The University of Edinburgh

College of Science and Engineering

Mathematics 3 Honours

MATH10069 Honours Algebra

Monday, 1st May 2017

14:30 – 17:30

Chairman of Examiners – Dr A Olde Daalhuis

External Examiner – Professor G Brown

Credit will be given for the best **FOUR** answers

In this examination, candidates are allowed to have three sheets of A4 paper with whatever notes they desire written or printed on one or both sides of the paper.

Magnifying devices to enlarge the contents of the sheets for viewing are not permitted.

No further notes, printed matter or books are allowed.

Calculators and other electronic aids

A scientific calculator is permitted in this examination.

It must not be a graphical calculator.

It must not be able to communicate with any other device.

This examination will be marked anonymously.

(1) Give an example of the following. [Explanations or justifications are not required.]

(a) A linearly independent subset of $\mathbb{R}^3$ with two elements. **[1 mark]**

(b) A spanning subset for $\mathbb{R}^2$ with three elements. **[1 mark]**

(c) The matrix of a non-identity rotation in $\mathbb{R}^2$. **[1 mark]**

(d) A vector space over $\mathbb{R}$ that properly contains $\mathbb{R}[X]$. **[1 mark]**

(e) A vector space with exactly 64 elements. **[1 mark]**

(f) A basis for the real vector space of quadratic polynomials $\mathbb{R}[X]_{<3} = \{a_0 + a_1X + a_2X^2\}$ that contains no constant polynomials. **[1 mark]**

(g) A linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$ whose rank is 2. **[1 mark]**

(h) A surjective linear mapping from a real vector space of dimension 3 to a real vector space of dimension 2. **[1 mark]**

(i) An isomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not the identity. **[1 mark]**

(j) A 3×3 matrix with entries in $\mathbb{R}$ all of whose entries are non-zero and which has 1 as an eigenvalue. **[1 mark]**

(k) A 2×2 matrix with entries in $\mathbb{Z}$ whose trace is 1, which is invertible in $\mathrm{Mat}(2; \mathbb{Z})$. **[1 mark]**

(l) A 2×2 matrix with entries in $\mathbb{Z}$ whose trace is 1, but which is not invertible in $\mathrm{Mat}(2; \mathbb{Z})$. **[1 mark]**

(m) An ideal I of $\mathbb{R}[X]$ which is neither $\{0\}$ nor $\mathbb{R}[X]$. **[1 mark]**

(n) A quadratic polynomial in $\mathbb{R}[X]$ with remainder 1 when divided by $X - 1$. **[1 mark]**

(o) An integral domain that is not a field. **[1 mark]**

(p) A ring that is not commutative. **[1 mark]**

(q) A ring with 14 elements. **[1 mark]**

(r) An infinite ring whose group of units is finite. **[1 mark]**

(s) A matrix $A \in \mathrm{Mat}(2; \mathbb{R})$ that has no real eigenvalues. **[1 mark]**

(t) A matrix $B \in \mathrm{Mat}(2; \mathbb{R})$ that has two distinct real eigenvalues. **[1 mark]**

(u) A positive definite symmetric bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. **[1 mark]**

(v) A non-zero alternating bilinear form $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. **[1 mark]**

(w) An inner product on $\mathbb{C}^2$. **[1 mark]**

(x) A non-diagonalisable matrix $C \in \mathrm{Mat}(2; \mathbb{C})$. **[1 mark]**

(y) A non-diagonal matrix $D \in \mathrm{Mat}(2; \mathbb{C})$ in Jordan Normal Form with characteristic polynomial $X^2 - 2X + 1$. **[1 mark]**

(2) In this question V and W denote finite dimensional vector spaces over a field F.

(a) Let $f : V \to W$ be a linear mapping. Prove that $\ker(f)$ and $\mathrm{im}(f)$ are subspaces of V and W respectively. State the Rank-Nullity Theorem for f, explaining the terms you use. **[7 marks]**

[Please turn over]

(b) Let $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the linear mapping defined as

$$g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_3 + x_4 \\ -2x_1 - 4x_2 - 2x_3 - 2x_4 \\ x_1 + x_2 + x_3 \\ x_2 + x_4 \end{pmatrix}$$

(i) Write down ${}_{\mathcal{S}(4)}[g]_{\mathcal{S}(4)}$, the matrix that represents g with respect to the standard basis $\mathcal{S}(4)$ of $\mathbb{R}^4$. **[2 marks]**

(ii) What is the dimension of $\mathrm{im}(g)$? Write down a basis for $\mathrm{im}(g)$. [You do not need to prove it is a basis.] **[4 marks]**

(iii) The vectors $(1, 0, -1, 0)^\mathsf{T}$ and $(0, 1, -1, -1)^\mathsf{T}$ belong to $\ker(g)$. Explain why they form a basis for this kernel. **[3 marks]**

(iv) Write down an ordered basis $\mathcal{A}$ for $\mathbb{R}^4$ that extends this basis of $\ker(g)$, and an ordered basis $\mathcal{B}$ for $\mathbb{R}^4$ that extends the basis of $\mathrm{im}(g)$ you have given in (ii) above. Calculate ${}_{\mathcal{B}}[g]_{\mathcal{A}}$, the matrix that represents g with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. **[6 marks]**

(c) Suppose that $h : V \to V$ is a linear mapping such that the n-fold composition of h with itself, h^n, is a bijection for some positive integer n. Prove that h^i is an isomorphism for all non-zero integers i. **[3 marks]**

(3) In this question R will denote a ring.

(a) What does it mean to say that R is an integral domain? Prove that if R is a finite integral domain then R must be a field. (State clearly any results that you assume in the proof.) **[7 marks]**

(b) Let $R = \mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field with 3 elements.

(i) Let I be the set $\{(x^2+1)p(x) : p(x) \in R\}$. Prove that I is an ideal of R. **[3 marks]**

(ii) The factor ring R/I has exactly 9 elements. What are they? **[4 marks]**

(iii) Show that the polynomial $x^2 + 1$ has no roots over $\mathbb{F}_3$, the field with three elements. **[2 marks]**

(iv) By showing that R/I is an integral domain, or otherwise, deduce that R/I is a field. **[3 marks]**

(c) In each case, explain clearly by an example why the following mappings are not ring homomorphisms.

(i) $f_1 : \mathbb{R} \to \mathbb{R}$ where $f_1(z) = -z$.

(ii) $f_2 : \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ where $f_2(a + 9\mathbb{Z}) = a + 12\mathbb{Z}$

(iii) $f_3 : \mathrm{Mat}(2; \mathbb{R}) \to \mathbb{R}[x]$ where $f_3(A) = \det(xI_2 - A)$, the characteristic polynomial of A. **[6 marks]**

(4) (a) Which of the following define inner products on the given vector spaces? For each give a brief proof, or state which condition of being an inner product fails and give a counterexample.

[Please turn over]

(i) On $\mathbb{R}^n$, the function
$$(\vec{v}, \vec{w}) = \vec{v}^{\mathsf{T}} A \vec{w},$$
where A is a real $n \times n$ diagonal matrix with strictly positive entries on the diagonal.

(ii) On $\mathbb{C}^2$, the function
$$(\vec{v}, \vec{w}) = v_1 \overline{w_1} + \overline{v_2} w_2,$$
where $\vec{v} = (v_1, v_2)^{\mathsf{T}}$ and $\vec{w} = (w_1, w_2)^{\mathsf{T}}$.

(iii) On $\mathbb{R}[X]$, the function
$$(P, Q) = \int_{-1}^{1} X P(X) Q(X) \, dX.$$
[6 marks]

(b) Consider $\mathbb{R}^3$ equipped with the usual inner product. By giving a basis, where appropriate, or otherwise, describe explicitly the elements $(x, y, z)^{\mathsf{T}}$ of the following subspaces:

(i) $\{(3, 2, 1)^{\mathsf{T}}\}^{\perp}$;

(ii) $\{(3, 2, 1)^{\mathsf{T}}, (-1, 1, 0)^{\mathsf{T}}, (1, 0, 0)^{\mathsf{T}}\}^{\perp}$;

(iii) $\{(x, y, z)^{\mathsf{T}} : x = y\}^{\perp}$.

[6 marks]

(c) Show that a subset M of a finite-dimensional inner product space satisfies $M = (M^{\perp})^{\perp}$ if and only if M is a subspace. **[4 marks]**

(d) Let V be a finite-dimensional real inner product space, and suppose T is an endomorphism of V.

(i) Show that $(T + T^*)/2$ is self-adjoint. (You may assume without proof that adjoints are unique and that $T = T^{**}$, but any other properties of adjoints you use must be justified.) **[2 marks]**

(ii) Hence, or otherwise, show that there is an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V consisting of eigenvectors of $(T + T^*)/2$, such that the eigenvalue corresponding to $\vec{v}_i$ is $(T\vec{v}_i, \vec{v}_i)$. **[7 marks]**

(5) (a) Let $A \in \mathrm{Mat}(3; \mathbb{C})$ be the matrix
$$A = \begin{pmatrix} 0 & -4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
Calculate the characteristic polynomial of A. What are the eigenvalues of A? What are the algebraic multiplicities of the eigenvalues you found? Their geometric multiplicities? **[7 marks]**

(b) State the Jordan Normal Form Theorem. **[5 marks]**

(c) Let $B \in \mathrm{Mat}(4; \mathbb{C})$ be the following matrix:
$$B = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -1 & 2 \end{pmatrix}.$$

There exists a matrix $P \in \mathrm{Mat}(4;\mathbb{C})$ such that

$$P^{-1}BP = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

[You do not need to prove this.]

(i) Find P. **[7 marks]**

(ii) Apart from the above, up to permutation there are two other Jordan normal forms for a matrix in $\mathrm{Mat}(4;\mathbb{C})$ with the same characteristic polynomial as B. What are they? **[2 marks]**

(d) Suppose that $C \in \mathrm{Mat}(n;\mathbb{C})$ is a matrix such that $C^2 = C$. Prove that the only possible eigenvalues of C are 0 and 1. **[4 marks]**

[End of Paper]