THE UNIVERSITY *of* EDINBURGH
School of Mathematics

Honours Algebra
MATH10069

Tuesday 12th May 2020

1300-1600 * †

* If you have extra time for in-person exams (according to your Schedule of Adjustments) then you are entitled to a fixed additional **1 hour** for this remote examination.

† You will have an additional **1 hour** to assemble and submit your PDF.

Attempt all questions

Important instructions

1. Start each question on a new sheet of paper.
2. Number your sheets of paper to help you scan them in order.
3. At the top of each page write the question number on the left and your student exam number BXXXXXX on the top right.
4. Only write on one side of each piece of paper.
5. If you have rough work to do, simply include it within your overall answer – put brackets at the start and end of it if you want to highlight that it is rough work.

This examination will be marked anonymously.

Honours Algebra

MATH10069 **Final exam** **May 2020**

1.

Give an example of the following. [Explanations or justifications are not required.]

(a) A subspace of the $\mathbb{C}$-vector space $\mathbb{C}[X]$ that is 3-dimensional.

(b) A subspace of the $\mathbb{C}$-vector space $\mathbb{C}[X]$ that is 3-dimensional and all of whose elements evaluated at $\sqrt{-1}$ produce 0.

(c) A basis for 2-by-2 matrices over the real numbers, $\mathrm{Mat}(2;\mathbb{R})$, considered as a vector space over the real numbers $\mathbb{R}$.

(d) A linear mapping $f\colon V \to V$ of nullity 1 where V is infinite dimensional.

(e) A surjective linear mapping from a vector space of dimension 30 to a vector space of dimension 2.

(f) A mapping from a vector space of dimension 1 to a vector space of dimension 30 that is not linear.

(g) A non-zero idempotent mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not the identity.

(h) A bilinear form $V \times V \to \mathbb{R}$ for some real vector space V that is neither alternating nor symmetric.

(i) A commutative ring that contains an element that is a zero divisor.

(j) A factor ring of $\mathbb{F}_2[X]$ that has 8 elements.

(k) A non-commutative ring that contains an element that is a zero divisor.

(l) A ring whose group of units is not cyclic.

(m) A matrix $A \in \mathrm{Mat}(4;\mathbb{R})$ that has four distinct real eigenvalues.

(n) A matrix $A \in \mathrm{Mat}(3;\mathbb{R})$ that has exactly one real eigenvalue.

(o) A matrix with entries in $\mathbb{R}$ that is not diagonal but that is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(p) A non-diagonalisable 3-by-3 matrix with entries in $\mathbb{C}$.

(q) An invertible (3×3)-matrix with entries in $\mathbb{C}$ and with trace 0.

(r) A (4×4)-matrix all of whose entries are non-zero real numbers and that has the eigenvalue 4 with algebraic multiplicity 1.

(s) An inner product on $\mathbb{R}^2$.

(t) A non self-adjoint operator on $\mathbb{R}^2$ with respect to the inner product you defined.

[20 marks]

2.

(a) Write down the matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ that represents the following linear mapping

$$f : \mathbb{R}^3 \to \mathbb{R}^2; (x, y, z)^T \mapsto (-2y + 3z, 6x - 4y)^T$$

with respect to the ordered basis $\mathcal{A} = ((1, 1, 1)^T, (1, 2, 1)^T, (0, 1, 2)^T)$ for $\mathbb{R}^3$ and the ordered basis $\mathcal{B} = ((2, 1)^T, (1, 2)^T)$ for $\mathbb{R}^2$. **[3 marks]**

(b) (i) Suppose $A \in \mathrm{Mat}(2, \mathbb{C})$ satisfies $\mathrm{Tr}(A)^2 = \mathrm{Tr}(A^2)$. Show that $\det(A) = 0$.

(ii) Let F be a field and suppose $A \in \mathrm{Mat}(2, F)$ satisfies $\mathrm{Tr}(A)^2 = \mathrm{Tr}(A^2)$. Is it always true that $\det(A) = 0$? Prove this, or give a counterexample.

[6 marks]

(c) In each case, explain clearly why the following mappings are not ring homomorphisms:

(i) $f_1 : \mathbb{Z} \to \mathrm{Mat}(2; \mathbb{Z})$ where $f_1(a) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

(ii) $f_2 : \mathbb{C}[X] \to \mathbb{C}$ where f_2 sends a polynomial $p(X) \in \mathbb{C}[X]$ to the coefficient of its highest degree term.

[6 marks]

(d) Let $V = \mathbb{R}[X]_{<n}$ be the vector space of real polynomials in 1 variable of degree at most $n - 1$. State, with a brief justification, whether the following functions $(\,\cdot\,, \cdot) : V \times V \to \mathbb{R}$ are inner products. If they are, give a proof; and if they are not, provide a counterexample to one of the axioms of an inner product space.

(i) $(P, Q) := \int_0^1 P(X)Q(X)dX$;

(ii) $(P, Q) := \int_0^1 \frac{dP}{dX}(X)\frac{dQ}{dX}(X)dX$;

(iii) Pick your favourite sequence $x_1, \ldots, x_n$ of distinct real numbers and take $(P, Q) := \sum_{j=1}^n P(x_j)Q(x_j)$.

[8 marks]

3.

(a) Let $T : V \to V$ be a self-adjoint linear mapping on a complex inner product space V.

(i) Prove that every eigenvalue of T is real.

(ii) Indicate clearly the precise statement in your proof above that goes wrong if T is not self-adjoint, explaining why it goes wrong.

(iii) Prove that if λ and μ are distinct eigenvalues of T with corresponding eigenvectors $\vec{v}$ and $\vec{w}$ respectively, then $(\vec{v}, \vec{w}) = 0$.

(iv) Give an example of a non self-adjoint T where the eigenvectors attached to different eigenvalues are not orthogonal.

[8 marks]

(b) Let R be a ring. A subset I of R is an ideal if it satisfies the following three conditions:

- $I \neq \emptyset$;
- I is closed under subtraction;
- For all $i \in I$ and $r \in R$, we have $ir, ri \in I$.

(i) Why does it follow from these conditions that if $x, y \in I$ then $x + y \in I$?

(ii) There is an equivalence relation $\sim_I$ on R given by $x \sim_I y$ if and only if $x - y \in I$ (you do not need to prove this). Show that the equivalence class of $x \in R$ is given by $x + I := \{x + i : i \in I\}$.

(iii) Prove that $(x + I)(y + I) = (xy + I)$ is a well-defined operation, stating exactly where you use any of the above conditions defining an ideal.

(iv) Suppose that R is commutative and I has the property that if $xy \in I$ then either $x \in I$ or $y \in I$, and that R/I is finite. Prove that R/I is a field, including the proofs of any significant theorems you use.

[11 marks]

4. Let α be a positive integer and let $f : \mathbb{Q}[X] \to \mathbb{R}$ be the ring homomorphism that is evaluation at $\sqrt{\alpha}$, meaning that

$$f(p_n X^n + \cdots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \cdots + p_1\sqrt{\alpha} + p_0.$$

(a) Show that the image of f is $\{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$. Show that kernel of f is a principal ideal, giving a specific generator for the ideal. **[12 marks]**

(b) Is the image of f a field? Justify your answer. **[6 marks]**

End of paper