

Student Solutions For Honors Algebra (MATH10069) Past Papers

May 5, 2022

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Exam 2014-2015**Question 1****Q1a**

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in \mathbb{R} .

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \dots, 15\}$

Q1c**Question 2****Q2a**

To show that \mathcal{B} forms a basis, consider the matrix that represents \mathcal{B}

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that $\dim B = 3$ so \mathcal{B} spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class $[v]$ for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$\text{can}(v) : V \rightarrow V/U = [v]$$

and therefore $\ker(\text{can}) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

Question 3**Question 4****Question 5**

Exam 2015-2016**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2016-2017**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2017-2018**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2018-2019**Question 1****Question 2****Question 3****Question 4**

Exam 2019-2020**Question 1****Question 2****Q2a**

To get the representing matrix of f , evaluate f at the standard basis.

$$\begin{aligned}f(1, 0, 0) &= (0, 6) \\f(0, 1, 0) &= (-2, -4) \\f(0, 0, 1) &= (3, 0)\end{aligned}$$

Lets call the representing matrix M

$$M = \begin{pmatrix} 0 & -2 & 3 \\ 6 & -4 & 0 \end{pmatrix}$$

Note that f currently is mapping from \mathbb{R}^3 to \mathbb{R}^2 w.r.t. to standard bases $S(3)$ and $S(2)$. So MA , where A is matrix formed by the ordered basis \mathcal{A}

$$\begin{aligned}A &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\MA &= \begin{pmatrix} 0 & -2 & 3 \\ 6 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\&= \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & -4 \end{pmatrix}\end{aligned}$$

represents $_{S(2)}[f]_{\mathcal{A}}$.

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

finally $_{\mathcal{B}}[f]_{\mathcal{A}}$ is then $B^{-1}MA$

$$\begin{aligned}&= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & -4 \end{pmatrix} \\&= \begin{pmatrix} 0 & 0 & 4 \\ 1 & -1 & -4 \end{pmatrix}\end{aligned}$$

Q2b

(i) Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned}Tr(A)^2 &= (a + d)^2 \\Tr(A^2) &= a^2 + bc + bc + d^2\end{aligned}$$

by construction, we have

$$\begin{aligned}Tr(A)^2 &= Tr(A^2) \\(a + d)^2 &= a^2 + 2bc + d^2 \\a^2 + 2ad + d^2 &= a^2 + 2bc + d^2\end{aligned}$$

which implies

$$ad = bc$$

since $\det A = ad - bc$, it follows that $\det A = 0$

(ii) idk

Q2c

(i) A mapping is a ring homomorphism if

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \end{aligned}$$

 f_1 violates the first property, consider

$$f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but

$$\begin{aligned} f(1) + f(-1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

so

$$f(0) \neq f(-1) + f(1)$$

(ii)

consider $X = 1 + x^2$ and $Y = 1 - x^2$

$$f(XY) =$$

Question 3**Q3a**

(i) Recall that the complex inner product has the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$

$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \quad (2)$$

$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

and T being self adjoint

$$(T\vec{x}, \vec{y}) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue λ .

$$\begin{aligned} \lambda(x, x) &= (x, \lambda x) \quad (\text{property 1}) \\ &= (x, Tx) \quad (\text{eigenvec property}) \\ &= (Tx, x) \quad (\text{self adjoint prop}) \\ &= (\lambda x, x) \quad (\text{eigenvec property}) \\ &= \overline{(x, \lambda x)} \quad (\text{property 2}) \\ &= (x, \bar{\lambda} x) \quad (\text{not sure why we can claim } x = \bar{x} \text{ here}) \\ &= \bar{\lambda}(x, x) \quad (\text{property 1}) \end{aligned}$$

implying that $\lambda = \bar{\lambda}$, which can only be true is $\lambda \in \mathbb{R}$.(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim $(x, Tx) = (Tx, x)$ unless T is self adjoint.

(iii)

By the self adjoint property

$$(Tv, w) = (v, Tw)$$

since $Tv = \lambda v$ and $Tw = \mu w$

$$(\lambda v, w) = (v, \mu w)$$

by property 1

$$(\lambda v, w) = \lambda(v, w)$$

$$(v, \mu w) = \mu(v, w)$$

thus

$$\lambda(v, w) = \mu(v, w)$$

which cannot be true unless $(v, w) = 0$ as λ and μ are distinct eigenvalues.

(iv)

Question 4

Q4a

$\sqrt{\alpha} \in \mathbb{Q}$ implies that $\alpha \in \mathbb{Q}$. Thus result of the mapping $f(p_n X^n + \cdots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \cdots + p_1 \sqrt{\alpha} + p_0$ can be simplified into the form $a + b\sqrt{\alpha}$, where $a, b \in \mathbb{Q}$. If n is even.

$$\begin{aligned} p_n(\sqrt{\alpha})^n + \cdots + p_1 \sqrt{\alpha} + p_0 &= p_0 + p_2 \alpha + p_4 \alpha^2 + \cdots + p_n \alpha^{\frac{n}{2}} \\ &\quad + p_1 \sqrt{\alpha} + p_3 \alpha \sqrt{\alpha} + \cdots + p_{n-1} \alpha^{\frac{n}{2}-1} \sqrt{\alpha} \\ &= \sum_{i=0}^{n/2} p_{2i} \alpha^i + \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1} \alpha^i \sqrt{\alpha} \end{aligned}$$

let

$$a = \sum_{i=0}^{n/2} p_{2i} \alpha^i$$

and

$$b = \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1} \alpha^i$$

to get

$$p_0 + p_2 \alpha + p_4 \alpha^2 + \cdots + p_n \alpha^{\frac{n}{2}} = a + b\sqrt{\alpha}$$

proof is same if n is odd except for an additional term. Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$, hence

$$\text{Im } f = \{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$$

Exam 2020-2021

Question 1

Q1a

F: A **noncommutative ring** is a ring such that

$$a \cdot b \neq b \cdot a$$

and an element in a ring is a **zero divisor** if there exists non-zero b such that

$$ab = 0 \quad \text{or} \quad ba = 0$$

An example of this is \mathbb{H} : The ring of quaternions. [proof if interested](#)

Q1b

F: Example: $V = \mathbb{R}$, $W = \mathbb{R}$

$$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Since $g \circ f(x) = x$ but $\dim V \neq \dim W$

Q1c

Question can be rephrased as: Do $n \times n$ matrices with odd n always have (a real) eigenvalue?

T: because the characteristic polynomial will have an odd degree, by the intermediate value theorem ([proof](#)), it must have at least one real root \implies matrix has at one real eigenvalue.

Q1d

F: The **group of units** R^\times in a ring R is the set of elements a with multiplicative inverse in R , or

$$R^\times = \{a \in R : \exists a^{-1} \in R \text{ s.t. } aa^{-1} = a^{-1}a = 1\}$$

Cyclic group is a group that can be generated by one element.

Note: $(\mathbb{Z}/m\mathbb{Z})^\times$ are elements in the ring that are relatively prime to m , i.e.: $(\mathbb{Z}/m\mathbb{Z})^\times = \{a \in (\mathbb{Z}/m\mathbb{Z}) : \gcd(a, m) = 1\}$

Note 2: only for prime m is $(\mathbb{Z}/m\mathbb{Z})^\times$ cyclic.

Consider $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$, the group cannot be generated by one element.

Q1e

T: Since A^T and A have the same characteristic polynomial \implies same eigenvalues.

Q1f

T: $R = \mathbb{R}[x]$: polynomials with real coefficients

$R < x^3 + 3x + 7 >$: think of it as resulting set of polynomials after multiplying every polynomial in R by $x^3 + 3x + 7$. More precisely:

$$R < x^3 + 3x + 7 > = \{(x^3 + 3x + 7) \cdot a : a \in R\}$$

The **Quotient (or Factor) Ring** R/I is then the cosets of I in R subject to special addition and multiplication¹ And the **equivalence relation** for cosets is defined to be

$$x \sim y \iff x - y \in I$$

for $x, y \in R$

Since,

$$((x^2 + 1) + I)((2x^2 + 3x) + I) = ((2x^4 + 3x^4 + 2x^2 + 3x) + I)$$

let

$$A = 2x^4 + 3x^4 + 2x^2 + 3x$$

and

$$B = -4x^2 - 20x - 21$$

$$A - B = 2x^4 + 3x^3 + 6x^2 + 23x + 21$$

since

$$\frac{A - B}{x^3 + 3x + 7} = 2x + 3$$

we have $A - B \in I$ and hence cosets $(A + I)$ and $(B + I)$ are equivalent

Q1g

F: An inner product must have the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$

$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x}) \quad (2)$$

$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

The proposed inner product violates property 3. Consider the polynomial

$$P(x) = \prod_{i=1}^{n-1} (x - i)$$

then

$$(P(x), P(x)) = 0$$

I dont see how it equals 0, maybe im misreading sth

¹See THEOREM 3.6.4, pg 53

Q1h

T: The complex inner product has the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$

$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \quad (2)$$

$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

Fairly straightforward by checking it satisfies the three properties. **I think you can probably claim properties 1 trivial/clear from definition.** Proving 2:

$$\begin{aligned} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) &= 4x_1\overline{y_1} - 2x_1\overline{y_2} - 2x_2\overline{y_1} + 3x_2\overline{y_2} \\ &= (2x_1 - x_2)\overline{(2y_1 - y_2)} + 2x_2\overline{y_2} \end{aligned}$$

proving property 3

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = |2x_1 - x_2|^2 + 2|x_2|^2$$

Q1i

F: The **Image** $\text{Im } f$ of a linear map $f : V \rightarrow W$ is $f(V) \in W$. (Everything in W that can be mapped to by f).

The **kernel** $\ker f$ is the set $\{v \in V : f(v) = 0\}$ (everything in V that is mapped to 0_W)

Consider the example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in both the image and kernel. Since

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{in kernel}) \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{in image}) \end{aligned}$$

Q1j

someone else explain it better pls

Question 2**Q2a**

$m_A(x)$ for $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is simply $x - \lambda$ because all the λ 's lie on the diagonal. so subtracting λI from A would equal 0.

$m_A(x)$ for $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, not exactly sure how besides calculating characteristic polynomial.

$$\begin{aligned} \chi_A(x) &= \det(A - xI) \\ &= \det \begin{pmatrix} \lambda - x & 1 \\ 0 & \lambda - x \end{pmatrix} \\ &= (\lambda - x)^2 \end{aligned}$$

Q2b

A subset I of a ring R is an ideal if

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$.

Clearly $m_A \in I_A$ since $\cdot m_A(A) = 0$. So 1 is true.

For any

$$q(x) \in F[x]$$

it follows that

$$q(x)m_A(x) \in I_A$$

since

$$\begin{aligned} q(A)m_A(A) &= q(A) \cdot 0 \\ &= 0 \end{aligned}$$

solutions claims it is closed under addition, not sure if the difference is meaningful, but we can prove that it is closed until subtraction via the following:

Consider $p, q \in F[x]$, clearly

$$p(x)m_A(x) \in I_A$$

and

$$q(x)m_A(x) \in I_A$$

Since

$$p(x) - q(x) \in F[x]$$

and

$$\begin{aligned} (p(A) - q(A))m_A(A) &= (p(A) - q(A))0 \\ &= 0 \end{aligned}$$

so

$$(p(x) - q(x))m_A(x) \in I_A$$

and therefore I_A is closed under subtraction

Question 3