# SIGNAL PROCESSING FUNDAMENTALS
# Through NCERT

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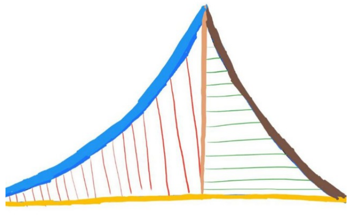


# Contents

# Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

# Chapter 1

# Harmonics

1.0.1 A circular disk of mass 10kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5s. The radius of the disc is 15cm. Determine the torsional spring constant of the wire. (Torsional spring constant $\alpha$ is defined by the relation J=-$\alpha\theta$, where J is the restoring couple and $\theta$ is the angle of twist).

**Solution:**

1.0.2 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0$ MHz.

(a) Determine, $B_0, \omega, k$ and $\lambda$

(b) Find expressions for $\mathbf{E}$ and $\mathbf{B}$

**Solution:**

Table 1.1: Input Parameters

| Symbol | Description | value |
|---|---|---|
| $f$ | frequency of source | 50.0 MHz |
| $E_0$ | Electric field amplitude | 120 N/C |
| $c$ | speed of light | 3 x $10^8$ m/s |
| $\mathbf{e_2}, \mathbf{e_3}$ | Standard Basis vectors | N/A |

Table 1.2: Formulae and Output

| Symbol | Description | Formula | Value |
|---|---|---|---|
| $\mathbf{E}$ | Electric field vector | $E_0 \sin(kx - 2\pi ft)\mathbf{e_2}$ | $120 \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_2}$ |
| $\mathbf{B}$ | Magnetic field vector | $B_0 \sin(kx - 2\pi ft)\mathbf{e_3}$ | $(4x10^{-7}) \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_3}$ |
| $B_0$ | Magnetic field strength | $\frac{E_0}{c}$ | 400nT |
| $\omega$ | Angular frequency | $2\pi f$ | 3.14 x $10^8$m/s |
| $k$ | Propagation constant | $\frac{2\pi f}{c}$ | 1.05rad/s |
| $\lambda$ | Wavelength | $\frac{c}{f}$ | 6.0m |

1.0.3 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

**Solution:**

| Symbol | Value | Description |
|---|---|---|
| $y(t)$ | $\cos(2\pi f_c t)$ | Wave equation of electro-magnetic wave |
| $f_c$ | $10^9$ | Frequency of electromagnetic wave |
| $t$ | seconds | Time |

Table 1.0.3: Variable description

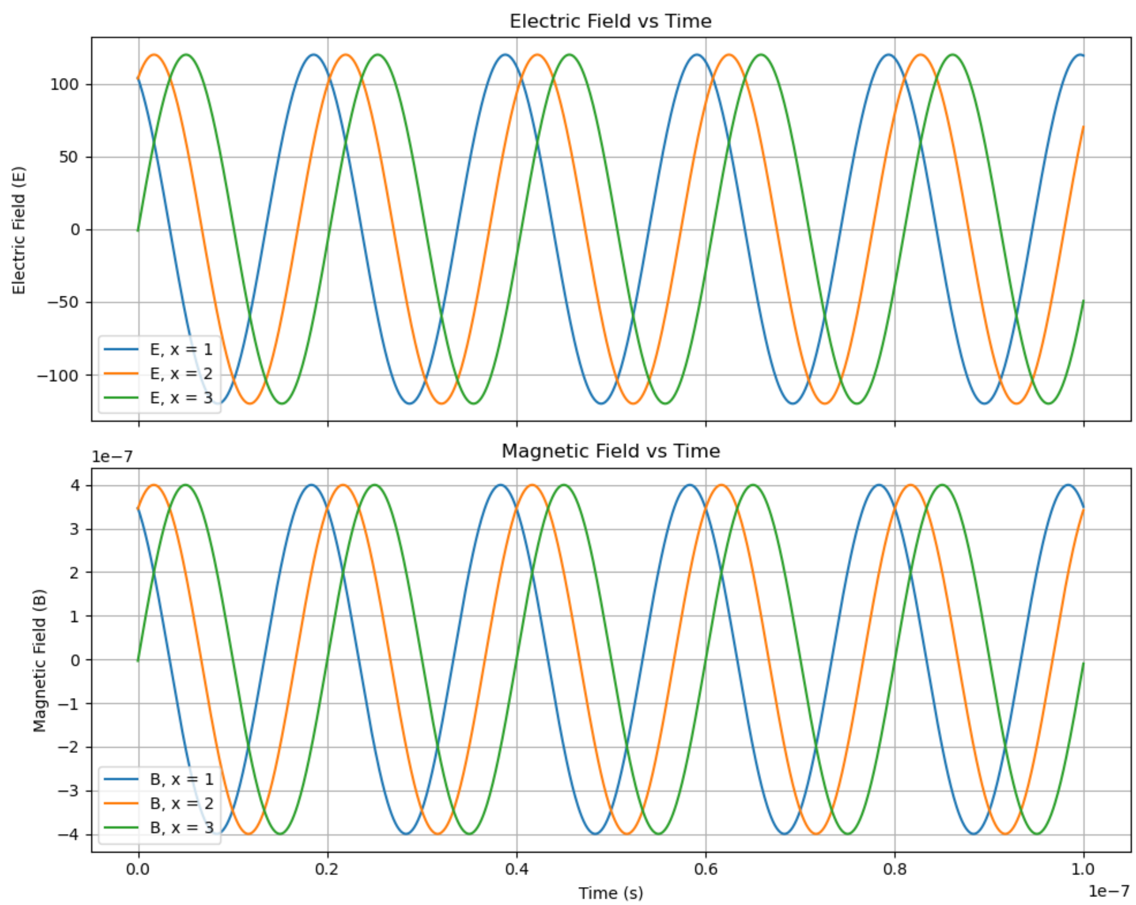



Figure 1.0.2: Graphs of **E** and **B**

1.0.4 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents $(i)$ travelling wave, $(ii)$ a stationary wave or $(iii)$ none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

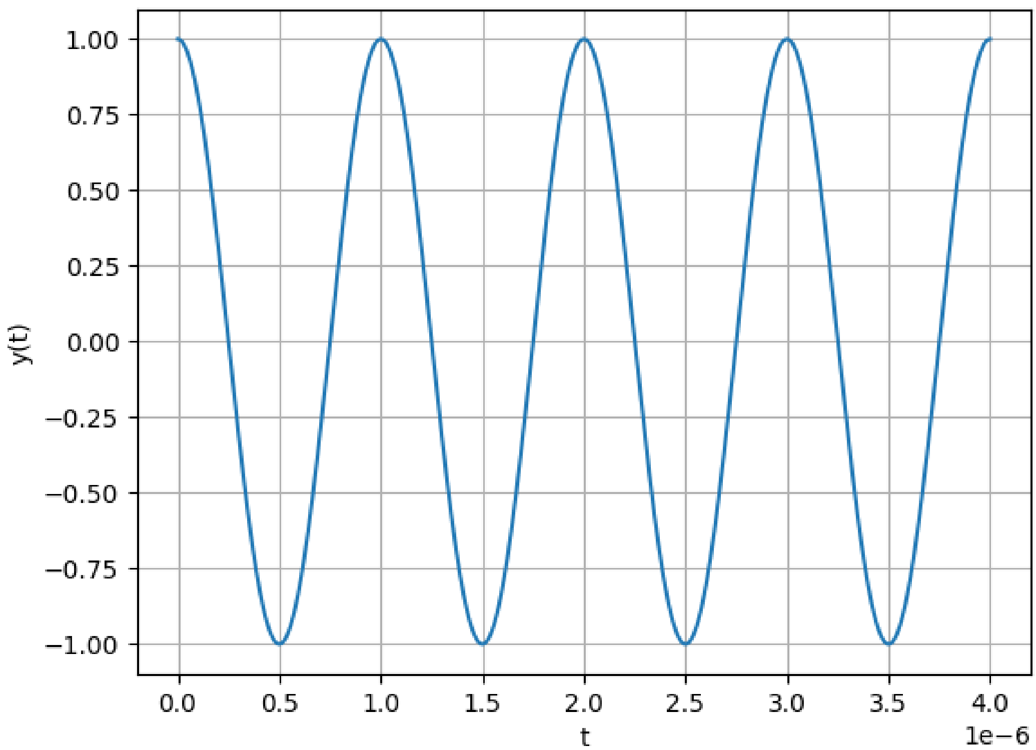



Figure 1.0.3: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

**Solution:**

Let us assume an equation:

$$y = A(x)\cos\left(\omega t + \phi\left(x\right)\right) \tag{1.1}$$

Fig. 1.0.4 and Fig. 1.0.4 are self explanatory for stationary and travelling waves.Fig. 1.0.4 and Fig. 1.0.4 are neither stationary nor travelling waves.

| TRAVELLING WAVE | STATIONARY WAVE |
|---|---|
| $y(x,t) = A\sin(kx \pm \omega t)$ | $y(x,t) = A\sin kx \cos \omega t$ |
| PARAMETERS | DEFINITION |
| $A$ | Amplitude |
| $\omega$ | Angular Velocity |
| $x$ | Position |
| $k$ | Wavenumber |

Table 1.0.4: Travelling wave *vs* Stationary wave

| STATIONARY WAVE CONDITION | TRAVELLING WAVE CONDITION |
|---|---|
| (1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t + \alpha)$ where $A_0$ is a constant, $k$ is the wavenumber, $x$ is the position and $\alpha$ is a phase constant. | (1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where $A_0$ is a constant number. |
| (2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant. | (2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx + \theta$ where k is the wavenumber and $\theta$ is the phaseconstant. |

Table 1.0.4: Travelling wave *vs* Stationary wave

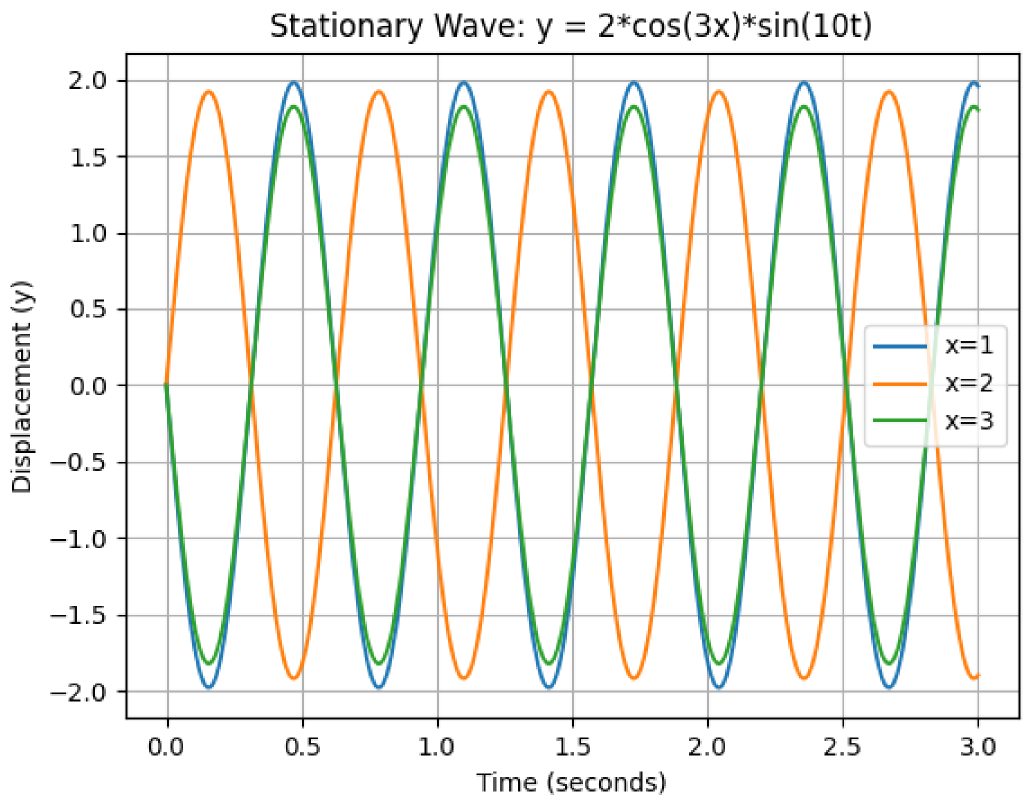



Figure 1.0.4: DIPLACEMENT $vs$ TIME-graph1

1.0.5 For the travelling harmonic wave $y\,(x,t) = 2.0\cos 2\pi\,(10t - 0.0080x + 0.35)$ where $x$ and $y$ are in $cm$ and $t$ in $s$. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

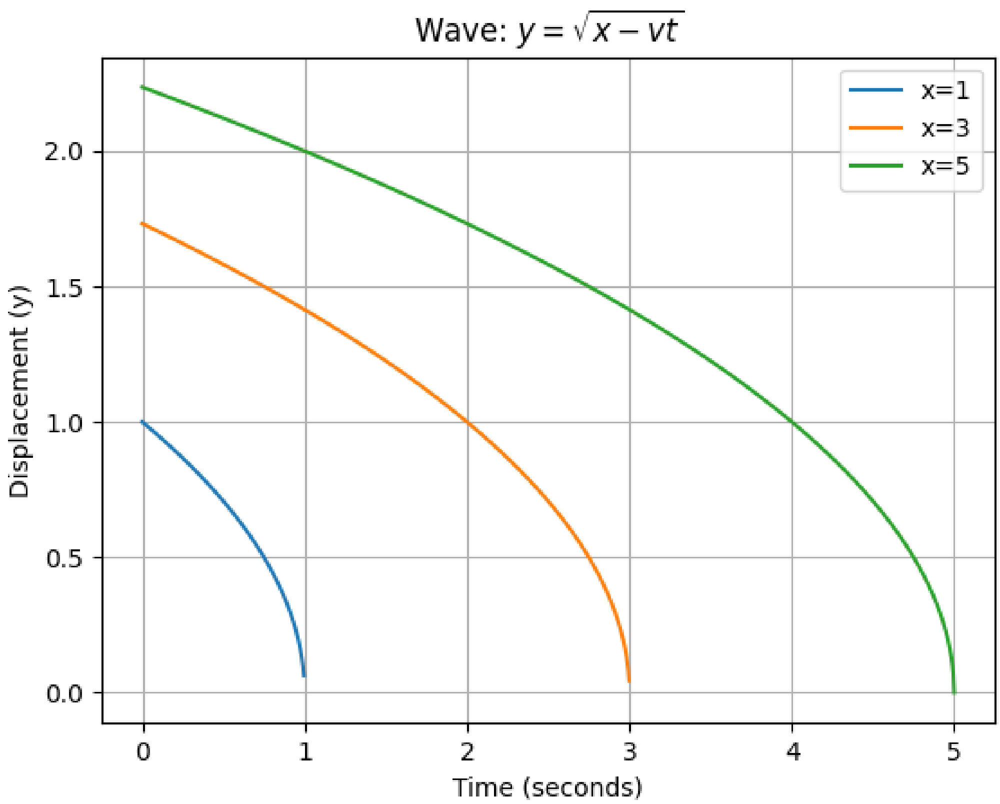



Figure 1.0.4: DIPLACEMENT $vs$ TIME-graph2

**Solution:**

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \tag{1.2}$$

$$= k\,(x_2 - x_1) \tag{1.3}$$

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $y(x_i, t)$ | equation of harmonic wave | $A\cos(2\pi ft - kx_i + \phi)$ |
| $k$ | angular wave number | $2\pi(0.008)$ |
| $\lambda = \frac{2\pi}{k}$ | wavelength | $125\,cm$ |
| $f$ | frequency | 10 |
| $A$ | amplitude | 2.0 |
| $\phi$ | phase constant | $2\pi(0.35)$ |
| $\theta_i$ | phase of $i^{th}$ harmonic wave | $(2\pi ft - kx + \phi)$ |
| $x_i$ | position of $i^{th}$ harmonic wave | |
| $t$ | time | |
| $x_2 - x_1$ | path difference | $400\,cm$ |
| | | $50\,cm$ |
| | | $\frac{\lambda}{2}$ |
| | | $\frac{3\lambda}{4}$ |

Table 1.0.5: Given parameters list

| **Parameter** | **Description** | **subquestion** | **Value** |
|---|---|---|---|
| $\Delta\theta$ | $\theta_1 - \theta_2$ | $(a)$ | $6.4\pi$ radians |
| | | $(b)$ | $0.8\pi$ radians |
| | | $(c)$ | $\pi$ radians |
| | | $(d)$ | $\frac{3\pi}{2}$ radians |

Table 1.0.5: Phase differences

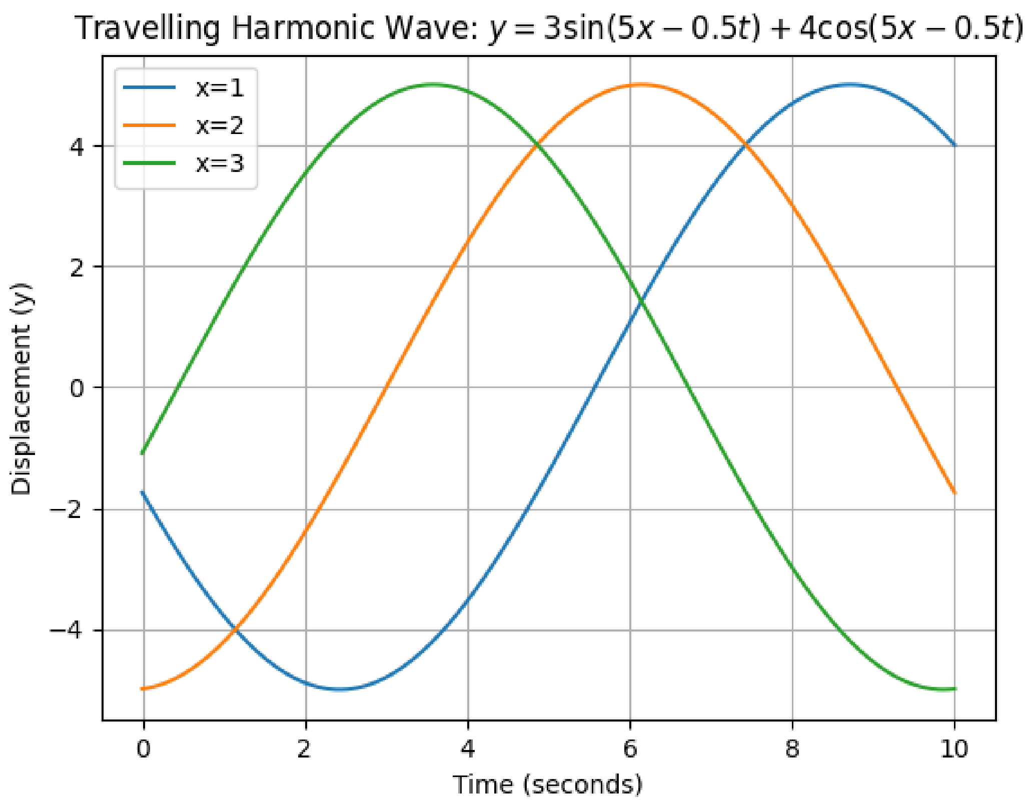



Figure 1.0.4: DIPLACEMENT *vs* TIME-graph3

1.0.6 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

**Solution:**

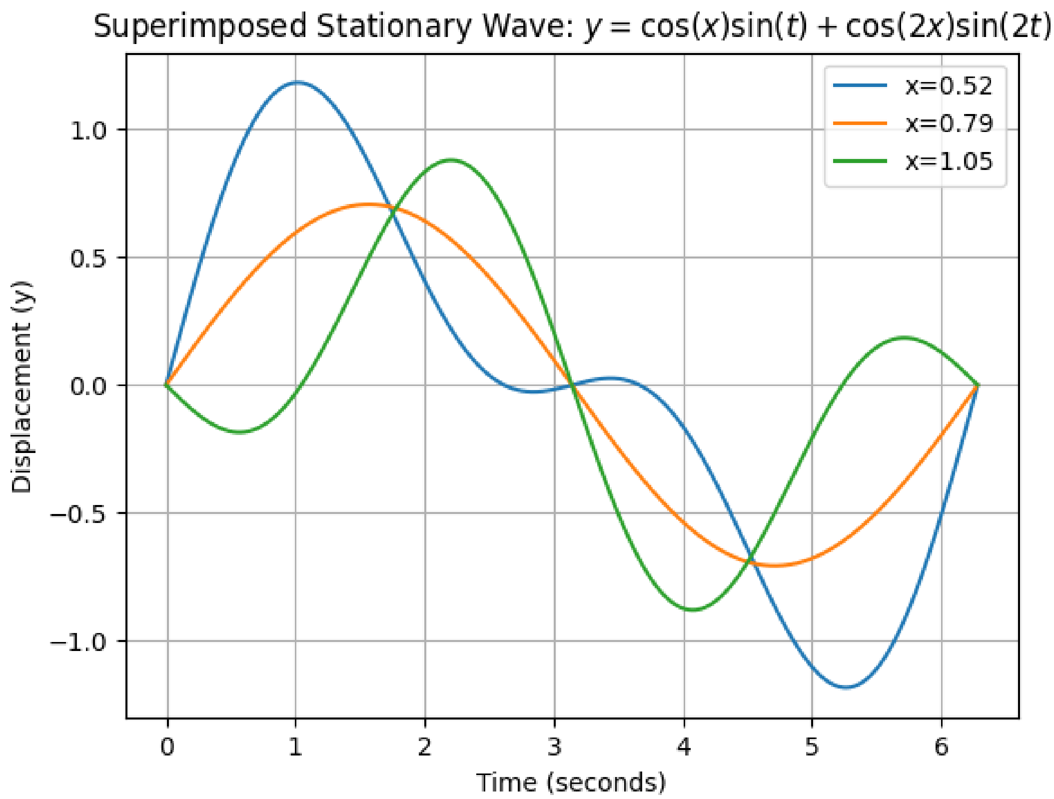



Figure 1.0.4: DIPLACEMENT *vs* TIME-graph4

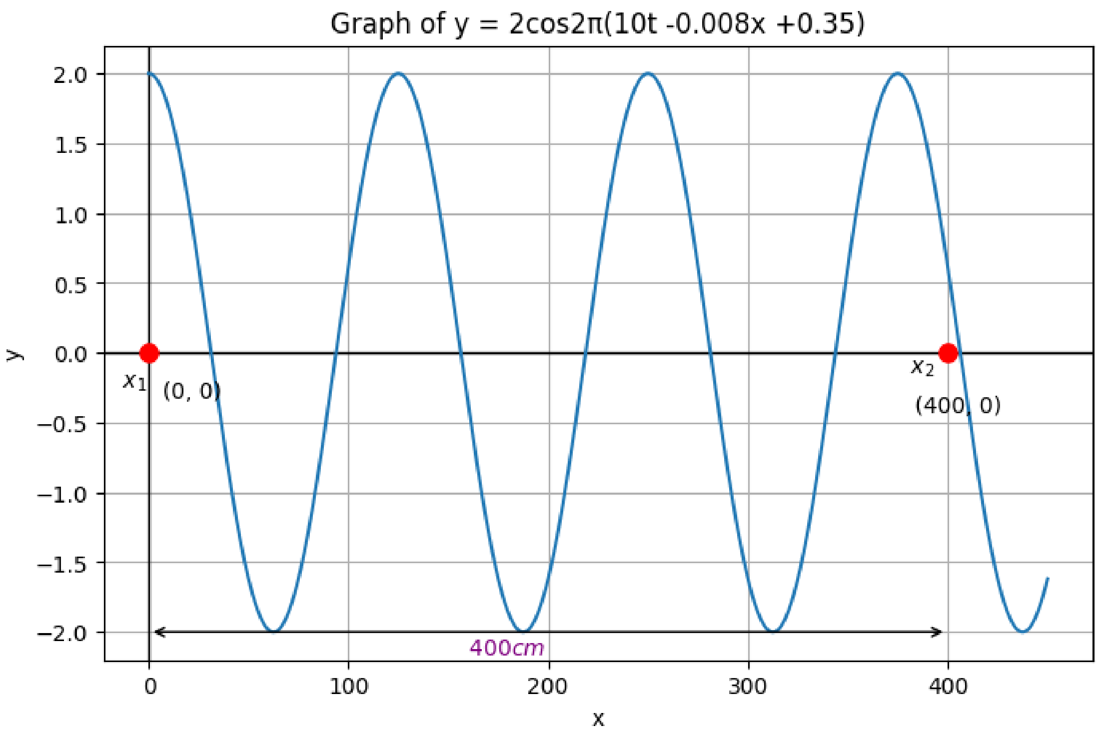



Figure 1.0.5:

(a)

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T [V(t)]^2\,dt \tag{1.4}$$

$$= f\int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi f t + \phi)\,dt \tag{1.5}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\int_0^{\frac{1}{f}} \cos(4\pi f t + 2\phi)\,dt\right) \tag{1.6}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.7}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin\left(4\pi + 2\phi\right) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.8}$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \tag{1.9}$$

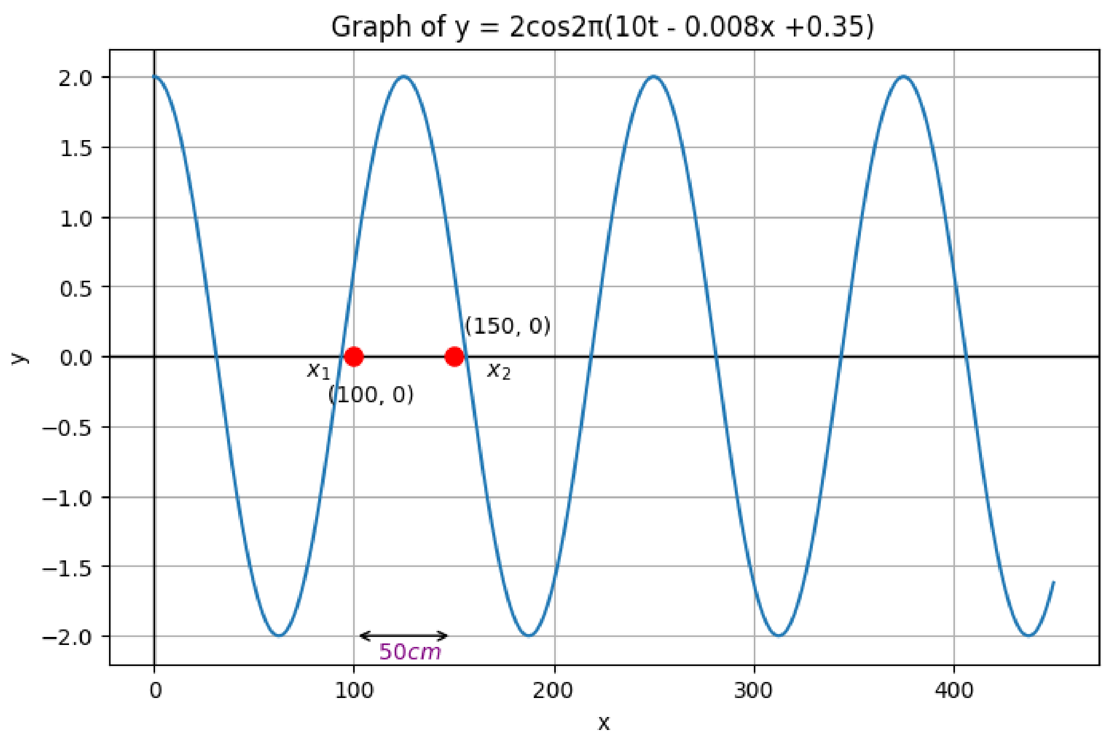



Figure 1.0.5:

To find the RMS voltage ($V_{\text{rms}}$) when the peak voltage ($V_0$) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \tag{1.10}$$

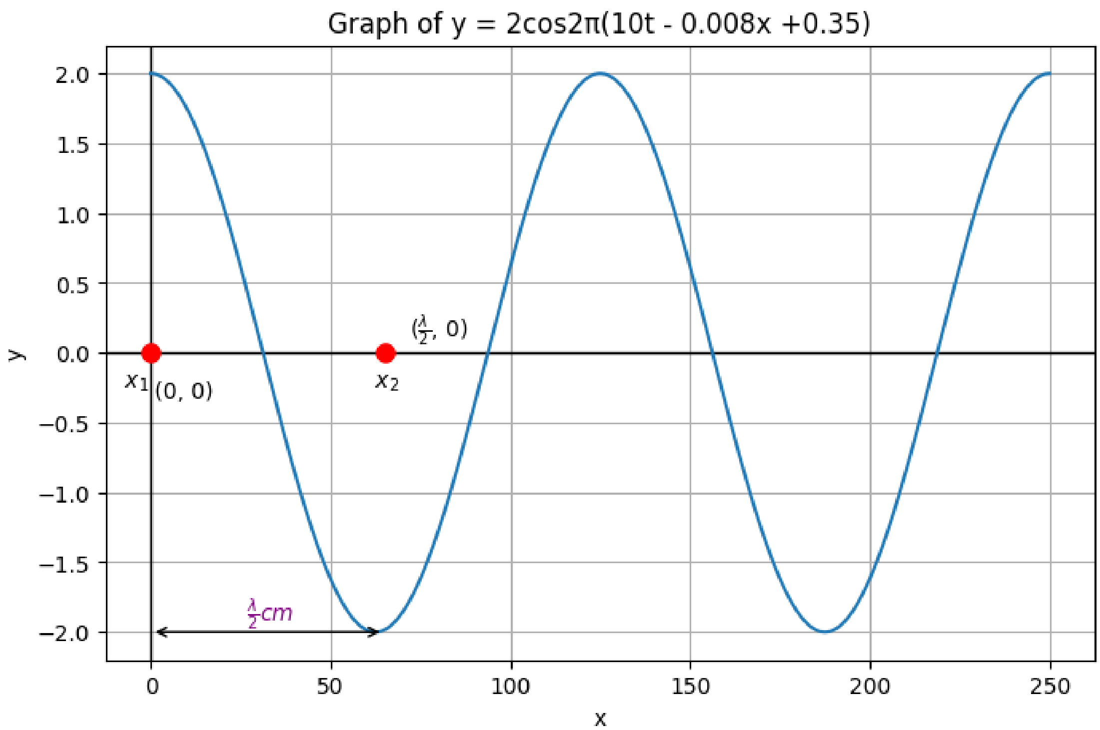



Figure 1.0.5:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T}\int_0^T [I(t)]^2 \, dt \tag{1.11}$$

$$= f\int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi f t + \phi) \, dt \tag{1.12}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.13}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin\left(4\pi + 2\phi\right) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.14}$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \tag{1.15}$$

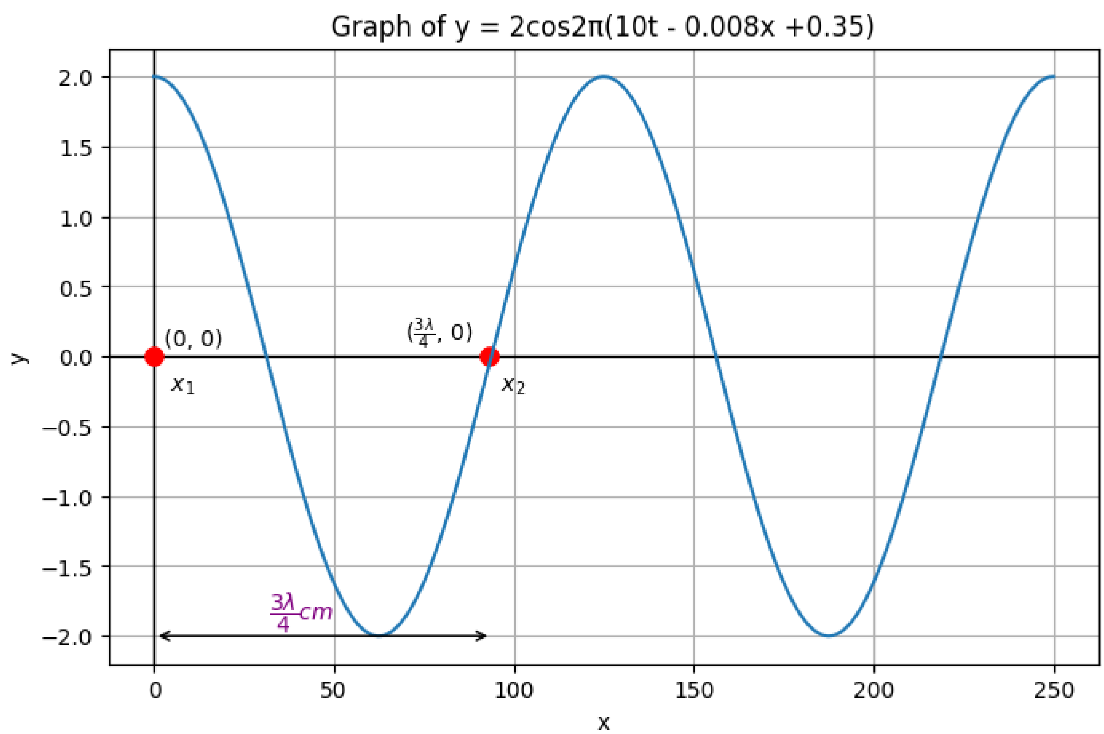


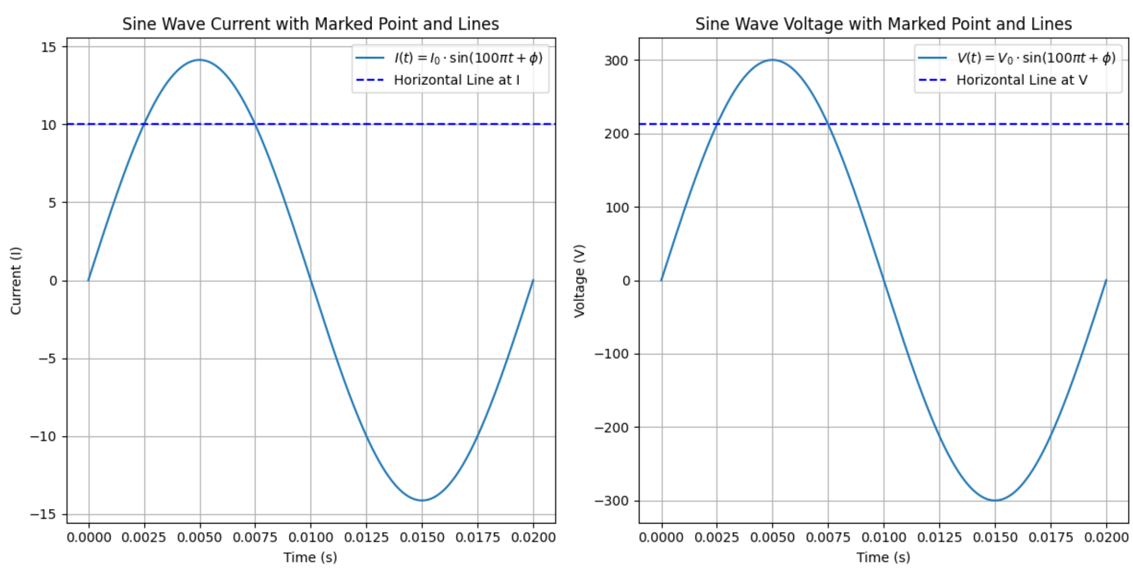

Figure 1.0.5:

To find the peak current ($I_0$) when the RMS current ($I_{\text{rms}}$) is given, you can use equation (1.15)

$$I_0 \approx 10\,\text{A} \times 1.414 \approx 14.14\,\text{A} \tag{1.16}$$

| parameter | value | description |
|---|---|---|
| $V(t)$ | $V_0 \cdot \sin(2\pi f t + \phi)$ | voltage in terms of time |
| $I(t)$ | $I_0 \cdot \sin(2\pi f t + \phi)$ | current in terms of time |
| $V_0$ | $300\,\mathrm{V}$ | peak voltage |
| $V_{\mathrm{rms}}$ | $\sqrt{\frac{1}{T}\int_0^T [V(t)]^2\,dt}$ | rms value of Voltage |
| $I_{\mathrm{rms}}$ | $10\,\mathrm{A}$ | rms value of current |
| $I_0$ | $\sqrt{2} \times I_{\mathrm{rms}}$ | peak current |
| $f$ | $50\,\mathrm{Hz}$ | frequency of the sinusoidal wave |
| $T$ | $0.02\,\mathrm{s}$ | time period of sinusoidal wave |

Table 1.0.6: Input Parameter Table



1.0.7 In Young's double-slit experiment using monochromatic light of wavelength $\lambda$, the intensity of light at a point on the screen where path difference is $\lambda$, is $K$ units. What is the intensity of light at a point where path difference is $\lambda/3$?

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $y_i(t)$ | Equation of light from $S_{i\text{th}}$ | $A\sin(\omega t - kx_i)$ |
| $k$ | Wave number | $\frac{2\pi}{\lambda}$ |
| $I$ | Intensity of wave | $\propto A^2$ |
| $\Delta x = x_1 - x_2$ | Path difference | $\lambda$ |
| | | $\frac{\lambda}{3}$ |
| $K$ | Intensity of light at $\Delta x = \lambda$ | |
| $A$ | Amplitude of wave from source | |
| $r$ | constant | $r \geq 0$ |

Table 1.0.7: Parameters

From Table 1.0.7:

$$y(t) = A\sin(2\pi ft - kx_1) + A\sin(2\pi ft - kx_2) \tag{1.17}$$

$$y(t) = 2A\cos\left(\frac{k\Delta x}{2}\right)\sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \tag{1.18}$$

From Table 1.0.7 and equation (1.18):

$$\therefore I \propto 4A^2\cos^2\left(\frac{k\Delta x}{2}\right) \tag{1.19}$$

From Table 1.0.7 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2\cos^2\left(\dfrac{2\pi}{2}\right)}{4A^2\cos^2\left(\dfrac{\pi}{3}\right)} \implies I_r = \frac{K}{4} \tag{1.20}$$

Hence, the Intensity of light at a point where path difference is $\frac{\lambda}{3}$ is $\frac{K}{4}$ units.

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $I_r$ | Net Intensity of light at $\Delta x = \frac{\lambda}{3}$ | $\frac{K}{4}$ |

Table 1.0.7:

Assuming $\Delta x = r\lambda$,

From equation (1.19):

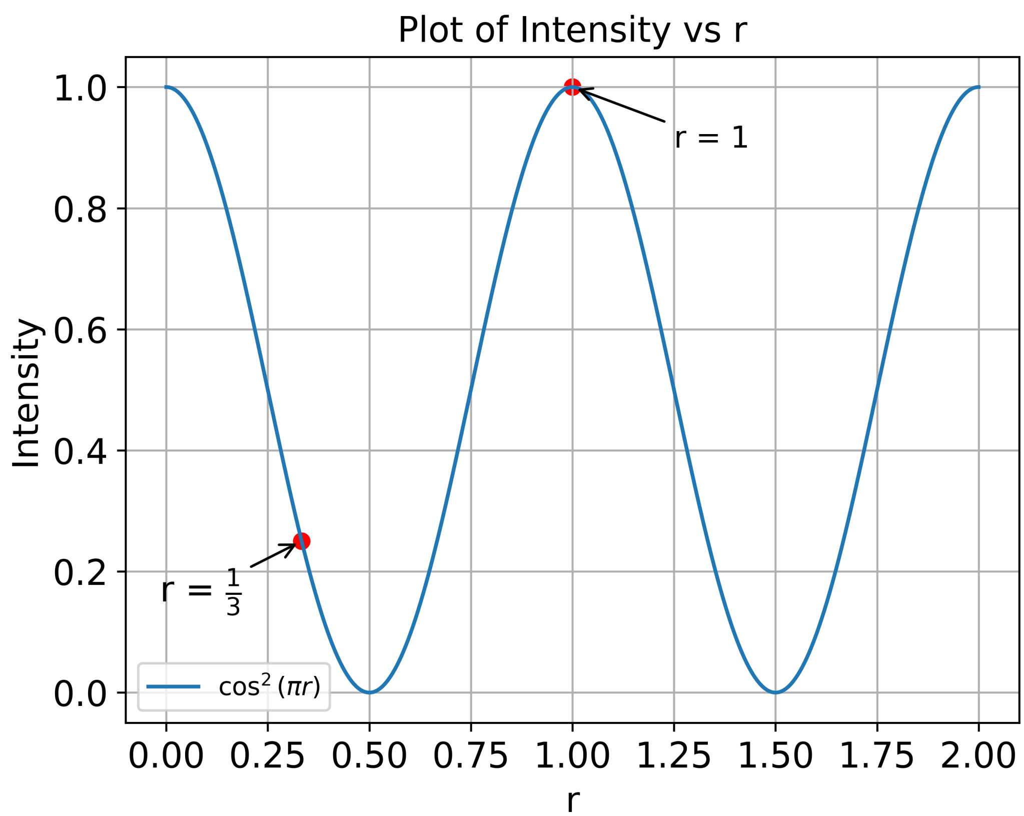

Plot of Intensity vs r
Intensity
r
r = 1
r = $\frac{1}{3}$
$\cos^2(\pi r)$
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Figure 1.0.7:

1.0.8 In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of 2.0 x $10^{10}$ Hz and amplitude 48 $Vm^{-1}$.

(a) What is the wavelength of the wave?

(b) What is the amplitude of the oscillating magnetic field?

(c) Show that the average energy density of the **E** field equals the average energy density of the **B** field. [$c = 3$ x $10^8 ms^{-1}$]

1.0.9 (a) For the wave on the string $y(x,t) = 0.06\sin(\frac{2\pi x}{3})\cos(120\pi t)$ , do all the points on the string oscillate with the same (a)frequency , (b)phase , (c)amplitude ? Explain your answers.

(b) What is the amplitude of a point 0.375m away from one end?

**Solution:**

1.0.10 A transverse harmonic wave on a string is described by

$$y(x,t) = 3.0 \sin\left(36t + 0.018x + \frac{\pi}{4}\right) \tag{1.21}$$

where $x$ and $y$ are in cm and $t$ in s. The positive direction of $x$ is from left to right.

(a) Is this a travelling wave or a stationary wave? If it is travelling, what are the speed and direction of its propogation?

(b) What are its amplitude and frequency?

(c) What is the initial phase at the origin?

(d) What is the least distance between two succesive crests in the wave?

**Solution:**

1.0.11 In deriving the single slit diffraction pattern, it was stated that the intensity is zero at angles of $\frac{n\lambda}{a}$. Justify this by suitably dividing the slit to bring out the cancellation.

**Solution:**

1.0.12 A 60 $\mu$ F capacitor is connected to a 110 V, 60 Hz ac supply. Determine the rms value of the current in the circuit.

**Solution:**

1.0.13 A charged $30\mu F$ capacitor is connected to a $27mH$ inductor. What is the angular frequency of free oscillations of the circuit?

**Solution:**

1.0.14 Obtain the resonance frequency of a series LCR circuit with $L = 2.0\,H$, $C = 32\,\mu F$, and $R = 10\,\Omega$. What is the Q-value of the circuit.

**Solution:**

1.0.15 A charged 30 $\mu$F capacitor is connected to a 27 mH inductor. Suppose the initial charge on the capacitor is 6mC.What is the total energy stored in the circuit initially? What is the total energy at later time?

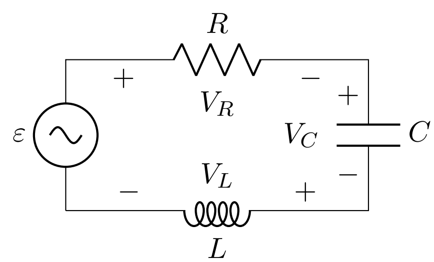
**Solution:**

1.0.16 A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is $3.5 \times 10^{-2}$ kg, and its linear mass density is $4.0 \times 10^{-2}$ kg/m. The length of the wire is 0.875 m. Determine the speed of a transverse wave on the string and the tension in the string.

**Solution:**

1.0.17 The given figure shows a series LCR circuit connected to a variable frequency 230 V source.

L = 5.0 H, C = 80 $\mu$F, R = 40 $\Omega$.




(a) Determine the source frequency which drives the circuit in resonance.

(b) Obtain the impedance of the circuit and the amplitude of current at the resonating frequency.

(c) Determine the rms potential drops across the three elements of the circuit. Show that the potential drop across the LC combination is zero at the resonating frequency.

**Solution:**

### 1.0.18 Q23) A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium.

(a) Does the pulse have a definite (i) frequency, (ii) wavelength, (iii) speed of propagation?

(b) If the pulse rate is 1 after every 20 s, (that is the whistle is blown for a split of second after every 20 s), Is the frequency of note produced by whistle equal to 1/20 or 0.05 Hz ?

**Solution:**

1.0.19 Suppose that the electric field part of an electromagnetic wave in vacuum given as

$\mathbf{E} = \{(3.1\text{N/C})\cos[(1.8\text{ rad/m})y + (5.4\times10^6\text{rad/s})t]\}\hat{\imath}$

(a) What is the direction of propagation ?

(b) What is the wavelength ?

(c) What is the frequency ?

(d) What is the amplitude of the magnetic field part of the wave?

(e) Write an expression for the magnetic field part of the wave.

**Solution:**

1.0.20 A 44 mH inductor is connected to 220 V, 50 Hz ac supply. Determine the rms value of the current in the circuit.

**Solution:**

1.0.21 The 6563 Å H$\alpha$ line emitted by hydrogen in a star is found to be redshifted by 15 Å. Estimate the speed with which the star is receding from the Earth. **Solution:**

1.0.22 The amplitude of the magnetic part of a harmonic elctromagnetic wave is $B_0 = 510\text{nT}$.What is the amplitude of the electric part of the electromagnetic wave.

**Solution:**

1.0.23 A $100\mu$F capacitor in series with a $40\Omega$ resistance is connected to a 110V, 60Hz supply.

($a$) What is the maximum current in the circuit?

($b$) What is the time lag between the current maximum and the voltage maximum?

**Solution:**

1.0.24 A 100 $\Omega$ resistor is connected to $220V$, $50Hz$ AC supply.

(1) What is the rms value of current in the circuit?

(2) What is the net power consumed over a full cycle?

**Solution:**

1.0.25 Two towers on top of two hills are 40 km apart.This line joining them passes 50 m above a hill halfway between the towers.What is the longest wavelength of radio waves,which can be sent between the towers without appereciable diffraction effects?

**Solution:**

1.0.26 A circuit containing a 80mH inductor and a 60$\mu$F capacitor in series is connected to a 230V, 50Hz supply. The resistance of the circuit is negligible.

(a) Obtain the current amplitude and rms value.

(b) Obtain the rms value of potential drops across each element.

(c) What is the average power transferred to the inductor ?

(d) What is the average power transferred to the capacitor ?

(e) What is the total average power absorbed by the circuit ? ('Average' implies 'averaged ov

**Solution:**

1.0.27 A coil of inductance 0.50 H and resistance 100 $\Omega$ is connected to a 240 V, 50 Hz ac supply.

(a) What is the maximum current in the coil?

(b) What is the time lag between the voltage maximum and the current maximum?

**Solution:**

1.0.28 A plane electromagnetic wave travels in vacuum along the $z$-direction. What can you say about the directions of its electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) field vectors? If the frequency of the wave is 30 MHz, what can you say about its wavelength?

**Solution:**

1.0.29 Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and longitudinal (P) sound waves. Typically the speed of S wave is about $4.0km/s$, and that of P wave is $8.0km/s$. A seismograph records P and S waves from an earthquake. The first P wave arrives $4min$ before the first S wave. Assuming the waves travel in straight line, at what distance does the earthquake occur ? **Solution:**

1.0.30 A hospital uses an ultrasonic scanner to locate tumors in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is 1.7 km/s? The operating frequency of the scanner is 4.2 MHz. **Solution:**

1.0.31 In double-slit experiment using light of wavelength $600nm$, the angular width of a fringe formed on a distant screen is $0.1°$. What is the spacing between the two slits?

**Solution:**

1.0.32 A spring having with a spring constant 1200 N$m^{-1}$ is mounted on a horizontal table as shown in Fig.A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass. **Solution:**

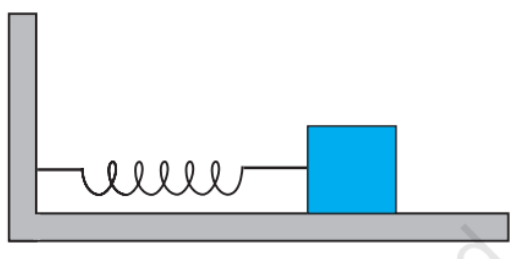


Figure 1.0.32:

1.0.33 A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does he disturbance take to reach the other end ? **Solution:**

1.0.34 In a Young's double-slit experiment, the slits ar e separated by 0.28 mm and the screen is placed $1.4m$ away. The distance between the central bright fringe and the fourth bright fringe is measured to be $1.2cm$. Determine the wavelength of light used in the experiment.

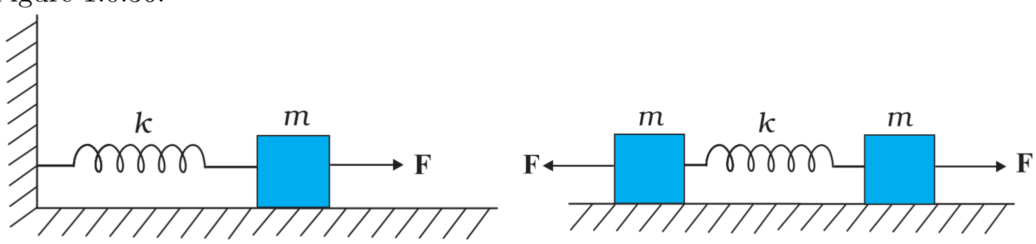
**Solution:**

1.0.35 A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of 360 km/hr. What is the frequency of sound reflected by the submarine? Take the speed of sound in water to be 1450 m/s.

**Solution:**

1.0.36 Figure 1.0.35 (a) shows a spring of force constant k clamped rigidly at one end and a mass $m$ attached to its free end. A force $F$ applied at the free end stretches the spring. Figure 1.0.35 (b) shows the same spring with both ends free and attached to a mass $m$ at either end. Each end of the spring in Fig. 1.0.35(b) is stretched by the same force $F$.

Figure 1.0.36:




(a) (b)

(a) What is the maximum extension of the spring in the two cases ?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case ?

**Solution:**

1.0.37 A bat is flitting about in a cave, navigating via ultrasonic beeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall ? NCERT Analog 11.15.27

**Solution:**

1.0.38 One end of a long string of linear mass density $8.0 \times 10^{-3}\,\mathrm{kg\ m^{-1}}$ is connected to an electrically driven tuning fork of frequency $256\,\mathrm{Hz}$. The other end passes over a pulley and is tied to a pan containing a mass of $90\,\mathrm{kg}$. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At $t = 0$, the left end (fork end) of the string $x = 0$ has zero transverse displacement ($y = 0$) and is moving along positive $y$-direction. The amplitude of the wave is $5.0\,\mathrm{cm}$. Write down the transverse displacement $y$ as a function of $x$ and $t$ that describes the wave on the string. NCERT Analog 11.15.24

**Solution:**

1.0.39 A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at 20 $°C = 343ms^{-1}$ NCERT Analog 11.15.3

**Solution:**

1.0.40 What is the Brewster angle for air to glass transition?(Refractive index of glass $=1.5$)

**Solution:**

1.0.41 A parallel beam of light with a wavelength of 500 nm falls on a narrow slit, and the resulting diffraction pattern is observed on a screen 1 m away. The distance to the first minimum from the center of the screen is 2.5 mm.

Find the width of the slit given that $y = 0.0025$ m, $L = 1$ m, and $\lambda = 5 \times 10^{-7}$ m.

**Solution:**

1.0.42 Two sitar strings A and B playing the note 'Ga' are slightly out of tune and produce beats of frequency 6Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3Hz. If the original frequency of A is 324Hz,what is the frequency of B?

**Solution:**

1.0.43 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion ($\omega$ is any positive constant):

(a) $\sin(\omega t) - \cos(\omega t)$

(b) $\sin^3(\omega t)$

(c) $3\cos\left(\frac{\pi}{4} - 2\omega t\right)$

(d) $\cos(\omega t) + \cos(3\omega t) + \cos(5\omega t)$

(e) $\exp\left(-\omega^2 t^2\right)$

(f) $1 + \omega t + \omega^2 t^2$

**Solution:**

# Chapter 2

# Filters

2.0.1 An LC circuit contains a $50\mu H$ inductor and a $50\mu F$ capacitor with an initial charge of $10mC$. The resistance of the circuit is negligible. Let the instant the circuit is closed by $t = 0$.

**a)** What is the total energy stored initially? Is it conserved during LC oscillations?

**b)** What is the natural frequency of the circuit?

**c)** At what time is the energy stored **(i)** completely electrical (i.e., stored in the capacitor)? **(ii)** completely magnetic (i.e., stored in the inductor)?

**d)** At what times is the total energy shared equally between the inductor and the capacitor?

**e)** If a resistor is inserted in the circuit, how much energy is eventually dissipated as heat?

(NCERT-Physics 12.7 12Q)

**Solution:**

2.0.2 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

**Solution:** Given parameters are:

| **Symbol** | **Value** | **Description** |
|---|---|---|
| $L$ | $3.0\,\mathrm{H}$ | Inductance |
| $C$ | $27\,\mu\mathrm{F}$ | Capacitance |
| R | $7.4\,\Omega$ | Resistance |
| Q | | Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance |
| $\omega_0$ | $\dfrac{1}{\sqrt{LC}}$ | Angular Resonant Frequency |

Table 2.1: Given Parameters

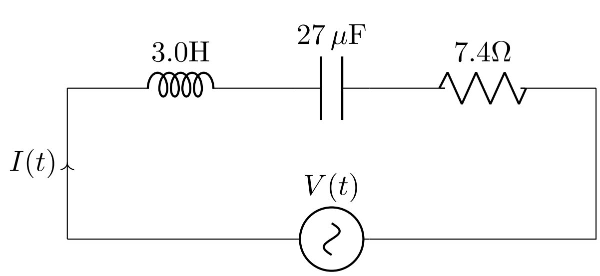



Figure 2.1: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{2.1}$$

Using reactances from Fig. 2.2,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{2.2}$$
$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{2.3}$$
$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{2.4}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

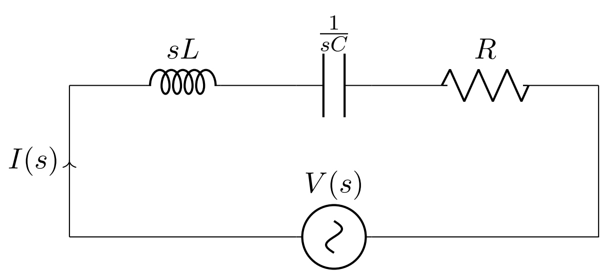



Figure 2.2: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{2.5}$$
$$\implies s = j\frac{1}{\sqrt{LC}} \tag{2.6}$$

$s$ can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{2.7}$$

Comparing (2.6) and (2.7), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.8}$$

(b) Quality Factor

i. Using voltage across inductor,

$$\begin{aligned} Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} &= \frac{|sLI(s)|}{|RI(s)|} && (2.9) \\ &= \frac{1}{\sqrt{LC}}\frac{L}{R} && (2.10) \\ &= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.11) \end{aligned}$$

ii. Using voltage across capacitor,

$$\begin{aligned} Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} &= \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} && (2.12) \\ &= \frac{\sqrt{LC}}{RC} && (2.13) \\ &= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.14) \end{aligned}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{2.15}$$

Using (2.4),

$$H(s) = R + sL + \frac{1}{sC} \tag{2.16}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{2.17}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{2.18}$$

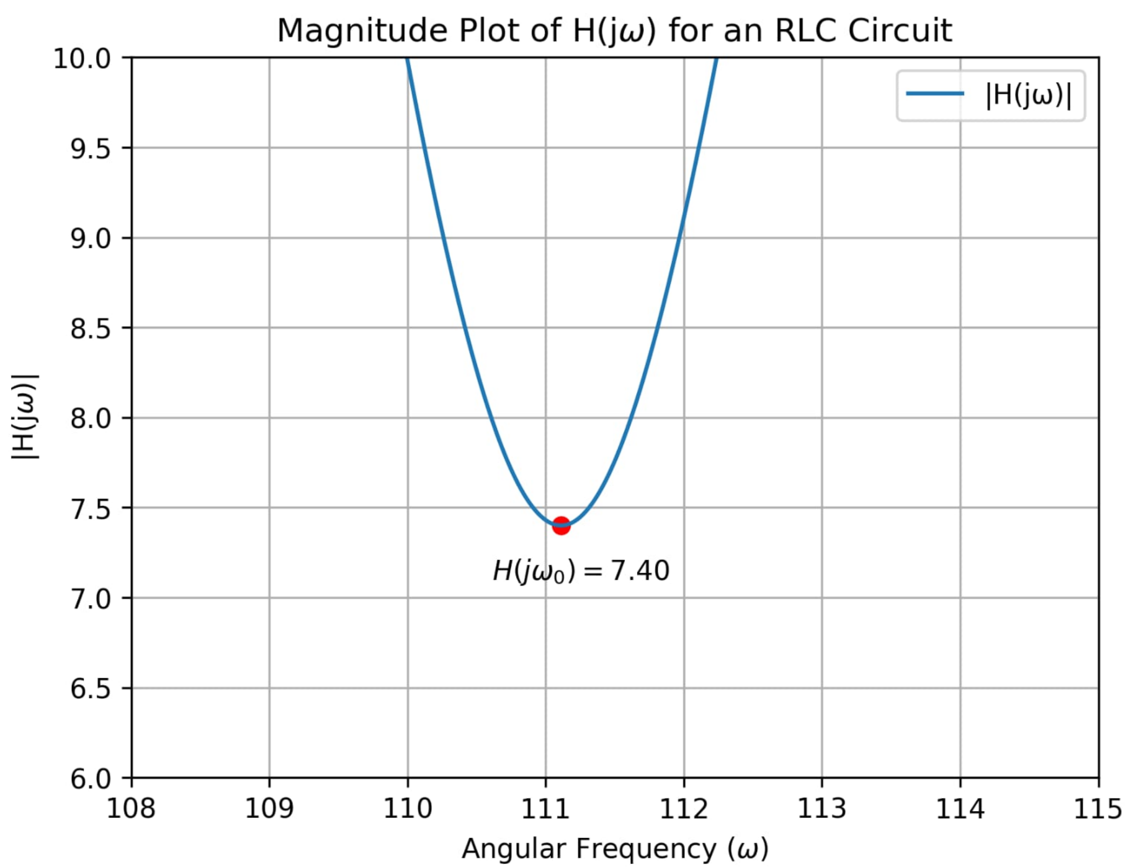



Figure 2.3: Impedance vs $\omega$ (using values in Table 2.1)

2.0.3 A circuit containing a $80mH$ inductor and a $60\mu F$ capacitor in series is connected to a $230V$, $50Hz$ supply. A resistance of $15\Omega$ is connected in series. Obtain the average power transferred to each element of the circuit, and the total power absorbed.

**Solution:**

2.0.4 A series LCR circuit with $L = 0.12H$ $C = 480nF$ $R = 23\Omega$ is connected to a $230V$ variable frequency supply.

(a) What is the source frequency for which current amplitude is maximum? Obtain this maximum value.

(b) What is the source frequency for which the average power absorbed by the circuit is maximum? Obtain the value of this maximum power.

(c) For which frequencies of the source is the power transferred to the circuit half the power at resonant frequency? What is the current amplitude at these frequencies?

(d) What is the Q-factor of the given circuit?

**Solution:**

2.0.5 A radio can tune over the frequency range of a portion of the MW broadcast band: (800 kHz to 1200 kHz). If its LC circuit has an effective inductance ($L$) and a variable capacitor with capacitance ($C$), what must be the range of $C$?

**Solution:**

# Chapter 3

# Z-transform

3.0.1 Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the $4^{th}$ by 18.

**Solution:**

| Symbols | Description | Values |
|---|---|---|
| $r$ | Common ratio of the GP | -2 |
| $x(n)$ | $(n+1)^{th}$ term of the Sequence | $x(0)r^n u(n)$ |
| $x(0)$ | First term of the GP | 3 |
| $x(2) - x(0)$ | First constraint | 9 |
| $x(1) - x(3)$ | Second constraint | 18 |

Table 3.1: Parameters, Descriptions, and Values

From the constraints given in 3.1:

$$x(0)\, r^2 - 9 = x(0) \tag{3.1}$$

$$x(0)\, r + 18 = x(0)\, r^3 \tag{3.2}$$

$$\implies x(0)(r^2 - 1) = 9 \tag{3.3}$$

$$\implies x(0)r(r^2 - 1) = 18 \tag{3.4}$$

By dividing (3.3) and (3.4) and solving ,we get:

$$\implies x(0) = 3 \tag{3.5}$$

$$\implies r = -2 \tag{3.6}$$

Z-Transform for x($n$) : Using (B.1.1) :

$$X(z) = \frac{1}{1 + 2z^{-1}}, \quad |z| > |2| \tag{3.7}$$

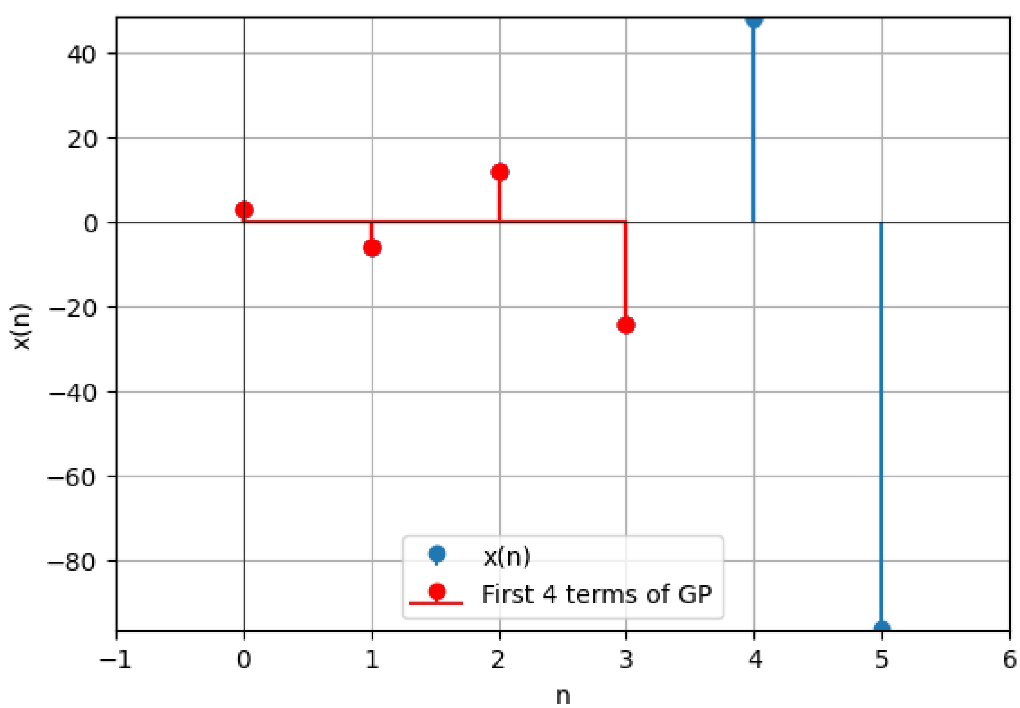



Figure 3.1: $x(n)$ vs n

3.0.2 The $4^{th}$ term of a G.P. is square of its second term, and the first term is -3. Determine its $7^{th}$ term.

**Solution:**

| **Variable** | **Description** | **value** |
|---|---|---|
| $x(0)$ | first term of G.P. | -3 |
| $r$ | Common ratio of G.P. | ? |
| $x(n)$ | general term of the G.P. | $x(0)\, r^n$ |
| $x(3)$ | fourth term | $[x(1)]^2$ |
| $u(n)$ | unit step function | - |

Table 3.2: A Table with input parameters

from Table 3.2

$$x(0)\, r^3 = \left(x(0)\, r^1\right)^2 \tag{3.8}$$

$$= x(0)^2\, r^2 \tag{3.9}$$

$$\implies r = x(0) \tag{3.10}$$

$$= -3 \tag{3.11}$$

general term

$$x(n) = x(0)\, r^n u(n) \tag{3.12}$$

$$= (-3)^{n+1}\, u(n) \tag{3.13}$$

The $7^{th}$ term of the sequence will be:

$$x(6) = (-3)(-3)^6 \tag{3.14}$$

$$= -2187 \tag{3.15}$$

Z transform of the given G.P is:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} = \frac{-3}{1 + 3z^{-1}}. \quad |z| > 3 \tag{3.16}$$

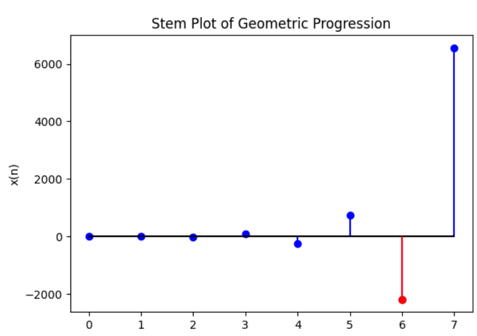



Figure 3.2: Graph showing first 8 terms of the GP

3.0.3 Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $n$ | Integer | .... -2,-1,0,1, 2, ... |
| $x_1(n)$ | General term of Numerator | $\left(n^3+5n^2+8n+4\right)\cdot u\left(n\right)$ |
| $x_2(n)$ | General Term of Denominator | $\left(n^3+4n^2+5n+2\right)\cdot u\left(n\right)$ |
| $y_1\left(n\right)$ | Sum of terms of numerator | ? |
| $y_2\left(n\right)$ | Sum of terms of denominator | ? |
| $U(z)$ | z-transform of $u(n)$ | $\frac{1}{1-z^{-1}},\{z\in\mathbb{C}:\lvert z\rvert>1\}$ |
| ROC | Region of convergence | $\left\{z:\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\rvert<\infty\right\}$ |

Table 1: Parameter Table

1. Analysis of Numerator:

$$X_1\left(z\right)=\sum_{n=-\infty}^{\infty}x_1\left(n\right)z^{-n} \tag{3.17}$$

$$=\sum_{n=-\infty}^{\infty}\left(n^3+5n^2+8n+4\right)u\left(n\right)z^{-n} \tag{3.18}$$

Using results of equations (B.3.2) to (B.3.5) we get:

$$\therefore X_1\left(z\right)=\frac{4+2z^{-1}}{\left(1-z^{-1}\right)^4},\lvert z\rvert>1 \tag{3.19}$$

From (A.3.2)

$$y_1(n) = x_1(n) * u(n) \tag{3.20}$$

$$Y_1(z) = X_1(z)\,U(z) \tag{3.21}$$

$$= \frac{4+2z^{-1}}{(1-z^{-1})^5}, |z| > 1 \tag{3.22}$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{(1-z^{-1})} + \frac{48z^{-2}}{(1-z^{-1})^2} + \frac{52z^{-3}}{(1-z^{-3})^3}, \tag{3.23}$$
$$+ \frac{28z^{-4}}{(1-z^{-1})^4} + \frac{6z^{-5}}{(1-z^{-1})^5} + 4, |z| > 1$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (3.23):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \tag{3.24}$$

$$= \frac{(3n+8)(n+1)(n+2)(n+3)}{12} u(n) \tag{3.25}$$

2. <u>Analysis of Denominator:</u>

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)\,z^{-n} \tag{3.26}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n)\,z^{-n} \tag{3.27}$$

Using results of equation (B.3.2) to (B.3.5) we get:

$$\therefore X_2(z) = \frac{2+4z^{-1}}{(1-z^{-1})^4}, |z| > 1 \tag{3.28}$$

From (A.3.2)

$$y_2(n) = x_2(n) * u(n) \tag{3.29}$$

$$Y_2(z) = X_2(z)\,U(z) \tag{3.30}$$

$$= \frac{2+4z^{-1}}{\left(1-z^{-1}\right)^5}, |z| > 1 \tag{3.31}$$

Using partial fractions:

$$\begin{aligned} Y_2(z) = {} & \frac{14z^{-1}}{\left(1-z^{-1}\right)} + \frac{36z^{-2}}{\left(1-z^{-1}\right)^2} + \frac{44z^{-3}}{\left(1-z^{-3}\right)^3} \\ & + \frac{26z^{-4}}{\left(1-z^{-1}\right)^4} + \frac{6z^{-5}}{\left(1-z^{-1}\right)^5} + 2, |z| > 1 \end{aligned} \tag{3.32}$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (3.32):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{3.33}$$

$$= \frac{(3n+4)(n+1)(n+2)(n+3)}{12} u(n) \tag{3.34}$$

As the sequence start from $n = 0$ , in RHS of question $n$ should be replaced by $n+1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n+8}{3n+4} \tag{3.35}$$

Hence Prooved.

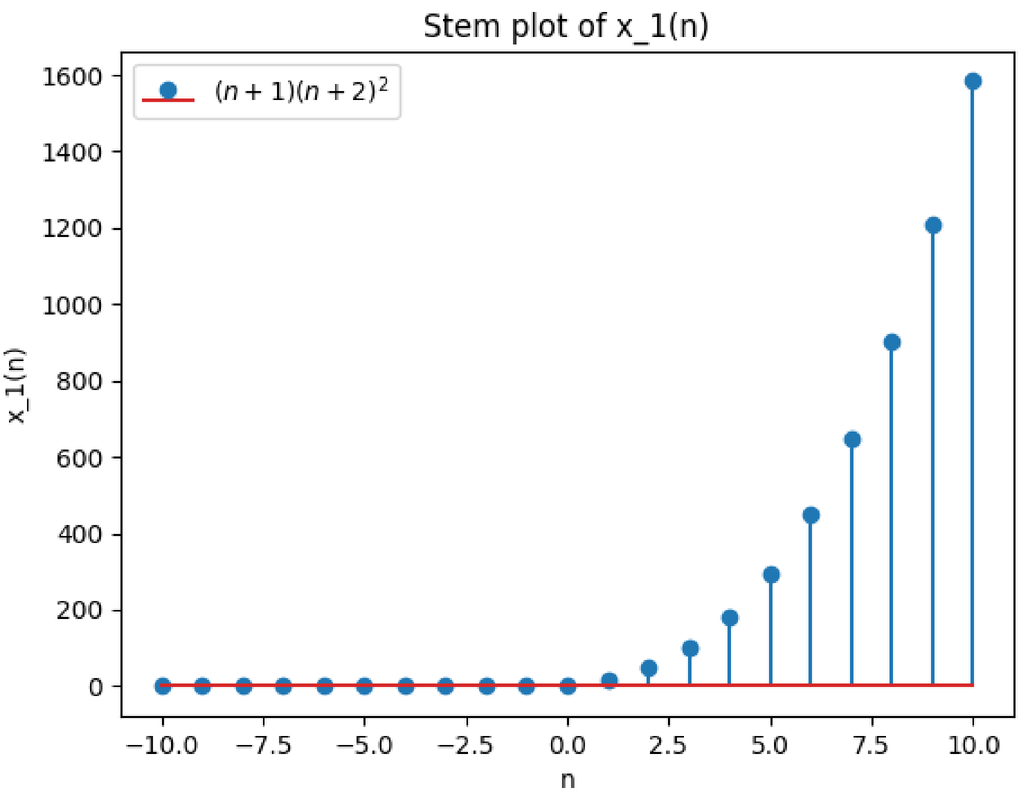



Figure 3.3: Stem Plot of $x_1(n)$

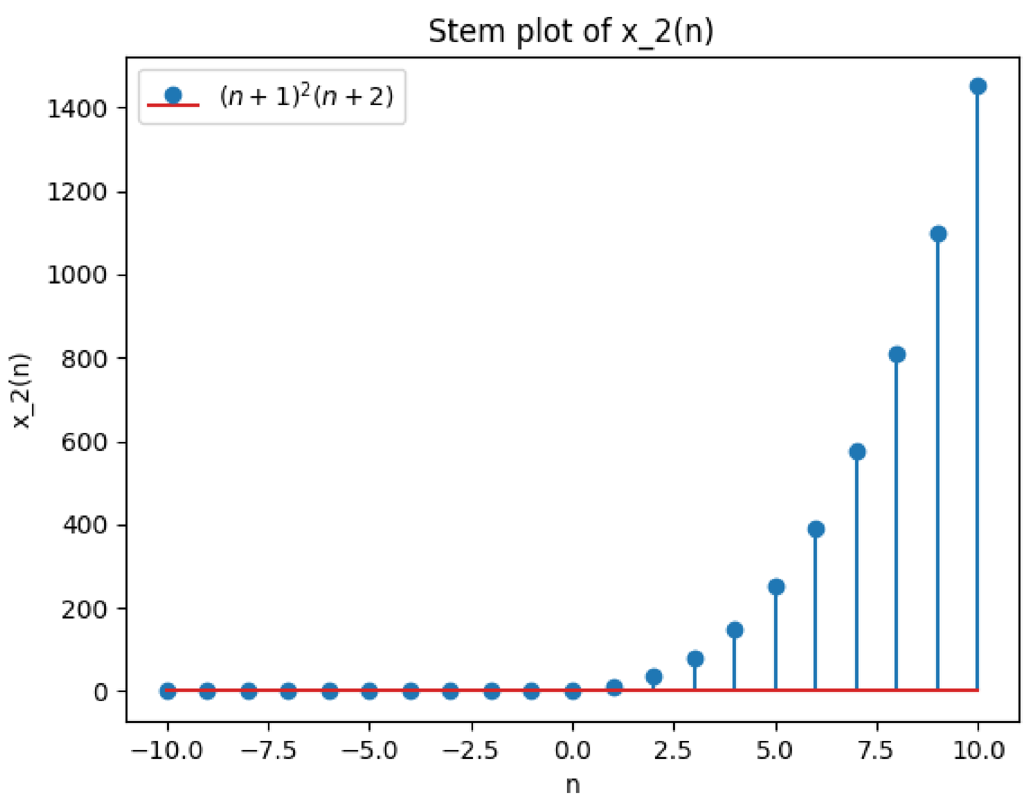



Figure 3.4: Stem Plot of $x_2\left(n\right)$

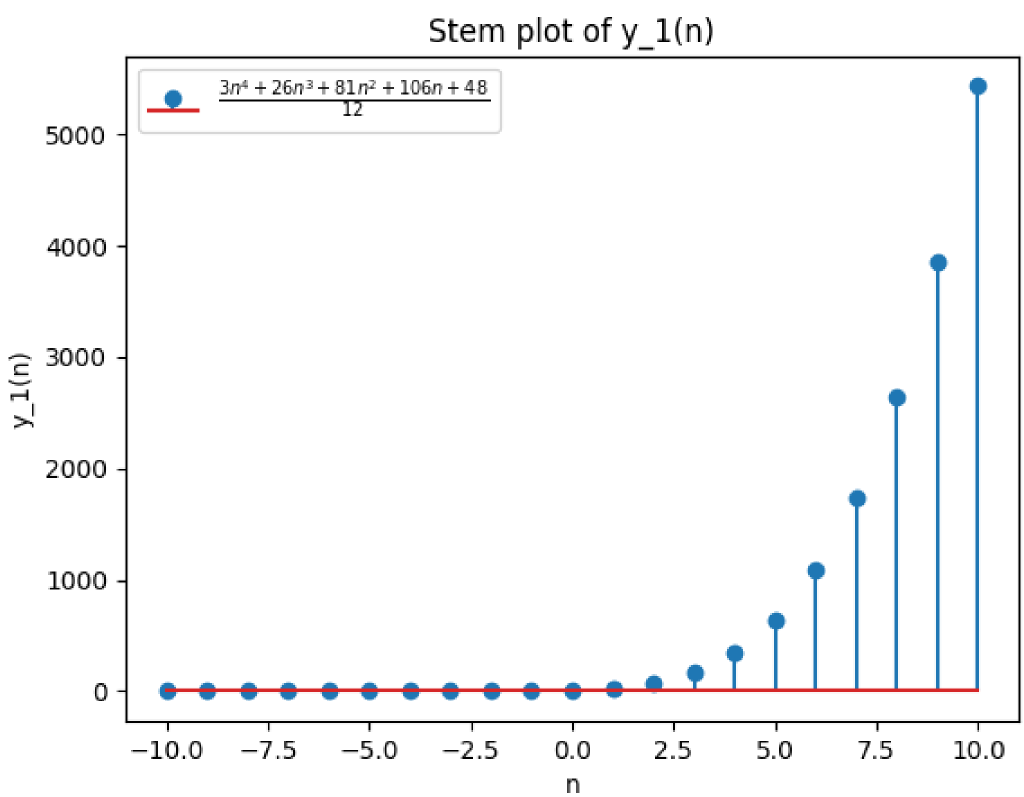



Figure 3.5: Stem Plot of $y_1(n)$

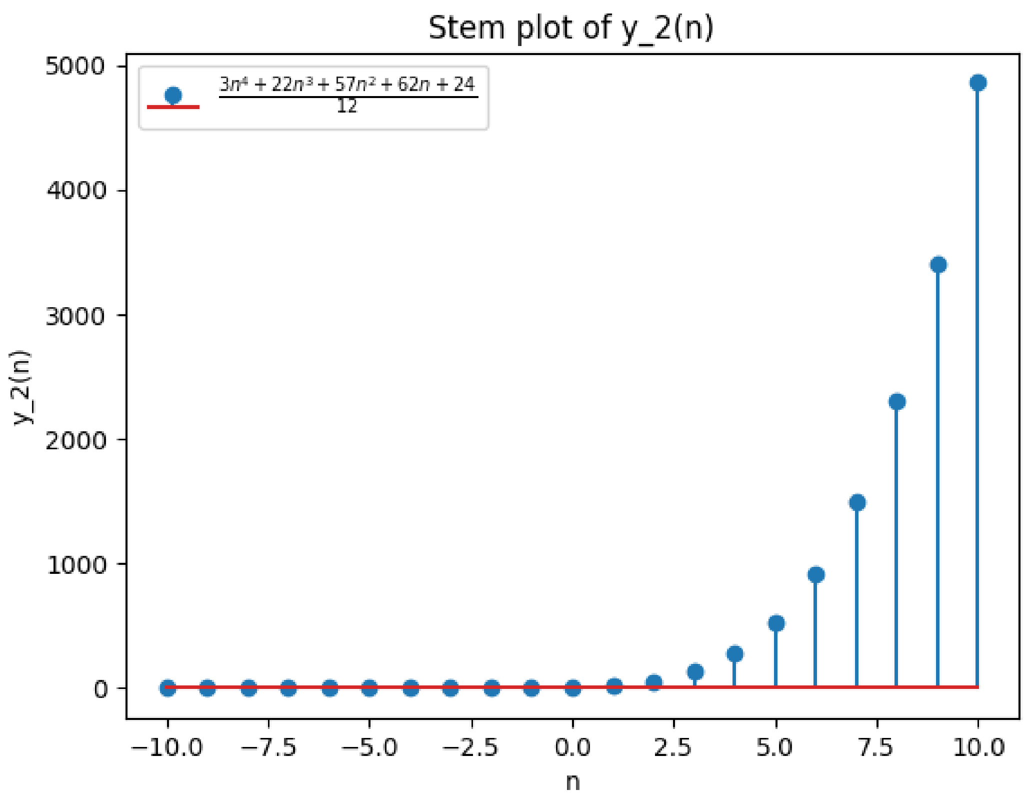



Figure 3.6: Stem Plot of $y_2(n)$

3.0.4 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \qquad n = 0 \tag{3.36}$$

$$= \frac{x(n-1)}{n}, \qquad n > 0 \tag{3.37}$$

$$= 0, \qquad n < 0 \tag{3.38}$$

**Solution:**

$$x(1) = \frac{x(0)}{1} = -1 \tag{3.39}$$
$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{3.40}$$
$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{3.41}$$
$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{3.42}$$
$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{3.43}$$
$$x(n) = \frac{-1}{n!}(u(n)) \tag{3.44}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{3.45}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.46}$$

using (3.44),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \tag{3.47}$$
$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{3.48}$$
$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{3.49}$$

| Symbol | Value | Description |
|---|---|---|
| $x(n)$ | $\frac{-1}{n!}$ | general term of the series |
| $X(z)$ | $-e^{z^{-1}}$ | Z-transform of x(n) |
| $u(n)$ | | unit step function |

Table 3.4: Parameters

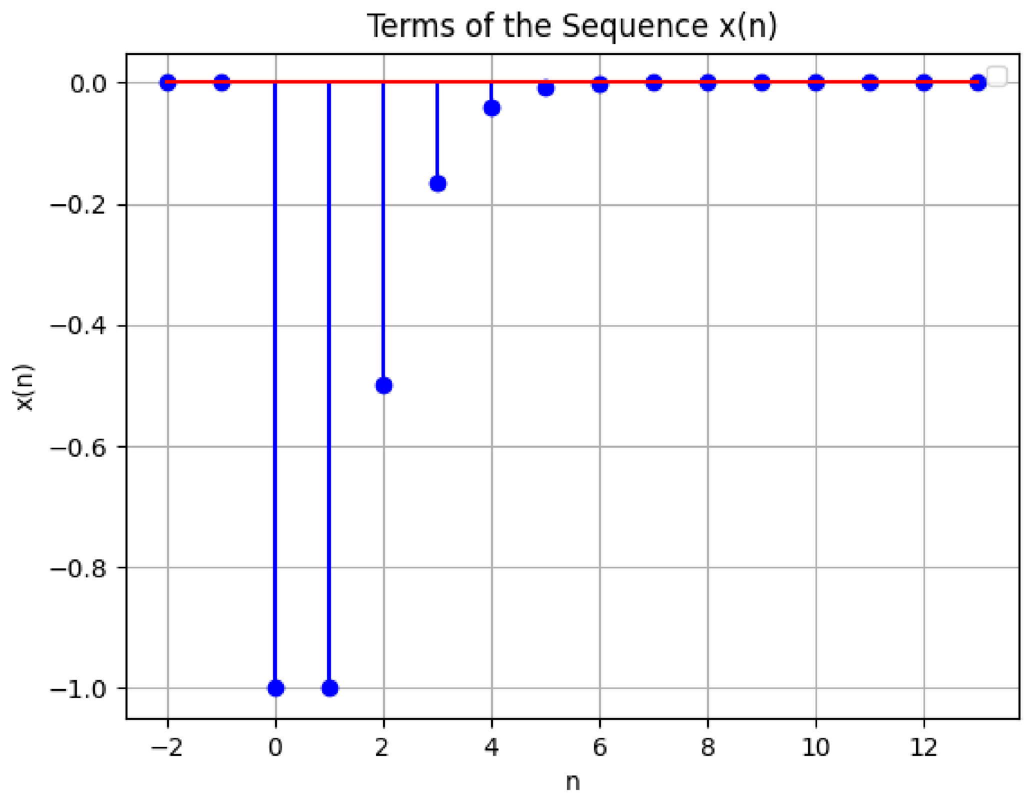



Figure 3.7: Plot of x(n) vs n

3.0.5 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

**Solution:**

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 5000 | Initial Income |
| $d$ | 200 | Annual Increment (Common Difference) |
| $x(n)$ | $(x(0) + nd)u(n)$ | $n^{th}$ term of the AP |

Table 3.5: Input Parameters

From the values given in Table 3.5:

$$7000 = 5000 + 200n \tag{3.50}$$

$$\implies 2000 = 200n \tag{3.51}$$

$$\therefore n = 10 \tag{3.52}$$

Let Z-transform of $x(n)$ be $X(z)$.

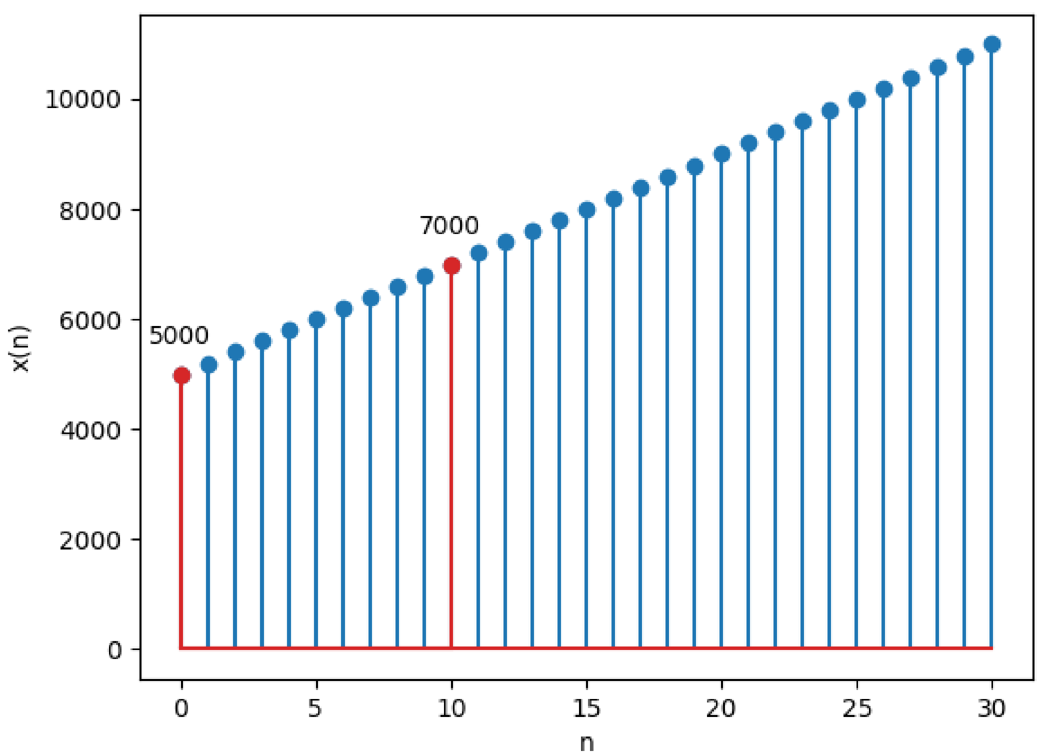



Figure 3.8: Plot of $x(n)$ vs $n$. See Table 3.5 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.53}$$

Using the values from Table 3.5:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.54}$$

3.0.6 Consider the sequence whose $n^{\text{th}}$ term is given by $2^n$. Find the first 6 terms of this sequence.

**Solution:**

| Variable | Description | Value |
|---|---|---|
| $x(n)$ | general term of sequence | $2^n u(n)$ |

Table 3.6: input parameters

$$X(Z) = \frac{1}{1 - 2z^{-1}} \quad |z| > |2| \tag{3.55}$$

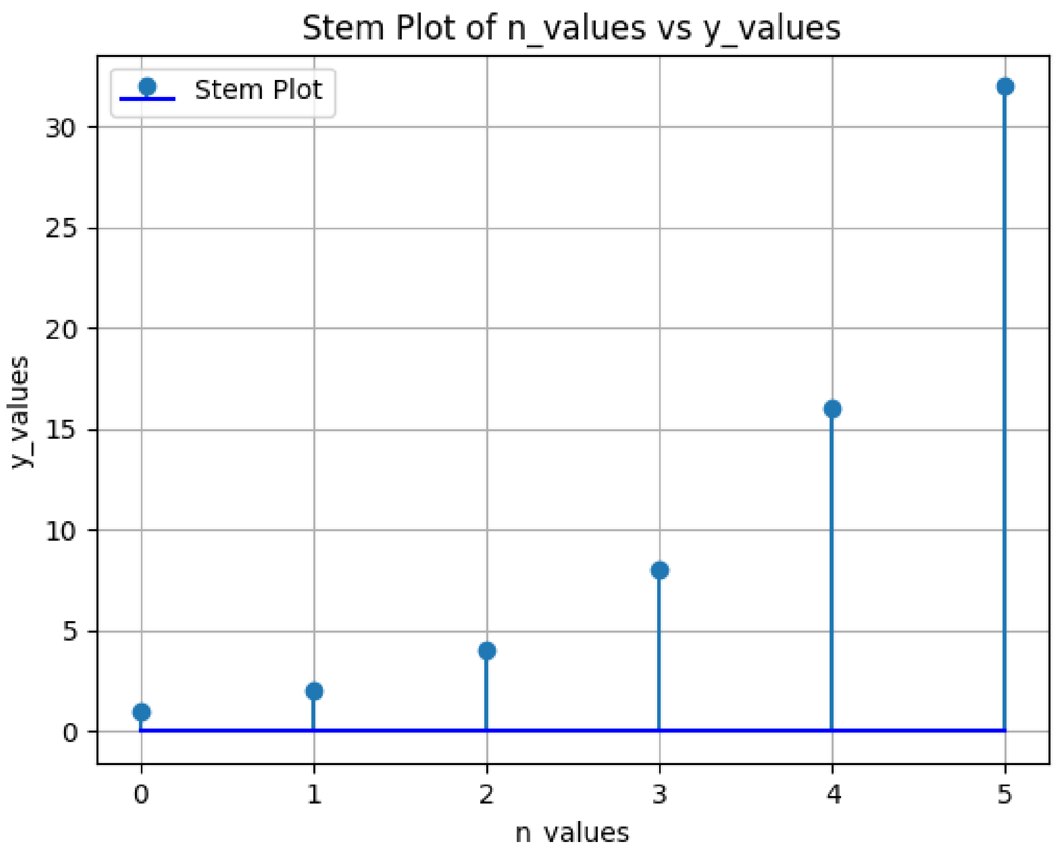



Figure 3.9: Six terms of given sequence

3.0.7 If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

**Solution:**

| **Variable** | **Description** |
|---|---|
| $x(0)$ | First term of the AP |
| $d$ | Common difference of the AP |
| $y(n)$ | Sum of $n+1$ terms of the AP |
| $x(n)$ | General term |

Table 3.7: Variables Used

$$y(n) = \frac{n+1}{2}\left(2x(0) + nd\right) u(n) \tag{3.56}$$

$$y(6) = 49 \tag{3.57}$$

$$y(16) = 289 \tag{3.58}$$

Then,

$$x(0) + 3d = 7 \tag{3.59}$$

$$x(0) + 8d = 17 \tag{3.60}$$

From equations 3.59 and 3.60, the augmented matrix is:

$$\begin{pmatrix} 1 & 3 & 7 \\ 1 & 8 & 17 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \tag{3.61}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{5} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 10 \end{pmatrix} \tag{3.62}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{3.63}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.64}$$

$$x(n) = (1 + 2n)\, u(n) \tag{3.65}$$

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \quad \{z \in \mathbb{C} : |z| > 1\} \tag{3.66}$$

$$y(n) = x(n) * u(n) \tag{3.67}$$

$$Y(z) = X(z)\ U(z) \tag{3.68}$$

$$\implies Y(z) = \left( \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \right) \left( \frac{1}{1 - z^{-1}} \right) \tag{3.69}$$

$$= \frac{1}{(1 - z^{-1})^2} + \frac{2z^{-1}}{(1 - z^{-1})^3} \tag{3.70}$$

$$(n+1)\, u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})^2} \{z \in \mathbb{C} : |z| > 1\} \tag{3.71}$$

$$n\,((n+1)\, u(n)) \xleftrightarrow{\mathcal{Z}} \frac{2z^{-1}}{(1 - z^{-1})^3} \{z \in \mathbb{C} : |z| > 1\} \tag{3.72}$$

From equations (B.11.1) and (B.11.2), taking the inverse Z Transform,

$$y(n) = (n+1)\,u(n) + n\left((n+1)\,u(n)\right) \tag{3.73}$$

$$\implies y(n) = (n+1)^2\; u(n) \tag{3.74}$$

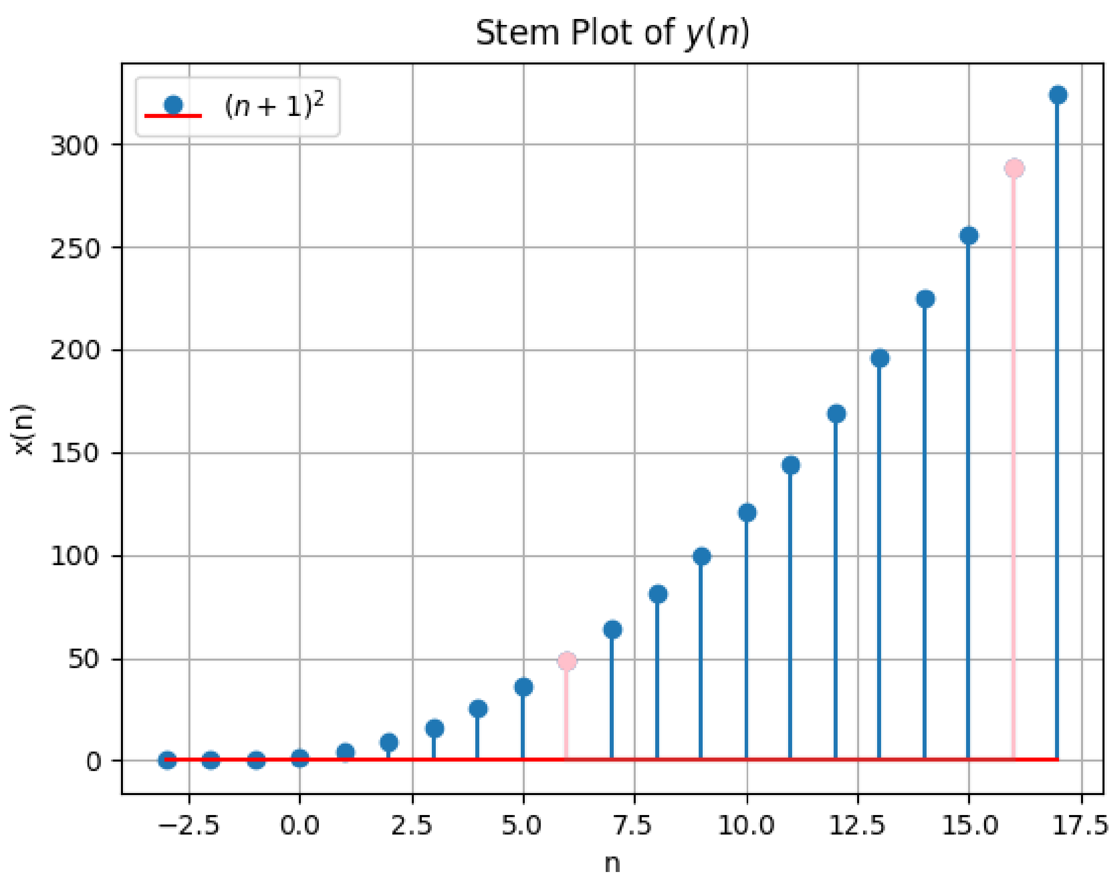



Figure 3.10: Stem Plot of y($n$)

3.0.8 Write the first five terms of the sequence and obtain the corresponding series:

$a_1 = a_2 = 2,\ a_n = a_{n-1} - 1,\ n > 2$

**Solution:**

| Parameter | Description | Value |
|---|---|---|
| $x(0)$ | First term | 2 |
| $x(1)$ | Second term | 2 |
| ROC | Region of convergence | $\{z : \lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\rvert < \infty\}$ |
| $x(n)$ | General term | $x(n) = \begin{cases} ? & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$ |

Table 1: Parameter Table

$$x(n) - x(n-1) = 2u(n) - 2u(n-1) - u(n-2) \tag{3.75}$$

$$X(z) - z^{-1}X(z) = \frac{2}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})} - \frac{2z^{-1}}{(1-z^{-1})} \tag{3.76}$$

$$X(z) = \frac{2 - 2z^{-1} - z^{-2}}{(1-z^{-1})^2}, |z| > 1 \tag{3.77}$$

Using partial fractions

$$X(z) = \frac{2z^{-1}}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})^2} + 2 \tag{3.78}$$

Taking inverse $Z$-transform by result of equation (B.4.6) in equation (3.78):

$$x(n) = 2u(n) + (1-n)u(n-1) \tag{3.79}$$

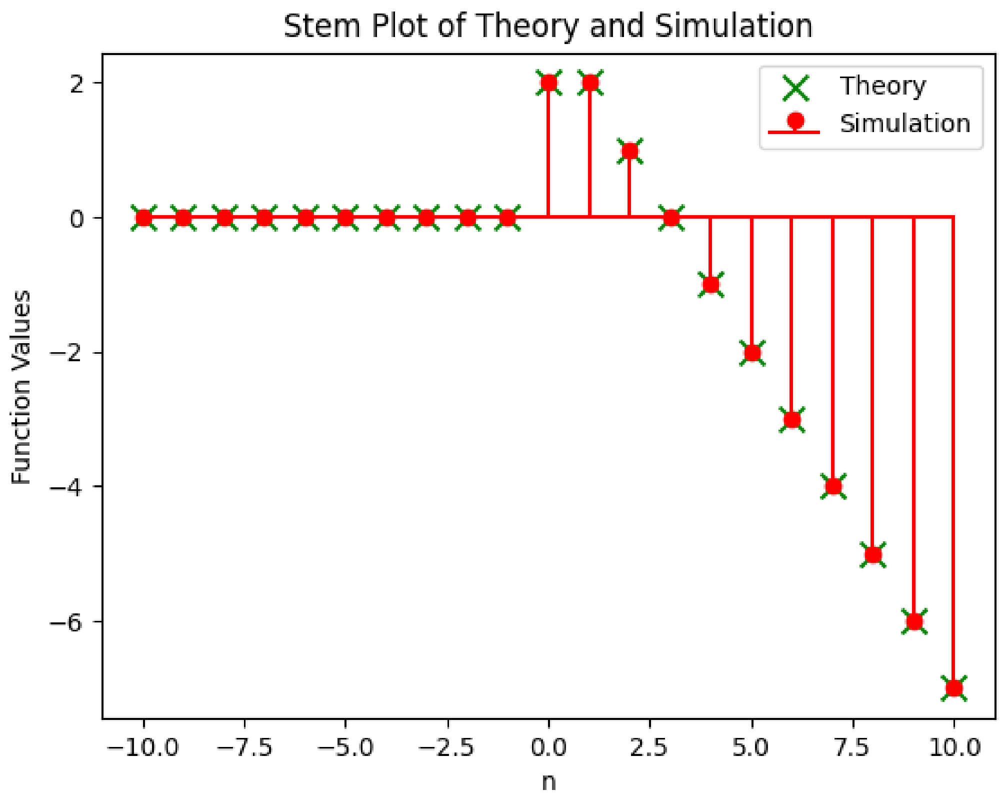



Figure 3.11: Comparison of Theory and Simulated Values

From the figureFig. 3.11 we can see that the theoretical and simulated values overlap.

3.0.9 Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $x(0)$ | First term of G.P. | 3 |
| $x(3)$ | Fourth term of G.P. | 81 |
| $r$ | common ratio of G.P. | r |

Table 3.9: input values

(a)

$$x(n) = x(0)\, r^n \tag{3.80}$$

from the values in Table 3.9

$$\frac{x(0)\, r^3}{x(0)} = 27 \tag{3.81}$$

$$r = 3 \tag{3.82}$$

$\therefore$ Required numbers are 9 and 27.

(b)

$$x(n) = 3^{n+1} u(n) \tag{3.83}$$

$$X(z) = \frac{3}{1 - 3z^{-1}} \quad |z| > 3 \tag{3.84}$$

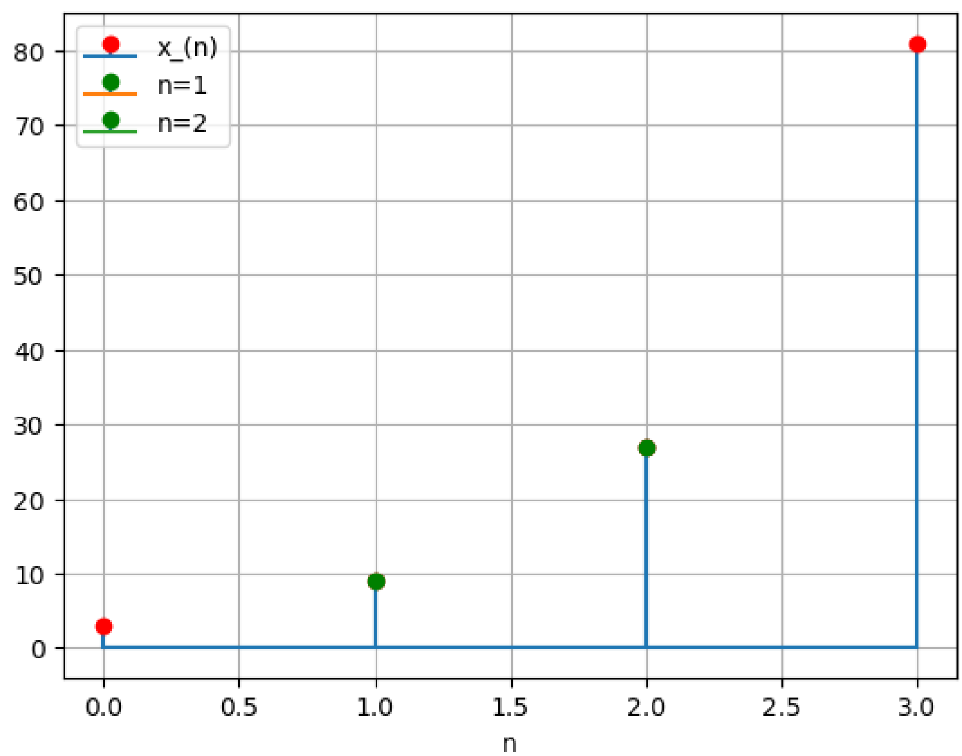



Figure 3.12: Graph of $x\,(n)$

3.0.10 What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

**Solution:**

The Z-transform of a sequence $x(n)$ is given by:

$$x(n) = 500(1.1)^n u(n) \tag{3.85}$$

$$X(Z) = \frac{500}{1-(1.1)z^{-1}}; |z| > 1.1 \tag{3.86}$$

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 500 | Principal amount before first year |
| $r$ | 1.1 | Common ratio of GP |
| $n$ | 10 | Number of years |
| $x(10)$ | $500(1.1)^{10} = 1296.87$ | Amount after 10 years |

Table 3.10: Parameter Table

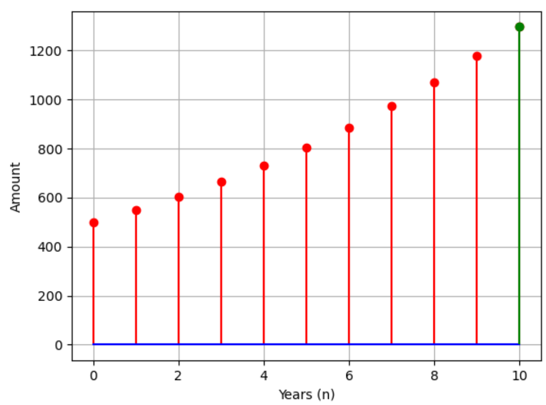



Figure 3.13: Plot of $x(n) = 500(1.1)^n$

3.0.11 Find the $20^{th}$ term from the last term of the AP: $3, 8, 13.....253$.

**Solution:** As the $20^{th}$ term is considered from last,

| Parameter | Description | Value |
|---|---|---|
| $x(0)$ | first term | 253 |
| $d$ | common difference | $3 - 8 = -5$ |
| $x(n)$ | $(n+1)^{th}$ term | $(x(0) + nd)u(n)$ |
| $u(n)$ | unit step function | |
| $x(n)$ | $20^{th}$ term | 158 |

Table 3.11: Input table

From equation (B.1.1) and (B.3.2):

$$X(z) = \frac{253}{1-z^{-1}} + \frac{-5z^{-1}}{(1-z^{-1})^2}; \{|z| > 1\} \tag{3.87}$$

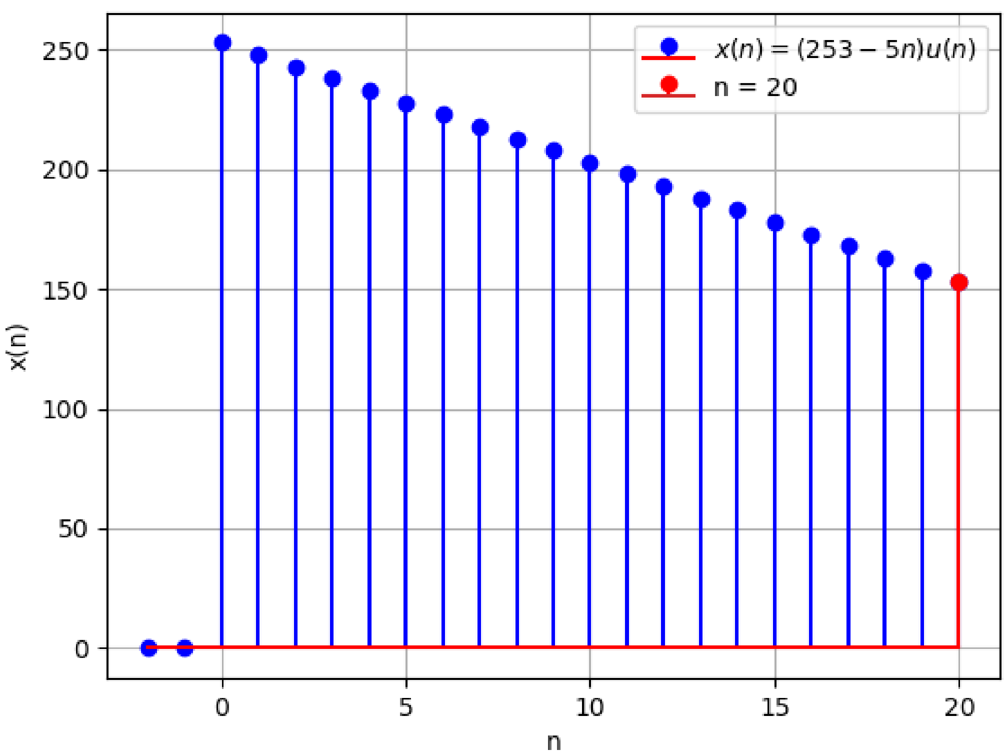



Figure 3.14:

3.0.12 Find the sum to $n$ terms of series , whose $n^{th}$ term is : $n(n+1)(n+4)$.

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $x(n)$ | $n^{th}$ term of series | $n(n+1)(n+4)u(n)$ |
| $y(n)$ | sum of n terms of series | |

Table 3.12: Given parameters

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \{|z|>1\} \tag{3.88}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3} \{|z|>1\} \tag{3.89}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4} \{|z|>1\} \tag{3.90}$$

$$n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5} \{|z|>1\} \tag{3.91}$$

From equation (B.3.2) to (B.3.4),

$$X(z) = \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4} + \frac{5z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3} + \frac{4z^{-1}}{\left(1-z^{-1}\right)^2} \{|z|>1\} \tag{3.92}$$

$$Y(z) = X(z)U(z) \tag{3.93}$$

$$= \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^5} + \frac{5z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^4} + \frac{4z^{-1}}{\left(1-z^{-1}\right)^3} \tag{3.94}$$

$$= \frac{1}{4}\left[\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^{5}}\right] + \frac{13}{6}\left[\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^{4}}\right] + \frac{19}{4}\left[\frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^{3}}\right] + \frac{17}{6}\left[\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}\right]\{|z|>1\} \quad (3.95)$$

Taking reverse z transform, using equations (3.88) to (3.91)

$$y(n) = \left(\frac{n^4}{4} + \frac{13n^3}{6} + \frac{19n^2}{4} + \frac{17n}{6}\right) u\,(n) \quad (3.96)$$

$$= \left(\frac{n^4}{4} + \frac{2n^3}{4} + \frac{10n^3}{6} + \frac{n^2}{4} + \frac{15n^2}{6} + \frac{4n^2}{2} + \frac{5n}{6} + \frac{4n}{2}\right) u\,(n) \quad (3.97)$$

$$= \left(\frac{n^4 + 2n^3 + n^4}{4}\right) u\,(n) + \left(\frac{10n^3 + 15n^2 + 5n}{6}\right) u\,(n) + \left(\frac{4n^2 + 4n}{2}\right) u\,(n) \quad (3.98)$$

$$= \left(\frac{n^2\,(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}\right) u\,(n) \quad (3.99)$$

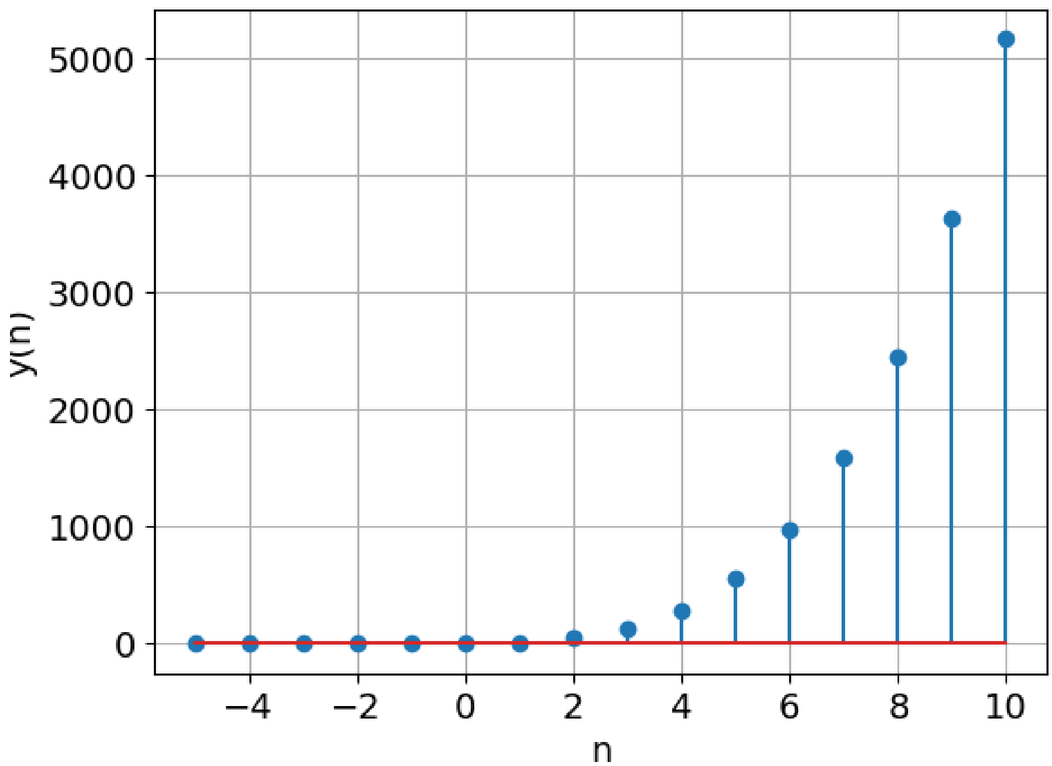



Figure 3.15: Sum of n terms of series

3.0.13 Find the indicated terms in the sequence whose nth terms is $a(n) = 4n-3$. Find $a(17)$ and $a(24)$.

**Solution:** In the question, following information is provided:

$$x(n) = (4n + 1)(u(n)) \tag{3.100}$$

| **Symbol** | **Value** | **Description** |
|---|---|---|
| $x(n)$ | $(4n+1)u(n)$ | The nth term of the sequence |
| $x(16)$ | ? | 17th term |
| $x(23)$ | ? | 24th term |

Table 3.13: Parameters

$$x(16) = 4 \times 16 + 1 = 65 \tag{3.101}$$

$$x(23) = 4 \times 23 + 1 = 93 \tag{3.102}$$

Using Z-Transform,

$$X(z) = 4\frac{z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{3.103}$$

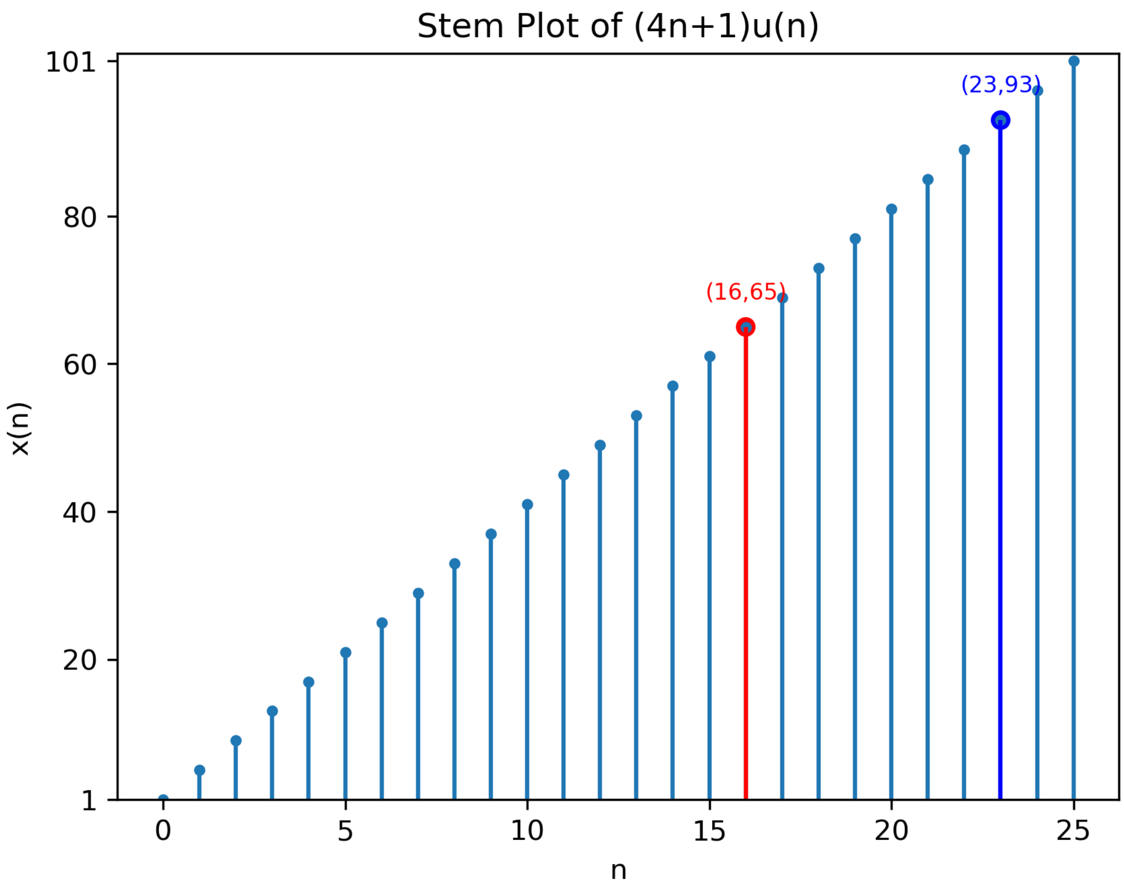



Figure 3.16: $x(n)$ vs $n$

3.0.14 The difference between any two cosecutive interior angles of a polygon is 5°.If the smallest angle is 120°,find the number of sides of polygon.

**Solution:**

| Variable | Description | Value |
|---|---|---|
| $x(0)$ | first term of AP | 120 |
| d | common difference of AP | 5 |
| $x(n)$ | general term of AP | none |

Table 3.14: input parameters

Sum of interior angles of a polygon with $n+1$ sides is given by

$$S = (n-1)180 \tag{3.104}$$

Sum of $n$ terms of AP is given by

$$y(n) = x(n) * u(n) \tag{3.105}$$

where $x(n) = 120 + 5n$

$$x(n) * u(n) = (n-1)180 \tag{3.106}$$

$$\begin{aligned} Y(z) &= X(z)U(z) && (3.107)\\ &= \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}\right)\frac{1}{1-z^{-1}} \quad |z| > 1 && (3.108)\\ &= \frac{120}{(1-z^{-1})^2} + \frac{5z^{-1}}{(1-z^{-1})^3} \quad |z| > 1 && (3.109)\end{aligned}$$

$$\begin{aligned} (n+1)\,u\,(n) &\overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{(1-z^{-1})^2}\right) \quad |z| > 1 && (3.110)\\ \frac{(n)\,(n-1)}{2}u\,(n-1) &\overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{z^{-1}}{(1-z^{-1})^3}\right) \quad |z| > 1 && (3.111)\end{aligned}$$

applying inverse Z-transform for each term and solving we get,

$$y\,(n) = \frac{n+1}{2}\,(240 + 5n)\,u(n) \tag{3.112}$$

now from (3.106)

$$y(n) = (n-1)180 \tag{3.113}$$

$$\frac{n+1}{2}\,(240+5n)\,u(n) = (n-1)180 \tag{3.114}$$

now replace $n$ by $n-1$:

$$n(235+5n) = (n-2)360 \tag{3.115}$$

$$5n^2 - 125n + 720 = 0 \tag{3.116}$$

$$n = 16, 9 \tag{3.117}$$

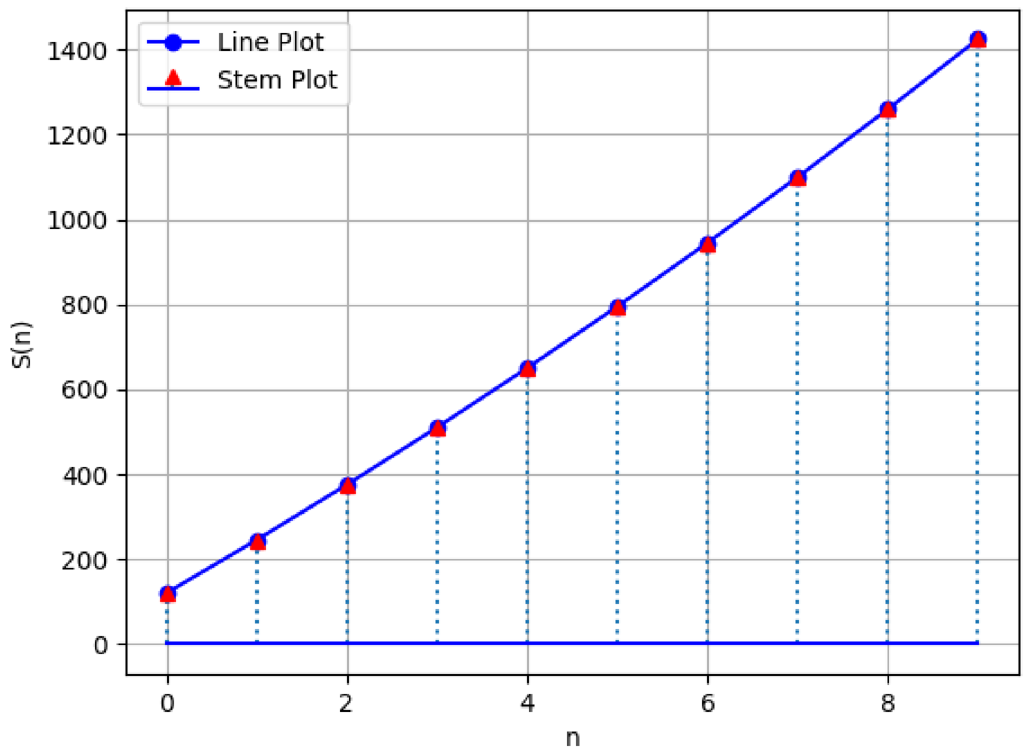



Figure 3.17: Plot of the sum of n terms taken from Python

3.0.15 The 5th,8th and 11th terms of a GP are p,q and s respectively .show that $q^2 = ps$

**Solution:**

| Symbol | Value | Description |
|---|---|---|
| $x(5)$ | $p$ | $x(0)\,r^5$ |
| $x(8)$ | $q$ | $x(0)\,r^8$ |
| $x(11)$ | $s$ | $x(0)\,r^{11}$ |
| $x(n)$ | | $x(0)\,r^n u(n)$ |
| $r$ | $\left(\frac{s}{p}\right)^{\frac{1}{6}}$ | common ratio |

Table 3.15: input parameters

From Table 3.15:

$$q^2 = x\,(0)\ r^8\,x\,(0)\ r^8 \tag{3.118}$$

$$= x\,(0)^2\ r^{16} \tag{3.119}$$

$$ps = x\,(0)\ r^5\,x(0)\,r^{11} \tag{3.120}$$

$$= x\,(0)^2\ r^{16} \tag{3.121}$$

$$\Longrightarrow\ q^2 = ps \tag{3.122}$$

now we will find r and x(0):

$$\frac{s}{p} = \frac{x\,(0)\,r^{11}}{x\,(0)\,r^5} \tag{3.123}$$

$$r = \left(\frac{s}{p}\right)^{\frac{1}{6}} \tag{3.124}$$

$$p = x\,(0)\left(\frac{s}{p}\right)^{\frac{5}{6}} \tag{3.125}$$

$$x\,(0) = \frac{p^{\frac{11}{6}}}{s^{\frac{5}{6}}} \tag{3.126}$$

Applying z-Transform:

$$X(z) = \frac{x\,(0)}{1 - r\,z^{-1}}\ \ , |z| > |r| \tag{3.127}$$

$$\Longrightarrow\ X(z) = \frac{p^3}{p^{\frac{7}{6}}\,s^{\frac{5}{6}} - q^2 z^{-1}}\ , |z| > \left|\left(\frac{q}{p}\right)^{\frac{1}{3}}\right| \tag{3.128}$$

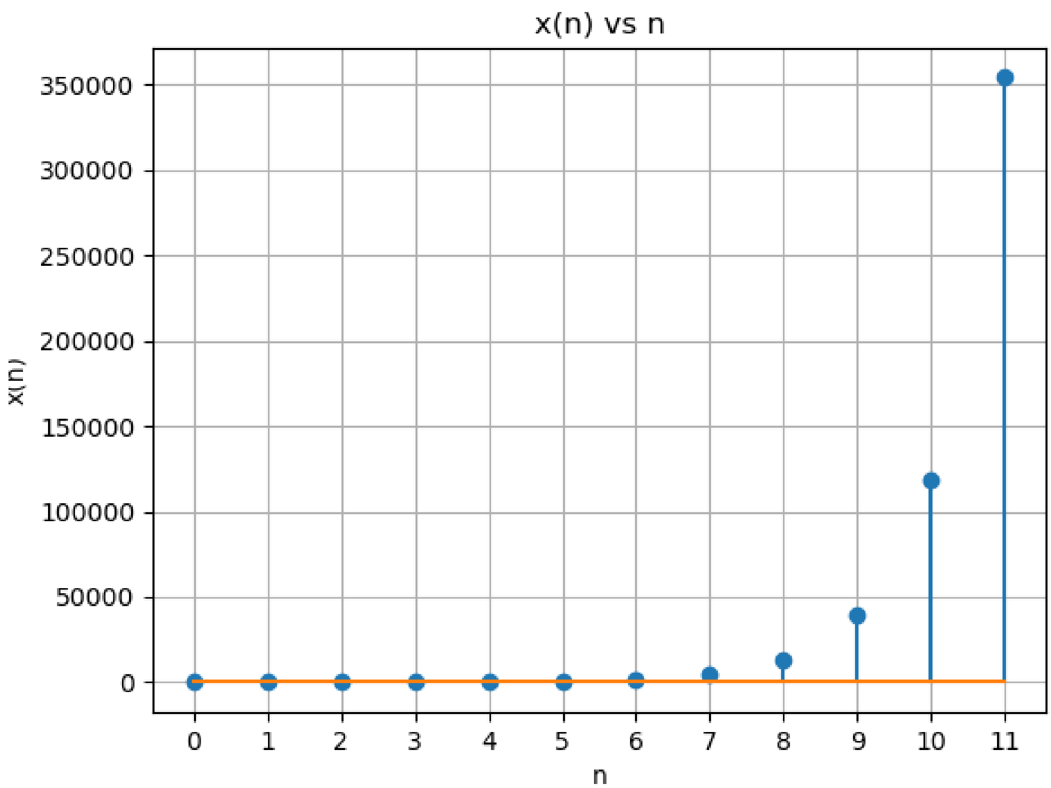



Figure 3.18: plot x($n$)vs n $p = 486, \quad q = 13122, \quad s = 354294, \quad r = 3$

3.0.16 The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

**Solution:**

$$y(n) = \left[\frac{(n+1)}{2}(2x(0) + nd)\right] u(n) \tag{3.129}$$

$$\implies y(3) = \frac{4}{2}(2x(0) + 3d) \tag{3.130}$$

$$\tag{3.131}$$

| Symbol | Value | Description |
|---|---|---|
| $x(0)$ | 11 | First term of AP |
| $y(3)$ | 56 | Sum of the first four terms of AP |
| $y(n) - y(n-4)$ | 112 | Sum of the last four terms of AP |

Table 3.16: Input Parameters

From Table 3.16:

$$\frac{4}{2}\left(2x\left(0\right) + 3d\right) = 56 \tag{3.132}$$

$$2x\left(0\right) + 3d = 28 \tag{3.133}$$

$$\implies d = 2 \tag{3.134}$$

$$y\left(n\right) - y\left(n-4\right) = \frac{4}{2}\left(2x\left(n\right) + 3\left(-d\right)\right) \tag{3.135}$$

From Table 3.16:

$$\frac{4}{2}\left(2x\left(n\right) + 3\left(-d\right)\right) = 112 \tag{3.136}$$

$$2x\left(n\right) - 3d = 56 \tag{3.137}$$

$$\implies x\left(n\right) = 31 \tag{3.138}$$

$$x\left(0\right) + \left(n\right)2 = 31 \tag{3.139}$$

$$\implies n = 10 \tag{3.140}$$

$$x\left(n\right) = \left(x\left(0\right) + 2n\right)u\left(n\right) \tag{3.141}$$

$$\implies X\left(z\right) = \frac{x\left(0\right)}{1 - z^{-1}} + 2\frac{z^{-1}}{\left(1 - z^{-1}\right)^2}. \quad |z| > 1 \tag{3.142}$$

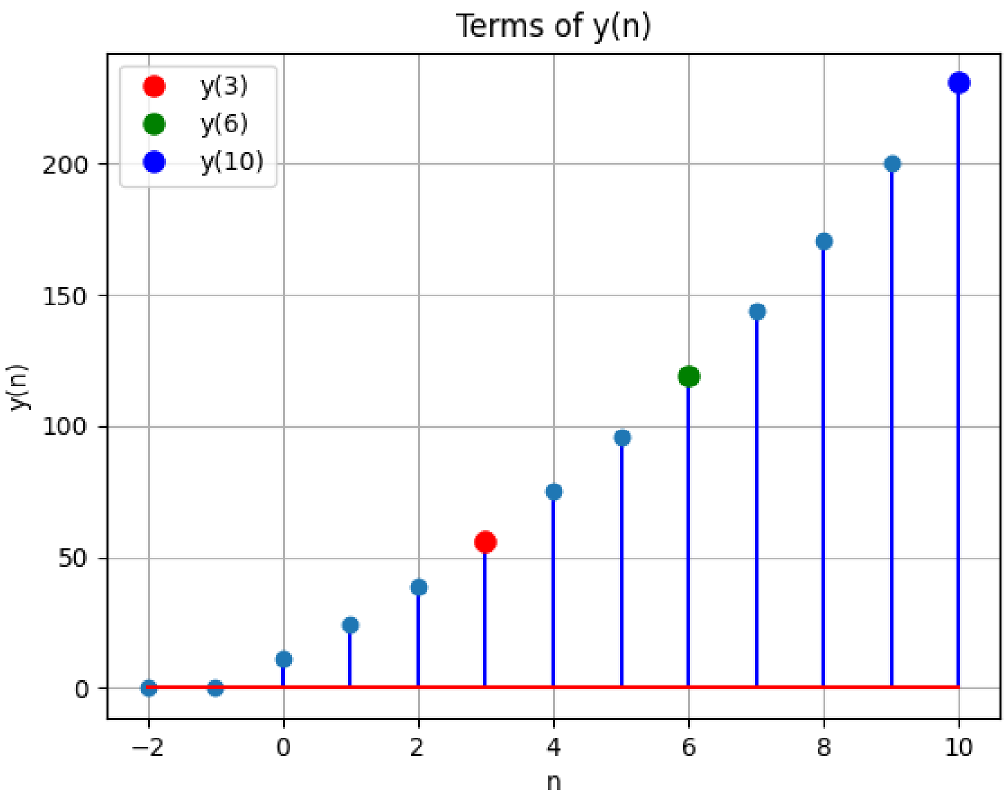



Figure 3.19: Plot y(n) vs n

3.0.17 Find the sum to $n$ terms of the series whose $n^{th}$ term is given by $(2n-1)^2$ ? **Solution:** Sum

| Variable | Description | Value |
|---|---|---|
| $x(n)$ | $n^{th}$ term of sequence | $(2n+1)^2u(n)$ |

Table 3.17: input parameters

of $n$ terms of AP is given by

$$y(n) = x(n) * u(n) \tag{3.143}$$

$$x(n) = (2n+1)^2u(n) \tag{3.144}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})} \quad |z|>1 \tag{3.145}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \tag{3.146}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \quad |z|>1 \tag{3.147}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4} \quad |z|>1 \tag{3.148}$$

$$\Longrightarrow X(z) = \frac{4z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{1}{(1-z^{-1})} + \frac{4z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \tag{3.149}$$

$$Y(z) = X(z)U(z) \tag{3.150}$$

$$= \left(\frac{4z^{-1}(z^{-1}+1)}{(1-z^{-1})^3} + \frac{1}{(1-z^{-1})} + \frac{4z^{-1}}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{3.151}$$

$$= \frac{4z^{-1}(z^{-1}+1)}{(1-z^{-1})^4} + \frac{1}{(1-z^{-1})^2} + \frac{4z^{-1}}{(1-z^{-1})^3} \tag{3.152}$$

$$\Longrightarrow Y(Z) = \frac{1}{(1-z^{-1})} + \frac{9z^{-1}}{(1-z^{-1})} + \frac{25z^{-2}}{(1-z^{-1})^2} + \frac{24z^{-3}}{(1-z^{-1})^3} + \frac{8z^{-4}}{(1-z^{-1})^4} \quad |z|>1 \tag{3.153}$$

Now from (3.145), (3.146), (3.147), (3.148), (3.153) By using inverse Z-transform pairs,

$$y\,(n) = u(n) + 9u(n-1) + 25(n-1)u(n-2) + 24\frac{(n-1)(n-2)}{2}u(n-3) + 8\frac{(n-1)(n-2)(n-3)}{6}u(n-4) \tag{3.154}$$

$$\Longrightarrow y\,(n) = \left(\frac{4n^3+12n^2+11n+3}{3}\right)u(n) \tag{3.155}$$

$\therefore$ Sum of $n$ terms of the series whose $n^{th}$ term is given by $(2n+1)^2$ is $\frac{4n^3+12n^2+11n+3}{3}$ .

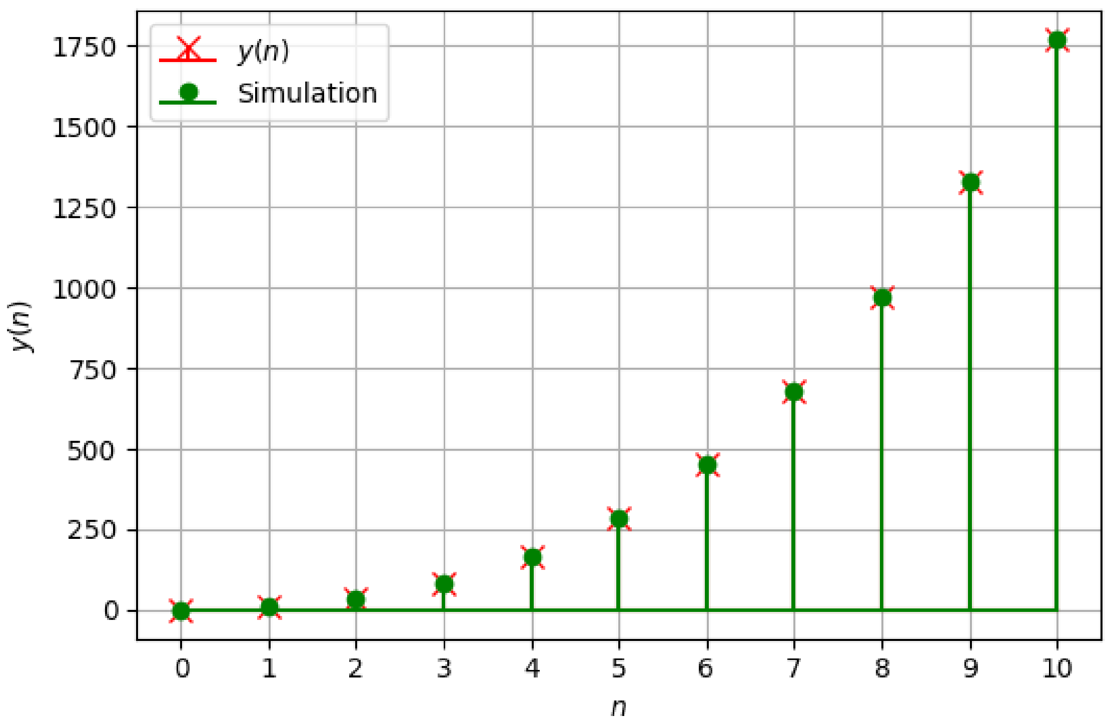



Figure 3.20: Theory vs Simulation

3.0.18 If the $4^{th}$, $10^{th}$ and $16^{th}$ terms of a G.P. are $x$, $y$, and $z$, respectively. Prove that $x,\ y,\ z$ are in G.P.

**Solution:**

| **Symbol** | **Value** | **Description** |
|---|---|---|
| $x$ | $x(0)\,r^4$ | $x(4)$ |
| $y$ | $x(0)\,r^{10}$ | $x(10)$ |
| $z$ | $x(0)\,r^{16}$ | $x(16)$ |
| $r$ | ? | $\frac{x(n)}{x(n-1)}$ |
| $x(0)$ | ? | First term |
| $x(n)$ | $x(0)\,r^n u(n)$ | General Term |

Table 3.18: **Given Information**

(a) From Table 3.0.18,

$$x = x(3) = x(0)\,r^3 \tag{3.156}$$

$$y = x(9) = x(0)\,r^9 \tag{3.157}$$

$$z = x(15) = x(0)\,r^{15} \tag{3.158}$$

Consider $\frac{x(9)}{x(3)}$ and $\frac{x(15)}{x(9)}$;

$$\frac{x(9)}{x(3)} = \frac{x(0)\,r^9}{x(0)\,r^3} = r^6 = \frac{x(15)}{x(9)} = \frac{x(0)\,r^{15}}{x(0)\,r^9} \tag{3.159}$$

From (3.159), $x(3)$, $x(9)$, $x(15)$ are in G.P.

$\therefore$ $x$, $y$, $z$ are in G.P.

(b) $x(0)$ and $r$ can be expressed in terms of $x$, $y$, and $z$ in the following manner.

$$\frac{y}{x} = r^6 \tag{3.160}$$

$$\Longrightarrow r = \sqrt[6]{\frac{y}{x}} = \left(\frac{y}{x}\right)^{\frac{1}{6}} \tag{3.161}$$

$$\Longrightarrow x(0) = \frac{x}{r^3} = x\left(\frac{x}{y}\right)^{\frac{3}{6}} \tag{3.162}$$

$$\therefore\ x(0) = x^{\frac{5}{3}} y^{-\frac{2}{3}} \text{ and } r = \left(\frac{y}{x}\right)^{\frac{1}{6}} = y^{\frac{1}{6}} x^{-\frac{1}{6}} \tag{3.163}$$

(c) From (B.5.4) Z-transform of a G.P. is

$$X(z) = \frac{x(0)}{1 - rz^{-1}}; |z| > |r| \tag{3.164}$$

Substituting $r$ and $x(0)$ from (3.163),

$$X(z) = \frac{x^{\frac{5}{3}} y^{-\frac{2}{3}}}{1 - \left(\frac{y}{x}\right)^{\frac{1}{6}} z^{-1}} \tag{3.165}$$

(d) Example Let $x(0) = 1$ and $r = 1.2$

$$x = x(3) = (1.2)^3 \tag{3.166}$$

$$y = x(9) == (1.2)^9 \tag{3.167}$$

$$z = x(15) = (1.2)^{15} \tag{3.168}$$

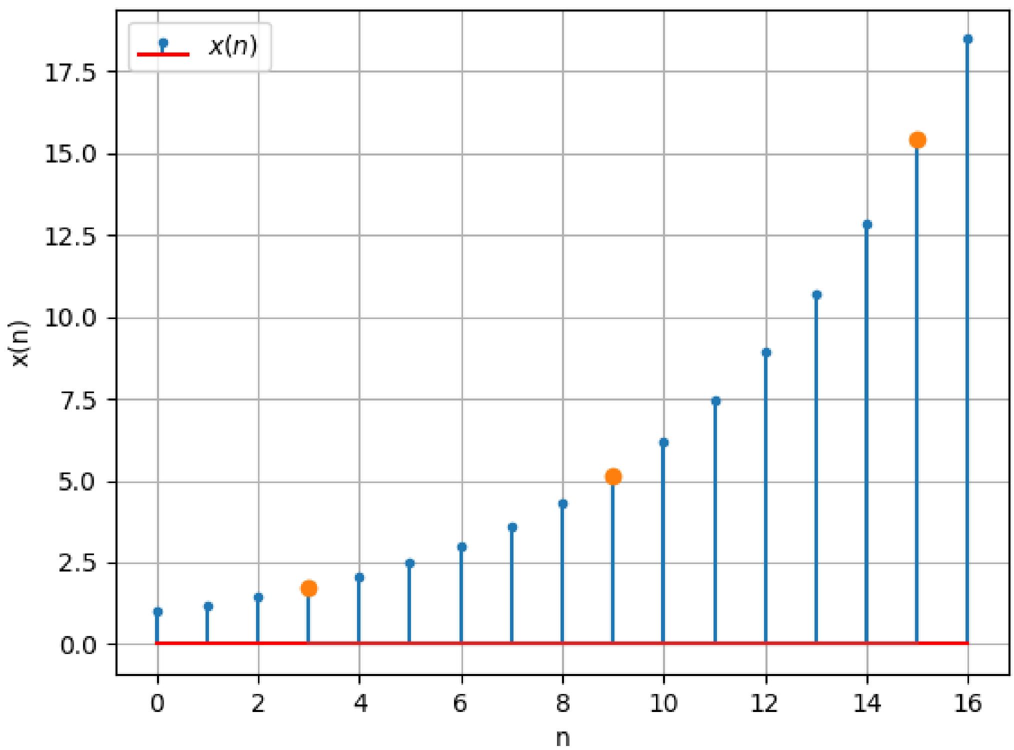



Figure 3.21: Stem Plot of $x(n)$ vs $n$

3.0.19 Show that the ratio of the sum of the first $n$ terms of a geometric progression (G.P.) to the sum of terms from $(n+1)$th to $(2n)$th term is $\frac{1}{r^n}$. **Solution:**

3.0.20 A G.P consists of an even number of terms. If the sum of all terms is 5 times the sum of terms occupying odd places, then find its common ratio.

**Solution:** Solving the Question in time domain:

| Parameter | Description | condition |
|---|---|---|
| $N$ | Number of terms in the G.P | - |
| $M$ | number of odd place terms | N=2M |
| $x(0)$ | first term in the G.P | - |
| $r$ | common ratio in the G.P | - |
| $x(n)$ | $n+1$ th term in the G.P | $x(n) = x(0)r^n$ |
| $y(n)$ | sum of G.P series | $y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n)$ |
| $x_o(n)$ | $n+1$ th term of G.P of odd places | $x_o(n) = x(0)r^{2n}$ |
| $y_o(n)$ | sum of terms in odd places | $y_o(n) = x(0)\left(\frac{r^{n+1}-1}{r^2-1}\right)u(n)$ |

Table 3.19: Input Parameters

$$x(n) = x(0)r^n \tag{3.169}$$

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{3.170}$$

The sum of terms in odd places:

$$x_o(n) = x(0)r^{2n} \tag{3.171}$$

$$y_o(n) = x(0)\left(\frac{r^{n+1}-1}{r^2-1}\right)u(n) \tag{3.172}$$

Then from (3.170) and (3.172)

$$x(0)\left(\frac{r^N-1}{r-1}\right)u(n) = 5\left(x(0)\left(\frac{r^{2M}-1}{r^2-1}\right)u(n)\right) \tag{3.173}$$

$$\frac{r^2-1}{r-1} = 5 \tag{3.174}$$

$$\text{as } r \neq 1, \quad \text{hence } r = 4 \tag{3.175}$$

$$\tag{3.176}$$

X,Y,Xo,Yo are frequency counterparts of the above GP

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \qquad |z| > |r| \tag{3.177}$$

$$X_o(z) = \frac{x(0)}{1 - r^2 z^{-1}} \tag{3.178}$$

$$Y(z) = \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \tag{3.179}$$

$$Y_o(z) = \frac{x(0)}{\left(1 - rz^{\frac{-1}{2}}\right)(1 - z^{-1})} \tag{3.180}$$

3.0.21 Find the sum to indicated number of terms in the geometric progression $x^3, x^5, x^7, ...n$ terms (if $x \neq \pm 1$).

**Solution:**

| **Input Parameters** | **Values** | **Description** |
|---|---|---|
| $x(0)$ | $x^3$ | Initial term |
| $r$ | $x^2$ | Common ratio |
| $x(n)$ | $x^{2n+3}u(n)$ | General term |

Table 3.20: Given inputs

From Table 3.20,

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \tag{3.181}$$

$$= \frac{x^3}{1 - x^2 z^{-1}} \quad |z| > x^2 \tag{3.182}$$

$$y(n) = x(n) * u(n) \tag{3.183}$$

$$Y(z) = X(z)U(z) \tag{3.184}$$

$$= \frac{x^3}{(1 - x^2 z^{-1})(1 - z^{-1})} \quad |z| > x^2 \cap |z| > 1 \tag{3.185}$$

$$= \frac{x^3}{x^2 - 1}\left(\frac{x^2}{1 - x^2 z^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{3.186}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{3.187}$$

$$x^{2n+2}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{x^2}{1 - x^2 z^{-1}} \quad |z| > x^2 \tag{3.188}$$

Taking inverse Z transform of $Y(z)$,

$$y(n) = x^3 \left(\frac{x^{2n+2} - 1}{x^2 - 1}\right) u(n) \tag{3.189}$$

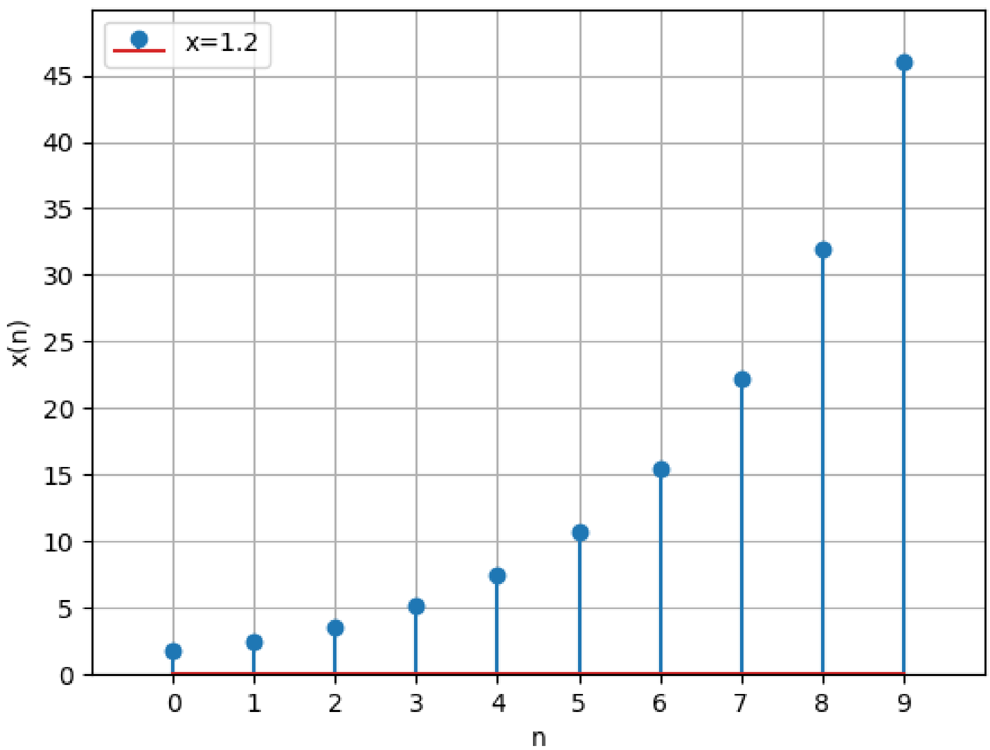



Figure 3.22: Plot of $x(n)$ for $x = 1.2$

3.0.22 Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

**Solution:**

| **Parameter** | **Value** | **Description** |
|---|---|---|
| $x(6) - x(4)$ | 12 | 7th term exceeds 5th by 12 |
| $x(2)$ | 16 | Third term |
| $d$ | ? | Common difference |
| $x(0)$ | ? | First term of AP |
| $x(n)$ | $(x(0) + nd)u(n)$ | General term |

Table 3.21: Input parameters table

From Table 3.21

$$x(0) + 6d - x(0) - 4d = 12 \tag{3.190}$$

$$\implies 2d = 12 \tag{3.191}$$

$$\implies d = 6 \tag{3.192}$$

Also,

$$x(0) + 2d = 16 \tag{3.193}$$

$$\implies x(0) + 2(6) = 16 \tag{3.194}$$

$$\implies x(0) = 4 \tag{3.195}$$

$$\therefore x(n) = 6n + 4 \tag{3.196}$$

From Table 3.21

$$X(z) = x(0)\frac{1}{1 - z^{-1}} + d\frac{z^{-1}}{(1 - z^{-1})^2} \tag{3.197}$$

$$= 4\frac{1}{1 - z^{-1}} + 6\frac{z^{-1}}{(1 - z^{-1})^2} \tag{3.198}$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{3.199}$$

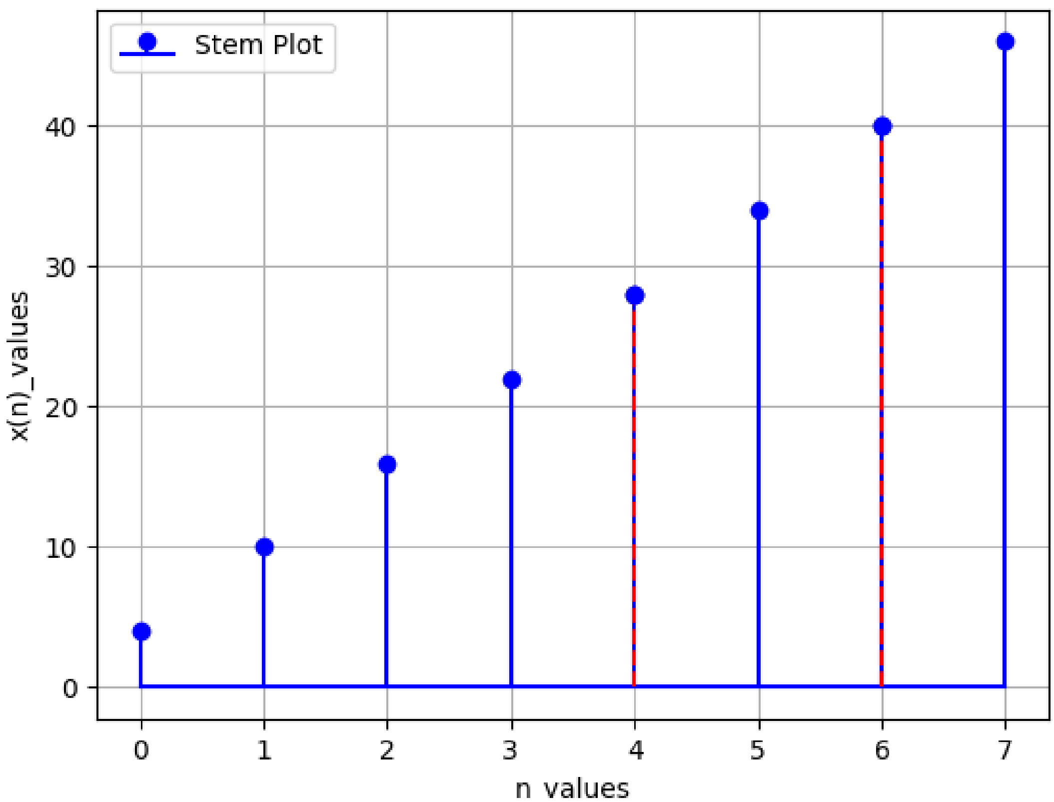



Figure 3.23: Given AP

3.0.23 Find the seventh term of the sequence where the nth term is given by $a_n = \frac{n^2}{2^n}$

**Solution:**

$$x(n) = \frac{(n+1)^2}{2^{(n+1)}} u(n) \tag{3.200}$$

| Parameter | Value |
|---|---|
| $x(n)$ | $\frac{(n+1)^2}{2^{(n+1)}} u(n)$ |
| $x(6)$ | ? |

Table 3.22: Input Parameters

$$x(6) = \frac{(6+1)^2}{2^{(6+1)}} \tag{3.201}$$

$$= \frac{49}{128} \tag{3.202}$$

(a) Scaling property:

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1 - az^{-1})}, \quad |z| > |a| \tag{3.203}$$

(b) Differentiation property:

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)\frac{dY(z)}{dz} \tag{3.204}$$

$$\implies nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{3.205}$$

$$\implies n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{3.206}$$

(c) Time shifting property:

$$y(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}Y(z) \tag{3.207}$$

The Z transform of $x(n)$ is given by:

from(4)

$$\frac{u(n)}{2^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-(2z)^{-1})}, \quad |z| > \frac{1}{2} \tag{3.208}$$

from(5)

$$\frac{n}{2^n}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(2z)^{-1}}{(1-(2z)^{-1})^2}, \quad |z| > \frac{1}{2} \tag{3.209}$$

$$\frac{n^2}{2^n}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(2z)^{-1}(1+(2z)^{-1})}{(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.210}$$

from(8)

$$\frac{(n+1)^2}{2^(n+1)}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (z)\frac{(2z)^{-1}(1+(2z)^{-1})}{(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.211}$$

$$X(z) = \frac{1+(2z)^{-1}}{2(1-(2z)^{-1})^3}, \quad |z| > \frac{1}{2} \tag{3.212}$$

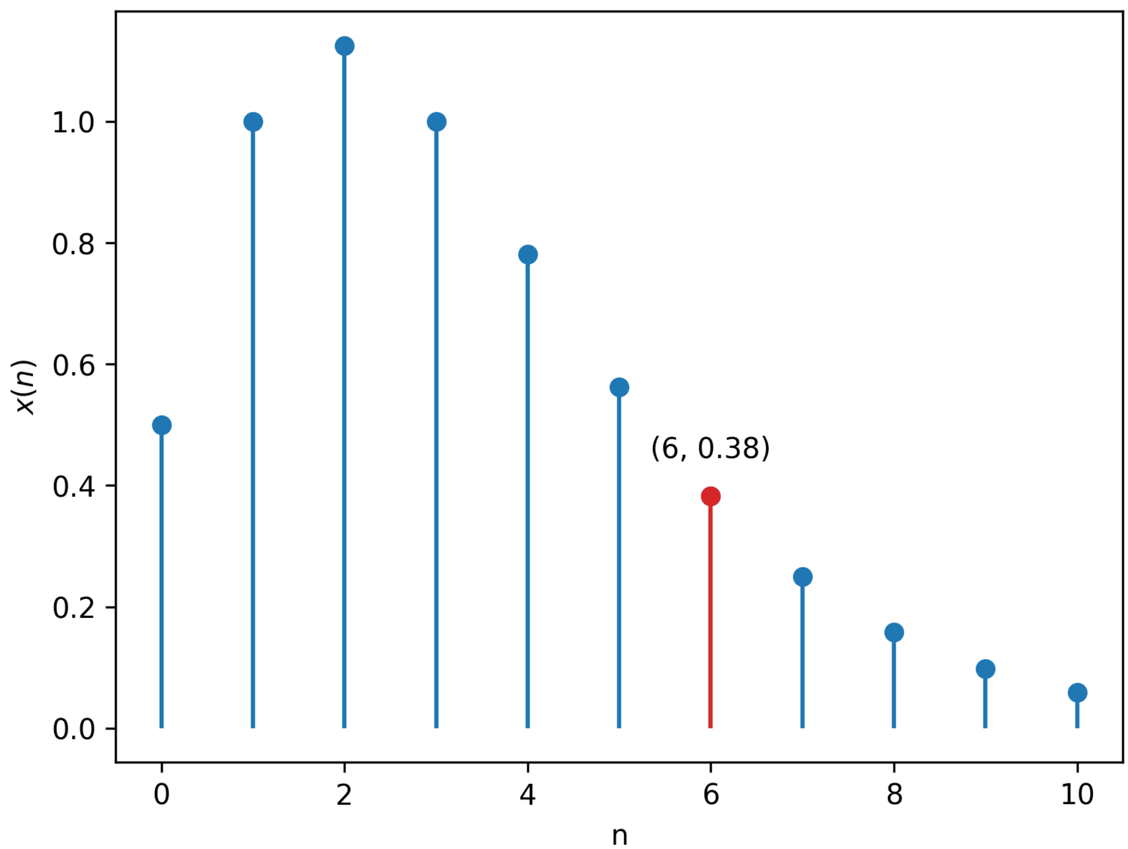



Figure 3.24: Stem plot of $x(n)$

3.0.24 Find sum to n terms of the following series:

$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\dots$ (NCERT 11.9.4.4) **Solution:**

3.0.25 1x2x3 + 2x3x4 + 3x4x5 +.....

**Solution:**

3.0.26 Find the sum of the following series up to $n$ terms:

(a) $5 + 55 + 555 + \ldots$

(b) $.6 + .66 + .666 + \ldots$

**Solution:**

3.0.27 Find $a_9$ in the sequence $a_n = (-1)^{n-1} n^3$

**Solution:**

3.0.28 Find the 20th term in this series.

$$2 \times 4 + 4 \times 6 + 6 \times 8 \cdots + n\,terms$$

**Solution:**

3.0.29 Q10) The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

**Solution:** Table of Parameters

| Input Variable | Condition |
|---|---|
| $x(0)$, $x(n)$ | first term and general term of a GP |
| $r$ | common ratio of a GP |
| $x(0), x(1), x(2)$ | three terms in GP |
| $x_i(n)$ | general term of i$^{\text{th}}$ GP sequence |
| $x_i(0)$ | first term of i$^{\text{th}}$ GP sequence |
| $r_i$ | common ratio of i$^{\text{th}}$ GP sequence |

(n+1)$^{\text{th}}$ term of GP x($n$) is given by:

$$x(n) = x(0)\, r^n \; u(n) \tag{3.213}$$

Then from given table of parameters,

$$x(0) + x(1) + x(2) = 56 \tag{3.214}$$

$$x(0) \implies \frac{56}{(1 + r + r^2)} \tag{3.215}$$

and from given another case following are in AP,

$$x(0) - 1, x(1) - 7, x(2) - 21$$

$$2(x(1) - 7) = x(0) - 1 + x(2) - 21 \tag{3.216}$$

$$x(0)\left(r^2 - 2r + 1\right) = 8 \tag{3.217}$$

and from (3.215)

$$\frac{56.\left(r^2 - 2r + 1\right)}{(1 + r + r^2)} = 8 \tag{3.218}$$

$$r_1 = 2\ ,\ r_2 = \frac{1}{2} \tag{3.219}$$

so from (3.215),

$$x_1(0) = 8\ ,\ x_2(0) = 32 \tag{3.220}$$

Then from (3.213)

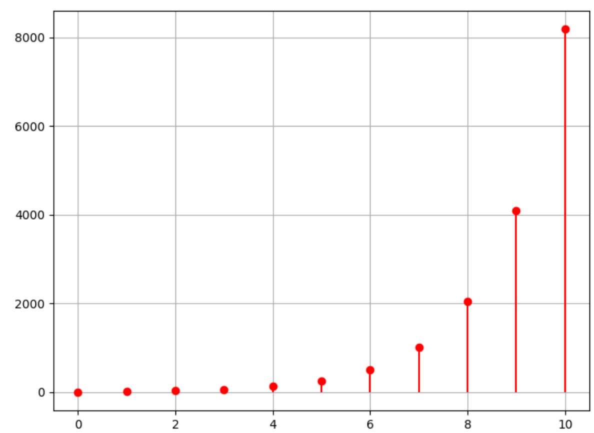
$$x_1(n) = 8.2^n = 2^{n+3}\ u(n) \tag{3.221}$$

$$x_2(n) = 32.\left(\frac{1}{2}\right)^n u(n) = 2^{5-n}\ u(n) \tag{3.222}$$

$x_1(0)$, $x_1(1)$ and $x_1(2)$ are 8, 16, 32 $(or)$ $x_2(0)$, $x_2(1)$ and $x_2(2)$ are 32, 16, 8 respectively

Graph of $x_1(n)$

$z$-transform of $x_1(n)$ is given by:



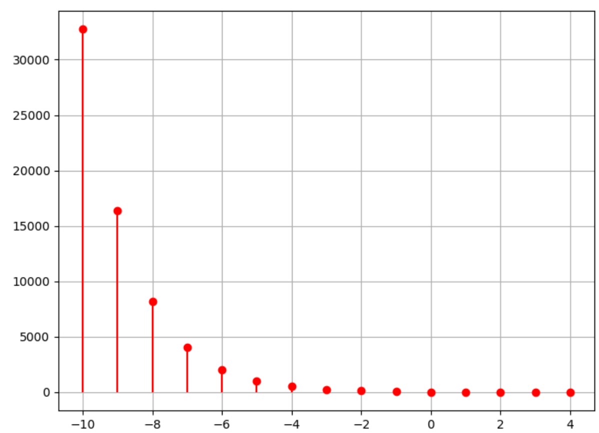

$$X_1(z) = \sum_{k=-\infty}^{\infty} x_1(k) \,.z^{-k} \tag{3.223}$$

from (3.221),

$$X_1(z) = \sum_{k=0}^{\infty} 2^{k+3} \; z^{-k} \tag{3.224}$$

Hence,

$$X_1(z) = \frac{8}{1 - 2z^{-1}}, \quad |2z^{-1}| < 1 \tag{3.225}$$




Graph of $x_2(n)$

and also from (3.222),

$$X_2(z) = \sum_{k=-\infty}^{\infty} x_2(k) .z^{-k} \tag{3.226}$$

$$X_2(z) = \sum_{k=0}^{\infty} 2^{5-k}\ z^{-k} \tag{3.227}$$

Hence,

$$X_2(z) = \frac{32}{1-(2z)^{-1}}, \quad |(2z)^{-1}| < 1 \tag{3.228}$$

3.0.30 Find the sum to n terms for the given series: $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$ **Solution:**

3.0.31 Find the sum to $n$ terms of the series:

$1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + ...$ (NCERT 11.9.4.7)

**Solution:**

| Variable | Description | Value |
|---|---|---|
| $y(n)$ | Sum of $n+1$ terms of the series | ? |
| $x(n)$ | General term | $(n+1)^2 u(n)$ |

Table 3.23: Variables Used

$$y(n) = 1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + ... \tag{3.229}$$

Let,

$$x(n) = (n+1)^2 u(n) \tag{3.230}$$

$$\implies y(n) = x(n) * u(n) * u(n) \tag{3.231}$$

$$Y(z) = X(z)\left(U(z)\right)^2 \tag{3.232}$$

From (B.3.3),

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \quad \{|z| > 1\} \tag{3.233}$$

Using (B.4.1),

$$(n+1)^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1 + z^{-1}}{\left(1 - z^{-1}\right)^3} \quad \{|z| > 1\} \tag{3.234}$$

From (3.234),

$$Y(z) = \left(\frac{1+z^{-1}}{(1-z^{-1})^3}\right)\left(\frac{1}{1-z^{-1}}\right)^2 \tag{3.235}$$

$$= \frac{1+z^{-1}}{(1-z^{-1})^5} \tag{3.236}$$

$$= \frac{1}{(1-z^{-1})^5} + \frac{z^{-1}}{(1-z^{-1})^5} \ \ \{|z|>1\} \tag{3.237}$$

From (B.4.9), using (B.4.1),

$$\frac{(n+1)(n+2)(n+3)(n+4)}{24}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})^5}\ \{|z|>1\} \tag{3.238}$$

$$\frac{(n)(n+1)(n+2)(n+3)}{24}u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^5}\ \{|z|>1\} \tag{3.239}$$

From (3.238) and (3.239), taking the Inverse Z Transform,

$$y(n) = \left(\frac{(n+1)(n+2)(n+3)(n+4)}{24}u(n)\right) + \left(\frac{(n)(n+1)(n+2)(n+3)}{24}u(n)\right) \tag{3.240}$$

$$\implies y(n) = \frac{(n+1)(n+2)^2(n+3)}{12}u(n) \tag{3.241}$$

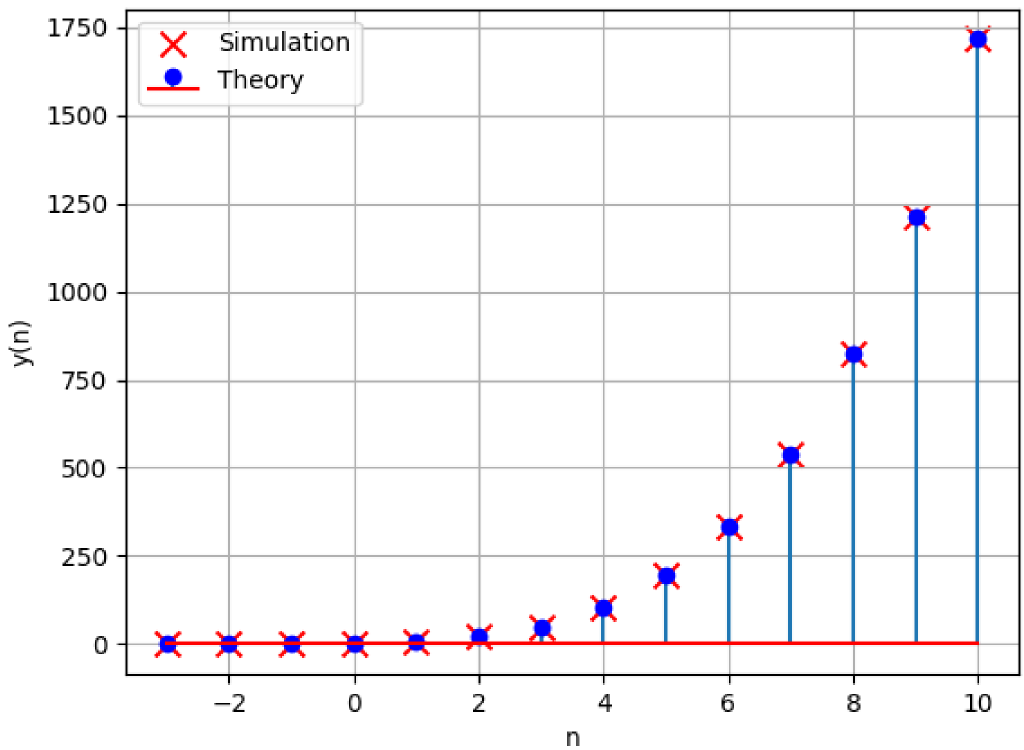



Figure 3.25: Stem Plot of $y\left(n\right)$

3.0.32 Find a GP for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

**Solution:** From Table 3.24:

$$x(0)r^4 = 4x(0)r^2 \tag{3.242}$$

$$\implies r = \pm 2 \tag{3.243}$$

From Table 3.24 and (3.243) :

| Parameter | Description | Value |
|---|---|---|
| $x(0)$ | First term of AP | $-$ |
| $r$ | Common ratio | $-$ |
| $x(n)$ | General term of given AP | $x(0)r^n u(n)$ |
| $x(0)+x(1)$ | sum of 1st and 2nd term | $-4$ |
| $\frac{x(4)}{x(2)}$ | Ratio of 5th and 3rd term | 4 |

Table 3.24: Input Parameters

$$x(0)r + x(0) = -4 \tag{3.244}$$

$$\implies x(0) = \frac{-4}{r+1} \tag{3.245}$$

$$x(0) = \begin{cases} \frac{-4}{3}, & r = +2 \\ 4, & r = -2 \end{cases} \tag{3.246}$$

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad , |z| > |r| \tag{3.247}$$

$$X(z) = \begin{cases} \frac{4}{3(2z^{-1}-1)}, & r = +2 \\ \frac{4}{1+2z^{-1}}, & r = -2 \end{cases} \tag{3.248}$$

$|z| > 2$

3.0.33 Find the sum of the following series up to n terms and obtain the Z-transform:

$$\ldots + 0 + \frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots$$

**Solution:**

3.0.34 Insert five numbers between 8 and 26 such that the resulting sequence is an A.P. and obtain the Z-transform of the sequence.

3.0.35 If $S_1$, $S_2$, $S_3$ are the sum of the first $n$ natural numbers, their squares, and their cubes, respectively, show that

$$9(S_2)^2 = (S_3)(1 + 8(S_1))$$

**Solution:**

3.0.36 If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P

**Solution:**

Table 3.25: Input Parameters

| Symbol | Remarks |
|---|---|
| $x(0)$ | $a$ |
| $x(1)$ | $b$ |
| $x(2)$ | $c$ |
| $x(3)$ | $d$ |
| $r$ | ratio of G.P a,b,c.... |
| $r_1$ | ratio of G.P $a^n + b^n, b^n + c^n, ....$ |
| $X(z)$ | z transform of G.P a,b,c.... |
| $Y(z)$ | z transform of G.P $a^n + b^n, b^n + c^n, ....$ |

From Table 3.25

$$r = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \tag{3.249}$$

From eq (3.249)

$$\frac{b^n+c^n}{a^n+b^n} = \frac{(ar)^n+\left(ar^2\right)^n}{(a)^n+(ar)^n} \tag{3.250}$$

$$= \frac{a^n r^n\left(1+r^n\right)}{a^n\left(1+r^n\right)} \tag{3.251}$$

$$= r^n \tag{3.252}$$

$$\frac{c^n+d^n}{b^n+c^n} = \frac{\left(ar^2\right)^n+\left(ar^3\right)^n}{(ar)^n+\left(ar^2\right)^n} \tag{3.253}$$

$$= \frac{a^n r^{2n}\left(1+r^n\right)}{a^n r^n\left(1+r^n\right)} \tag{3.254}$$

$$= r^n \tag{3.255}$$

$$\frac{b^n+c^n}{a^n+b^n} = \frac{c^n+d^n}{b^n+c^n} \tag{3.256}$$

Hence proved they are in in G.P

$$x(n) = a\left(\frac{b}{a}\right)^n u(n) \tag{3.257}$$

$$X(z) = \frac{a}{1-\left(\frac{b}{a}\right)z^{-1}}, \quad |z| > \left|\frac{b}{a}\right| \tag{3.258}$$

$$r_1 = \frac{b^n+c^n}{a^n+b^n} = \frac{c^n+d^n}{b^n+c^n} \tag{3.259}$$

From eq (3.259)

$$y\left(n\right)=\left(a^{n}+b^{n}\right)\left(\frac{b^{n}+c^{n}}{a^{n}+b^{n}}\right)^{n}u\left(n\right) \tag{3.260}$$

$$Y\left(z\right)=\frac{a^{n}+b^{n}}{1-\left(\frac{b^{n}+c^{n}}{a^{n}+b^{n}}\right)z^{-1}},\quad \left|z\right|>\left|\frac{b^{n}+c^{n}}{a^{n}+b^{n}}\right| \tag{3.261}$$

3.0.37 Write the first five terms of the sequence whose $n^{th}$ terms $a_n = \frac{n}{n+1}$

**Solution:**

| Term | Value | Description |
|---|---|---|
| $x(n)$ | $\frac{n+1}{n+2}u(n)$ | General term |

Table 3.26: Input Parameters

Here, Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n} \tag{3.262}$$

$$= \sum_{n=-\infty}^{\infty} \frac{n+1}{n+2}.u(n).z^{-n} \tag{3.263}$$

$$= \sum_{n=-\infty}^{\infty} u(n).z^{-n} - \frac{1}{n+2}u(n).z^{-n} \tag{3.264}$$

Now,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{3.265}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty} -\frac{1}{n+2}u(n).z^{-n} &= -\frac{1}{2} - \frac{z^{-1}}{3} - \frac{z^{-2}}{4}... \\ &= z^2[-z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3}...] + z \\ &= z + z^2\log(1-z^{-1})\end{aligned}$$

$$\frac{-1}{n+2}.u(n) \overset{Z}{\longleftrightarrow} \frac{1}{z^{-1}} + \frac{\log(1-z^{-1})}{z^{-2}}, \quad |z| > 1 \tag{3.266}$$

$$X(z) = \frac{1}{1-z^{-1}} + \frac{1}{z^{-1}} + \frac{\log(1-z^{-1})}{z^{-2}}, \quad |z| > 1 \tag{3.267}$$

3.0.38 150 workers were engaged to finish a job in a certain number of days, 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

**Solution:**

3.0.39 The Fibonacci sequence is defined by $1 = \mathrm{a}1 = \mathrm{a}_2$ and $\mathrm{a}_n = \mathrm{a}n - 1 + \mathrm{a}_{n-2}$ , $\mathrm{n} > 2$

Find $\dfrac{a_{n+1}}{a_n}$, for n = 1, 2, 3, 4, 5

**Solution:**

3.0.40 A farmer buys a used tractor for Rs 12000. He pays Rs 6000 in cash and agrees to pay the balance in annual installments of Rs 500 plus 12 % interest on the unpaid amount. How much will the tractor cost him? **Solution:**

3.0.41 1) Find the sum to n terms for the given series: $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$ **Solution:**

3.0.42 The sum of the first $n$ terms of two arithmetic progressions (AP) is in the ratio $5n+4 : 9n+6$.

Find the ratio of their 18th terms.

**Solution:**

3.0.43 How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636? **Solution:**

3.0.44 Find the sum to indicated number of terms in the geometric progression:

$1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).

**Solution:**

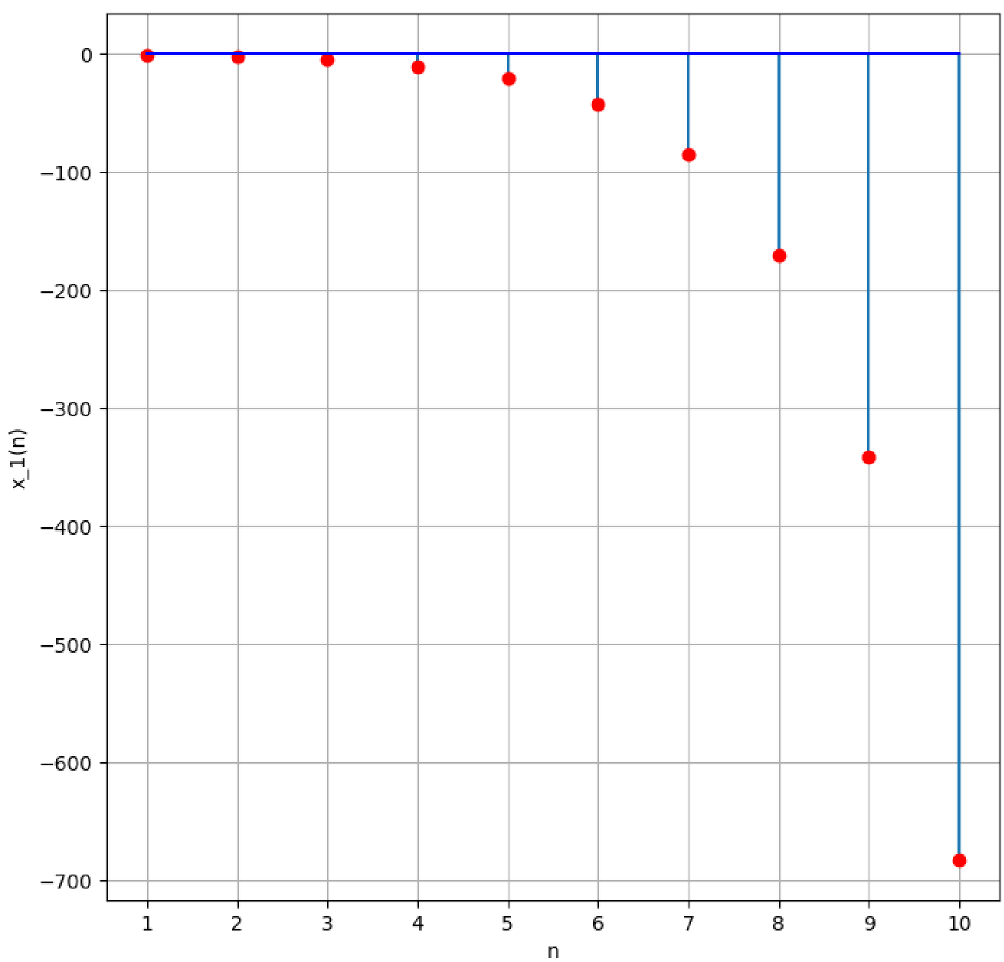



Figure 3.26: Representation of x(n) for $r = 2$

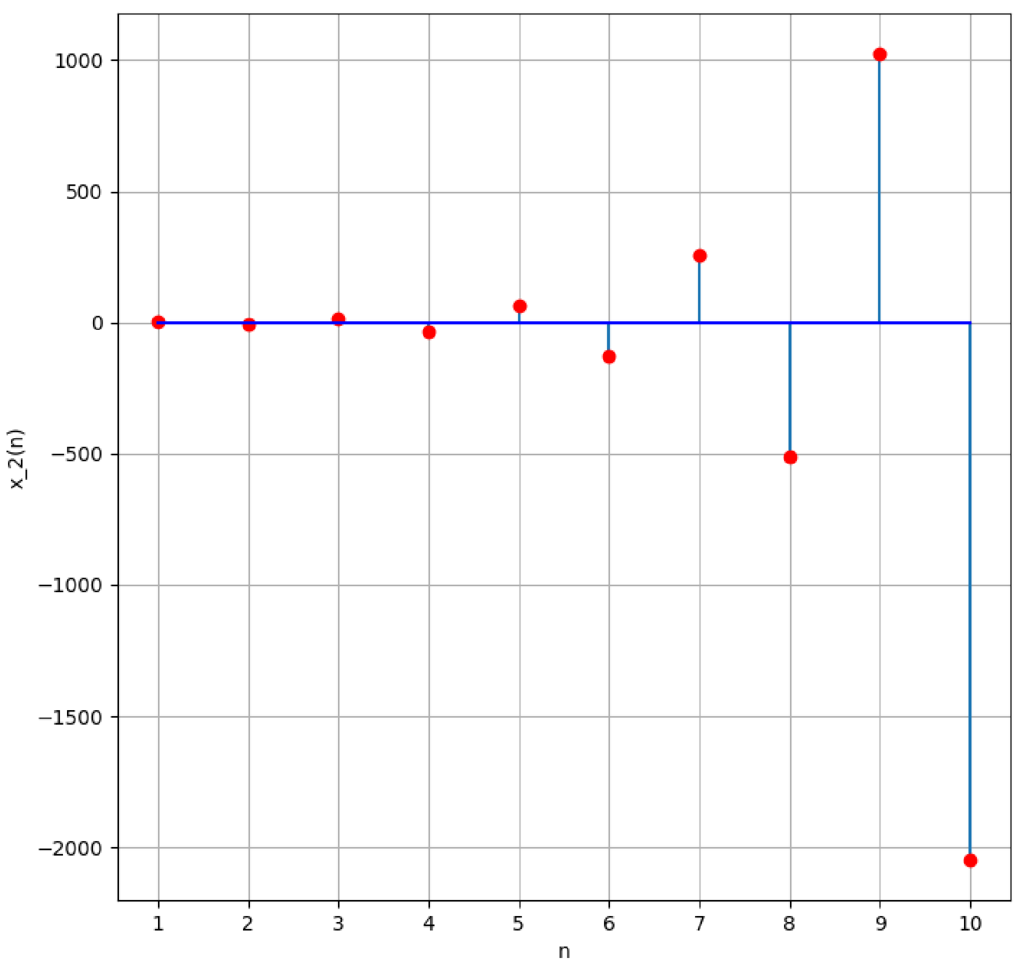



Figure 3.27: Representation of x(n) for $r = -2$

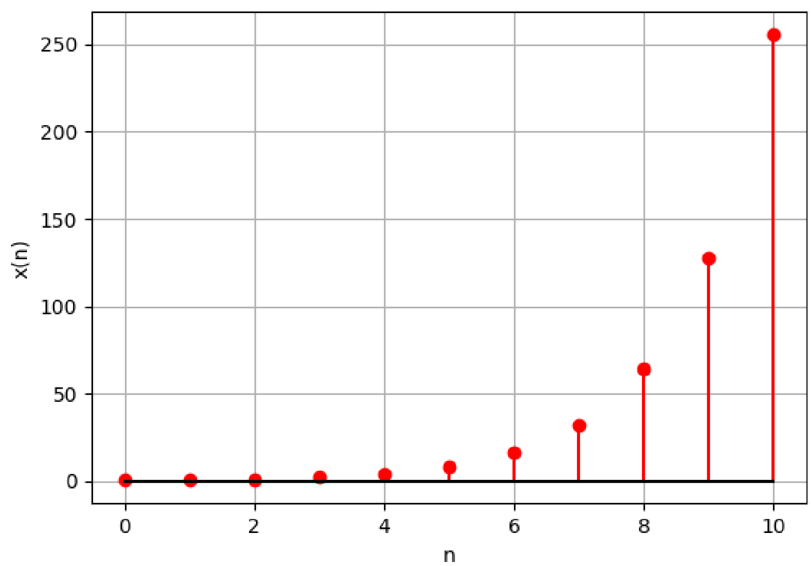



Figure 3.28: Stem Plot of $x(n) = (0.25)2^n u(n)$, $a = 0.25, r = 2$

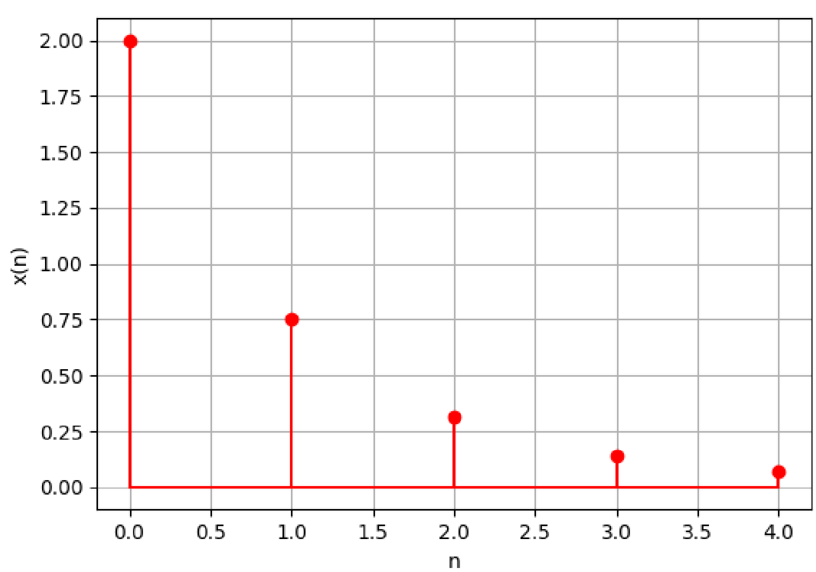



Figure 3.29: Stem Plot of $x(n) = (0.25^n + 0.5^n)\, u(n)$, $a = 0.25, b = 0.5$

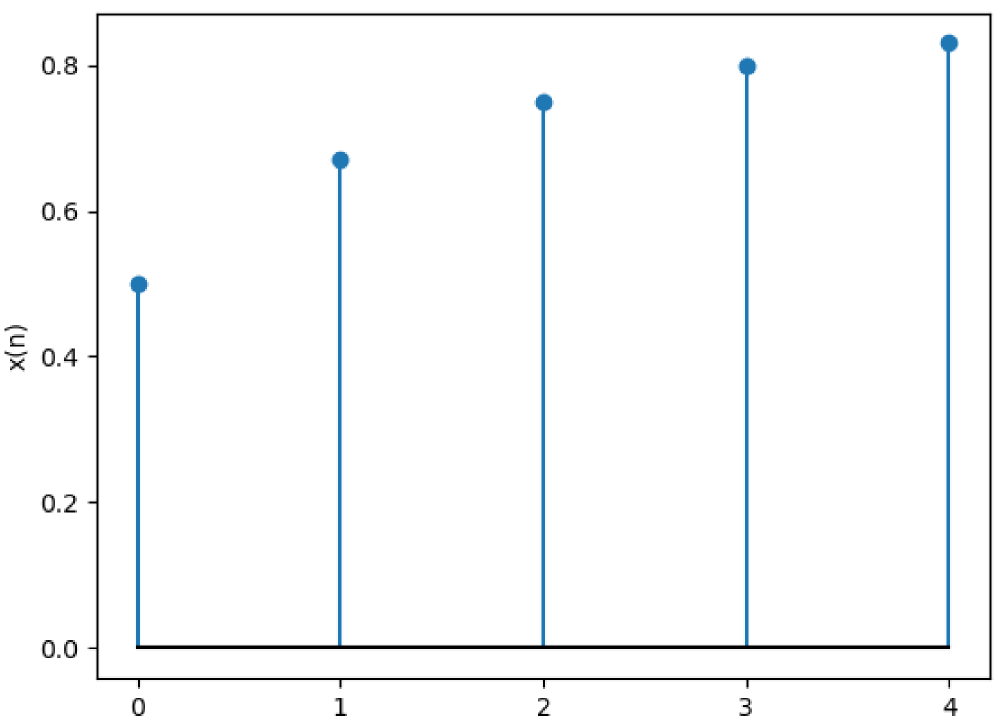



Figure 3.30: Stem plot for x($n$)

# Chapter 4

# Sequences

4.0.1 Find the number of terms in each of the following APs.

(a) 7, 13, 19, ... 205

(b) 18, $15\frac{1}{2}$, 13, ... -47

**Solution:** The number of terms in the AP x($n$) is given by:

| **Parameter** | **Used to denote** | **Values** |
|---|---|---|
| $x_i(n)$ | $n^{th}$ term of $i^{th}$ series $(i = (1, 2))$ | $(x_i(0) + nd_i)\, u(n)$ |
| $x_i(0)$ | First term of $i^{th}$ AP | $x_1(0) = 7$ <br> $x_2(0) = 18$ |
| $d_i$ | Commmon difference of $i^{th}$ AP | $d_1 = 6$ <br> $d_2 = -2.5$ |

Table 4.1: Parameter Table

$$\frac{x(n) - x(0)}{d} + 1 \tag{4.1}$$

$$X_i(z) = \frac{x_i\,(0)}{1 - z^{-1}} + d_i \frac{z^{-1}}{(1 - z^{-1})^2} \text{ , for i=1,2} \tag{4.2}$$

$$\text{ROC : } |z| > 1 \text{ as it is an AP} \tag{4.3}$$

(a)

$$x_1\,(n) = (7 + (n)\,6)\,u\,(n) \tag{4.4}$$

Using the values in Table 4.1 and equation (4.1),

$$k_1 = \frac{205 - 7}{6} + 1 = 34 \tag{4.5}$$

Using the values in Table 4.1 and equation (4.2) :

$$X_1\,(z) = \frac{7 - z^{-1}}{(1 - z^{-1})^2} \tag{4.6}$$

ROC is $|z| > 1$

(b)

$$x_2\,(n) = (18 + n\,(-2.5)\,u\,(n)) \tag{4.7}$$

Using the values in Table 4.1 and equation (4.1),

$$k_2 = \frac{-47 - 18}{-2.5} + 1 = 27 \tag{4.8}$$

Using the values in Table 4.1 and equation (4.2) :

$$X_2(z) = \frac{18 - (20.5)\, z^{-1}}{(1 - z^{-1})^2} \tag{4.9}$$

ROC is $|z| > 1$.

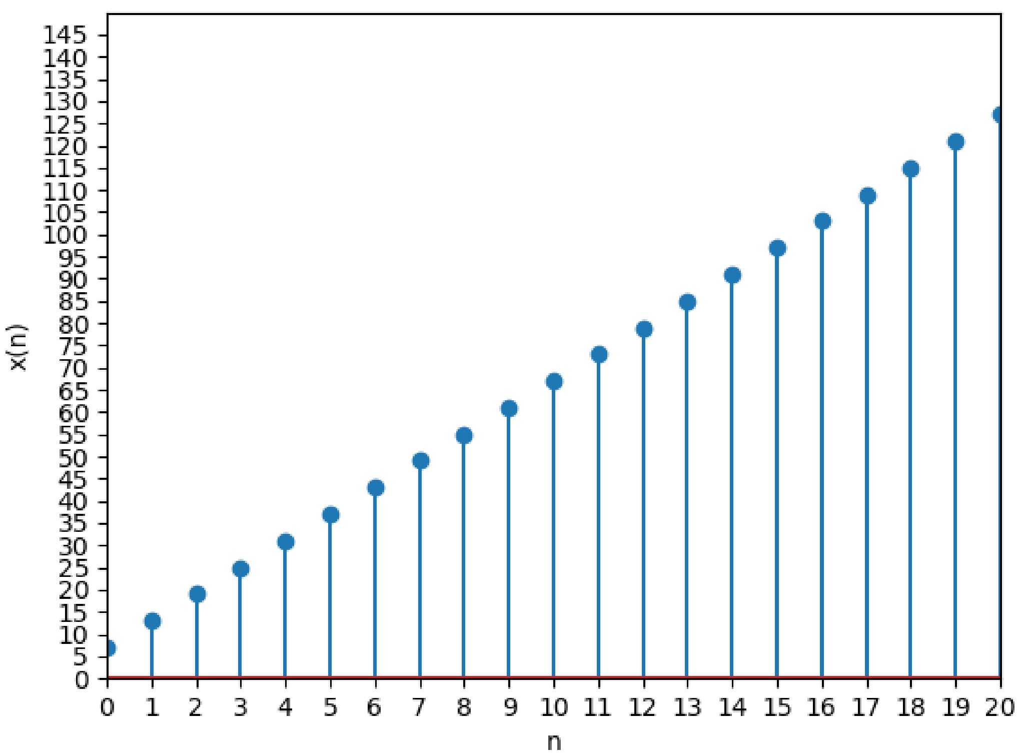


Figure 4.1: Plot of $x_1(n)$

4.0.2 For what value of $n$, are the $nth$ terms of two A.Ps: 63, 65, 67,... and 3, 10, 17,... equal?

**Solution:**

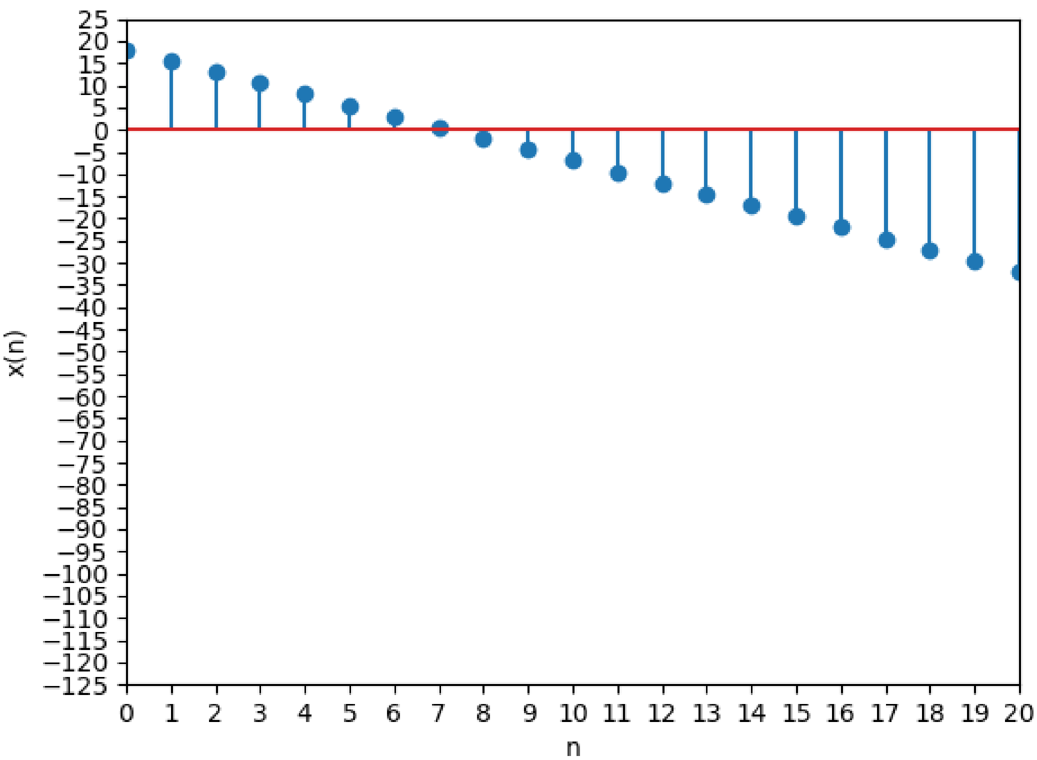



Figure 4.2: Plot of $x_2(n)$

$$x_i(n) = x(0)\,u(n) + dnu(n) \tag{4.10}$$

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{4.11}$$

(a)

$$x_1(n) = 63u(n) + 2nu(n) \tag{4.12}$$

$$X_1(z) = \frac{63}{1-z^{-1}} + \frac{2z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{4.13}$$

| Parameter | Sub-question | Description | Value |
|---|---|---|---|
| $x_i(0)$ | $x_1(0)$ | $1^{st}$ term of $1^{st}$ A.P. | 63 |
| | $x_2(0)$ | $1^{st}$ term of $2^{nd}$ A.P. | 3 |
| $d_i$ | $d_1$ | Common difference of $1^{st}$ A.P. | 2 |
| | $d_2$ | Common difference of $2^{nd}$ A.P. | 7 |

Table 4.2: input values

(b)

$$x_2(n) = 3u(n) + 7nu(n) \tag{4.14}$$

$$X_2(z) = \frac{3}{1-z^{-1}} + \frac{7z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{4.15}$$

(c) given,

$$x_1(n) = x_2(n) \tag{4.16}$$

$$\therefore 63 + 2n = 7n + 3 \tag{4.17}$$

$$\implies n = 12 \tag{4.18}$$

4.0.3 Two APs have the same common difference.The difference between their 100th terms is 100,what is the difference between their 1000th terms?

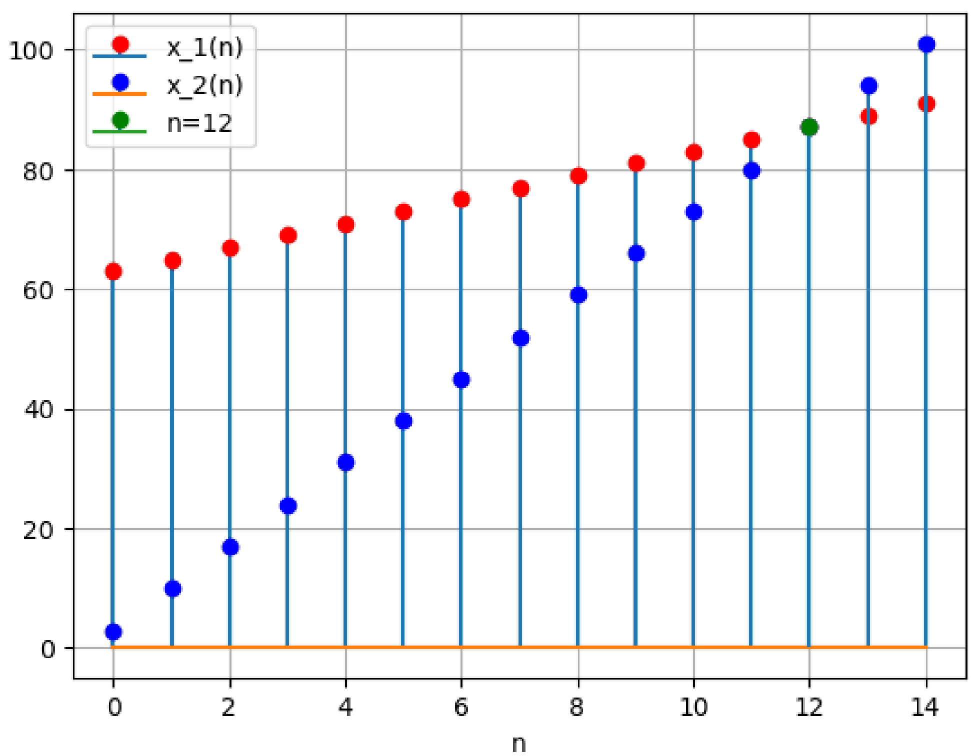



Figure 4.3: Graphs of $x_1(n)$ and $x_2(n)$ and both are equal at $n = 12$

**Solution:**

$$x(n) = \{x(0) + nd\}u(n) \tag{4.19}$$

$$x(99) - y(99) = 100 \tag{4.20}$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \tag{4.21}$$

$$\implies x(0) - y(0) = 100 \tag{4.22}$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \tag{4.23}$$
$$= x(0) - y(0) \tag{4.24}$$
$$= 100 \tag{4.25}$$
$$\implies x(999) - y(999) = 100 \tag{4.26}$$

| **Variable** | **Description** | **Value** |
|---|---|---|
| $x(n)$ | $n^{th}$ term of X | none |
| $y(n)$ | $n^{th}$ term of Y | none |
| $d$ | common difference between the terms of AP | none |
| $x(99) - y(99)$ | difference of $99^{th}$ terms of X and Y | 100 |

Table 4.3: input parameters

Let

$$x(n) = \{101, 106, 111, ...\} \tag{4.27}$$
$$y(n) = \{1, 6, 11, ...\} \tag{4.28}$$

4.0.4 Check whether -150 is a term of the AP: 11,8,5,2,....

**Solution:**

$$x(n) = x(0) + nd \tag{4.29}$$
$$n = \frac{x(n) - x(0)}{d} \tag{4.30}$$

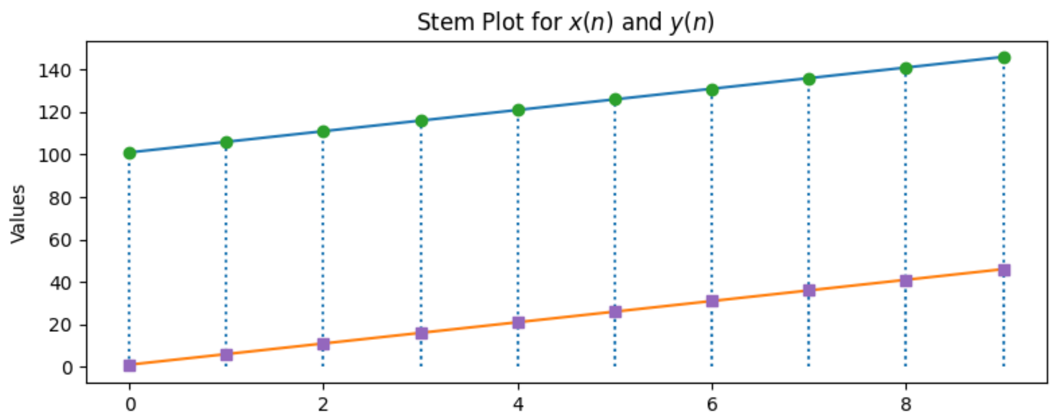



Figure 4.4:

$$x(n) - x(0) \equiv 0 \pmod{d} \tag{4.31}$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \tag{4.32}$$

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \tag{4.33}$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.34}$$

| Variable | Description | Value |
|---|---|---|
| $x(0)$ | First term of AP | 11 |
| $d$ | Common difference | -3 |
| $x(n)$ | General term of given AP | None |

Table 4.4: Input parameters

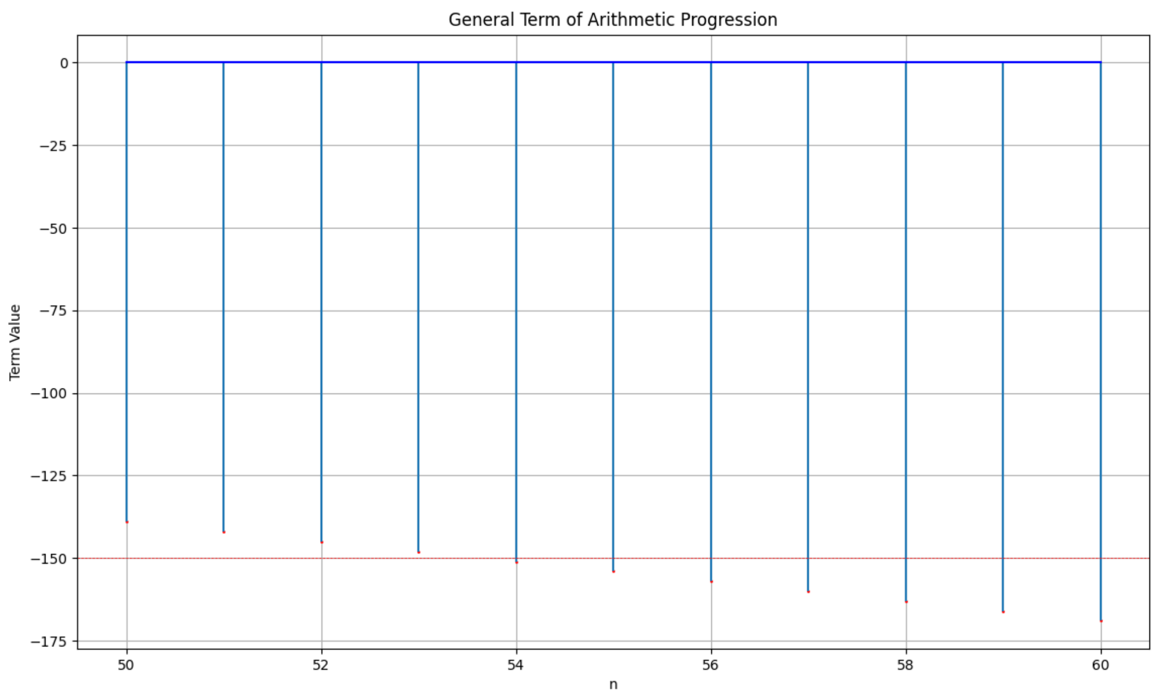



Figure 4.5: Representation of x(n)

4.0.5 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

**Solution:**

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4}\right) u(n) \tag{4.35}$$

$$n^k u(n) \xleftrightarrow{\mathcal{Z}} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{4.36}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{4.37}$$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+z^{-1})}{(1-z^{-1})^3} \quad |z| > 1 \tag{4.38}$$

$$n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \quad |z| > 1 \tag{4.39}$$

Referencing the equations from (4.37), (4.38), and (4.39).

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{4(1-z^{-1})^4} + \frac{3(z^{-1})(1+z^{-1})}{4(1-z^{-1})^3} + \frac{2z^{-1}}{(1-z^{-1})^2} + \frac{3}{2(1-z^{-1})} \quad |z| > 1 \tag{4.40}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{3}{2(1-z^{-1})^3} + \frac{3z^{-2}}{2(1-z^{-1})^4} \quad |z| > 1 \tag{4.41}$$

4.0.6 (a) 30th term of the AP: 10, 7, 4, ... is

(b) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

**Solution:**

| **Parameter** | **value** | **Description** |
|---|---|---|
| $x_i(0)$ | 10 | First term |
| | -3 | |
| $d_i$ | -3 | Common difference |
| | $\frac{5}{2}$ | |
| $x_1(29)$ | ? | 30th term |
| $x_2(10)$ | ? | 11th term |

Table 4.5: Input Parameters

$$x_i(n) = [x_i(0) + nd_i]\, u(n) \tag{4.42}$$

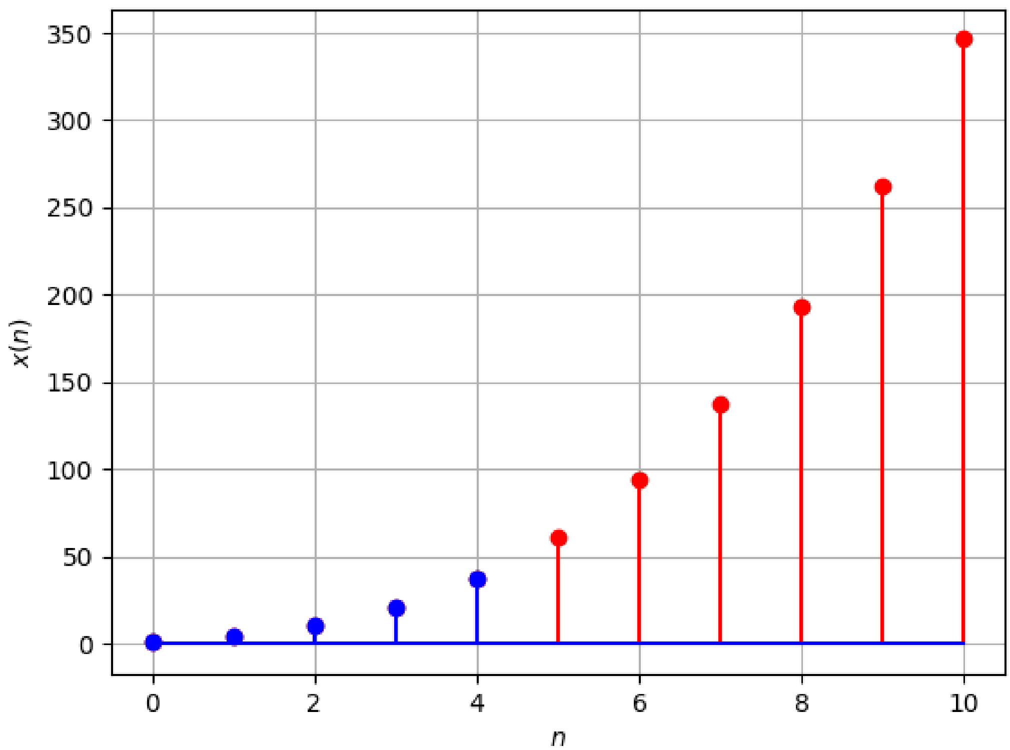



Figure 4.6: Plot of equation(4.35)

(a) From (4.42) Table 4.5 :

$$x_1(n) = [10 - 3n]\, u(n) \tag{4.43}$$

$$x_1(29) = -77 \tag{4.44}$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.45}$$

(b) From (4.42) and Table 4.5 :

$$x_2(n) = \left[-3 + \frac{5}{2}n\right] u(n) \tag{4.46}$$

$$x_2(10) = 22 \tag{4.47}$$

$$X_2(z) = \frac{5.5z^{-1} - 3}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.48}$$

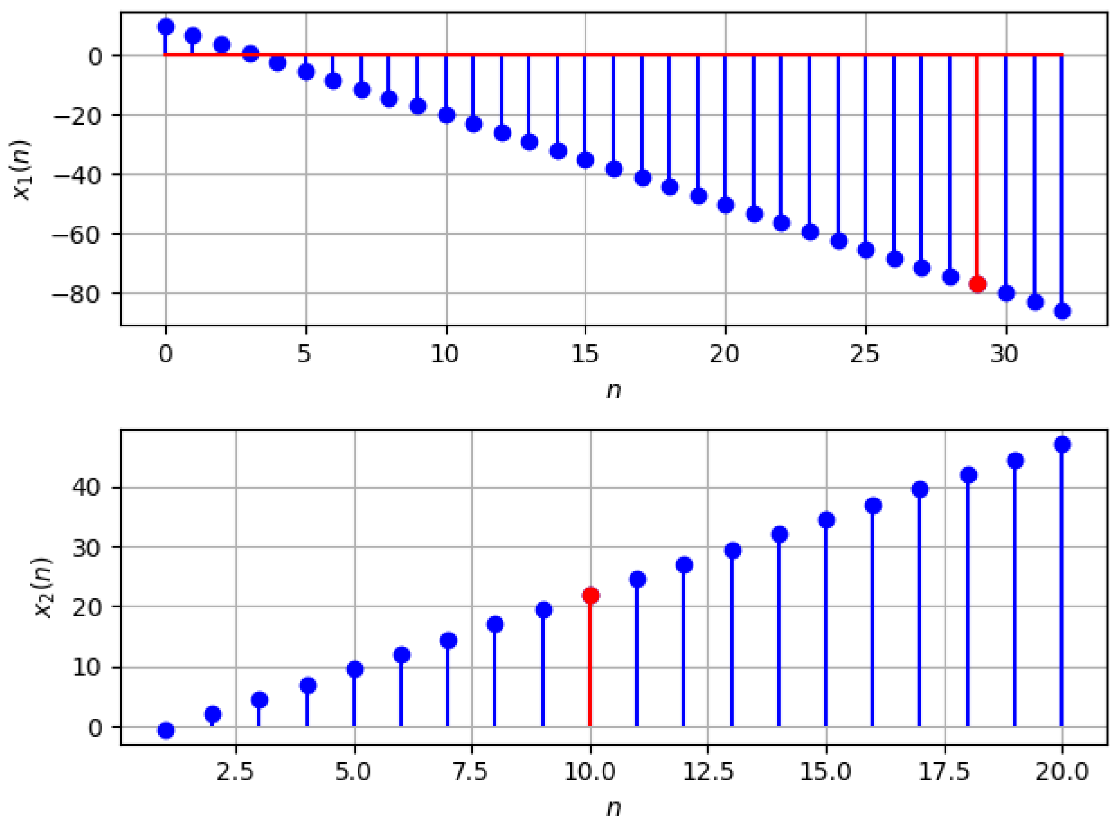



Figure 4.7: stem plots of $x_1(n)$ and $x_2(n)$

4.0.7 Write the first five terms of the sequence whose nth term is $\frac{2n-3}{6}$ and obtain the Z transform of the series **Solution:**

$$x(n) = \frac{2n-1}{6}(u(n)) \tag{4.49}$$

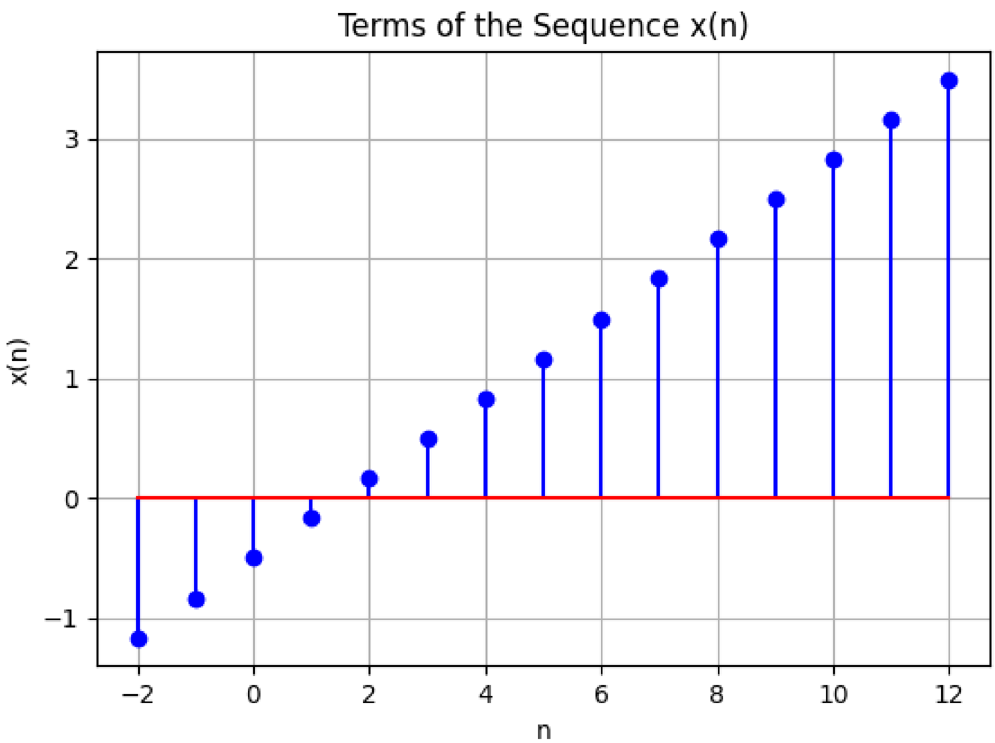



Figure 4.8: Plot of x(n) vs n

$$X(z) = \frac{3z^{-1}-1}{6(1-z^{-1})^2} \quad |z| > 1 \tag{4.50}$$

4.0.8 For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P ?

**Solution:** Let $r$ be the common ratio

| **Variable** | **Description** | **Value** |
|---|---|---|
| $x(0)$ | First term of the GP | $-\left(\frac{2}{7}\right)$ |
| $x(1)$ | Second term of the GP | $x$ |
| $x(2)$ | Third term of the GP | $-\left(\frac{7}{2}\right)$ |
| $r$ | Common ratio of the GP | |
| $x(n)$ | General term | $x(0)\,r^n\,u(n)$ |

Table 4.6: Variables Used

From Table 4.6:

$$\implies \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \tag{4.51}$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \tag{4.52}$$

$$x = \pm 1 \tag{4.53}$$

$$\implies r = \pm\frac{7}{2} \tag{4.54}$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right)\left(\pm\frac{7}{2}\right)^n u(n) \tag{4.55}$$

Applying z-Transform :

$$\implies X_1(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}+2}\right) \quad |z| > \frac{7}{2} \tag{4.56}$$

$$\implies X_2(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}-2}\right) \quad |z| > \frac{7}{2} \tag{4.57}$$

4.0.9 Find the $20^{th}$ and $n^{th}$ terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

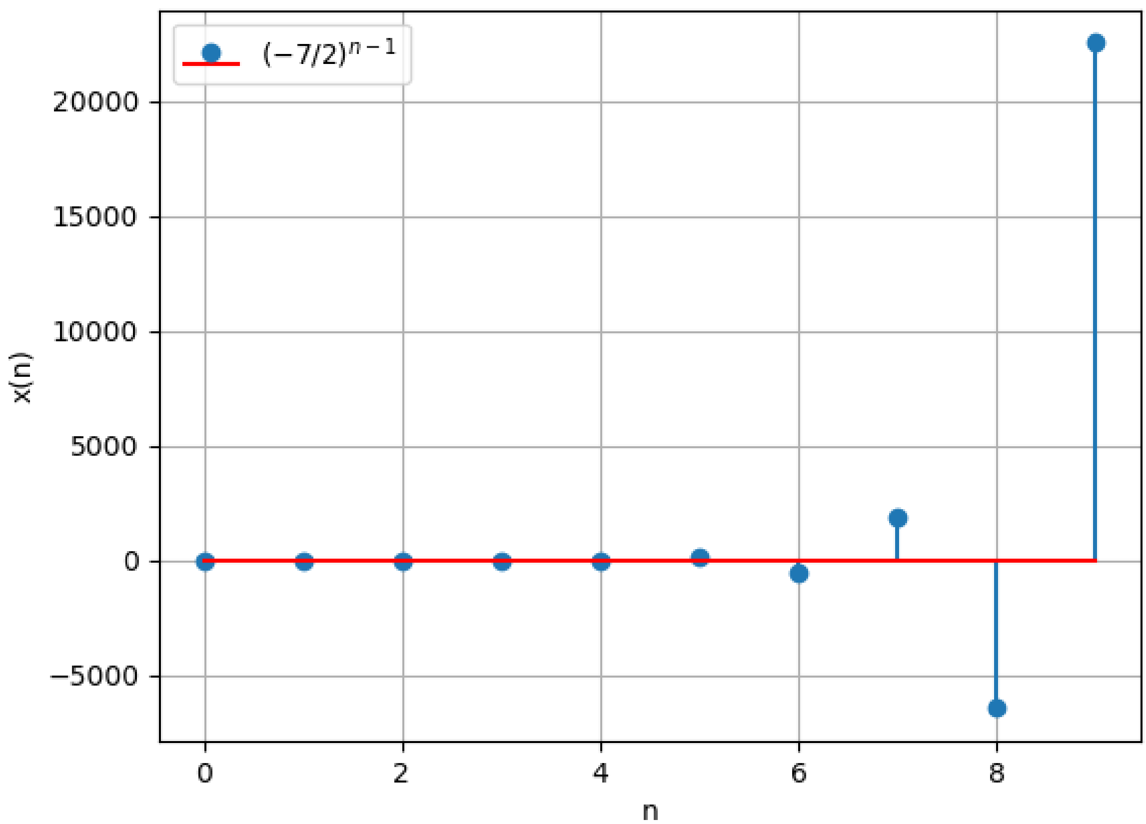



Figure 4.9: Stem Plot of $x_1$(n)

**Solution:**

From Table 4.7: $Z$-Transform of $x(n)$:

$$\implies X(z) = \frac{5}{2}\left(\frac{1}{1-\frac{z^{-1}}{2}}\right); \left\{z \in \mathbb{C} : |z| > \frac{1}{2}\right\} \tag{4.58}$$

4.0.10 Which term of the following sequences:

(a) $2,2\sqrt{2},4\ldots$ is 128 (b) $\sqrt{3},3,3\sqrt{3}\ldots$ is 729

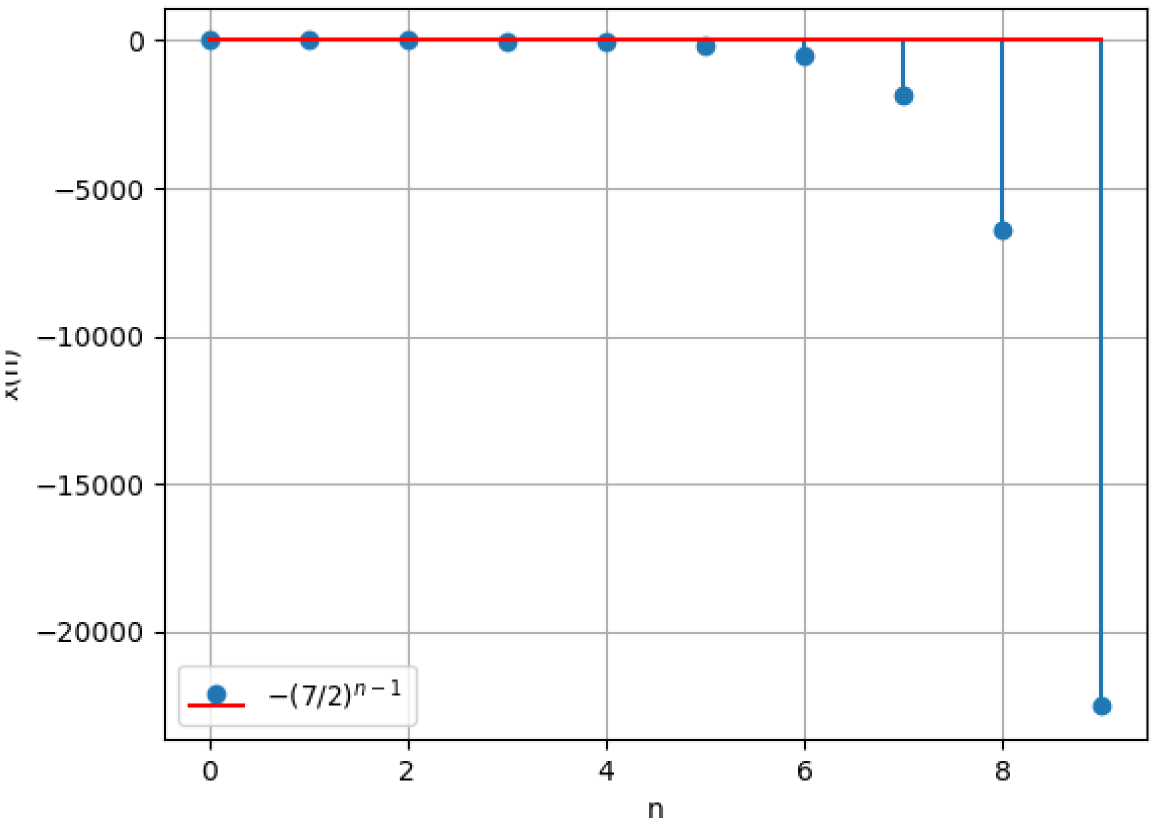



Figure 4.10: Stem Plot of $x_2(n)$

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \dots$ is $\frac{1}{19683}$

**Solution:** For a general GP series and $k > 0$,

$$x(k) = x(0)\, r^k \tag{4.59}$$

$$\therefore k = \log_r \frac{x(k)}{x(0)} \tag{4.60}$$

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $x(0)$ | First Term | $\frac{5}{2}$ |
| $r = \frac{x(1)}{x(0)}$ | Common Ratio | $\frac{1}{2}$ |
| $x(n)$ | $n^{th}$ Term | $\frac{5}{2}\left(\frac{1}{2}\right)^n \cdot u(n)$ |
| $x(19)$ | $20^{th}$ Term | $\frac{5}{2}\left(\frac{1}{2}\right)^{19}$ |
| $u(n)$ | Unit step function | |

Table 4.7: Parameters

And the Z-transform $X(z)$:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \tag{4.61}$$

(a) By Table 4.8, (4.60) and Table 4.8:

$$x_1(n) = x_1(0)\, r_1^n u(n) \tag{4.62}$$

$$k_1 = \log_{r_1} \frac{128}{x_1(0)} \tag{4.63}$$

$$\therefore k_1 = 12 \tag{4.64}$$

$$X_1(z) = \frac{2}{1 - \sqrt{2}z^{-1}} \quad |z| > \sqrt{2} \tag{4.65}$$

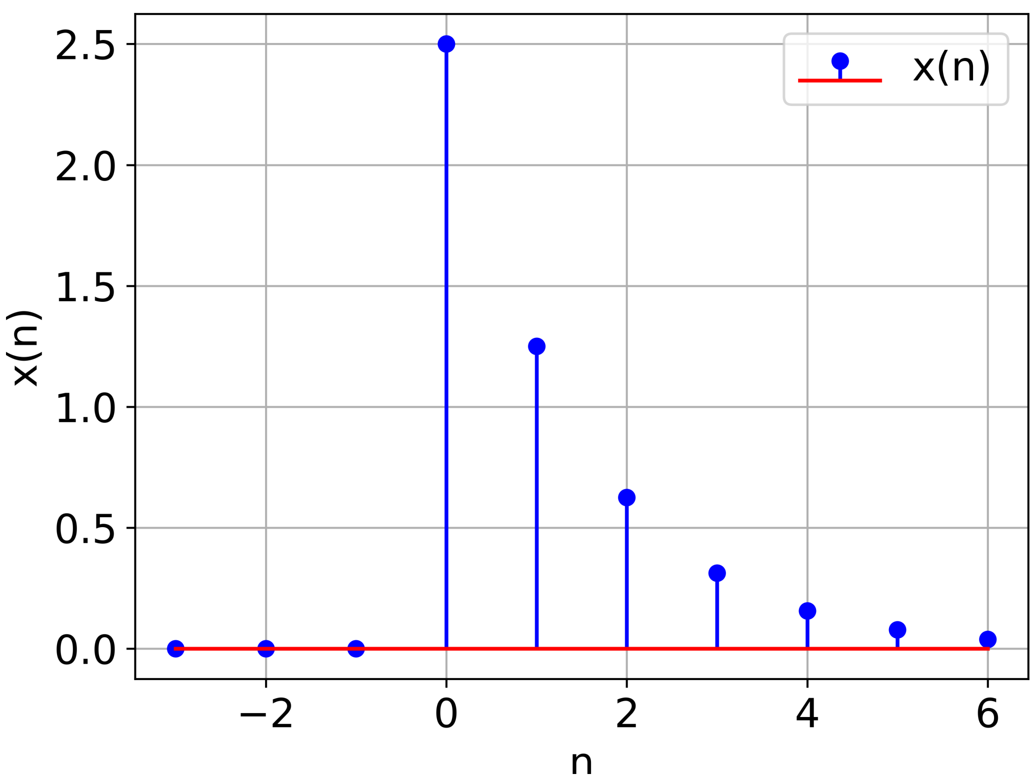



Figure 4.11:

(b) By (4.60), (4.61) and Table 4.8:

$$x_2(n) = x_2(0)\, r_2^n u(n) \tag{4.66}$$

$$k_2 = \log_{r_2} \frac{729}{x_2(0)} \tag{4.67}$$

$$\therefore k_2 = 11 \tag{4.68}$$

$$X_2(z) = \frac{\sqrt{3}}{1 - \sqrt{3} z^{-1}} \quad |z| > \sqrt{3} \tag{4.69}$$

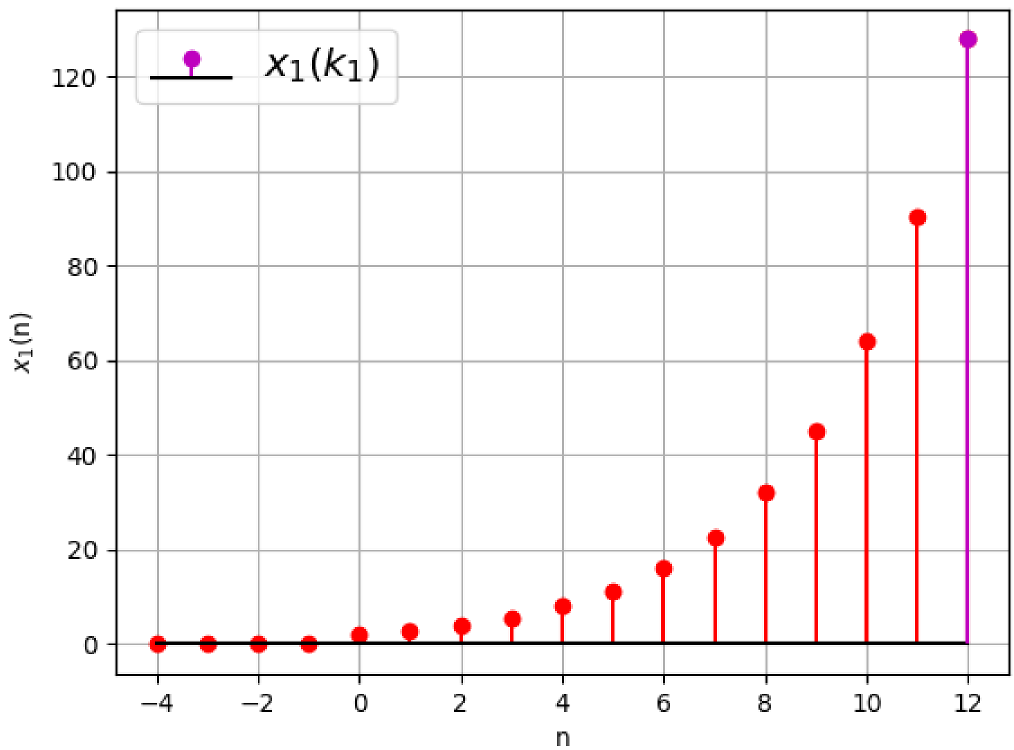



Figure 1: Plot of $x_1(n)$ vs n. See Table 4.8

(c) By (4.60), (4.61) and Table 4.8:

$$x_3(n) = x_3(0)\, r_3^n u(n) \tag{4.70}$$

$$k_3 = \log_{r_3} \frac{1}{19683 x_3(0)} \tag{4.71}$$

$$\therefore k_3 = 8 \tag{4.72}$$

$$X_3(z) = \frac{1}{3 - z^{-1}} \quad |z| > \frac{1}{3} \tag{4.73}$$

Find the $20^{th}$ and $n^{th}$ terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

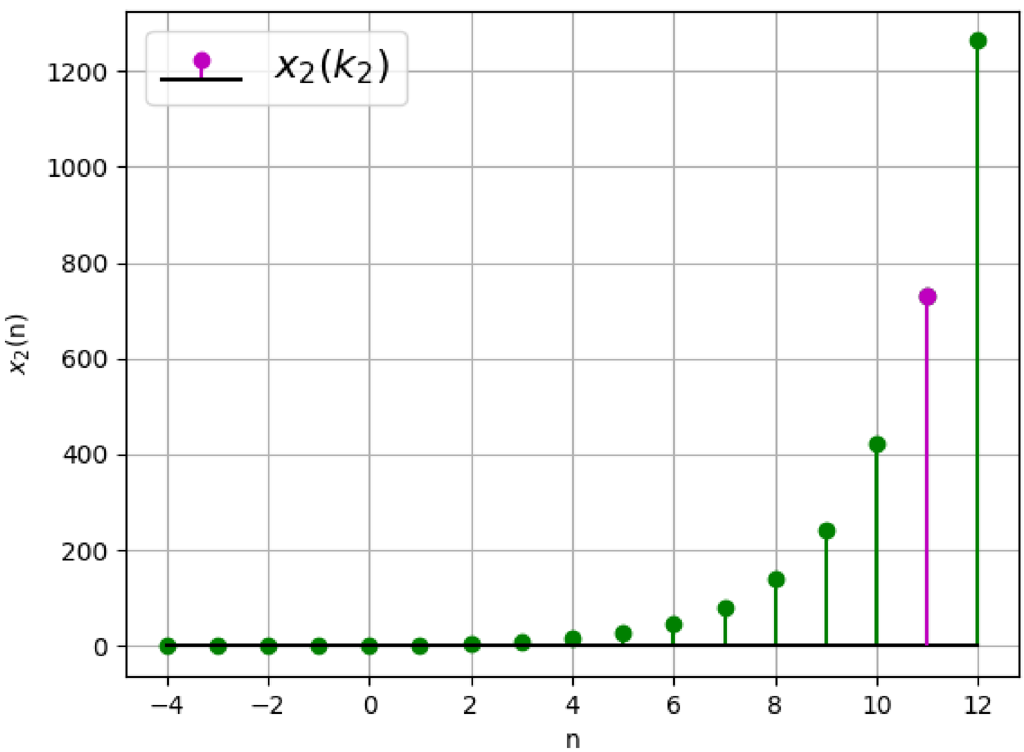



Figure 2: Plot of $x_2(n)$ vs n. See Table 4.8

| Parameter | Description | Value |
|---|---|---|
| $r_i$ | Common ratio of G.P (a),(b),(c) | $\sqrt{2}, \sqrt{3}, \frac{1}{3}$ |
| $x_i(0)$ | Initial Values | $2, \sqrt{3}, \frac{1}{3}$ |
| $x_i(k_i)$ | Given Values | $128, 729, \frac{1}{19683}$ |
| $k_i$ | Desired index | $12, 11, 8$ |
| $x_i\left(n\right)$ | Series | $x_i\left(0\right) r_i^n u\left(n\right)$ |
| $X_i\left(z\right)$ | Z-Transform of $x_i\left(n\right)$ | $\frac{x(0)}{1-rz^{-1}}$ |

Table 4.8: Table of parameters

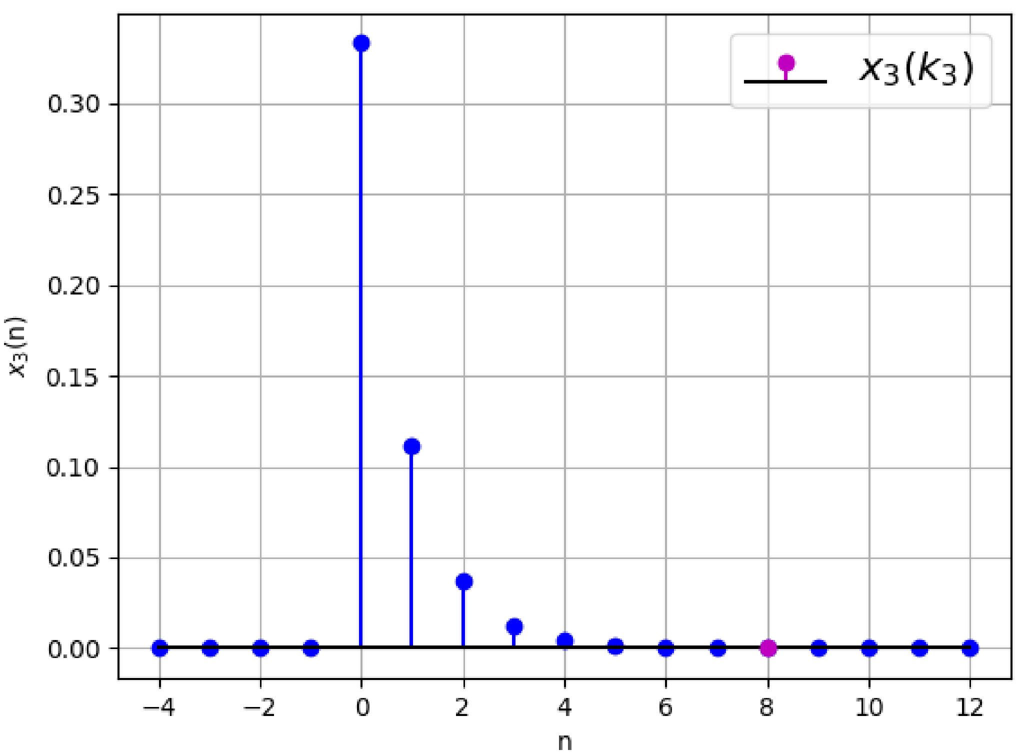



Figure 3: Plot of $x_3(n)$ vs n. See Table 4.8

4.0.11 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of $2^{nd}$ hour, $4^{th}$ hour and $n^{th}$ hour?

**Solution:** From Table 4.9:

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 30 | Initial no. of bacteria |
| $r$ | 2 | Ratio of no. of bacteria at end of hour to start of hour (Common Ratio) |
| $x(n)$ | $r^n x(0)u(n)$ | $n^{th}$ term of the GP |

Table 4.9: Input Parameters

$$x(2) = 120 \tag{4.74}$$

$$x(4) = 480 \tag{4.75}$$

$$x(n) = 30(2^n)u(n) \tag{4.76}$$

$$X(z) = \frac{30z^{-1}}{1-2z^{-1}} \quad |z| > 2 \tag{4.77}$$

4.0.12 Ramkali saved Rs 5 in the first week of a year and then increased her weekly savings by Rs 1.75. If in the $n$th week, her weekly savings become Rs 20.75, find $n$.

**Solution:**

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 5 | First term of AP |
| $d$ | 1.75 | Common difference of AP |
| $x(n)$ | 20.75 | $n^{th}$ term of AP |

Table 4.10: Parameter List

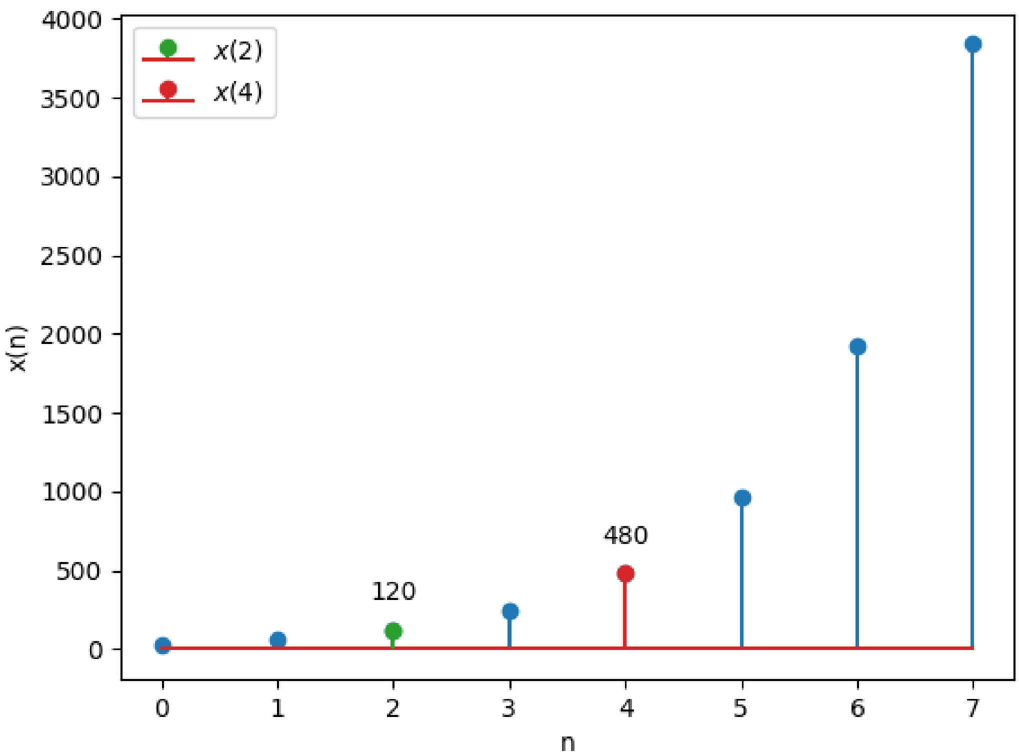



Figure 4.15: Plot of $x(n)$ vs $n$. See Table 4.9 for details.

$$x(n) = x(0) + (n)(d) \tag{4.78}$$

$$20.75 = 5 + (n)(1.75) \tag{4.79}$$

$$\implies 15.75 = (n)(1.75) \tag{4.80}$$

$$\implies n = \frac{15.75}{1.75} \tag{4.81}$$

$$\implies n = 9 \tag{4.82}$$

$$x(n) = 5u(n) + 1.75nu(n) \tag{4.83}$$

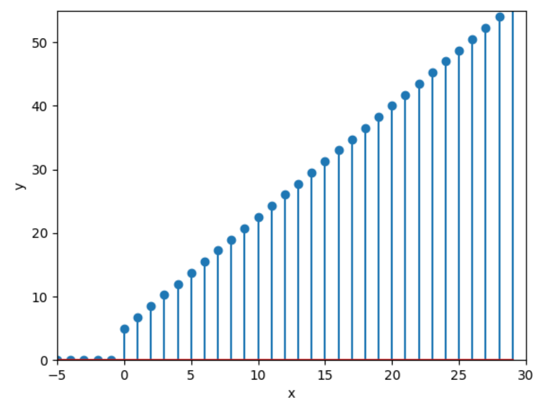



Figure 4.16: Plot of $x(n) = 5 + 1.75n$

The Z-transform of a sequence $x(n)$ is given by:

$$X(z) = \frac{5z^{-1}}{1 - z^{-1}} + \frac{1.75z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{4.84}$$

4.0.13 Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an $A.P.$, is equal to twice the $m^{th}$ terms.

**Solution:**

For an $AP$,

$$x\left(n\right) = [x\left(0\right) + nd]u\left(n\right) \tag{4.85}$$

$$\implies x\left(m+n\right) + x\left(m-n\right) = [x\left(0\right) + \left(m+n\right)d] + [x\left(0\right) + \left(m-n\right)d] \tag{4.86}$$

$$= 2[x\left(0\right) + md] \tag{4.87}$$

$$\therefore x\left(m+n\right) + x\left(m-n\right) = 2x\left(m\right) \tag{4.88}$$

| PARAMETER | VALUE | DESCRIPTION |
|---|---|---|
| $x(0)$ | $x(0)$ | First term |
| $d$ | $d$ | common difference |
| $x(n)$ | $[x(0)+nd]u(n)$ | General term of the series |

Table 4.11: Parameter Table1

4.0.14 The sum of the first three terms of a G.P is 39/10 and their product is 1. Find the common ratio and the terms.

**Solution:**

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{4.89}$$

From Table 4.13 and (4.89) :

$$y(2) = x(0)\left(\frac{r^3-1}{r-1}\right) \tag{4.90}$$

$$\frac{39}{10} = x(0)\left(r^2+r+1\right) \tag{4.91}$$

$$\implies \frac{39r}{10} = r^2+r+1 \quad (\because x(0)r = 1) \tag{4.92}$$

$$\implies (2r-5)(5r-2) = 0 \tag{4.93}$$

$$\implies r = \frac{2}{5} \quad or \quad \frac{5}{2} \tag{4.94}$$

(a) If $r = \frac{2}{5}$, then terms are $\frac{5}{2}$, 1, $\frac{2}{5}$.

(b) If $r = \frac{5}{2}$, then terms are $\frac{2}{5}$, 1, $\frac{5}{2}$.

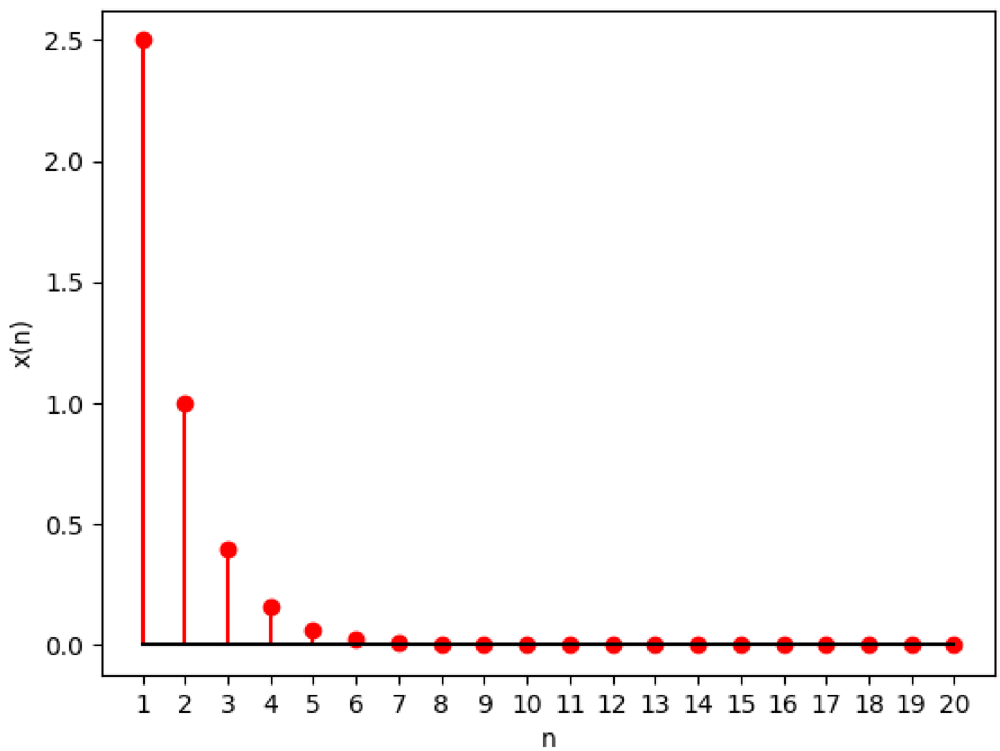



Figure 4.17: stem plots of GP if $r = \frac{2}{5}$

4.0.15 The ratio of the A.M and G.M of two positive numbers $a$ and $b$ is $m : n$. Show that $a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$.

**Solution:** Expressing A.M and G.M in terms of $a$ and $b$:

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n} \tag{4.95}$$

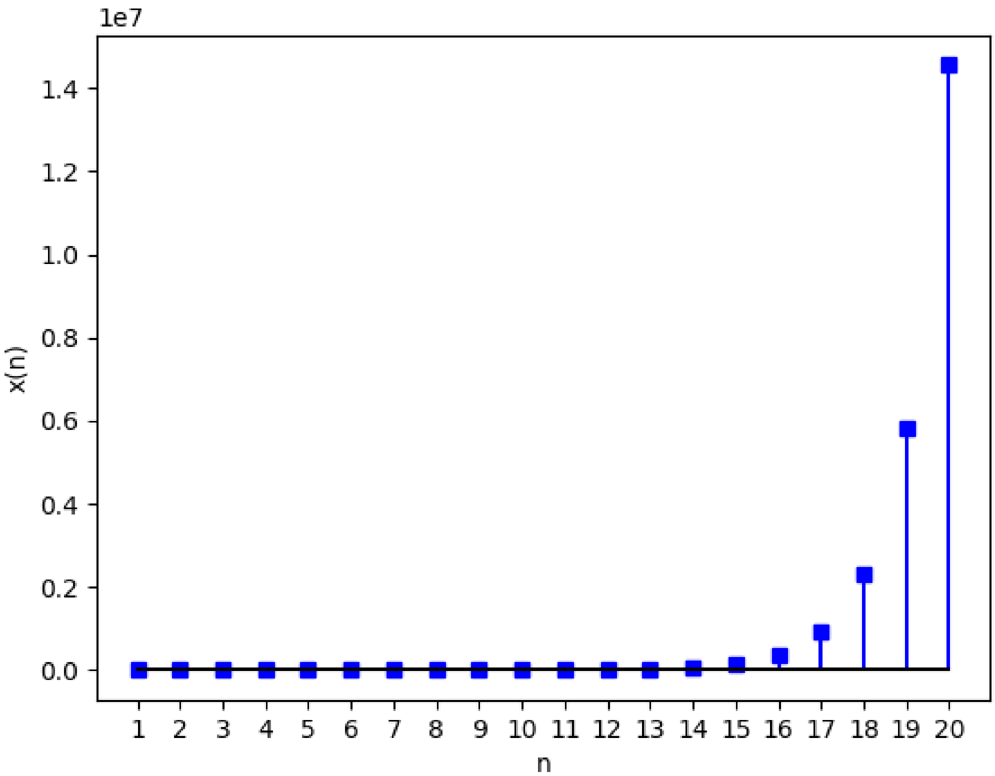



Figure 4.18: stem plots of GP if $r = \frac{5}{2}$

Let's assume that $x = \sqrt{\frac{a}{b}}$. Then, we have:

$$\frac{a}{b} = x^2 \tag{4.96}$$

Substituting the value of x in equation (4.95):

$$\frac{1+x^2}{2x} = \frac{m}{n} \tag{4.97}$$

$$\frac{1}{x} + x = \frac{2m}{n} \tag{4.98}$$

$$x^2 - \frac{2m}{n}x + 1 = 0 \tag{4.99}$$

$$\implies x = \frac{m}{n} \pm \frac{\sqrt{m^2 - n^2}}{n} \tag{4.100}$$

Since $x = \sqrt{\frac{a}{b}}$, $x$ must be positive.

$$x = \frac{m + \sqrt{m^2 - n^2}}{n} \tag{4.101}$$

Referencing the value of $x$ from equation(4.96).

$$\frac{a}{b} = \left(\frac{m + \sqrt{m^2 - n^2}}{n}\right)^2 \tag{4.102}$$

Multiplying both the numerator and denominator with $\left(m - \sqrt{m^2 - n^2}\right)$:

$$\frac{a}{b} = \frac{1}{n^2} \frac{\left(m + \sqrt{m^2 - n^2}\right)^2 \left(m - \sqrt{m^2 - n^2}\right)}{\left(m - \sqrt{m^2 - n^2}\right)} \tag{4.103}$$

$$\implies a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right) \tag{4.104}$$

nth term of the AP :

$$y(n) = [a + n(b - a)]\, u(n) \tag{4.105}$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{4.106}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1 - z^{-1})} \quad |z| > |1| \tag{4.107}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \tag{4.108}$$

Referencing the equations from (4.107),(4.108).

$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{a}{(1 - z^{-1})} + \frac{(b - a)\, z^{-1}}{(1 - z^{-1})^2} \quad |z| > |1| \tag{4.109}$$

nth term of the GP :

$$y(n) = a \left(\frac{b}{a}\right)^n u(n) \tag{4.110}$$

$$r^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1 - rz^{-1})} \quad |z| > |r| \tag{4.111}$$

Referencing the equation from (4.111).

$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{a^2 z^{-1}}{(a - bz^{-1})} \quad |z| > \left|\frac{b}{a}\right| \tag{4.112}$$

4.0.16 The sum of three numbers in an arithmetic progression (AP) is 24 and the product of those three numbers is 440, find the values of the three numbers.

**Solution:** The following information is provided in the question:

Let the three numbers in the arithmetic progression be denoted as $x(0)$, $x(1)$, and $x(2)$.

From Table 4.14

$$x(0) + x(1) + x(2) = 24 \tag{4.113}$$

$$(x(1) - d) + x(1) + (x(1) + d) = 24 \tag{4.114}$$

$$3x(1) = 24 \tag{4.115}$$

$$\implies x(1) = 8 \tag{4.116}$$

$$x(0) \cdot x(1) \cdot x(2) = 440 \tag{4.117}$$

$$(8 - d) \cdot (8) \cdot (8 + d) = 440 \tag{4.118}$$

$$(8 - d) \cdot (8 + d) = 55 \tag{4.119}$$

$$64 - d^2 = 55 \tag{4.120}$$

$$\implies d = 3 \tag{4.121}$$

$$\implies x(0) = 5 \tag{4.122}$$

$$x(n) = (5 + 3n)\, u(n) \tag{4.123}$$

From equation (B.3.2):

$$X(z) = \frac{5 - 8z^{-1}}{(1 - z^{-1})^2}; \quad |z| > |1| \tag{4.124}$$

Therefore, the required three numbers in AP are 5, 8, and 11.

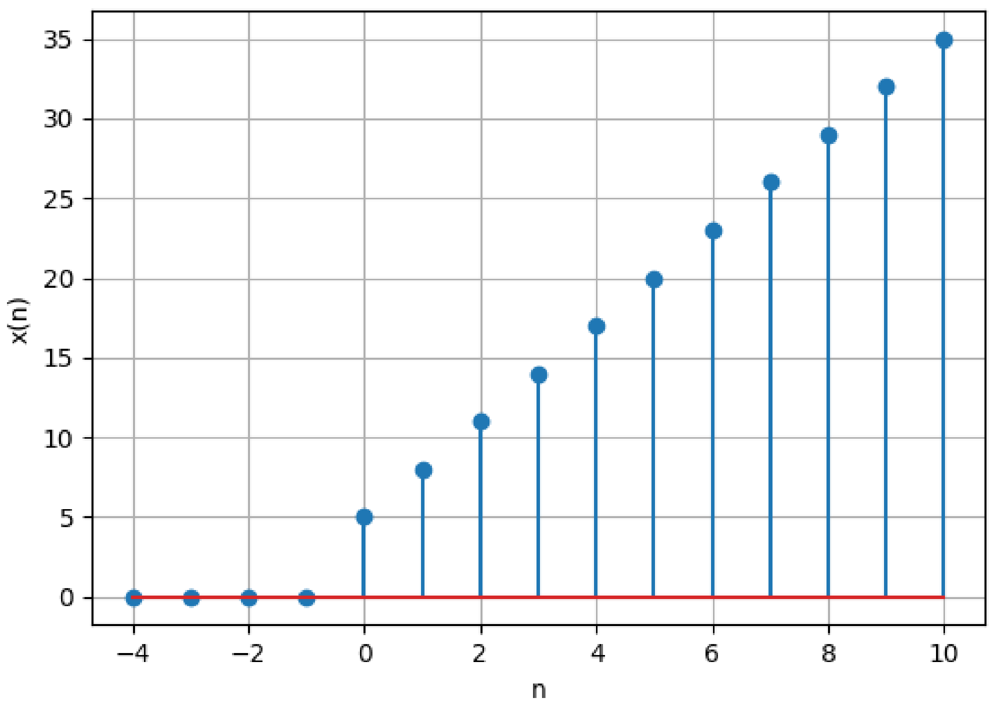



Figure 4.19: stem plots of x(n)

4.0.17 The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2 , respectively. Find the last term and the number of terms.

**Solution:**

$$x(n) = x(0)r^n u(n) \tag{4.125}$$

From (B.5.4)

$$X(z) = \frac{5}{1-2z^{-1}} \quad |z| > |2| \tag{4.126}$$

By contour integration:

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{4.127}$$

$$315 = 5\left(2^{n+1}-1\right) \tag{4.128}$$

$$\implies n = 5 \tag{4.129}$$

The number of terms is $n + 1 = 6$

From (4.125):

$$x(5) = 5\left(2^{5}\right) \tag{4.130}$$

$$= 160 \tag{4.131}$$

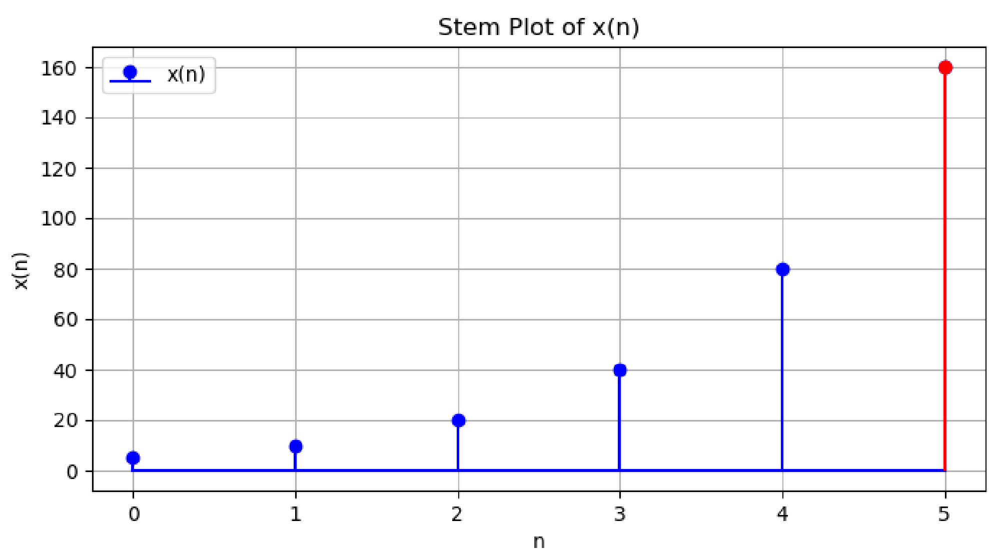



Figure 4.20: Stem plot of x(n)

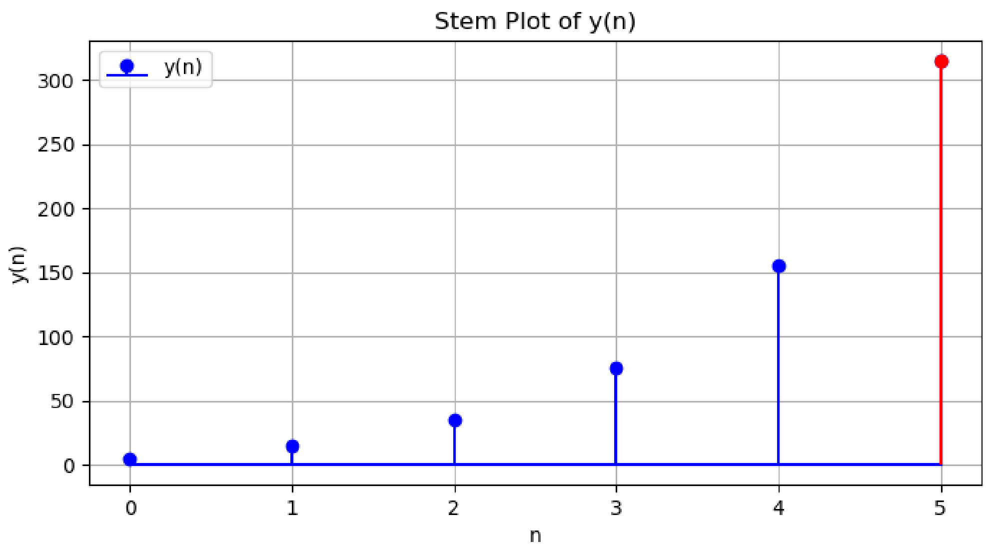



Figure 4.21: Stem plot of y(n)

4.0.18 Find the sum of n terms of the A.P. whose kth term is $5k+1$.

**Solution:**

Apply the Z-transform to $x(n)$:

$$X(z) = \frac{5z^{-1}}{\left(1-z^{-1}\right)^2} + \frac{1}{\left(1-z^{-1}\right)} \quad |z| > 1 \tag{4.132}$$

Sum of First $n$ Terms:

$$y(n) = x(n) * u(n) \tag{4.133}$$

Applying Z transform on both sides:

$$Y(z) = X(z)\,U(z) \tag{4.134}$$

$$= \frac{1}{\left(1-z^{-1}\right)^2} + \frac{5}{2} \cdot \frac{2z^{-1}}{\left(1-z^{-1}\right)^3} \tag{4.135}$$

Now we can compare the above pairs as;

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \tag{4.136}$$

$$u(n) \overset{Z}{\longleftrightarrow} \frac{1}{\left(1-z^{-1}\right)} \tag{4.137}$$

$$n(n-1)u(n) \stackrel{Z}{\longleftrightarrow} \frac{2z^{-1}}{(1-z^{-1})^3} \tag{4.138}$$

On referring the above equations and comparing, we can obtain the Z transform inverse as follows:

$$y(n) = (n+1)u(n) + \frac{5}{2}n(n-1)u(n) \tag{4.139}$$

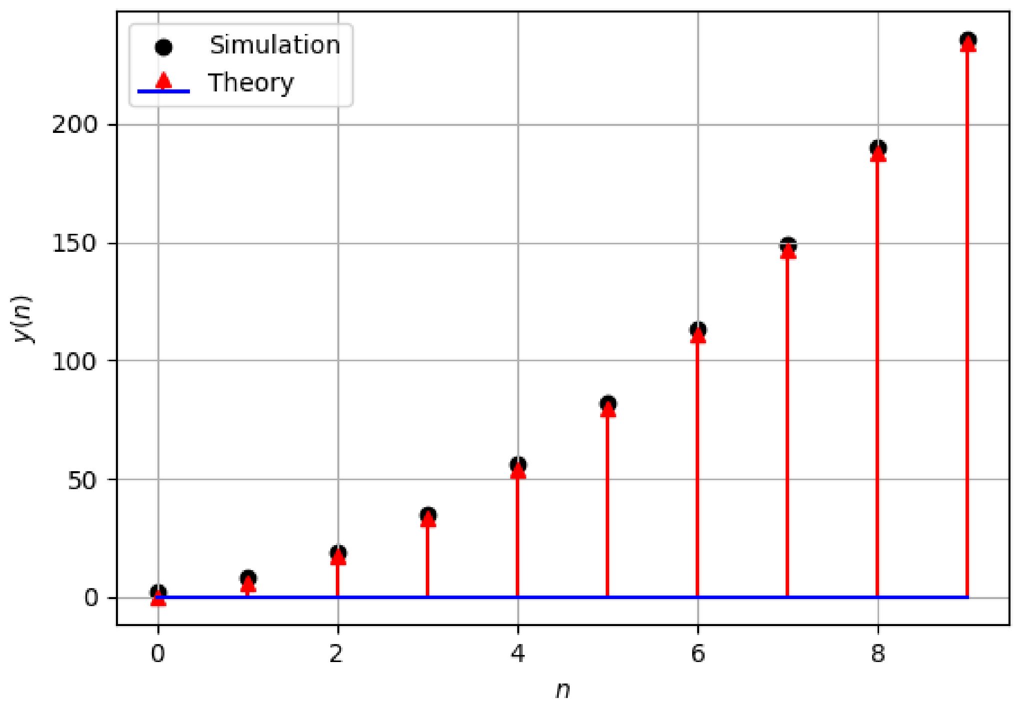
$$= \left(n+1+\frac{5}{2}n(n-1)\right)u(n) \tag{4.140}$$

Since we are taking n starting from 0 we replace n with n+1 to make our simulation match with the theory

Therefore, we have got the sum of n terms as:

$$y(n) = \left(n+2+\frac{5}{2}n(n+1)\right)u(n+1) \tag{4.141}$$

The stem plot is given as




4.0.19 How many 3 digit numbers are divisible by 7?

**Solution:**

4.0.20 A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

**Solution:**

4.0.21 If $a$, $b$, $c$ are in A.P.; $b$, $c$, $d$ are in G.P and $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$ are in A.P. prove that $a$, $c$, $e$ are in G.P.

**Solution:**

4.0.22 Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

**Solution:**

From Table 4.17

$$x(0) + 10d = 38 \tag{4.142}$$

$$x(0) + 15d = 73 \tag{4.143}$$

From equations 4.142 and 4.143, the augmented matrix is:

$$\begin{pmatrix} 1 & 10 & 38 \\ 1 & 15 & 73 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 10 & 38 \\ 0 & 5 & 35 \end{pmatrix} \tag{4.144}$$

$$\xleftrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 5 & 35 \end{pmatrix} \tag{4.145}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 1 & 7 \end{pmatrix} \tag{4.146}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} -32 \\ 7 \end{pmatrix} \tag{4.147}$$

The general term and the Z-transform are given by

$$x(n) = (-32 + 7n)\, u(n) \tag{4.148}$$

$$\tag{4.149}$$

The 31st term of this A.P. is

$$x(30) = 178 \tag{4.150}$$

From (B.10.6), the Z-Transform of $x(n)$ is given by

$$X(z) = \frac{-32}{1 - z^{-1}} + \frac{7z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{4.151}$$

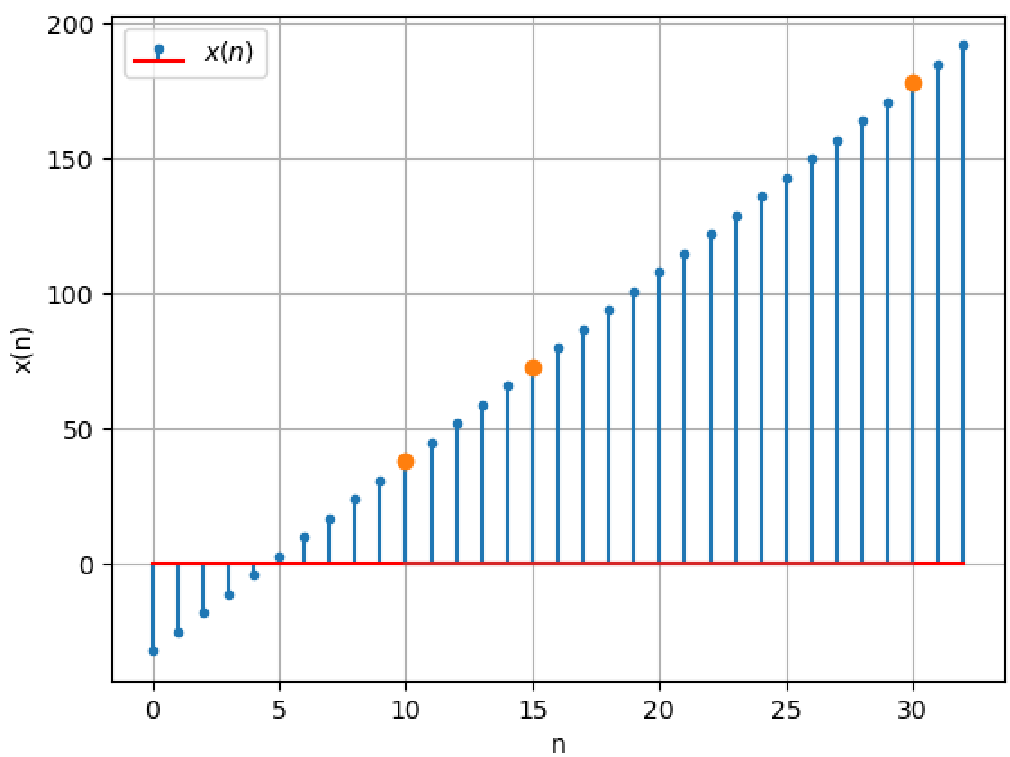



Figure 4.22: Stem plot of $x(0)$ v/s $n$

4.0.23 If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in arithmetic progression (AP), prove that $a$, $b$, $c$ are also in AP.

**Solution:** Common difference can be written as:

$$b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right) \tag{4.152}$$

$$\implies (b-a)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=(c-b)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{4.153}$$

$$\implies b-a=c-b \tag{4.154}$$

Hence proved that $a$, $b$, $c$ are in AP.

From table Table 4.18

$$X(z)=x(0)\left(\frac{1}{1-z^{-1}}\right)+d\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) \tag{4.155}$$

$$=a\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1-z^{-1}}\right)+(b-a)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) \quad |z|>1 \tag{4.156}$$

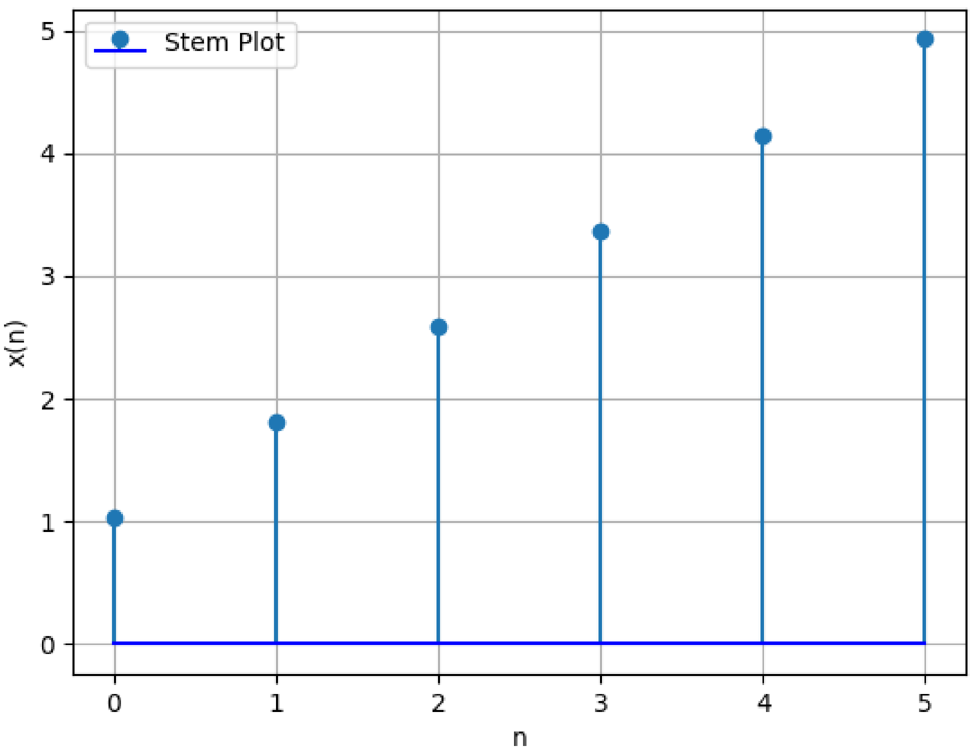



Figure 4.23: graph with value of $a = 3, b = 5, c = 7$

4.0.24 If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is A.M between a and b, then find value of n.

**Solution:**

4.0.25 The 17th term of ap exceeds its 10th term by 7. FInd its common difference?

**Solution:**

4.0.26 If $p^{th}, q^{th}, r^{th}$ term of a GP are $a, b$ and $c$ respectively Prove that

$$a^{q-r} b^{r-p} c^{p-q} = 1$$

**Solution:**

4.0.27 Write the first five terms of the sequence whose $n^{th}$ term is : $x(n) = (-1)^{n-1}5^{n+1}$.

**Solution:**

4.0.28 The ratio of sums of m and n terms of an A.P. is $m^2 : n^2$.Show that the ratio of $m^{th}$ and $n^{th}$ term is (2m-1):(2n-1).

**Solution:**

4.0.29 If $a$ and $b$ are the roots of $x^2 - 3x + p = 0$ and $c$ , $d$ are roots of $x^2 - 12x + q = 0$ where $a, b, c, d$ form a G.P. Prove that $(q + p) : (q - p) = 17{:}15$ .

**Solution:**

4.0.30 Write the first five terms in the sequence defined recursively as follows:

$$a_0 = 3$$

$$a_n = 3a_{n-1} + 2 \quad \text{for } n > 0$$

**Solution:**

4.0.31

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} \tag{4.157}$$

then show that a,b,c,d are in G.P

**Solution:**

4.0.32 Sum of the first p, q and r terms of an A.P. are a, b and c, respectively.

Prove that $\frac{a}{p}\left(q-r\right)+\frac{b}{q}\left(r-p\right)+\frac{c}{r}\left(p-q\right)=0$ NCERT-discrete 11.9.2.11

**Solution:**

$$y\left(n\right)=\frac{n+1}{2}\left(2x\left(0\right)+nd\right)u\left(n\right) \tag{4.158}$$

Using y($n$),

$$a=\frac{p}{2}\left(2x\left(0\right)+\left(p-1\right)d\right) \tag{4.159}$$
$$b=\frac{q}{2}\left(2x\left(0\right)+\left(q-1\right)d\right) \tag{4.160}$$
$$c=\frac{r}{2}\left(2x\left(0\right)+\left(r-1\right)d\right) \tag{4.161}$$

which can be represented as,

$$p.x\left(0\right)+\frac{p\left(p-1\right)}{2}.d+a.\left(-1\right)=0 \tag{4.162}$$
$$q.x\left(0\right)+\frac{q\left(q-1\right)}{2}.d+b.\left(-1\right)=0 \tag{4.163}$$
$$r.x\left(0\right)+\frac{r\left(r-1\right)}{2}.d+c.\left(-1\right)=0 \tag{4.164}$$

resulting in the matrix equation,

$$\begin{pmatrix} p & \frac{p(p-1)}{2} & a \\ q & \frac{q(q-1)}{2} & b \\ r & \frac{r(r-1)}{2} & c \end{pmatrix}\mathbf{x}=0 \tag{4.165}$$

where,

$$\mathbf{x} = \begin{pmatrix} x(0) \\ d \\ -1 \end{pmatrix} \tag{4.166}$$

solving the equations (4.159),(4.160) and (4.161) by row reducing the matrix in (4.165),

$$\begin{pmatrix} p & \frac{p(p-1)}{2} & a \\ q & \frac{q(q-1)}{2} & b \\ r & \frac{r(r-1)}{2} & c \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{R_1}{p}, R_2 \leftarrow \frac{R_2}{q}]{R_3 \leftarrow \frac{R_3}{r}} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 1 & \frac{q-1}{2} & \frac{b}{q} \\ 1 & \frac{r-1}{2} & \frac{c}{r} \end{pmatrix} \tag{4.167}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 0 & \frac{q-p}{2} & \frac{b}{q} - \frac{a}{p} \\ 0 & \frac{r-p}{2} & \frac{c}{r} - \frac{a}{p} \end{pmatrix} \tag{4.168}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{\frac{q-p}{2}}} \begin{pmatrix} 1 & \frac{p-1}{2} & \frac{a}{p} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right) \frac{2}{q-p} \\ 0 & \frac{r-p}{2} & \frac{c}{r} - \frac{a}{p} \end{pmatrix} \tag{4.169}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{p-1}{2} R_2]{R_3 \leftarrow R_3 - \frac{r-p}{2} R_2} \begin{pmatrix} 1 & 0 & \frac{a}{p} - \frac{\left(\frac{b}{q} - \frac{a}{p}\right)(p-1)}{q-p} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right) \frac{2}{q-p} \\ 0 & 0 & \left(\frac{c}{r} - \frac{a}{p}\right) - \frac{\left(\frac{b}{q} - \frac{a}{p}\right)(r-p)}{q-p} \end{pmatrix} \tag{4.170}$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & \frac{aq(q-1) - bp(p-1)}{pq(q-p)} \\ 0 & 1 & \left(\frac{b}{q} - \frac{a}{p}\right) \frac{2}{q-p} \\ 0 & 0 & \frac{\frac{a}{p}(r-q) + \frac{b}{q}(p-r) + \frac{c}{r}(q-p)}{q-p} \end{pmatrix} \tag{4.171}$$

After row reduction of matrix we get,

$$x(0) = \left(\frac{aq(q-1) - bp(p-1)}{pq(q-p)}\right) \tag{4.172}$$

$$d = \left(\frac{b}{q} - \frac{a}{p}\right)\frac{2}{q-p} \tag{4.173}$$

$$\frac{\frac{a}{p}(r-q) + \frac{b}{q}(p-r) + \frac{c}{r}(q-p)}{q-p} = 0 \tag{4.174}$$

$$\therefore \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0 \tag{4.175}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{4.176}$$

$$X(z) = \frac{aq(q-1) - bp(p-1)}{pq(q-p)(1-z^{-1})} + \frac{2\left(\frac{b}{q} - \frac{a}{p}\right)z^{-1}}{(q-p)(1-z^{-1})^2} \tag{4.177}$$

$$R.O.C\,(|z| > 1) \tag{4.178}$$

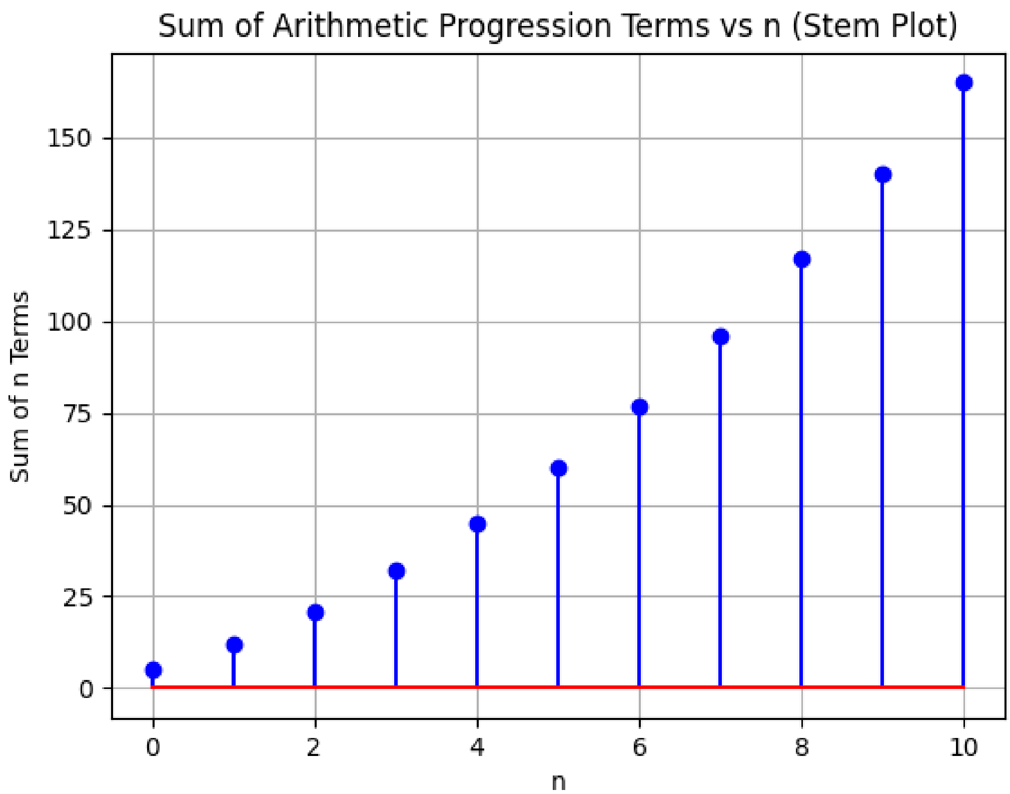



Figure 4.24: Plot of x(n) *vs* n

4.0.33 The pth, qth and rth terms of an AP are a,b,c respectively. Show that

$$(q-r)a+(r-p)b+(p-q)c=0$$

**Solution:**

4.0.34 Find the sum to indicated number of term in each of the geometric progressions in $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, ....n$ terms

**Solution:**

$$X(z) = x(0)\left(\frac{1}{1 - rz^{-1}}\right), \quad |rz^{-1}| < 1 \tag{4.179}$$

$$y(n) = x(n) * u(n) \tag{4.180}$$

$$Y(z) = X(z)\,U(z) \tag{4.181}$$

$$= \sqrt{7}\left(\frac{1}{1 - \sqrt{3}z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right), \quad |z| > \sqrt{3} \tag{4.182}$$

$$= \left(\frac{\sqrt{7}}{\sqrt{3} - 1}\right)\left(\left(\frac{\sqrt{3}}{1 - \sqrt{3}z^{-1}}\right) - \left(\frac{1}{1 - z^{-1}}\right)\right) \tag{4.183}$$

$$\frac{1}{1 - rz^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} r^n u(n), \quad |z| > r \tag{4.184}$$

$$y(n) = \sqrt{7}\left(\frac{\sqrt{3}^{n+1} - 1}{\sqrt{3} - 1}\right)u(n), \quad |z| > \sqrt{3} \tag{4.185}$$

4.0.35 How many multiples of 4 lie between 10 and 250?

**Solution:**

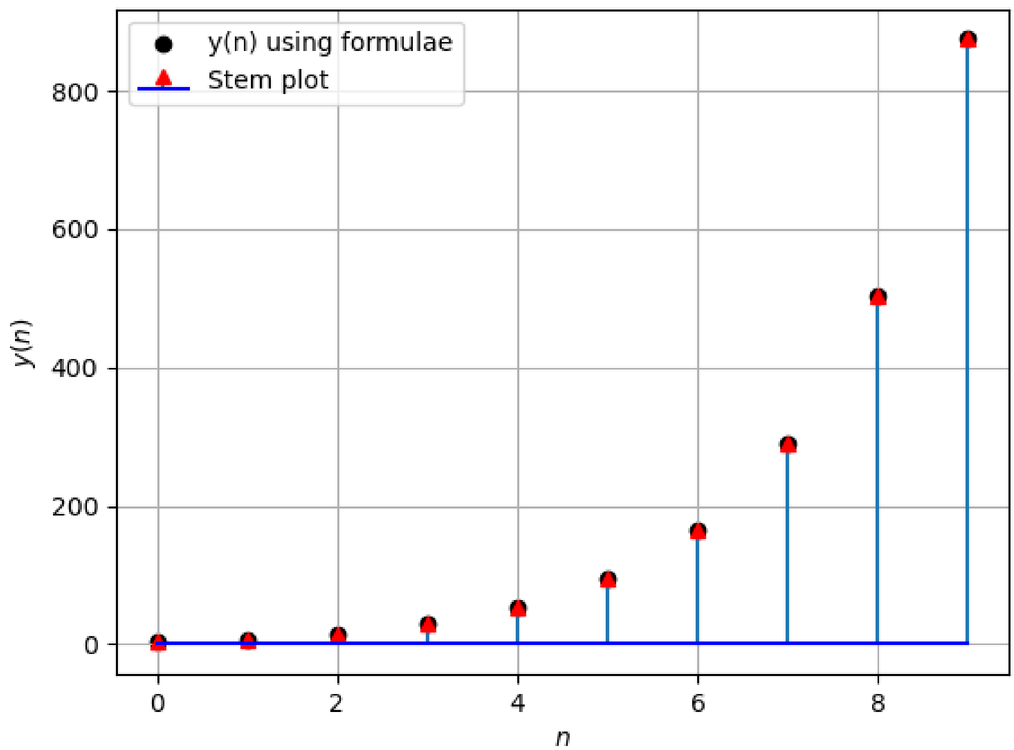



Figure 4.25: STEM PLOT OF $y(n)$

4.0.36 if $a, b, c$ and $d$ are in GP then show that $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

**Solution:**

4.0.37 In an A.P. the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that $20^{\text{th}}$ term is $-112$. (NCERT MATHS 11.9.2.3)

**Solution:**

General term can be written as

$$x(n) = (x(0) + nd)\, u(n) \tag{4.186}$$

By referreing (B.8.2)

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{4.187}$$

Taking the inverse Z-transform by contour integration by refering (B.10.6),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{4.188}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\} u(n) \tag{4.189}$$

Therefore,

$$y(4) = 5x(0) + 10d \tag{4.190}$$

$$y(9) = 10x(0) + 45d \tag{4.191}$$

Given,

$$\sum_{n=0}^{4} x(n) = \frac{1}{4}\sum_{n=5}^{9} x(n) \tag{4.192}$$

Simplifying:

$$y(4) = \frac{1}{4}(y(9) - y(4)) \tag{4.193}$$

$$\implies 5x(0) + 10d = \frac{1}{4}(5x(0) + 35d) \tag{4.194}$$

$$x(0) = \frac{-d}{3} \tag{4.195}$$

$$\implies d = -6 \tag{4.196}$$

From (4.196) and Table 1:

$$x(n) = (2 - 6n)\,u(n) \tag{4.197}$$

From (4.197):

$$x(19) = x(0) + 19d \tag{4.198}$$

$$= -112 \tag{4.199}$$

From (4.197) and (4.187):

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{6z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.200}$$

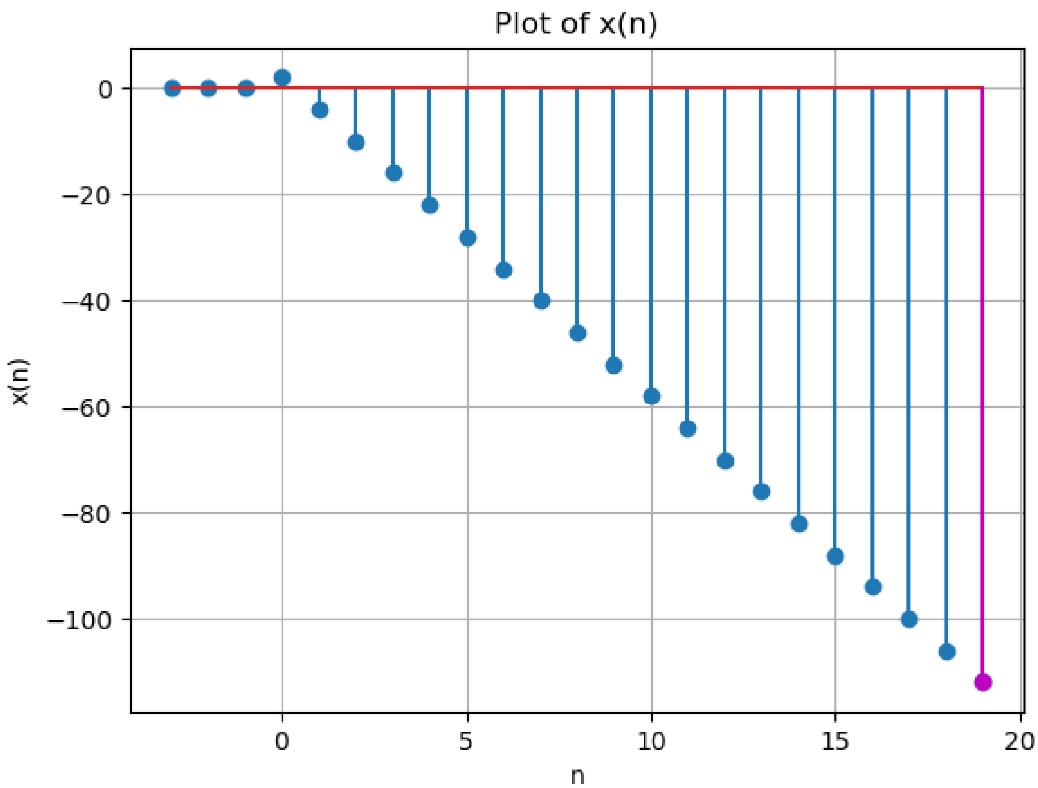



Figure 1: graph of $x(n) = 2 - 6n$

4.0.38 If the 3rd and the 9th terms of an AP are 4 and -8, respectively, which term of this AP is zero?

**Solution:**

4.0.39 Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$. **Solution:**

4.0.40 The 17th term of ap exceeds its 10th term by 7. FInd its common difference?

**Solution:**

$$x(n) = \{x(0) + nd\}u(n) \tag{4.201}$$

$$x(17) - x(10) = 7 \tag{4.202}$$

$$\implies x(0) + 17d - x(0) + 10d = 7 \tag{4.203}$$

$$\implies 17d - 10d = 7 \tag{4.204}$$

$$\implies 7d = 7 \tag{4.205}$$

$$\implies d = 1 \tag{4.206}$$

Taking Z-Transform:

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}\{|z| > 1\} \tag{4.207}$$

(b) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}\{|z| > 1\} \tag{4.208}$$

0 Taking Z-Transform of (4.201) using (4.207)and (4.208)

$$X(n) = 100\frac{1}{1 - z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2} \tag{4.209}$$

Let

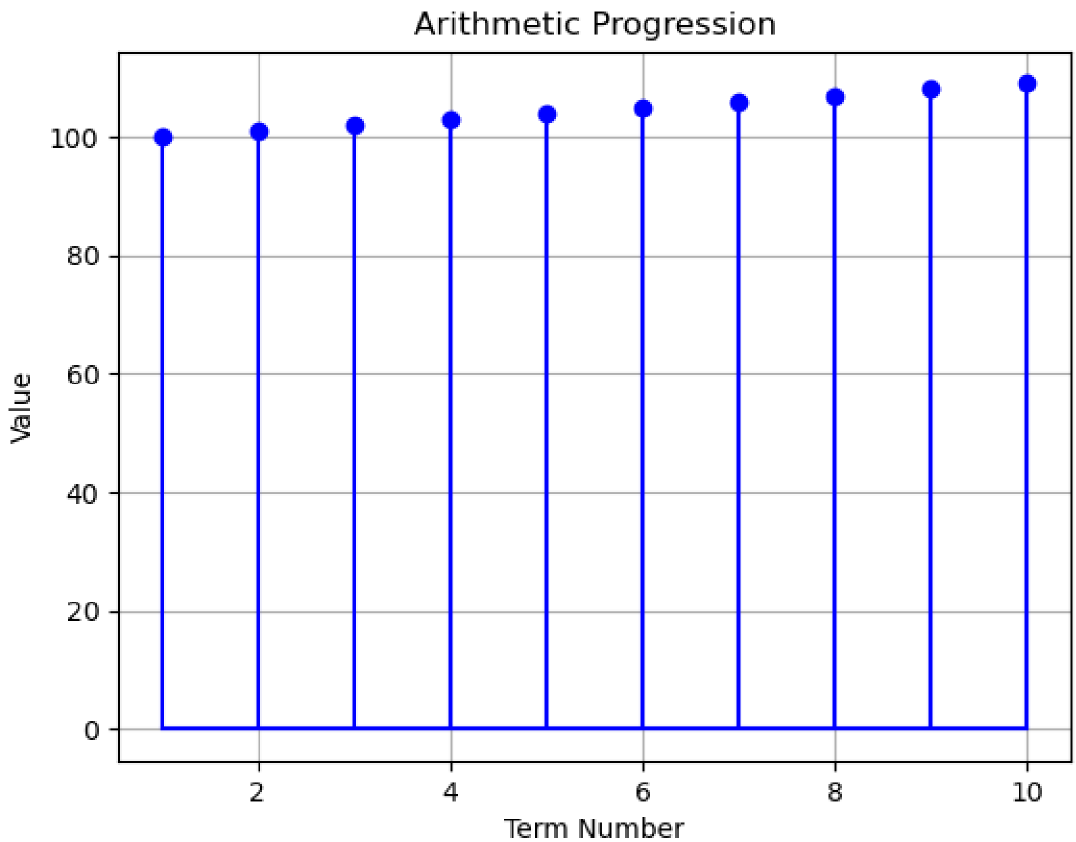



Figure 4.27:

$$x(n) = \{101, 102, 103, ...\} \tag{4.210}$$

4.0.41 If $p^{th}, q^{th}, r^{th}$ term of a GP are $a, b$ and $c$ respectively Prove that

$$a^{q-r}b^{r-p}c^{p-q} = 1$$

**Solution:**

$$x(n) = (x(0)d^n)u(n) \tag{4.211}$$

$$a = x(p) = (x(0)d^p) \tag{4.212}$$

$$b = x(q) = (x(0)d^q) \tag{4.213}$$

$$c = x(r) = (x(0)d^r) \tag{4.214}$$

$$a^{q-r}b^{r-p}c^{p-q} = x(0)^{q-r}d^{p(q-r)}x(0)^{r-p}d^{q(r-p)}x(0)^{p-q}d^{r(p-q)} \tag{4.215}$$

$$= x(0)^{q-r+r-p+p-q}d^{p(q-r)+q(r-p)+r(p-q)} \tag{4.216}$$

$$= x(0)^0 d^0 \tag{4.217}$$

$$a^{q-r}b^{r-p}c^{p-q} = 1 \tag{4.218}$$

Taking Z-Transform:

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}}\{|z| > 1\} \tag{4.219}$$

(b) $\mathcal{Z}\{d^n u(n)\}$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-dz^{-1})}\{|z| > |d|\} \tag{4.220}$$

Taking Z-Transform of (4.211) using (4.219)and (4.220)

$$X(z) = \frac{x(0)}{1 - dz^{-1}} \qquad |z| > |d| \tag{4.221}$$

4.0.42 An AP consists of 50 terms of which $3^{rd}$ term is 12 and the last term is 106. Find the $29^{th}$ term.

**Solution:**

| **Parameter** | **Value** | **description** |
|---|---|---|
| $x(2)$ | 12 | Third term |
| $x(49)$ | 106 | Last term |
| $x(0)$ | | First term |
| $d$ | | Common difference |
| $x(n)$ | $(x(0)+nd)u(n)$ | general term |

Table 4.25: Input parameters

$$\begin{pmatrix} x(2) \\ x(49) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 49 \end{pmatrix} \begin{pmatrix} x(0) \\ d \end{pmatrix} \tag{4.222}$$

$$\begin{pmatrix} 12 \\ 106 \end{pmatrix} = \begin{pmatrix} x(0)+2d \\ x(0)+49d \end{pmatrix} \tag{4.223}$$

$$\text{converting to augmented matrix} \tag{4.224}$$

$$= \begin{pmatrix} x(0)+2d & |12 \\ x(0)+49d & |106 \end{pmatrix} \tag{4.225}$$

$$R_2 \rightarrow R_2 - R_1 \tag{4.226}$$

$$\xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} x(0)+2d & |12 \\ 47d & |94 \end{pmatrix} \tag{4.227}$$

From (4.227), we get

$$\implies x(0) = 8 \tag{4.228}$$

$$\implies d = 2 \tag{4.229}$$

From the Table 4.25 :

$$\implies \; x(n) = (8 + 2n)u(n) \tag{4.230}$$

Finding $x(28)$ :

$$x(28) = x(0) + 28(2) \tag{4.231}$$

$$\implies \; x(28) = 64 \tag{4.232}$$

Z-transform :

$$\implies X(z) = \frac{8 - 6z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{4.233}$$

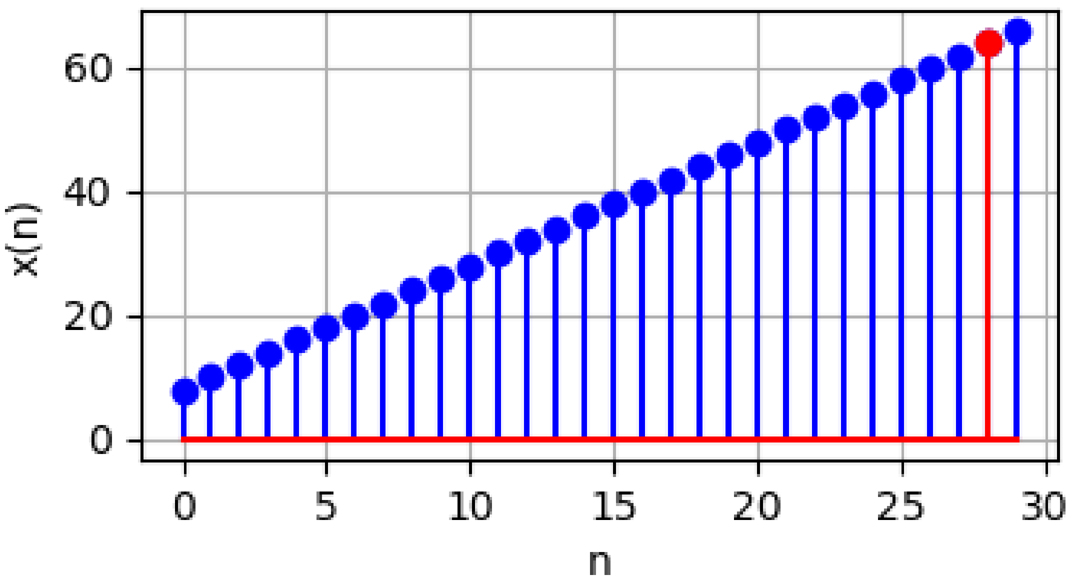



Figure 4.28: graph of the given AP

| | |
|---|---|
| $x(0)$ | 3 |
| $d$ | 2 |
| $m$ | 6 |
| $n$ | 2 |
| $x(m+n)$ | 19 |
| $x(m-n)$ | 11 |
| $x(m)$ | 15 |

Table 4.12: Verified Values

| **Parameter** | **Value** | **Description** |
|---|---|---|
| $x(0)$ | | First term |
| $r$ | | Common ratio |
| $x(0)^3r^3$ | 1 | Product of terms |
| $x(0) + x(0)r + x(0)r^2$ | $\frac{39}{10}$ | Sum of terms |

Table 4.13: Input Parameters

| Parameter | Value | Description |
|---|---|---|
| $x(n)$ | $(x(0) + n \cdot d)\, u(n)$ | $(n+1)$th term |
| $d$ | 3 | common difference |
| $x(0) + x(1) + x(2)$ | 24 | sum of the terms |
| $x(0) \cdot x(1) \cdot x(2)$ | 440 | product of terms |

Table 4.14: Parameters

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 5 | First term |
| $r$ | 2 | Common ratio |
| $y(n)$ | 315 | Sum of $n+1$ terms |
| $x(n)$ | ? | Last term |

Table 4.15: Input Parameters

| Symbol | Value | Parameter |
|---|---|---|
| $x(0)$ | 1 | First Term |
| $x(n)$ | $(5n+1)u(n)$ | kth Term |
| $d$ | 5 | Common Difference |

Table 4.16: Given Parameters

| Symbol | Value | Description |
|---|---|---|
| $x\,(0)$ | -32 | First term |
| $x\,(10)$ | 38 | 11th term |
| $x\,(15)$ | 73 | 16th term |
| $d$ | 7 | Common Difference |
| $x\,(n)$ | $x(0) + nd$ | $(n+1)$th term |

Table 4.17: Given Values

| parameter | value | description |
|---|---|---|
| $x(0)$ | $a\left(\frac{1}{b} + \frac{1}{c}\right)$ | First Term of given AP |
| $d$ | (b - a)$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ | Common Difference of given AP |
| $x(n)$ | $(x(0) + nd)u(n)$ | General Term of given AP |

Table 4.18: Input Parameter Table

| Symbol | Value | Description |
|---|---|---|
| $x(n)$ | $(x(0) + nd)u(n)$ | $n^{th}$ term of an A.P |
| $x(0)$ | $x(0)$ | $1^{st}$ term of the A.P |
| $d$ | $d$ | Common difference |
| $y(n)$ | $x(n) * u(n)$ | Sum of n terms of an AP |
| $a$ | $y(p-1)$ | Sum of first p terms of the AP |
| $b$ | $y(q-1)$ | Sum of first q terms of the AP |
| $c$ | $y(r-1)$ | Sum of first r terms of the AP |

Table 4.19: Variable description

| | |
|---|---|
| $x(0)$ | 5 |
| $d$ | 2 |
| $p$ | 8 |
| $q$ | 10 |
| $r$ | 4 |
| $a$ | 96 |
| $b$ | 140 |
| $c$ | 32 |

Table 4.20: Verified Values

| variable | value | description |
|---|---|---|
| $x(0)$ | $\sqrt{7}$ | first term of the geometric progession |
| $r$ | $\sqrt{3}$ | common ratio of the geometeric progression |
| $x(n)$ | $\sqrt{7(3^n)}u\left(n\right)$ | $n^{th}$ term of the geometric progession |
| $y(n)$ | $\frac{x(0)(r^{n+1}-1)}{r-1}u\left(n\right)$ | Sum of the n term of the geometric progression |

Table 4.21: Input parameters

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $x(0)$ | First term | 2 |
| $x(19)$ | $20^{\text{th}}$ term | $-112$ |
| $y(n)$ | sum upto $n^{\text{th}}$ term | |

Table 1: Input data

| **Variable** | **Description** | **Value** |
|---|---|---|
| $x(n)$ | $n^{th}$ term of AP | none |
| $d$ | common difference between the terms of AP | none |
| $x(17)-x(10)$ | difference of $17^{th}$ and $10^{th}$ term of X | 7 |

Table 4.23: input parameters

| **Variable** | **Description** | **Value** |
|---|---|---|
| $x(n)$ | $n^{th}$ term of GP | none |
| $d$ | common ratio between the terms of GP | none |
| $x(p)$ | a | $x(0)d^p$ |
| $x(q)$ | b | $x(0)d^q$ |
| $x(r)$ | c | $x(0)d^r$ |

Table 4.24: input parameters

# Chapter 5

# Contour Integration

5.1 In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato,and the other potatoes are placed 3 m apart in a straight line. There are ten potatoes in the line. A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?

(NCERT-Maths 10.5.3.20Q)

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $x(0)$ | First term | 10 |
| $d$ | Common Difference | 6 |
| $y(9)$ | Total distance covered | ? |

Table 1: Parameter Table

From (B.8.2) :

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.1}$$

$$= \frac{10}{1-z^{-1}} + \frac{6z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.2}$$

$$= \frac{10-4z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{5.3}$$

From (A.3.2)

$$y(n) = \sum_{k=0}^{n} x(k) = x(n) * u(n) \tag{5.4}$$

Taking z transform :

$$Y(z) = X(z)U(z) \tag{5.5}$$

$$\Longrightarrow Y(z) = \frac{10-4z^{-1}}{\left(1-z^{-1}\right)^3} \quad |z| > 1 \tag{5.6}$$

Taking inverse z transform :

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{n-1} dz \tag{5.7}$$

$$y(9) = \frac{1}{2\pi j} \oint_C Y(z)\, z^8 \, dz \tag{5.8}$$

$$= \frac{1}{2\pi j} \oint_C \frac{10z^{11} - 4z^{10}}{(z-1)^3} \, dz \tag{5.9}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left(f\left(z\right)\right) \tag{5.10}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(10z^{11} - 4z^{10}\right) \tag{5.11}$$

$$= 370 \tag{5.12}$$

$$\therefore y\left(9\right) = 370 \tag{5.13}$$

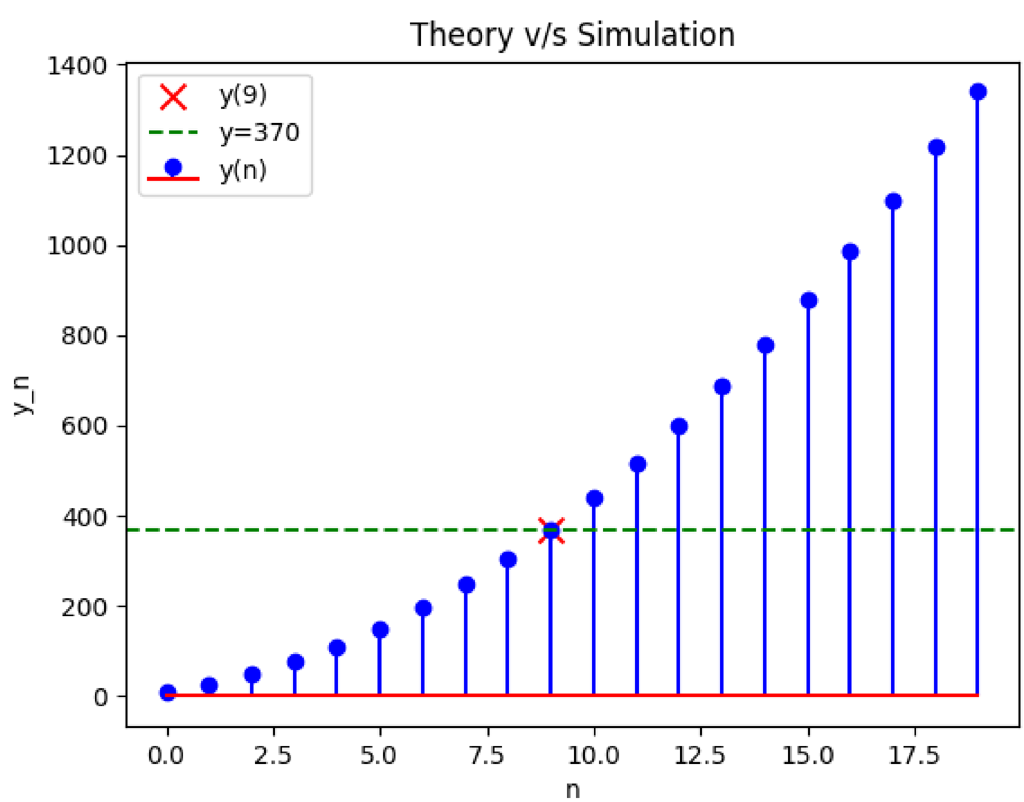



Figure 5.1: Theory matches with the simulated values

5.2 Find the sum of the first 15 multiples of 8.

**Solution:**

| PARAMETER | VALUE | DESCRIPTION |
|---|---|---|
| $x(0)$ | 8 | First term |
| $d$ | 8 | common difference |
| $x(n)$ | $[8+8n]u(n)$ | General term of the series |

Table 5.2: Parameter Table1

For an $AP$,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{\left(1 - z^{-1}\right)^2} \tag{5.14}$$

$$\implies X(z) = \frac{8}{1 - z^{-1}} + \frac{8z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{5.15}$$

$$= \frac{8}{\left(1 - z^{-1}\right)^2}, \quad |z| > 1 \tag{5.16}$$

$$y(n) = x(n) * u(n) \tag{5.17}$$

$$\implies Y(z) = X(z) U(z) \tag{5.18}$$

$$Y(z) = \left(\frac{8}{\left(1 - z^{-1}\right)^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \tag{5.19}$$

$$= \frac{8}{\left(1 - z^{-1}\right)^3}, \quad |z| > 1 \tag{5.20}$$

Using Contour Integration to find the inverse $Z$-transform,

$$y(14) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{13}\, dz \tag{5.21}$$

$$= \frac{1}{2\pi j} \oint_C \frac{8z^{13}}{\left(1 - z^{-1}\right)^3}\, dz \tag{5.22}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z)\right) \tag{5.23}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z - 1)^3 \frac{8z^{16}}{(z - 1)^3}\right) \tag{5.24}$$

$$= 4 \lim_{z \to 1} \frac{d^2}{dz^2} (z^{16}) \tag{5.25}$$

$$= 960 \tag{5.26}$$

$$\therefore \boxed{y(14) = 960} \tag{5.27}$$

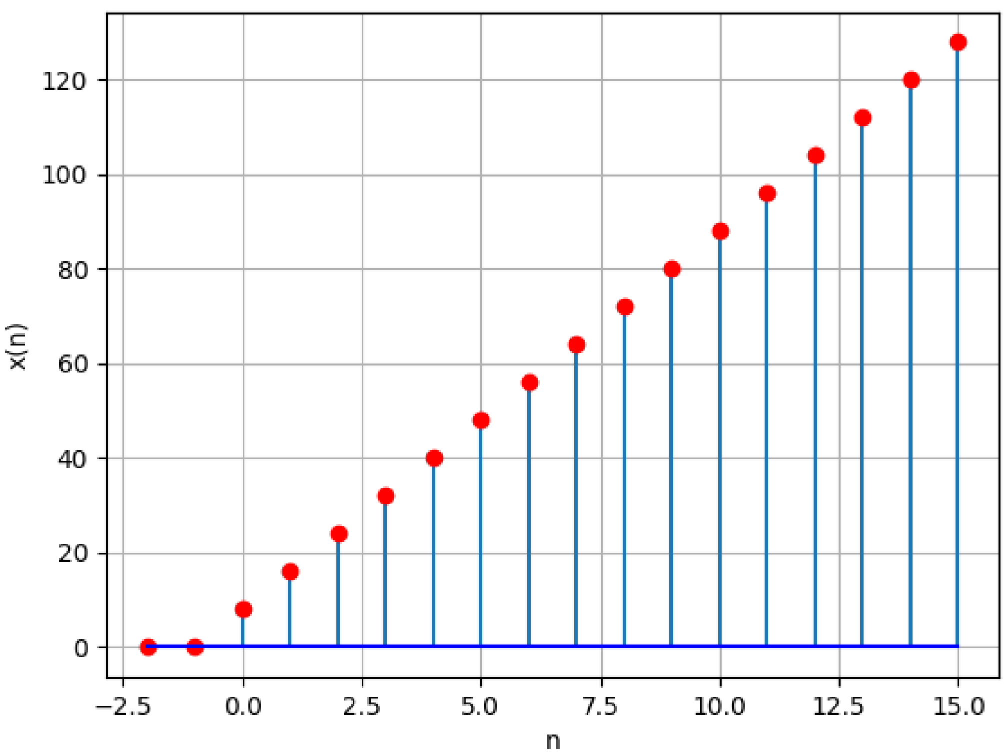



Figure 5.2: Plot of x(n) *vs* n

ss

5.3 If the sum of $n$ terms of an A.P. is $3n^2 + 5n$ and its $m^{th}$ term is 164, find the value of $m$.

**Solution:**

$$Y(z) = \sum_{n=0}^{\infty} y(n) z^{-n} \tag{5.28}$$

$$= \frac{2\left(4 - z^{-1}\right)}{\left(1 - z^{-1}\right)^3}, \qquad |z| > 1 \tag{5.29}$$

$$U(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1 \tag{5.30}$$

$$X(z) = \frac{Y(z)}{U(z)} \tag{5.31}$$

$$= 2\left(\frac{1}{1 - z^{-1}}\right) + 6\left(\frac{1}{\left(1 - z^{-1}\right)^2}\right) \tag{5.32}$$

$$= \frac{8z^2 - 2z}{(z-1)^2} \tag{5.33}$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{5.34}$$

$$= \frac{1}{2\pi j} \oint_C \frac{\left(8z^{n+1} - 2z^n\right) dz}{(z-1)^2} \tag{5.35}$$

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.36}$$

$$= \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{8z^{n+1} - 2z^n}{(z-1)^2}\right) \tag{5.37}$$

$$= \lim_{z \to 1} \left(8(n+1) z^n - 2nz^{n-1}\right) \tag{5.38}$$

$$\implies x(n) = (6n + 8)(u(n)) \tag{5.39}$$

$$164 = (6m + 8)(u(m)) \tag{5.40}$$

$$\implies m = 26 \tag{5.41}$$

| Symbol | Remarks |
| --- | --- |
| $y\left(n\right)=\left(3n^{2}+11n+8\right)\left(u\left(n\right)\right)$ | Sum of $n$ terms |
| $x(m-1)$ | 164 |
| $y\left(n\right)$ | $x\left(n\right)*u\left(n\right)$ |

Table 5.3: Parameters

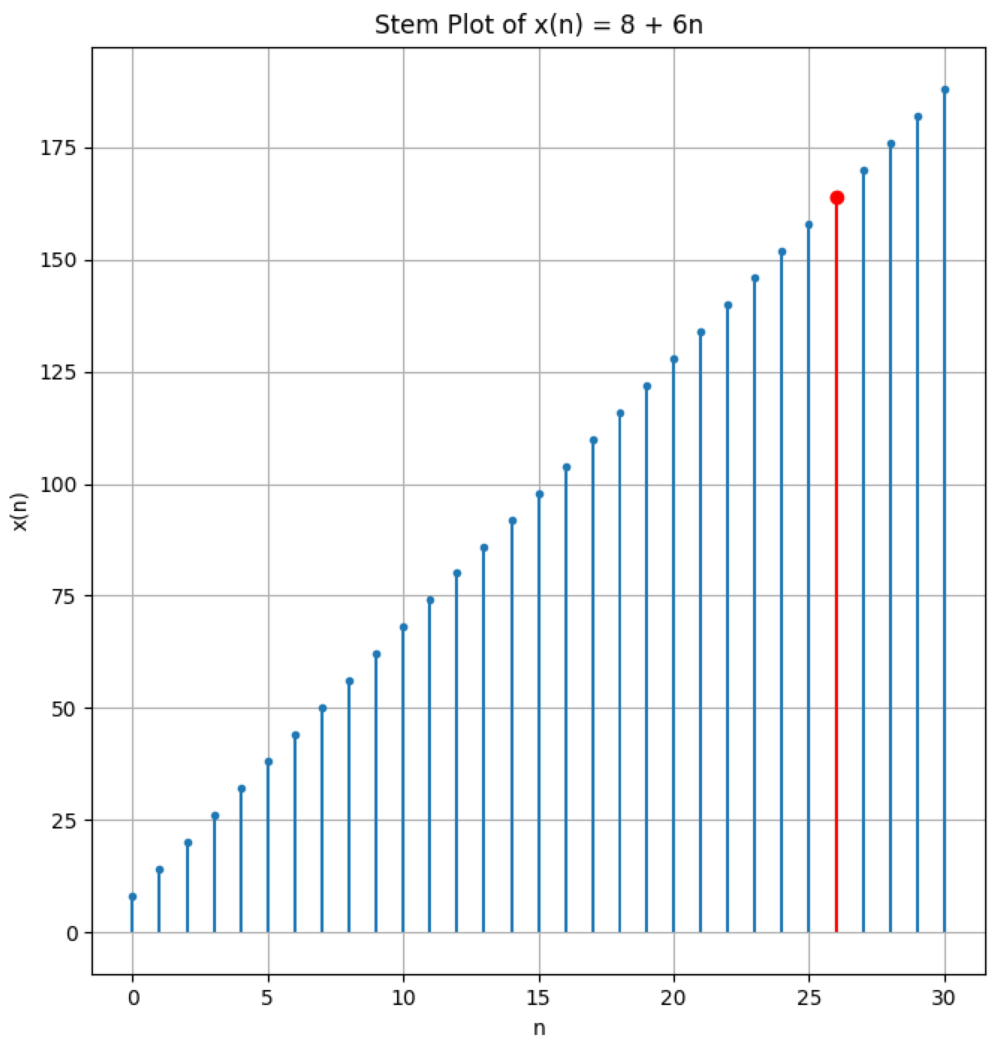



Figure 5.3: Plot of x(n) vs n

5.4 Find the sums given below:

(a) $7 + \frac{21}{2} + 14... + 84$

(b) $34 + 32 + 30... + 10$

(c) $-5 + -8 + -11... - 230$

**Solution:**

| Symbols | Description | Values |
|---|---|---|
| $d_i$ | Common Difference for $i^{th}$ AP | 3.5 |
| | | -2 |
| | | -3 |
| $x_i(n)$ | $n^{th}$ term for $i^{th}$ Sequence | $(7 + \frac{7n}{2})u_{(n)}$ |
| | | $(34 - 2n)u_{(n)}$ |
| | | $(-5 + -3n)u_{(n)}$ |
| $x_i(0)$ | First term for $i^{th}$ AP | 7 |
| | | 34 |
| | | -5 |

Table 5.4: Parameters , Descriptions And Values

(a) $7 + \frac{21}{2} + 14... + 84$

$$x_1(n) = (x_1(0) + nd_1)\, u_{(n)} \tag{5.42}$$

$$\implies 84 = 7 + \frac{7n}{2} \tag{5.43}$$

$$\implies n = 22 \tag{5.44}$$

i. z-Transform of $x_1(n)$ : Using (B.1.1)

$$X_1(z) = \frac{7z}{z-1} + \frac{7z}{2(z-1)^2}, \quad |z| > |1| \tag{5.45}$$

ii. Z-Transform of $y_1(n)$ :

$$y_1(n) = x_1(n) * h(n) \tag{5.46}$$

$$h(n) = u(n) \tag{5.47}$$

$$H(z) = \frac{z}{z-1} \tag{5.48}$$

$$Y_1(z) = X_1(z) * H(z) \tag{5.49}$$

$$= \left(\frac{7z}{z-1} + \frac{7z}{2(z-1)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.50}$$

iii. Inversion of $Y_1(z)$ : Using Contour Integration :

$$y_1(22) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{21}\, dz \tag{5.51}$$

$$\implies = \frac{1}{2\pi j}\oint_C \left(\frac{7z^{23}}{(z-1)^2} + \frac{7z^{23}}{2(z-1)^3}\right) dz \tag{5.52}$$

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.53}$$

For $R_1$ , $m = 2$ , where $m$ corresponds to number of repeated poles .

$$R_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{7z^{23}}{(z-1)^2}\right) \tag{5.54}$$

$$= 7\lim_{z\to 1}\frac{d}{dz}(z^{23}) \tag{5.55}$$

$$= 161 \tag{5.56}$$

For $R_2$ , $m = 3$

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{(7z^{13})}{2(z-1)^3} \right) \tag{5.57}$$

$$= \left( \frac{7}{4} \right) \lim_{z \to 1} \frac{d^2}{dz^2} (z^{23}) \tag{5.58}$$

$$= \frac{1771}{2} \tag{5.59}$$

$$R_1 + R_2 = \frac{2093}{2} \tag{5.60}$$

$$\implies y_1(22) = \frac{2093}{2} \tag{5.61}$$

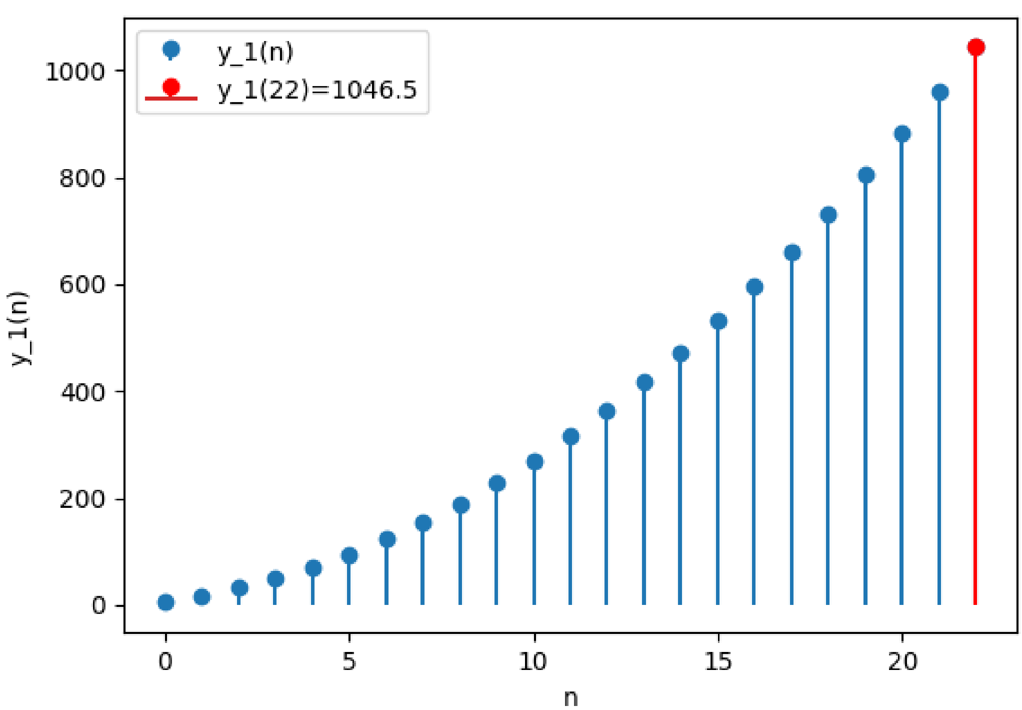



Figure 5.4: $y_1(n)$ vs n

(b) 34 + 32 + 30 ... + 10

$$x_2\left(n\right) = \left(x_2\left(0\right) + nd_2\right)u_{(n)} \tag{5.62}$$

$$\implies 10 = 34 - 2n \tag{5.63}$$

$$\implies n = 12 \tag{5.64}$$

i. Z-Transform of $x_2\left(n\right)$ : Using (B.1.1)

$$X_2(z) = \frac{34z}{z-1} - \frac{2z}{\left(z-1\right)^2}, \quad |z| > |1| \tag{5.65}$$

ii. Z-Transform of $y_2\left(n\right)$ :

$$y_2\left(n\right) = x_2\left(n\right) * h\left(n\right) \tag{5.66}$$

$$h\left(n\right) = u\left(n\right) \tag{5.67}$$

$$Y_2\left(z\right) = X_2\left(z\right) * H\left(z\right) \tag{5.68}$$

$$= \left(\frac{34z}{\left(z-1\right)^1} - \frac{2z}{\left(z-1\right)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.69}$$

iii. Inversion of $Y_2\left(z\right)$ : Using Contour Integration :

$$y_2(12) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{11}\, dz \tag{5.70}$$

$$\implies = \frac{1}{2\pi j}\oint_C \left(\frac{34z^{13}}{\left(z-1\right)^2} - \frac{2z^{13}}{\left(z-1\right)^3}\right) dz \tag{5.71}$$

Using (5.53) For $R_1$ , $m = 2$ :

$$R_1 = \frac{1}{(1)!} \lim_{z\to 1} \frac{d}{dz} \left( (z-1)^2 \frac{34z^{13}}{(z-1)^2} \right) \tag{5.72}$$

$$= 34 \lim_{z\to 1} \frac{d}{dz} (z^{13}) \tag{5.73}$$

$$= 442 \tag{5.74}$$

For $R_2$ , $m = 3$ :

$$R_2 = \frac{1}{(2)!} \lim_{z\to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{(-2z^{13})}{(z-1)^3} \right) \tag{5.75}$$

$$= - \lim_{z\to 1} \frac{d^2}{dz^2} (z^{13}) \tag{5.76}$$

$$= -156 \tag{5.77}$$

$$R_1 + R_2 = 286 \tag{5.78}$$

$$\implies y_2(12) = 286 \tag{5.79}$$

(c) -5 + -8 + -11 ... -230

$$x_3(n) = (x_3(0) - 3n)\, u_{(n)} \tag{5.80}$$

$$\implies -230 = -5 - 3n \tag{5.81}$$

$$\implies n = 75 \tag{5.82}$$

i. Z-Transform of $x_3(n)$ : Using (B.1.1)

$$X_3(z) = \frac{-5z}{(z-1)^1} - \frac{3z}{(z-1)^2}, \quad |z| > |1| \tag{5.83}$$

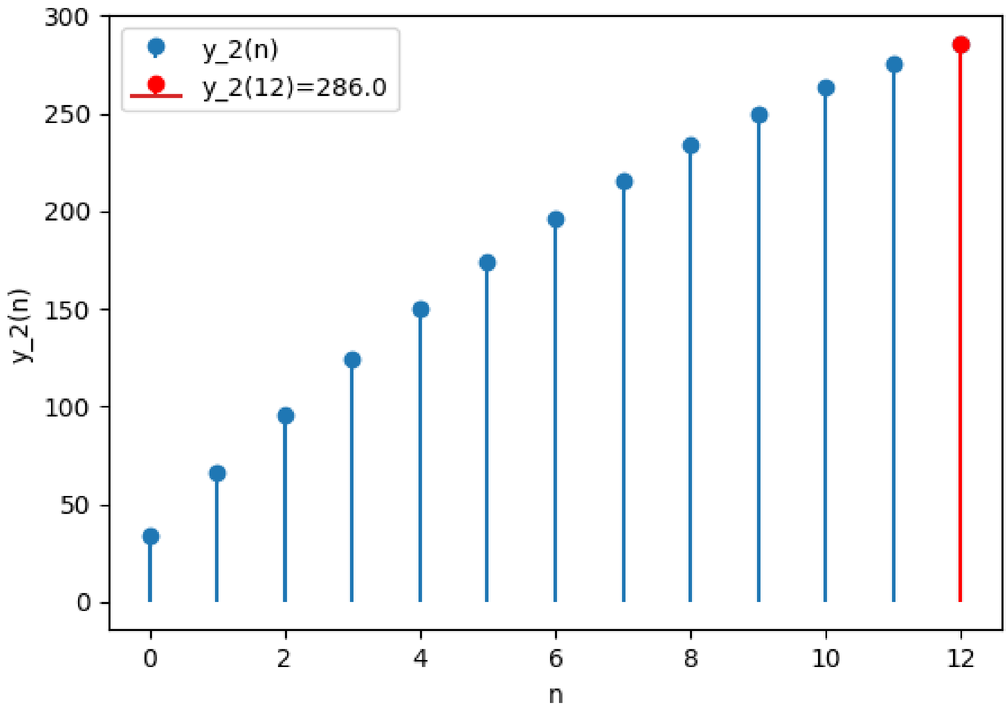



Figure 5.5: $y_2\left(n\right)$ vs n

ii. Z-Transform of $y_3\left(n\right)$ :

$$y_3\left(n\right) = x_3\left(n\right) * h\left(n\right) \tag{5.84}$$

$$h\left(n\right) = u\left(n\right) \tag{5.85}$$

$$Y_3\left(z\right) = X_3\left(z\right) * H\left(z\right) \tag{5.86}$$

$$= \left(\frac{-5z}{\left(z-1\right)^1} - \frac{3z}{\left(z-1\right)^2}\right)\left(\frac{z}{z-1}\right), \quad |z| > |1| \tag{5.87}$$

iii. Inversion of $Y_3(z)$ : Using Contour Integration :

$$y_1(75) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{74}\, dz \tag{5.88}$$

$$\implies = \frac{1}{2\pi j} \oint_C \left( \frac{-5z^{76}}{(z-1)^2} - \frac{3z^{76}}{(z-1)^3} \right) dz \tag{5.89}$$

Using (5.53) For $R_1$ , $m = 2$ :

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left( (z-1)^2 \frac{-5z^{76}}{(z-1)^2} \right) \tag{5.90}$$

$$= -5 \lim_{z \to 1} \frac{d}{dz} (z^{76}) \tag{5.91}$$

$$= -380 \tag{5.92}$$

For $R_2$ , $m = 3$ :

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{3z^{76}}{(z-1)^3} \right) \tag{5.93}$$

$$= 1.5 \lim_{z \to 1} \frac{d^2}{dz^2} (z^{76}) \tag{5.94}$$

$$= -8550 \tag{5.95}$$

$$R_1 + R_2 = -8930 \tag{5.96}$$

$$\implies y_3(75) = -8930 \tag{5.97}$$

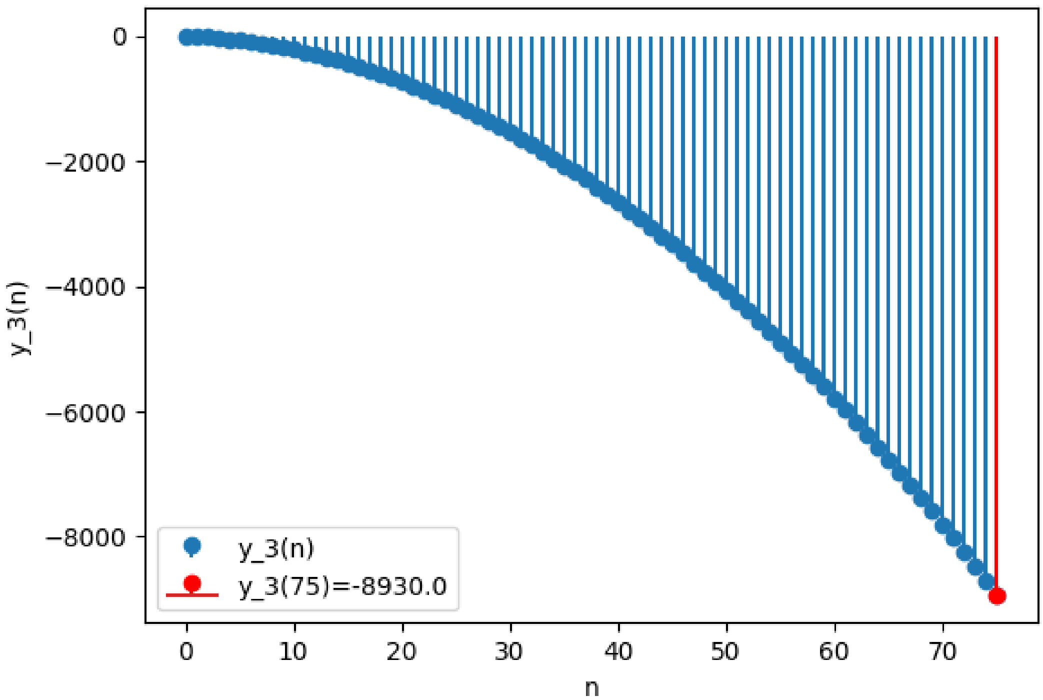



Figure 5.6: $y_3(n)$ vs n

5.5 Show that $a_0$ , $a_1$ , $a_2$ , . . ., $a_n$ , . . . form an AP where an is defined as below :

(a) $a_n = (3 + 4n)$

(b) $a_n = (9 - 5n)$

Also find the sum of the first 15 terms in each case. **Solution:**

| Parameter | Description | Value |
|---|---|---|
| $x_i(n)$ | $i^{th}$Discrete signal | $(3+4n)u(n)$ |
| | | $(9-5n)u(n)$ |
| $x_i(0)$ | First term of $i^{th}$AP | 3 |
| | | 9 |
| $d_i$ | common difference of $i^{th}$AP | 4 |
| | | $-5$ |

Table 5.5: Given parameters

(a) From equation (B.10.6)

$$X(z) = \frac{3}{1-z^{-1}} + \frac{4.z^{-1}}{(1-z^{-1})^2}; |z| > 1 \tag{5.98}$$

$$\because y(n) = x(n) * u(n) \tag{5.99}$$

$$Y(z) = X(z)U(z) \tag{5.100}$$

$$= \left[\frac{3}{(1-z^{-1})^2} + \frac{4z^{-1}}{(1-z^{-1})^3}\right] \tag{5.101}$$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{5.102}$$

$$= \frac{1}{2\pi j}\int \frac{3.z^{15}}{(z-1)^2}dz + \frac{1}{2\pi j}\int \frac{4.z^{15}}{(z-1)^3}dz \tag{5.103}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.104}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{3.z^{15}}{(z-1)^2}\right) \tag{5.105}$$

$$= 45 \tag{5.106}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{4.z^{15}}{(z-1)^3}\right) \tag{5.107}$$

$$= 420 \tag{5.108}$$

$$\implies y(14) = R_1 + R_2 \tag{5.109}$$

$$= 465 \tag{5.110}$$

(b) From equation (B.10.6)

$$X(z) = \frac{9}{1-z^{-1}} - \frac{5.z^{-1}}{(1-z^{-1})^2}; |z| > 1 \tag{5.111}$$

$$\because y(n) = x(n) * u(n) \tag{5.112}$$

$$Y(z) = X(z)U(z) \tag{5.113}$$

$$= \left[\frac{9}{(1-z^{-1})^2} - \frac{5z^{-1}}{(1-z^{-1})^3}\right] \tag{5.114}$$

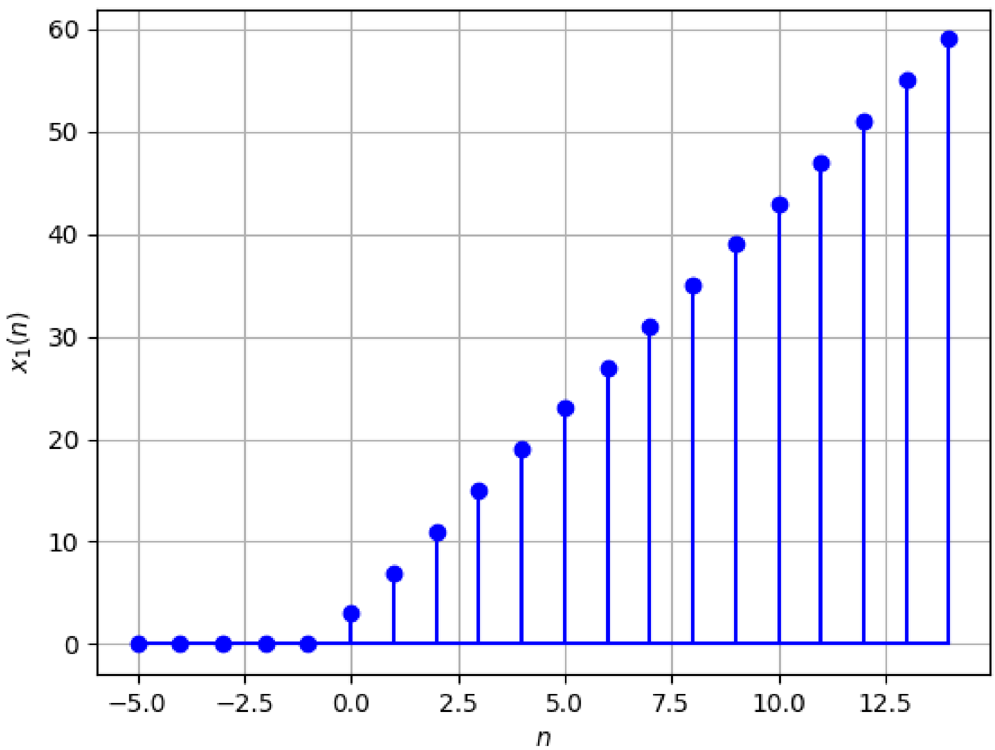



Figure 5.7: $x_1(n) = (3 + 4n)u(n)$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{5.115}$$

$$= \frac{1}{2\pi j}\int \frac{9.z^{15}}{(z-1)^2}dz - \frac{1}{2\pi j}\int \frac{5.z^{15}}{(z-1)^3}dz \tag{5.116}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.117}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{9.z^{15}}{(z-1)^2}\right) \tag{5.118}$$

$$= 135 \tag{5.119}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{5.z^{15}}{(z-1)^3}\right) \tag{5.120}$$

$$= 525 \tag{5.121}$$

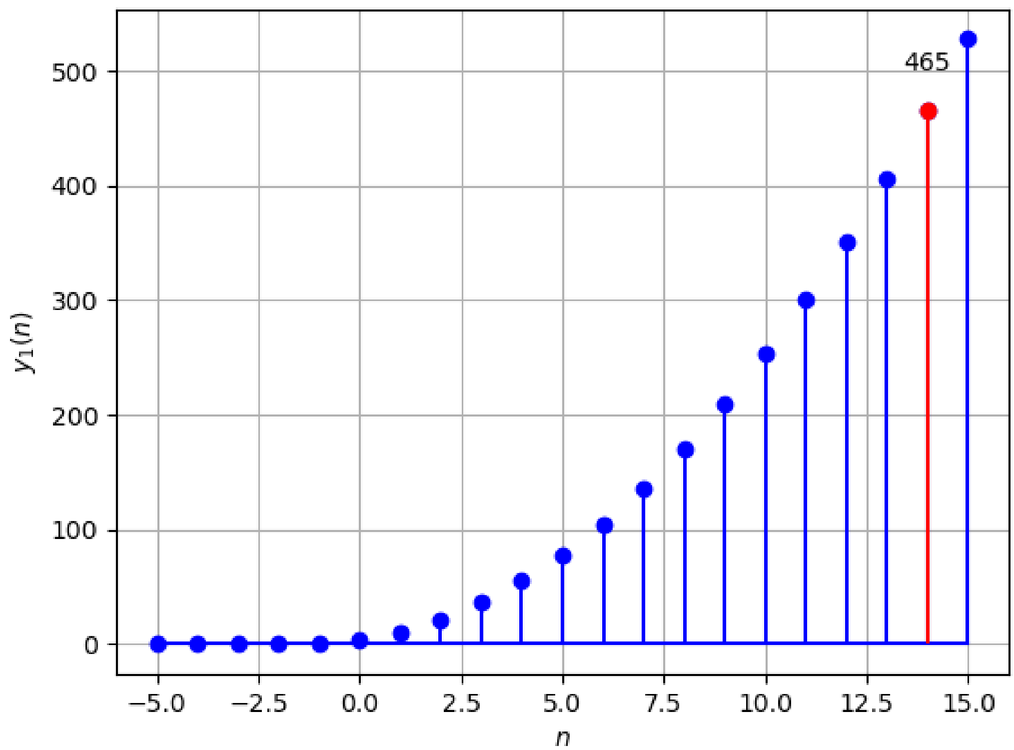



Figure 5.8: $x_1(n) = (2n^2 + 5n + 3)u(n)$

5.6 If the sum of $n$ terms of an AP is $(pn + qn^2)$, where $p$ and $q$ are constants, find the common difference. **Solution:**

| **Symbol** | **Value** | **Description** |
|---|---|---|
| $y(n)$ | $(pn + qn^2)$ | Sum of $n$ terms |
| $x(n)$ | | $n^{th}$ term of AP |
| $d$ | $x(n+1) - x(n)$ | Common Difference |

Table 5.6: Given Parameters

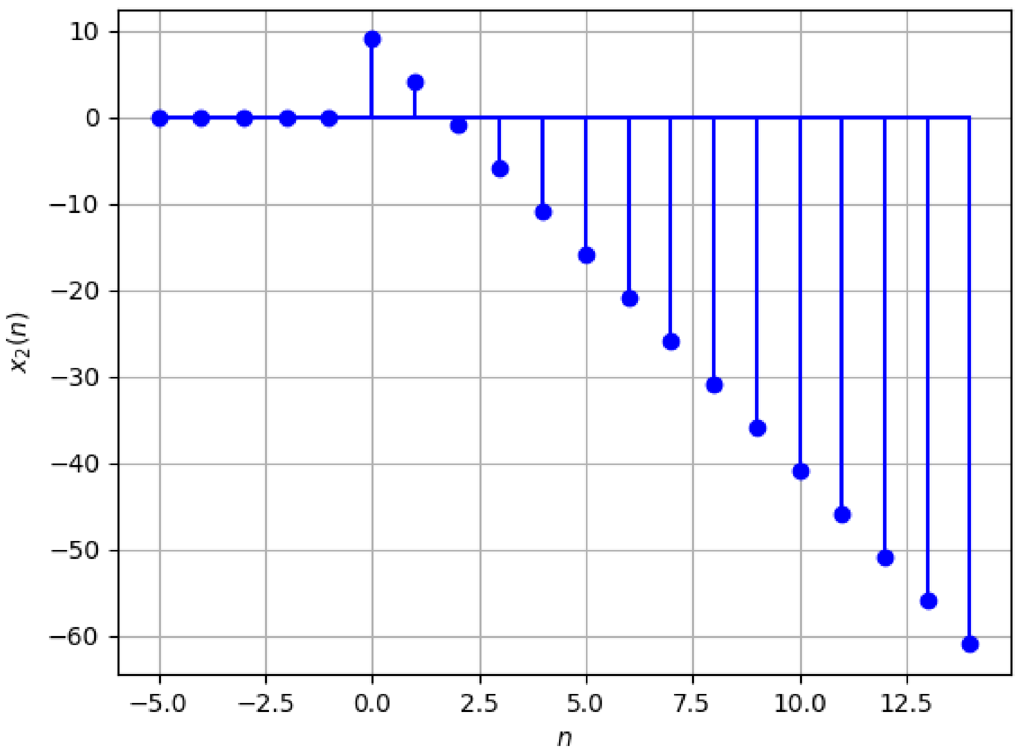



Figure 5.9: $x_2(n) = (9 - 5n)u(n)$

Sum of $n$ terms, as a discrete signal:

$$y(n) = (pn + qn^2)u(n) \tag{5.124}$$

Taking the $Z$-Transform,

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}\{|z| > 1\} \tag{5.125}$$

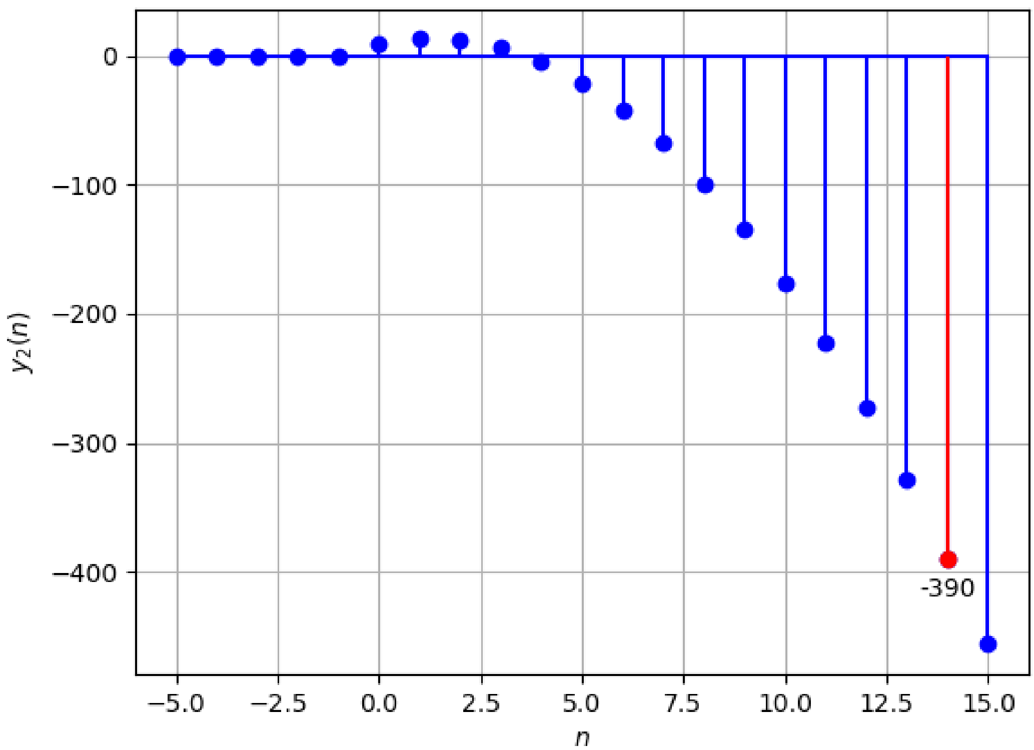



Figure 5.10: $x_2(n) = (-5n^2 + 13n + 18)u(n)$

(b) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \, \{|z| > 1\} \tag{5.126}$$

(c) $\mathcal{Z}\{n^2u(n)\}$

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \, \{|z| > 1\} \tag{5.127}$$

Taking the Z-Transform of (5.124) using (5.126) and (5.127)

$$Y(z) = p\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}\right) \tag{5.128}$$

Now,

$$y(n) = x(n) * u(n) \tag{5.129}$$

$$\implies Y(z) = X(z)U(z) \tag{5.130}$$

$$\implies X(z) = \frac{Y(z)}{U(z)} \tag{5.131}$$

Using (5.125) in (5.131),

$$X(z) = p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right) \tag{5.132}$$

Using contour integration for inverse Z-Transform:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \tag{5.133}$$

$$= \frac{1}{2\pi j}\oint_C \left[p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right)\right] z^{n-1}dz \tag{5.134}$$

Calculating the residues $R_1$ and $R_2$ at pole $z = 1$:

$$R_1 = \frac{1}{0!} \lim_{z \to 1} (z-1) \left( p \left( \frac{z^{-1}}{1 - z^{-1}} \right) \right) z^{n-1} \tag{5.135}$$

$$= p \tag{5.136}$$

$$R_2 = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left( (z-1)^2 q \left( \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \right) z^{n-1} \tag{5.137}$$

$$= q \lim_{z \to 1} \frac{d}{dz} \left( z^n + z^{n-1} \right) \tag{5.138}$$

$$= q(2n - 1) \tag{5.139}$$

$$\implies x(n) = R_1 + R_2 \tag{5.140}$$

$$= p + q(2n - 1) \tag{5.141}$$

Writing x(n) as a discrete signal we get:

$$x(n) = (p - q)u(n) + 2qnu(n) \tag{5.142}$$

To simplify, use $n = 0$:

$$y(0) = x(0) \tag{5.143}$$

$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{5.144}$$

$$\implies p = q \tag{5.145}$$

$\therefore$ (5.142) an be written as:

$$x(n) = 2qnu(n) \tag{5.146}$$

Common difference is given by:

$$d = x(n+1) - x(n) \tag{5.147}$$

$$= 2q \tag{5.148}$$

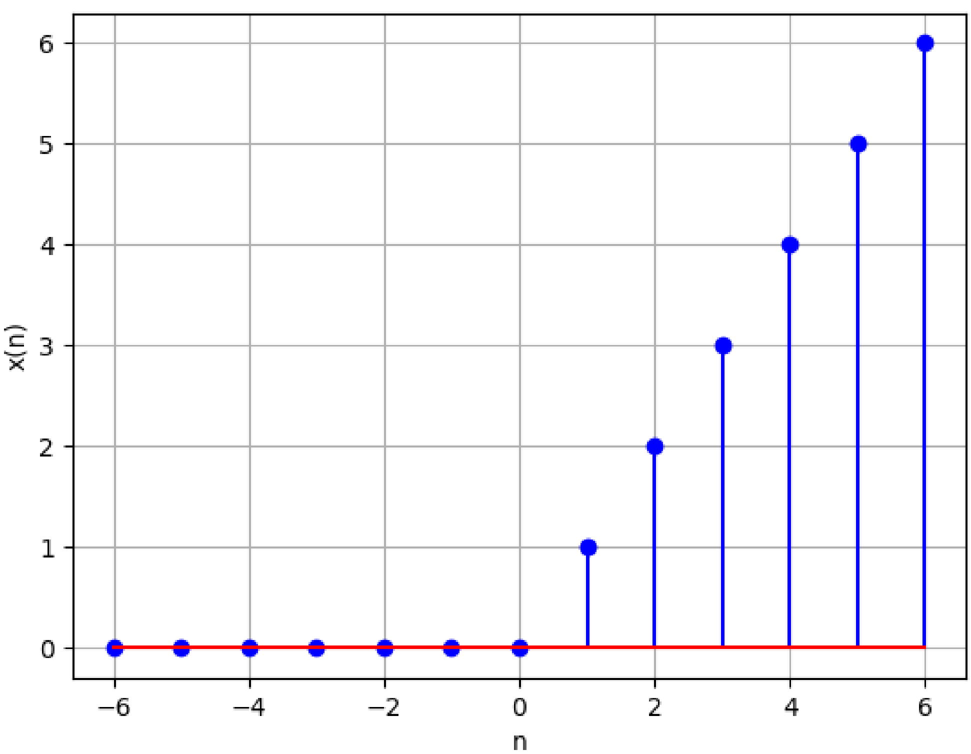



Figure 5.11: Plot of x(n) vs n for $p = q = 0.5$

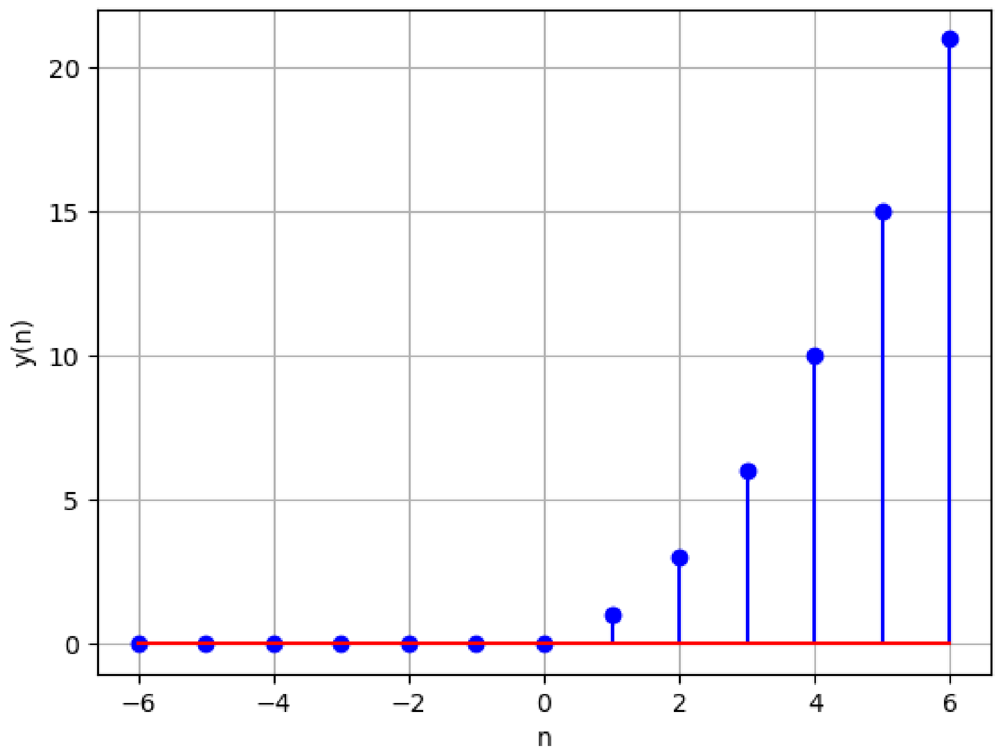



Figure 5.12: Plot of y(n) vs n for $p = q = 0.5$

5.7 Find the sum of the first 40 positive integers divisible by 6

**Solution:**

| Parameter | Description | Value |
|---|---|---|
| x(0) | First Term | 6 |
| d | Common Difference | 6 |

Table 5.7: Parameter Table 10.5.3.12

$$x(n) = (6 + 6n)(u(n)) \tag{5.149}$$

$$\implies X(z) = \frac{6}{1 - z^{-1}} + \frac{6z^{-1}}{(1 - z^{-1})^2} \quad \text{(B.10.6)} \tag{5.150}$$

$$\implies X(z) = \frac{6}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{5.151}$$

$$y(n) = x(n) * u(n) \tag{5.152}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.153}$$

$$= \frac{6}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{5.154}$$

Using contour integration to find the inverse Z-transform:

$$\implies y(39) = \frac{1}{2\pi j} \oint_C Y(z)\ z^{38}\ dz \tag{5.155}$$

$$= \frac{1}{2\pi j} \oint_C \frac{6z^{41}}{(z-1)^3}\ dz \tag{5.156}$$

We can observe that there is only a three times repeated pole at z=1,

$$\implies R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.157}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{6z^{41}}{(z-1)^3} \right) \tag{5.158}$$

$$= 3 \lim_{z \to 1} \frac{d^2}{dz^2} \left( z^{41} \right) \tag{5.159}$$

$$= 4920 \tag{5.160}$$

$$\therefore \boxed{y(39) = 4920} \tag{5.161}$$

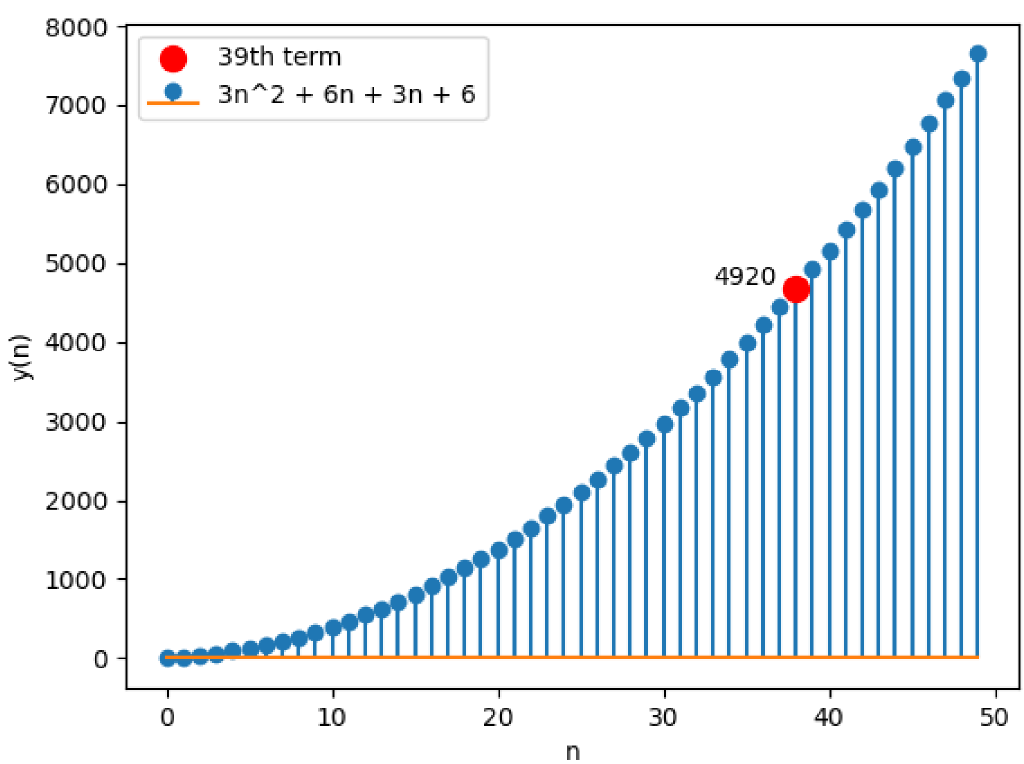



Figure 5.13: Plot of y$(n)$ $vs$ n

5.8 If the sum of certain number of terms in a AP 25,22,19,... is 116. Find the last term.

**Solution:**

| **Symbol** | **Value** | **Description** |
|---|---|---|
| $x(0)$ | 25 | first term of AP |
| $d$ | $-3$ | common difference |
| $x(n)$ | $(25-3n)u(n)$ | $n$-th term of AP |
| $y(n)$ | 116 | sum of terms |

Table 5.8: Input Parameters

$$x(n) = (25-3n)u(n) \tag{5.162}$$

Applying Z transform:

$$x(z) = \frac{25}{1-z^{-1}} - \frac{3z^{-1}}{(1-z^{-1})^2} \tag{5.163}$$

$$= \frac{25-28z^{-1}}{(1-z^{-1})^2} \tag{5.164}$$

Region of Convergence or R.O.C :

$$|z| > 1 \tag{5.165}$$

For AP, the sum of first n+1 terms can be written as :

$$y(n) = x(n) * u(n) \tag{5.166}$$

Applying Z transform on both sides

$$Y(z) = X(z)U(z) \tag{5.167}$$

$$= \frac{25}{(1-z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})^3} \tag{5.168}$$

Using contour integration to find inverse Z transform:

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z) z^{n-1} dz \tag{5.169}$$

$$= \frac{1}{2\pi j} \oint_C \left[ \frac{25}{(1-z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})^3} \right] z^{n-1}\, dz \tag{5.170}$$

The sum of the terms of the sequence is computed using the residue theorem, expressed as $R_i$, which represents the residue of the Z-transform at $z = 1$ for the expression $Y(z)$.

$$R_i = R_1 + R_2 \tag{5.171}$$

$R_1$ and $R_2$ are residues calculated at the poles of the Z-transform.

$$R_1 = \frac{1}{(2-1)!} \left. \frac{d(25z^{n+1})}{dz} \right|_{z=1} \tag{5.172}$$

$$= 25(n+1) \tag{5.173}$$

$$R_2 = \frac{1}{(3-1)!} \left. \frac{d^2(-3z^{n+1})}{dz^2} \right|_{z=1} \tag{5.174}$$

$$= \frac{-3}{2}(n+1)(n) \tag{5.175}$$

The sum of terms is given by $R_i$:

$$25(n+1) + \frac{-3}{2}n(n+1) = 116 \tag{5.176}$$

Solving the equation gives :

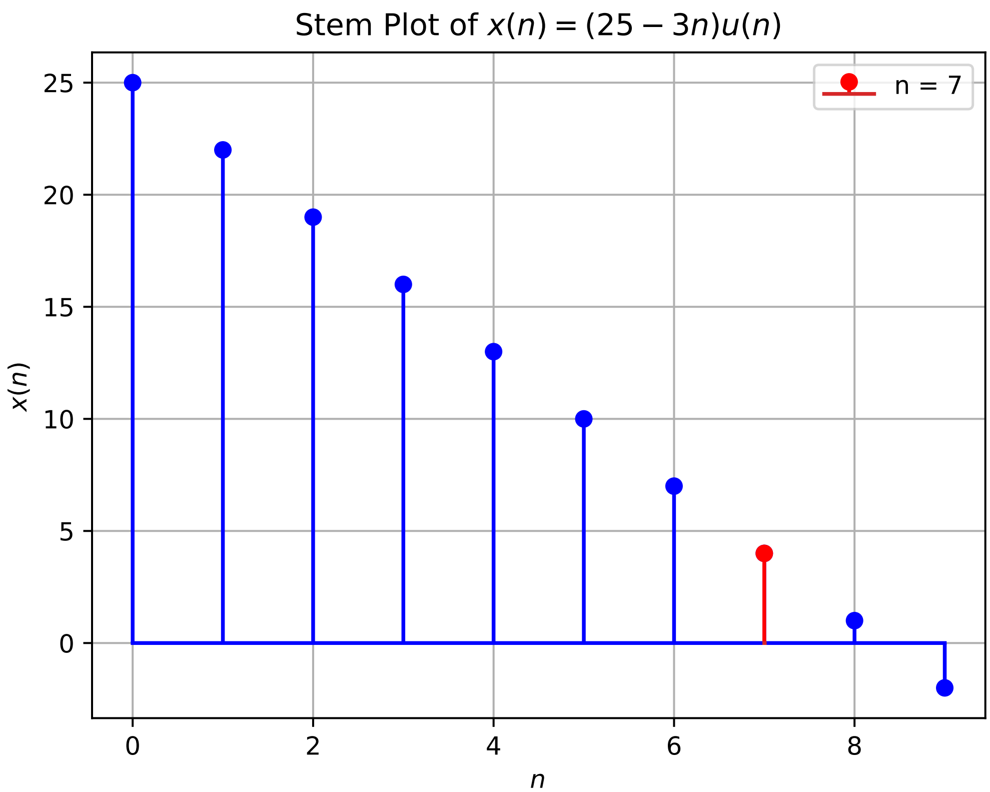
$$n = 7 \tag{5.177}$$

$$n = 8.667 \tag{5.178}$$

Since n can take only integer values, $n = 8.667$ is rejected. Upon substituting the value of n in equation (5.162):

$$x(7) = 4 \tag{5.179}$$

Hence the last term of the given AP is 4.




5.9 The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

**Solution:**

$$x(n) = (x(0) + nd)u(n) \tag{5.180}$$

$$x(l) = (17 + 9l)u(l) \tag{5.181}$$

Thus,

$$l = 37 \tag{5.182}$$

| Parameters in expression | | |
|---|---|---|
| Symbol | Description | Value |
| $x(n)$ | $n^{th}$ term of series | |
| $x(l)$ | Last($l^{th}$) term of series | 350 |
| $x(0)$ | Starting ($0^{th}$) term of series | 17 |
| d | Common difference of AP | 9 |

Table 5.9: Parameters

Using (B.10.6),

$$X(z) = \frac{(17 - 8z^{-1})}{(1 - z^{-1})^2}, \quad |z| > |1| \tag{5.183}$$

$$y(n) = x(n) * u(n) \tag{5.184}$$

$$\implies Y(z) = X(z)U(z) \tag{5.185}$$

$$= \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} \tag{5.186}$$

Using contour integral to find Z transform, we get

$$y(37) = \frac{1}{2\pi j} \oint_C Y(z) z^{36} dz \tag{5.187}$$

$$= \frac{1}{2\pi j} \oint_C \frac{(17 - 8z^{-1})}{(1 - z^{-1})^3} z^{36} dz \tag{5.188}$$

Now, using Cauchy's residual theorem and observing the fact that 3 repeated poles exist at z = 1,

$$
\begin{aligned}
R &= \frac{1}{(k-1)!}\lim_{z\to c}\frac{d^{k-1}}{dz^{k-1}}((z-c)^k f(z)) && (5.189)\\
&= \frac{1}{2!}\lim_{z\to 1}\frac{d^{k-1}}{dz^{k-1}}((z-1)^3\frac{(17-8z^{-1})}{(1-z^{-1})^3}z^{36}) && (5.190)\\
&= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}(17z^{39}-8z^{38}) && (5.191)\\
&= 6973 && (5.192)
\end{aligned}
$$

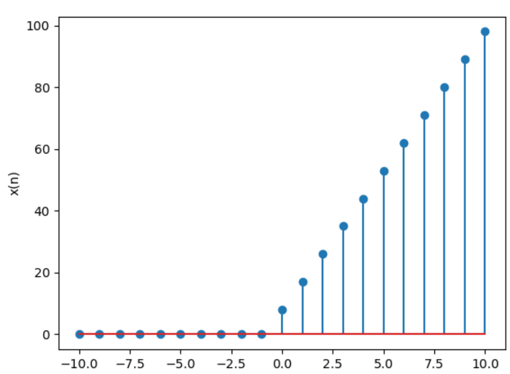



Figure 5.14: Stem Plot of $x(n)$ v/s n

5.10 A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.Each step has a rise of 1/4 m and a tread of 1/2 m. Calculate the total volume of concrete required to build the terrace. [Hint: Volume of concrete required to build the first step=

$$V = \frac{1}{4} \cdot \frac{1}{2} \cdot 50 \tag{5.193}$$

**Solution:** here

| parameter | description | value |
|---|---|---|
| $x(0)$ | first term | 6.25 |
| $d$ | common difference | 6.25 |
| $n$ | no of terms $-1$ | 14 |
| $x(n)$ | volume of $(n+1)$th step | $(6.25 + 6.25n)\, u(n)$ |

Table 5.10: formula parameters

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \qquad |z| > |1| \tag{5.194}$$

$$\implies X(z) = \left( \frac{6.25}{(1 - z^{-1})^2} \right) \qquad |z| > |1| \tag{5.195}$$

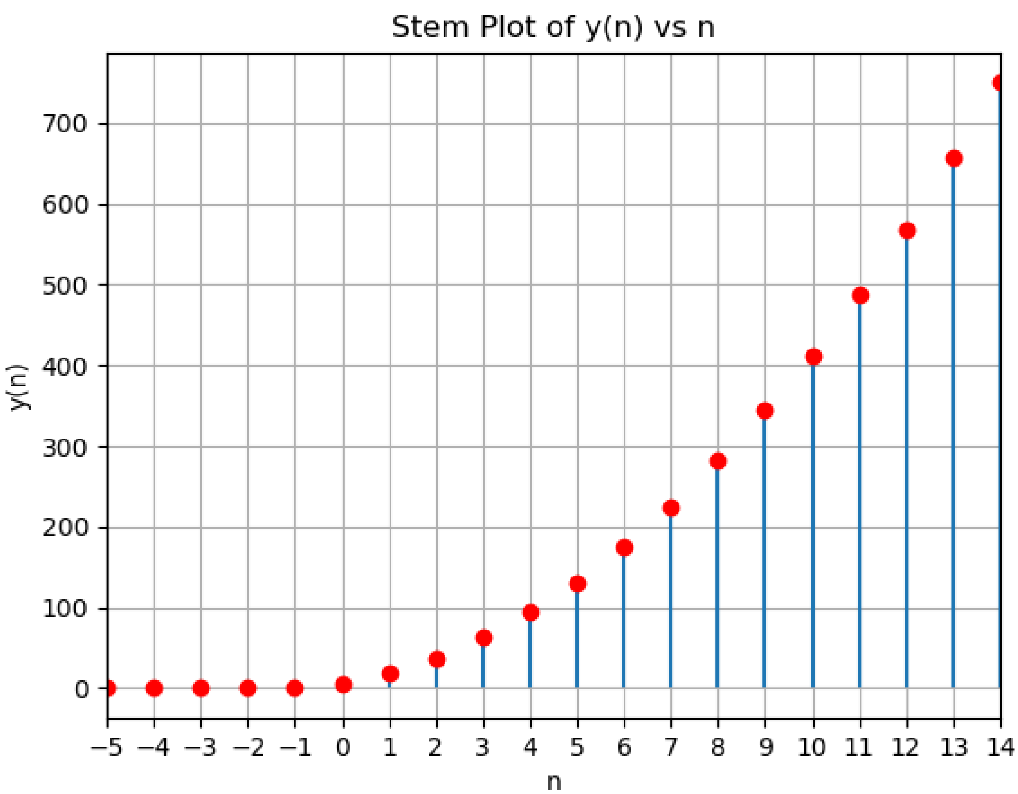



Figure 5.15: plot y($n$) vs n

$$y(n) = x(n) * u(n) \tag{5.196}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.197}$$

$$U(z) = \frac{1}{1 - z^{-1}} \quad |z| > |1| \tag{5.198}$$

$$Y(z) = \left(\frac{6.25}{1 - z^{-1}} + \frac{6.25z^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |1| \tag{5.199}$$

$$Y(z) = \frac{6.25z^3}{(z-1)^3} \quad |z| > |1| \tag{5.200}$$

contour integration to find inverse z transform

$$y(14) = \frac{1}{2\pi j}\oint_c Y(z)\, z^{13} dz \tag{5.201}$$

$$= \frac{1}{2\pi j}\oint_c \frac{6.25z^{16}}{(z-1)^3} \tag{5.202}$$

pole at 1 repeated 3 times

$$\implies m = 3 \tag{5.203}$$

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.204}$$

$$\implies y(14) = \frac{1}{(2!)}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{6.25z^{16}}{(z-1)^3}\right) \tag{5.205}$$

$$= 3.125\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{16}\right) \tag{5.206}$$

$$y(14) = 750 \tag{5.207}$$

5.11 Given a GP with a = 729 and $7^{th}$ term 64, find $S_7$.

**Solution:**

| **Parameter** | **Description** | **Value** |
|---|---|---|
| x(0) | First Term | 729 |
| r | Common Ratio | |
| x($n$) | $(n+1)^{th}$ Term | $x(0)\,r^n u(n)$ |
| x(6) | $7^{th}$ Term | 64 |
| y($k$) | Sum of first $(k+1)$ terms | |

Table 5.11: Parameter Table

from Table 5.11 :

$$x(6) = x(0)\,r^6 \tag{5.208}$$

$$\implies 64 = 729r^6 \tag{5.209}$$

$$\therefore r = \frac{2}{3} \tag{5.210}$$

using Table 5.11 and equation (5.210)

$$X(z) = \frac{729}{1 - \frac{2}{3}z^{-1}}, |z| > \frac{2}{3} \tag{5.211}$$

using Table 5.11 and equation (5.211)

$$Y(z) = \frac{729}{\left(1 - \frac{2}{3}z^{-1}\right)\left(1 - z^{-1}\right)} \tag{5.212}$$

$$= 2187\left(\frac{1}{1 - z^{-1}} - \frac{\frac{2}{3}}{1 - \frac{2}{3}z^{-1}}\right), |z| > 1 \tag{5.213}$$

Using contour integration for inverse z transform,

$$y(6) = \frac{1}{2\pi j}\oint Y(z)\, z^5 dz \tag{5.214}$$

Using equation (5.213) in (5.214) :

$$y(6) = \frac{1}{2\pi j}\left(\oint \frac{2187z^6}{z - 1}dz - \oint \frac{1458z^6}{z - \frac{2}{3}}dz\right) \tag{5.215}$$

$$\frac{1}{2\pi j}\left(\oint \frac{2187z^6}{z - 1}dz\right) = 2187 \tag{5.216}$$

$$\frac{1}{2\pi j}\left(\oint \frac{1458z^6}{z - \frac{2}{3}}dz\right) = 128 \tag{5.217}$$

using equations (5.216) and (5.217) in (5.215):

$$y(6) = 2187 - 128 \tag{5.218}$$

$$= 2059 \tag{5.219}$$

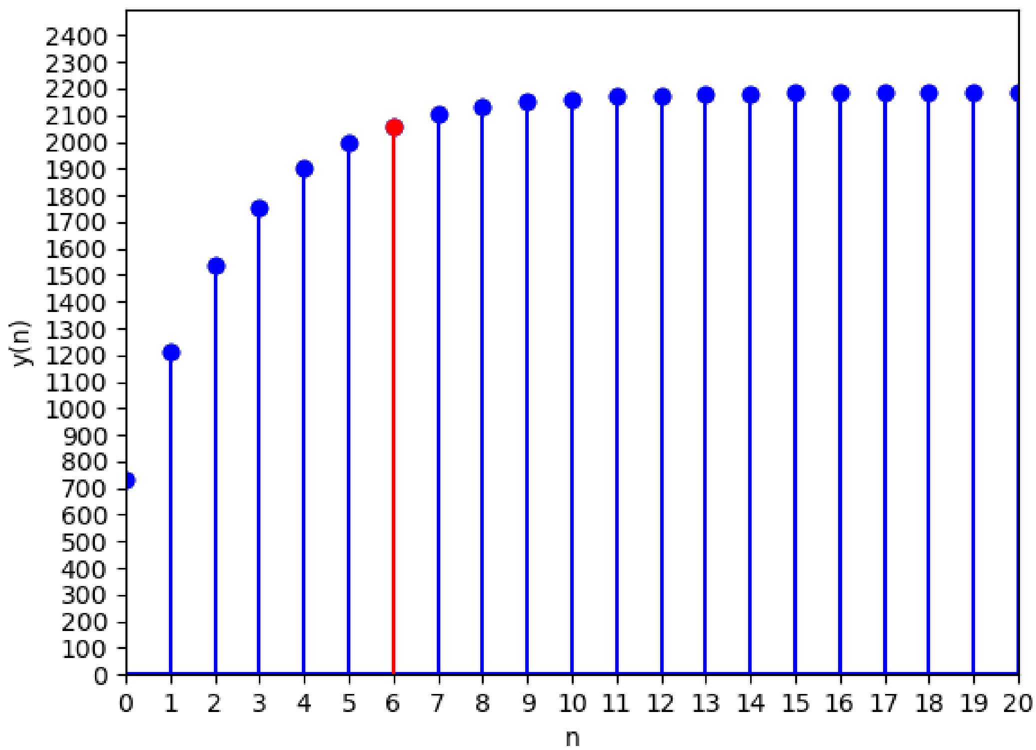



Figure 5.16: Plot of $y(n)$

5.12 Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

**Solution:**

$$x(n) = (105 + 5n)(u(n)) \tag{5.220}$$

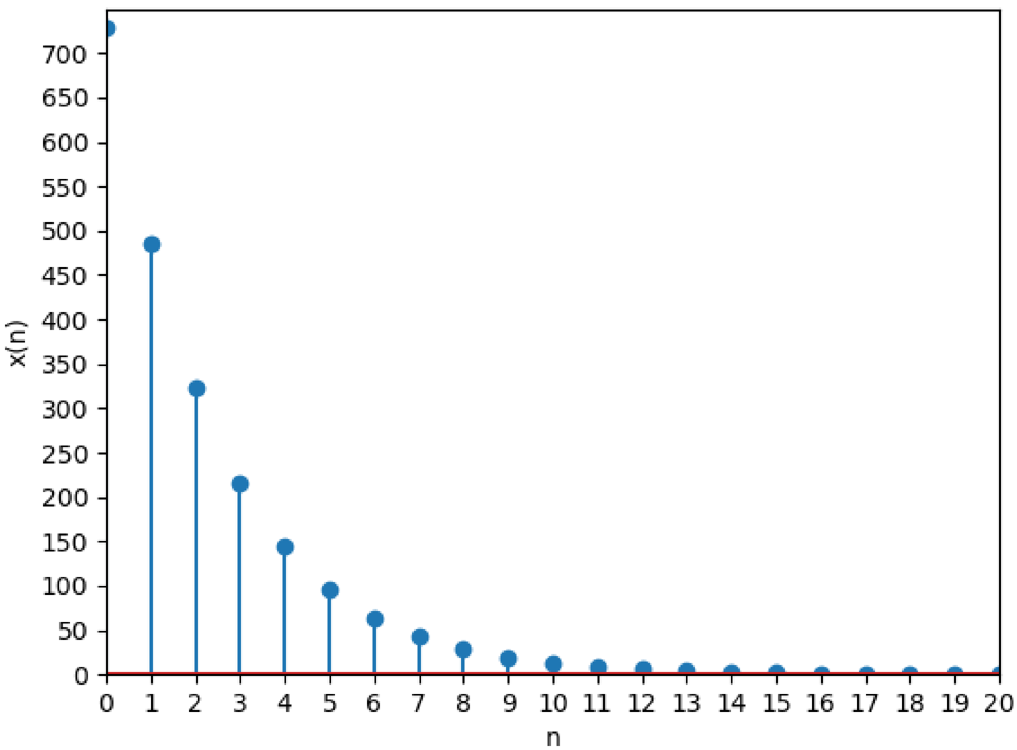



Figure 5.17: Plot of $x\,(n)$

| Parameter | Description | Value |
|---|---|---|
| $x(0)$ | First Term | 105 |
| $d$ | Common Difference | 5 |
| $n$ | Total terms | 179 |
| $x(178)$ | Last Term | 995 |
| $m$ | No of poles | 3 |

Table 5.12: Given Parameters

On taking Z transform

$$X(z) = \frac{x(0)}{(1-z^{-1})} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{5.221}$$

$$= \frac{105}{1-z^{-1}} + \frac{5z^{-1}}{(1-z^{-1})^2} \tag{5.222}$$

$$\implies X(z) = \frac{105-100z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{5.223}$$

$$y(n) = x(n) * u(n) \tag{5.224}$$

$$\implies Y(z) = X(z)\,U(z) \tag{5.225}$$

$$= \frac{105-100z^{-1}}{(1-z^{-1})^2}\frac{1}{(1-z^{-1})} \tag{5.226}$$

$$= \frac{105-100z^{-1}}{(1-z^{-1})^3} \quad |z| > 1 \tag{5.227}$$

Using contour integration to find the inverse Z-transform:

$$\implies y(178) = \frac{1}{2\pi j}\oint_C Y(z)\; z^{177}\; dz \tag{5.228}$$

$$= \frac{1}{2\pi j}\oint_C \frac{\left(105-100z^{-1}\right) z^{177}}{(1-z^{-1})^3}\; dz \tag{5.229}$$

We can observe that there is only a 3 times repeated pole at $z = 1$,

$$\implies R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.230}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{\left(105 - 100z^{-1}\right) z^{180}}{(z-1)^3} \right) \tag{5.231}$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} \left(105z^{180} - 100z^{179}\right) \tag{5.232}$$

$$= 98450 \tag{5.233}$$

$$\therefore y\,(178) = 98450 \tag{5.234}$$

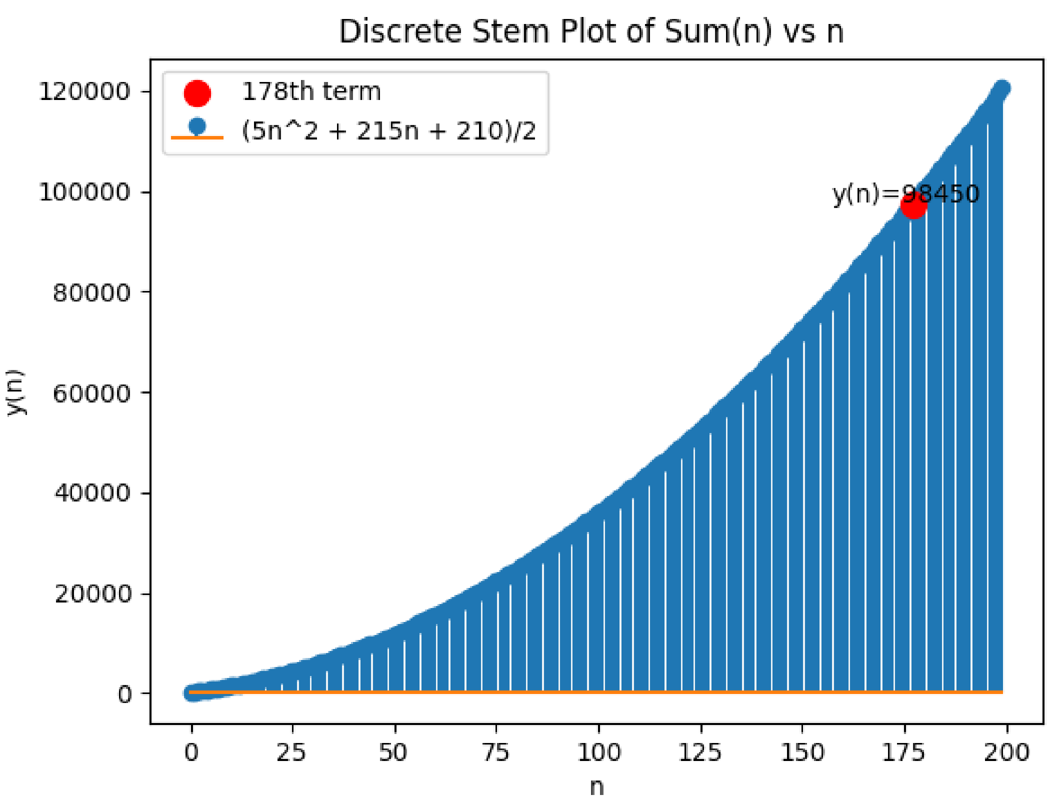



Figure 5.18: Plot of $x(n)$ $vs$ $n$

5.13 Find the sum of odd numbers between 0 and 50.

**Solution:**

5.14 A ladder has rungs 25cm apart.The rungs decrease uniformly in length from 45cm at the bottom to 25cm at the top.If the top and bottom rungs are 2 and 1/2 meter apart.what is length of wood required for the rungs?

5.15 Q2) The sum of the third and the seventh terms of AP is 6 and their product is 8.

Find the sum of first sixteen terms of the AP

**Solution:**

5.16 A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii $0.5cm$, $1.0cm$, $1.5cm$, $2.0cm$, . . . What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)

**Solution:**

5.17 In an A.P., if the $p$-th term is $\frac{1}{q}$ and $q$-th term is $\frac{1}{p}$, prove that the sum of the first $pq$ terms is $\frac{1}{2}(pq+1)$, where $p \neq q$. **Solution:**

5.18 Find the sum to indicated number of terms in each of the geometric progressions in $0.15, 0.015, 0.0015 \ldots 20 terms$.

**Solution:**

| Parameter | Description | Value |
|---|---|---|
| $n$ | No. of terms in the G.P | 20 |
| $x(0)$ | first term in the G.P | 0.15 |
| $r$ | common ratio in the G.P | 0.1 |

Table 5.13: Variables and their descriptions

$$x(n) = x(0)r^n \tag{5.235}$$

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \qquad |z| > |r| \tag{5.236}$$

$$U(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1 \tag{5.237}$$

$$y(n) = x(n) * u(n) \tag{5.238}$$

$$Y(z) = X(z)U(z) \tag{5.239}$$

$$= \left(\frac{0.15}{1 - 0.1z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > 1 \tag{5.240}$$

Using Contour integration

$$y(20) = \frac{1}{2\pi j} \oint_C \frac{0.15z^2}{(z-1)(z-0.1)} z^{19} dz \tag{5.241}$$

$$= \frac{1}{2\pi j} \oint_C \frac{0.15}{0.9} \left( \frac{1}{z-1} - \frac{1}{z-0.1} \right) z^{21} dz \tag{5.242}$$

$$= \frac{1}{6} \left( \left( \lim_{z \to 1} \frac{z^{21}}{z-1}(z-1) \right) - \left( \lim_{z \to 0.1} \frac{z^{21}}{z-0.1}(z-0.1) \right) \right) \tag{5.243}$$

$$= \frac{1}{6}(1 - 0.1^{21}) \tag{5.244}$$

$$= 0.16667 \tag{5.245}$$

$\therefore$ Sum of 20 terms of the given GP is 0.16667

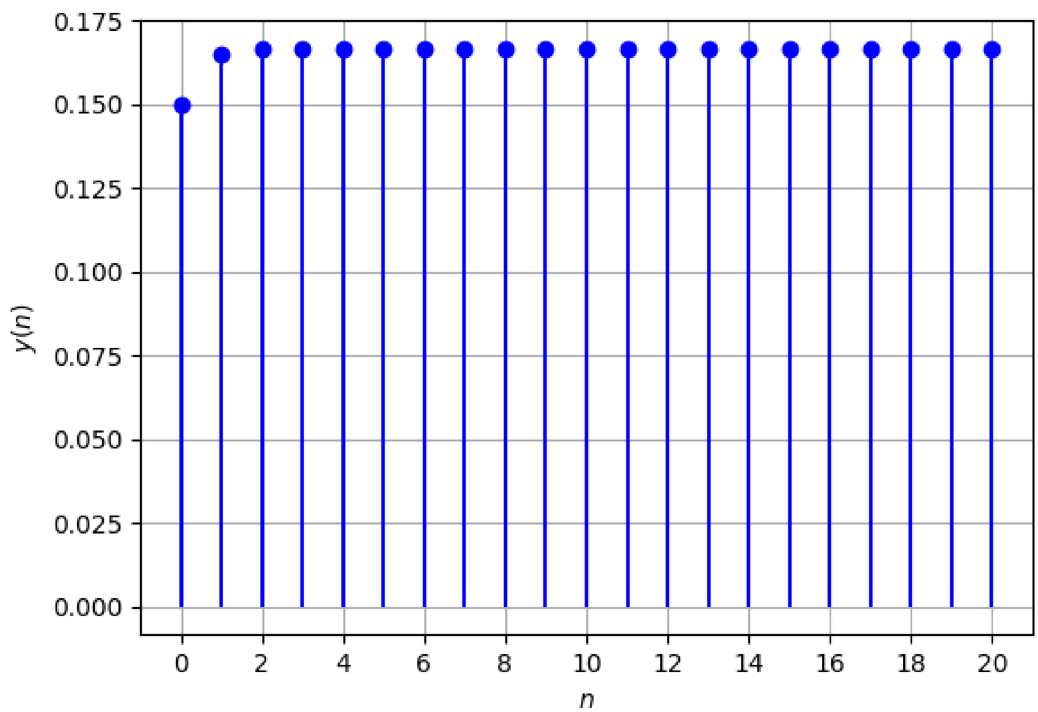



Figure 5.18: Stem plot of y(n)

5.19 A man deposited Rs 10000 in a bank at the rate of 5% simple interest anually. Find the amount in $15^{\text{th}}$ year since he deposited the amount and also calculate the total amount after 20 years.

**Solution:**

| **Parameter** | **Value/Formula** | **description** |
|---|---|---|
| $x(0)$ | Rs.10000 | Total amount deposited |
| $r$ | 5 | Rate of interest |
| $x(n)$ | $(x(0) + nd)\,u(n)$ | amount at the start of $(n+1)^{\text{th}}$ year |
| $d$ | ? | common difference |

Table 1: Input data

$$\text{Interest in one year} = \frac{10000 \times 5 \times 1}{100} \tag{5.246}$$

$$d = 500 \tag{5.247}$$

From (5.247) and Table 1:

$$x(n) = (10000 + 500n)u(n) \tag{5.248}$$

From (B.8.2)

$$X(z) = \frac{10000}{1 - z^{-1}} + \frac{500z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{5.249}$$

Amount in $15^{\text{th}}$ year is

$$x(14) = x(0) + 14 \times d \tag{5.250}$$

$$\implies x(14) = 17000 \tag{5.251}$$

Total amount after 20 years is

$$x(20) = x(0) + 20 \times 500 \tag{5.252}$$

$$\implies x(20) = 20000 \tag{5.253}$$

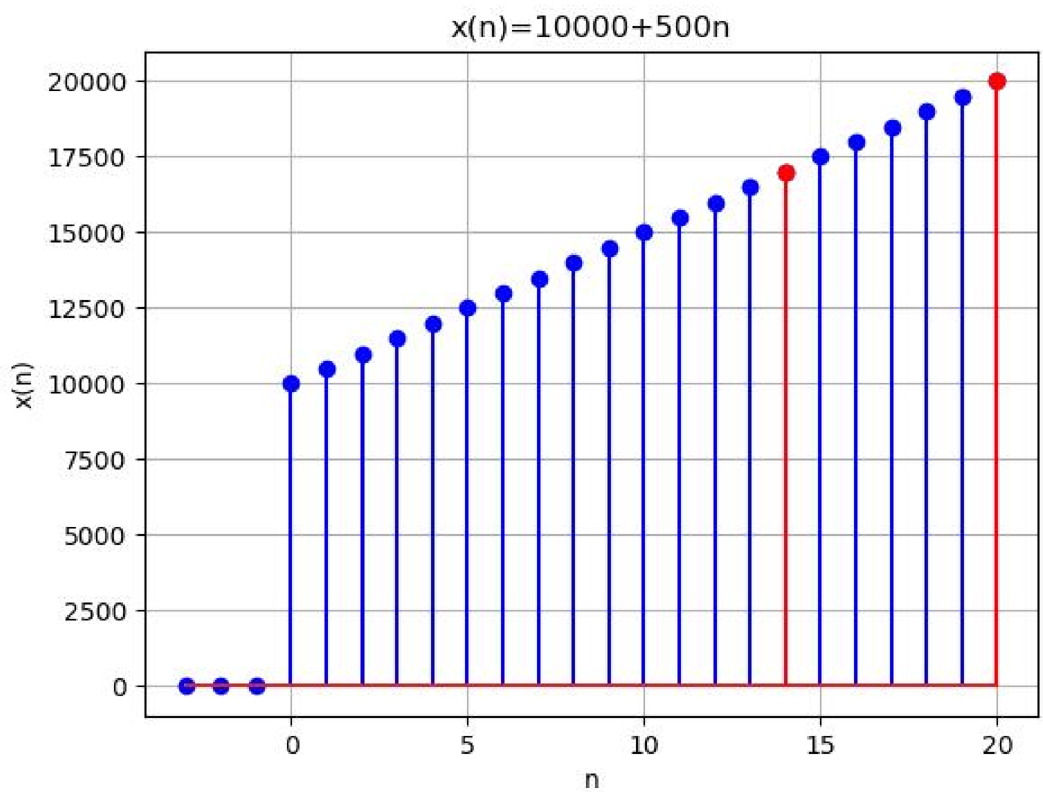



Figure 1: graph for $x(n) = 10000 + 500n$

5.20 The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered is equal to the sum of the numbers of the houses following it. Find this value of $x$.

Hint:$S_{x-1} = S_{49} - S_x$

**Solution:**

| Parameter | Value | Description |
|---|---|---|
| $x(0)$ | 1 | First house |
| $d$ | 1 | Common difference |
| $x(n)$ | $(n+1)\,u(n)$ | $(n+1)\,th$ house |
| $y(n)$ | $\left(\frac{n+1}{2}\right)(n+2)\,u(n)$ | Sum of $n+1$ number of houses. |
| $x_2(n)$ | $(49-n)\,u(n)$ | $(n+1)\,th$ house from last house |
| $y_2(n)$ | $\left[49n - \left(\frac{n}{2}\right)(n+1)\right]u(n)$ | Sum of $n+1$ houses from last house. |

Table 5.20: Input Parameters

For an AP:

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{5.254}$$

$$\implies X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2} \tag{5.255}$$

$$= \frac{1}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.256}$$

$$\therefore y\left(n\right)=\frac{\left(n+1\right)}{2}\left(n+2\right) \tag{5.257}$$

$$y\left(x-2\right)=y\left(n-1\right)-y\left(x-1\right) \tag{5.258}$$

From Table 5.20:

$$\left(\frac{x-1}{2}\right)x = \frac{n}{2}(n+1) - \frac{x}{2}(x+1) \tag{5.259}$$

$$(x-1) + x(x+1) = n(n+1) \tag{5.260}$$

$$2x^2 = n(n+1) \tag{5.261}$$

$$x = \sqrt{\frac{n}{2}(n+1)} \tag{5.262}$$

$$x = 35 \tag{5.263}$$

**Result Confirmation:**

To prove:

$$y(33) = y_2(13) \tag{5.264}$$

**LHS:**

$$y(n) = x(n) * u(n) \tag{5.265}$$

$$\implies Y(z) = X(z) \times U(z) \tag{5.266}$$

$$Y(z) = \left(\frac{1}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{5.267}$$

$$= \frac{1}{(1-z^{-1})^3}, \quad |z| > 1 \tag{5.268}$$

$$\tag{5.269}$$

Using Contour Integration to find inverse Z-transform,

$$y(33) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{32}\, dz \tag{5.270}$$

$$= \frac{1}{2\pi j}\oint_C \frac{z^{32}}{(1-z^{-1})^3}\, dz \tag{5.271}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{5.272}$$

$$= \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{35}\right) \tag{5.273}$$

$$= 595 \tag{5.274}$$

**RHS:**

From Table 5.20:

$$X_2(z) = \frac{49 - 50z^{-1}}{(1 - z^{-1})^2} \tag{5.275}$$

$$y_2(n) = x_2(n) * u(n) \tag{5.276}$$

$$\implies Y_2(z) = X_2(z) \times U(z) \tag{5.277}$$

$$y_2(z) = \frac{49 - 50z^{-1}}{(1 - z^{-1})^3} \tag{5.278}$$

$$\tag{5.279}$$

Using Contour Integration to find inverse Z-transform,

$$y_2(13) = \frac{1}{2\pi j} \oint_C Y(z)\ z^{12}; dz \tag{5.280}$$

$$= \frac{1}{2\pi j} \oint_C \frac{49 - 50z^{-1}}{(1 - z^{-1})^3} \left(z^{12}\right)\ dz \tag{5.281}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.282}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left(49z^{15} - 50z^{14}\right) \tag{5.283}$$

$$= 49.15.14 - 50.14.13 \tag{5.284}$$

$$= 595 \tag{5.285}$$

$$LHS = RHS$$

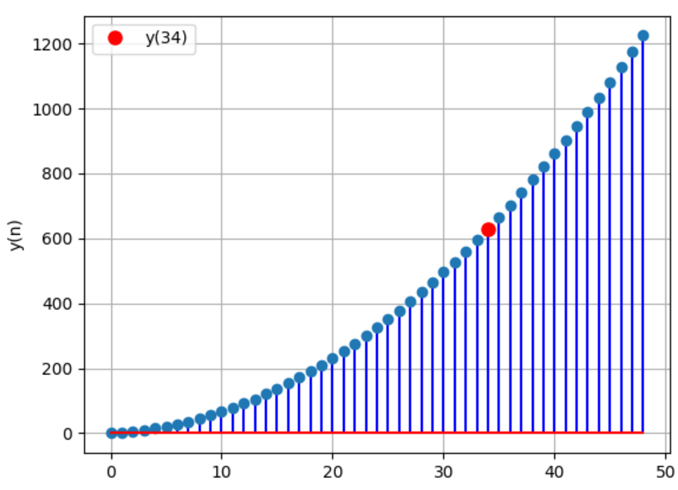



Figure 5.20: Plot y(n) vs n

5.21 Find the sum of all two digit numbers which when divided by 4,yields 1 as reminder?

**Solution:**

# Chapter 6

# Laplace Transform

6.0.1 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of

(a) The spring constant $K$

(b) The damping constant $b$ for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

**Solution:** The parameters are :

| **Parameter** | **Value(SI)** | **Description** |
|---|---|---|
| $x_0$ | 0.15 | Initial spring compression |
| $m$ | 750 | Mass |
| $g$ | 9.8 | Gravitational acc |
| $k$ | $mg/x_0$ | Spring constant |
| $b$ | | Damping constant |

Table 6.1: Input Parameters

| **Parameter** | **Value(SI)** | **Description** |
|---|---|---|
| $x$ | | Spring Extension |
| $F_1$ | $kx$ | Spring Force |
| $F_2$ | $b\frac{dx}{dt}$ | Damping Force |
| $s$ | | Complex Frequency |
| $s_1, s_2$ | | Values of s |

Table 6.2: Intermediate Parameters

Initially the automobile is in rest, so we can use,

$$mg = kx_0 \tag{6.1}$$

$$\implies k = \frac{mg}{x_0} \tag{6.2}$$

Now, as the oscillation begins, from the FBD, we have net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{6.3}$$

$$\implies -m\frac{d^2x(t)}{dt^2} = kx(t) + b\frac{dx(t)}{dt} + mgu(t) \tag{6.4}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = -gu(t) \tag{6.5}$$

Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{b}{m}sX(s) + \frac{k}{m}X(s) = -\frac{g}{s} \tag{6.6}$$

$$\implies X(s) = -\frac{g}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \tag{6.7}$$

$$\implies X(s) = -\frac{g}{s(s-s_1)(s-s_2)} \tag{6.8}$$

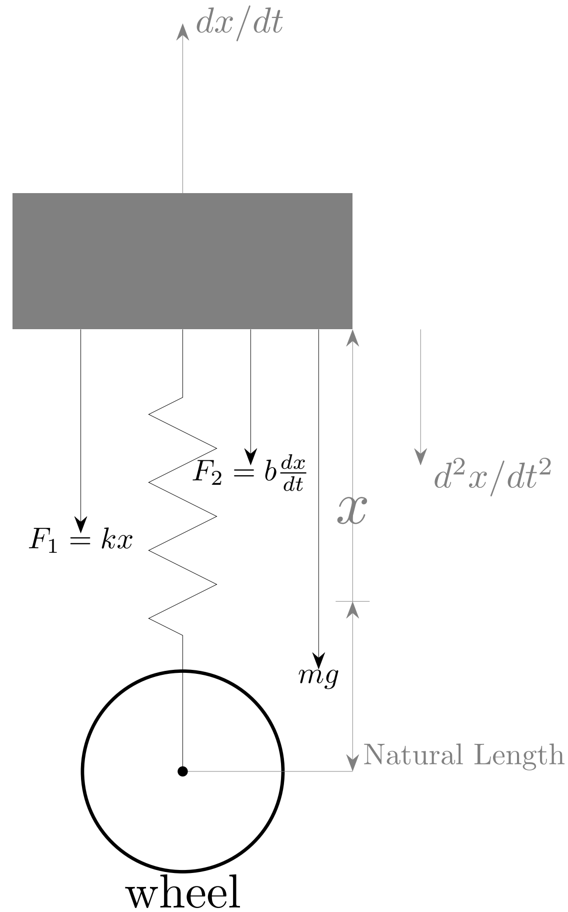



Figure 6.1: FBD of the damped oscillation system

Where

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{6.9}$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{6.10}$$

From (6.8) we get,

$$\Longrightarrow X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2(s-s_2)} - \frac{1}{s_1(s-s_1)}\right] + \frac{g}{s_1 s_2}\left(\frac{1}{s}\right) \tag{6.11}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = \frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2}e^{s_2 t} - \frac{1}{s_1}e^{s_1 t}\right]u(t) \tag{6.12}$$

$$\Longrightarrow x(t) = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bt/2m}u(t) \sin\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}{gb}\right)\right) + \frac{mg}{k}u(t) \tag{6.13}$$

(Substituting the values of $s_1$ and $s_2$ from (6.9) and (6.10))

From (6.13) we have the amplitude after one time period $T$,

$$\frac{1}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bT/2m} \tag{6.14}$$

$$\Longrightarrow e^{\pi b/\sqrt{mk}} = 2 \tag{6.15}$$

$$\Longrightarrow b = \frac{\sqrt{mk}\ln 2}{\pi} \tag{6.16}$$

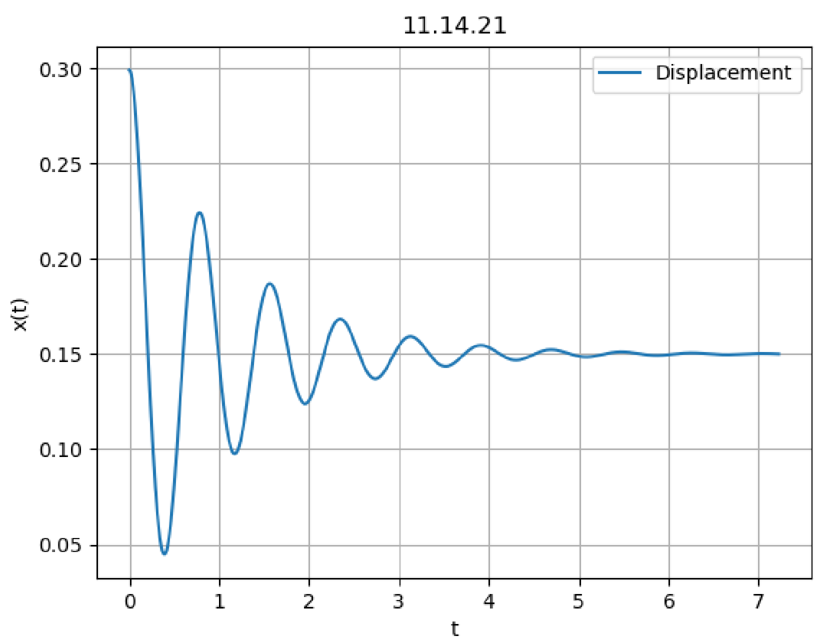



Figure 6.2: Displacement Vs. Time Graph

# Chapter 7

# Systems

7.0.1 A simple pendulum of length $l$ and having a bob of mass $M$ is suspended in a car. The car is moving in a circular track of radius $R$ with a uniform speed $v$. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

7.0.2 A spring balance has a scale that that reads from 0 to 50 kg.The length of the scale is 20 cm.A body is suspended from this balance,when displaced and released, oscillates with a period of 0.6 s.What is weight of the body?

# Appendix A

# Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{A.1.1}$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

A.3 If

$$x(n) = 0, \quad n < 0, \tag{A.3.1}$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{A.3.2}$$

# Appendix B

# Z-transform

B.1 The $Z$-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n) z^{-n} \tag{B.1.1}$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \tag{B.2.1}$$

$$P(z) = P_1(z) P_2(z) \tag{B.2.2}$$

B.3

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zX'(z) \tag{B.3.1}$$

From (B.3.1)

$$\implies nu\left(n\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}},|z|>1 \tag{B.3.2}$$

$$\implies n^{2}u\left(n\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^{3}},|z|>1 \tag{B.3.3}$$

$$\implies n^{3}u\left(n\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^{4}},|z|>1 \tag{B.3.4}$$

$$\implies n^{4}u\left(n\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^{5}} \tag{B.3.5}$$

$$\text{where}\,|z|>1$$

B.4

$$x(n-k)\overset{\mathcal{Z}}{\longleftrightarrow}z^{-k}X(z) \tag{B.4.1}$$

Using (B.4.1):

$$nu\left(n-1\right)\overset{\mathcal{Z}}{\longleftrightarrow}z\frac{2z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.2}$$

Now ,

$$\frac{\left(n-1\right)}{2}u\left(n-2\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.3}$$

$$\frac{\left(n-1\right)\left(n-2\right)}{6}u\left(n-3\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-3}}{\left(1-z^{-1}\right)^{3}} \tag{B.4.4}$$

$$\vdots$$

$$\frac{\left(n-1\right)\left(n-2\right)\ldots\left(n-k+1\right)}{\left(k-1\right)!}u\left(n-k\right)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-k}}{\left(1-z^{-1}\right)^{k}} \tag{B.4.5}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^{2}}\right]=(n-1)\,u\,(n-1) \tag{B.4.6}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^{3}}\right]=\frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{B.4.7}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^{4}}\right]=\frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{B.4.8}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^{5}}\right]=\frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{B.4.9}$$

$$u\,(n-1)$$

B.5 For a Geometric progression

$$x\,(n)=x\,(0)\,r^{n}u\,(n)\,, \tag{B.5.1}$$

$$\Longrightarrow X\,(z)=\sum_{n=-\infty}^{\infty}x\,(n)\,z^{-n}=\sum_{n=0}^{\infty}x\,(0)\,r^{n}z^{-n} \tag{B.5.2}$$

$$=\sum_{n=0}^{\infty}x\,(0)\left(rz^{-1}\right)^{n} \tag{B.5.3}$$

$$=\frac{x\,(0)}{1-rz^{-1}},\quad |z|>|r| \tag{B.5.4}$$

B.6 Substituting $r=1$ in (B.5.4),

$$u(n)\overset{\mathcal{Z}}{\longleftrightarrow}U(z)=\frac{1}{1-z^{-1}},\quad |z|>1 \tag{B.6.1}$$

B.7 From (B.3.1) and (B.6.1),

$$nu(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})^{2}},\quad |z|>1 \tag{B.7.1}$$

B.8 For an AP,

$$x(n) = [x(0) + nd]\, u(n) = x(0)u(n) + dnu(n) \tag{B.8.1}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.8.2}$$

upon substituting from (B.6.1) and (B.7.1).

B.9 From (A.3.2), the sum to $n$ terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{B.9.1}$$

where $x(n)$ is defined in (B.5.1). From (B.2.2), (B.5.4) and (B.6.1),

$$Y(z) = X(z)\, U(z) \tag{B.9.2}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right) \left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > |1| \tag{B.9.3}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{B.9.4}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1} \left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{B.9.5}$$

using partial fractions. Again, from (B.5.4) and (B.6.1), the inverse of the above can be expressed as

$$y(n) = x(0) \left(\frac{r^{n+1} - 1}{r - 1}\right) u(n) \tag{B.9.6}$$

B.10 For the AP $x(n)$, the sum of first $n+1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{B.10.1}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{B.10.2}$$

$$= x(n) * u(n) \tag{B.10.3}$$

Taking the Z-transform on both sides, and substituting (B.8.2) and (B.6.1),

$$Y(z) = X(z)U(z) \tag{B.10.4}$$

$$\implies Y(z) = \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}\right)\frac{1}{1-z^{-1}} \quad |z| > 1 \tag{B.10.5}$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \tag{B.10.6}$$

B.11 From (B.4.1) and (B.7.1),

$$(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \tag{B.11.1}$$

From (B.11.1) and (B.3.1),

$$n(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \tag{B.11.2}$$

B.12 Taking the inverse Z-transform of (B.10.6),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{B.12.1}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\}u(n) \tag{B.12.2}$$

# Appendix C

# Contour Integration

C.1

$$x(n) \xrightarrow{\text{Z}} X(z) \tag{C.1.1}$$

$$\implies X(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \tag{C.1.2}$$

Multiplying both side with $z^{k-1}$ and integrating on a contour integral enclosing the region of convergence. Where $C$ is a counter-clockwise closed contour in region of convergence.

$$\frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz = \frac{1}{2\pi j}\oint_C \sum_{k=-\infty}^{\infty} x(k)\, z^{-n+k-1} dz \tag{C.1.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, \frac{1}{2\pi j}\oint_C z^{-n+k-1}.dz \tag{C.1.4}$$

From cauchy's integral theorem

$$\frac{1}{2\pi j}\oint_C z^{-k} dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \tag{C.1.5}$$

$$= \delta(1-k) \tag{C.1.6}$$

So eq (C.1.4) becomes

$$\frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz = \sum_{k=-\infty}^{\infty} x(k)\,\delta(k-n) \tag{C.1.7}$$

$$\implies x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.8}$$

Contour integrals like (C.1.8) can be evaluated using Cauchy's residue theorem.

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.9}$$

$$= \sum \left[\text{Residue of } X(z)\, z^{n-1} \text{ at poles inside } C\right] \tag{C.1.10}$$

C.2 Question: Find the sum of n terms of an AP where common difference $= d$ using Contour Integration.

**Solution:**

By performing inverse Z transform on S($z$) using contour integration

$$s(n) = \frac{1}{2\pi j}\oint_C S(z)\ z^{n-1}\ dz \tag{C.2.1}$$

$$s(n) = \frac{1}{2\pi j}\oint_C \left(\frac{x(0)\, z^{n-1}}{\left(1-z^{-1}\right)^2} + \frac{dz^{n-2}}{\left(1-z^{-1}\right)^3}\right)\ dz \tag{C.2.2}$$

For $R_1$ we can observe that the pole has been repeated twice.

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{C.2.3}$$

$$R_1 = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{x(0)\, z^{n+1}}{(z-1)^2}\right) \tag{C.2.4}$$

$$= x(0)\,(n+1)\lim_{z\to 1}\left(z^n\right) \tag{C.2.5}$$

$$= x(0)\,(n+1) \tag{C.2.6}$$

For $R_2$ we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{dz^{n+1}}{(z-1)^3} \right) \tag{C.2.7}$$

$$= \frac{d(n+1)}{2} \lim_{z \to 1} \frac{d}{dz} (z^n) \tag{C.2.8}$$

$$= \frac{d(n+1)(n)}{2} \lim_{z \to 1} \left( z^{n-1} \right) \tag{C.2.9}$$

$$= \frac{d(n)(n+1)}{2} \tag{C.2.10}$$

$$\implies R = R_1 + R_2 \tag{C.2.11}$$

Using (C.2.6) and (C.2.10) in (C.2.11)

$$R = x(0)(n+1) + \frac{d(n)(n+1)}{2} \tag{C.2.12}$$

Finally,

$$s(n) = x(0)(n+1)u(n) + d\left( \frac{n(n+1)}{2} \right) u(n) \tag{C.2.13}$$

$$= \frac{n+1}{2} (2x(0) + nd) u(n) \tag{C.2.14}$$

C.3 Question: Find the sum of n terms of GP where common ratio is $r$ using Contour Integration.

**Solution:**

| Symbol | Value | Description |
|---|---|---|
| $x(n)$ | $x(0)r^n u(n)$ | $n^{th}$ $n^{th}$ term of gp G.P |
| $x(0)$ | $x(0)$ | $1^{st}$ term of the G.P |
| $d$ | $r$ | Common ratio |
| $s(n)$ | $\sum_{k=0}^{n} x\left(k\right)$ | Sum of n terms of GP |

$$X\left(z\right) = \sum_{n=-\infty}^{\infty} x\left(n\right) z^{-n} \tag{C.3.1}$$

$$= \sum_{n=-\infty}^{\infty} x(0)r^n u(n) z^{-n} \tag{C.3.2}$$

$$= \sum_{n=0}^{\infty} x(0) r^n z^{-n} \tag{C.3.3}$$

$$= \frac{x(0)}{1 - rz^{-1}} \tag{C.3.4}$$

$$U\left(z\right) = \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{C.3.5}$$

Now we will perform inverse Z transform on S($z$) using contour integration to find s($n$)

$$s(n) = \frac{1}{2\pi j} \oint_C S(z)\ z^{n-1}\ dz \tag{C.3.10}$$

$$= \frac{1}{2\pi j} \oint_C \frac{x(0)z^{n-1}}{(1 - rz^{-1})(1 - z^{-1})}\ dz \tag{C.3.11}$$

$$= \frac{1}{2\pi j} \oint_C \frac{x(0)z^{n+1}}{(z - r)(z - 1)}\ dz \tag{C.3.12}$$

$$= \frac{x(0)}{r - 1} \left( \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - r}\ dz - \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - 1} \right)\ dz \tag{C.3.13}$$

we already know;

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f\,(z)\right) \tag{C.3.14}$$

Now for first contour integral,

$$R_1 = \frac{1}{(1-1)!} \lim_{z \to a} \left((z - a) f\,(z)\right) \tag{C.3.15}$$

$$= \lim_{z \to r} \left( (z - r) \frac{z^{n+1}}{z - r} \right) \tag{C.3.16}$$

$$= \lim_{z \to r} \left( z^{n+1} \right) \tag{C.3.17}$$

$$= r^{n+1} \tag{C.3.18}$$

for second contour integral,

$$R_2 = \frac{1}{(1-1)!} \lim_{z \to a} ((z-a)f(z)) \tag{C.3.19}$$

$$= \lim_{z \to 1} \left( (z-1) \frac{z^{n+1}}{z-1} \right) \tag{C.3.20}$$

$$= \lim_{z \to 1} \left( z^{n+1} \right) \tag{C.3.21}$$

$$= 1 \tag{C.3.22}$$

So finally the sum of n terms of the GP is given by:

$$s(n) = \frac{x(0)}{r-1} (R_1 - R_2) \tag{C.3.23}$$

$$= \frac{x(0)}{r-1} \left( r^{n+1} - 1 \right) \tag{C.3.24}$$