Student Solutions For Honors Algebra (MATH10069) Past Papers

April 21, 2022

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Exam 2014-2015

Question 1

Q1a

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in $\mathbb{R}$.

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \ldots, 15\}$

Q1c

Question 2

Q2a

To show that $\mathcal{B}$ forms a bsis, consider the matrix that represents $\mathcal{B}$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that $\dim B = 3$ so $\mathcal{B}$ spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class $[v]$ for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$can(v) : V \to V/U = [v]$$

and therefore $\ker(can) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

Question 3

Question 4

Question 5

Exam 2015-2016

Question 1

Question 2

Question 3

Question 4

Question 5

Exam 2016-2017

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Question 4

Question 5

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Exam 2019-2020

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Q3a

(i) Recall that the complex inner product has the following properties:

$$(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \tag{1}$$
$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \tag{2}$$
$$(\vec{x}, \vec{x}) \geq 0 \tag{3}$$

and T being self adjoint

$$(T\vec{x}, y) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue λ.

$$\begin{aligned}
\lambda(x, x) &= (x, \lambda x) \quad \textit{(property 1)} \\
&= (x, Tx) \quad \textit{(eigenvec property)} \\
&= (Tx, x) \quad \textit{(self adjoint prop)} \\
&= (\lambda x, x) \quad \textit{(eigenvec propety)} \\
&= \overline{(x, \lambda x)} \quad \textit{(property 2)} \\
&= (x, \overline{\lambda} x) \quad \textit{(not sure why we can claim } x = \bar{x} \textit{ here)} \\
&= \overline{\lambda}(x, x) \quad \textit{(property 1)}
\end{aligned}$$

implying that $\lambda = \overline{\lambda}$, which can only be true is $\lambda \in \mathbb{R}$.

(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim $(x, Tx) = (Tx, x)$ unless T is self adjoint.

(iii)

By the self adjoint property

$$(Tv, w) = (v, Tw)$$

since $Tv = \lambda v$ and $Tw = \mu w$

$$(\lambda v, w) = (v, \mu w)$$

by property 1

$$(\lambda v, w) = \lambda(v, w)$$
$$(v, \mu w) = \mu(v, w)$$

thus

$$\lambda(v, w) = \mu(v, w)$$

which cannot be true unless $(v, w) = 0$ as λ and μ are distinct eigenvalues.

(iv)

Question 4

Q4a

$\sqrt{\alpha} \in \mathbb{Q}$ implies that $\alpha \in \mathbb{Q}$. Thus result of the mapping $f(p_n X^n + \cdots + p_1 X + p_0) = p_n(\sqrt{\alpha})^n + \cdots + p_1\sqrt{\alpha} + p_0$ can be simplied into the form $a + b\sqrt{\alpha}$, where $a, b \in \mathbb{Q}$. If n is even.

$$\begin{aligned} p_n(\sqrt{\alpha})^n + \cdots + p_1\sqrt{\alpha} + p_0 &= p_0 + p_2\alpha + p_4\alpha^2 + \cdots + p_n\alpha^{\frac{n}{2}} \\ &\quad + p_1\sqrt{\alpha} + p_3\alpha\sqrt{a} + \cdots + p_{n-1}\alpha^{\frac{n}{2}-1}\sqrt{\alpha} \\ &= \sum_{i=0}^{n/2} p_{2i}\alpha^i + \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1}\alpha^i\sqrt{\alpha} \end{aligned}$$

let

$$a = \sum_{i=0}^{n/2} p_{2i}\alpha^i$$

and

$$b = \sum_{i=0}^{\frac{n}{2}-1} p_{2i+1}\alpha^i$$

to get

$$p_0 + p_2\alpha + p_4\alpha^2 + \cdots + p_n\alpha^{\frac{n}{2}} = a + b\sqrt{\alpha}$$

proof is same if n is odd except for an additional term. Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$, hence

$$\operatorname{Im} f = \{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$$

Exam 2020-2021

Question 1

Q1a

F: A **noncommutative ring** is a ring such that

$$a \cdot b \neq b \cdot a$$

and an element in a ring is a **zero divisor** if there exists non-zero b such that

$$ab = 0 \quad \text{or} \quad ba = 0$$

An example of this is $\mathbb{H}$: The ring of quaternions. proof if interested

Q1b

F: Example: $V = \mathbb{R}$, $W = \mathbb{R}$

$$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$g\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Since $g \circ f(x) = x$ but $\dim V \neq \dim W$

Q1c

Question can be rephrased as: Do $n \times n$ matrices with odd n always have (a real) eigenvalue?
T: because the characteristic polynomial will have an odd degree, by the intermediate value theorem (proof), it must have at least one real root $\implies$ matrix has at one real eigenvalue.

Q1d

F:

Question 2

Question 3