The University of Edinburgh

College of Science and Engineering

Mathematics 3 Honours

MATH10069 Honours Algebra

Monday, 28th April 2014

9:30am – 12:30pm

Chairman of Examiners – Professor J M Figueroa-O'Farrill

External Examiner – Professor J Greenlees

Credit will be given for the best **FOUR** answers

Calculators approved by the College of Science and Engineering for use in examinations.

Make and Model

Casio fx85 (any version, e.g. fx85WA, fx85MS)
Casio fx83 (any version, e.g. fx83ES)
Casio fx82 (any version)

This examination will be marked anonymously.

Honours Algebra

1. Given an example of the following. [Explanations or justifications are not required.]

  (a) An infinite dimensional vector space over a field.

  (b) A vector space with exactly 16 elements.

  (c) A basis for the real vector space of 2×2-matrices with real entries, $\mathrm{Mat}(2; \mathbb{R})$.

  (d) A linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ whose nullity is 1.

  (e) An injective linear mapping from a vector space of dimension 1 to a vector space of dimension 2.

  (f) A mapping from a vector space of dimension 1 to a vector space of dimension 2 that is not linear.

  (g) A linear mapping between vector spaces defined without writing a matrix.

  (h) A linear mapping that is neither injective nor surjective.

  (i) A symmetric bilinear form.

  (j) An alternating multilinear form on $V \times V \times V \times V \to \mathbb{R}$ for some non-zero real vector space V.

  (k) An integral domain.

  (l) A commutative ring in which all non-zero elements are invertible.

  (m) A noncommutative ring in which all non-zero elements are invertible.

  (n) A commutative ring in which not every ideal is a principal ideal.

  (o) A ring with infinitely many elements whose group of units is finite.

  (p) A matrix $A \in \mathrm{Mat}(2; \mathbb{R})$ that has two distinct real eigenvalues.

  (q) A matrix $A \in \mathrm{Mat}(2; \mathbb{R})$ that has no real eigenvalues.

  (r) A matrix with entries in $\mathbb{C}$ that is diagonalisable but not diagonal.

  (s) An invertible matrix $A \in \mathrm{Mat}(2; \mathbb{Z})$ that is not the identity.

  (t) An matrix $A \in \mathrm{Mat}(3; \mathbb{R})$ that is non-invertible and has trace 1.

  (u) A matrix $A \in \mathrm{Mat}(3; \mathbb{C})$ in Jordan Normal Form.

(v) A matrix $A \in \mathrm{Mat}(4; \mathbb{R})$ all of whose entries are non-zero and all of whose eigenvalues are real.

(w) An inner product on $\mathbb{R}^2$.

(x) A self-adjoint operator on $\mathbb{R}^2$ with respect to the inner product you defined.

(y) A non self-adjoint operator on $\mathbb{R}^2$ with respect to the inner product you defined.

[25 marks]

2. Throughout this question U, V and W denote vector spaces over a field F.

(a) Let $\mathcal{S} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis of $V = \mathbb{R}^3$ and let $\mathcal{B} = (\vec{v}_1 = \vec{e}_1, \vec{v}_2 = \vec{e}_1 + \vec{e}_2, \vec{v}_3 = \vec{e}_1 + \vec{e}_2 + \vec{e}_3)$. Show that $\mathcal{B}$ is a basis of V. Now suppose that a linear mapping $f : V \to V$ is represented with respect to $\mathcal{S}$ by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the matrix B that represents f with respect to $\mathcal{B}$. Write down an equation that expresses the relationship between A and B. **[8 marks]**

(b) (i) Assume that $U \subseteq V$ is a subspace of the vector space V. Define the quotient vector space V/U including stating the definition of addition and scalar multiplication in V/U. Define the canonical mapping $\mathsf{can} : V \to V/U$. What is the kernel of can? **[7 marks]**

(ii) Let $f : V \to W$ be a linear mapping such that $U \subseteq \ker(f)$. Show that the mapping $\overline{f} : V/U \to W$ given by $\overline{f}(v + U) = f(v)$ is a well-defined linear mapping. Show that

$$\ker(\overline{f}) = \mathsf{can}(\ker(f))$$

where $\mathsf{can}(\ker(f)) = \{\mathsf{can}(x) : x \in \ker(f)\}$. **[6 marks]**

(iii) State the Rank-Nullity Theorem carefully and clearly. Apply it in the situation of (ii) above to give a formula for the nullity of f in terms of the nullity of $\overline{f}$ and the dimension of U. **[4 marks]**

3. Throughout this question R will denote a ring.

(a) Let S be a nonempty subset of a ring R. Define what is meant by S being a subring of R. State the Test for a Subring. Show that $S = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \text{ is odd}\}$ is a subring of $\mathbb{Q}$, the rational numbers. **[8 marks]**

(b) Explain why each of the following mappings is not a ring homomorphism.

(i) $f_1 : \mathbb{Z} \to \mathbb{Z}$ where $f_1(z) = z^2$.

(ii) $f_2 : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$ where $f_2([a]) = [4a]$

(iii) $f_3 : \mathrm{Mat}(2, \mathbb{R}) \to \mathrm{Mat}(2; \mathbb{R})$ where $f_3(A) = A^{\mathsf{T}}$.

[6 marks]

(c) Let $R = \mathbb{C}[X]$ be the ring of polynomials with coefficients in $\mathbb{C}$.

(i) Define the degree of a non-zero element $P \in R$.

(ii) Let $P, Q \in R$ with $Q \neq 0$. Prove that there exist $A, B \in R$ such that

$$P = AQ + B$$

and $\deg(B) < \deg(Q)$ or $B = 0$.

(iii) Suppose that $I \trianglelefteq R$ is a non-zero ideal of R. Let Q be a non-zero polynomial of minimal degree belonging to I. Show that $I = {}_R\langle Q \rangle$, the ideal generated by Q.

(iv) Hence, or otherwise, show that

$$ {}_R\langle X - 1 \rangle \cap {}_R\langle X + 1 \rangle = {}_R\langle X^2 - 1 \rangle.$$

[11 marks]

4. Let V be a finite dimensional real vector space.

(a) State the defining properties of an inner product on V. Let U be a subspace of V. Define the orthogonal complement $U^{\perp}$ of U in V and prove that $V = U \oplus U^{\perp}$, stating clearly any theorems you assume during the proof. **[11 marks]**

(b) In each of the following cases state whether the given formula defines an inner product on $\mathbb{R}^2$. For any that are *not* inner products, prove that they are not; for any that *are* inner products, a proof is not required. In each formula $\vec{x} = (x_1, x_2)^{\mathsf{T}}$ and $\vec{y} = (y_1, y_2)^{\mathsf{T}}$.

(i) $(\vec{x}, \vec{y}) = x_1 + y_1 + 3x_2y_2$.

(ii) $(\vec{x}, \vec{y}) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$.

(iii) $(\vec{x}, \vec{y}) = x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$.

[5 marks]

(c) Let $\mathbb{R}^4$ have the standard inner product and let U be the subspace of $\mathbb{R}^4$ with basis $\{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = (0, 1, 0, -1)^{\mathsf{T}} \text{ and } \vec{u}_2 = (1, 2, 1, -4)^{\mathsf{T}}.$$

Using the Gram-Schmidt process, construct an orthonormal basis for U. Find the orthogonal projection of $(2, 2, -1, 1)^{\mathsf{T}}$ onto U.

[5 marks]

(d) Let $T : V \to V$ be a linear mapping and assume that T is self-adjoint.

(i) State the Spectral Theorem for T.

(ii) Suppose that each eigenvalue of T is greater than or equal to $\alpha \in \mathbb{R}$. Prove that

$$(T\vec{x}, \vec{x}) \geq \alpha(\vec{x}, \vec{x}).$$

[4 marks]

5.

(a) Let $A \in \mathrm{Mat}(n; F)$ with F a field and let $\lambda \in F$. Define $E(\lambda, A)$, the eigenspace of A with eigenvalue λ, and $E^{\mathrm{gen}}(\lambda, A)$, the generalised eigenspace of A with eigenvalue λ. Define too the algebraic and geometric multiplicities of A with eigenvalue λ. Define $\chi_A(x)$, the characteristic polynomial of A. Prove that the eigenvalues of A are exactly the roots of $\chi_A(x) \in F[x]$. **[8 marks]**

(b) Let $A \in \mathrm{Mat}(4; \mathbb{C})$ be the following matrix:

$$A = \begin{pmatrix} 10 & -5 & -13 & 8 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ -6 & 3 & 11 & -8 \end{pmatrix}$$

There exists a matrix $P \in \mathrm{Mat}(4; \mathbb{C})$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[You do not need to prove this.]

(i) What is $\chi_A(x)$, the characteristic polynomial of A?

(ii) Calculate P.

(iii) Give two other possible Jordan Normal Forms for a matrix in $\mathrm{Mat}(4; \mathbb{C})$ whose characteristic polynomial equals $\chi_A(x)$.

[13 marks]

(c) Let $\mathbb{F}_2$ be the field with two elements and let $B \in \mathrm{Mat}(2n; \mathbb{F}_2)$ be the $(2n \times 2n)$-matrix all of whose entries are 1.

(i) Show that $B^2 = 0$.

(ii) What are the geometric and algebraic multiplicities of B with eigenvalue 0?

[4 marks]