THE UNIVERSITY *of* EDINBURGH
School of Mathematics

Honours Algebra
MATH10069

Friday 11th December 2020

1300-1500 † *

† **All students**: you have an additional **1 hour** to assemble and submit your PDF.

Final submission deadline: 16:00.

* **Students with a Schedule of Adjustment**: You are entitled to a further fixed additional **1 hour** for this remote examination.

Final submission deadline: 17:00

Attempt all questions

Important instructions

1. Start each question on a new sheet of paper.
2. Number your sheets of paper to help you scan them in order.
3. Only write on one side of each piece of paper.
4. If you have rough work to do, simply include it within your overall answer – put brackets at the start and end of it if you want to highlight that it is rough work.

In each of Questions 1, 2 and 3 you may use an earlier part of that question when answering the later parts, even if you have not managed to complete the earlier parts.

(1) For each part state whether the statement is True or False, giving a justification or counterexample for your answer. (Each part is 3 marks.)

(a) Every noncommutative ring has a zero divisor.

(b) If $f : V \to W$ and $g : W \to V$ are linear maps such that $g \circ f = \mathrm{Id}_V$ then V and W have the same dimension.

(c) If $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a linear mapping then f has an eigenvalue.

(d) The group of units $(\mathbb{Z}/m\mathbb{Z})^{\times}$ is cyclic. (Here $m \in \mathbb{Z}_{>0}$.)

(e) Let $A \in \mathrm{Mat}(n, F)$. The eigenvalues of A and A^T are the same.

(f) Let $R = \mathbb{R}[x]$ and $I = {}_R < x^3 + 3x + 7 >$. Then

$$\left((x^2+1)+I\right)\left((2x^2+3x)+I\right) = (-4x^2-20x-21)+I.$$

(g) The following defines an inner product on $\mathbb{R}[x]_{<n} := \{P \in \mathbb{R}[x] \mid \deg(P) < n\}$,

$$(P, Q) = \sum_{i=1}^{n-1} P(i)Q(i), \qquad \forall P, Q \in \mathbb{R}[x]_{<n}.$$

(h) The following defines an inner product on $\mathbb{C}^2$,

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = 4x_1\overline{y_1} - 2x_1\overline{y_2} - 2x_2\overline{y_1} + 3x_2\overline{y_2},$$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{C}^2$.

(i) A nonzero vector $\vec{v} \in V$ cannot be both in the kernel and image of a linear mapping $f : V \to V$.

(j) There exist linear maps $f : F^n \to F^n$ such that $\ker f = 0$ and $\ker f^2 \neq 0$.

[30 marks]

(2) Let $A \in \mathrm{Mat}(n, F)$, where F is an arbitrary field. Recall that if $P(x) = p_n x^n + p_{n-1}x^{n-1} + \ldots + p_0 \in F[x]$ then $P(A) = p_n A^n + p_{n-1}A^{n-1} + \ldots + p_0 \, \mathrm{Id}_n$.

Let $m_A(x)$ be a nonzero monic polynomial of minimal degree such that $m_A(A) = 0$.

(a) Calculate $m_A(x)$ for $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in F$. **[5 marks]**

(b) Show that $I_A := \{Q(x) \in F[x] \,|\, Q(A) = 0\}$ is an ideal in $F[x]$ generated by $m_A(x)$. **[5 marks]**

(c) Prove that $m_A(x) = m_{P^{-1}AP}(x)$ for any invertible $P \in \mathrm{Mat}(n, F)$. **[5 marks]**

(d) If $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a block matrix prove that $m_A(x) = \mathsf{lcm}(m_{A_1}(x), m_{A_2}(x))$, where lcm means "*the least common multiple*". **[7 marks]**

(e) Prove that $m_A(\lambda) = 0$, that is $(x - \lambda) | m_A(x)$, if and only if λ is an eigenvalue of A. **[7 marks]**

(f) If $p(x) = \prod_{i=1}^{m} (x - \lambda_i)^{d_i}$ (where the $\lambda_i \in F$ are distinct) find a matrix A such that $m_A(x) = p(x)$. **[6 marks]**

(3) (a) Let $A \in \mathrm{Mat}(n, \mathbb{R})$ and define a mapping

$$\alpha_A : \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}, \quad \text{by} \quad \alpha_A(\vec{v}) = \vec{v}^T A \vec{v}.$$

(i) Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Show that $\mathrm{Im}\, \alpha_A = (0, \infty)$. **[4 marks]**

(ii) Suppose that $A^T = -A$. Prove that $\mathrm{Im}\, \alpha_A = \{0\}$ and that $\mathrm{Id}_n + A$ is invertible. **[6 marks]**

(iii) Suppose that $A = M^T M$ for some $M \in \mathrm{Mat}(n, \mathbb{R})$. Prove that $\mathrm{Im}\, \alpha_A \subseteq [0, \infty)$ and that $\mathrm{Im}\, \alpha_A \subseteq (0, \infty)$ if M is invertible. **[6 marks]**

(iv) Suppose that $A^T = A$. Then by the Spectral Theorem A is diagonalisable with real eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that

$$\mathrm{Im}\, \alpha_A = \left\{ \sum_{i=1}^{n} a_i \lambda_i \,|\, a_i \geq 0 \;\text{ with } (a_1, \ldots, a_n) \neq (0, \ldots, 0) \right\}.$$

[6 marks]

(v) Show that the following are equivalent for $A^T = A$,

(i) $\mathrm{Im}\, \alpha_A = (0, \infty)$,

(ii) All the eigenvalues of A are positive,

(iii) $A = M^T M$ for some invertible $M \in \mathrm{Mat}(n, \mathbb{R})$.

[7 marks]

(b) Let $A \in \mathrm{Mat}(n, F)$ be such that $A^2 = A$. Prove that A is diagonalizable. **[6 marks]**

[End of Paper]