Student Solutions For Honors Algebra (MATH10069) Past Papers

April 21, 2022

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Exam 2014-2015

Question 1

Q1a

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in $\mathbb{R}$.

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \ldots, 15\}$

Q1c

Question 2

Q2a

To show that $\mathcal{B}$ forms a bsis, consider the matrix that represents $\mathcal{B}$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that $\dim B = 3$ so $\mathcal{B}$ spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class $[v]$ for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$can(v) : V \to V/U = [v]$$

and therefore $\ker(can) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

Question 3

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Exam 2019-2020

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Q3a

(i) Recall that the complex inner product has the following properties:

$$(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \tag{1}$$
$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \tag{2}$$
$$(\vec{x}, \vec{x}) \geq 0 \tag{3}$$

and T being self adjoint

$$(T\vec{x}, y) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue λ.

$$\begin{aligned}
\lambda(x,x) &= (x, \lambda x) \quad \textit{(property 1)} \\
&= (x, Tx) \quad \textit{(eigenvec property)} \\
&= (Tx, x) \quad \textit{(self adjoint prop)} \\
&= (\lambda x, x) \quad \textit{(eigenvec propety)} \\
&= \overline{(x, \lambda x)} \quad \textit{(property 2)} \\
&= (x, \overline{\lambda} x) \quad \textit{(not sure why we can claim } x = \bar{x} \textit{ here)} \\
&= \overline{\lambda}(x,x) \quad \textit{(property 1)}
\end{aligned}$$

implying that $\lambda = \overline{\lambda}$, which can only be true is $\lambda \in \mathbb{R}$.

(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim $(x, Tx) = (Tx, x)$ unless T is self adjoint.

(iii)

By the self adjoint property

$$(Tv, w) = (v, Tw)$$

since $Tv = \lambda v$ and $Tw = \mu w$

$$(\lambda v, w) = (v, \mu w)$$

by property 1

$$(\lambda v, w) = \lambda(v, w)$$
$$(v, \mu w) = \mu(v, w)$$

thus

$$\lambda(v, w) = \mu(v, w)$$

which cannot be true unless $(v, w) = 0$ as λ and μ are distinct eigenvalues.

(iv)

Question 4

Q4a

$\sqrt{\alpha} \in \mathbb{Q}$ implies that $\alpha \in \mathbb{Q}$. Thus result of the mapping $f(p_n X^n + \cdots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \cdots + p_1\sqrt{\alpha} + p_0$ can be simplied into the form $a + b\sqrt{\alpha}$, where $a, b \in \mathbb{Q}$. If n is even.

$$\begin{aligned} p_n(\sqrt{\alpha})^n + \cdots + p_1\sqrt{\alpha} + p_0 &= p_0 + p_2\alpha + p_4\alpha^2 + \cdots + p_n\alpha^{\frac{n}{2}} \\ &\quad + p_1\sqrt{\alpha} + p_3\alpha\sqrt{a} + \cdots + p_{n-1}\alpha^{\frac{n}{2}-1}\sqrt{\alpha} \\ &= \sum_{i=0}^{n/2} p_{2i}\alpha^i + \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1}\alpha^i\sqrt{\alpha} \end{aligned}$$

let

$$a = \sum_{i=0}^{n/2} p_{2i}\alpha^i$$

and

$$b = \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1}\alpha^i$$

to get

$$p_0 + p_2\alpha + p_4\alpha^2 + \cdots + p_n\alpha^{\frac{n}{2}} = a + b\sqrt{\alpha}$$

proof is same if n is odd except for an additional term. Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$, hence

$$\operatorname{Im} f = \{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$$

Exam 2020-2021

Question 1

Q1a

F: A **noncommutative ring** is a ring such that

$$a \cdot b \neq b \cdot a$$

and an element in a ring is a **zero divisor** if there exists non-zero b such that

$$ab = 0 \quad \text{or} \quad ba = 0$$

An example of this is $\mathbb{H}$: The ring of quaternions. proof if interested

Q1b

F: Example: $V = \mathbb{R}$, $W = \mathbb{R}$

$$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$g\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Since $g \circ f(x) = x$ but $\dim V \neq \dim W$

Q1c

Question can be rephrased as: Do $n \times n$ matrices with odd n always have (a real) eigenvalue?
T: because the characteristic polynomial will have an odd degree, by the intermediate value theorem (proof), it must have at least one real root $\implies$ matrix has at one real eigenvalue.

Q1d

F: The group of units $R^\times$ in a ring R is the set of elements a with multiplicative inverse in R, or

$$R^\times = \{a \in R : \exists\, a^{-1} \in R \text{ s.t. } aa^{-1} = a^{-1}a = 1\}$$

Cyclic group is a group that can be generated by one element.
Note: $(\mathbb{Z}/m\mathbb{Z})^\times$ are elements in the ring that are relatively prime to m, i.e.: $(\mathbb{Z}/m\mathbb{Z})^\times = \{a \in (\mathbb{Z}/m\mathbb{Z}) : gcd(a,m) = 1\}$
Note 2: only for prime m is $(\mathbb{Z}/m\mathbb{Z})^\times$ cyclic.

Consider $(\mathbb{Z}/8\mathbb{Z})^\times = \{1,3,5,7\}$, the group cannot be generated by one element.

Q1e

T: Since A^T and A have the same characteristic polynomial $\implies$ same eigenvalues.

Q1f

T: $R = \mathbb{R}[x]$: polynomials with real coefficients
$_R < x^3 + 3x + 7 >$: think of it as resulting set of polynomials after multiplying every polynomial in R by $x^3 + 3x + 7$. More precisely:

$$_R < x^3 + 3x + 7 >= \{(x^3 + 3x + 7) \cdot a : a \in R\}$$

The **Quotient (or Factor) Ring** R/I is then the cosets of I in R subject to special addition and multiplication[1] And the **equivalence relation** for cosets is defined to be

$$x \sim y \iff x - y \in I$$

for $x, y \in R$
Since,

$$((x^2 + 1) + I)((2x^2 + 3x) + I) = ((2x^4 + 3x^4 + 2x^2 + 3x) + I)$$

let

$$A = 2x^4 + 3x^4 + 2x^2 + 3x$$

and

$$B = -4x^2 - 20x - 21$$
$$A - B = 2x^4 + 3x^3 + 6x^2 + 23x + 21$$

since

$$\frac{A - B}{x^3 + 3x + 7} = 2x + 3$$

we have $A - B \in I$ and hence cosets $(A + I)$ and $(B + I)$ are equivalent

Q1g

F: An inner product must have the following properties:

$$(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \tag{1}$$
$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x}) \tag{2}$$
$$(\vec{x}, \vec{x}) \geq 0 \tag{3}$$

The proposed inner product violates property 3. Consider the polynomial

$$P(x) = \prod_{i=1}^{n-1} (x - i)$$

then

$$(P(x), P(x)) = 0$$

I dont see how it equals 0, maybe im misreading sth

[1]See THEOREM 3.6.4, pg 53

Q1h

T: The complex inner product has the following properties:

$$(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$
$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \quad (2)$$
$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

Fairly straightforward by checking it satisfies the three properties. I think you can probably claim properties 1 trivial/clear from definition. Proving 2:

$$\begin{aligned}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) &= 4x_1\overline{y_1} - 2x_1\overline{y_2} - 2x_2\overline{y_1} + 3x_2\overline{y_2} \\ &= (2x_1 - x_2)\overline{(2y_1 - y_2)} + 2x\end{aligned}$$

proving property 3

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = |2x_1 - x_2|^2 + 2|x_2|^2$$

Q1i

F: The **Image** $\operatorname{Im} f$ of a linear map $f : V \to W$ is $f(V) \in W$. (Everything in W that can be mapped to by f).
The **kernel** $\ker f$ is the set $\{v \in V : f(v) = 0\}$ (everything in V that is mapped to 0_W)

Consider the example $A = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$. The vector $v = \begin{pmatrix}1\\0\end{pmatrix}$ is in both the image and kernel. Since

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{(in kernel)}$$
$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{(in image)}$$

Q1j

someone else explain it better pls

Question 2

Q2a

First notice that the polynomial $m_A(x)$ here is the characteristic polynomial.
$m_A(x)$ for $A = \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}$ is simply $x - \lambda$ because all the λ's lie on the diagonal. so subtracting λI from A would equal 0.
Calculating $m_A(x)$ for $A = \begin{pmatrix}\lambda & 1\\0 & \lambda\end{pmatrix}$

$$\begin{aligned}\chi_A(x) &= \det(A - xI) \\ &= \det\begin{pmatrix}\lambda - x & 1\\0 & \lambda - x\end{pmatrix} \\ &= (\lambda - x)^2\end{aligned}$$

Q2b

A subset I of a ring R is an ideal if

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$.

Clearly $m_A \in I_A$ since $\cdot m_A(A) = 0$. So 1 is true.

For any

$$q(x) \in F[x]$$

it follows that

$$q(x)m_A(x) \in I_A$$

since

$$\begin{aligned} q(A)m_A(A) &= q(A) \cdot 0 \\ &= 0 \end{aligned}$$

solutions claims it is closed under addition, not sure if the difference is meaningful, but we can prove that it is closed until substraction via the following:

Question 3