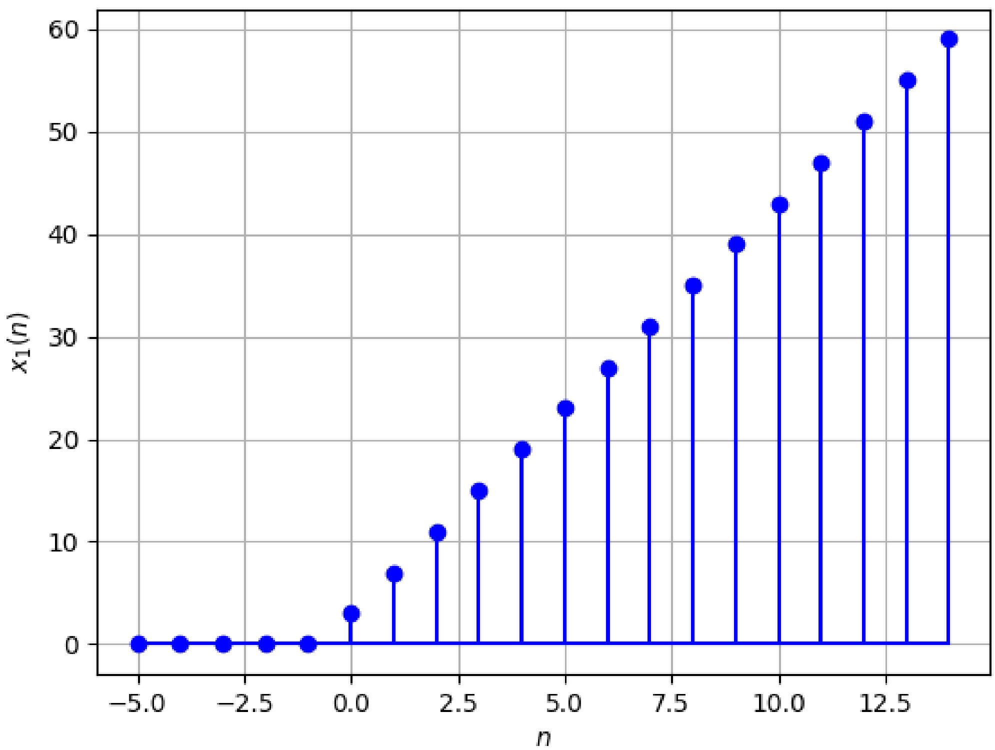


Figure 5.7: $x_1(n) = (3 + 4n)u(n)$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{5.115}$$

$$= \frac{1}{2\pi j}\int \frac{9.z^{15}}{(z-1)^2}dz - \frac{1}{2\pi j}\int \frac{5.z^{15}}{(z-1)^3}dz \tag{5.116}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.117}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{9.z^{15}}{(z-1)^2}\right) \tag{5.118}$$

$$= 135 \tag{5.119}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{5.z^{15}}{(z-1)^3}\right) \tag{5.120}$$

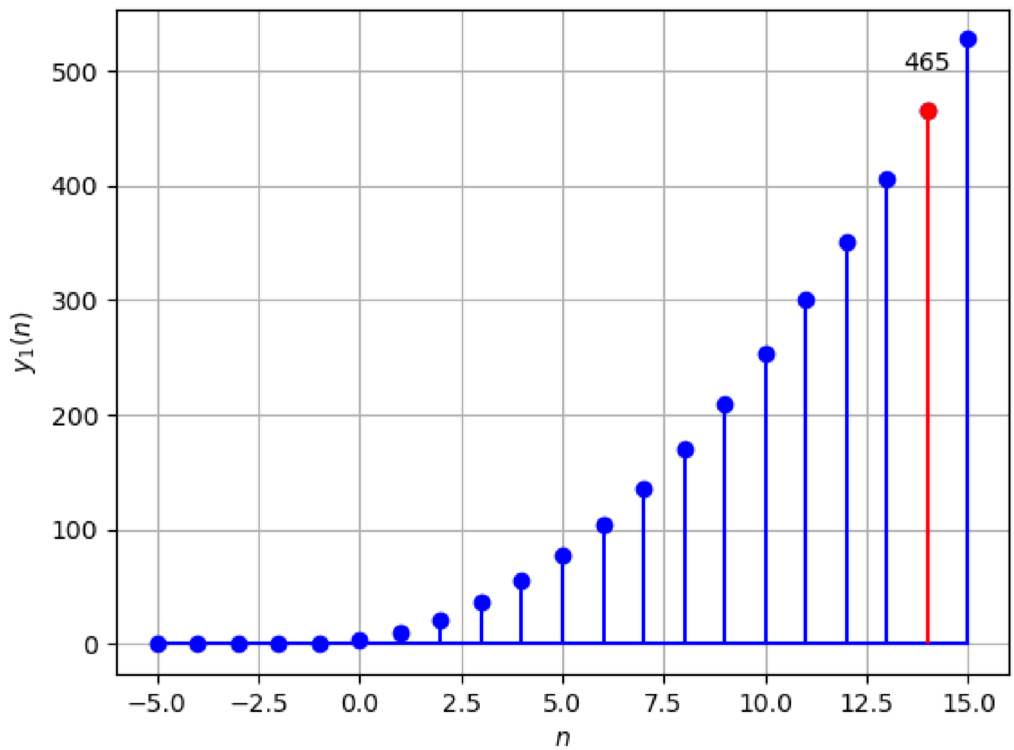


Figure 5.8: $x_1(n) = (2n^2 + 5n + 3)u(n)$

5.6 If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference. **Solution:**

Symbol	Value	Description
$y(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

Table 5.6: Given Parameters

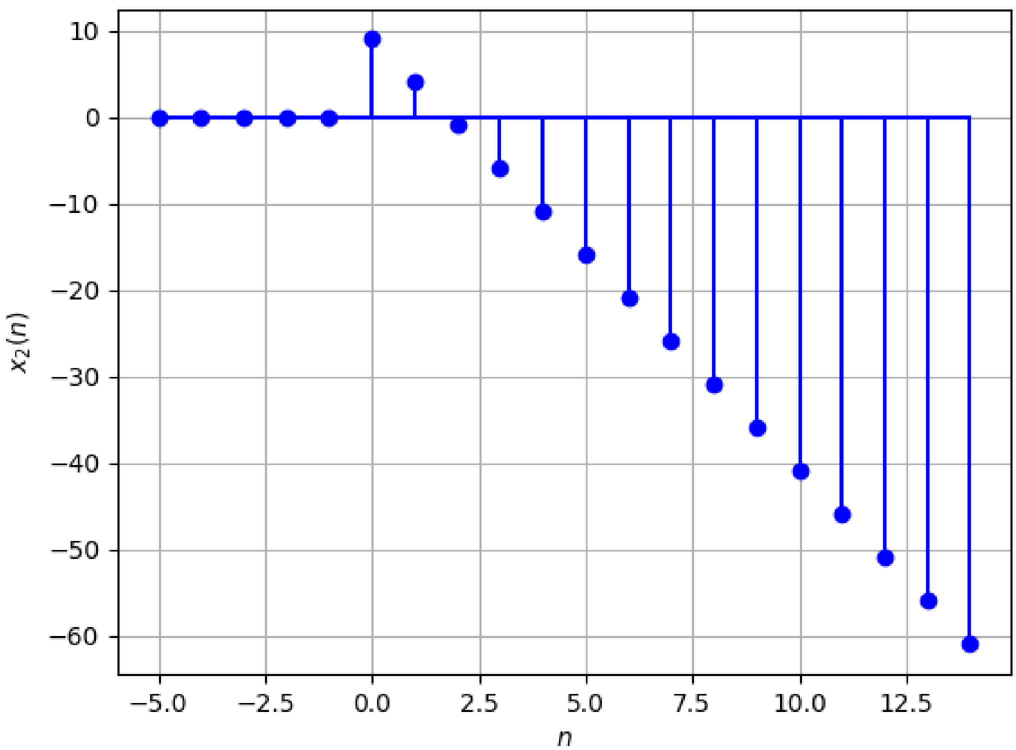


Figure 5.9: $x_2(n) = (9 - 5n)u(n)$

Sum of n terms, as a discrete signal:

$$y(n) = (pn + qn^2)u(n) \tag{5.124}$$

Taking the Z-Transform,

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}\{|z| > 1\} \tag{5.125}$$

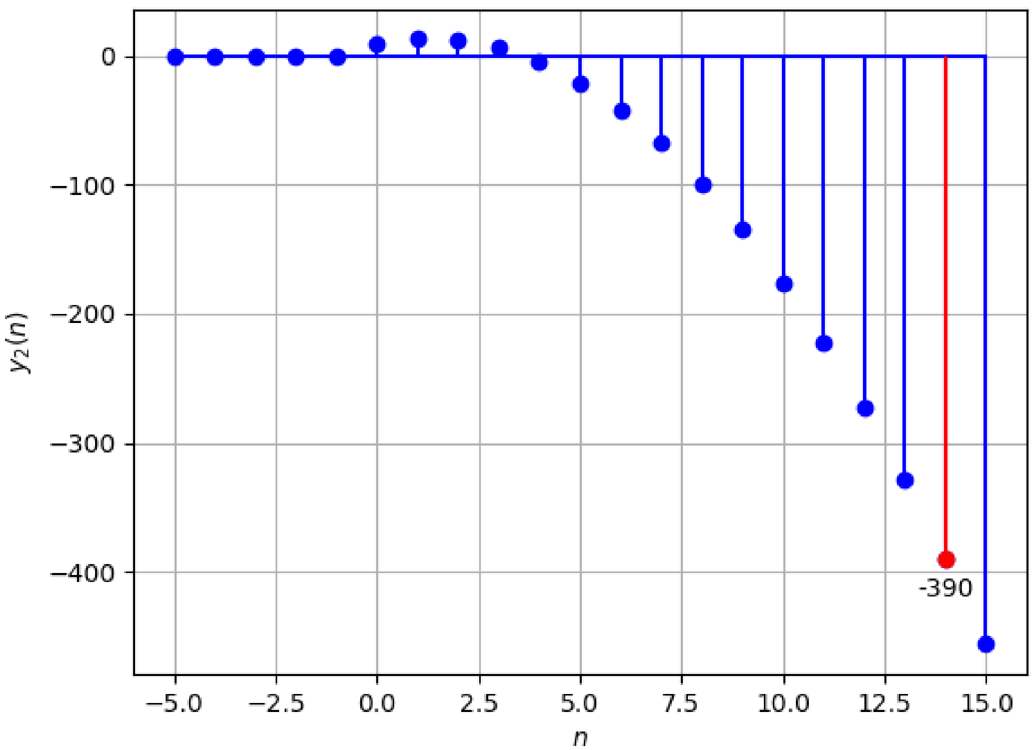


Figure 5.10: $x_2(n) = (-5n^2 + 13n + 18)u(n)$

(b) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \; \{|z| > 1\} \tag{5.126}$$

(c) $\mathcal{Z}\{n^2u(n)\}$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \; \{|z| > 1\} \tag{5.127}$$

Taking the Z-Transform of (**??**) using (**??**) and (**??**)

$$Y(z) = p\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}\right) \tag{5.128}$$

Now,

$$y(n) = x(n) * u(n) \tag{5.129}$$

$$\implies Y(z) = X(z)U(z) \tag{5.130}$$

$$\implies X(z) = \frac{Y(z)}{U(z)} \tag{5.131}$$

Using (**??**) in (**??**),

$$X(z) = p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right) \tag{5.132}$$

Using contour integration for inverse Z-Transform:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \tag{5.133}$$

$$= \frac{1}{2\pi j}\oint_C \left[p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right)\right] z^{n-1}dz \tag{5.134}$$

Calculating the residues R_1 and R_2 at pole $z = 1$:

$$R_1 = \frac{1}{0!} \lim_{z \to 1} (z-1) \left(p \left(\frac{z^{-1}}{1 - z^{-1}} \right) \right) z^{n-1} \tag{5.135}$$

$$= p \tag{5.136}$$

$$R_2 = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 q \left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \right) z^{n-1} \tag{5.137}$$

$$= q \lim_{z \to 1} \frac{d}{dz} \left(z^n + z^{n-1} \right) \tag{5.138}$$

$$= q(2n - 1) \tag{5.139}$$

$$\implies x(n) = R_1 + R_2 \tag{5.140}$$

$$= p + q(2n - 1) \tag{5.141}$$

Writing x(n) as a discrete signal we get:

$$x(n) = (p - q)u(n) + 2qnu(n) \tag{5.142}$$

To simplify, use $n = 0$:

$$y(0) = x(0) \tag{5.143}$$

$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{5.144}$$

$$\implies p = q \tag{5.145}$$

$\therefore$ (**??**) an be written as:

$$x(n) = 2qnu(n) \tag{5.146}$$

Common difference is given by:

$$d = x(n+1) - x(n) \tag{5.147}$$

$$= 2q \tag{5.148}$$

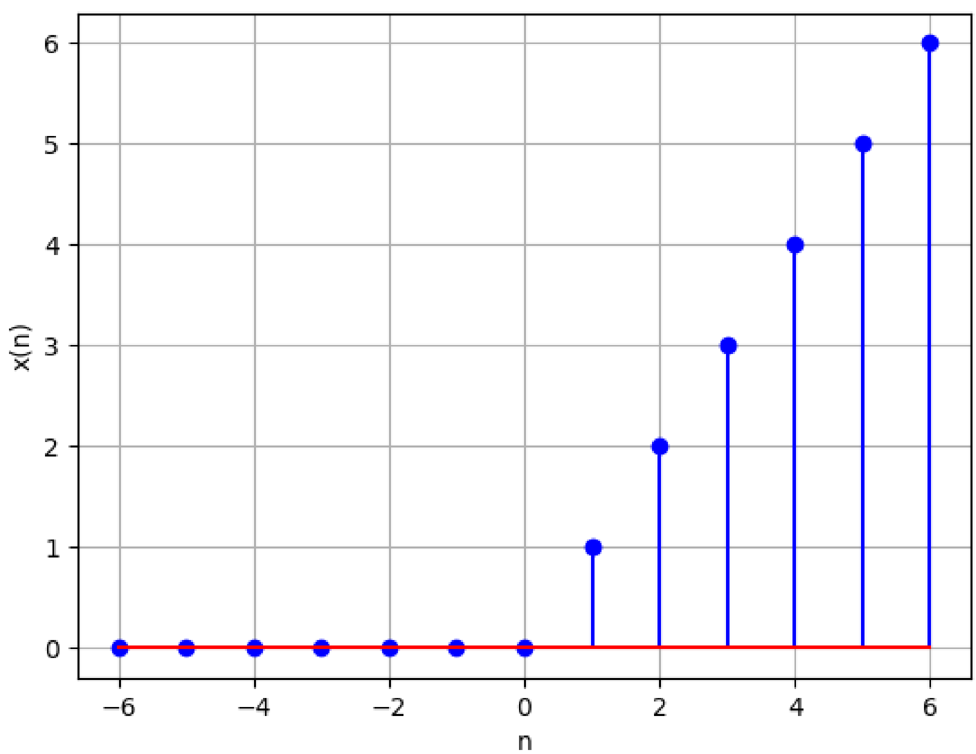


Figure 5.11: Plot of x(n) vs n for $p = q = 0.5$

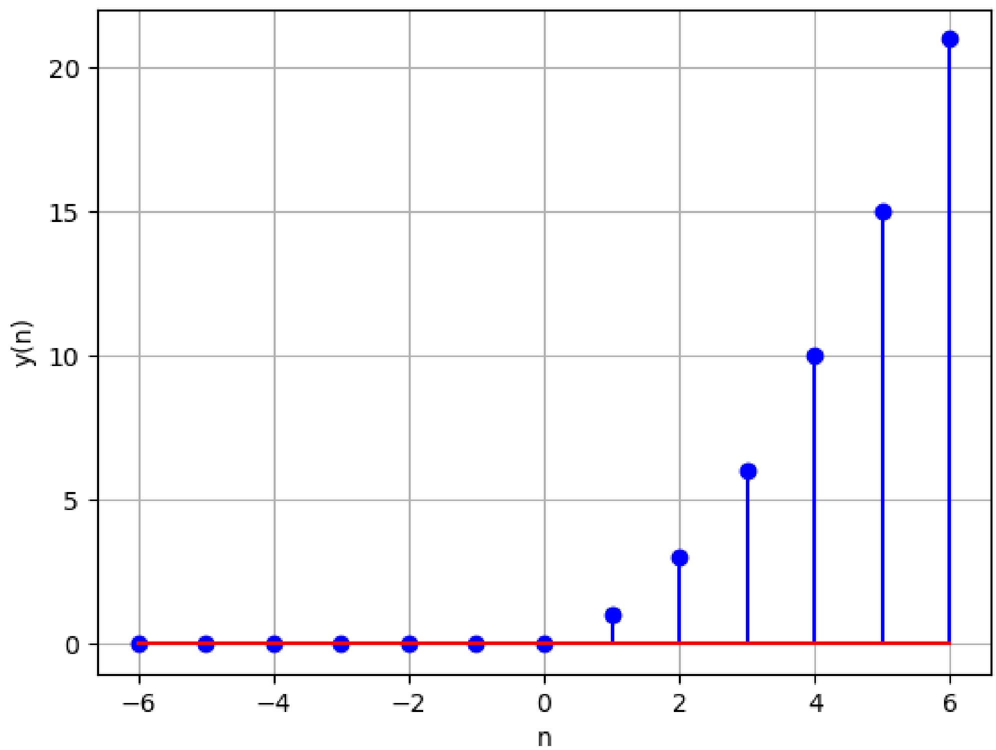


Figure 5.12: Plot of y(n) vs n for $p = q = 0.5$

5.7 Find the sum of the first 40 positive integers divisible by 6

Solution:

Parameter	Description	Value
x(0)	First Term	6
d	Common Difference	6

Table 5.7: Parameter Table 10.5.3.12

$$x(n) = (6+6n)(u(n)) \tag{5.149}$$

$$\implies X(z) = \frac{6}{1-z^{-1}} + \frac{6z^{-1}}{(1-z^{-1})^2} \quad (\textbf{??}) \tag{5.150}$$

$$\implies X(z) = \frac{6}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.151}$$

$$y(n) = x(n) * u(n) \tag{5.152}$$

$$\implies Y(z) = X(z)U(z) \tag{5.153}$$

$$= \frac{6}{(1-z^{-1})^3}, \quad |z| > 1 \tag{5.154}$$

Using contour integration to find the inverse Z-transform:

$$\implies y(39) = \frac{1}{2\pi j}\oint_C Y(z)\ z^{38}\ dz \tag{5.155}$$

$$= \frac{1}{2\pi j}\oint_C \frac{6z^{41}}{(z-1)^3}\ dz \tag{5.156}$$

We can observe that there is only a three times repeated pole at z=1,

$$\implies R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.157}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{6z^{41}}{(z-1)^3} \right) \tag{5.158}$$

$$= 3 \lim_{z \to 1} \frac{d^2}{dz^2} \left(z^{41} \right) \tag{5.159}$$

$$= 4920 \tag{5.160}$$

$$\therefore \boxed{y\,(39) = 4920} \tag{5.161}$$

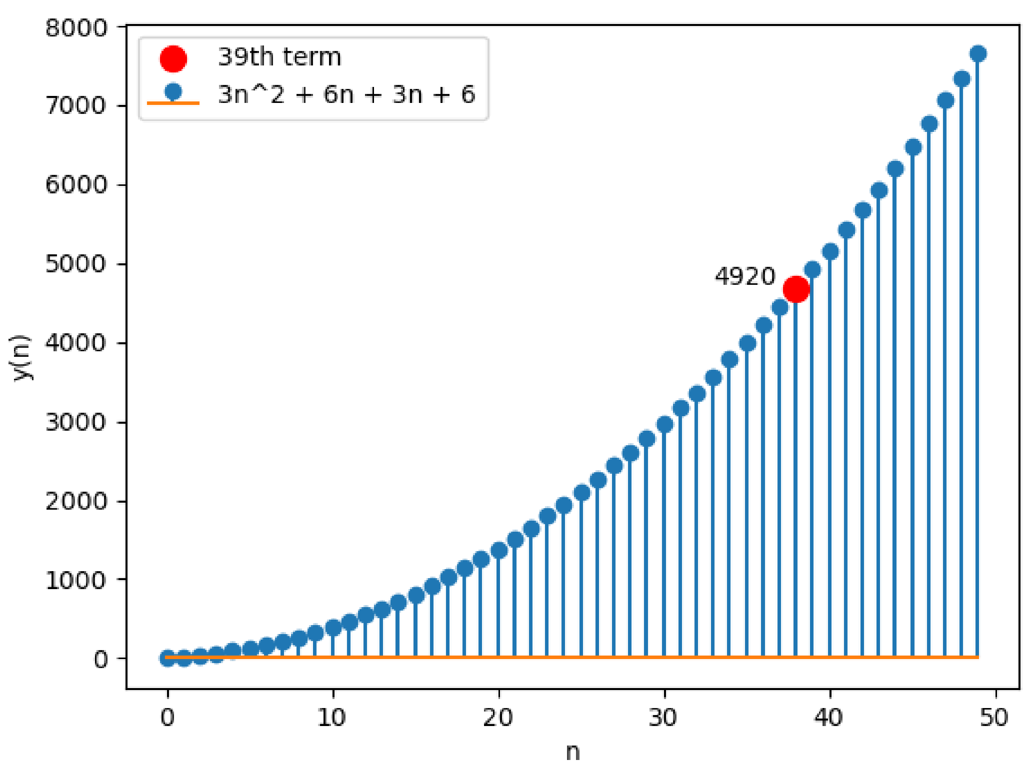


Figure 5.13: Plot of y(n) vs n

5.8 If the sum of certain number of terms in a AP 25,22,19,... is 116. Find the last term.

Solution:

Symbol	**Value**	**Description**
$x(0)$	25	first term of AP
d	-3	common difference
$x(n)$	$(25-3n)u(n)$	n-th term of AP
$y(n)$	116	sum of terms

Table 5.8: Input Parameters

$$x(n) = (25-3n)u(n) \tag{5.162}$$

Applying Z transform:

$$x(z) = \frac{25}{1-z^{-1}} - \frac{3z^{-1}}{(1-z^{-1})^2} \tag{5.163}$$

$$= \frac{25-28z^{-1}}{(1-z^{-1})^2} \tag{5.164}$$

Region of Convergence or R.O.C :

$$|z| > 1 \tag{5.165}$$

For AP, the sum of first n+1 terms can be written as :

$$y(n) = x(n) * u(n) \tag{5.166}$$

Applying Z transform on both sides

$$Y(z) = X(z)U(z) \tag{5.167}$$

$$= \frac{25}{(1-z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})^3} \tag{5.168}$$

Using contour integration to find inverse Z transform:

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z) z^{n-1} dz \tag{5.169}$$

$$= \frac{1}{2\pi j} \oint_C \left[\frac{25}{(1-z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})^3} \right] z^{n-1}\, dz \tag{5.170}$$

The sum of the terms of the sequence is computed using the residue theorem, expressed as R_i, which represents the residue of the Z-transform at $z = 1$ for the expression $Y(z)$.

$$R_i = R_1 + R_2 \tag{5.171}$$

R_1 and R_2 are residues calculated at the poles of the Z-transform.

$$R_1 = \frac{1}{(2-1)!} \left. \frac{d(25z^{n+1})}{dz} \right|_{z=1} \tag{5.172}$$

$$= 25(n+1) \tag{5.173}$$

$$R_2 = \frac{1}{(3-1)!} \left. \frac{d^2(-3z^{n+1})}{dz^2} \right|_{z=1} \tag{5.174}$$

$$= \frac{-3}{2}(n+1)(n) \tag{5.175}$$

The sum of terms is given by R_i:

$$25(n+1) + \frac{-3}{2}n(n+1) = 116 \tag{5.176}$$

Solving the equation gives :

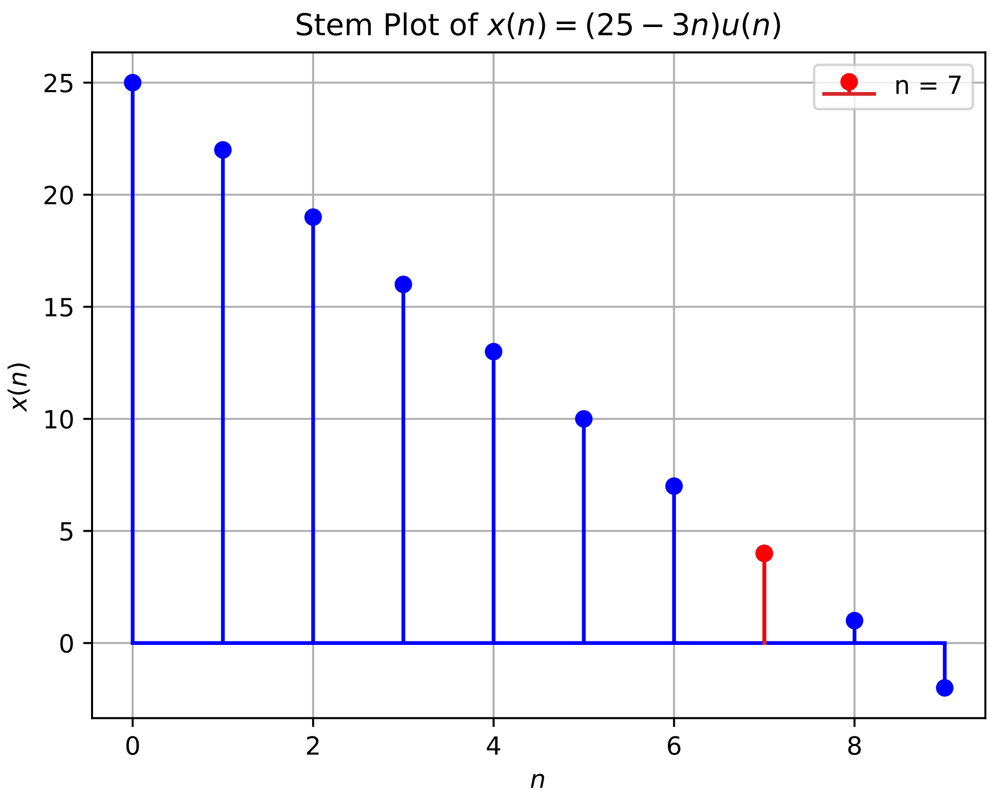
$$n = 7 \tag{5.177}$$

$$n = 8.667 \tag{5.178}$$

Since n can take only integer values, $n = 8.667$ is rejected. Upon substituting the value of n in equation (**??**):

$$x(7) = 4 \tag{5.179}$$

Hence the last term of the given AP is 4.



5.9 The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

Solution:

$$x(n) = (x(0) + nd)u(n) \tag{5.180}$$

$$x(l) = (17 + 9l)u(l) \tag{5.181}$$

Thus,

$$l = 37 \tag{5.182}$$

Parameters in expression		
Symbol	Description	Value
$x(n)$	n^{th} term of series	
$x(l)$	Last(l^{th}) term of series	350
$x(0)$	Starting (0^{th}) term of series	17
d	Common difference of AP	9

Table 5.9: Parameters

Using (**??**),

$$X(z) = \frac{(17 - 8z^{-1})}{(1 - z^{-1})^2}, \quad |z| > |1| \tag{5.183}$$

$$y(n) = x(n) * u(n) \tag{5.184}$$

$$\implies Y(z) = X(z)U(z) \tag{5.185}$$

$$= \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} \tag{5.186}$$

Using contour integral to find Z transform, we get

$$y(37) = \frac{1}{2\pi j} \oint_C Y(z) z^{36} dz \tag{5.187}$$

$$= \frac{1}{2\pi j} \oint_C \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} z^{36} dz \tag{5.188}$$

Now, using Cauchy's residual theorem and observing the fact that 3 repeated poles exist at z = 1,

$$R = \frac{1}{(k-1)!} \lim_{z \to c} \frac{d^{k-1}}{dz^{k-1}}((z-c)^k f(z)) \tag{5.189}$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^{k-1}}{dz^{k-1}}((z-1)^3 \frac{(17-8z^{-1})}{(1-z^{-1})^3} z^{36}) \tag{5.190}$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2}(17z^{39} - 8z^{38}) \tag{5.191}$$

$$= 6973 \tag{5.192}$$

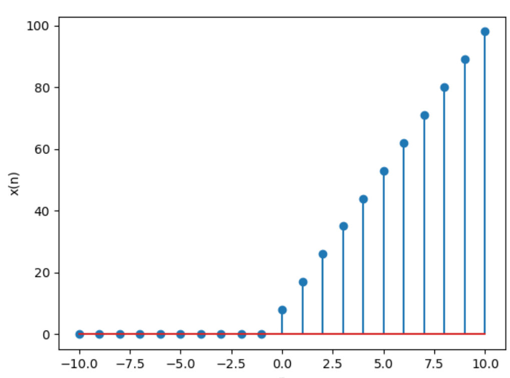


Figure 5.14: Stem Plot of $x(n)$ v/s n

5.10 A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.Each step has a rise of 1/4 m and a tread of 1/2 m. Calculate the total volume of concrete required to build the terrace. [Hint: Volume of concrete required to build the first step=

$$V = \frac{1}{4} \cdot \frac{1}{2} \cdot 50 \tag{5.193}$$

Solution: here

parameter	**description**	**value**
$x(0)$	first term	6.25
d	common difference	6.25
n	no of terms -1	14
$x(n)$	volume of $(n+1)$th step	$(6.25 + 6.25n)\, u(n)$

Table 5.10: formula parameters

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \qquad |z| > |1| \tag{5.194}$$

$$\implies X(z) = \left(\frac{6.25}{(1 - z^{-1})^2} \right) \qquad |z| > |1| \tag{5.195}$$

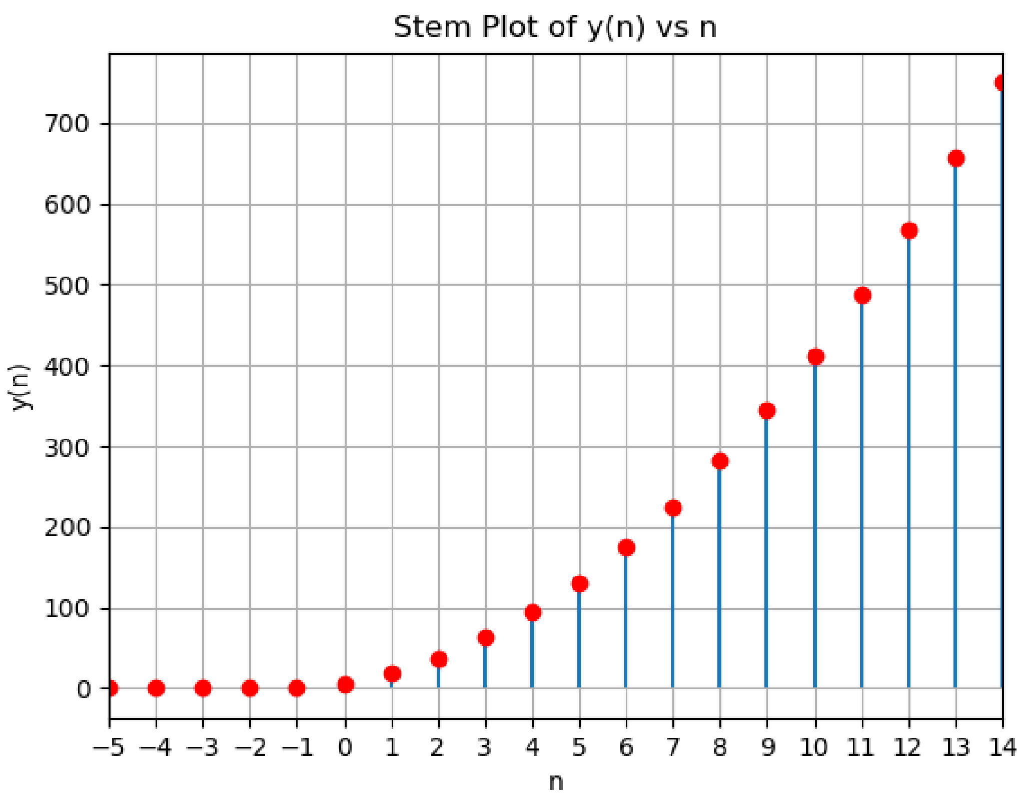


Figure 5.15: plot y(n) vs n

$$y(n) = x(n) * u(n) \tag{5.196}$$

$$\implies Y(z) = X(z)U(z) \tag{5.197}$$

$$U(z) = \frac{1}{1 - z^{-1}} \quad |z| > |1| \tag{5.198}$$

$$Y(z) = \left(\frac{6.25}{1 - z^{-1}} + \frac{6.25z^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |1| \tag{5.199}$$

$$Y(z) = \frac{6.25z^3}{(z-1)^3} \quad |z| > |1| \tag{5.200}$$

contour integration to find inverse z transform

$$y(14) = \frac{1}{2\pi j}\oint_c Y(z)\, z^{13} dz \tag{5.201}$$
$$= \frac{1}{2\pi j}\oint_c \frac{6.25z^{16}}{(z-1)^3} \tag{5.202}$$

pole at 1 repeated 3 times

$$\implies m = 3 \tag{5.203}$$
$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.204}$$
$$\implies y(14) = \frac{1}{(2!)}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{6.25z^{16}}{(z-1)^3}\right) \tag{5.205}$$
$$= 3.125\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{16}\right) \tag{5.206}$$
$$y(14) = 750 \tag{5.207}$$

5.11 Given a GP with a = 729 and 7^{th} term 64, find S_7.

Solution:

Parameter	**Description**	**Value**
x(0)	First Term	729
r	Common Ratio	
x(n)	$(n+1)^{th}$ Term	$x(0)\, r^n u(n)$
x(6)	7^{th} Term	64
y(k)	Sum of first $(k+1)$ terms	

Table 5.11: Parameter Table

from Table **??** :

$$x(6) = x(0)\, r^6 \tag{5.208}$$

$$\implies 64 = 729 r^6 \tag{5.209}$$

$$\therefore r = \frac{2}{3} \tag{5.210}$$

using Table **??** and equation (**??**)

$$X(z) = \frac{729}{1 - \frac{2}{3} z^{-1}}, |z| > \frac{2}{3} \tag{5.211}$$

using Table **??** and equation (**??**)

$$Y(z) = \frac{729}{\left(1 - \frac{2}{3}z^{-1}\right)\left(1 - z^{-1}\right)} \tag{5.212}$$

$$= 2187\left(\frac{1}{1 - z^{-1}} - \frac{\frac{2}{3}}{1 - \frac{2}{3}z^{-1}}\right), |z| > 1 \tag{5.213}$$

Using contour integration for inverse z transform,

$$y(6) = \frac{1}{2\pi j}\oint Y(z)\, z^5 dz \tag{5.214}$$

Using equation (**??**) in (**??**) :

$$y(6) = \frac{1}{2\pi j}\left(\oint \frac{2187z^6}{z - 1}dz - \oint \frac{1458z^6}{z - \frac{2}{3}}dz\right) \tag{5.215}$$

$$\frac{1}{2\pi j}\left(\oint \frac{2187z^6}{z - 1}dz\right) = 2187 \tag{5.216}$$

$$\frac{1}{2\pi j}\left(\oint \frac{1458z^6}{z - \frac{2}{3}}dz\right) = 128 \tag{5.217}$$

using equations (**??**) and (**??**) in (**??**):

$$y(6) = 2187 - 128 \tag{5.218}$$

$$= 2059 \tag{5.219}$$

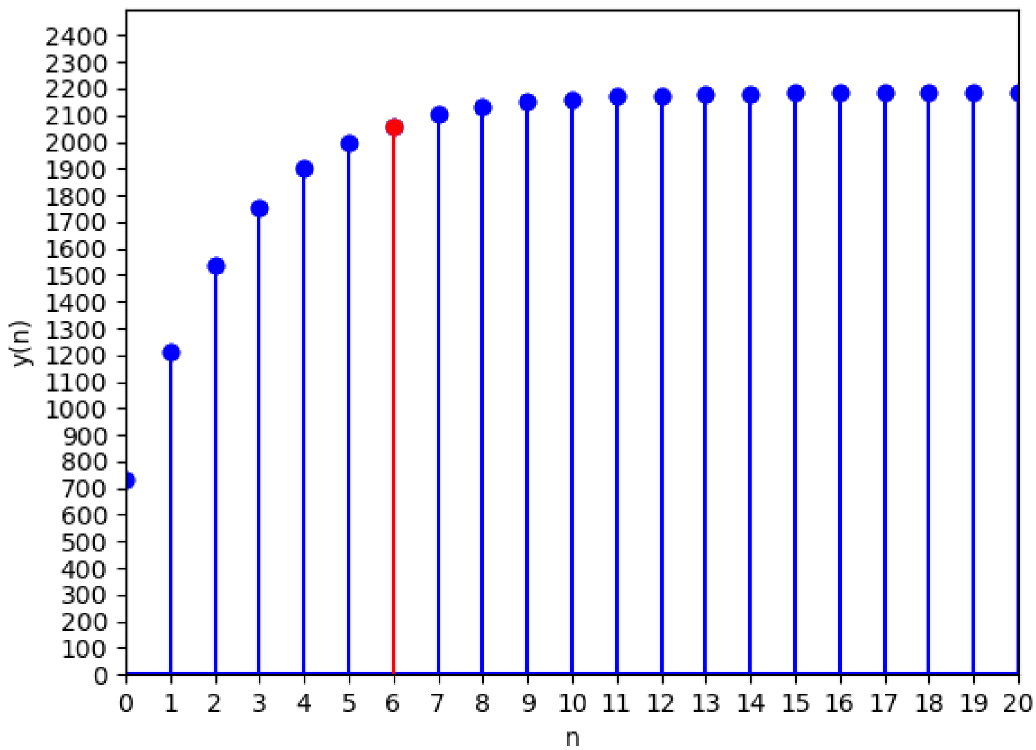


Figure 5.16: Plot of $y\left(n\right)$

5.12 Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Solution:

$$x\left(n\right)=\left(105+5n\right)\left(u(n)\right) \tag{5.220}$$

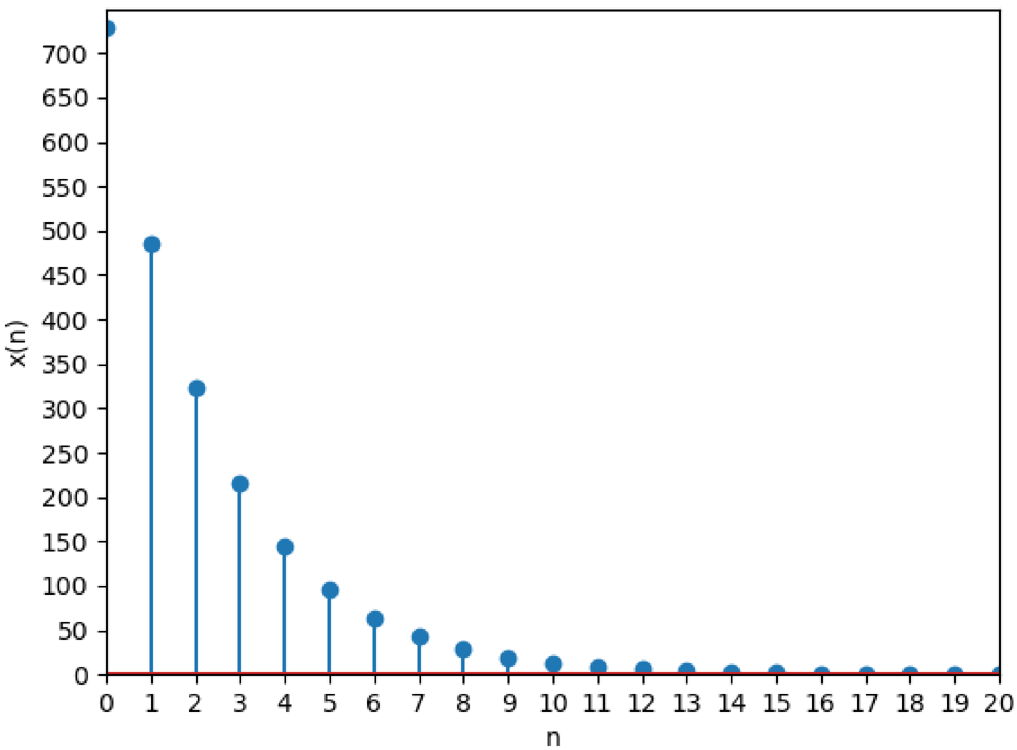


Figure 5.17: Plot of $x(n)$

Parameter	Description	Value
$x(0)$	First Term	105
d	Common Difference	5
n	Total terms	179
$x(178)$	Last Term	995
m	No of poles	3

Table 5.12: Given Parameters

On taking Z transform

$$X(z) = \frac{x(0)}{(1-z^{-1})} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{5.221}$$

$$= \frac{105}{1-z^{-1}} + \frac{5z^{-1}}{(1-z^{-1})^2} \tag{5.222}$$

$$\implies X(z) = \frac{105-100z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{5.223}$$

$$y(n) = x(n) * u(n) \tag{5.224}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.225}$$

$$= \frac{105-100z^{-1}}{(1-z^{-1})^2} \frac{1}{(1-z^{-1})} \tag{5.226}$$

$$= \frac{105-100z^{-1}}{(1-z^{-1})^3} \quad |z| > 1 \tag{5.227}$$

Using contour integration to find the inverse Z-transform:

$$\implies y(178) = \frac{1}{2\pi j} \oint_C Y(z)\; z^{177}\, dz \tag{5.228}$$

$$= \frac{1}{2\pi j} \oint_C \frac{\left(105-100z^{-1}\right) z^{177}}{(1-z^{-1})^3}\, dz \tag{5.229}$$

We can observe that there is only a 3 times repeated pole at $z = 1$,

$$\implies R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.230}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{\left(105 - 100z^{-1}\right) z^{180}}{(z-1)^3} \right) \tag{5.231}$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} \left(105z^{180} - 100z^{179}\right) \tag{5.232}$$

$$= 98450 \tag{5.233}$$

$$\therefore y\,(178) = 98450 \tag{5.234}$$

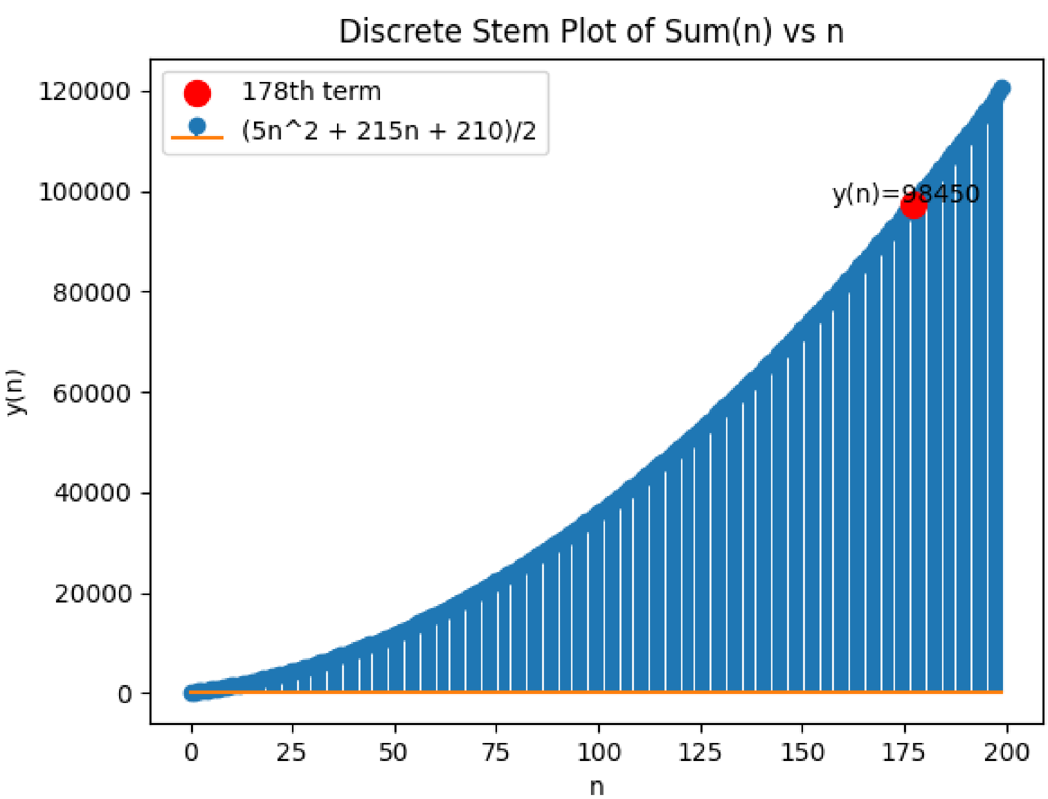


Figure 5.18: Plot of $x(n)$ vs n

5.13 Find the sum of odd numbers between 0 and 50.

Solution:

Symbol	Value	Description
$x(0)$	1	first term of AP
d	2	common difference
$x(n)$	$(1+2n)u(n)$	n-th term of AP

Table 5.13: Given Parameters

Last term of the given sequence is 49.

$$x(n) = (2n+1)u(n) \tag{5.235}$$

$$\therefore (2n+1) = 49 \tag{5.236}$$

$$\implies n = 24 \tag{5.237}$$

Applying Z transform From equation (**??**):

$$X(z) = \frac{1+z^{-1}}{(1-z^{-1})^2}, \quad |z| > |1| \tag{5.238}$$

For AP, the sum of first n+1 terms can be written as

$$y(n) = x(n) * u(n) \tag{5.239}$$

$$Y(z) = X(z)U(z) \tag{5.240}$$

$$= \frac{1}{(1-z^{-1})^2} + \frac{2z^{-1}}{(1-z^{-1})^3}, \quad |z| > |1| \tag{5.241}$$

Using contour integration to find inverse Z transform

$$y(n) = \frac{1}{2\pi j}\oint_C Y(z)z^{n-1}dz \tag{5.242}$$

$$y(24) = \frac{1}{2\pi j}\int Y(z)z^{23}dz \tag{5.243}$$

$$= \frac{1}{2\pi j}\int \frac{1.z^{25}}{(z-1)^2}dz - \frac{1}{2\pi j}\int \frac{2.z^{25}}{(z-1)^3}dz \tag{5.244}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f\left(z\right)\right) \tag{5.245}$$

For R1 , $m = 2$, where m corresponds to number of repeated poles

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{1.z^{25}}{(z-1)^2}\right) \tag{5.246}$$

$$= 25 \tag{5.247}$$

For R2 , $m = 3$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{2.z^{25}}{(z-1)^3}\right) \tag{5.248}$$

$$= 600 \tag{5.249}$$

$$\implies y(24) = R_1 + R_2 \tag{5.250}$$

$$= 625 \tag{5.251}$$

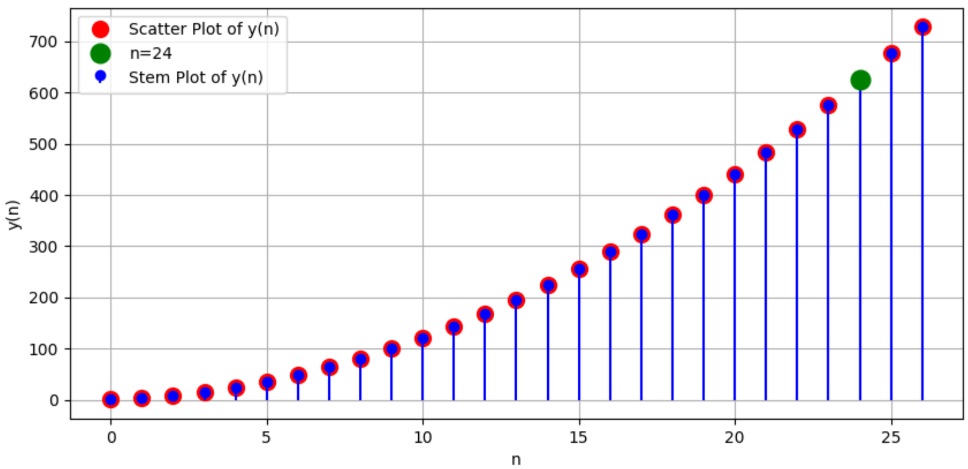


Figure 5.19: Combination of stem and scatter plot of y(n)

5.14 A ladder has rungs 25cm apart.The rungs decrease uniformly in length from 45cm at the bottom to 25cm at the top.If the top and bottom rungs are 2 and 1/2 meter apart.what is length of wood required for the rungs?

Solution: Total number of rungs:

parameter	value
n	10
$x(0)$	45
$x(10)$	25

Table 5.14: Description of Parameters

$$\frac{\left(\frac{5}{2}\right)100}{25} + 1 = 11 \tag{5.252}$$

As the length of rungs decreases uniformly, it is in A.P:

$$x(n) = (x(0) + nd)\, u(n) \tag{5.253}$$

$$25 = 45 + 10d \tag{5.254}$$

$$d = -2 \tag{5.255}$$

The sum of the lengths of all rungs gives the total length of wood required. So, finding the sum of A.P using Z-transform:

$$X(z) = \frac{45}{1 - z^{-1}} + \frac{-2z^{-1}}{(1 - z^{-1})^2} \tag{5.256}$$

$$y(n) = x(n) * u(n) \tag{5.257}$$

$$Y(z) = X(z)U(z) \tag{5.258}$$

$$Y(z) = \frac{45}{(1 - z^{-1})^2} + \frac{-2z^{-1}}{(1 - z^{-1})^3} \tag{5.259}$$

Taking the inverse of the Z-transform using counter-integration

From (**??**),

$$y(10) = 45\,((10 + 1)) + \frac{-2}{2}\,(10\,(10 + 1)) \tag{5.260}$$

$$y(10) = 385 \tag{5.261}$$

The length of wood required for the rungs is 385 cm.

5.15 Q2) The sum of the third and the seventh terms of AP is 6 and their product is 8.

Find the sum of first sixteen terms of the AP

Solution: Table of Parameters

Input Variables	Input Condition
x(2)+x(6)	6
x(2).x(6)	8
$x_i(n)$	general term of AP sequence
$y_i(n)$	sum of first n terms of AP sequence
$x_i(0)$	first term of AP sequence
d_i	common difference of AP sequence

Then from table of parameters,

$$x^2(6) - 6.x(6) + 8 = 0 \tag{5.262}$$

$$x(6) = 2 \text{ or } 4 \tag{5.263}$$

(a)

$$(\ x_i(2), x_i(6)\ ) = \begin{cases} (2,4) & \text{if } i = 1 \\ (4,2) & \text{if } i = 2 \end{cases} \tag{5.264}$$

(b)

$$(\ x_i(0), d_i\ ) = \begin{cases} \left(1, \frac{1}{2}\right) & \text{if } i = 1 \\ \left(5, -\frac{1}{2}\right) & \text{if } i = 2 \end{cases} \tag{5.265}$$

(c)

$$x_i(n) = \begin{cases} \left(\frac{n+2}{2}\right) u(n) & \text{if } i = 1 \\ \left(\frac{10-n}{2}\right) u(n) & \text{if } i = 2 \end{cases} \tag{5.266}$$

z-Transform of x_1 (n), x_2 (n) are given by:

$$X_1(z) = \frac{1 - \frac{z^{-1}}{2}}{\left(1 - z^{-1}\right)^2}, \quad \left|z^{-1}\right| < 1 \tag{5.267}$$

$$X_2(z) = \frac{5 - \frac{11z^{-1}}{2}}{\left(1 - z^{-1}\right)^2}, \quad \left|z^{-1}\right| < 1 \tag{5.268}$$

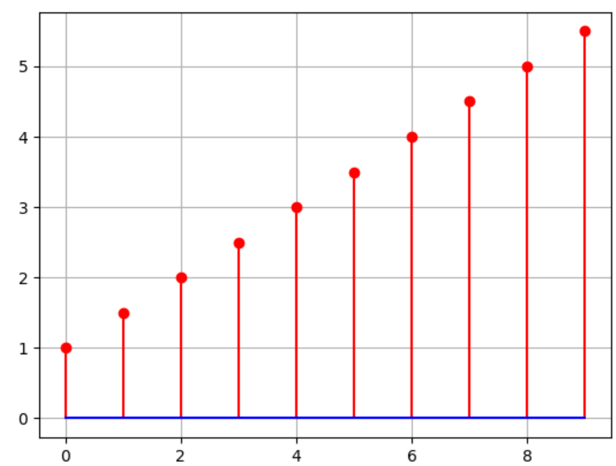
Similarly for sum of first n terms of AP,

$$y_i(n) = x_1(n) * u(n) \tag{5.269}$$

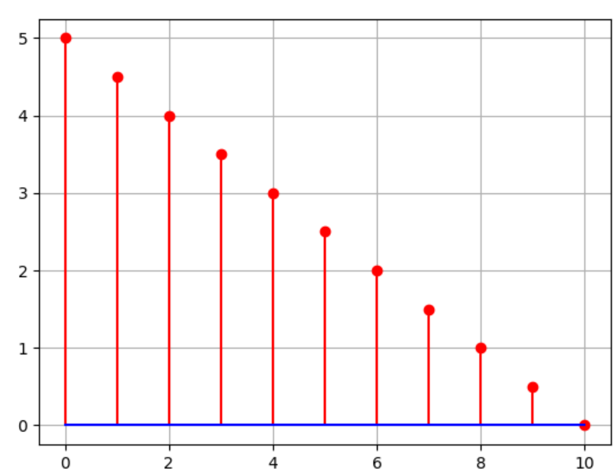
$$Y_i(z) = \frac{X_i(z)}{\left(1 - z^{-1}\right)} \tag{5.270}$$

$$Y_1(z) = \frac{1 - \frac{z^{-1}}{2}}{\left(1 - z^{-1}\right)^3}, \quad |z| > 1 \tag{5.271}$$

$$Y_2(z) = \frac{5 - \frac{11z^{-1}}{2}}{\left(1 - z^{-1}\right)^3}, \quad |z| > 1 \tag{5.272}$$



Graph of $x_1\,(n)$



Graph of $x_2\,(n)$

Inverse z-transform by counter integral method for $y_1(z)$,

Since n starts from 0 to n-1 for $x_1(n)$ so, n $\rightarrow$ n-1 so that $y_1(n)$ starts from 1 to n for given n,

$$y_1(16) = \oint_C \frac{z^3 \left(1 - \frac{z^{-1}}{2}\right)}{(z-1)^3} \, z^{14} \, dz \tag{5.273}$$

$$y_1(16) = \frac{1}{2!} \left(\frac{d^2}{dz^2} z^{17} - \frac{1}{2} \frac{d^2}{dz^2} z^{16} \right)_{z=1} \tag{5.274}$$

$$y_1(16) = 76 \tag{5.275}$$

Similarly for $y_2(z)$,

$$y_2(16) = \oint_C \frac{z^3 \left(5 - \frac{11z^{-1}}{2}\right)}{(z-1)^3} \, z^{14} \, dz \tag{5.276}$$

$$y_2(16) = \frac{1}{2!} \left(5 \, \frac{d^2}{dz^2} z^{17} - \frac{11}{2} \frac{d^2}{dz^2} z^{16} \right)_{z=1} \tag{5.277}$$

$$y_2(16) = 20 \tag{5.278}$$

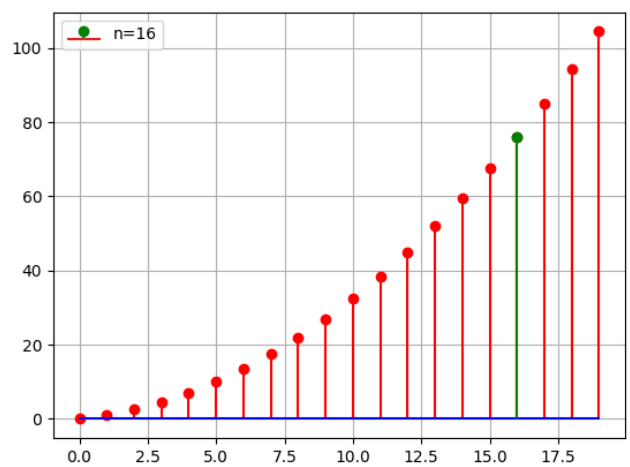
In fact,

(a)

$$y_1(n) = \left(\frac{n(n+3)}{4}\right) u(n) \tag{5.279}$$

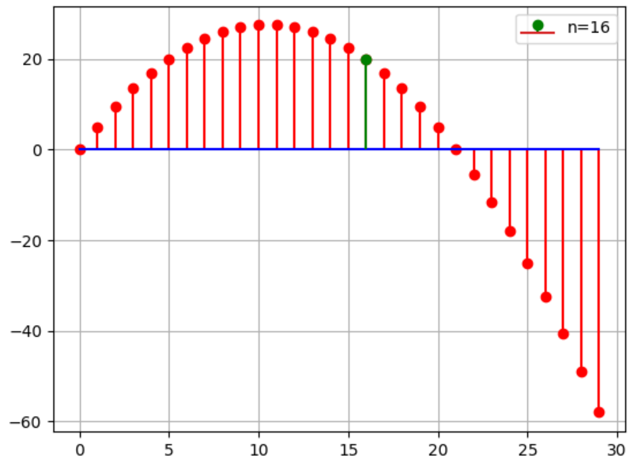
(b)

$$y_2(n) = \left(\frac{n(21-n)}{4}\right) u(n) \tag{5.280}$$



Graph of $y_1(n)$

Graph of $y_2(n)$



5.16 A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii $0.5cm$, $1.0cm$, $1.5cm$, $2.0cm$, . . . What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)

Solution:

Variable	Description	Value
x(0)	First term	0.5
d	common difference	0.5
y(n)	Sum of $n+1$ terms	-
C_n	Length of n^{th} semicircle	$\pi x\left(n\right)$

Table 1: Variables Used

General Term can be written as

$$x\left(n\right) = x\left(0\right) + nd \tag{5.281}$$

Sum upto $n+1$ terms is given by

$$y(n) = x(n) * u(n) \tag{5.282}$$

The corresponding Z-Transform is given by (**??**). Referring to Table **??**, substituting the values in (**??**),

$$Y(z) = \frac{0.5}{(1-z^{-1})^2} + \frac{0.5z^{-1}}{(1-z^{-1})^3} \quad ROC(|z| > 1) \tag{5.283}$$

Finding $y(n)$ by Contour Integration,

$$y(12) = \frac{1}{2\pi j}\oint_C \left(\frac{0.5z^{12-1}}{(1-z^{-1})^2} + \frac{0.5z^{12-2}}{(1-z^{-1})^3}\right) dz \tag{5.284}$$

Using Residue Theorem to evaluate the integral, let

$$Y(z) = S_1 + S_2 \tag{5.285}$$

S_1 has 2 poles,

$$S_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{0.5z^{12+1}}{(z-1)^2}\right) \tag{5.286}$$

$$S_1 = 0.5\,(12+1)\lim_{z\to 1}\left(z^1 2\right) \tag{5.287}$$

$$S_1 = 0.5\,(12+1) \tag{5.288}$$

Similarly, S_2 has 3 poles,

$$S_2 = \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{0.5z^{12+1}}{(z-1)^3}\right) \tag{5.289}$$

$$= \frac{0.5\,(12+1)}{2}\lim_{z\to 1}\frac{d}{dz}\left(z^1 2\right) \tag{5.290}$$

$$= \frac{0.5\,(12+1)\,(12)}{2}\lim_{z\to 1}\left(z^{12-1}\right) \tag{5.291}$$

$$= \frac{0.5\,(12)\,(12+1)}{2} \tag{5.292}$$

Finally,

$$y\,(12) = 0.5\,(12+1) + \frac{0.5\,(12)\,(12+1)}{2} \tag{5.293}$$

$$y\,(12) = 45.5 \tag{5.294}$$

$$\sum_{n=0}^{12} C_n = \pi y\,(12) \tag{5.295}$$

$$\sum_{n=0}^{12} C_n = \pi\,(45.5) \tag{5.296}$$

$$= 143 \tag{5.297}$$

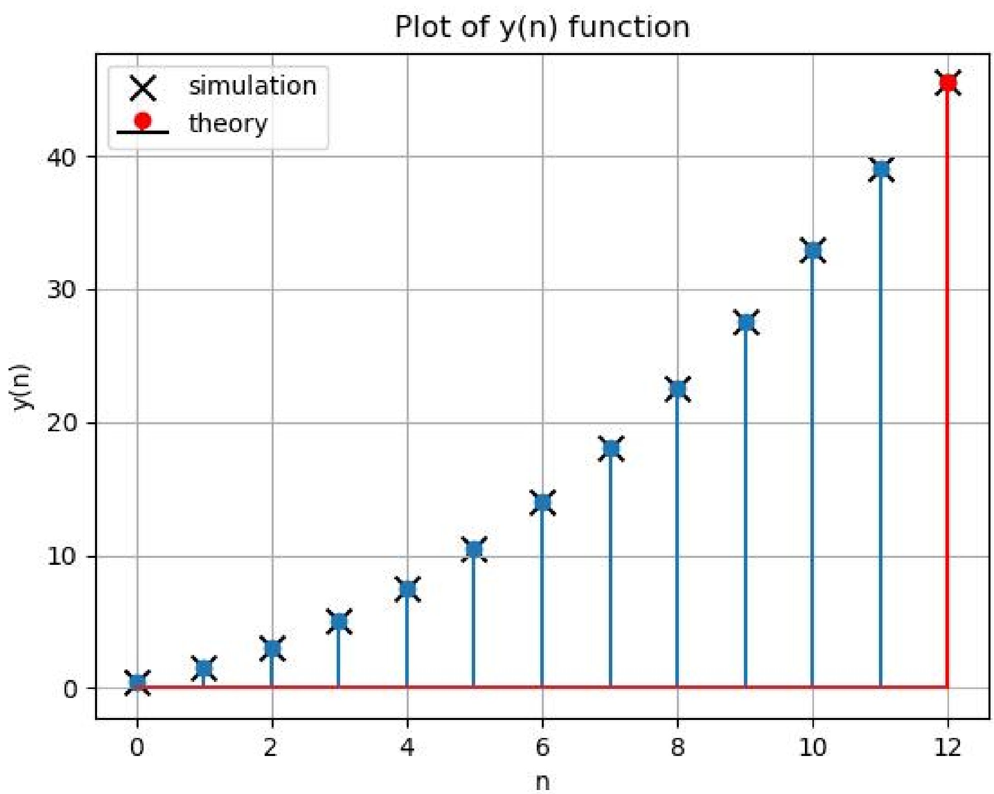


Figure 1: Plot of Sum of n terms taken from Python3

5.17 Find the sum to indicated number of terms in each of the geometric progressions in $0.15, 0.015, 0.0015 \ldots 20 terms.$

Solution:

Parameter	Description	Value
n	No. of terms in the G.P	20
$x(0)$	first term in the G.P	0.15
r	common ratio in the G.P	0.1

Table 5.16: Variables and their descriptions

$$x(n) = x(0)r^n \tag{5.298}$$

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \qquad |z| > |r| \tag{5.299}$$

$$U(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1 \tag{5.300}$$

$$y(n) = x(n) * u(n) \tag{5.301}$$

$$Y(z) = X(z)U(z) \tag{5.302}$$

$$= \left(\frac{0.15}{1 - 0.1z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > 1 \tag{5.303}$$

Using Contour integration

$$y(20) = \frac{1}{2\pi j}\oint_C \frac{0.15z^2}{(z-1)(z-0.1)} z^{19} dz \tag{5.304}$$

$$= \frac{1}{2\pi j}\oint_C \frac{0.15}{0.9}\left(\frac{1}{z-1} - \frac{1}{z-0.1}\right) z^{21} dz \tag{5.305}$$

$$= \frac{1}{6}\left(\left(\lim_{z\to 1} \frac{z^{21}}{z-1}(z-1)\right) - \left(\lim_{z\to 0.1} \frac{z^{21}}{z-0.1}(z-0.1)\right)\right) \tag{5.306}$$

$$= \frac{1}{6}(1 - 0.1^{21}) \tag{5.307}$$

$$= 0.16667 \tag{5.308}$$

∴ Sum of 20 terms of the given GP is 0.16667

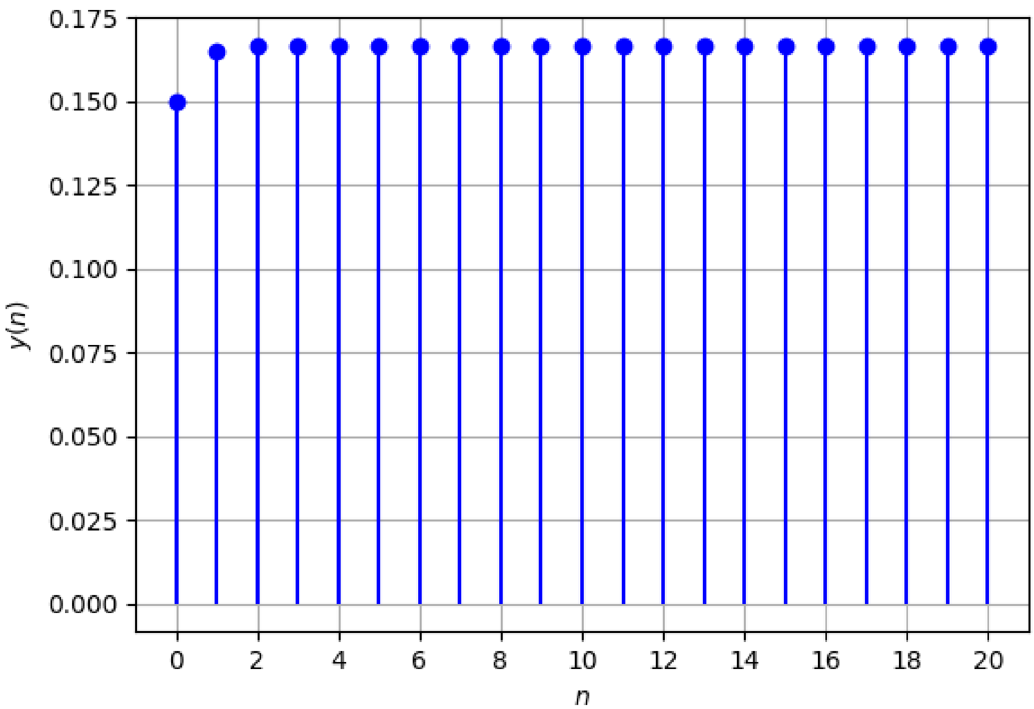


Figure 5.17: Stem plot of y(n)

5.18 A man deposited Rs 10000 in a bank at the rate of 5% simple interest anually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

Solution:

$$\text{Interest in one year} = \frac{10000 \times 5 \times 1}{100} \tag{5.309}$$

$$d = 500 \tag{5.310}$$

Parameter	**Value/Formula**	**description**
$x(0)$	Rs.10000	Total amount deposited
r	5	Rate of interest
$x(n)$	$(x(0)+nd)\,u(n)$	amount at the start of $(n+1)^{\text{th}}$ year
d	?	common difference

Table 1: Input data

From (**??**) and Table **??**:

$$x(n) = (10000 + 500n)u(n) \tag{5.311}$$

From (**??**)

$$X(z) = \frac{10000}{1-z^{-1}} + \frac{500z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{5.312}$$

Amount in 15^{th} year is

$$x(14) = x(0) + 14 \times d \tag{5.313}$$

$$\implies x(14) = 17000 \tag{5.314}$$

Total amount after 20 years is

$$x(20) = x(0) + 20 \times 500 \tag{5.315}$$

$$\implies x(20) = 20000 \tag{5.316}$$

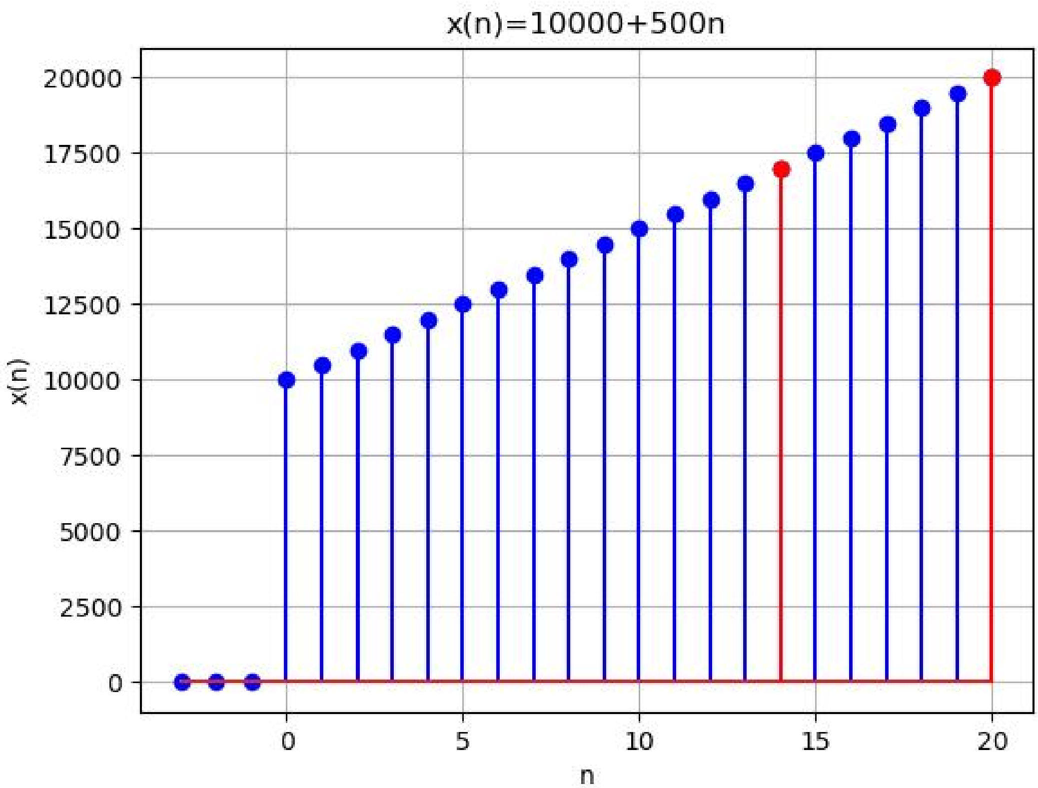


Figure 1: graph for $x(n) = 10000 + 500n$

5.19 A manufacturer reckons that the value of a machine, which costs him Rs.15625, will depreciate each year by 20%.Find the estimated value at the end of 5 years.

Solution:

$$x(n) = 15625\left(1 - \frac{1}{5}\right)^n u(n) \tag{5.317}$$

Result:

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1 - az^{-1})} \quad |z| > a \tag{5.318}$$

Symbol	Value	Description
$x(0)$	15625	Initial Cost
r	$\frac{4}{5}$	common ratio of GP
n	5	Number of years
$x(5)$	$15625(\frac{4}{5})^5$	Estimated value after 5 years

Table 1: Parameter Table

From (**??**)

$$X(Z) = \frac{15625}{1 - \left(1 - \frac{1}{5}\right) z^{-1}} \quad |z| > \frac{4}{5} \tag{5.319}$$

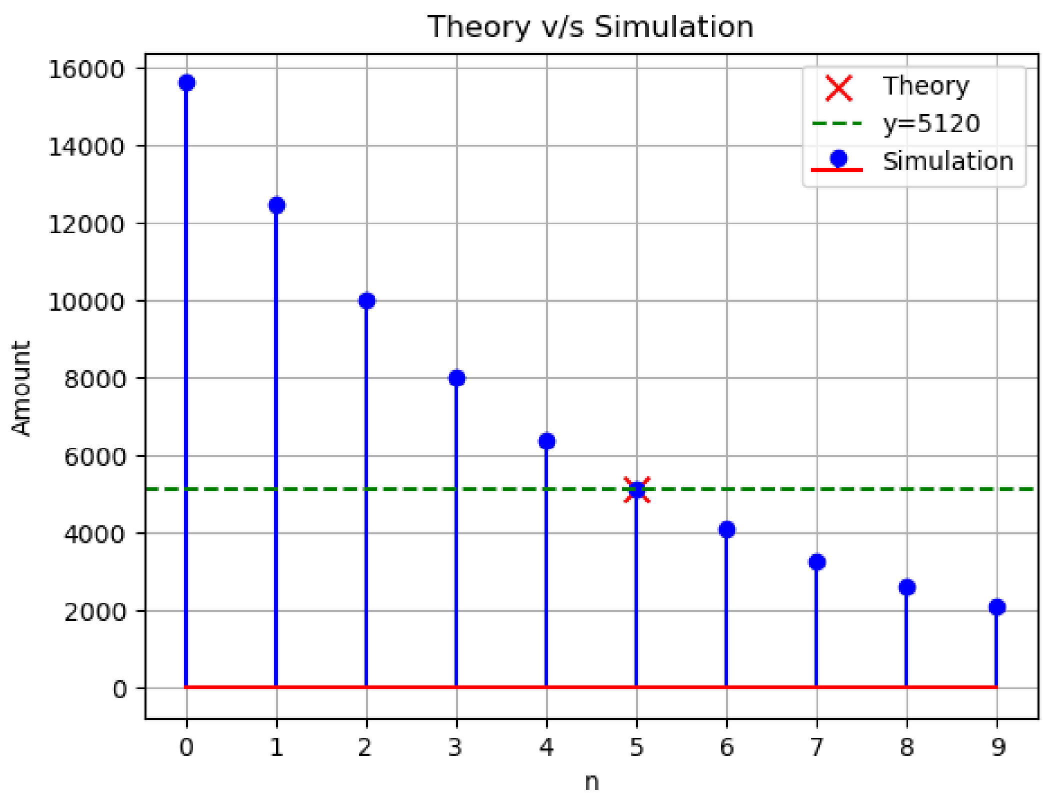


Figure 5.19: Theory matches with simulated values

5.20 The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered is equal to the sum of the numbers of the houses following it. Find this value of x.

Hint:$S_{x-1} = S_{49} - S_x$

Solution:

Parameter	Value	Description
$x(0)$	1	First house
d	1	Common difference
$x(n)$	$(n+1)\,u(n)$	$(n+1)\,th$ house
$y(n)$	$\left(\frac{n+1}{2}\right)(n+2)\,u(n)$	Sum of $n+1$ number of houses.
$x_2(n)$	$(49-n)\,u(n)$	$(n+1)\,th$ house from last house
$y_2(n)$	$\left[49n-\left(\frac{n}{2}\right)(n+1)\right]u(n)$	Sum of $n+1$ houses from last house.

Table 5.20: Input Parameters

For an AP:

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{5.320}$$

$$\implies X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2} \tag{5.321}$$

$$= \frac{1}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.322}$$

$$\therefore y(n) = \frac{(n+1)}{2}(n+2) \tag{5.323}$$

$$y(x-2) = y(n-1) - y(x-1) \tag{5.324}$$

From Table **??**:

$$\left(\frac{x-1}{2}\right)x = \frac{n}{2}\left(n+1\right) - \frac{x}{2}\left(x+1\right) \tag{5.325}$$

$$\left(x-1\right) + x\left(x+1\right) = n\left(n+1\right) \tag{5.326}$$

$$2x^2 = n\left(n+1\right) \tag{5.327}$$

$$x = \sqrt{\frac{n}{2}\left(n+1\right)} \tag{5.328}$$

$$x = 35 \tag{5.329}$$

Result Confirmation:

To prove:

$$y\left(33\right) = y_2\left(13\right) \tag{5.330}$$

LHS:

$$y(n) = x(n) * u(n) \tag{5.331}$$

$$\implies Y(z) = X(z) \times U(z) \tag{5.332}$$

$$Y(z) = \left(\frac{1}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{5.333}$$

$$= \frac{1}{(1-z^{-1})^3}, \quad |z| > 1 \tag{5.334}$$

$$\tag{5.335}$$

Using Contour Integration to find inverse Z-transform,

$$y(33) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{32}\, dz \tag{5.336}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^{32}}{(1-z^{-1})^3}\, dz \tag{5.337}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.338}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(z^{35}\right) \tag{5.339}$$

$$= 595 \tag{5.340}$$

RHS:

From Table **??**:

$$X_2(z) = \frac{49 - 50z^{-1}}{(1 - z^{-1})^2} \tag{5.341}$$

$$y_2(n) = x_2(n) * u(n) \tag{5.342}$$

$$\implies Y_2(z) = X_2(z) \times U(z) \tag{5.343}$$

$$y_2(z) = \frac{49 - 50z^{-1}}{(1 - z^{-1})^3} \tag{5.344}$$

$$\tag{5.345}$$

Using Contour Integration to find inverse Z-transform,

$$y_2(13) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{12}; dz \tag{5.346}$$

$$= \frac{1}{2\pi j} \oint_C \frac{49 - 50z^{-1}}{(1 - z^{-1})^3} \left(z^{12}\right)\, dz \tag{5.347}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.348}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(49z^{15} - 50z^{14}\right) \tag{5.349}$$

$$= 49.15.14 - 50.14.13 \tag{5.350}$$

$$= 595 \tag{5.351}$$

$$LHS = RHS$$

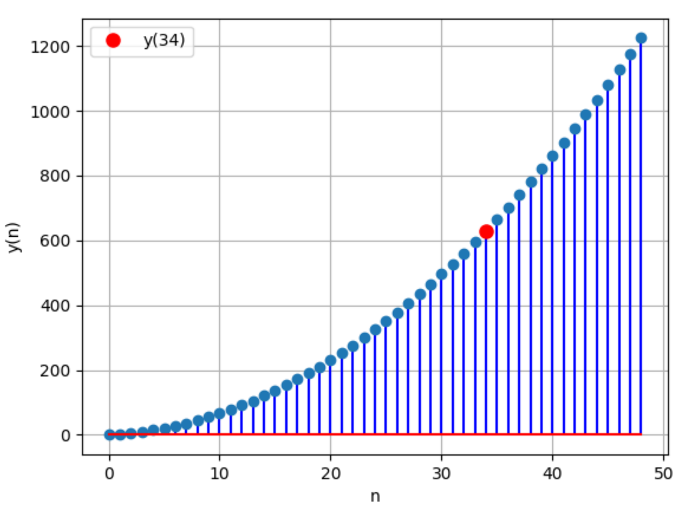


Figure 5.20: Plot y(n) vs n

5.21 A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹200 for the first day, ₹250 for the second day, ₹300 for the third day, etc., the penalty for each succeeding day being ₹50 more than for the preceding day.How much money the contractor has to pay as penalty, if he has delayed the work by 30 days?

Solution:

Variable	**Description**	**Value**
x(0)	First term of AP	200
d	common difference in the AP	50
x(n)	General term	$(200 + n \times 50)$u(n)

Table 5.21: Variables Used

From equation(**??**)

$$\implies X(z) = \frac{200}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \qquad |z| > |1| \tag{5.352}$$

$$y(n) = x(n) * u(n) \tag{5.353}$$

$$Y(z) = X(z)\, U(z) \tag{5.354}$$

$$\implies Y(z) = \left(\frac{200}{1 - z^{-1}} + \frac{50z^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \tag{5.355}$$

$$= \frac{200}{(1 - z^{-1})^2} + \frac{50z^{-1}}{(1 - z^{-1})^3} \tag{5.356}$$

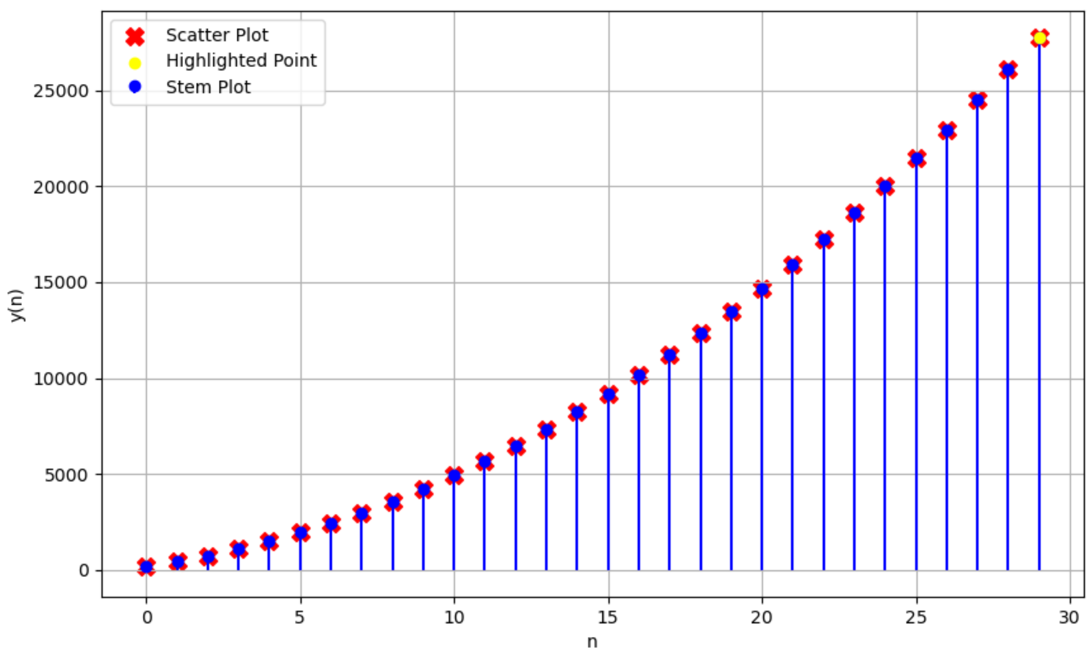


Figure 5.21: Combination of scatter plot and stem plot

Contour integration to find z transform

$$y(29) = \frac{1}{2\pi j}\oint_c Y\left(Z\right)z^{28}dz \tag{5.357}$$

$$= \frac{1}{2\pi j}\oint_c \frac{(200-150z^{-1})z^{28}}{(1-z^{-1})^3} \tag{5.358}$$

pole at 1 repeated 3 times

$$\therefore m = 3 \tag{5.359}$$

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((za)^m f(z)\right) \tag{5.360}$$

$$= \frac{1}{(2!)} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{(200 - 150z^{-1})z^{28}}{(1-z^{-1})^3}\right) \tag{5.361}$$

$$= \lim_{z \to 1} \frac{d^2}{dz^2} \left(100 - 75z^{-1}\right) z^{31} \tag{5.362}$$

$$\implies y(n) = 27750 \tag{5.363}$$

5.22 Find the sum of all two digit numbers which when divided by 4,yields 1 as reminder?

Solution: Input parameters are:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	13	First term
d	4	common difference
$x(n)$	$[13+4n]u(n)$	General term of the series

Table 5.22: INPUT PARAMETER TABLE

$$\text{x(n)} = x(0) + n\text{d} \tag{5.364}$$

$$n = \frac{97-13}{4} = 21 \tag{5.365}$$

$$\tag{5.366}$$

From **??**

$$X(z) = \frac{13-9z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{5.367}$$

$$y(n) = x(n) * u(n) \tag{5.368}$$

$$Y(z) = X(z)U(z) \tag{5.369}$$

$$\implies Y(z) = \frac{13-9z^{-1}}{(1-z^{-1})^3}, |z| > 1 \tag{5.370}$$

Using contour integration to find the inverse z-transform,

$$y(n) = \frac{1}{2\pi j}\oint_C Y(z)z^{n-1}dz \tag{5.371}$$

$$y(21) = \frac{1}{2\pi j}\oint_C \frac{(13-9z^{-1})z^{20}}{(1-z^{-1})^3} \tag{5.372}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.373}$$

$$= \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left(13z^{23} - 9z^{22}\right) \tag{5.374}$$

$$R = 1210 \tag{5.375}$$

$$\therefore y(21) = 1210 \tag{5.376}$$

Therefore, the sum of all two-digit numbers that, when divided by 4, yield a remainder of 1 is 1210.

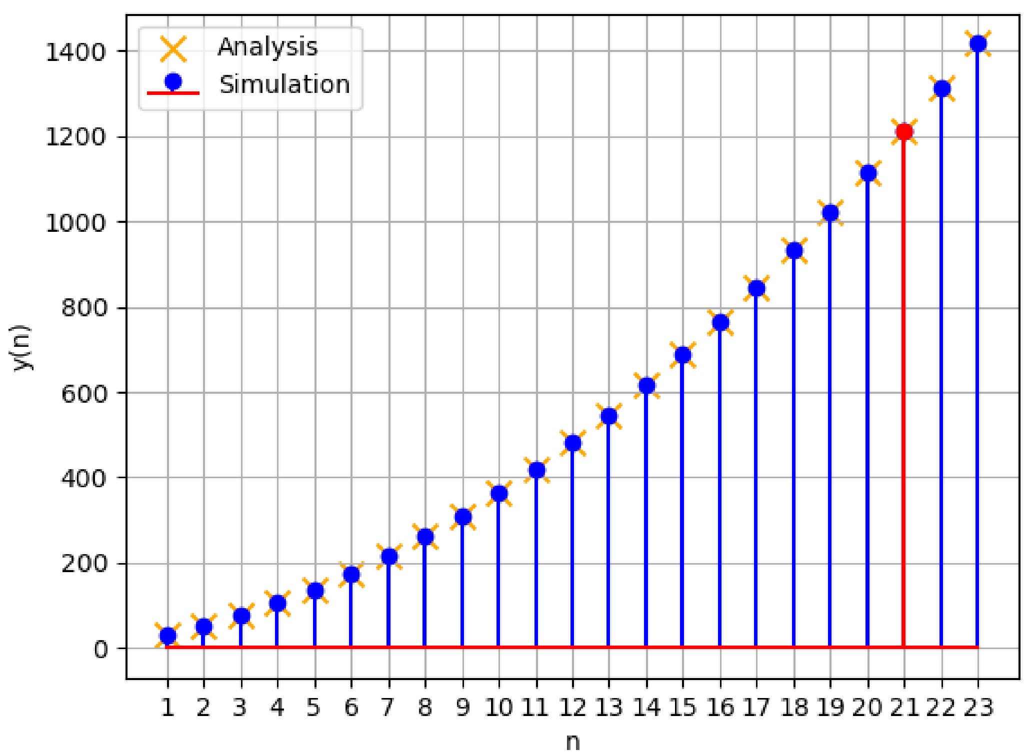
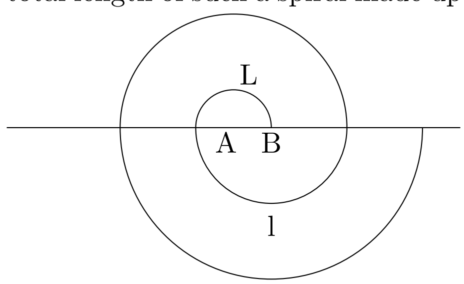


Figure 5.22: y(n) $= 13 + 15n + 2n^2$

5.23 A spiral is made up of successive semicircles,with centres alternately at A and B,starting with centre at A,of radii $0.5cm, 1.0cm, 1.5cm, 2.0cm$,... as shown in Fig.5.4.what is the total length of such a spiral made up of thirteen consecutive semicircles?(Take $\pi = \frac{22}{7}$)



Solution:

5.24 Evaluate $\sum_{k=1}^{11}(2+3^k)$

Solution:

5.25 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row,18 in the row next to it and so on (see Fig **??**). In how many rows are the 200 logs placed and how many logs are in the top row? **Solution:**

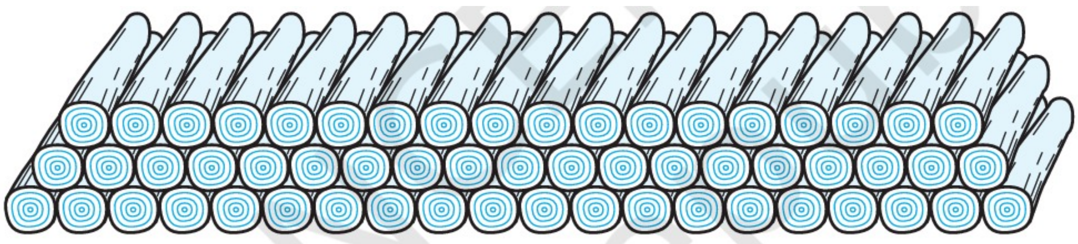


Figure 5.25:

Solution:

Symbol	Value	Description
$x(0)$	20	first term of AP
d	-1	common difference
$x(n)$		$(x(0) + nd)\, u(n)$
$y(n)$	200	

Table 1: input parameters

by the differentiation property:

$$x(n) \overset{z}{\longleftrightarrow} (-z)\frac{dX(z)}{dz} \tag{5.377}$$

$$\implies nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{5.378}$$

$$\implies n^2u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, |z| > 1 \tag{5.379}$$

$$\implies Z^{-1}\left[\frac{1}{(1-z^{-1})^2}\right] = (n+1)\,u(n) \tag{5.380}$$

$$\implies Z^{-1}\left[\frac{z^{-1}}{(1-z^{-1})^3}\right] = \frac{n(n+1)}{2}u(n) \tag{5.381}$$

from (**??**)

$$\implies X(Z) = \frac{20}{1-z^{-1}} - \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{5.382}$$

from (**??**)

$$y(n) = x(n) * u(n) \tag{5.383}$$

$$\implies Y(Z) = X(Z)\,U(Z) \tag{5.384}$$

$$= \frac{20}{(1-z^{-1})^2} - \frac{z^{-1}}{(1-z^{-1})^3}, |z| > 1 \tag{5.385}$$

substituting (**??**), (**??**):

$$y(n) = \left(20(n+1) - \frac{n(n+1)}{2}\right)u(n) \tag{5.386}$$

$$= \frac{(n+1)(40-n)}{2} \tag{5.387}$$

since number of logs=200

$$200 = \frac{(n+1)(40-n)}{2} \tag{5.388}$$

$$n = 24, 15 \tag{5.389}$$

for n=24

$$x(24) = 20 - 24 \tag{5.390}$$

$$= -4 \tag{5.391}$$

but logs can't be negative

for n=15

$$x(15) = 20 - 15 \tag{5.392}$$

$$= 5 \tag{5.393}$$

so number of rows=15

number of logs=5

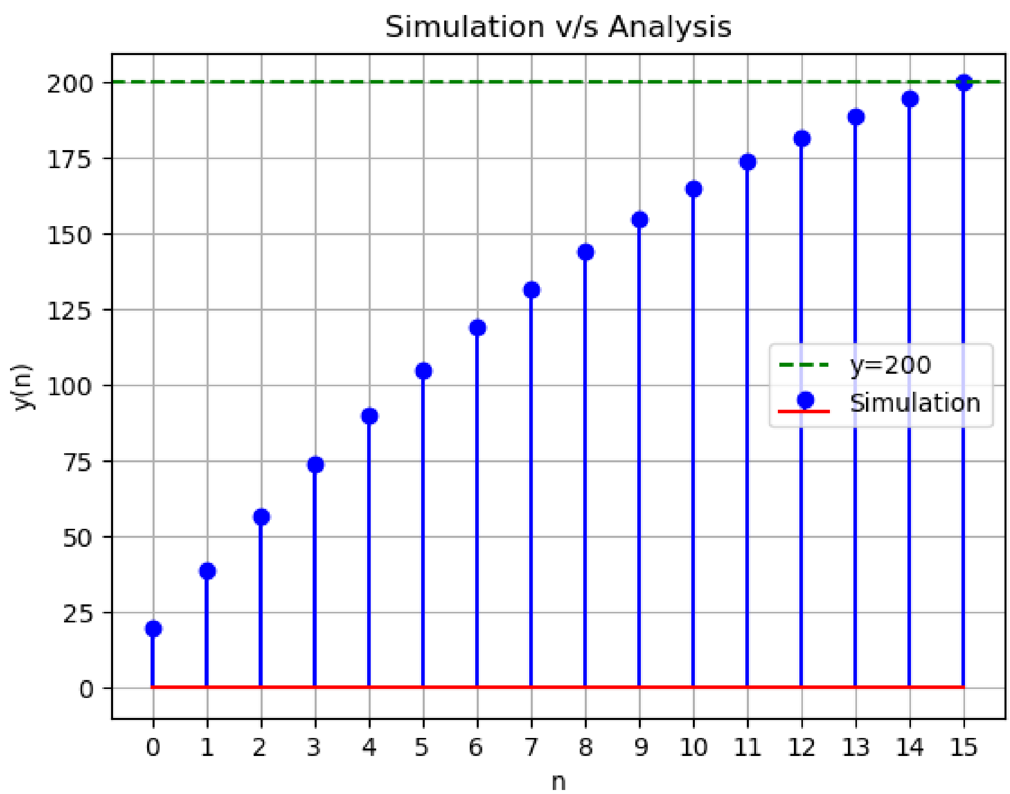


Figure 1: plot of y(n) v/s n

5.26 In an AP:

(a) given a = 5, d = 3,a_n = 5, find n and S_n.

(b) given a = 7, a_{13}=35, find d and S_{13}.

(c) given a_{12} = 37, d = 3, find a and S_{12}.

(d) given a_3= 15, S_{10} = 125, find d and a_{10}.

(e) given d = 5, S_9= 75, find a and a_9.

(f) given a = 2, d = 8, S_n= 90, find n and a_n.

(g) given a = 8, a_n= 62, S_n= 210, find n and d.

(h) given an= 4, d = 2, S_n = −14, find n and a.

(i) given a = 3, n = 8, S = 192, find d.

(j) given l = 28, S = 144, and there are total 9 terms. Find a.

Solution:

5.27 Find the sum of n terms of the series: $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

Solution:

5.28 Find the sum of the first 22 terms of an AP in which $d = 7$ and the 22^{nd} term is 149.

Solution:

Parameter	**Description**	**Formulae/Value**
$u(n)$	Unit step function	$\begin{cases} 0, & \text{if } n < 0, \\ 1, & \text{if } n \geq 0. \end{cases}$
$x(0)$	First term of A.P	-
d	Commom difference	7
n	Count of terms starting from '0'	-
$x(n)$	$(n+1)^{th}$ term of the A.P	$(x(0) + nd)\, u(n)$
$x(21)$	Value of 22^{nd} term	149

Table 5.28: Parameters

Now, the 22^{nd} term means $x(21)$, so

$$x(21) = (x(0) + nd)\, u(21) \tag{5.394}$$

$$149 = (x(0) + 21(7))(1) \tag{5.395}$$

$$x(0) = 2 \tag{5.396}$$

The standard z transforms,

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, |z| > 1 \tag{5.397}$$

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \tag{5.398}$$

The general term is $x(n) = (2 + 7n)\, u\,(n)$, The z transform of the general term is

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{5.399}$$
$$= \frac{2}{1 - z^{-1}} + \frac{7z^{-1}}{(1 - z^{-1})^2} \tag{5.400}$$
$$= \frac{2 + 5z^{-1}}{(1 - z^{-1})^2}; \quad |z| > 1 \tag{5.401}$$

On convolution for finding the sum

$$y(n) = x(n) * u(n) \tag{5.402}$$

On z-transform,

$$Y(z) = X(z) \cdot U(z) \tag{5.403}$$
$$= \left(\frac{2 + 5z^{-1}}{(1 - z^{-1})^2} \right) \cdot \frac{1}{1 - z^{-1}} \tag{5.404}$$
$$\implies Y(z) = \frac{2 + 5z^{-1}}{(1 - z^{-1})^3}; \qquad |z| > 1 \tag{5.405}$$

Using Contour integration to find the inverse z-transform,

$$y(n) = \oint_c Y(z) \cdot z^{n-1} dz \tag{5.406}$$
$$y(21) = \oint_c \frac{2 + 5z^{-1}}{(1 - z^{-1})^3} \cdot z^{20} dz \tag{5.407}$$

We can observe there are three poles and thus m = 3,

$$y\left(21\right) = \frac{1}{\left(m-1\right)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left(\left(z-a\right)^m f\left(z\right)\right) \tag{5.408}$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left(\left(z-1\right)^3\cdot\frac{2+5z^{-1}}{\left(1-z^{-1}\right)^3}\cdot\left(z^{20}\right)\right) \tag{5.409}$$

$$= \frac{1}{2}\left(1012+2310\right) \tag{5.410}$$

$$\implies y\left(21\right) = 1661 \tag{5.411}$$

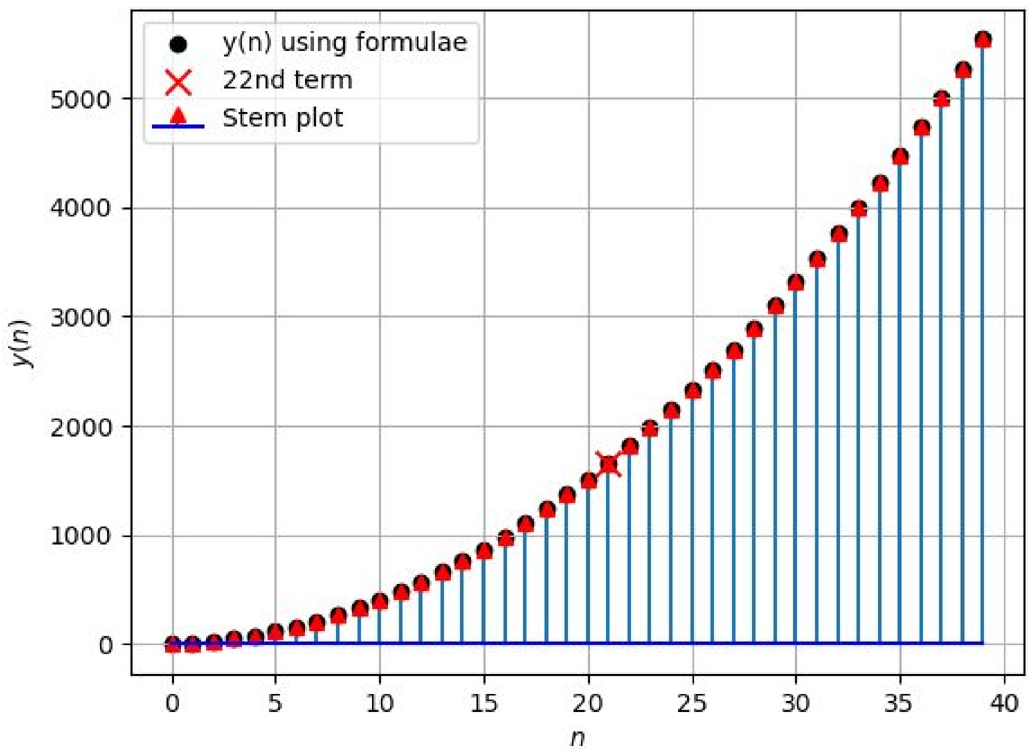


Figure 5.28: Simulation v/s theoretical

5.29 Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Solution:

5.30 In a school, students thought of planting trees in and around the school to reduce air

pollution. It was decided that the number of trees, that each section of each class will

plant, will be the same as the class, in which they are studying, e.g., a section of Class I

will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are

three sections of each class. How many trees will be planted by the students?

Solution:

5.31 A sum of Rs.700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is Rs.20 less than its preceding prize, find the value of each of the prizes.

Solution:

Parameter	Description	Value
x(n)	n^{th} term of discrete signal	(x(0)+nd)u(n)
x(0)	1^{st} term of the AP	?
d	Common Difference of the AP	-20

Table 5.31: Given parameters

From (**??**)

$$X(z) = \frac{x(0)}{1 - z^{-1}} - \frac{20.z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{5.412}$$

$$\because y(n) = x(n) * u(n) \tag{5.413}$$

$$Y(z) = X(z)U(z) \tag{5.414}$$

$$\implies Y(z) = \frac{x(0)}{(1 - z^{-1})^2} - \frac{20.z^{-1}}{(1 - z^{-1})^3}; |z| > 1 \tag{5.415}$$

Using contour integration for inverse Z transformation,

$$y(6) = \frac{1}{2\pi j}\oint_c Y(z)z^5 dz \tag{5.416}$$

$$= \frac{1}{2\pi j}\int \frac{x(0).z^7}{(z-1)^2}dz - \frac{1}{2\pi j}\int \frac{20.z^7}{(z-1)^3}dz \tag{5.417}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.418}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{x(0).z^7}{(z-1)^2}\right) \tag{5.419}$$

$$= 7x(0) \tag{5.420}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{-20.z^7}{(z-1)^3}\right) \tag{5.421}$$

$$= -420 \tag{5.422}$$

$$\implies y(6) = R_1 + R_2 \tag{5.423}$$

$$700 = -420 + 7x(0) \tag{5.424}$$

$$\implies x(0) = 160 \tag{5.425}$$

The value of each of the prizes is 160,140,120,100,80,60,40.

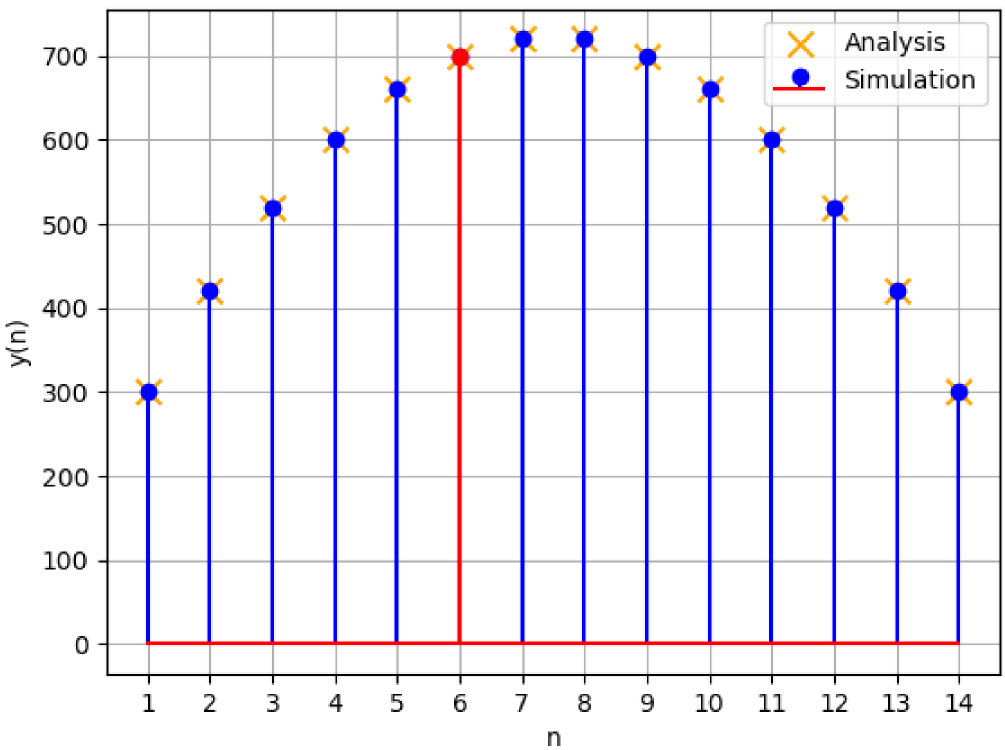


Figure 5.31: y(n) $= 170n - 10n^2$

5.32 Find the sum of all numbers between 200 and 400 which are divisible by 7.

Solution:

Parameter	Description	Value
$x(n)$	n^{th} term of the AP	(x(0)+nd)u(n)
$x(0)$	1^{st} term of the AP	?
d	Common Difference of the AP	7

Table 5.32: Input parameters

The first and last term of the AP are 203 and 399 respectively.

$$\implies x(n) = (203 + 7n)u(n) \tag{5.426}$$

To calculate the number of terms in the AP,

$$399 = 203 + 7n \tag{5.427}$$

$$\implies n = 28 \tag{5.428}$$

From (**??**)

$$X(z) = \frac{203}{1 - z^{-1}} + \frac{7.z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{5.429}$$

$$\because y(n) = x(n) * u(n) \tag{5.430}$$

$$Y(z) = X(z)U(z) \tag{5.431}$$

$$\implies Y(z) = \frac{203}{(1 - z^{-1})^2} + \frac{7.z^{-1}}{(1 - z^{-1})^3}; |z| > 1 \tag{5.432}$$

Using Contour integration for inverse Z transform,

$$y(28) = \frac{1}{2\pi j}\oint_c Y(z)z^{27}dz \tag{5.433}$$

$$= \frac{1}{2\pi j}\int \frac{203.z^{29}}{(z-1)^2}dz + \frac{1}{2\pi j}\int \frac{7.z^{29}}{(z-1)^3}dz \tag{5.434}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.435}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{203.z^{29}}{(z-1)^2}\right) \tag{5.436}$$

$$= 203 \times 29 = 5887 \tag{5.437}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{7.z^{29}}{(z-1)^3}\right) \tag{5.438}$$

$$= \frac{7\times 29\times 28}{2!} = 2842 \tag{5.439}$$

$$y(28) = R_1 + R_2 \tag{5.440}$$

$$\implies y(28) = 8729 \tag{5.441}$$

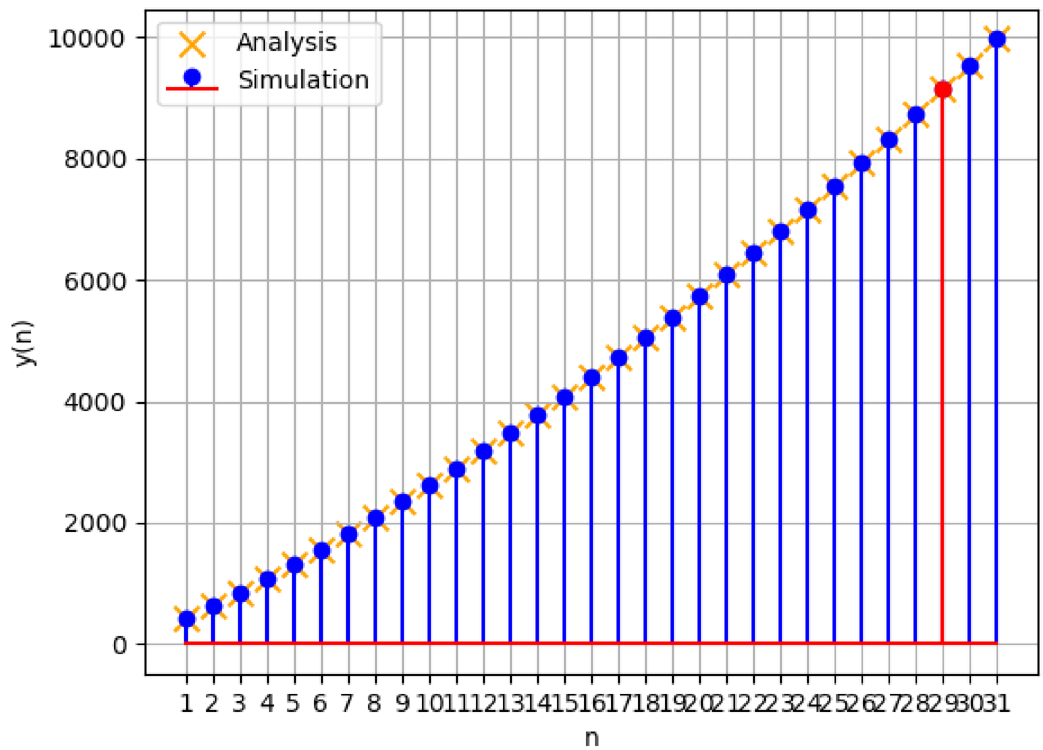


Figure 5.32: y(n) = $199.5n + 3.5n^2$

5.33 Find the 20th term in this series.

$$2 \times 4 + 4 \times 6 + 6 \times 8 \cdots + n\, terms$$

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
$x(n)$	$4(n+1)n+2u(n)$	General term of the series

Table 5.33: Table of parameters

Using Z- transform,

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3}, |z|>1 \tag{5.442}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^2}, |z|>1 \tag{5.443}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1-z^{-1}\right)}, |z|>1 \tag{5.444}$$

$$X(z) = \sum_{n=-\infty}^{n=\infty} 4(n+1)(n+2)u(n)z^{-n} \tag{5.445}$$

$$X(z) = \sum_{n=-\infty}^{n=\infty} 4\left(n^2+3n+2\right)u(n)z^{-n} \tag{5.446}$$

$$X(z) = \frac{8}{\left(1 - z^{-1}\right)^3}, |z| > 1 \tag{5.447}$$

$$y(n) = x(n) * u(n) \tag{5.448}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.449}$$

$$Y(z) = \left(\frac{8}{\left(1 - z^{-1}\right)^3}\right)\left(\frac{1}{1 - z^{-1}}\right) \tag{5.450}$$

$$= \frac{8}{\left(1 - z^{-1}\right)^4}, \quad |z| > 1 \tag{5.451}$$

Using Contour Integration to find the inverse Z-transform,

$$y(19) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{18}\, dz \tag{5.452}$$

$$= \frac{1}{2\pi j} \oint_C \frac{8z^{18}}{\left(1 - z^{-1}\right)^4}\, dz \tag{5.453}$$

We can observe that the pole is repeated 4 times and thus $m = 4$,

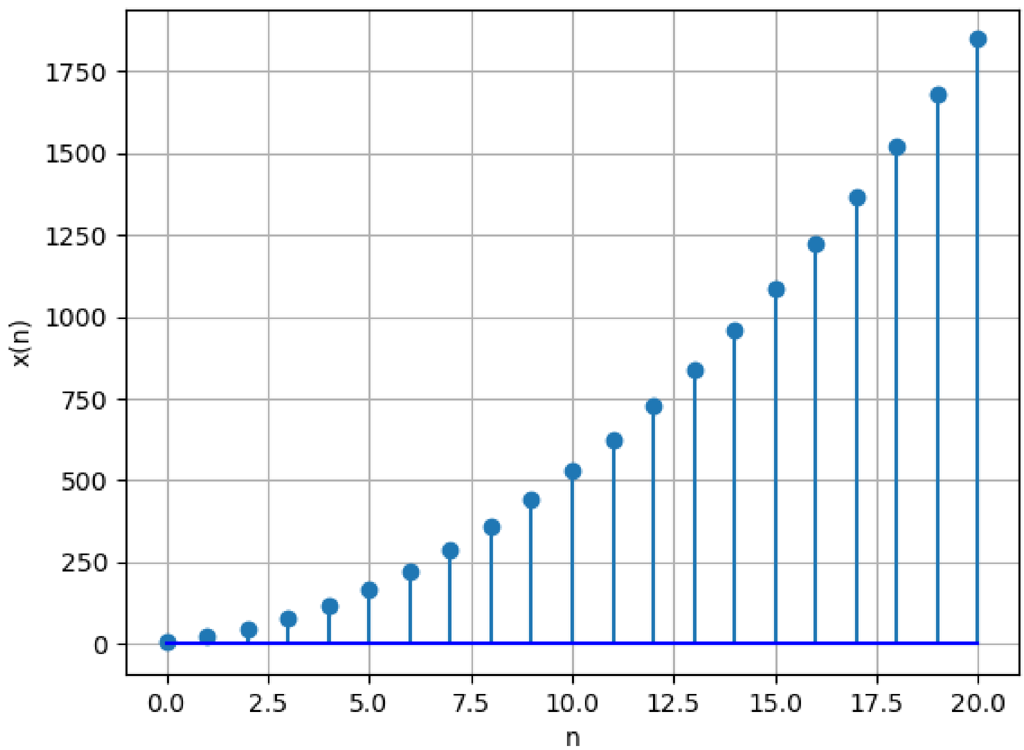
$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.454}$$

$$= \frac{1}{(3)!} \lim_{z \to 1} \frac{d^3}{dz^3} \left((z-1)^4 \frac{8z^{22}}{\left(z-1\right)^4}\right) \tag{5.455}$$

$$= \frac{4}{3} \lim_{z \to 1} \frac{d^3}{dz^3} (z^{22}) \tag{5.456}$$

$$= 12320 \tag{5.457}$$

$$\therefore \boxed{y(19) = 12320} \tag{5.458}$$



5.34 Find the sum of the following APs:

(a) $2, 7, 12, \ldots$ to 10 terms.

(b) $-37, -33, -29, \ldots$ to 12 terms.

(c) $0.6, 1.7, 2.8, \ldots$ to 100 terms.

(d) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots$ to 11 terms.

Solution: We have the general terms,

$$x(n) = [x(0) + nd]\, u(n) \tag{5.459}$$

Input Parameters	Values	Description
$x(0)$	$2, -37, 0.6, \frac{1}{15}$	First term of AP
d	$x(1) - x(0)$	Common difference of AP
$x(n)$	$[x(0) + nd]u(n)$	General term of AP
$y(n-1)$		Sum to n terms of AP

Table 5.34: Parameters

Now for sum,

$$y(n) = x(n) * u(n) \tag{5.460}$$

$$Y(z) = X(z)\,U(z) \tag{5.461}$$

From (**??**), we get $Y(z)$ as,

$$Y(z) = \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3} \tag{5.462}$$

Now using contour integration for each case,

(a)

$$x(0) = 2 \tag{5.463}$$

$$d = 5 \tag{5.464}$$

$$y(n) = \frac{1}{2\pi j}\oint_C Y(z)\,z^{n-1}dz \tag{5.465}$$

$$y(n) = \frac{1}{2\pi j}\oint_C \left(\frac{2z^{n-1}}{(1-z^{-1})^2} + \frac{5z^{n-2}}{(1-z^{-1})^3}\right)dz \tag{5.466}$$

$$\tag{5.467}$$

For R_1 we can observe that the pole has been repeated twice.

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{2z^{n+1}}{(z-1)^2} \right) \tag{5.468}$$

$$= 2(n+1) \tag{5.469}$$

For R_2 we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{5z^{n+1}}{(z-1)^3} \right) \tag{5.470}$$

$$= \frac{5(n)(n+1)}{2} \tag{5.471}$$

$$\implies R = R_1 + R_2 \tag{5.472}$$

Using (**??**) and (**??**),

$$y(n) = \frac{n+1}{2} (4+5n) u(n) \tag{5.473}$$

$$y(9) = 245 \tag{5.474}$$

(b)

$$x(0) = -37 \tag{5.475}$$

$$d = 4 \tag{5.476}$$

$$y(n) = \frac{1}{2\pi j} \oint_C \left(\frac{-37z^{n-1}}{(1-z^{-1})^2} + \frac{4z^{n-2}}{(1-z^{-1})^3} \right) dz \tag{5.477}$$

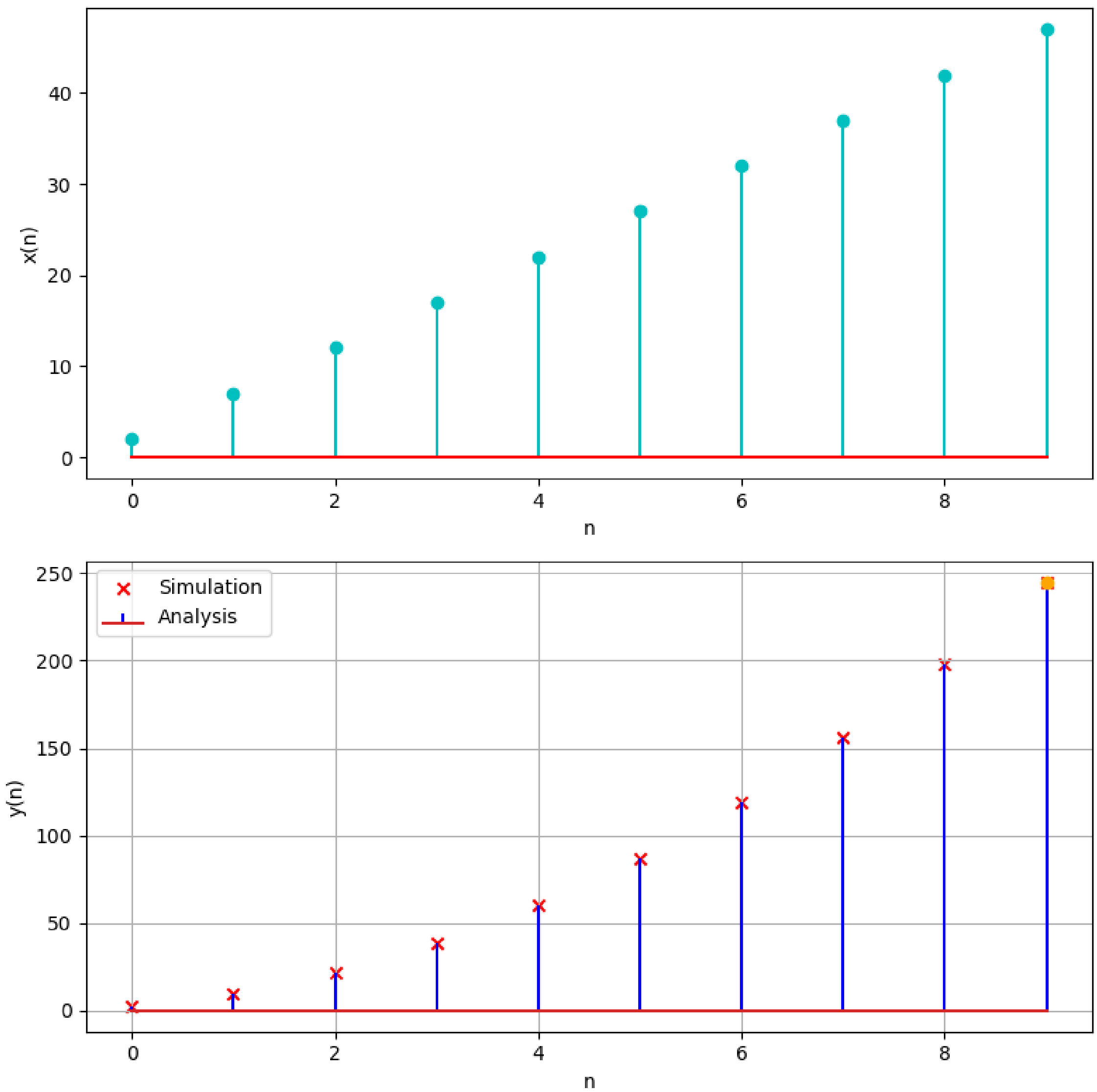


Figure 5.34: $1st$ AP

For R_1 the pole has been repeated twice.

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{-37 z^{n+1}}{(z-1)^2} \right) \tag{5.478}$$

$$= -37 \left(n + 1 \right) \tag{5.479}$$

For R_2 the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{4z^{n+1}}{(z-1)^3} \right) \tag{5.480}$$

$$= \frac{4(n)(n+1)}{2} \tag{5.481}$$

Using (**??**) and (**??**),

$$y(n) = \frac{n+1}{2} (-74 + 4n) u(n) \tag{5.482}$$

$$y(11) = -180 \tag{5.483}$$

(c)

$$x(0) = 0.6 \tag{5.484}$$

$$d = 1.1 \tag{5.485}$$

$$y(n) = \frac{1}{2\pi j} \oint_C \left(\frac{0.6z^{n-1}}{(1-z^{-1})^2} + \frac{1.1z^{n-2}}{(1-z^{-1})^3} \right) dz \tag{5.486}$$

For R_1 the pole is repeated twice.

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{0.6z^{n+1}}{(z-1)^2} \right) \tag{5.487}$$

$$= 0.6(n+1) \tag{5.488}$$

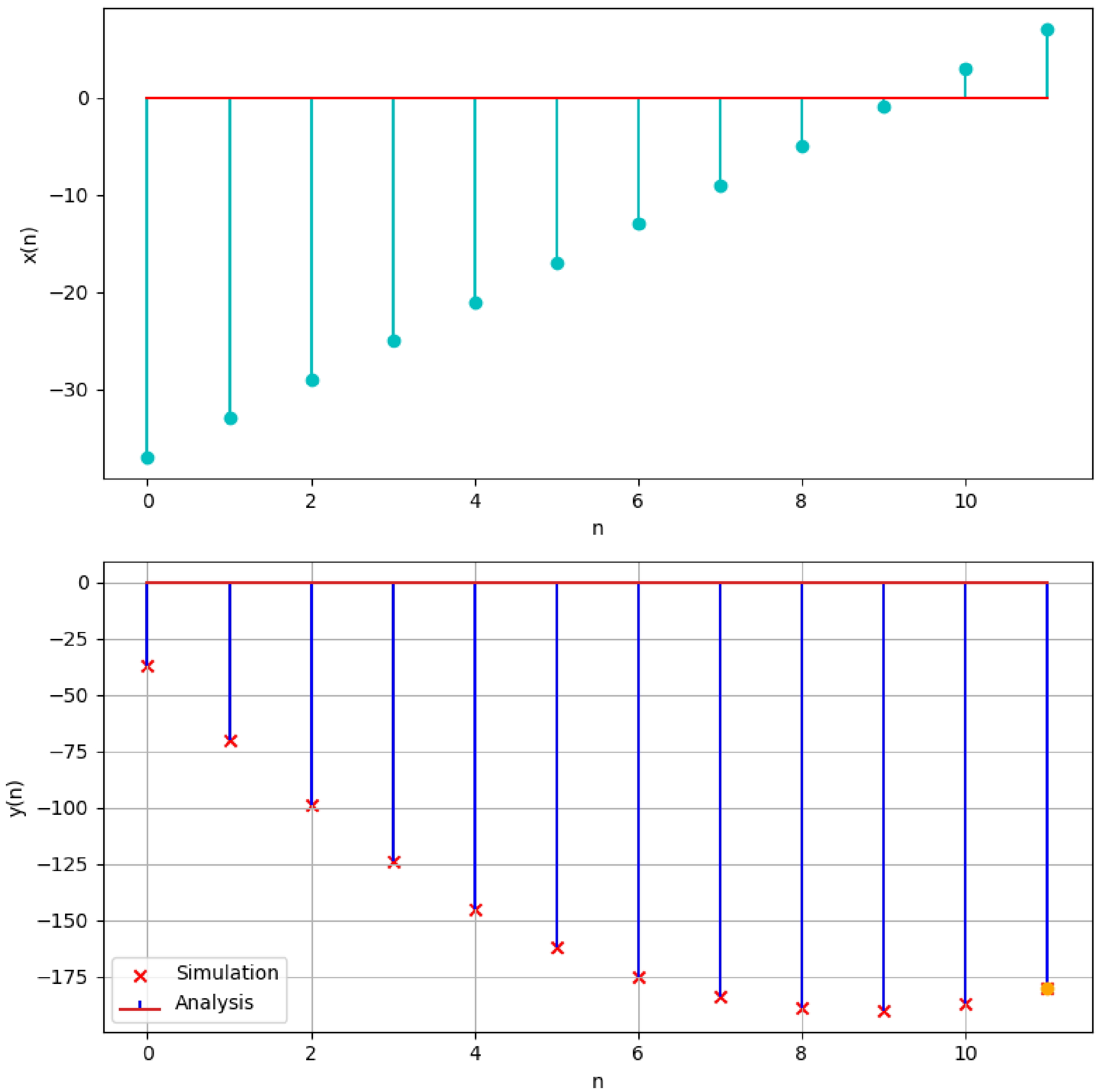


Figure 5.34: $2nd$ AP

For R_2 the pole is repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{1.1 z^{n+1}}{(z-1)^3} \right) \tag{5.489}$$

$$= \frac{1.1\,(n)\,(n+1)}{2} \tag{5.490}$$

Using (**??**) and (**??**),

$$y(n) = \frac{n+1}{2}(1.2 + 1.1n)\,u(n) \tag{5.491}$$

$$y(99) = 5505 \tag{5.492}$$

(d)

$$x(0) = \frac{1}{15} \tag{5.493}$$

$$d = \frac{1}{60} \tag{5.494}$$

$$y(n) = \frac{1}{2\pi j}\oint_C \left(\frac{\frac{1}{15}z^{n-1}}{(1-z^{-1})^2} + \frac{1.1z^{n-2}}{(1-z^{-1})^3}\right) dz \tag{5.495}$$

For R_1 the pole is repeated twice.

$$R_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{\frac{1}{15}z^{n+1}}{(z-1)^2}\right) \tag{5.496}$$

$$= \frac{1}{15}(n+1) \tag{5.497}$$

For R_2 the pole is repeated thrice.

$$R_2 = \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{\frac{1}{60}z^{n+1}}{(z-1)^3}\right) \tag{5.498}$$

$$= \frac{(n)(n+1)}{120} \tag{5.499}$$

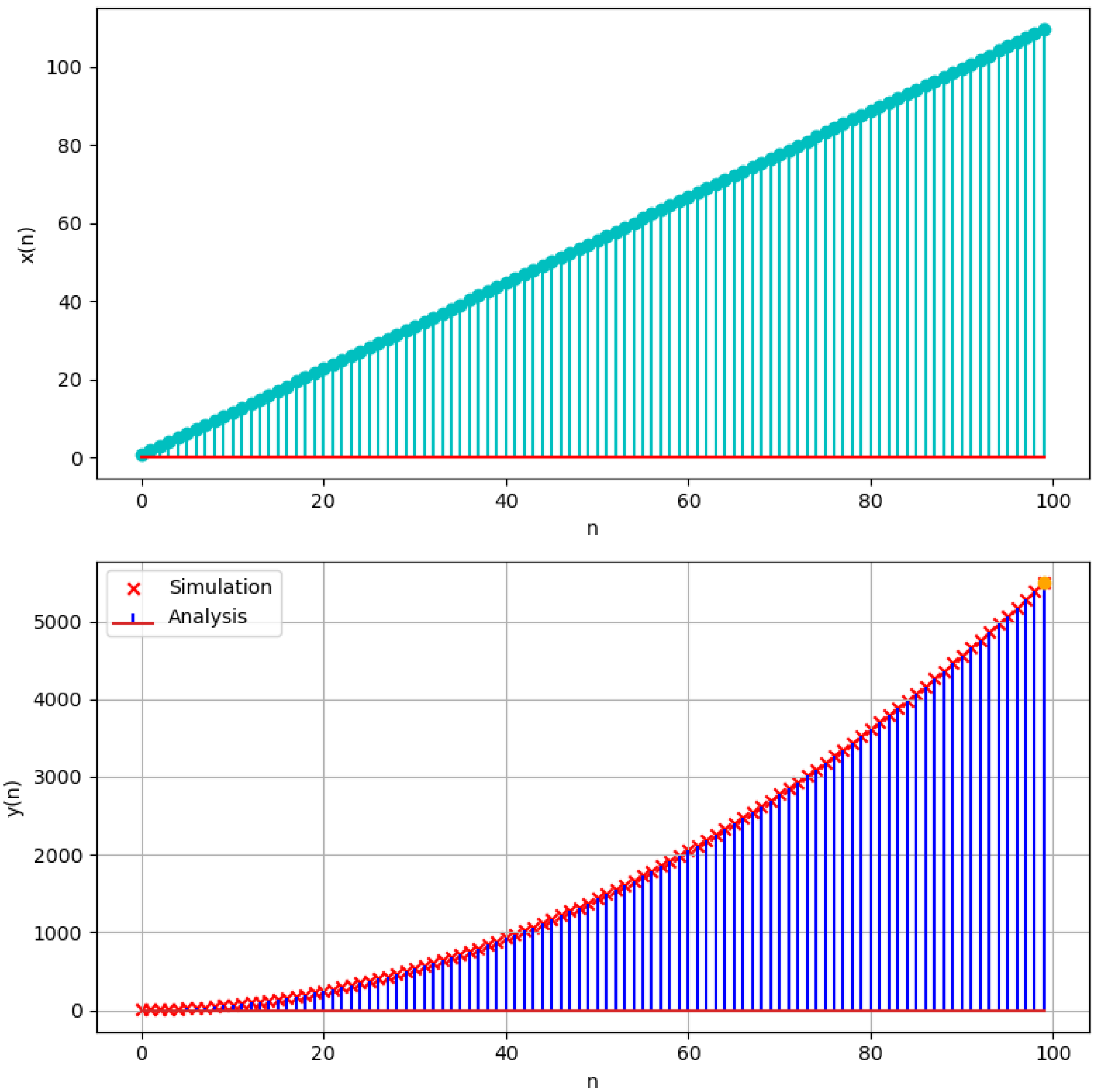


Figure 5.34: $4th$ AP

Using (??) and (??),

$$y(n) = \frac{n+1}{2}\left(\frac{2}{15} + \frac{n}{60}\right)u(n) \tag{5.500}$$

$$y(10) = 1.65 \tag{5.501}$$

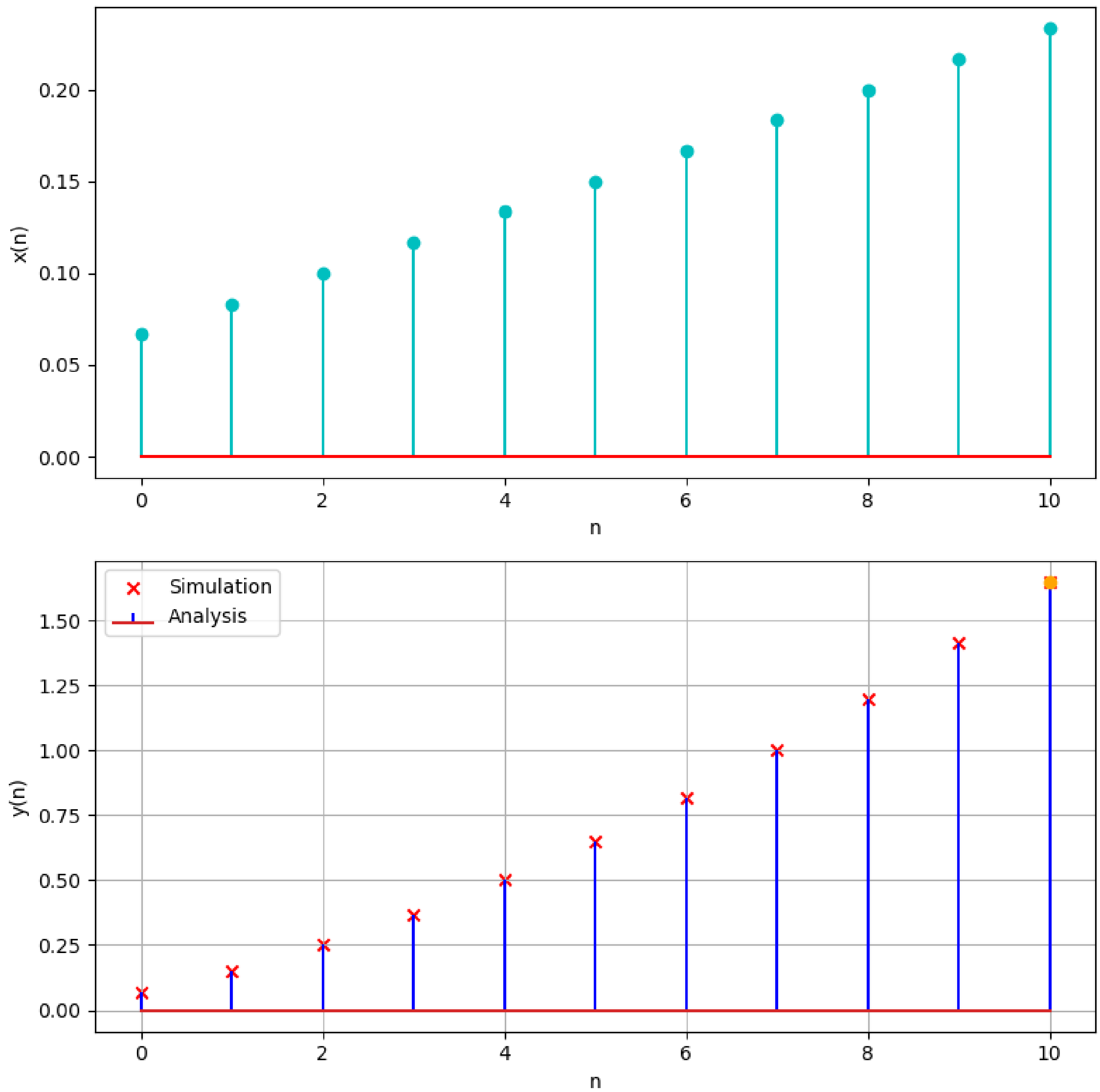


Figure 5.34: $4th$ AP

Chapter 6

Laplace Transform

6.0.1 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of

(a) The spring constant K

(b) The damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

Solution: The parameters are :

Parameter	**Value(SI)**	**Description**
x_0	0.15	Initial spring compression
m	750	Mass
g	9.8	Gravitational acc
k	mg/x_0	Spring constant
b		Damping constant

Table 6.1: Input Parameters

Parameter	**Value(SI)**	**Description**
x		Spring Extension
F_1	kx	Spring Force
F_2	$b\frac{dx}{dt}$	Damping Force
s		Complex Frequency
s_1, s_2		Values of s

Table 6.2: Intermediate Parameters

Initially the automobile is in rest, so we can use,

$$mg = kx_0 \tag{6.1}$$

$$\implies k = \frac{mg}{x_0} \tag{6.2}$$

Now, as the oscillation begins, from the FBD, we have net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{6.3}$$

$$\implies -m\frac{d^2x(t)}{dt^2} = kx(t) + b\frac{dx(t)}{dt} + mgu(t) \tag{6.4}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = -gu(t) \tag{6.5}$$

Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{b}{m}sX(s) + \frac{k}{m}X(s) = -\frac{g}{s} \tag{6.6}$$

$$\implies X(s) = -\frac{g}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \tag{6.7}$$

$$\implies X(s) = -\frac{g}{s(s-s_1)(s-s_2)} \tag{6.8}$$

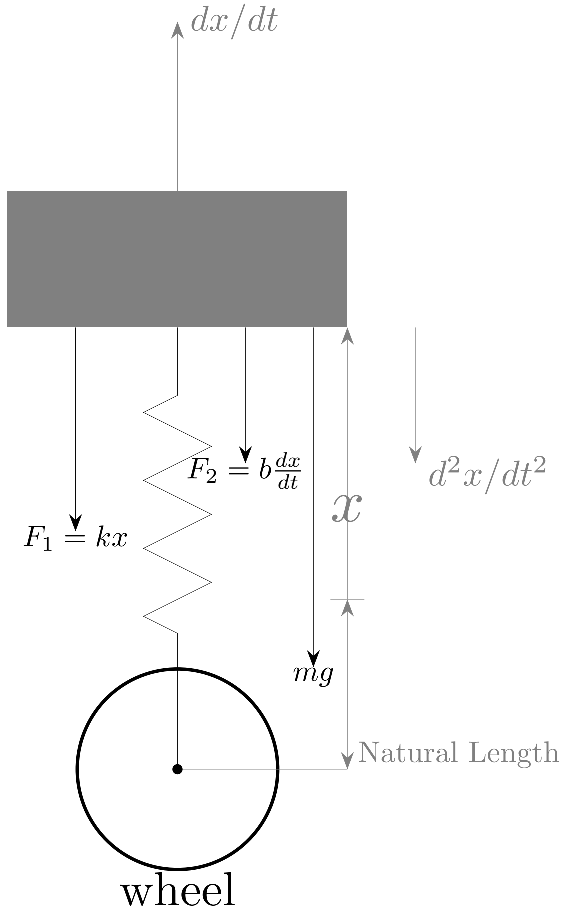


Figure 6.1: FBD of the damped oscillation system

Where

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{6.9}$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{6.10}$$

From (**??**) we get,

$$\implies X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2(s-s_2)} - \frac{1}{s_1(s-s_1)}\right] \\ + \frac{g}{s_1 s_2}\left(\frac{1}{s}\right) \tag{6.11}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = \frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2}e^{s_2 t} - \frac{1}{s_1}e^{s_1 t}\right]u(t) \tag{6.12}$$

$$\implies x(t) = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bt/2m}u(t) \\ \sin\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}{gb}\right)\right) \\ + \frac{mg}{k}u(t) \tag{6.13}$$

(Substituting the values of s_1 and s_2 from (**??**) and (**??**))

From (**??**) we have the amplitude after one time period T,

$$\frac{1}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} = \\ \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bT/2m} \tag{6.14}$$

$$\implies e^{\pi b/\sqrt{mk}} = 2 \tag{6.15}$$

$$\implies b = \frac{\sqrt{mk}\ln 2}{\pi} \tag{6.16}$$

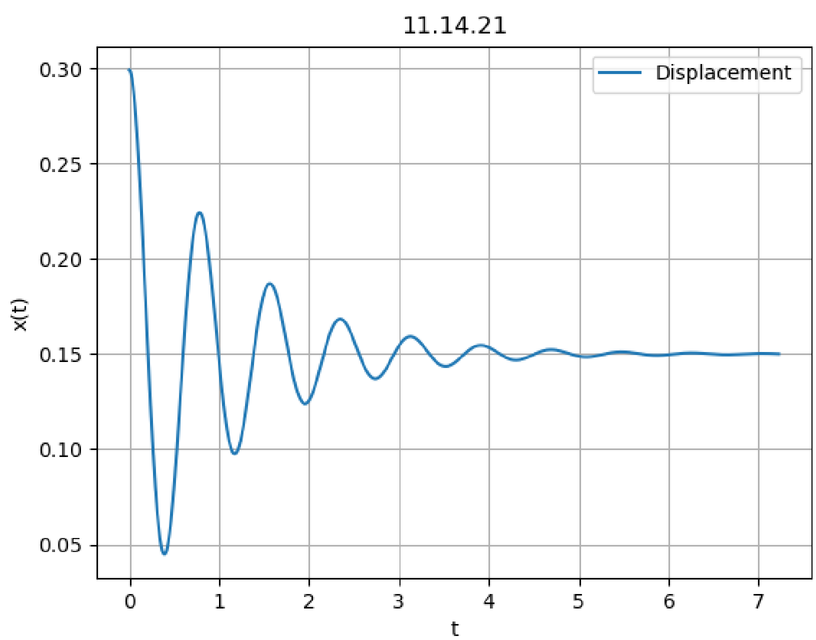


Figure 6.2: Displacement Vs. Time Graph

6.0.2 A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0, and v_0 . [Hint : Start with the equation $x = a\cos(\omega t + \theta)$ and note that the initial velocity is negative.]

Solution:

Chapter 7

Systems

7.0.1 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving in a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period? **Solution:**

Parameter	Description
v	Speed
R	Radius of circular track
M	Mass of bob
g	Acceleration due to gravity
a_c	Centrifugal acceleration
g_e	Effective gravitational acceleration $\sqrt{g^2+a^2}$
Residual Formula for $m=1$	$\lim_{s\to a}\left(\left(s-a\right)\theta\left(t\right)e^{st}\right)$

Table 7.1: Parameters

From the figure, restoring force:

$$F_r = -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + \left(Mg\right)^2}\right)\sin\theta\left(t\right) \tag{7.1}$$

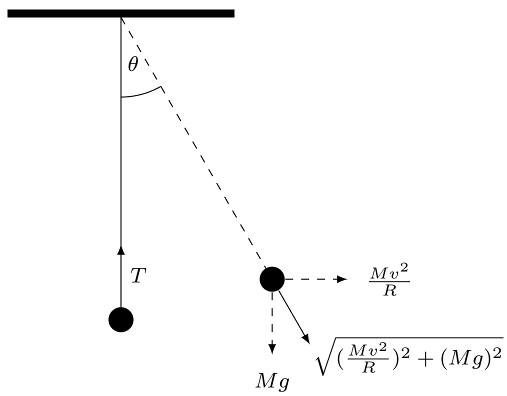


Figure 7.1: Free Body Diagram

For small oscillations, $\theta(t) << 1$.

$$\implies \sin\theta(t) \approx \theta(t) \tag{7.2}$$

$$\implies F_r \approx -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + (Mg)^2}\right)\theta(t) \tag{7.3}$$

$$\implies a = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta(t) \tag{7.4}$$

$$l\frac{d^2\theta(t)}{dt^2} = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta(t) \tag{7.5}$$

$$\text{Let } k = \frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right) \tag{7.6}$$

$$\frac{d^2\theta(t)}{dt^2} + k\theta(t) = 0 \tag{7.7}$$

$$\tag{7.8}$$

Taking Laplace transform:

$$s^2\Theta\left(s\right) - s\theta(0) - \theta'(0) + k\Theta\left(s\right) = 0 \tag{7.9}$$

Assuming $\theta(0) = 0$:

$$s^2\Theta\left(s\right) - \theta'(0) + k\Theta\left(s\right) = 0 \tag{7.10}$$

$$\implies \Theta\left(s\right) = \frac{\theta'(0)}{s^2 + k} \tag{7.11}$$

Taking inverse laplace transform using Bromwich integral:

$$\theta(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Theta(s)e^{st}dt, c > 0 \tag{7.12}$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{\theta'(0)e^{st}}{s^2 + k}dt \tag{7.13}$$

Since the poles $s = j\sqrt{k}$ and $s = -j\sqrt{k}$ (both non repeated, i.e. $m = 1$) lie inside a semicircle for some $c > 0$. Using Jordans lemma and method of residues from **??**:

$$R_1 = \lim_{s \to j\sqrt{k}} \left(\left(s - j\sqrt{k}\right) \left(\frac{\theta'(0)}{s^2 + k}\right) e^{st} \right) \tag{7.14}$$

$$= \left(\frac{\theta'(0) e^{j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{7.15}$$

$$R_2 = \lim_{s \to -j\sqrt{k}} \left(\left(s + j\sqrt{k}\right) \left(\frac{\theta'(0)}{s^2 + k}\right) e^{st} \right) \tag{7.16}$$

$$= \left(\frac{-\theta'(0) e^{-j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{7.17}$$

$$\theta(t) = R_1 + R_2 \tag{7.18}$$

$$\theta(t) = \left(\frac{\theta'(0)\left(e^{j\sqrt{k}t} - e^{-j\sqrt{k}t}\right)}{2j\sqrt{k}} \right) \tag{7.19}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{k}} \sin\left(\sqrt{k}t\right) \tag{7.20}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)}} \sin\left(\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)}\, t\right) \tag{7.21}$$

$$\frac{2\pi}{T} = \sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)} \tag{7.22}$$

$$\implies T = 2\pi \sqrt{\frac{\ell}{\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}}} \tag{7.23}$$

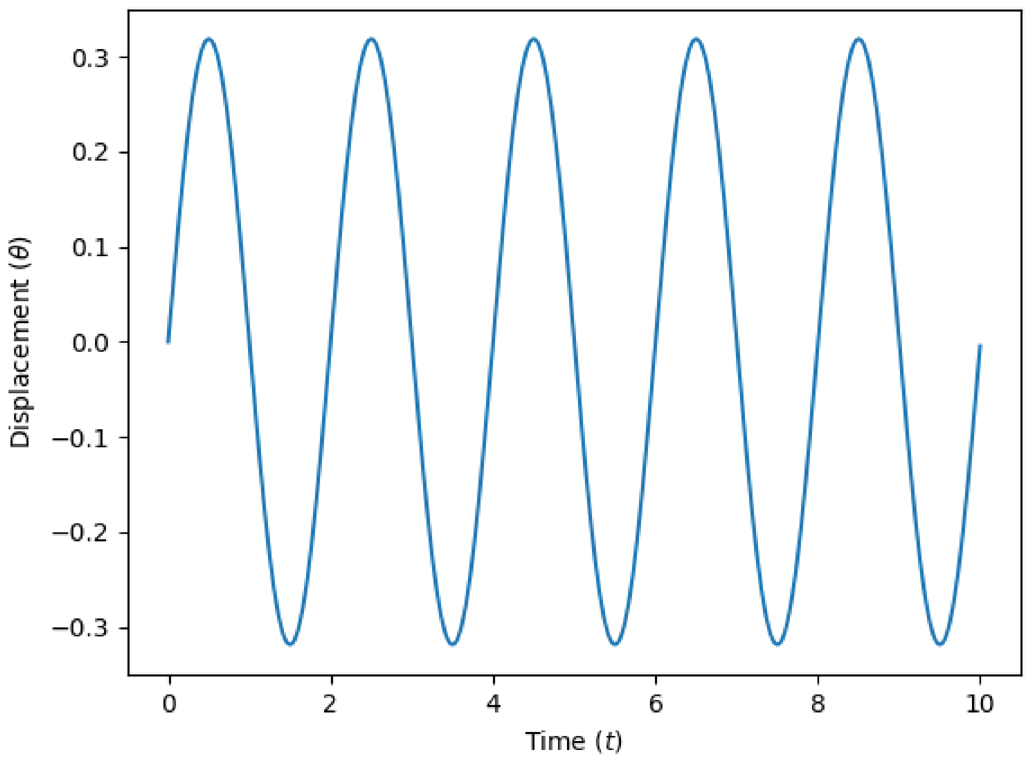


Figure 7.2: $\theta(t)$ vs t for $v = 1, R = 1, \ell = 1, g = 9.8, \theta'(0) = 1$

7.0.2 A spring balance has a scale that that reads from 0 to 50 kg.The length of the scale is 20 cm.A body is suspended from this balance,when displaced and released, oscillates with a period of 0.6 s.What is weight of the body? **Solution:**

$$F = ma = -kx \tag{7.24}$$

This equation can be rearranged as:

Table 7.2: Input Parameters

Parameter	**Value**	**Description**
M	50 kg	Mass of block
l	0.2 m	Maximum displacement of spring
T	0.6 s	Time period of oscillation
F	490 N	Force
k	2450 N/m	Spring Constant

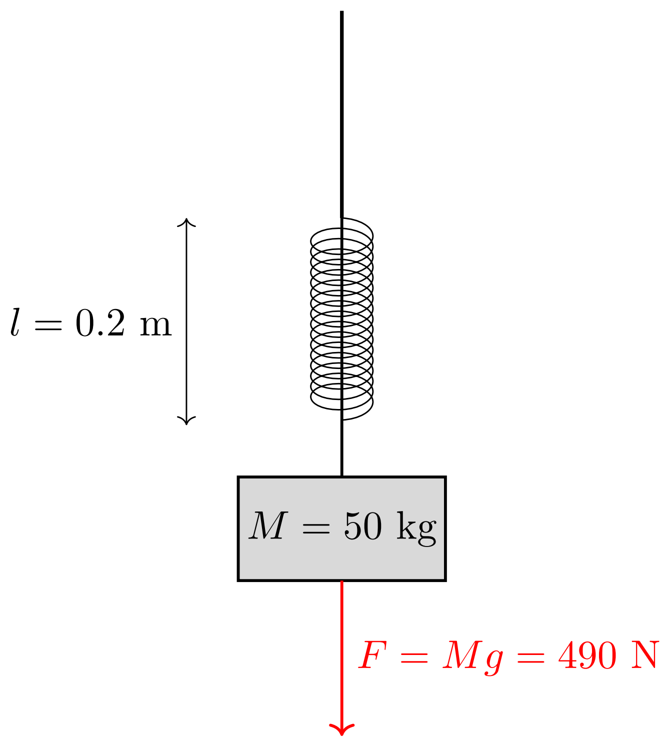


Figure 7.3: spring-mass system

$$ma = -kx \tag{7.25}$$

$$m\frac{d^2x}{dt^2} = -kx \tag{7.26}$$

Derivation of Simple Harmonic Motion Period using Laplace Transform

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{7.27}$$

Take the Laplace transform of both sides:

$$s^2X(s) - sx(0) - x'(0) + \frac{k}{m}X(s) = 0 \tag{7.28}$$

Rearrange terms to solve for $X(s)$:

$$X(s)(s^2 + \frac{k}{m}) = sx(0) + x'(0) \tag{7.29}$$

Solve for $X(s)$:

$$X(s) = \frac{sx(0) + x'(0)}{s^2 + \frac{k}{m}} \tag{7.30}$$

Find the roots of the characteristic equation:

$$s^2 + \frac{k}{m} = 0 \tag{7.31}$$

Let $\omega^2 = \frac{k}{m}$, then $s = \pm j\omega$.

Express ω in terms of T:

$$T = \frac{2\pi}{\omega} \tag{7.32}$$

Solve for ω in terms of T:

$$\omega = \frac{2\pi}{T} \tag{7.33}$$

Substitute ω back into the characteristic equation:

$$s = \pm j\frac{2\pi}{T} \tag{7.34}$$

Now, the Laplace transform solution $X(s)$ becomes:

Express s in terms of ω:

$$X(s) = \frac{s(x_0) + x'(0)}{(j\omega)^2 + \omega^2} \tag{7.35}$$

Simplify and take the inverse Laplace transform to obtain the displacement $x(t)$ in the time domain:

$$x(t) = A\cos(\omega t + \phi) \tag{7.36}$$

$$A = l = 0.2m \tag{7.37}$$

$$\phi = 0 \tag{7.38}$$

$$\omega = \frac{2\pi}{T} = 10.47\text{rad/s} \tag{7.39}$$

Where A is the amplitude, ω is the angular frequency, and ϕ is the phase angle. The period T is related to the angular frequency by $T = \frac{2\pi}{\omega}$, giving the desired result:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{7.40}$$

The weight of th body is defined as:

$$\text{Weight} = mg = 22.36 \times 9.8 = 219.16\,\text{N} \tag{7.41}$$

Therefore, the weight of the body is approximately 219 N.

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{A.1.1}$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

A.3 If

$$x(n) = 0, \quad n < 0, \tag{A.3.1}$$

from (??),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{A.3.2}$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n) z^{-n} \tag{B.1.1}$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \tag{B.2.1}$$

$$P(z) = P_1(z) P_2(z) \tag{B.2.2}$$

B.3

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zX'(z) \tag{B.3.1}$$

From (**??**)

$$\implies nu\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}, |z| > 1 \tag{B.3.2}$$

$$\implies n^{2}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^{3}}, |z| > 1 \tag{B.3.3}$$

$$\implies n^{3}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^{4}}, |z| > 1 \tag{B.3.4}$$

$$\implies n^{4}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^{5}} \tag{B.3.5}$$

where $|z| > 1$

B.4

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \tag{B.4.1}$$

Using (**??**):

$$nu\left(n-1\right) \overset{\mathcal{Z}}{\longleftrightarrow} z\frac{2z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.2}$$

Now ,

$$\frac{\left(n-1\right)}{2}u\left(n-2\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.3}$$

$$\frac{\left(n-1\right)\left(n-2\right)}{6}u\left(n-3\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{\left(1-z^{-1}\right)^{3}} \tag{B.4.4}$$

$$\vdots$$

$$\frac{\left(n-1\right)\left(n-2\right)\dots\left(n-k+1\right)}{(k-1)!}u\left(n-k\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{\left(1-z^{-1}\right)^{k}} \tag{B.4.5}$$

$$\implies Z^{-1}\left[\frac{z^{-2}}{\left(1-z^{-1}\right)^{2}}\right]=(n-1)\,u\,(n-1) \tag{B.4.6}$$

$$\implies Z^{-1}\left[\frac{z^{-3}}{\left(1-z^{-1}\right)^{3}}\right]=\frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{B.4.7}$$

$$\implies Z^{-1}\left[\frac{z^{-4}}{\left(1-z^{-1}\right)^{4}}\right]=\frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{B.4.8}$$

$$\implies Z^{-1}\left[\frac{z^{-5}}{\left(1-z^{-1}\right)^{5}}\right]=\frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{B.4.9}$$

$$u\,(n-1)$$

B.5 For a Geometric progression

$$x\,(n)=x\,(0)\,r^{n}u\,(n)\,, \tag{B.5.1}$$

$$\implies X\,(z)=\sum_{n=-\infty}^{\infty}x\,(n)\,z^{-n}=\sum_{n=0}^{\infty}x\,(0)\,r^{n}z^{-n} \tag{B.5.2}$$

$$=\sum_{n=0}^{\infty}x\,(0)\left(rz^{-1}\right)^{n} \tag{B.5.3}$$

$$=\frac{x\,(0)}{1-rz^{-1}},\quad |z|>|r| \tag{B.5.4}$$

B.6 Substituting $r=1$ in (**??**),

$$u(n)\overset{\mathcal{Z}}{\longleftrightarrow}U(z)=\frac{1}{1-z^{-1}},\quad |z|>1 \tag{B.6.1}$$

B.7 From (**??**) and (**??**),

$$nu(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})^{2}},\quad |z|>1 \tag{B.7.1}$$

B.8 For an AP,

$$x(n) = [x(0) + nd]\, u(n) = x(0)u(n) + dnu(n) \tag{B.8.1}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.8.2}$$

upon substituting from (**??**) and (**??**).

B.9 From (**??**), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{B.9.1}$$

where $x(n)$ is defined in (**??**). From (**??**), (**??**) and (**??**),

$$Y(z) = X(z)\, U(z) \tag{B.9.2}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > |1| \tag{B.9.3}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{B.9.4}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1}\left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{B.9.5}$$

using partial fractions. Again, from (**??**) and (**??**), the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^{n+1} - 1}{r - 1}\right) u(n) \tag{B.9.6}$$

B.10 For the AP $x(n)$, the sum of first $n+1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{B.10.1}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{B.10.2}$$

$$= x(n) * u(n) \tag{B.10.3}$$

Taking the Z-transform on both sides, and substituting (**??**) and (**??**),

$$Y(z) = X(z)U(z) \tag{B.10.4}$$

$$\implies Y(z) = \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}\right)\frac{1}{1-z^{-1}} \quad |z| > 1 \tag{B.10.5}$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \tag{B.10.6}$$

B.11 From (**??**) and (**??**),

$$(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \tag{B.11.1}$$

From (**??**) and (**??**),

$$n(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \tag{B.11.2}$$

B.12 Taking the inverse Z-transform of (**??**),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{B.12.1}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\}u(n) \tag{B.12.2}$$

Symbol	Description
x(n)	$\frac{1}{n+c}u\left(n\right), c \geq 2$
$w\left(n\right)$	$\frac{1}{n+1}u\left(n\right)$
$D\left(z\right)$	$z^2 log\left(1-z^{-1}\right)$

Table B.13.1: Notations

B.13

$$\delta\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} 1 \tag{B.13.1}$$

$$\delta\left(n+k\right) \overset{\mathcal{Z}}{\longleftrightarrow} z^k, \forall k \in \mathbb{R} \tag{B.13.2}$$

$$\log\left(1-x\right) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \tag{B.13.3}$$

Let $x\left(n\right)$ and $w\left(n\right)$ be some expression as described in the table in discrete time domain, whose Z transform, $X\left(z\right)$ and $W\left(z\right)$ respectively, shall be obtained for future reference.

$$\text{For } x\left(n\right) = \frac{1}{n+c}u\left(n\right), \forall c \geq 2, c \in \mathbb{N}$$

$$X\left(z\right) = \sum_{n=-\infty}^{n=+\infty} x\left(n\right) z^{-n} \tag{B.13.4}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-n} \tag{B.13.5}$$

$$= z^c \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-(n+c)} \tag{B.13.6}$$

Using, (**??**)

$$X(z) = z^c\Bigg(-\log\left(1-z^{-1}\right) - z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3} - \ldots - \frac{z^{-(c-1)}}{c-1}\Bigg)\ \forall c \geq 2, c \in \mathbb{N} \tag{B.13.7}$$

For $w(n) = \dfrac{1}{n+1}u(n)$,

$$W(z) = \sum_{n=-\infty}^{n=+\infty} w(n)\, z^{-n} \tag{B.13.8}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-n} \tag{B.13.9}$$

$$= z\sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-(n+1)} \tag{B.13.10}$$

Using (**??**),

$$W(z) = -z\log\left(1-z^{-1}\right) \tag{B.13.11}$$

Let $D(z)$ be some expression as described in the table in Z domain, whose inverse Z transform, $d(n)$ shall be obtained for future reference.

For $D(z) = z^k \log\left(1 - z^{-1}\right) \forall k \geq 1, k \in \mathbb{Z}$

$$D(z) = \left(-z^{k-1} - \frac{1}{2}z^{k-2} - \frac{1}{3}z^{k-3} - \frac{1}{4}z^{k-4} - \dots\right) \tag{B.13.12}$$

Using (**??**),

$$\begin{aligned} d(n) = &\left(-\delta(n+k-1) - \frac{1}{2}\delta(n+k-2) - \frac{1}{3}\delta(n+k-3)\right. \\ &\left.- \frac{1}{4}\delta(n+k-4) - \dots - \frac{1}{n+k}\delta(0) - \dots\right) &\text{(B.13.13)} \\ = &\frac{-1}{n+k}u(n) &\text{(B.13.14)} \end{aligned}$$

Appendix C

Contour Integration

C.1

$$x(n) \xrightarrow{Z} X(z) \tag{C.1.1}$$

$$\implies X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{C.1.2}$$

Multiplying both side with z^{k-1} and integrating on a contour integral enclosing the region of convergence. Where C is a counter-clockwise closed contour in region of convergence.

$$\frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz = \frac{1}{2\pi j} \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{-n+k-1} dz \tag{C.1.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \frac{1}{2\pi j} \oint_C z^{-n+k-1}.dz \tag{C.1.4}$$

From cauchy's integral theorem

$$\frac{1}{2\pi j} \oint_C z^{-k} dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \tag{C.1.5}$$

$$= \delta(1-k) \tag{C.1.6}$$

So eq (**??**) becomes

$$\frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz = \sum_{k=-\infty}^{\infty} x(k)\, \delta(k-n) \tag{C.1.7}$$

$$\implies x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.8}$$

Contour integrals like (**??**) can be evaluated using Cauchy's residue theorem.

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.9}$$

$$= \sum \left[\text{Residue of } X(z)\, z^{n-1} \text{ at poles inside } C\right] \tag{C.1.10}$$

C.2 Question: Find the sum of n terms of an AP where common difference $= d$ using Contour Integration.

Solution:

By performing inverse Z transform on S(z) using contour integration

$$s(n) = \frac{1}{2\pi j}\oint_C S(z)\ z^{n-1}\ dz \tag{C.2.1}$$

$$s(n) = \frac{1}{2\pi j}\oint_C \left(\frac{x(0)\, z^{n-1}}{(1-z^{-1})^2} + \frac{dz^{n-2}}{(1-z^{-1})^3}\right)\ dz \tag{C.2.2}$$

For R_1 we can observe that the pole has been repeated twice.

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{C.2.3}$$

$$R_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{x(0)\, z^{n+1}}{(z-1)^2}\right) \tag{C.2.4}$$

$$= x(0)\,(n+1)\lim_{z\to 1}(z^n) \tag{C.2.5}$$

$$= x(0)\,(n+1) \tag{C.2.6}$$

For R_2 we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{dz^{n+1}}{(z-1)^3} \right) \tag{C.2.7}$$

$$= \frac{d(n+1)}{2} \lim_{z \to 1} \frac{d}{dz} (z^n) \tag{C.2.8}$$

$$= \frac{d(n+1)(n)}{2} \lim_{z \to 1} \left(z^{n-1}\right) \tag{C.2.9}$$

$$= \frac{d(n)(n+1)}{2} \tag{C.2.10}$$

$$\implies R = R_1 + R_2 \tag{C.2.11}$$

Using (**??**) and (**??**) in (**??**)

$$R = x(0)(n+1) + \frac{d(n)(n+1)}{2} \tag{C.2.12}$$

Finally,

$$s(n) = x(0)(n+1)u(n) + d\left(\frac{n(n+1)}{2}\right)u(n) \tag{C.2.13}$$

$$= \frac{n+1}{2}(2x(0) + nd)u(n) \tag{C.2.14}$$

C.3 Question: Find the sum of n terms of GP where common ratio is r using Contour Integration.

Solution:

Symbol	Value	Description
$x(n)$	$x(0)r^n u(n)$	n^{th} n^{th} term of gp G.P
$x(0)$	$x(0)$	1^{st} term of the G.P
d	r	Common ratio
$s(n)$	$\sum_{k=0}^{n} x(k)$	Sum of n terms of GP

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{C.3.1}$$

$$= \sum_{n=-\infty}^{\infty} x(0)r^n u(n) z^{-n} \tag{C.3.2}$$

$$= \sum_{n=0}^{\infty} x(0)r^n z^{-n} \tag{C.3.3}$$

$$= \frac{x(0)}{1 - rz^{-1}} \tag{C.3.4}$$

$$U(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{C.3.5}$$

Now we will perform inverse Z transform on S(z) using contour integration to find s(n)

$$
\begin{aligned}
s(n) &= \frac{1}{2\pi j} \oint_C S(z)\, z^{n-1}\, dz && \text{(C.3.10)}\\
&= \frac{1}{2\pi j} \oint_C \frac{x(0) z^{n-1}}{(1 - rz^{-1})(1 - z^{-1})}\, dz && \text{(C.3.11)}\\
&= \frac{1}{2\pi j} \oint_C \frac{x(0) z^{n+1}}{(z - r)(z - 1)}\, dz && \text{(C.3.12)}\\
&= \frac{x(0)}{r - 1} \left(\frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - r}\, dz - \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - 1} \right) dz && \text{(C.3.13)}
\end{aligned}
$$

we already know;

$$
R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z) \right) \tag{C.3.14}
$$

Now for first contour integral,

$$
\begin{aligned}
R_1 &= \frac{1}{(1-1)!} \lim_{z \to a} \left((z - a) f(z) \right) && \text{(C.3.15)}\\
&= \lim_{z \to r} \left((z - r) \frac{z^{n+1}}{z - r} \right) && \text{(C.3.16)}\\
&= \lim_{z \to r} \left(z^{n+1} \right) && \text{(C.3.17)}\\
&= r^{n+1} && \text{(C.3.18)}
\end{aligned}
$$

for second contour integral,

$$
\begin{aligned}
R_2 &= \frac{1}{(1-1)!} \lim_{z \to a} \left((z-a) f(z)\right) && \text{(C.3.19)} \\
&= \lim_{z \to 1} \left((z-1) \frac{z^{n+1}}{z-1}\right) && \text{(C.3.20)} \\
&= \lim_{z \to 1} \left(z^{n+1}\right) && \text{(C.3.21)} \\
&= 1 && \text{(C.3.22)}
\end{aligned}
$$

So finally the sum of n terms of the GP is given by:

$$
\begin{aligned}
s(n) &= \frac{x(0)}{r-1} \left(R_1 - R_2\right) && \text{(C.3.23)} \\
&= \frac{x(0)}{r-1} \left(r^{n+1} - 1\right) && \text{(C.3.24)}
\end{aligned}
$$

Appendix D

Fourier Series

D.1 Complex Fourier Series

Consider,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nft} \tag{D.1.1}$$

where c_n is the exponential fourier coefficient.

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-j2\pi nft}\,dt \tag{D.1.2}$$

where T is the time period of function $x(t)$.

D.2 Trigonometric Fourier Series

We can write:

$$e^{j2\pi nft} = cos(2\pi nftt) + jsin(2\pi nft) \tag{D.2.1}$$

Substituting (**??**) in (**??**)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \left(\cos(2\pi nft) + j\sin(2\pi nft)\right) \tag{D.2.2}$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi nft)) + (b_n \sin(2\pi nft)) \tag{D.2.3}$$

where a_0, a_n and b_n are trigonometric fourier series.

$$a_0 = c_0 \tag{D.2.4}$$

$$= \frac{1}{T}\int_0^T x(t)\,dt \tag{D.2.5}$$

$$a_n = 2Re(c_n) \tag{D.2.6}$$

$$= \frac{2}{T}\int_0^T x(t)\cos(2\pi nft)\,dt \tag{D.2.7}$$

$$b_n = -2Im(c_n) \tag{D.2.8}$$

$$= \frac{2}{T}\int_0^T x(t)\sin(2\pi nft)\,dt \tag{D.2.9}$$