

Student Solutions For Honors Algebra (MATH10069) Past Papers

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Exam 2014-2015**Question 1****Q1a**

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in \mathbb{R} .

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \dots, 15\}$

Q1c**Question 2****Q2a**

To show that \mathcal{B} forms a basis, consider the matrix that represents \mathcal{B}

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that $\dim B = 3$ so \mathcal{B} spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class $[v]$ for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$can(v) : V \rightarrow V/U = [v]$$

and therefore $\ker(can) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

Question 3**Question 4****Question 5**

Exam 2015-2016**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2016-2017**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2017-2018**Question 1****Question 2****Question 3****Question 4****Question 5**

Exam 2018-2019**Question 1****Question 2****Question 3****Question 4**

Exam 2019-2020

Question 1

Question 2

Question 3

Q3a

(i) Recall that the complex inner product has the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$

$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \quad (2)$$

$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

and T being self adjoint

$$(T\vec{x}, \vec{y}) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue λ .

$$\begin{aligned} \lambda(x, x) &= (x, \lambda x) \quad (\text{property 1}) \\ &= (x, Tx) \quad (\text{eigenvec property}) \\ &= (Tx, x) \quad (\text{self adjoint prop}) \\ &= (\lambda x, x) \quad (\text{eigenvec property}) \\ &= \overline{(x, \lambda x)} \quad (\text{property 2}) \\ &= (x, \bar{\lambda}x) \quad (\text{not sure why we can claim } x = \bar{x} \text{ here}) \\ &= \bar{\lambda}(x, x) \quad (\text{property 1}) \end{aligned}$$

implying that $\lambda = \bar{\lambda}$, which can only be true is $\lambda \in \mathbb{R}$.

(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim $(x, Tx) = (Tx, x)$ unless T is self adjoint.

(iii)

By the self adjoint property

$$(Tv, w) = (v, Tw)$$

since $Tv = \lambda v$ and $Tw = \mu w$

$$(\lambda v, w) = (v, \mu w)$$

by property 1

$$(\lambda v, w) = \lambda(v, w)$$

$$(v, \mu w) = \mu(v, w)$$

thus

$$\lambda(v, w) = \mu(v, w)$$

which cannot be true unless $(v, w) = 0$ as λ and μ are distinct eigenvalues.

(iv)

Question 4**Q4a**

$\sqrt{\alpha} \in \mathbb{Q}$ implies that $\alpha \in \mathbb{Q}$. Thus result of the mapping $f(p_n X^n + \cdots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \cdots + p_1 \sqrt{\alpha} + p_0$ can be simplified into the form $a + b\sqrt{\alpha}$, where $a, b \in \mathbb{Q}$. If n is even.

$$\begin{aligned} p_n(\sqrt{\alpha})^n + \cdots + p_1 \sqrt{\alpha} + p_0 &= p_0 + p_2 \alpha + p_4 \alpha^2 + \cdots + p_n \alpha^{\frac{n}{2}} \\ &\quad + p_1 \sqrt{\alpha} + p_3 \alpha \sqrt{\alpha} + \cdots + p_{n-1} \alpha^{\frac{n}{2}-1} \sqrt{\alpha} \\ &= \sum_{i=0}^{n/2} p_{2i} \alpha^i + \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1} \alpha^i \sqrt{\alpha} \end{aligned}$$

let

$$a = \sum_{i=0}^{n/2} p_{2i} \alpha^i$$

and

$$b = \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1} \alpha^i$$

to get

$$p_0 + p_2 \alpha + p_4 \alpha^2 + \cdots + p_n \alpha^{\frac{n}{2}} = a + b\sqrt{\alpha}$$

proof is same if n is odd except for an additional term. Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$, hence

$$\text{Im } f = \{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$$

Exam 2020-2021

Question 1

Question 2

Question 3