The University of Edinburgh

College of Science and Engineering

School of Mathematics

Mathematics 3 Honours

MATH10069 Honours Algebra

Wednesday 8th May 2019

2.30pm – 5.30pm

Chairman of Examiners – Professor A Olde Daalhuis

External Examiner – Professor G Brown

Answer ALL questions

In this examination, candidates are allowed to have three sheets of A4 paper with whatever notes they desire written or printed on one or both sides of the paper. Magnifying devices to enlarge the contents of the sheets for viewing are not permitted. No further notes, printed matter or books are allowed in this exam.

Calculators and other electronic aids

A scientific calculator is permitted in this examination.

It must not be a graphical calculator.

It must not be able to communicate with any other device.

This examination will be marked anonymously.

(1) Give an example of the following. [Explanations or justifications are not required.]

(a) An infinite dimensional vector space.

(b) A vector space with exactly 32 elements.

(c) A basis for the complex numbers $\mathbb{C}$, considered as a vector space over the real numbers $\mathbb{R}$.

(d) A linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ whose rank is 1.

(e) An injective linear mapping from a vector space of dimension 1 to a vector space of dimension 2.

(f) A mapping from a vector space of dimension 2 to a vector space of dimension 2 that is not linear.

(g) A non-zero alternating bilinear form on $V \times V \to \mathbb{R}$ for some real vector space V.

(h) A commutative ring that is not an integral domain.

(i) A ring that is not commutative.

(j) A ring whose group of units is infinite.

(k) A matrix $A \in \mathrm{Mat}(2; \mathbb{R})$ that has two distinct real eigenvalues.

(l) A matrix $A \in \mathrm{Mat}(2; \mathbb{R})$ that has no real eigenvalues.

(m) A matrix with entries in $\mathbb{R}$ that is not diagonal but that is diagonalisable.

(n) A non-diagonalisable matrix with entries in $\mathbb{C}$.

(o) An invertible (3×3)-matrix with trace 1.

(p) A non-invertible matrix whose determinant is not zero. (You must explain which ring of matrices you are considering.)

(q) A (4×4)-matrix all of whose entries are non-zero real numbers and that has the eigenvalue 1 with algebraic multiplicity 1.

(r) An inner product on $\mathbb{C}^2$.

(s) A self-adjoint operator on $\mathbb{C}^2$ with respect to the inner product you defined.

(t) A non self-adjoint operator on $\mathbb{C}^2$ with respect to the inner product you defined.

[20 marks]

(2) (a) Let $\mathcal{S}(2) = (\vec{e}_1, \vec{e}_2)$ be the standard basis of $\mathbb{R}^2$ and let $\mathcal{B} = (\vec{v}_1 = -3\vec{e}_1 + 2\vec{e}_2, \vec{v}_2 = 2\vec{e}_1 - \vec{e}_2)$. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$. Now suppose that a linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented with respect to $\mathcal{S}(2)$ by the matrix

$$A = \begin{pmatrix} -6 & -9 \\ 4 & 6 \end{pmatrix}$$

Find the matrix B that represents f with respect to $\mathcal{B}$. Write down an explicit equation that expresses the relationship between A and B. **[7 marks]**

(b) In each of the following cases state whether the given formula defines an inner product on $\mathbb{R}^2$. For any that are *not* inner products, give a counterexample; for any that *are* inner products, a proof is not required. In each formula $\vec{x} = (x_1, x_2)^\mathsf{T}$ and $\vec{y} = (y_1, y_2)^\mathsf{T}$.

(i) $(\vec{x}, \vec{y}) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$.

(ii) $(\vec{x}, \vec{y}) = x_1^2y_1^2 + 5x_2y_2$.

(iii) $(\vec{x}, \vec{y}) = x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$.

[5 marks]

(c) Let $A \in \mathrm{Mat}(4; \mathbb{C})$ be the following matrix:

$$A = \begin{pmatrix} 2 & -1 & -3 & 2 \\ 4 & -4 & -4 & 4 \\ -4 & 4 & 4 & -4 \\ -6 & 5 & 7 & -6 \end{pmatrix}$$

There exists a matrix $P \in \mathrm{Mat}(4; \mathbb{C})$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

[You do not need to prove this.] Find P. **[8 marks]**

(3) (a) Define what it means for a ring to be an integral domain. State and prove the cancellation theorem for integral domains. Prove that $\mathbb{Z}[X]$ is an integral domain. **[5 marks]**

(b) Let V be a finite dimensional complex inner product space. Define the norm on V and state and prove the Cauchy-Schwarz inequality. **[8 marks]**

(c) Let A be a matrix with entries in a field F. Define the determinant $\det(A)$, explaining any terminology you use. Prove that $\det(A^{\mathsf{T}}) = \det(A)$. Define $\chi_A(x) \in F[x]$, the characteristic polynomial of A and show that the eigenvalues of A in F are exactly the roots of the polynomial $\chi_A(x)$. **[7 marks]**

(4) (a) Let $f : R \to S$ be a ring homomorphism. Define the kernel and image of f and prove that the image of f is a subring of S. State the First Isomorphism Theorem for Rings. **[8 marks]**

(b) Let $A \in \mathrm{Mat}(3; \mathbb{F}_2)$. Show that the mapping $f_A : \mathbb{F}_2[X] \to \mathrm{Mat}(3; \mathbb{F}_2)$, defined by

$$p_0 + p_1X + p_2X^2 + \cdots + p_nX^n \mapsto p_0I + p_1A + p_2A^2 + \cdots + p_nA^n$$

for any $p_0 + p_1X + p_2X^2 + \cdots + p_nX^n \in \mathbb{F}_2[X]$, is a ring homomorphism. Using the First Isomorphism Theorem or otherwise, prove that the image of f_A is a field with 8 elements when

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

[12 marks]

[End of Paper]