Logistic Regression

Boston University CS 506 - Lance Galletti

Logistic Regression

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Assume we have **2 classes**.

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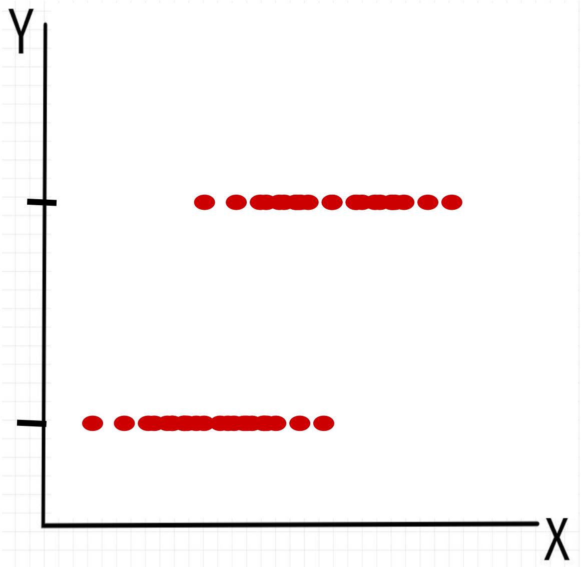
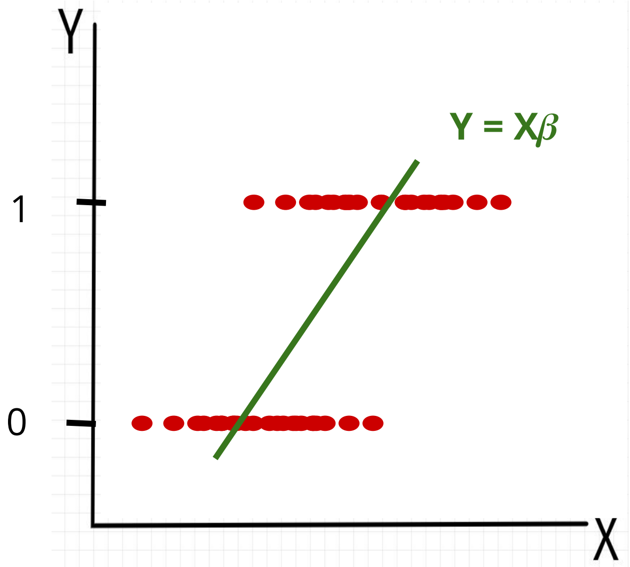


Logistic Regression

What will a linear model look like?



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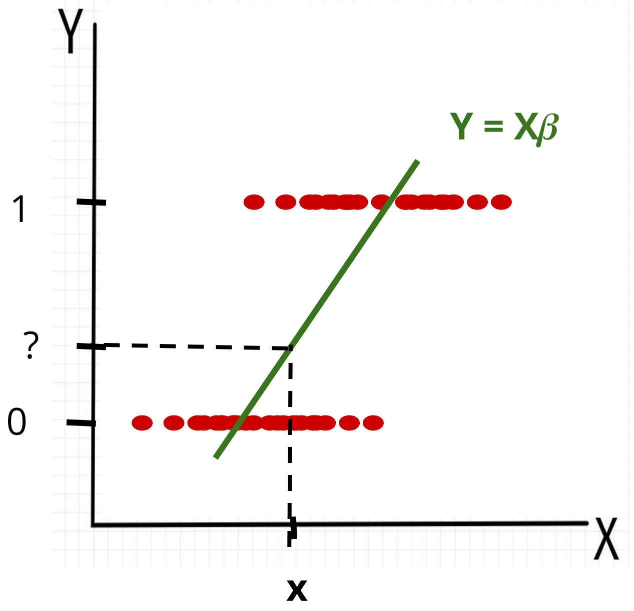
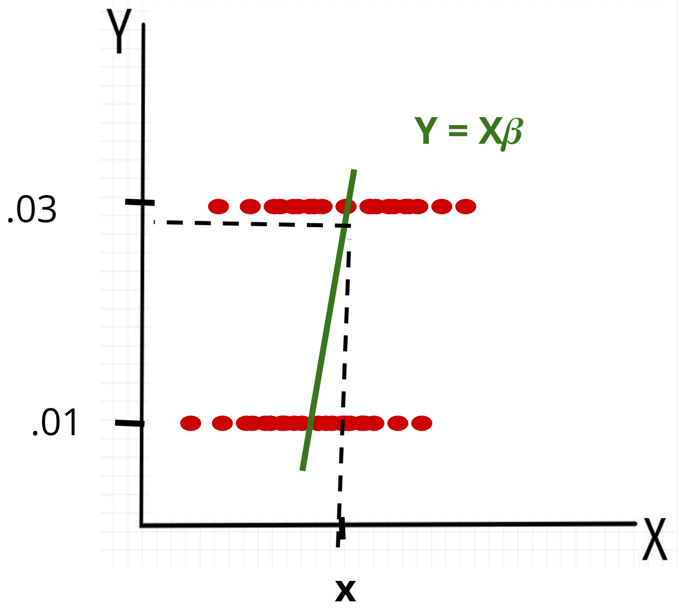


Logistic Regression

What if the numerical values of the classes change?



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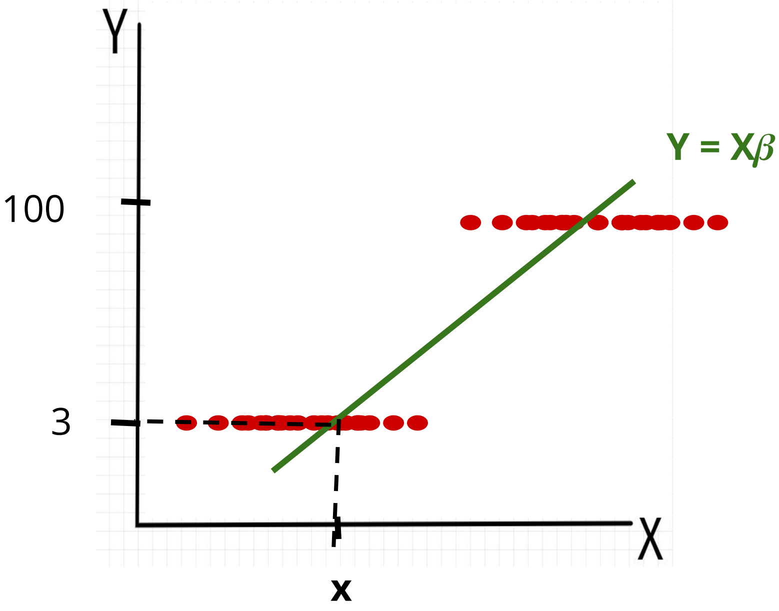

Logistic Regression

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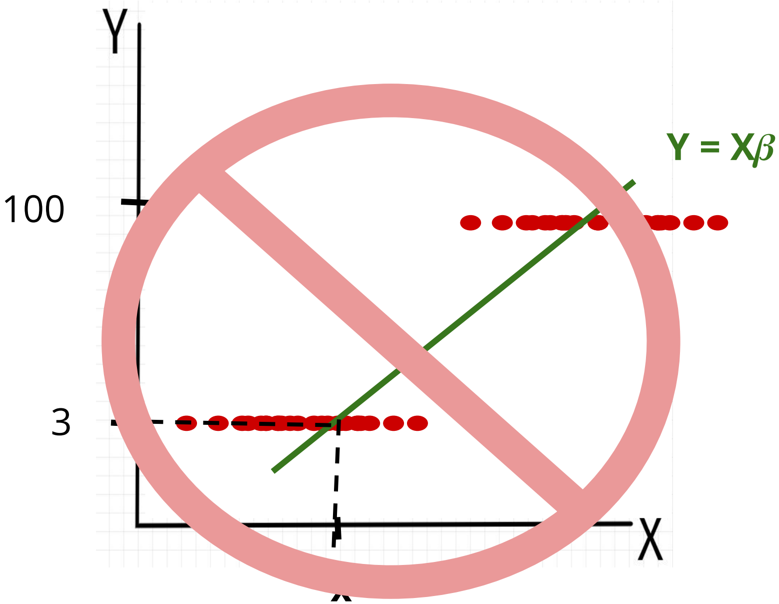

Logistic Regression

The numerical values associated with the class are **arbitrary numbers**. A model based on these numbers would be **meaningless…**

So we **should NOT model the class itself** with a linear model.

Logistic Regression

Notice that a linear function will predict a **continuum** of values. So we should find an interpretation / transformation of the class that is **continuous** for us to predict.

Logistic Regression

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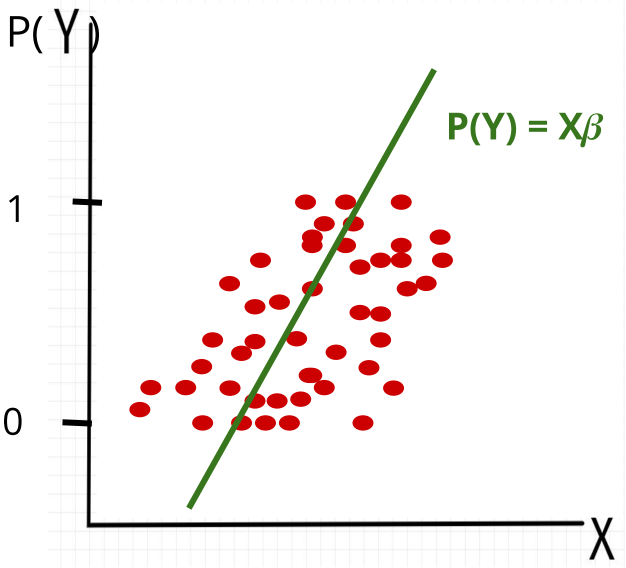

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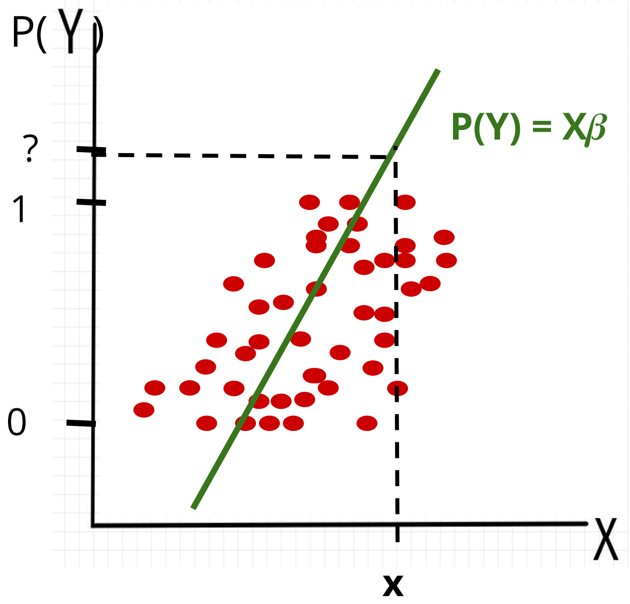

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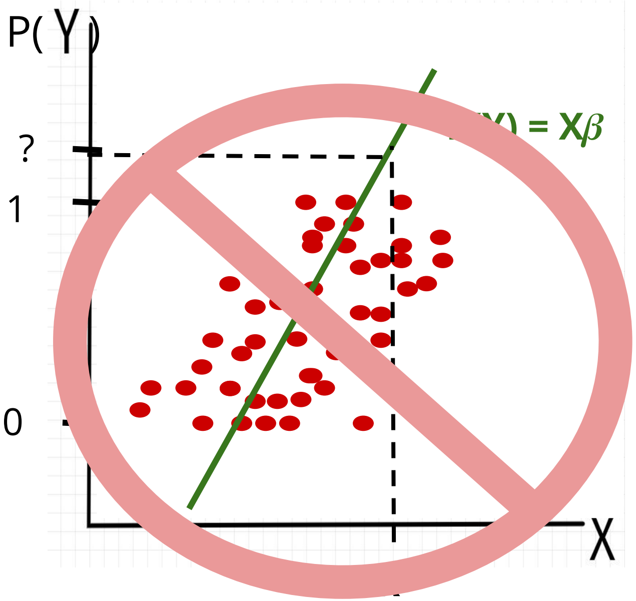


Logistic Regression

So it’s not just a continuum of values - the range of values needs to be (-∞, ∞)!

Define the odds = p / 1 - p where p = P(Y = class 1 | X)

Now the range of $X\boldsymbol{\beta}_{LS}$ is [0, ∞)

In order to get (-∞, ∞), let’s take the log of the odds! This is also convenient numerically because in the odds format, tiny variations in p have large effects on the odds!

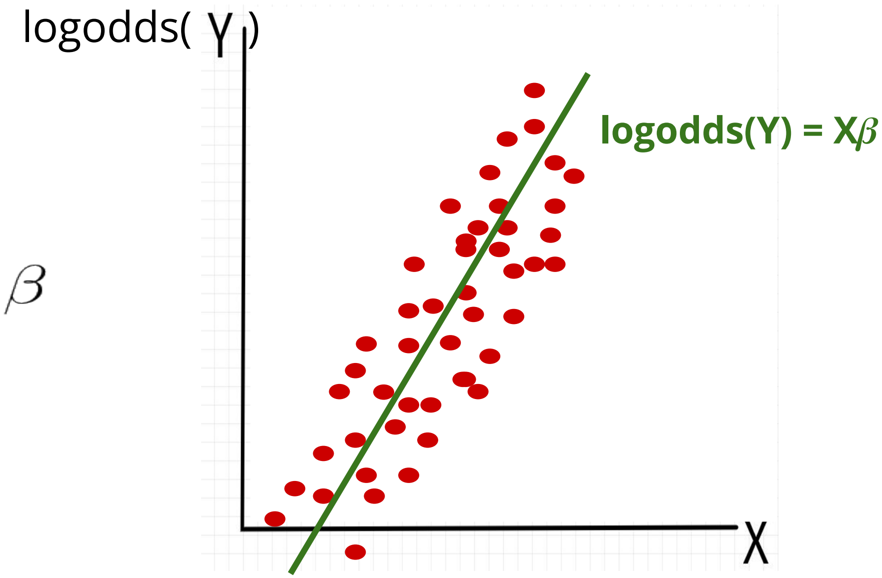
Logistic Regression

Our goal is to fit a linear model to **the log-odds of being in one of our classes** (in the 2-class case) i.e.

$$\log(\frac{P(Y=1|X)}{1-P(Y=1|X)}) = X\beta$$

Logistic Regression

$$\log\left(\frac{P(Y=1|X)}{1-P(Y=1|X)}\right) = X\beta$$



How do we make a prediction with this model?

DECISION RULE:

IF $P(Y=1|X) > ½$ THEN 1 ELSE 0

Logistic Regression

Suppose we have such a model. How do we recover the P(Y=1|X)?

$$\log(\frac{P(Y=1|X)}{1-P(Y=1|X)}) = \alpha + \beta X$$

$$\frac{P(Y=1|X)}{1-P(Y=1|X)} = e^{\alpha+\beta X}$$

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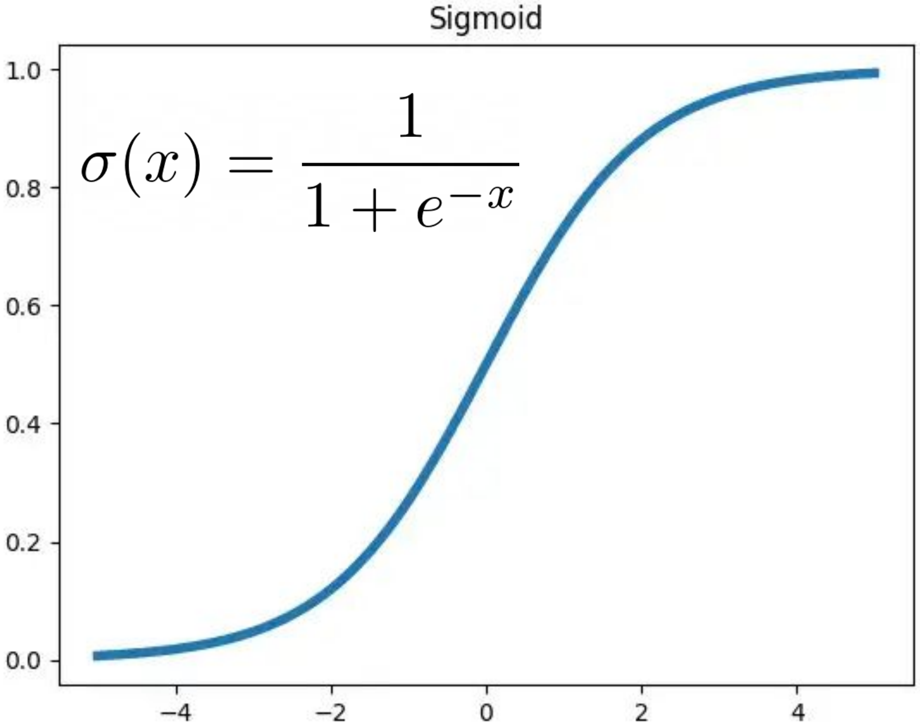
$$P(Y=1|X) = \frac{e^{\alpha+\beta X}}{1+e^{\alpha+\beta X}}$$

The function we apply to our probability to obtain the log odds is called the **logit** function. The function used to retrieve our probability from the log odds is called **logit^{-1}** or **sigmoid**

$$\log(\frac{P(Y=1|X)}{1-P(Y=1|X)}) = \alpha + \beta X$$

$$P(Y=1|X) = \sigma(\alpha + \beta x)$$

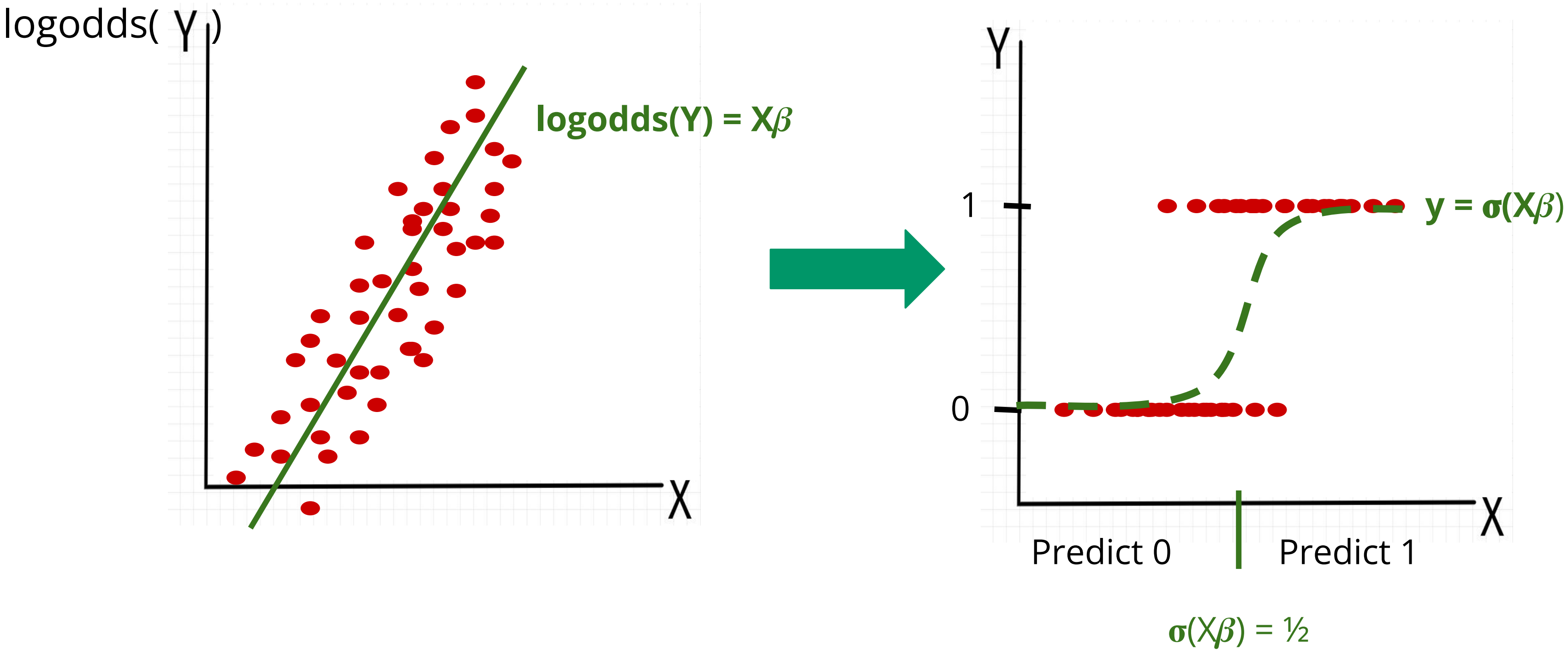


Sigmoid

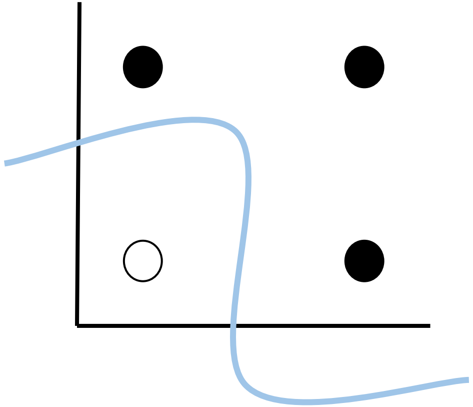
$$\sigma(x) = \frac{1}{1+e^{-x}}$$

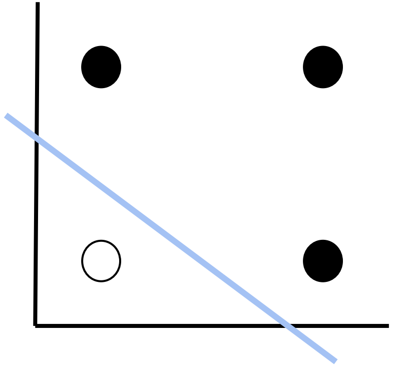
DECISION RULE:

IF P(Y=1|X) > ½ THEN 1 ELSE 0



What does the decision boundary look like?

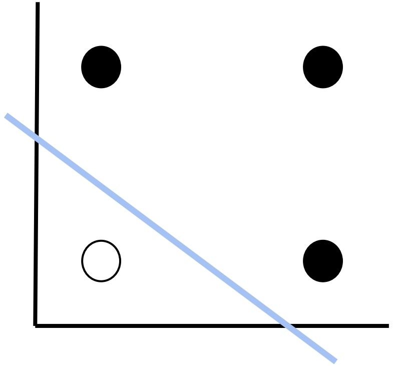




What does the decision boundary look like?

Decision Boundary is where **P(Y = 1 | X) = ½**

$$P(Y = 1|X) = \frac{e^{\alpha+\beta X}}{1 + e^{\alpha+\beta X}}$$



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Decision Boundary is where **e^{wx+b} = 1**

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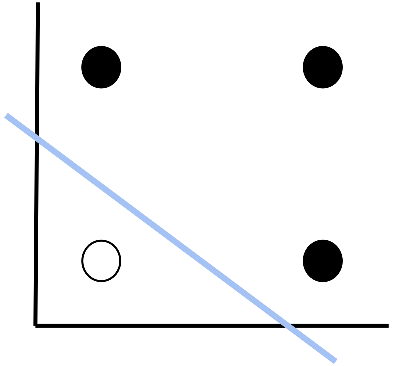

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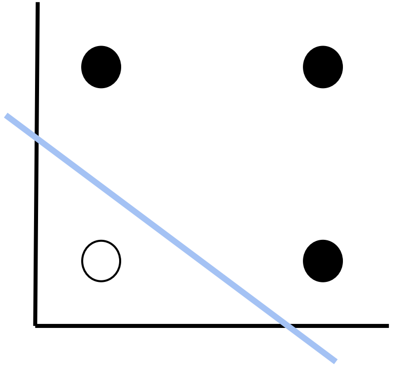
Decision Boundary is where **wx+b = 0**

$$P(Y = 1|X) = \frac{e^{\alpha+\beta X}}{1 + e^{\alpha+\beta X}}$$



Worksheet a) -> c)

Maximum Likelihood Estimator

How do we learn our model? I.e. the α and β parameters.

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We know:

$$P(y_i|x_i) = \begin{cases} \sigma(\alpha + \beta x_i) & \text{if } y_i = 1 \\ 1 - \sigma(\alpha + \beta x_i) & \text{if } y_i = 0 \end{cases}$$

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$$P(y_i|x_i) = \sigma(\alpha + \beta x_i)^{y_i}(1 - \sigma(\alpha + \beta x_i))^{1-y_i}$$

Maximum Likelihood Estimator

So we can define the probability of having seen the data we saw:

$$L(\alpha, \beta) = \prod_{i=1}^{n} P(y_i | x_i)$$

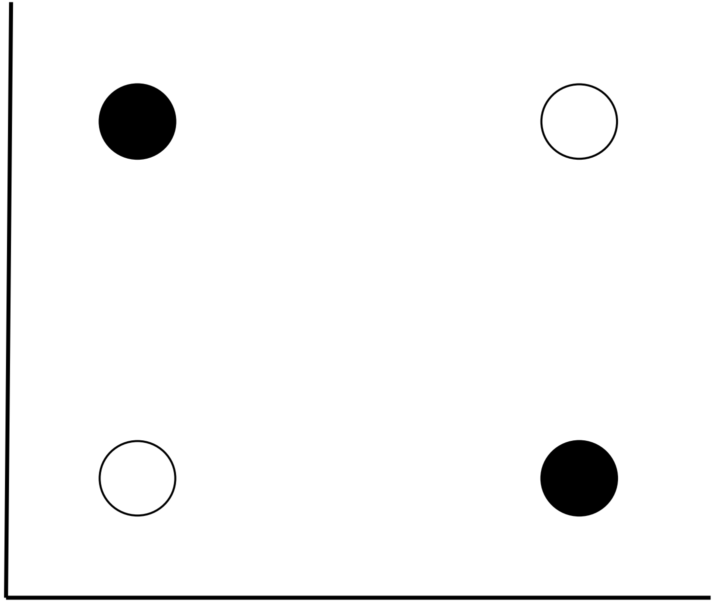
$$= \prod_{i=1}^{n} \sigma(\alpha + \beta x_i)^{y_i} (1 - \sigma(\alpha + \beta x_i))^{1-y_i}$$

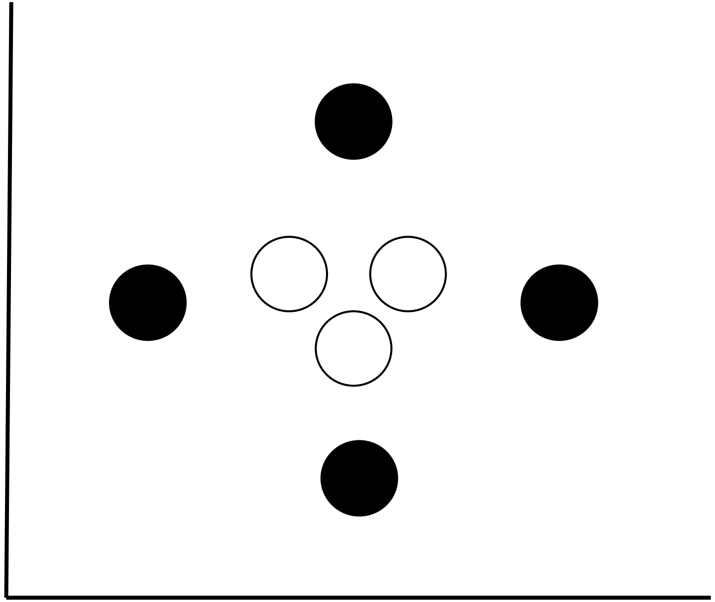
And try to maximize this quantity!

Maximum Likelihood Estimator

We will learn how to solve this next week - it’s not as simple as linear regression

What if the data is not linearly separable





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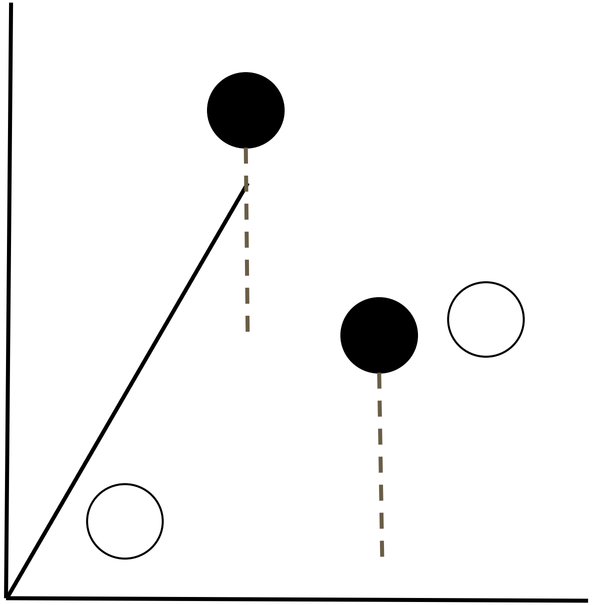

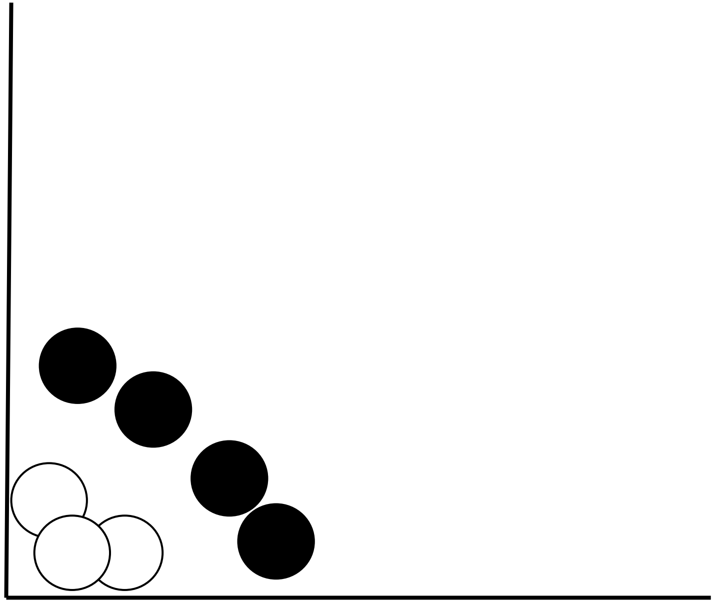

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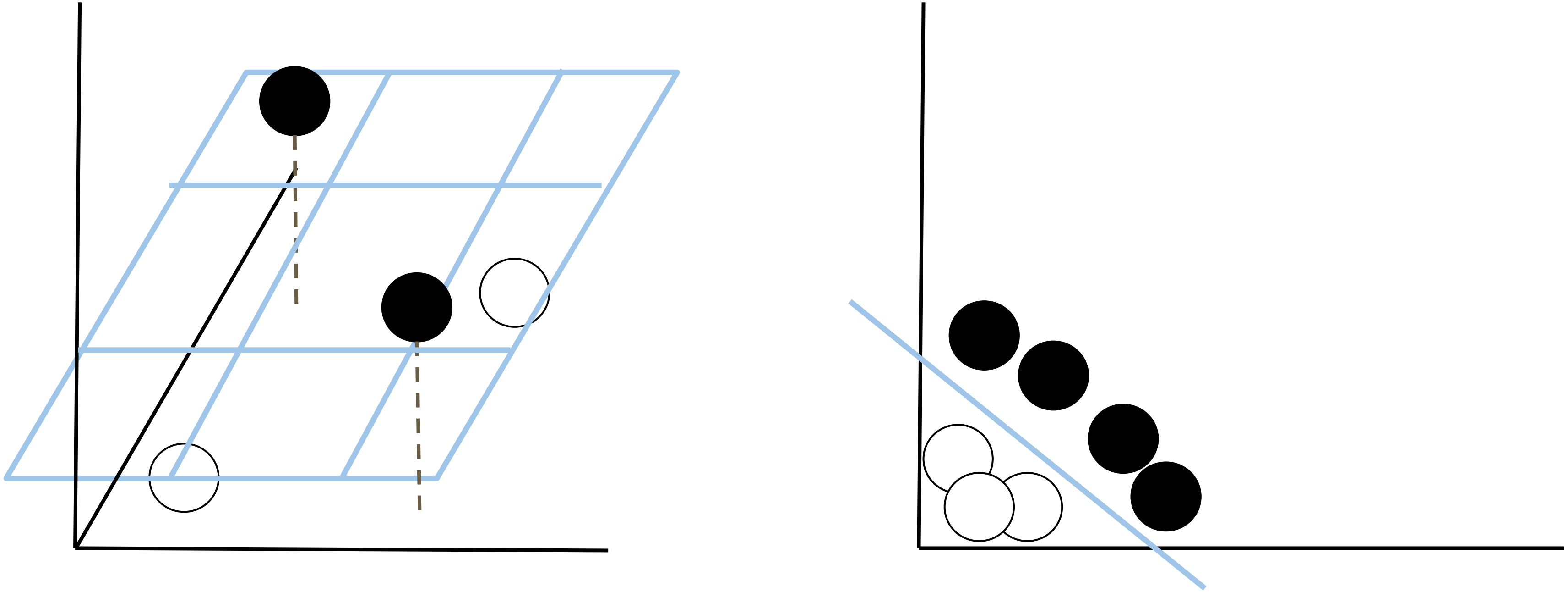
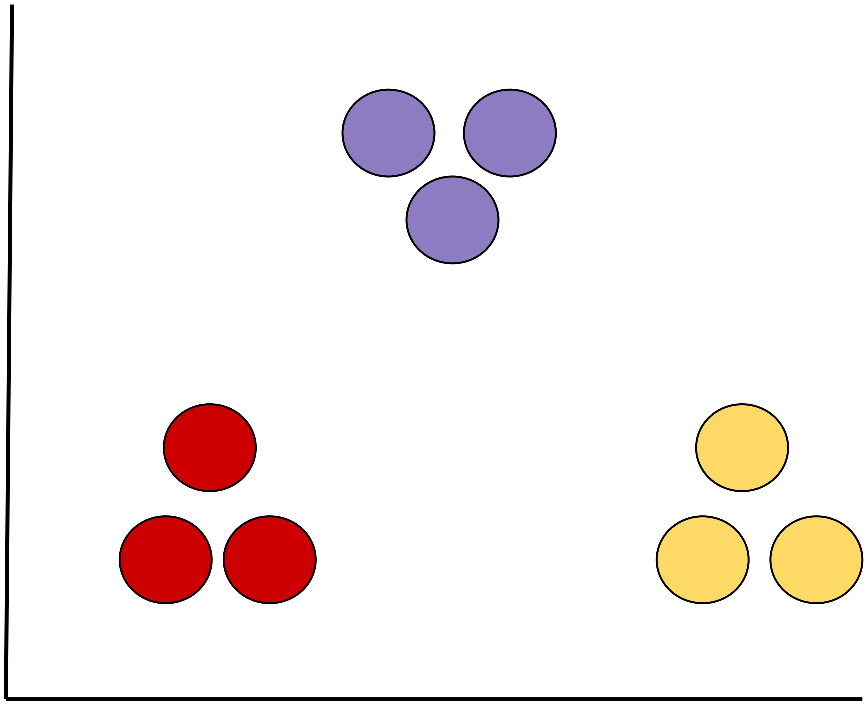


Worksheet d) -> h)

What if there are 3 classes



What if there are 3 classes

Setup:

$$\log\left(\frac{P(Y=0|X)}{P(Y=2|X)}\right)=\beta_0 X$$

$$\log\left(\frac{P(Y=1|X)}{P(Y=2|X)}\right)=\beta_1 X$$

$$P(Y=2|X)=1-(P(Y=1|X)+P(Y=0|X))$$

What if there are 3 classes

$$P(Y = 0|X) = P(Y = 2|X)e^{\beta_0 X}$$

$$P(Y = 1|X) = P(Y = 2|X)e^{\beta_1 X}$$

$$P(Y = 2|X) = 1 - (P(Y = 1|X) + P(Y = 0|X))$$

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$$P(Y=2|X) = \frac{1}{1+e^{\beta_0 X}+e^{\beta_1 X}}$$

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$$P(Y = 0|X) = \frac{e^{\beta_0 X}}{1 + e^{\beta_0 X} + e^{\beta_1 X}}$$

$$P(Y = 1|X) = \frac{e^{\beta_1 X}}{1 + e^{\beta_0 X} + e^{\beta_1 X}}$$

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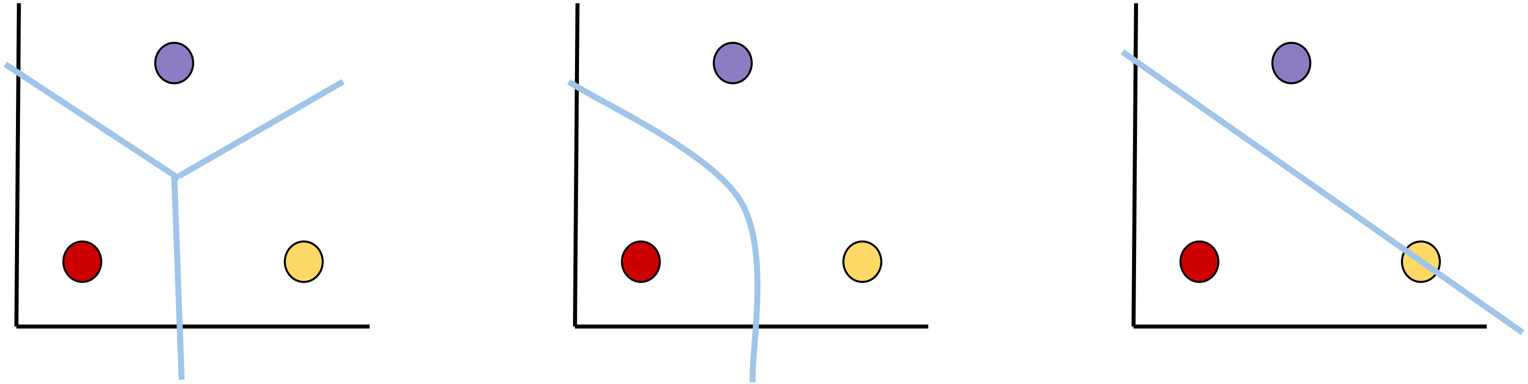

Worksheet i) ->