The University of Edinburgh

College of Science and Engineering

Mathematics 3 Honours

MATH10069 Honours Algebra

Monday, 30th April 2018

09:30 – 12:30

Chairman of Examiners – Dr A Olde Daalhuis

External Examiner – Professor G Brown

Credit will be given for the best **FOUR** answers

In this examination, candidates are allowed to have three sheets of A4 paper with whatever notes they desire written or printed on one or both sides of the paper.

Magnifying devices to enlarge the contents of the sheets for viewing are not permitted.

No further notes, printed matter or books are allowed.

Calculators and other electronic aids

A scientific calculator is permitted in this examination.

It must not be a graphical calculator.

It must not be able to communicate with any other device.

This examination will be marked anonymously.

(1) Give an example of the following. [Explanations or justifications are not required.]

(a) An infinite dimensional vector space. **[1 mark]**

(b) A vector space with exactly 64 elements. **[1 mark]**

(c) A linearly independent subset of $\mathbb{C}^4$ with three elements. **[1 mark]**

(d) A spanning subset for $\mathbb{R}^2$ with four elements. **[1 mark]**

(e) A mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not linear. **[1 mark]**

(f) A non-zero linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is neither injective nor surjective. **[1 mark]**

(g) A linear mapping that can be defined without writing a matrix. **[1 mark]**

(h) A mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$ that has rank 1. **[1 mark]**

(i) A 4×4 matrix with entries in $\mathbb{R}$ all of whose entries are non-zero and which has 1 as an eigenvalue. **[1 mark]**

(j) A 2×2 matrix with entries in $\mathbb{Z}$ whose determinant is non-zero but that is not invertible in $\operatorname{Mat}(2;\mathbb{Z})$. **[1 mark]**

(k) A 2×2 matrix with entries in $\mathbb{Z}$ whose trace is 0 and that is invertible in $\operatorname{Mat}(2;\mathbb{Z})$. **[1 mark]**

(l) An ideal I of $\mathbb{C}[X]$ that is neither $\{0\}$ nor $\mathbb{C}[X]$. **[1 mark]**

(m) An integral domain that is not a field. **[1 mark]**

(n) An integral domain that is not a field and that is not isomorphic to your previous answer. **[1 mark]**

(o) A commutative ring that is not an integral domain. **[1 mark]**

(p) A division ring. **[1 mark]**

(q) A division ring that it is not isomorphic to your previous answer. **[1 mark]**

(r) A ring whose group of units is infinite. **[1 mark]**

(s) A matrix $A \in \operatorname{Mat}(3;\mathbb{R})$ that has one real eigenvalue, and whose algebraic multiplicity is one. **[1 mark]**

(t) A non-diagonalisable matrix in $\operatorname{Mat}(3;\mathbb{C})$. **[1 mark]**

(u) A symmetric bilinear form. **[1 mark]**

(v) A non-zero alternating bilinear form $V \times V \to \mathbb{R}$ for some real vector space V. **[1 mark]**

(w) An inner product on $\mathbb{C}^2$. **[1 mark]**

(x) A self-adjoint endomorphism on $\mathbb{C}^2$ with respect to the inner product you defined. **[1 mark]**

(y) A non self-adjoint endomorphism on $\mathbb{C}^2$ with respect to the inner product you defined. **[1 mark]**

(2) Let T, U, V and W be finite dimensional vector spaces.

(a) Let $\mathcal{S}(3) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis of $T = \mathbb{R}^3$ and let $\mathcal{B} = (\vec{v}_1 = \vec{e}_1 + \vec{e}_3, \vec{v}_2 = \vec{e}_2 + \vec{e}_3, \vec{v}_3 = \vec{e}_1 + \vec{e}_2 + 3\vec{e}_3)$. Show that $\mathcal{B}$ is a basis of T. Now suppose that a linear mapping $f : T \to T$ is represented with respect to $\mathcal{S}(3)$ by the matrix

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \\ 6 & 9 & -3 \end{pmatrix}.$$

Find the matrix B that represents f with respect to $\mathcal{B}$. Write down an explicit equation that expresses the relationship between A and B. **[8 marks]**

(b) Assume that $U \subseteq V$ is a subspace of the vector space V. Define the quotient vector space V/U including the definition of addition and scalar multiplication in V/U. Define the canoncial mapping $\mathsf{can} : V \to V/U$. Prove that U is the kernel of can. **[7 marks]**

(c) Give a basis $\mathcal{C}$ for $T/\ker(f)$ in (a). Write down the matrix for the mapping $T/\ker(f) \to T$ induced by f, with respect to the bases $\mathcal{C}$ of $T/\ker(f)$ and $\mathcal{B}$ of T. **[5 marks]**

(d) Let $g : U \to V$ and $h : V \to W$ be linear mappings. Prove that the composite linear mapping $h \circ g : U \to W$ satistfies $h \circ g = 0$ if and only if $\mathrm{im}(g) \subseteq \ker(h)$. Use the Rank–Nullity theorem to prove that if $h \circ g = 0$ then

$$\dim U - \dim V \leq \dim \ker(g) - \dim \mathrm{im}(h).$$

[5 marks]

(3) Let $R = \mathbb{F}_2[X]$ denote the ring of polynomials with coefficients from $\mathbb{F}_2$, the field with two elements.

(a) Define what it means for a ring to be an integral domain. State and prove the cancellation theorem for integral domains. Prove that $R = \mathbb{F}_2[X]$ is an integral domain. **[8 marks]**

(b) Prove that $I = \{P \in R : P \text{ is divisible by } X^3 + X^2 + X + 1\}$ is an ideal of R. **[4 marks]**

(c) (i) The factor ring R/I has exactly 8 elements. What are they? **[4 marks]**

(ii) Show that R/I is not a field. Which elements of R/I from your list in (c)(i) are units? **[5 marks]**

(d) In each case, explain clearly why the following mappings are not ring homomorphisms.

(i) $f_1 : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ where $f_1([a]) = [a]^4$. **[2 marks]**

(ii) $f_2 : \mathrm{Mat}(2; \mathbb{C}) \to \mathrm{Mat}(2; \mathbb{C})$ where $f_2(A) = \frac{1}{2}(A + A^T)$. **[2 marks]**

(4) (a) Define the following concepts in a real inner product space: inner product norm, orthogonality, orthonormal family, orthonormal basis. Prove that every finite dimensional real inner product space has an orthonormal basis. **[11 marks]**

(b) Which of the following define inner products on the given real vector space? For each give a brief proof, or state which condition of being an inner product fails and give a counter-example.

(i) On $\mathbb{R}^2$, the function
$$\langle \vec{x}, \vec{y} \rangle = x_1 + y_1 + 2x_2y_2,$$
where $\vec{x} = (x_1, x_2)^\top$ and $\vec{y} = (y_1, y_2)^\top$.

(ii) On Mat(n; $\mathbb{R}$), the function
$$\langle A, B \rangle = \operatorname{tr}(A^\top B),$$
where tr denotes the trace.

(iii) On $\mathbb{R}[X]$, the function
$$\langle P, Q \rangle = \int_{-1}^{1} X^3 P(X) Q(X) \mathrm{d}X.$$

[4 marks]

(c) Consider $\mathbb{R}^4$ with the standard inner product. Let U be the two dimensional subspace of $\mathbb{R}^4$ generated by the vectors $(1,1,1,1)^\top$ and $(3,0,0,1)^\top$.

(i) Find an orthonormal basis of U containing $\frac{1}{2}(1,1,1,1)^\top$ with respect to the standard inner product on $\mathbb{R}^4$.

(ii) Find the orthonormal projection of $(2,1,2,1)^\top$ to U.

[5 marks]

(d) Let $U \in \mathsf{Mat}(n;\mathbb{C})$ be a unitary matrix. Show that $\langle U\vec{x}, U\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, where U acts by the usual matrix multiplication and $\langle -, - \rangle$ is the standard inner product on $\mathbb{C}^n$. **[5 marks]**

(5) (a) Let $A \in \mathrm{Mat}(3;\mathbb{C})$ be the matrix
$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$
Calculate the characteristic polynomial of A. What are the eigenvalues of A? What are the algebraic multiplicities of the eigenvalues you found? Their geometric multiplicities? **[7 marks]**

(b) State the Jordan Normal Form Theorem. **[5 marks]**

(c) Let $B \in \mathrm{Mat}(4;\mathbb{C})$ be the following matrix:
$$B = \begin{pmatrix} 0 & -1 & 2 & -1 \\ -2 & -2 & 4 & -1 \\ -3 & -4 & 7 & -2 \\ -3 & -4 & 6 & -1 \end{pmatrix}.$$
There exists a matrix $P \in \mathrm{Mat}(4;\mathbb{C})$ such that
$$P^{-1}BP = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
[You do not need to prove this.]

(i) Find P. **[7 marks]**

(ii) There is a polynomial $P(X) = X^4 + aX^3 + bX^2 + cX + d \in \mathbb{R}[X]$ such that $P(B)$, obtained by replacing each X by B, is the zero matrix. What are a, b, c and d? **[3 marks]**

(iii) Given an expression for B^{-1} in terms only of B^2, B and I. **[3 marks]**

[End of Paper]