Student Solutions For Honors Algebra (MATH10069) Past Papers

April 21, 2022

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Student Solutions EXAM 2014-2015

Exam 2014-2015

Question 1

Q1a

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in \mathbb{R} .

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \dots, 15\}$

Q1c

Question 2

$\mathbf{Q2a}$

To show that $\mathcal B$ forms a bsis, consider the matrix that represents $\mathcal B$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that dim B = 3 so \mathcal{B} spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class [v] for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$can(v):V\to V/U=[v]$$

and therefore $\ker(can) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

Question 3

Question 4

Student Solutions EXAM 2015-2016

Exam 2015-2016

 ${\bf Question} \ {\bf 1}$

Question 2

 ${\bf Question} \ {\bf 3}$

Question 4

Student Solutions EXAM 2016-2017

Exam 2016-2017

 ${\bf Question} \ {\bf 1}$

Question 2

 ${\bf Question} \ {\bf 3}$

Question 4

Student Solutions EXAM 2017-2018

Exam 2017-2018

 ${\bf Question} \ {\bf 1}$

Question 2

 ${\bf Question} \ {\bf 3}$

Question 4

Student Solutions EXAM 2018-2019

Exam 2018-2019

 ${\bf Question} \ {\bf 1}$

Question 2

Question 3

Student Solutions EXAM 2019-2020

Exam 2019-2020

Question 1

Question 2

Question 3

Q3a

(i) Recall that the complex inner product has the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \tag{1}$$

$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \tag{2}$$

$$(\vec{x}, \vec{x}) \ge 0 \tag{3}$$

and T being self adjoint

$$(T\vec{x}, y) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue λ .

$$\lambda(x,x) = (x,\lambda x) \quad (property \ 1)$$

$$= (x,Tx) \quad (eigenvec \ property)$$

$$= (Tx,x) \quad (self \ adjoint \ prop)$$

$$= (\lambda x,x) \quad (eigenvec \ propety)$$

$$= \overline{(x,\lambda x)} \quad (property \ 2)$$

$$= (x,\overline{\lambda}x) \quad (not \ sure \ why \ we \ can \ claim \ x = \overline{x} \ here)$$

$$= \overline{\lambda}(x,x) \quad (property \ 1)$$

implying that $\lambda = \overline{\lambda}$, which can only be true is $\lambda \in \mathbb{R}$.

(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim (x, Tx) = (Tx, x) unless T is self adjoint.

(iii)

By the self adjoint property

$$(Tv, w) = (v, Tw)$$

since $Tv = \lambda v$ and $Tw = \mu w$

$$(\lambda v, w) = (v, \mu w)$$

by property 1

$$(\lambda v, w) = \lambda(v, w)$$

$$(v, \mu w) = \mu(v, w)$$

thus

$$\lambda(v, w) = \mu(v, w)$$

which cannot be true unless (v, w) = 0 as λ and μ are distinct eigenvalues.

(iv)

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Question 4

Q4a

 $\sqrt{\alpha} \in \mathbb{Q}$ implies that $\alpha \in \mathbb{Q}$. Thus result of the mapping $f(p_n X^n + \dots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \dots + p_1 \sqrt{\alpha} + p_0$ can be simplied into the form $a + b\sqrt{\alpha}$, where $a, b \in \mathbb{Q}$. If n is even.

$$p_{n}(\sqrt{\alpha})^{n} + \dots + p_{1}\sqrt{\alpha} + p_{0} = p_{0} + p_{2}\alpha + p_{4}\alpha^{2} + \dots + p_{n}\alpha^{\frac{n}{2}} + p_{1}\sqrt{\alpha} + p_{3}\alpha\sqrt{a} + \dots + p_{n-1}\alpha^{\frac{n}{2}-1}\sqrt{\alpha}$$
$$= \sum_{i=0}^{n/2} p_{2i}\alpha^{i} + \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1}\alpha^{i}\sqrt{\alpha}$$

let

$$a = \sum_{i=0}^{n/2} p_{2i} \alpha^i$$

and

$$b = \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1} \alpha^i$$

to get

$$p_0 + p_2\alpha + p_4\alpha^2 + \dots + p_n\alpha^{\frac{n}{2}} = a + b\sqrt{\alpha}$$

proof is same if n is odd except for an additional term. Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$, hence

$$\operatorname{Im} f = \{ a + b\sqrt{\alpha} : a, b \in \mathbb{Q} \}$$

Student Solutions EXAM 2020-2021

Exam 2020-2021

Question 1

Q1a

F: A noncommutative ring is a ring such that

$$a \cdot b \neq b \cdot a$$

and an element in a ring is a zero divisor if there exists non-zero b such that

$$ab = 0$$
 or $ba = 0$

An example of this is H: The ring of quaternions. proof if interested

Q1b

F: Example: $V = \mathbb{R}, W = \mathbb{R}$

$$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$g \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Since $g \circ f(x) = x$ but $\dim V \neq \dim W$

Q1c

Question can be rephrased as: Do $n \times n$ matrices with odd n always have (a real) eigenvalue? **T:** because the characteristic polynomial will have an odd degree, by the intermediate value theorem (proof), it must have at least one real root \implies matrix has at one real eigenvalue.

Q1d

F:

Question 2