The University of Edinburgh

College of Science and Engineering

Mathematics 3 Honours

MATH10069 Honours Algebra

Monday, 2nd May 2016

9:30am – 12:30pm

Chairman of Examiners – Professor J M Figueroa-O'Farrill

External Examiner – Professor J Greenlees

Credit will be given for the best **FOUR** answers

Calculators and other electronic aids

A scientific calculator is permitted in this examination.

It must not be a graphical calculator.

It must not be able to communicate with any other device.

This examination will be marked anonymously.

(1) Give an example of the following. [Explanations or justifications are not required.]

(a) An infinite dimensional vector space.

(b) A finite dimensional subspace of the vector space you used for (a).

(c) A spanning set for $\mathbb{R}^2$ that is not a basis.

(d) A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not linear.

(e) A linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ whose nullity is 2.

(f) A linear map that is neither surjective nor injective.

(g) A linear map $f : V \to V$ for some vector space V that is injective but not surjective.

(h) A bilinear form on $\mathbb{R}^2$ that is symmetric.

(i) A bilinear form on $\mathbb{R}^2$ that is alternating.

(j) A bilinear form on $\mathbb{R}^2$ that is neither symmetric nor alternating.

(k) A ring with 4 elements.

(l) An ideal in a ring that is not a principal ideal.

(m) An integral domain that is not a field.

(n) A ring that is not commutative.

(o) A ring that is not commutative and not isomorphic to your answer in (n).

(p) A matrix in $\mathrm{Mat}(3;\mathbb{R})$ that has exactly 2 distinct eigenvalues.

(q) A matrix in $\mathrm{Mat}(2;\mathbb{Z})$ that is invertible in $\mathrm{Mat}(2;\mathbb{Z})$ and all of whose entries are non-zero.

(r) A cubic polynomial in $\mathbb{R}[X]$ whose remainder is 1 on division by $(X+1)$.

(s) A non-diagonalisable matrix in Jordan Normal form whose characteristic polynomial is $X^2 + 2X + 1$.

(t) A non-diagonal matrix in $\mathrm{Mat}(2;\mathbb{R})$ that is diagonalisable.

(u) An equivalence relation on a finite set whose equivalence classes do not all have the same number of elements.

(v) A theorem that guarantees a matrix is diagonalisable.

(w) A ring that is not a field, but whose group of units is infinite.

(x) An inner product on the complex vector space $\mathbb{C}^3$.

(y) A self-adjoint operator on $\mathbb{C}^3$ with respect to the inner product you defined in (x).

[25 marks]

(2) (a) Let V, W be vector spaces over a field F and let $f\colon V \to W$ be a linear map.

(i) Define the kernel of f and show that it is a vector subspace of V.

(ii) Define the image of f and show that it is a vector subspace of W.

(iii) State the first isomorphism theorem for V, W, f.

(iv) State the rank-nullity theorem. Give a proof of it using the first isomorphism theorem, stating clearly any other results that you use.

[10 marks]

(b) (i) Work out the matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ for the linear map

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2; (x, y, z) \mapsto (-x - y + 2z, 2x + 2y - 3z)$$

with respect to the basis $\mathcal{A} = ((0,3,2)^T, (1,1,1)^T, (1,2,2)^T)$ of $\mathbb{R}^3$ and the basis $\mathcal{B} = ((1,0)^T, (0,1)^T)$ for $\mathbb{R}^2$.

(ii) Write down a basis for the kernel of f.

(iii) Write down a basis for $\mathbb{R}^3/\mathrm{Ker}(f)$ and prove that it is a basis.

[10 marks]

(c) Let V be a finite-dimensional vector space over a field F and let $f\colon V \to V$ be an endomorphism. Using the rank-nullity theorem twice, or otherwise, show that if $\ker(f \circ f) = \ker(f)$, then $f : \mathrm{im}(f) \to \mathrm{im}(f \circ f)$ is an isomorphism.

[5 marks]

(3) (a) For each of the following mappings between rings, state the property of a ring homomorphism that is not satisfied and give a example demonstrating that this property is not satisfied.

(i) $f_1 : \mathbb{R}[x] \to \mathbb{R}[x]$ where $p(x) \mapsto xp(x)$.

(ii) $f_2 : \mathrm{Mat}(2,\mathbb{R}) \to \mathrm{Mat}(2,\mathbb{R})$ where $A \mapsto \frac{1}{2}\left(A + A^T\right)$.

(iii) $f_3 : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$ where $[a] \mapsto [a^2]$.

[6 marks]

(b) Let R be a ring.

(i) State the definition for R to be an integral domain and the Cancellation Law for Integral Domains. [You do not need to prove it.]

(ii) Prove that a finite integral domain is actually a field.

[7 marks]

(c) Let $R = \mathbb{Z}$ and $m \in \mathbb{Z}$. Show that ${}_{\mathbb{Z}}\langle m \rangle$ is an ideal in $\mathbb{Z}$. [3 marks]

(d) Now assume that R is commutative. A proper ideal P in R is called *prime* if whenever $rs \in P$, then $r \in P$ or $s \in P$.

(i) Show that P is prime if and only if the factor ring R/P is an integral domain.

(ii) Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree 2. Using the euclidean algorithm, or otherwise, show that ${}_{R}\langle q(x) \rangle$ is prime if and only if $q(x)$ has no real roots.

[9 marks]

(4) (a) State the definition of an inner product for a vector space V over $\mathbb{C}$. In each of the following cases determine whether the given formula defines an inner product on $\mathbb{C}^2$. For any that are not inner products, state the defining property of the inner product that does not hold and give a counterexample; for any that are inner products, a proof is not required. In each formula, we have $\vec{v} = (v_1, v_2)^T$ and $\vec{w} = (w_1, w_2)^T$, where $v_1, v_2, w_1, w_2 \in \mathbb{C}$.

(i) $(\vec{v}, \vec{w}) = v_1\overline{w_1} + v_2\overline{w_2}$.

(ii) $(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2$.

(iii) $(\vec{v}, \vec{w}) = v_1 + \overline{w_1} + v_2 + \overline{w_2}$.

[5 marks]

(b) Let V be a finite dimensional real inner product space.

(i) Prove that V has an orthonormal basis.

(ii) Let $L = \{v_1, \dots, v_r\}$ be a set of pairwise orthogonal non-zero vectors in V. Show that L is linearly independent.

[8 marks]

(c) Let U be the subspace of $\mathbb{R}^4$ with basis

$$\{(0,1,0,1)^T, (1,2,3,4)^T\}.$$

Recall that the standard inner product on $\mathbb{R}^4$ is defined to be

$$((x_1, x_2, x_3, x_4)^T, (y_1, y_2, y_3, y_4)^T) = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

(i) Find an orthonormal basis of U containing $\left\{\frac{1}{\sqrt{2}}(0,1,0,1)^T\right\}$ with respect to the standard inner product on $\mathbb{R}^4$.

(ii) Find the orthogonal projection of $(2,1,3,1)$ to U.

[4 marks]

(d) Let V be a finite dimensional complex inner product space.

(i) Let T be an endomorphism of V. Give the definition of the adjoint T^* to T.

(ii) State the Spectral Theorem for Hermitian Matrices.

(iii) Let A be a Hermitian matrix. Show that if A satisfies $A^k = 0$ for some k, then $A = 0$.

[8 marks]

(5) Let $A \in \mathrm{Mat}(n; \mathbb{C})$ be an n by n matrix with entries in the complex numbers, $\mathbb{C}$.

(a) Let $\lambda \in \mathbb{C}$. Define $E(\lambda, A)$, the eigenspace of A with eigenvalue λ, and $E^{\text{gen}}(\lambda, A)$, the generalised eigenspace of A with eigenvalue λ. Define $\chi_A(x)$, the characteristic polynomial of A. Prove that the eigenvalues of A are exactly the roots of the polynomial $\chi_A(x)$. [6 marks]

(b) State the Jordan Normal Form theorem. [4 marks]

(c) Let $A \in \mathrm{Mat}(4; \mathbb{C})$ be the following matrix:

$$A = \begin{pmatrix} 2 & -1 & -3 & 2 \\ 4 & -4 & -4 & 4 \\ -4 & 4 & 4 & -4 \\ -6 & 5 & 7 & -6 \end{pmatrix}.$$

You may assume there exists a matrix $P \in \mathrm{Mat}(4; \mathbb{C})$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

(i) What is $\chi_A(x)$, the characteristic polynomial of A in this example?

(ii) Find P.

(iii) Apart from the above, there are two other possible Jordan normal forms for a matrix in $\mathrm{Mat}(4; \mathbb{C})$ with characteristic polynomial equal to $\chi_A(x)$. What are they? [11 marks]

(d) Let $B \in \mathrm{Mat}(n; \mathbb{F}_2)$ be the matrix all of whose entries are one. By considering B^2, prove that B is diagonalisable if and only if n is odd. [4 marks]

[End of Paper]