SIGNAL PROCESSING

FUNDAMENTALS

Through NCERT

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Contents

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Analog

1.1. Harmonics

1.1.1 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0$ MHz.

(a) Determine, B_0, ω, k and λ

(b) Find expressions for $\mathbf{E}$ and $\mathbf{B}$

Solution:

Table 1.1: Input Parameters

Symbol	Description	value
f	frequency of source	50.0 MHz
E_0	Electric field amplitude	120 N/C
c	speed of light	3×10^8 m/s
$\mathbf{e_2}, \mathbf{e_3}$	Standard Basis vectors	N/A

Table 1.2: Formulae and Output

Symbol	Description	Formula	Value
$\mathbf{E}$	Electric field vector	$E_0 \sin(kx - 2\pi ft)\mathbf{e_2}$	$120 \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e_2}$
$\mathbf{B}$	Magnetic field vector	$B_0 \sin(kx - 2\pi ft)\mathbf{e_3}$	$(4x10^{-7}) \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e_3}$
B_0	Magnetic field strength	$\frac{E_0}{c}$	400nT
ω	Angular frequency	$2\pi f$	3.14 x 10^8m/s
k	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
λ	Wavelength	$\frac{c}{f}$	6.0m

Figure 1.1.1: Graphs of **E** and **B**

1.1.2 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.1.2: Variable description

1.1.3 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

TRAVELLING WAVE	STATIONARY WAVE
$y(x,t) = A\sin(kx \pm \omega t)$	$y(x,t) = A\sin kx \cos \omega t$
PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.1.3: Travelling wave vs Stationary wave
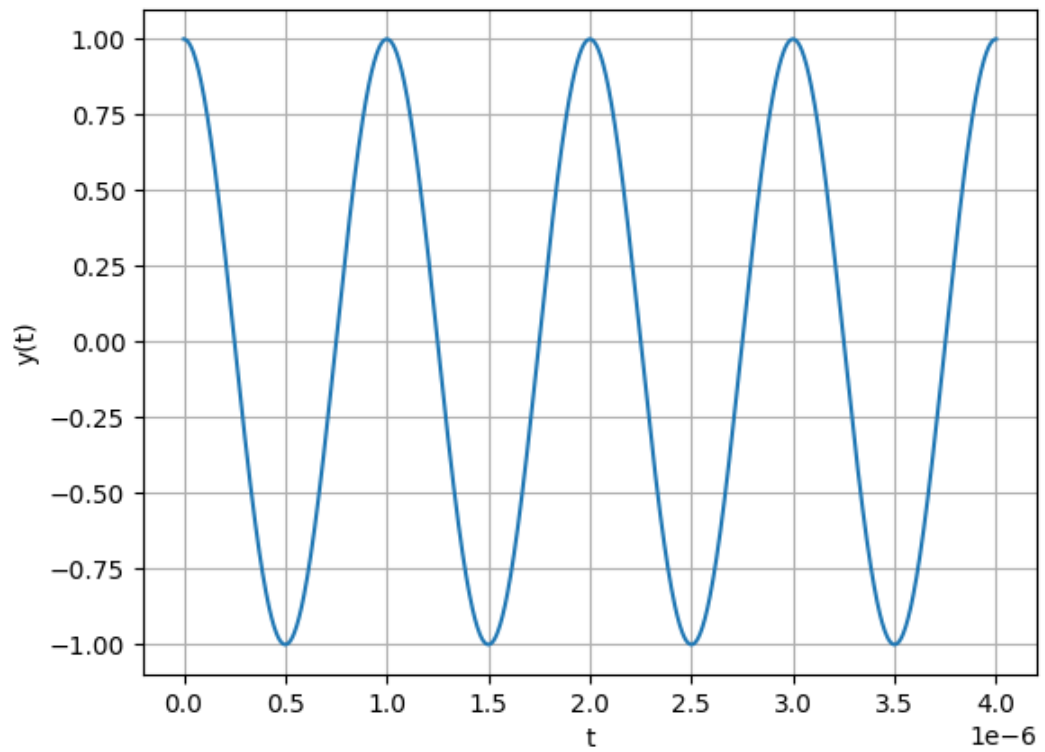
Figure 1.1.2: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

Let us assume an equation:

$$y = A(x)\cos\left(\omega t + \phi\left(x\right)\right) \tag{1.1}$$

Fig. 1.1.3 and Fig. 1.1.3 are self explanatory for stationary and travelling waves.Fig. 1.1.3 and Fig. 1.1.3 are neither stationary nor travelling waves.

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STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t + \alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx + \theta$ where k is the wavenumber and θ is the phaseconstant.

Table 1.1.3: Travelling wave vs Stationary wave

1.1.4 For the travelling harmonic wave $y(x,t) = 2.0 \cos 2\pi (10t - 0.0080x + 0.35)$ where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \tag{1.2}$$

$$= k(x_2 - x_1) \tag{1.3}$$

Figure 1.1.3: DIPLACEMENT *vs* TIME-graph1

1.1.5   (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

Solution:

Parameter	Description	Value
$y\left(x_i, t\right)$	equation of harmonic wave	$A\cos\left(2\pi ft - kx_i + \phi\right)$
k	angular wave number	$2\pi\left(0.008\right)$
$\lambda = \dfrac{2\pi}{k}$	wavelength	$125\,cm$
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	$2\pi\left(0.35\right)$
θ_i	phase of ith harmonic wave	$\left(2\pi ft - kx + \phi\right)$
x_i	position of ith harmonic wave	
t	time	
$x_2 - x_1$	path difference	$400\,cm$
		$50\,cm$
		$\dfrac{\lambda}{2}$
		$\dfrac{3\lambda}{4}$

Table 1.1.4: Given   parameters list

Parameter	Description	subquestion	Value
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\dfrac{3\pi}{2}$ radians

Table 1.1.4: Phase   differences

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi ft + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi ft + \phi)$	current in terms of time
V_0	$300\,\text{V}$	peak voltage
V_{rms}	$\sqrt{\frac{1}{T}\int_0^T [V(t)]^2\,dt}$	rms value of Voltage
I_{rms}	$10\,\text{A}$	rms value of current
I_0	$\sqrt{2}\times I_{\text{rms}}$	peak current
f	$50\,\text{Hz}$	frequency of the sinusoidal wave
T	$0.02\,\text{s}$	time period of sinusoidal wave

Table 1.1.5: Input Parameter Table

Figure 1.1.3: DIPLACEMENT *vs* TIME-graph2

Figure 1.1.3: DIPLACEMENT vs TIME-graph3

(a)

$$V_{\text{rms}}^2 = \frac{1}{T} \int_0^T [V(t)]^2 \, dt \tag{1.4}$$

$$= f \int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi ft + \phi) \, dt \tag{1.5}$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \int_0^{\frac{1}{f}} \cos(4\pi ft + 2\phi) \, dt \right) \tag{1.6}$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \left[\frac{\sin(4\pi ft + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \tag{1.7}$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \tag{1.8}$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \qquad 10 \tag{1.9}$$

Figure 1.1.3: DIPLACEMENT *vs* TIME-graph4

To find the RMS voltage (V_{rms}) when the peak voltage (V_0) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \tag{1.10}$$

Figure 1.1.4:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T} \int_0^T [I(t)]^2 \, dt \qquad (1.11)$$

$$= f \int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi f t + \phi) \, dt \qquad (1.12)$$

$$= \frac{1}{2} I_0^2 \left(1 - \frac{1}{f} \left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \qquad (1.13)$$

$$= \frac{1}{2} I_0^2 \left(1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \qquad (1.14)$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \qquad (1.15)$$

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Figure 1.1.4:

To find the peak current (I_0) when the RMS current (I_{rms}) is given, you can use equation (1.15)

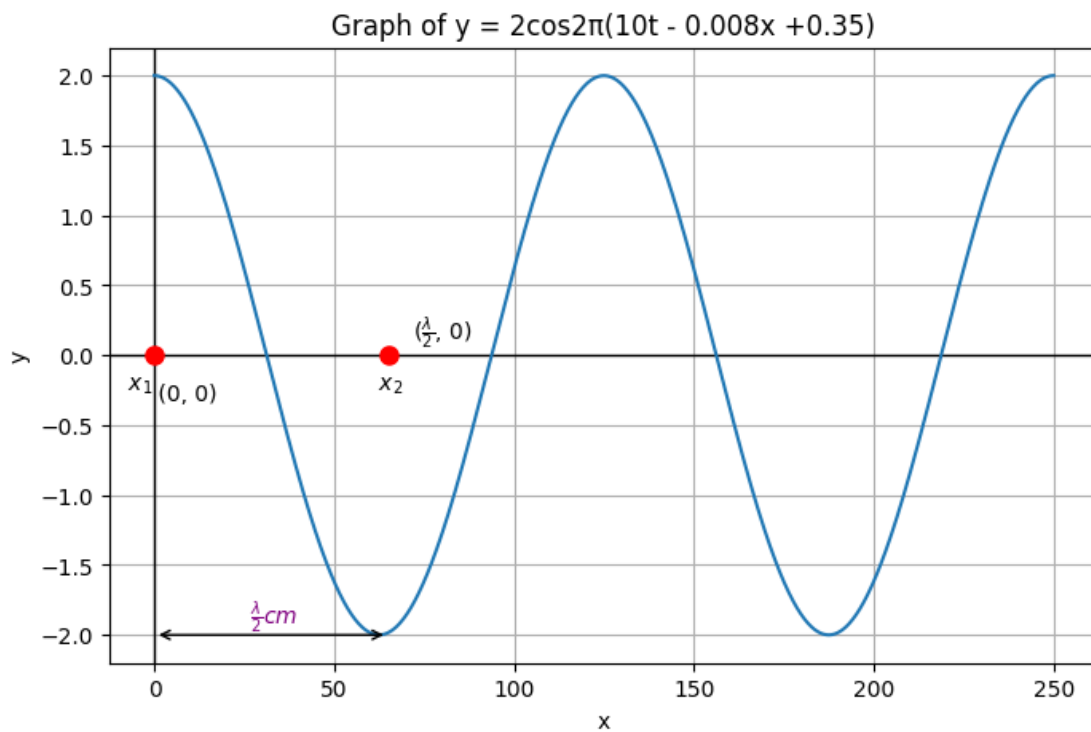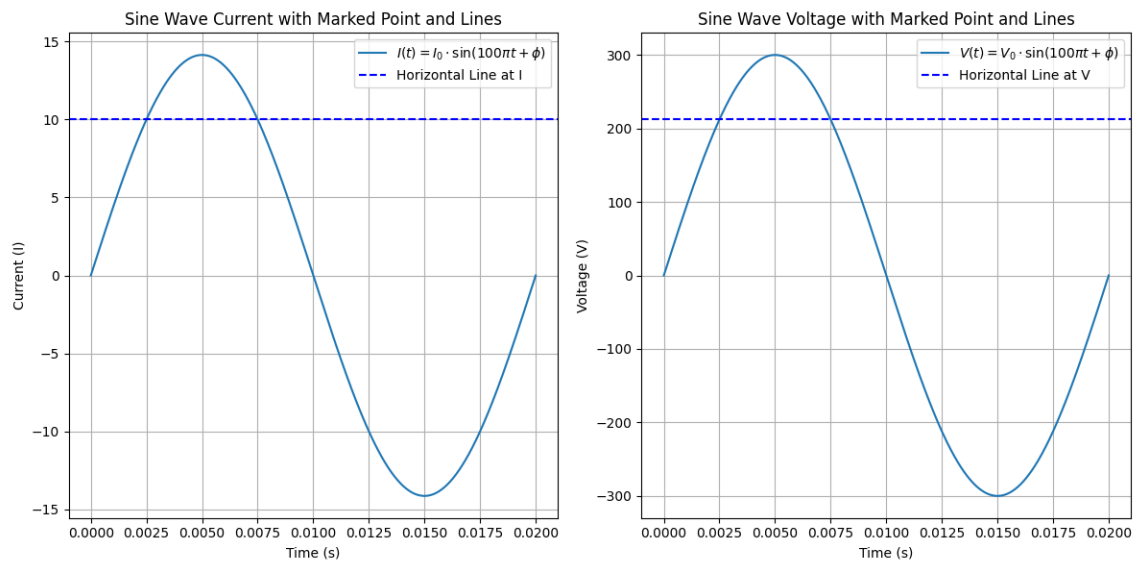$$I_0 \approx 10\,\text{A} \times 1.414 \approx 14.14\,\text{A} \tag{1.16}$$

Graph of y = 2cos2π(10t - 0.008x +0.35)

Figure 1.1.4:

Sine Wave Current with Marked Point and Lines

$I(t) = I_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at I

Sine Wave Voltage with Marked Point and Lines

$V(t) = V_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at V

Graph of y = 2cos2π(10t - 0.008x +0.35)

Figure 1.1.4:

1.1.6 In Young's double-slit experiment using monochromatic light of wavelength λ, the intensity of light at a point on the screen where path difference is λ, is K units. What is the intensity of light at a point where path difference is $\lambda/3$?

Solution:

From Table 1.1.6:

$$y(t) = A\sin(2\pi ft - kx_1) + A\sin(2\pi ft - kx_2) \tag{1.17}$$

$$y(t) = 2A\cos\left(\frac{k\Delta x}{2}\right)\sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \tag{1.18}$$

Parameter	Description	Value
$y_i(t)$	Equation of light from $S_{i\text{th}}$	$A\sin(\omega t - kx_i)$
k	Wave number	$\dfrac{2\pi}{\lambda}$
I	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	λ
		$\dfrac{\lambda}{3}$
K	Intensity of light at $\Delta x = \lambda$	
A	Amplitude of wave from source	
r	constant	$r \geq 0$

Table 1.1.6: Parameters

From Table 1.1.6 and equation (1.18):

$$\therefore I \propto 4A^2 \cos^2\left(\frac{k\Delta x}{2}\right) \tag{1.19}$$

From Table 1.1.6 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2 \cos^2\left(\dfrac{2\pi}{2}\right)}{4A^2 \cos^2\left(\dfrac{\pi}{3}\right)} \implies I_r = \frac{K}{4} \tag{1.20}$$

Hence, the Intensity of light at a point where path difference is $\dfrac{\lambda}{3}$ is $\dfrac{K}{4}$ units.

Parameter	Description	Value
I_r	Net Intensity of light at $\Delta x = \dfrac{\lambda}{3}$	$\dfrac{K}{4}$

Table 1.1.6:

Assuming $\Delta x = r\lambda$,
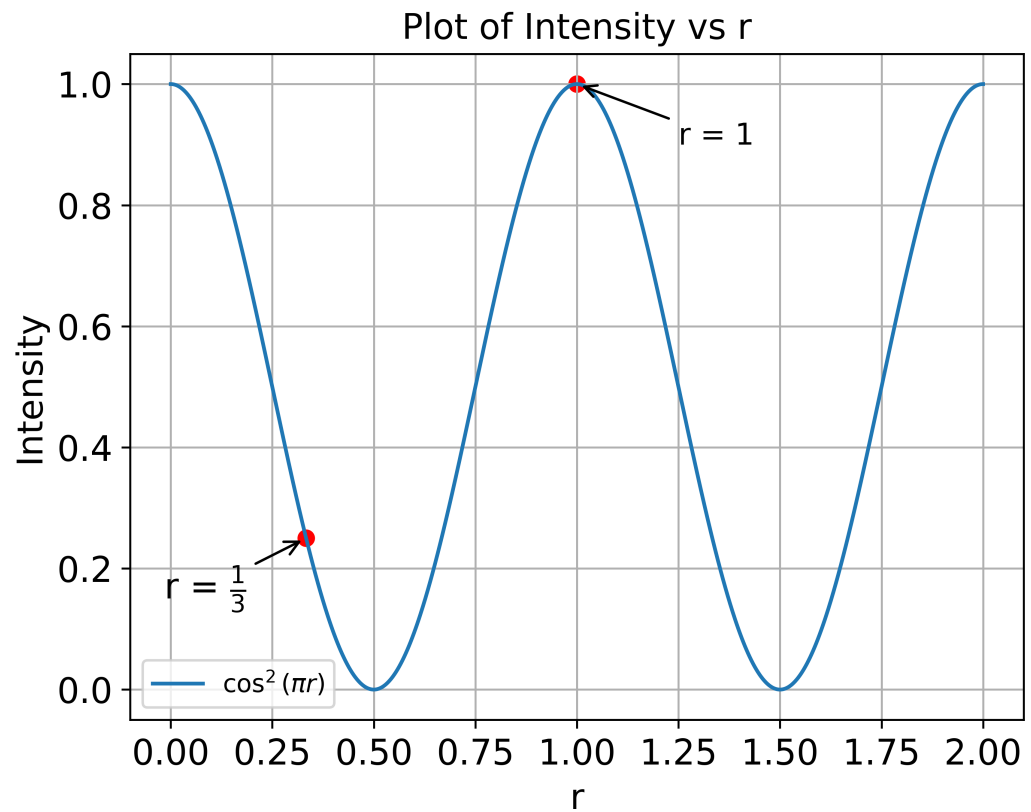
From equation (1.19):

Figure 1.1.6:

1.2. Filters

1.2.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a

suitable way.

Solution: Given parameters are:

Symbol	Value	Description
L	$3.0\,\mathrm{H}$	Inductance
C	$27\,\mu\mathrm{F}$	Capacitance
R	$7.4\,\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\dfrac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 1.11: Given Parameters
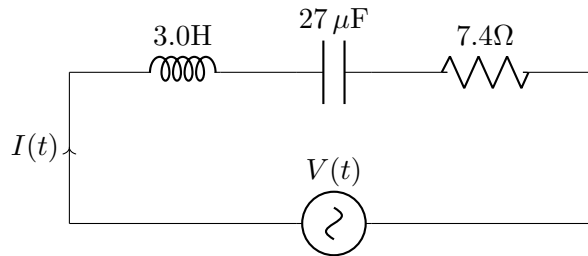
Figure 1.12: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{1.21}$$

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Using reactances from Fig. 1.13,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{1.22}$$

$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{1.23}$$

$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \dfrac{1}{sC}\right)} \tag{1.24}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

Figure 1.13: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{1.25}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{1.26}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{1.27}$$

Comparing (1.26) and (1.27), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.28}$$

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(b) Quality Factor

    i. Using voltage across inductor,

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{1.29}$$

$$= \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{1.30}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.31}$$

    ii. Using voltage across capacitor,

$$Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} = \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} \tag{1.32}$$

$$= \frac{\sqrt{LC}}{RC} \tag{1.33}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.34}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{1.35}$$

Using (1.24),

$$H(s) = R + sL + \frac{1}{sC} \tag{1.36}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{1.37}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{1.38}$$
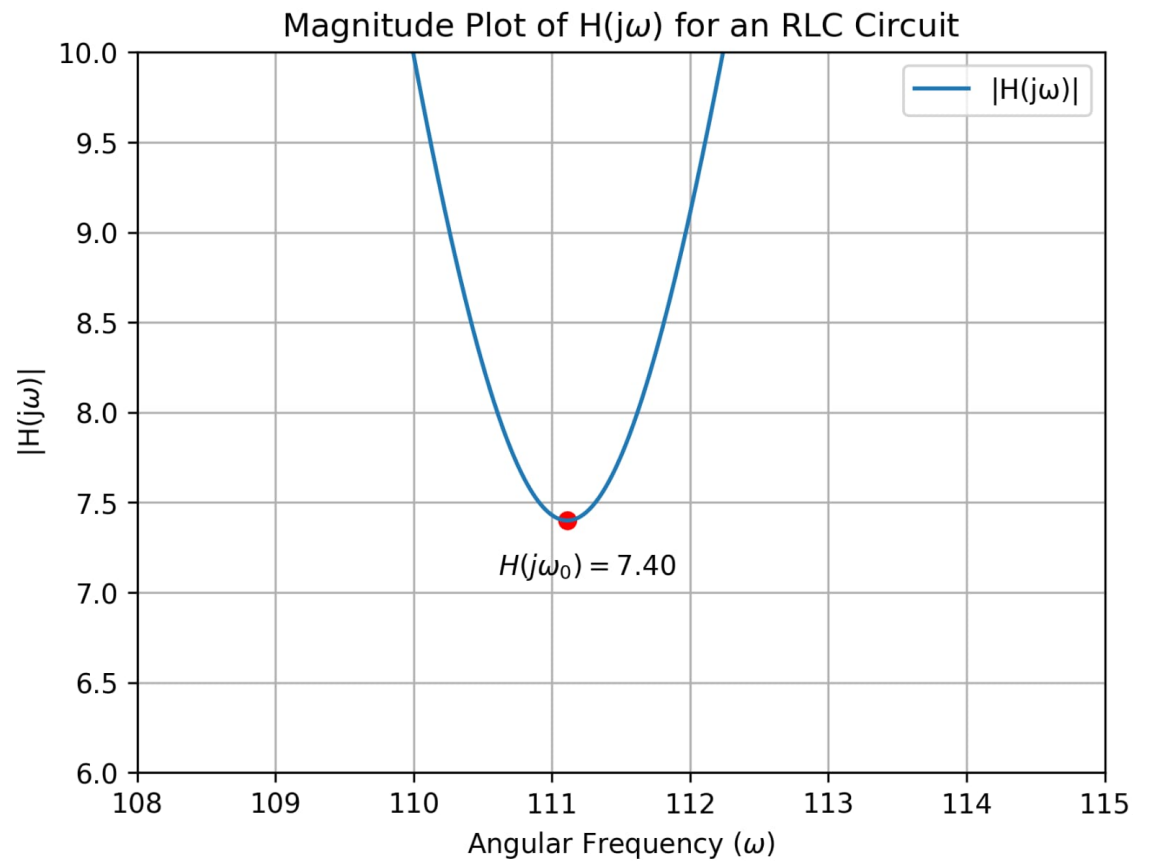
Figure 1.14: Impedance vs ω (using values in Table 1.11)

Chapter 2

Discrete

2.1. Z-transform

2.1.1 Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3 + 5n^2 + 8n + 4\right) \cdot u\left(n\right)$
$x_2(n)$	General Term of Denominator	$\left(n^3 + 4n^2 + 5n + 2\right) \cdot u\left(n\right)$
$y_1\left(n\right)$	Sum of terms of numerator	?
$y_2\left(n\right)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : |z| > 1\}$
ROC	Region of convergence	$\left\{z : \left|\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right| < \infty\right\}$

Table 1: Parameter Table

1. <u>Analysis of Numerator:</u>

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \tag{2.1}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 8n + 4\right) u(n) z^{-n} \tag{2.2}$$

Using results of equations (2.0.1.5) to (2.0.1.8) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \tag{2.3}$$

From (1.0.1.4)

$$y_1(n) = x_1(n) * u(n) \tag{2.4}$$

$$Y_1(z) = X_1(z) U(z) \tag{2.5}$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \tag{2.6}$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{(1 - z^{-1})} + \frac{48z^{-2}}{(1 - z^{-1})^2} + \frac{52z^{-3}}{(1 - z^{-3})^3}, \tag{2.7}$$
$$+ \frac{28z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 4, |z| > 1$$

Substituting results of equation (2.0.1.14) to (2.0.1.17) in equation (2.7):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \tag{2.8}$$

$$= \frac{(3n + 8)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.9}$$

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2. <u>Analysis of Denominator:</u>

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \tag{2.10}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n) z^{-n} \tag{2.11}$$

Using results of equation (2.0.1.5) to (2.0.1.8) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{\left(1 - z^{-1}\right)^4}, |z| > 1 \tag{2.12}$$

From (1.0.1.4)

$$y_2(n) = x_2(n) * u(n) \tag{2.13}$$

$$Y_2(z) = X_2(z) U(z) \tag{2.14}$$

$$= \frac{2 + 4z^{-1}}{\left(1 - z^{-1}\right)^5}, |z| > 1 \tag{2.15}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{\left(1 - z^{-1}\right)} + \frac{36z^{-2}}{\left(1 - z^{-1}\right)^2} + \frac{44z^{-3}}{\left(1 - z^{-3}\right)^3} \tag{2.16}$$

$$+ \frac{26z^{-4}}{\left(1 - z^{-1}\right)^4} + \frac{6z^{-5}}{\left(1 - z^{-1}\right)^5} + 2, |z| > 1$$

Substituting results of equation (2.0.1.14) to (2.0.1.17) in equation (2.16):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{2.17}$$

$$= \frac{(3n + 4)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.18}$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n + 1$:

$$\frac{y_1 (n)}{y_2 (n)} = \frac{3n + 8}{3n + 4} \tag{2.19}$$

Hence Prooved.

Figure 2.1: Stem Plot of $x_1 (n)$
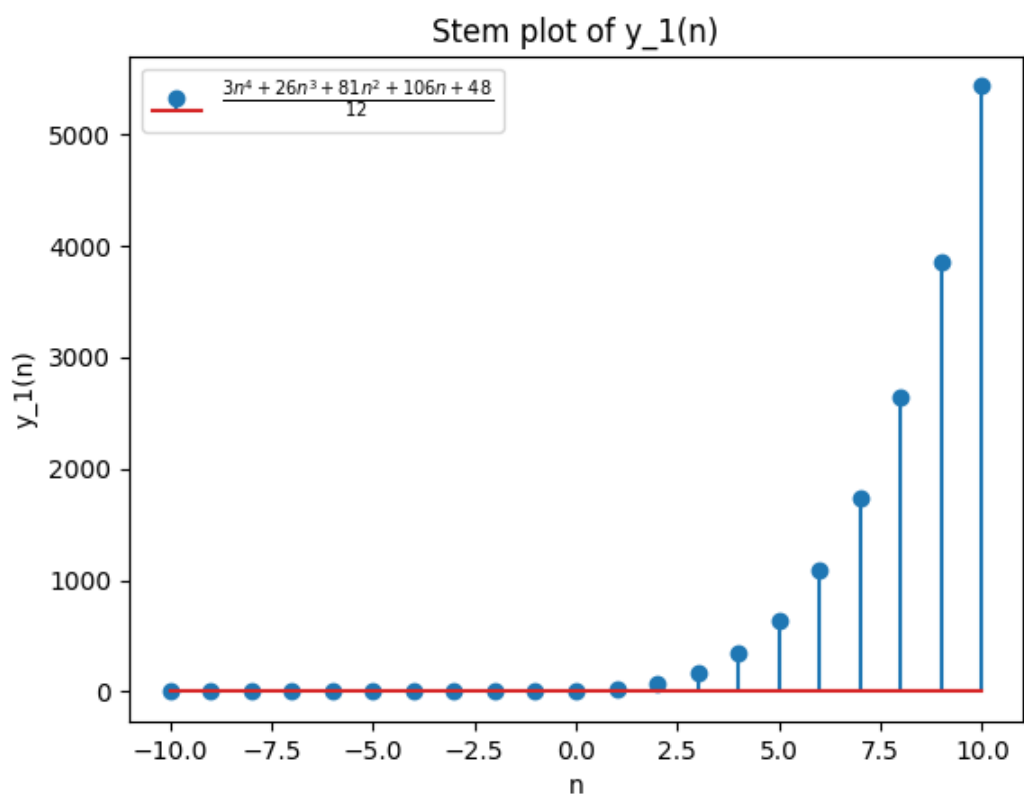
Figure 2.2: Stem Plot of $x_2\left(n\right)$

Figure 2.3: Stem Plot of $y_1(n)$

Figure 2.4: Stem Plot of $y_2 (n)$

2.1.2 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$x (n) = -1, \qquad\qquad n = 0 \qquad\qquad (2.20)$$

$$= \frac{x (n - 1)}{n}, \qquad\qquad n > 0 \qquad\qquad (2.21)$$

$$= 0, \qquad\qquad n < 0 \qquad\qquad (2.22)$$

Solution:

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$$x(1) = \frac{x(0)}{1} = -1 \tag{2.23}$$

$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{2.24}$$

$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{2.25}$$

$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{2.26}$$

$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{2.27}$$

$$x(n) = \frac{-1}{n!}(u(n)) \tag{2.28}$$

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \tag{2.29}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{2.30}$$

using (2.28),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \tag{2.31}$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{2.32}$$

$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{2.33}$$

Symbol	Value	Description
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 2.2: Parameters

Figure 2.5: Plot of x(n) vs n

2.1.3 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of the AP

Table 2.3: Input Parameters

From the values given in Table 2.3:

$$7000 = 5000 + 200n \tag{2.34}$$

$$\implies 2000 = 200n \tag{2.35}$$

$$\therefore n = 10 \tag{2.36}$$

Let Z-transform of $x(n)$ be $X(z)$.
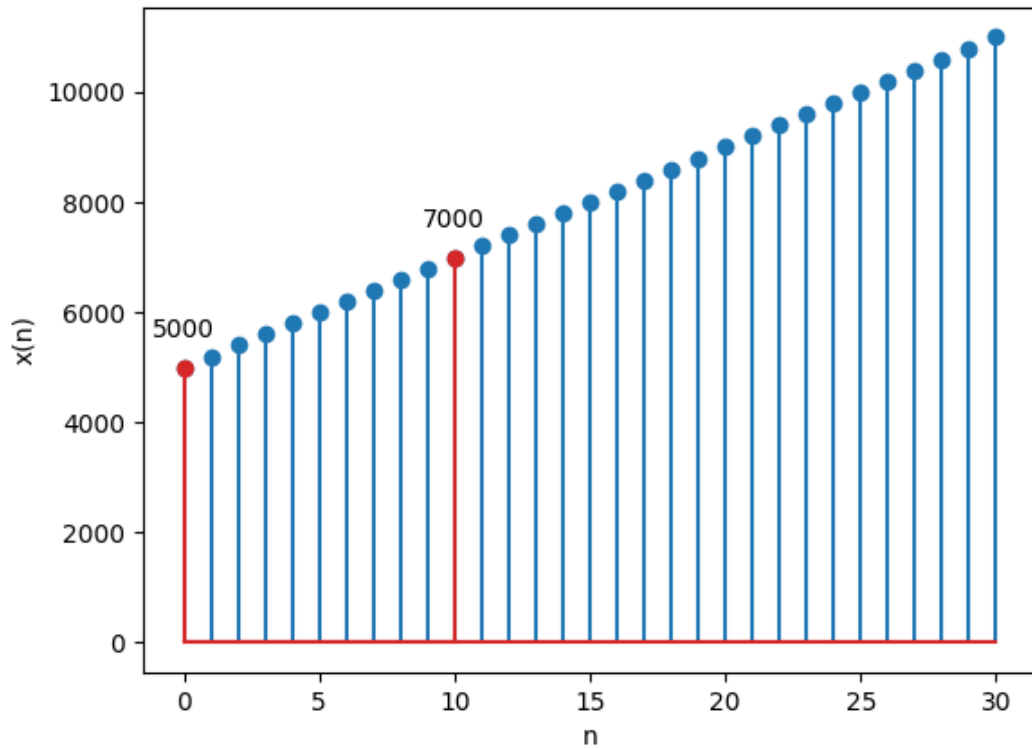
Figure 2.6: Plot of $x(n)$ vs n. See Table 2.3 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.37}$$

Using the values from Table 2.3:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.38}$$

2.2. Sequences

2.2.1 For what value of n, are the nth terms of two A.Ps: 63, 65, 67,... and 3, 10, 17,... equal?

Solution:

Parameter	Sub-question	Description	Value
$x_i(0)$	$x_1(0)$	1st term of 1st A.P.	63
	$x_2(0)$	1st term of 2nd A.P.	3
d_i	d_1	Common difference of 1st A.P.	2
	d_2	Common difference of 2nd A.P.	7

Table 2.4: input values

$$x_i(n) = x(0)u(n) + dnu(n) \tag{2.39}$$

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.40}$$

(a)

$$x_1(n) = 63u(n) + 2nu(n) \tag{2.41}$$

$$X_1(z) = \frac{63}{1 - z^{-1}} + \frac{2z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.42}$$

(b)

$$x_2\left(n\right) = 3u\left(n\right) + 7nu\left(n\right) \tag{2.43}$$

$$X_2\left(z\right) = \frac{3}{1 - z^{-1}} + \frac{7z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{2.44}$$

(c) given,

$$x_1\left(n\right) = x_2\left(n\right) \tag{2.45}$$

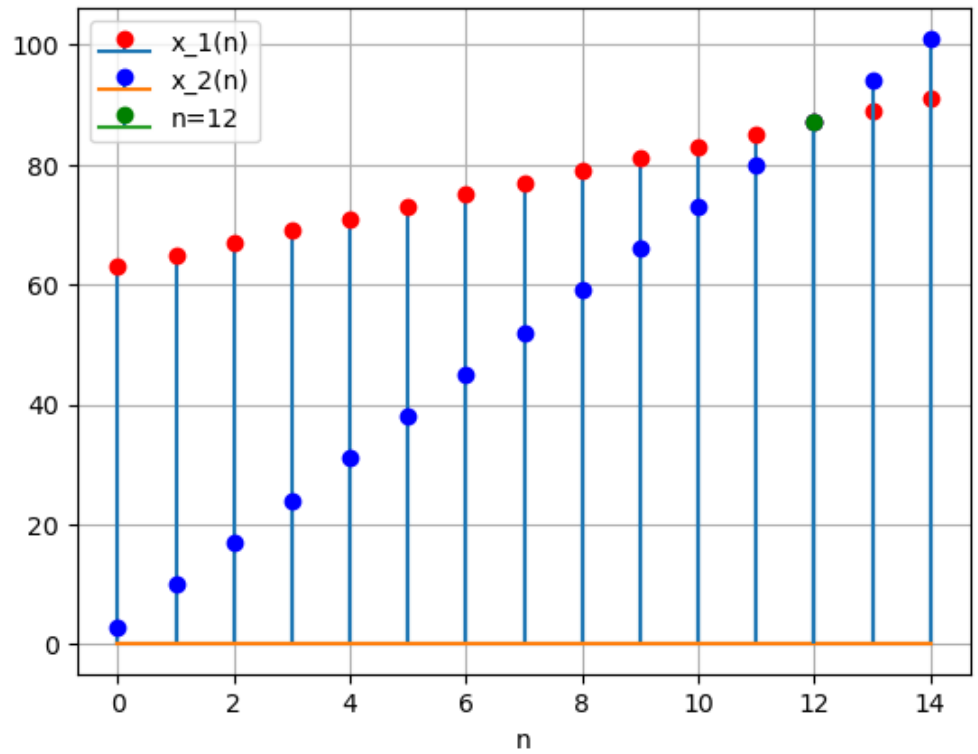$$\therefore 63 + 2n = 7n + 3 \tag{2.46}$$

$$\implies n = 12 \tag{2.47}$$

Figure 2.7: Graphs of $x_1(n)$ and $x_2(n)$ and both are equal at $n = 12$

2.2.2 Two APs have the same common difference.The difference between their 100th terms is 100,what is the difference between their 1000th terms?

Solution:

$$x(n) = \{x(0) + nd\}u(n) \tag{2.48}$$

$$x(99) - y(99) = 100 \tag{2.49}$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \tag{2.50}$$

$$\implies x(0) - y(0) = 100 \tag{2.51}$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \tag{2.52}$$

$$= x(0) - y(0) \tag{2.53}$$

$$= 100 \tag{2.54}$$

$$\implies x(999) - y(999) = 100 \tag{2.55}$$

Variable	Description	Value
$x(n)$	n^{th} term of X	none
$y(n)$	n^{th} term of Y	none
d	common difference between the terms of AP	none
$x(99) - y(99)$	difference of 99th terms of X and Y	100

Table 2.5: input parameters

Let

$$x(n) = \{101, 106, 111, ...\} \tag{2.56}$$

$$y(n) = \{1, 6, 11, ...\} \tag{2.57}$$

2.2.3 Check whether -150 is a term of the AP: 11,8,5,2,....

Solution:

$$x(n) = x(0) + nd \tag{2.58}$$

$$n = \frac{x(n) - x(0)}{d} \tag{2.59}$$
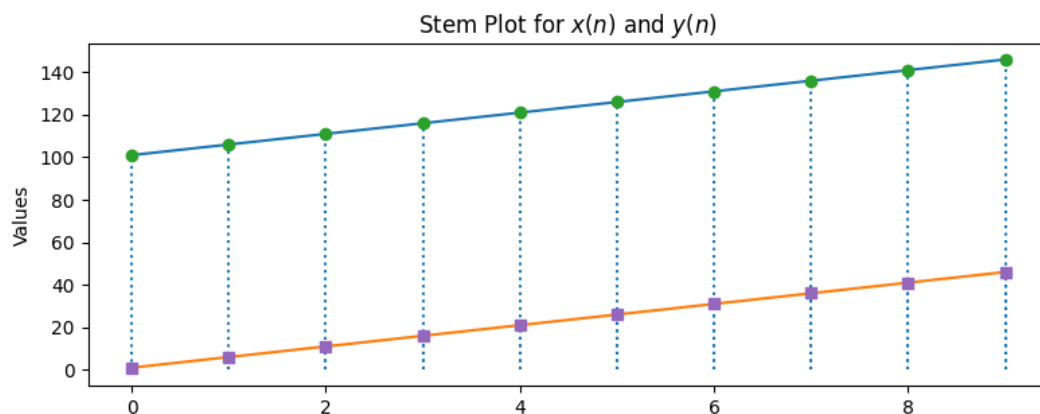
Figure 2.8:

$$x(n) - x(0) \equiv 0 \pmod{d} \tag{2.60}$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \tag{2.61}$$

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \tag{2.62}$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.63}$$

Variable	Description	Value
$x(0)$	First term of AP	11
d	Common difference	-3
$x(n)$	General term of given AP	None

Table 2.6: Input parameters
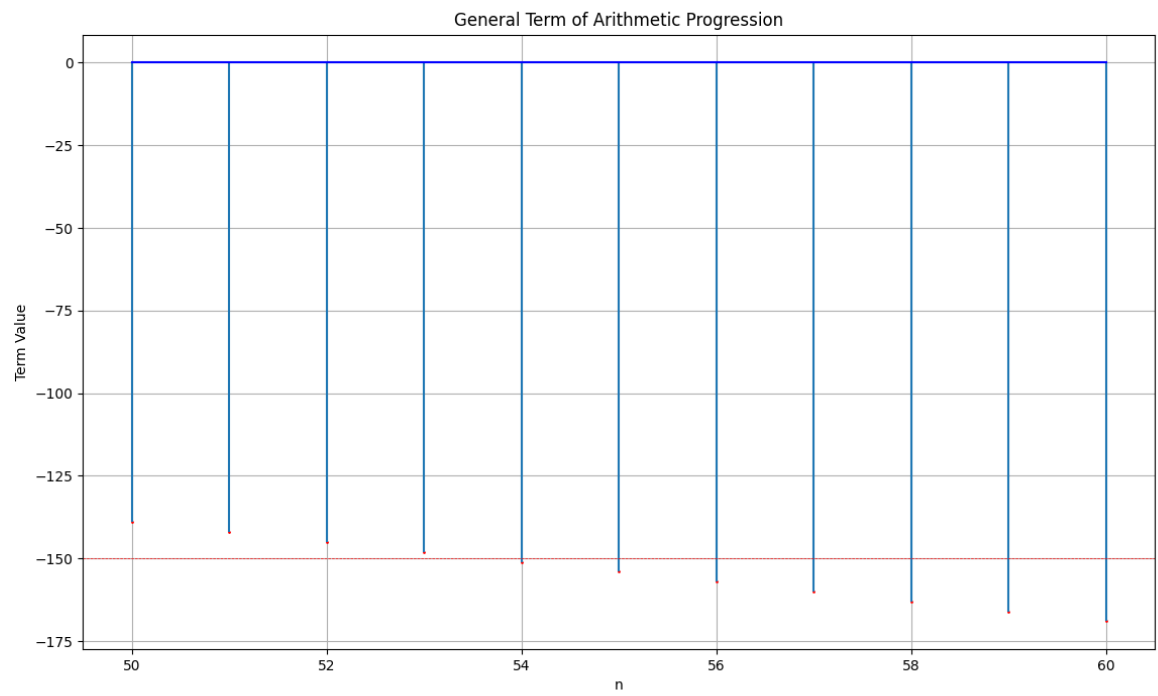
Figure 2.9: Representation of x(n)

2.2.4 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

Solution:

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4}\right) u(n) \qquad (2.64)$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \qquad (2.65)$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \qquad (2.66)$$

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + z^{-1})}{(1 - z^{-1})^3} \quad |z| > 1 \qquad (2.67)$$

$$n^3 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \quad |z| > 1 \qquad (2.68)$$

Referencing the equations from (4.3), (4.4), and (4.5).

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{4(1 - z^{-1})^4} + \frac{3(z^{-1})(1 + z^{-1})}{4(1 - z^{-1})^3} + \frac{2z^{-1}}{(1 - z^{-1})^2} + \frac{3}{2(1 - z^{-1})} \quad |z| > 1 \qquad (2.69)$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{3}{2(1 - z^{-1})^3} + \frac{3z^{-2}}{2(1 - z^{-1})^4} \quad |z| > 1 \qquad (2.70)$$
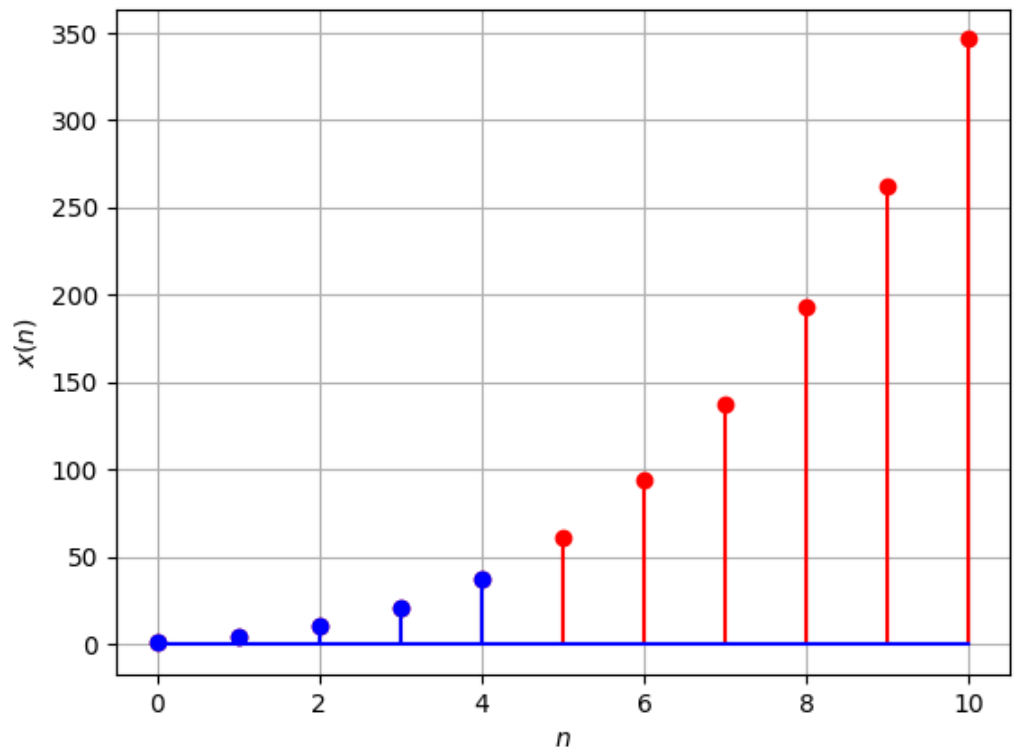
Figure 2.10: Plot of equation(4.1)

2.2.5 (a) 30th term of the AP: 10, 7, 4, . . . is

(b) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

Solution:

$$x_i(n) = [x_i(0) + nd_i] u(n) \tag{2.71}$$

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Parameter	value	Description
$x_i(0)$	10	First
	-3	term
d_i	-3	Common
	$\frac{5}{2}$	difference
$x_1(29)$	?	30th term
$x_2(10)$	?	11th term

Table 2.7: Input Parameters

(a) From (2.71) Table 2.7 :

$$x_1(n) = [10 - 3n]\, u(n) \tag{2.72}$$

$$x_1(29) = -77 \tag{2.73}$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.74}$$

(b) From (2.71) and Table 2.7 :

$$x_2(n) = \left[-3 + \frac{5}{2}n\right] u(n) \tag{2.75}$$

$$x_2(10) = 22 \tag{2.76}$$
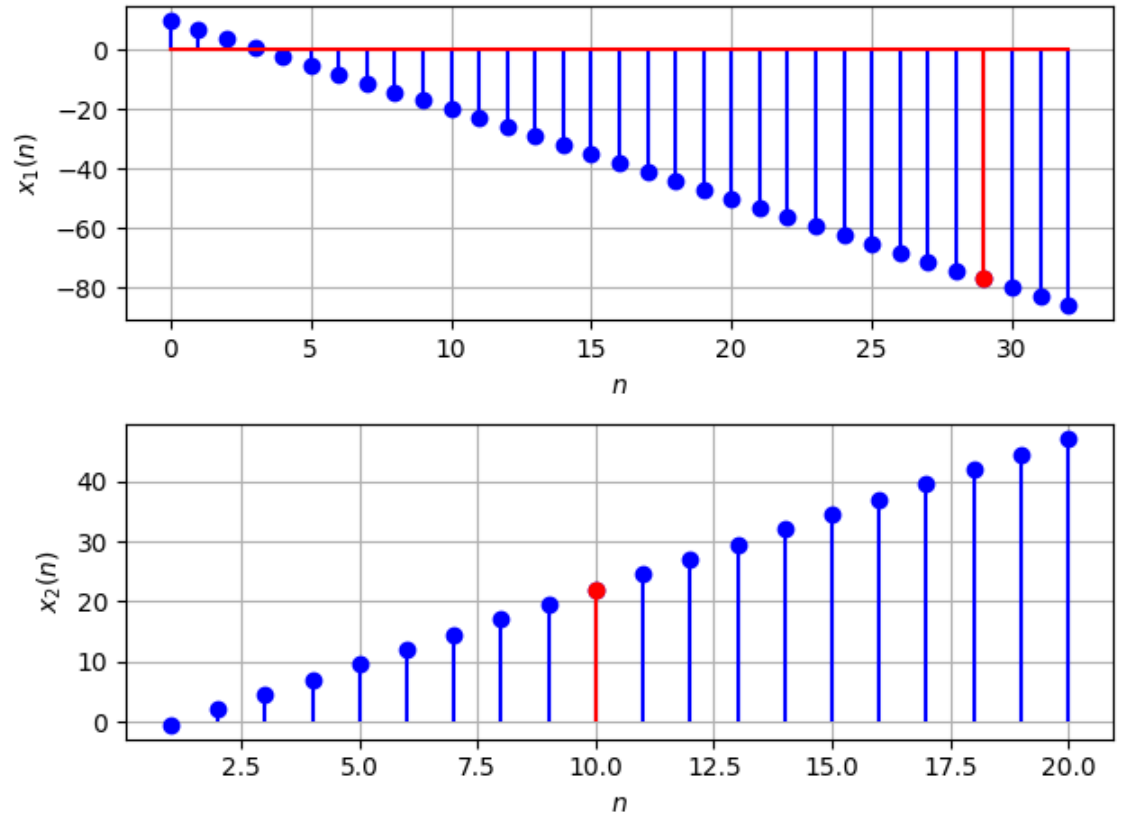
Figure 2.11: stem plots of $x_1(n)$ and $x_2(n)$

2.2.6 Write the first five terms of the sequence whose nth term is $\frac{2n-3}{6}$ and obtain the Z transform of the series **Solution:**

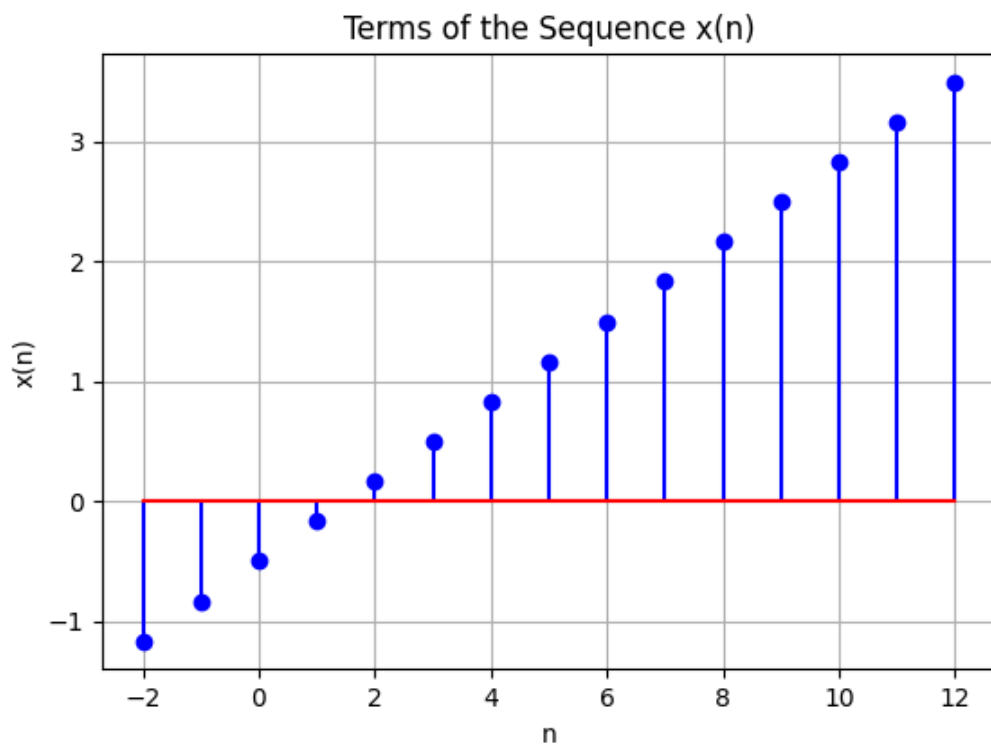$$x(n) = \frac{2n-1}{6}(u(n)) \tag{2.78}$$

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Figure 2.12: Plot of x(n) vs n

$$X(z) = \frac{3z^{-1} - 1}{6(1 - z^{-1})^2} \quad |z| > 1 \qquad (2.79)$$

2.2.7 For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P ?

Solution: Let r be the common ratio

Variable	Description	Value
$x(0)$	First term of the GP	$-\left(\frac{2}{7}\right)$
$x(1)$	Second term of the GP	x
$x(2)$	Third term of the GP	$-\left(\frac{7}{2}\right)$
r	Common ratio of the GP	
$x(n)$	General term	$x(0)\, r^n\, u(n)$

Table 2.8: Variables Used

From Table 2.8:

$$\implies \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \tag{2.80}$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \tag{2.81}$$

$$x = \pm 1 \tag{2.82}$$

$$\implies r = \pm\frac{7}{2} \tag{2.83}$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right)\left(\pm\frac{7}{2}\right)^n u(n) \tag{2.84}$$

Applying z-Transform :

$$\implies X_1(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}+2}\right) \quad |z| > \frac{7}{2} \tag{2.85}$$

$$\implies X_2(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}-2}\right) \quad |z| > \frac{7}{2} \tag{2.86}$$
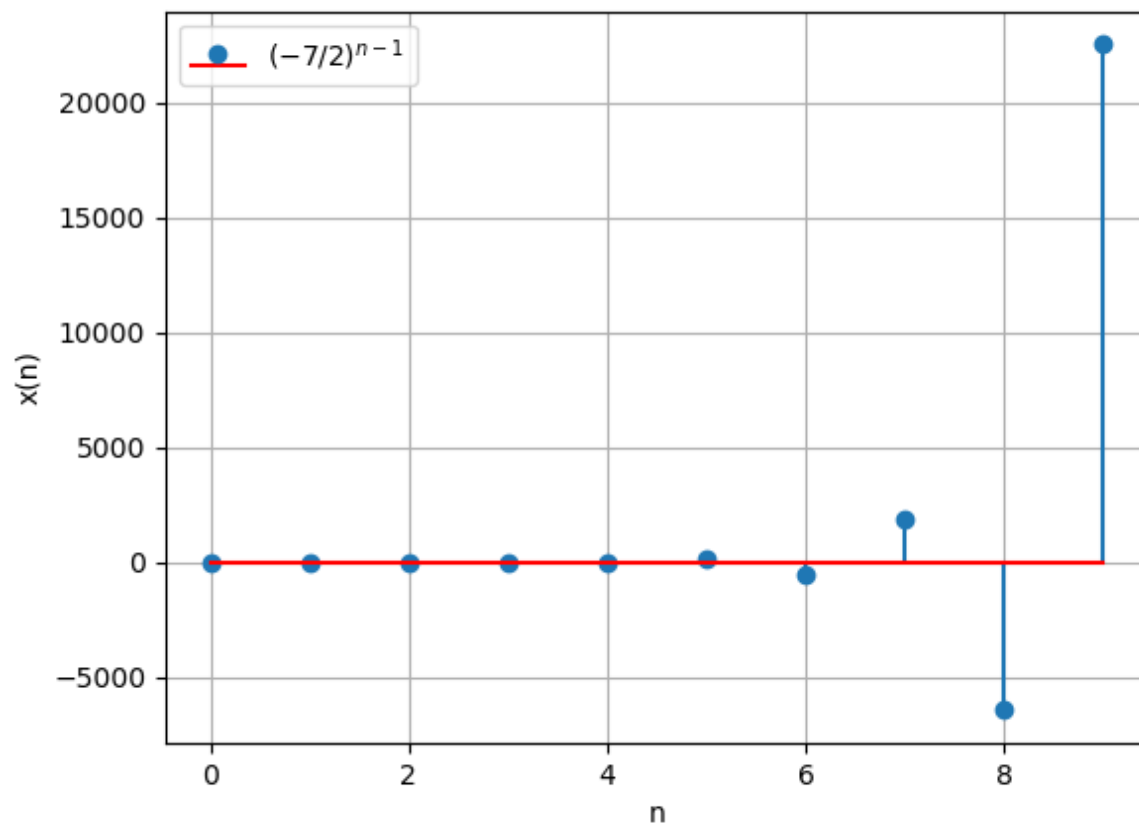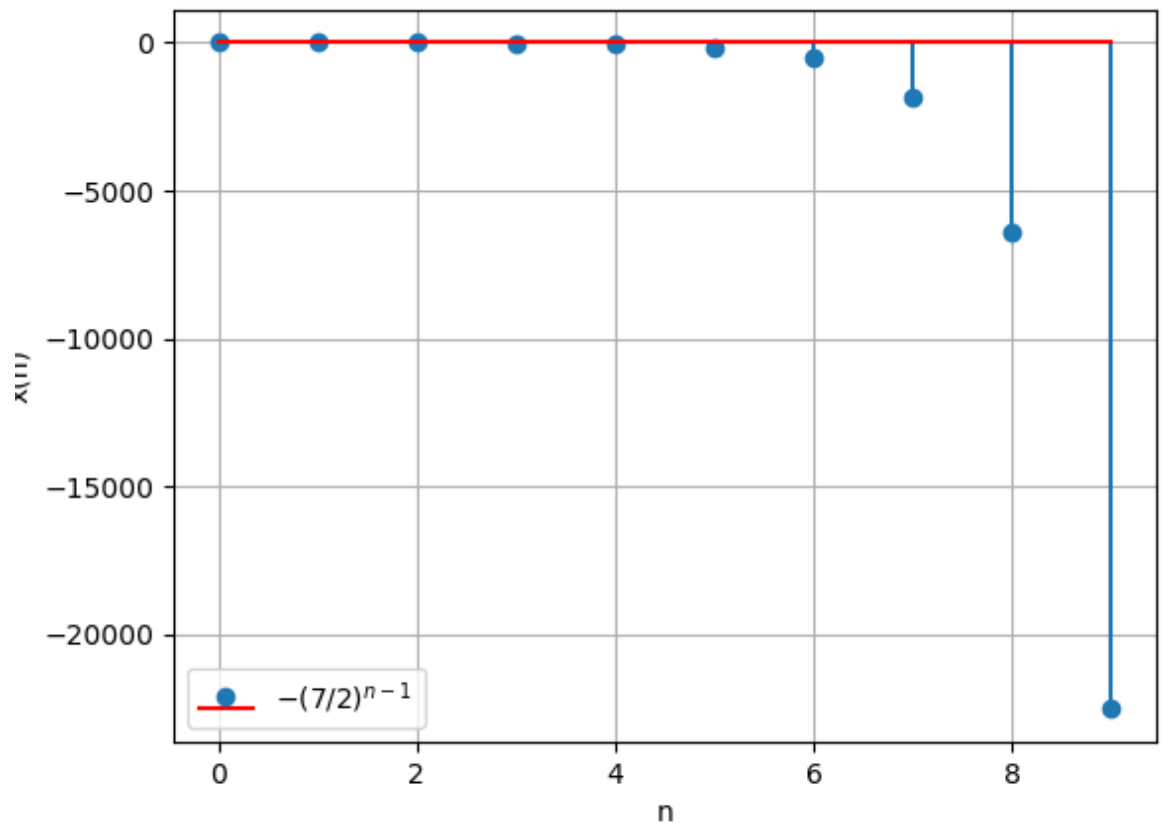
45

Figure 2.13: Stem Plot of $x_1(\text{n})$

Figure 2.14: Stem Plot of $x_2(n)$

2.2.8 Find the 20th and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

Solution:

From Table 2.9: Z-Transform of $x(n)$:

$$\implies X(z) = \frac{5}{2}\left(\frac{1}{1 - \frac{z^{-1}}{2}}\right) ; \left\{z \in \mathbb{C} : |z| > \frac{1}{2}\right\} \tag{2.87}$$

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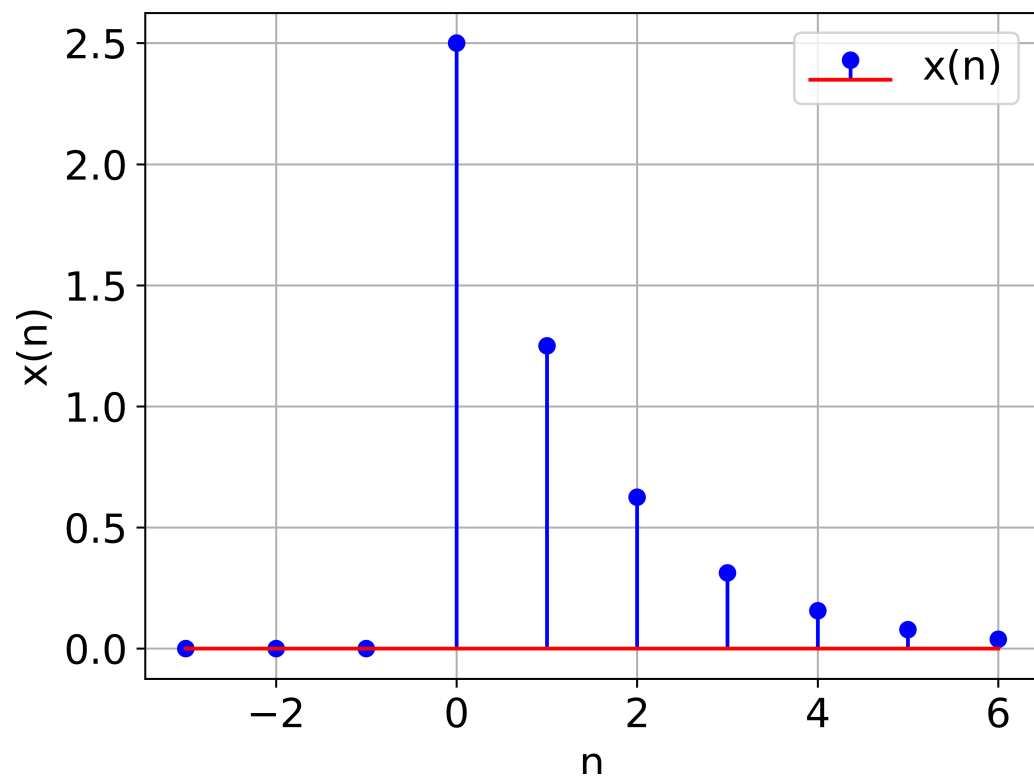| Parameter	Description	Value
$x(0)$	First Term	$\dfrac{5}{2}$
$r = \dfrac{x(1)}{x(0)}$	Common Ratio	$\dfrac{1}{2}$
$x(n)$	n^{th} Term	$\dfrac{5}{2}\left(\dfrac{1}{2}\right)^{n}\cdot u(n)$
$x(19)$	20th Term	$\dfrac{5}{2}\left(\dfrac{1}{2}\right)^{19}$
$u(n)$	Unit step function	

Table 2.9: Parameters

Figure 2.15:

Chapter 3

Contour Integration

3.1 Find the sum of the first 15 multiples of 8.

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
d	8	common difference
$x(n)$	$[8 + 8n]u(n)$	General term of the series

Table 3.1: Parameter Table1

For an AP,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{3.1}$$

$$\implies X(z) = \frac{8}{1 - z^{-1}} + \frac{8z^{-1}}{(1 - z^{-1})^2} \tag{3.2}$$

$$= \frac{8}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{3.3}$$

$$y(n) = x(n) * u(n) \tag{3.4}$$

$$\implies Y(z) = X(z) U(z) \tag{3.5}$$

$$Y(z) = \left(\frac{8}{(1 - z^{-1})^2} \right) \left(\frac{1}{1 - z^{-1}} \right) \tag{3.6}$$

$$= \frac{8}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{3.7}$$

Using Contour Integration to find the inverse Z-transform,

$$y(14) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{13}\, dz \tag{3.8}$$

$$= \frac{1}{2\pi j} \oint_C \frac{8z^{13}}{(1 - z^{-1})^3}\, dz \tag{3.9}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z) \right) \tag{3.10}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z - 1)^3 \frac{8z^{16}}{(z - 1)^3} \right) \tag{3.11}$$

$$= 4 \lim_{z \to 1} \frac{d^2}{dz^2} (z^{16}) \tag{3.12}$$

$$= 960 \tag{3.13}$$

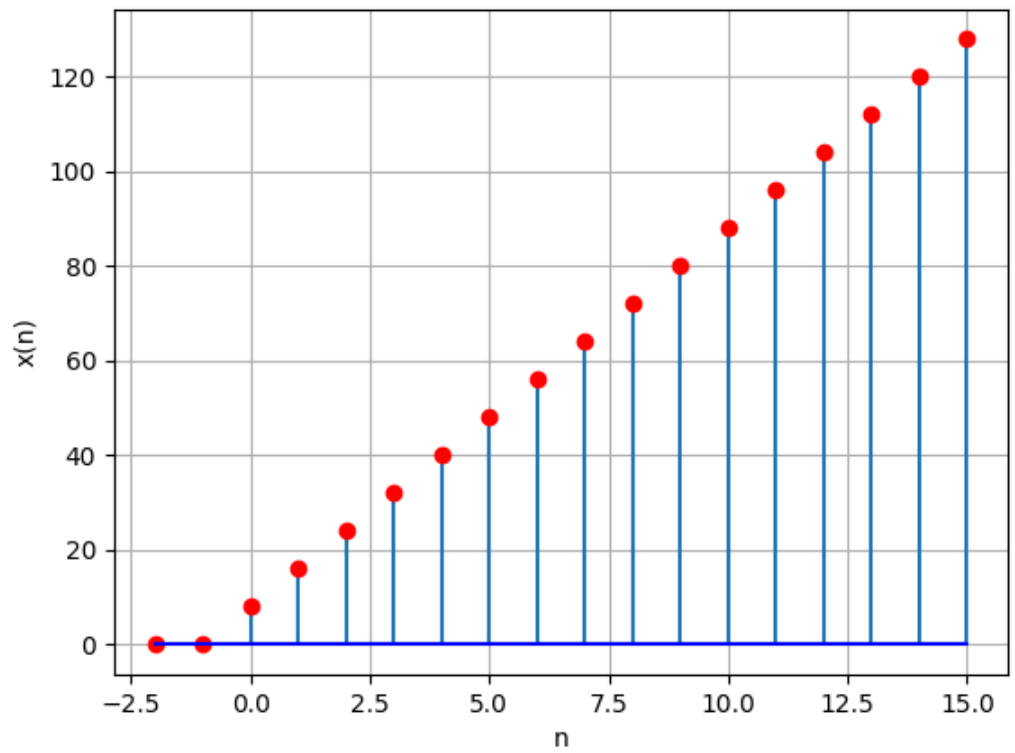$$\therefore \boxed{y(14) = 960} \qquad\qquad (3.14)$$

Figure 3.1: Plot of x(n) *vs* n

Chapter 4

Laplace Transform

4.0.1 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of

(a) The spring constant K

(b) The damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

Solution: Parameters:

Parameter	Value(SI)	Description
x_0	0.15	Initial spring compression
m	750	Mass
g	9.8	Gravitational acc
k	mg/x_0	Spring constant
b		Damping constant

Table 4.1: Input Parameters

Parameter	Value(SI)	Description
x		Spring Extension
F_1	kx	Spring Force
F_2	$b\frac{dx}{dt}$	Damping Force
s		Complex Frequency
s_1, s_2		Values of s

Table 4.2: Intermediate Parameters

Initially the automobile is in rest, so we can use,

$$mg = kx_0 \tag{4.1}$$

$$\implies k = \frac{mg}{x_0} \tag{4.2}$$

Now, as the oscillation begins, from the Fig. 4.1 we net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{4.3}$$

$$\implies -m\frac{d^2x(t)}{dt^2} = kx(t) + b\frac{dx(t)}{dt} + mgu(t) \tag{4.4}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = -gu(t) \tag{4.5}$$

Now, taking the Laplace transform on both sides,

$$s^2 X(s) + \frac{b}{m}sX(s) + \frac{k}{m}X(s) = -\frac{g}{s} \tag{4.6}$$

$$\implies X(s) = -\frac{g}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \tag{4.7}$$

$$\implies X(s) = -\frac{g}{s(s - s_1)(s - s_2)} \tag{4.8}$$

Where

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{4.9}$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{4.10}$$

From (4.8) we get,

$$\begin{aligned}\implies X(s) &= \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2(s - s_2)} - \frac{1}{s_1(s - s_1)}\right] \\ &+ \frac{g}{s_1 s_2}\left(\frac{1}{s}\right)\end{aligned} \tag{4.11}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = \frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2}e^{s_2 t} - \frac{1}{s_1}e^{s_1 t}\right]u(t) \tag{4.12}$$

$$\begin{aligned}\implies x(t) &= \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bt/2m}u(t) \\ &\sin\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}{gb}\right)\right) \\ &+ \frac{mg}{k}u(t)\end{aligned} \tag{4.13}$$

(Substituting the values of s_1 and s_2 from (4.9) and (4.10))

From (4.13) we have the amplitude after one time period T,

$$\frac{1}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} =$$
$$\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}e^{-bT/2m} \tag{4.14}$$

$$\implies e^{\pi b/\sqrt{mk}} = 2 \tag{4.15}$$
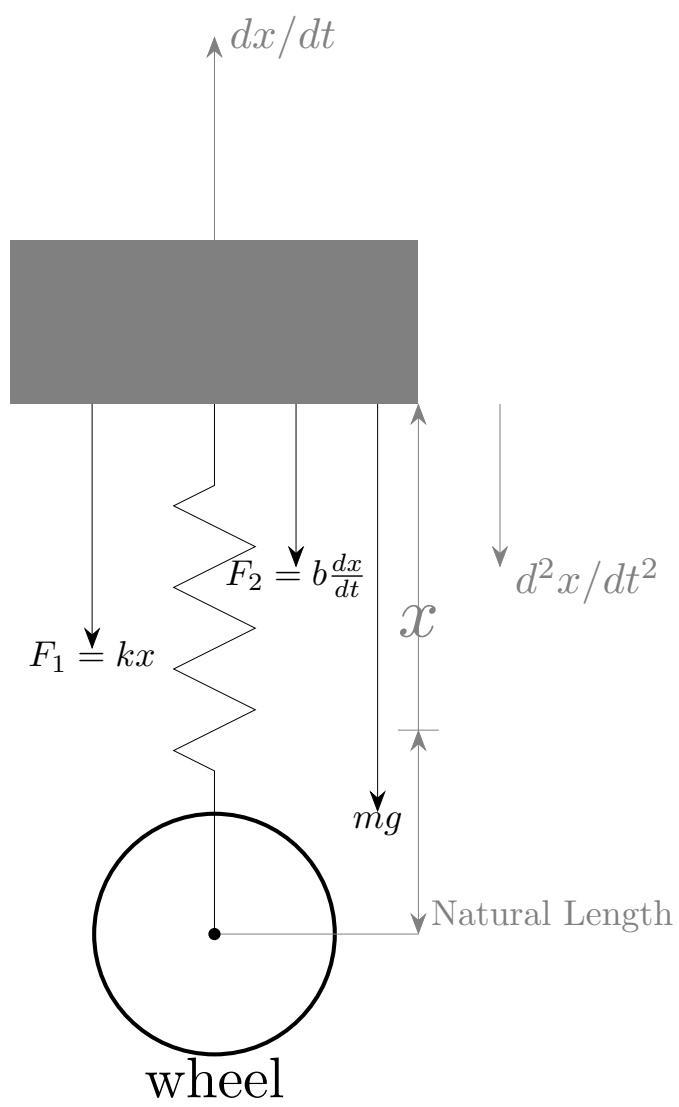$$\implies b = \frac{\sqrt{mk}\ln 2}{\pi} \tag{4.16}$$

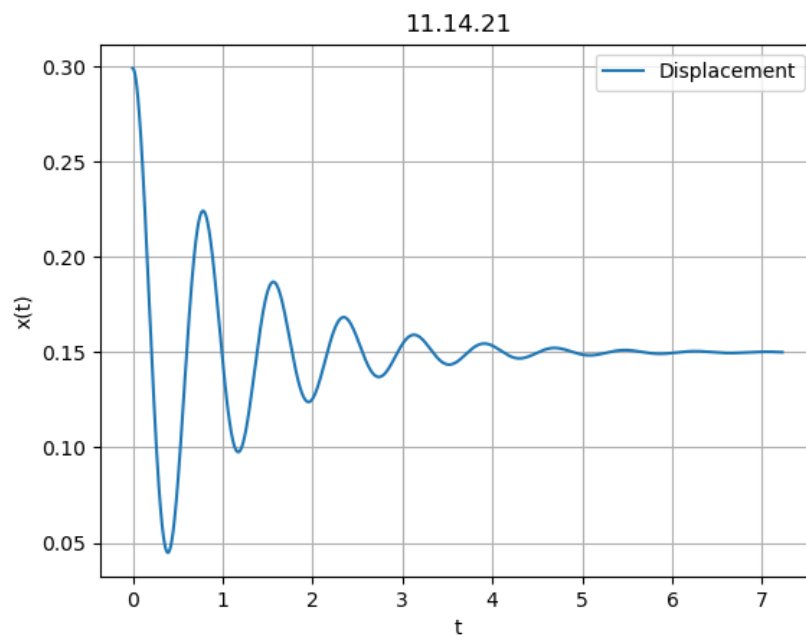Figure 4.1: FBD of the damped oscillation system

Figure 4.2: Displacement Vs. Time Graph

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad (1.0.1.1)$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (1.0.1.2)$$

A.3 If

$$x(n) = 0, \quad n < 0, \qquad (1.0.1.3)$$

from (1.0.1.1),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \qquad (1.0.1.4)$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n)z^{-n} \qquad (2.0.1.1)$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \qquad (2.0.1.2)$$

$$P(z) = P_1(z)P_2(z) \qquad (2.0.1.3)$$

B.3

$$nx(n) \xleftrightarrow{\mathcal{Z}} -zX'(z) \qquad (2.0.1.4)$$

From (2.0.1.4)

$$\implies nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{2.0.1.5}$$

$$\implies n^2 u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, |z| > 1 \tag{2.0.1.6}$$

$$\implies n^3 u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, |z| > 1 \tag{2.0.1.7}$$

$$\implies n^4 u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \tag{2.0.1.8}$$

where $|z| > 1$

B.4

$$x(n-k) \xleftrightarrow{\mathcal{Z}} z^{-k} X(z) \tag{2.0.1.9}$$

Using (2.0.1.9):

$$nu(n-1) \xleftrightarrow{\mathcal{Z}} z\frac{2z^{-2}}{(1-z^{-1})^2} \tag{2.0.1.10}$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \xleftrightarrow{\mathcal{Z}} \frac{z^{-2}}{(1-z^{-1})^2} \tag{2.0.1.11}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \xleftrightarrow{\mathcal{Z}} \frac{z^{-3}}{(1-z^{-1})^3} \tag{2.0.1.12}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\ldots(n-k+1)}{(k-1)!}u(n-k) \xleftrightarrow{\mathcal{Z}} \frac{z^{-k}}{(1-z^{-1})^k} \tag{2.0.1.13}$$

$$\implies Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)\,u\,(n-1) \tag{2.0.1.14}$$

$$\implies Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{2.0.1.15}$$

$$\implies Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{2.0.1.16}$$

$$\implies Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^5}\right] = \frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{2.0.1.17}$$

$$u\,(n-1)$$

B.5 For a Geometric progression

$$x\,(n) = x\,(0)\,r^n u\,(n)\,, \tag{2.0.1.18}$$

$$\implies X\,(z) = \sum_{n=-\infty}^{\infty} x\,(n)\,z^{-n} = \sum_{n=0}^{\infty} x\,(0)\,r^n z^{-n} \tag{2.0.1.19}$$

$$= \sum_{n=0}^{\infty} x\,(0)\,\left(rz^{-1}\right)^n \tag{2.0.1.20}$$

$$= \frac{x\,(0)}{1-rz^{-1}}, \quad |z| > |r| \tag{2.0.1.21}$$

B.6 Substituting $r = 1$ in (2.0.1.21),

$$u(n) \xleftrightarrow{\mathcal{Z}} U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{2.0.1.22}$$

B.7 From (2.0.1.4) and (2.0.1.22),

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{2.0.1.23}$$

B.8 For an AP,

$$x(n) = [x(0) + nd] \, u(n) = x(0)u(n) + dnu(n) \tag{2.0.1.24}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{2.0.1.25}$$

upon substituting from (2.0.1.22) and (2.0.1.23).

B.9 From (1.0.1.4), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{2.0.1.26}$$

where $x(n)$ is defined in (2.0.1.18). From (2.0.1.3), (2.0.1.21) and (2.0.1.22),

$$Y(z) = X(z) U(z) \tag{2.0.1.27}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}} \right) \left(\frac{1}{1 - z^{-1}} \right) \quad |z| > |r| \cap |z| > |1| \tag{2.0.1.28}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{2.0.1.29}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1} \left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}} \right) \tag{2.0.1.30}$$

using partial fractions. Again, from (2.0.1.21) and (2.0.1.22), the inverse of the above can be expressed as

$$y(n) = x(0) \left(\frac{r^{n+1} - 1}{r - 1} \right) u(n) \tag{2.0.1.31}$$

B.10 For the AP $x(n)$, the sum of first $n+1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{2.0.1.32}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{2.0.1.33}$$

$$= x(n) * u(n) \tag{2.0.1.34}$$

Taking the Z-transform on both sides, and substituting (2.0.1.25) and (2.0.1.22),

$$Y(z) = X(z)U(z) \tag{2.0.1.35}$$

$$\implies Y(z) = \left(\frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \right) \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{2.0.1.36}$$

$$= \frac{x(0)}{(1 - z^{-1})^2} + \frac{dz^{-1}}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{2.0.1.37}$$

B.11 From (2.0.1.9) and (2.0.1.23),

$$(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})^2}, \quad |z| > 1, \tag{2.0.1.38}$$

From (2.0.1.38) and (2.0.1.4),

$$n(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^3}, \quad |z| > 1, \tag{2.0.1.39}$$

B.12 Taking the inverse Z-transform of (2.0.1.37),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{2.0.1.40}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\}u(n) \tag{2.0.1.41}$$