Student Solutions For Honors Algebra (MATH10069) Past Papers

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Exam 2014-2015

Question 1

Q1a

An example of infinite dimensional vector space over a field is $\mathbb{R}[x]$, the set of polynomials with coefficients in $\mathbb{R}$.

Q1b

An vector space with exactly 16 elements is $\mathbb{Z}/16\mathbb{Z} = \{0, 1, 2, \dots, 15\}$

Q1c

Question 2

Q2a

To show that $\mathcal{B}$ forms a bsis, consider the matrix that represents $\mathcal{B}$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using gaussian elimination we find that

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or that $\dim B = 3$ so $\mathcal{B}$ spans $V = \mathbb{R}^3$

Q2b

i)

Denote the equivalence class $[v]$ for $v \in V$ by

$$[v] = \{v + u : u \in U\}$$

and addition and multiplication is defined as follows

$$k[n] = [kn]$$

for all $k \in \mathbb{R}$, and

$$[v_1] + [v_2] = [v_1 + v_2]$$

thus the canonical mapping is simply

$$can(v) : V \to V/U = [v]$$

and therefore $\ker(can) = 0$ as

$$[0] = \{0 + u : u \in U\}$$

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Q2a

Here provides a faster method:

$$\begin{aligned} f(1,1,1) &= (1,2) &&= 0(2,1) + 1(1,2) \\ f(1,2,1) &= (-1,-2) &&= 0(2,1) - 1(1,2) \\ f(0,1,2) &= (4,-4) &&= 4(2,1) - 4(1,2) \end{aligned}$$

Hence,

$$_{\mathcal{B}}[f]_{\mathcal{A}} = \begin{pmatrix} 0 & 0 & 4 \\ 1 & -1 & -4 \end{pmatrix}$$

Standard method:
To get the representing matrix of f, evaluate f at the standard basis.

$$\begin{aligned} f(1,0,0) &= (0,6) \\ f(0,1,0) &= (-2,-4) \\ f(0,0,1) &= (3,0) \end{aligned}$$

Lets call the representing matrix M

$$M = \begin{pmatrix} 0 & -2 & 3 \\ 6 & -4 & 0 \end{pmatrix}$$

Note that f currently is mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ w.r.t. to standard bases $S(3)$ and $S(2)$. So MA, where A is matrix formed by the ordered basis $\mathcal{A}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\begin{aligned} MA &= \begin{pmatrix} 0 & -2 & 3 \\ 6 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & -4 \end{pmatrix} \end{aligned}$$

represents $_{S(2)}[f]_{\mathcal{A}}$.

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

finally $_{\mathcal{B}}[f]_{\mathcal{A}}$ is then $B^{-1}MA$

$$\begin{aligned} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 4 \\ 2 & -2 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 4 \\ 1 & -1 & -4 \end{pmatrix} \end{aligned}$$

Q2b

(i) Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} Tr(A)^2 &= (a+d)^2 \\ Tr(A^2) &= a^2 + bc + bc + d^2 \end{aligned}$$

by construction, we have

$$\begin{aligned} Tr(A)^2 &= Tr(A^2) \\ (a+d)^2 &= a^2 + 2bc + d^2 \\ a^2 + 2ad + d^2 &= a^2 + 2bc + d^2 \end{aligned}$$

which implies

$$ad = bc$$

since $\det A = ad - bc$, it follows that $\det A = 0$

Alternative method:
By Cayley-Hamilton theorem,

$$A^2 - \operatorname{Tr} A + \det A \mathbb{I}_2 = 0$$

Take trace on both side:

$$\begin{aligned} \operatorname{Tr} A^2 - (\operatorname{Tr} A)^2 + 2 \det A &= 0 \\ 2 \det A &= 0 \end{aligned}$$

Since 2 is a unit in $\mathbb{C}$, so $\det A = 0$.

(ii) No. Notice in the previous proof requires a condition is 2 is a unit. So let $F = \mathbb{F}_2$, then let $A = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\operatorname{Tr} A^2 = (\operatorname{Tr} A)^2$ but $\det A = 1 \neq 0$

Q2c

A mapping is a ring homomorphism if

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \end{aligned}$$

(i)

f_1 violates the first property, consider

$$f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but

$$\begin{aligned} f(1) + f(-1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

so

$$f(0) \neq f(-1) + f(1)$$

(ii) f_2 violates the first property, consider $p_1(X) = X + 1$ and $p_2(X) = -X$.

$$\begin{aligned} f_2(p_1(X)) &= 1 \\ f_2(p_2(X)) &= -1 \\ f_2(p_1(X) + p_2(X)) &= f_2(1) = 1 \\ f_2(p_1(X)) + f_2(p_2(X)) &\neq f_2(p_1(X) + p_2(X)) \end{aligned}$$

Q2d

(i) **Yes**. See Exercise 80.

(ii) **No**. Let $P = Q$ be constant and it violates property 3.

(iii) **Yes**. Linearity and symmetric is trivial. Check property 3:

$$(P, P) = \sum_{j=1}^{n} P(x_j)^2 \geq 0$$

Equality holds only when $P(x_j) = 0$ for all j. However, $P(X)$ has at most $n-1$ roots so the only possible case is $P(X) = 0$.

Question 3

Q3a

See THEOREM 5.3.7.

(i) Recall that the complex inner product has the following properties:

$$(\lambda \vec{x} + \mu \vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \tag{1}$$
$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})} \tag{2}$$
$$(\vec{x}, \vec{x}) \geq 0 \tag{3}$$

and T being self adjoint

$$(T\vec{x}, y) = (\vec{x}, T\vec{y})$$

Thus, let v be an eigenvector with eigenvalue $lambda$.

$$\begin{aligned}
\lambda(\vec{v}, \vec{v}) &= (\lambda \vec{v}, \vec{v}) \quad \textit{(property 1)} \\
&= (T\vec{v}, \vec{v}) \quad \textit{(eigenvector property)} \\
&= (\vec{v}, T\vec{v}) \quad \textit{(self adjoint property)} \\
&= (\vec{v}, \lambda \vec{v}) \quad \textit{(eigenvector property)} \\
&= \overline{(\lambda \vec{v}, \vec{v})} \quad \textit{(property 2)} \\
&= \overline{\lambda}\overline{(\vec{v}, \vec{v})} \quad \textit{(property 1)} \\
&= \overline{\lambda}(\vec{v}, \vec{v}) \quad \textit{(property 2)} \\
\lambda(\vec{v}, \vec{v}) = \overline{\lambda}(\vec{v}, \vec{v}) &\implies \lambda = \overline{\lambda} \\
&\implies \lambda \in \mathbb{R}
\end{aligned}$$

(ii) The proof in (i) fails if T is not self adjoint due to the fact that we cannot claim $(x, Tx) = (Tx, x)$ unless T is self adjoint.

(iii) Both λ and μ are real by part (i).

$$\begin{aligned}
\lambda(\vec{v}, \vec{w}) &= (T\vec{v}, \vec{w}) \\
&= (\vec{v}, T\vec{w}) \\
&= \mu(\vec{v}, \vec{w})
\end{aligned}$$

which cannot be true unless $(v, w) = 0$ as λ and μ are distinct eigenvalues.

(iv) By constructive method,

$$\begin{aligned}
\lambda &= 1 \\
\mu &= 2 \\
\lambda &\neq \mu \\
\vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\vec{w} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(\vec{v}, \vec{w}) &\neq 0 \\
D &= \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
P &= \begin{pmatrix} \vec{v} & \vec{w} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
T &= PDP^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}
\end{aligned}$$

Q3b

(i)

(ii)

(iii)

(iv)

Question 4

Q4a

$$\begin{aligned}
\sum_{i=0}^{n} p_i(\sqrt{\alpha})^i &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i}(\sqrt{\alpha})^{2i} + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i+1}(\sqrt{\alpha})^{2i+1} \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i}\alpha^i + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i+1}\alpha^i\sqrt{\alpha} \\
&= \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i}\alpha^i \right) + \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_{2i+1}\alpha^i \right) \sqrt{\alpha}
\end{aligned}$$

Therefore, f maps every rational polynomial to a real number of the form $a + b\sqrt{\alpha}$ for some $a, b \in \mathbb{Q}$
Now we show it is surjective by $f(a + bX) = a + b\sqrt{\alpha}$.
Claim:

$$\ker f = \begin{cases} {}_{\mathbb{Q}[X]}\langle X - \sqrt{\alpha} \rangle & \sqrt{\alpha} \in \mathbb{Q} \\ {}_{\mathbb{Q}[X]}\langle X^2 - \alpha \rangle & \text{otherwise} \end{cases}$$

If $\sqrt{\alpha} \in \mathbb{Q}$, $p(X) = q(X)(X - \sqrt{\alpha}) + a$.

$$\begin{aligned} p(X) \in \ker f &\iff f(p(X)) = 0 \\ &\iff f(q(X)(X - \sqrt{\alpha}) + a) = 0 \\ &\iff f(a) = 0 \\ &\iff a = 0 \\ &\iff (X - \sqrt{\alpha}) | p(X) \\ &\implies \ker f = {}_{\mathbb{Q}[X]}\langle X - \sqrt{\alpha} \rangle \end{aligned}$$

If $\sqrt{\alpha} \notin \mathbb{Q}$, $p(X) = q(X)(X^2 - \alpha) + a + bX$.

$$\begin{aligned} p(X) \in \ker f &\iff f(p(X)) = 0 \\ &\iff f(q(X)(X^2 - \alpha) + a + bX) = 0 \\ &\iff f(a + bX) = 0 \\ &\iff a + b\sqrt{\alpha} = 0 \\ &\iff a = b = 0 \quad (\alpha \text{ is irrational}) \\ &\iff (X^2 - \alpha) + a + bX) | p(X) \\ &\implies \ker f = {}_{\mathbb{Q}[X]}\langle X^2 - \alpha \rangle \end{aligned}$$

Hence $\ker f$ is a principle ideal.

Q4b

Yes. If $\sqrt{\alpha}$ is rational, then $\operatorname{im} f = \mathbb{Q}$ which is definitely a field. If $\sqrt{\alpha}$ is irrational, then using high school math,

$$(a + b\sqrt{\alpha})^{-1} = \frac{a - b\sqrt{\alpha}}{a^2 - \alpha b^2}$$

Other properties of field can be verified easily. So $\operatorname{im} f$ is a field.

Exam 2020-2021

Question 1

Q1a

F: A **noncommutative ring** is a ring such that

$$a \cdot b \neq b \cdot a$$

and an element in a ring is a **zero divisor** if there exists non-zero b such that

$$ab = 0 \quad \text{or} \quad ba = 0$$

An example of this is $\mathbb{H}$: The ring of quaternions. proof if interested

Q1b

F: Example: $V = \mathbb{R}$, $W = \mathbb{R}$

$$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$g\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Since $g \circ f(x) = x$ but $\dim V \neq \dim W$

Q1c

Question can be rephrased as: Do $n \times n$ matrices with odd n always have (a real) eigenvalue?
T: because the characteristic polynomial will have an odd degree, by the intermediate value theorem (proof), it must have at least one real root $\implies$ matrix has at one real eigenvalue.

Q1d

F: The group of units $R^\times$ in a ring R is the set of elements a with multiplicative inverse in R, or

$$R^\times = \{a \in R : \exists\, a^{-1} \in R \text{ s.t. } aa^{-1} = a^{-1}a = 1\}$$

Cyclic group is a group that can be generated by one element.
Note: $(\mathbb{Z}/m\mathbb{Z})^\times$ are elements in the ring that are relatively prime to m, i.e.: $(\mathbb{Z}/m\mathbb{Z})^\times = \{a \in (\mathbb{Z}/m\mathbb{Z}) : gcd(a, m) = 1\}$
Note 2: only for prime m is $(\mathbb{Z}/m\mathbb{Z})^\times$ cyclic.

Consider $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$, the group cannot be generated by one element.

Q1e

T: Since A^T and A have the same characteristic polynomial $\implies$ same eigenvalues.

Q1f

T: $R = \mathbb{R}[x]$: polynomials with real coefficients
$_R < x^3 + 3x + 7 >$: think of it as resulting set of polynomials after multiplying every polynomial in R by $x^3 + 3x + 7$. More precisely:

$$_R < x^3 + 3x + 7 >= \{(x^3 + 3x + 7) \cdot a : a \in R\}$$

The **Quotient (or Factor) Ring** R/I is then the cosets of I in R subject to special addition and multiplication[1] And the **equivalence relation** for cosets is defined to be

$$x \sim y \iff x - y \in I$$

for $x, y \in R$
Since,

$$((x^2 + 1) + I)((2x^2 + 3x) + I) = ((2x^4 + 3x^4 + 2x^2 + 3x) + I)$$

let

$$A = 2x^4 + 3x^4 + 2x^2 + 3x$$

and

$$B = \ -4x^2 - 20x - 21$$
$$A - B = 2x^4 + 3x^3 + 6x^2 + 23x + 21$$

since

$$\frac{A - B}{x^3 + 3x + 7} = 2x + 3$$

we have $A - B \in I$ and hence cosets $(A + I)$ and $(B + I)$ are equivalent

Q1g

F: An inner product must have the following properties:

$$(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z}) \quad (1)$$
$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x}) \quad (2)$$
$$(\vec{x}, \vec{x}) \geq 0 \quad (3)$$

The proposed inner product violates property 3. Consider the polynomial

$$P(x) = \prod_{i=1}^{n-1} (x - i)$$

then

$$(P(x), P(x)) = 0$$

I dont see how it equals 0, maybe im misreading sth

[1]See THEOREM 3.6.4, pg 53

Q1h

T: The complex inner product has the following properties:

$$(\lambda\vec{x}+\mu\vec{y},\vec{z}) = \lambda(\vec{x},\vec{z})+\mu(\vec{y},\vec{z}) \tag{1}$$
$$(\vec{x},\vec{y}) = \overline{(\vec{y},\vec{x})} \tag{2}$$
$$(\vec{x},\vec{x}) \geq 0 \tag{3}$$

Fairly straightforward by checking it satisfies the three properties. I think you can probably claim properties 1 trivial/clear from definition. Proving 2:

$$\begin{aligned}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right) &= 4x_1\overline{y_1}-2x_1\overline{y_2}-2x_2\overline{y_1}+3x_2\overline{y_2}\\ &= (2x_1-x_2)\overline{(2y_1-y_2)}+2x_2\overline{y_2}\end{aligned}$$

proving property 3

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = |2x_1-x_2|^2+2|x_2|^2$$

Q1i

F: The **Image** $\operatorname{Im} f$ of a linear map $f: V\to W$ is $f(V)\in W$. (Everything in W that can be mapped to by f).
The **kernel** $\ker f$ is the set $\{v\in V : f(v)=0\}$ (everything in V that is mapped to 0_W)

Consider the example $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. The vector $v=\begin{pmatrix}1\\0\end{pmatrix}$ is in both the image and kernel. Since

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \text{(in kernel)}$$
$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \quad \text{(in image)}$$

Q1j

someone else explain it better pls

Question 2

Q2a

$m_A(x)$ for $A=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$ is simply $x-\lambda$ because all the λ's lie on the diagonal. so subtracting λI from A would equal 0.
$m_A(x)$ for $A=\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}$, not exactly sure how besides calculating characteristic polynomial.

$$\begin{aligned}\chi_A(x) &= \det(A-xI)\\ &= \det\begin{pmatrix}\lambda-x&1\\0&\lambda-x\end{pmatrix}\\ &= (\lambda-x)^2\end{aligned}$$

Q2b

A subset I of a ring R is an ideal if

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$.

Clearly $m_A \in I_A$ since $\cdot m_A(A) = 0$. So 1 is true.

For any

$$q(x) \in F[x]$$

it follows that

$$q(x)m_A(x) \in I_A$$

since

$$\begin{aligned} q(A)m_A(A) &= q(A) \cdot 0 \\ &= 0 \end{aligned}$$

solutions claims it is closed under addition, not sure if the difference is meaningful, but we can prove that it is closed until subtraction via the following:

Consider $p, q \in F[x]$, clearly

$$p(x)m_A(x) \in I_A$$

and

$$q(x)m_A(x) \in I_A$$

Since

$$p(x) - q(x) \in F[x]$$

and

$$\begin{aligned} (p(A) - q(A))m_A(A) =& (p(A) - q(A))0 \\ =& 0 \end{aligned}$$

so

$$(p(x) - q(x))m_A(x) \in I_A$$

and therefore I_A is closed under subtraction

Question 3