H Exercises

Exercise III.1 Compute the probability of measuring $|0\rangle$ and $|1\rangle$ for each of the following quantum states:

1. $0.6|0\rangle + 0.8|1\rangle$.
2. $\frac{1}{\sqrt{3}}|0\rangle + \sqrt{2/3}|1\rangle$.
3. $\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$.
4. $-\frac{1}{25}(24|0\rangle - 7|1\rangle)$.
5. $-\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/6}}{\sqrt{2}}|1\rangle$.

Exercise III.2 Compute the probability of the four states if the following are measured in the computational basis:

1. $(e^i|00\rangle + \sqrt{2}|01\rangle + \sqrt{3}|10\rangle + 2e^{2i}|11\rangle)/\sqrt{10}$.
2. $\frac{1}{2}(-|0\rangle + |1\rangle) \otimes (e^{\pi i}|0\rangle + e^{-\pi i}|1\rangle)$.
3. $$(\sqrt{1/3}|0\rangle - \sqrt{2/3}|1\rangle) \otimes \sqrt{2}\left(\frac{e^{\pi i/4}}{2}|0\rangle + \frac{e^{\pi i/2}}{2}|1\rangle\right).$$

Exercise III.3 Suppose that a two-qubit register is in the state

$$|\psi\rangle = \frac{3}{5}|00\rangle - \frac{\sqrt{7}}{5}|01\rangle + \frac{e^{i\pi/2}}{\sqrt{5}}|10\rangle - \frac{2}{5}|11\rangle.$$

1. Suppose we measure just the first qubit. Compute the probability of measuring a $|0\rangle$ or a $|1\rangle$ and the resulting register state in each case.
2. Do the same, but supposing instead that we measure just the second qubit.

Exercise III.6 Prove that $U(t) \stackrel{\text{def}}{=} \exp(-iHt/\hbar)$ is unitary.

Exercise III.7 Use spectral decomposition to show that $K = -i\log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Exercise III.8 Show that the commutators ($[L, M]$ and $\{L, M\}$) are bilinear (linear in both of their arguments).

Exercise III.9 Show that $[L, M]$ is anticommutative, i.e., $[M, L] = -[L, M]$, and that $\{L, M\}$ is commutative.

Exercise III.10 Show that $LM = \frac{[L,M]+\{L,M\}}{2}$.

Exercise III.11 In Sec. B.5 we proved the no-cloning theorem with single ancillary constant qubit. Prove that the cloning is still impossible if multiple ancillary qubits are provided. That is, show that we cannot have a unitary operator $U(|\psi\rangle \otimes |C\rangle) = |\psi\rangle|\psi\rangle \otimes |D\rangle$, where $|C\rangle$ is an $n > 1$ dimensional vector and $|D\rangle$ is an $n-1$ dimensional vector. (Note that $|D\rangle$ might depend on $|\psi\rangle$.)

Exercise III.12 Show that the four Bell states are orthonormal (i.e., both orthogonal and normalized).

Exercise III.13 Prove that $|\beta_{11}\rangle$ is entangled.

Exercise III.14 Prove that $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is entangled.

Exercise III.15 What is the effect of Y (imaginary definition) on the computational basis vectors? What is its effect if you use the real definition (C.2.a, p. 106)?

Exercise III.16 Prove that I, X, Y, and Z are unitary. Use either the imaginary or real definition of Y (C.2.a, p. 106).

Exercise III.17 What is the matrix for H in the *sign basis*?

Exercise III.19 Prove the following useful identities:

$$HXH = Z, HYH = -Y, HZH = X.$$

Exercise III.20 Show (using the real definition of Y, C.2.a, p. 106):
$|0\rangle\langle 0| = \frac{1}{2}(I+Z), |0\rangle\langle 1| = \frac{1}{2}(X+Y), |1\rangle\langle 0| = \frac{1}{2}(X-Y), |1\rangle\langle 1| = \frac{1}{2}(I-Z).$

Exercise III.21 Prove that the Pauli matrices span the space of 2×2 matrices.

Exercise III.22 Prove $|\beta_{xy}\rangle = (P \otimes I)|\beta_{00}\rangle$, where $xy = 00, 01, 11, 10$ for $P = I, X, Y, Z$, respectively.

Exercise III.23 Suppose that P is one of the Pauli operators, but you don't know which one. However, you are able to pick a 2-qubit state $|\psi_0\rangle$ and operate on it, $|\psi_1\rangle = (P \otimes I)|\psi_0\rangle$. Further, you are able to select a unitary operation U to apply to $|\psi_1\rangle$, and to measure the 2-qubit result, $|\psi_2\rangle = U|\psi_1\rangle$, in the computational basis. Select $|\psi_0\rangle$ and U so that you can determine with certainty the unknown Pauli operator P.

Exercise III.24 What is the matrix for CNOT in the standard basis? Prove your answer.

Exercise III.25 Show that CNOT does not violate the No-cloning Theorem by showing that, in general, $\text{CNOT}|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$. Under what conditions does the equality hold?

Exercise III.26 What quantum state results from

$$\text{CNOT}(H \otimes I)\ \frac{1}{2}(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle)?$$

Express the result in the computational basis.

Exercise III.27 Compute $(Y \otimes I)\text{CNOT}(H \otimes I)|00\rangle$. Show your work.

Exercise III.28

3. Apply Z to the state resulting from measuring $|10\rangle$.

Exercise III.29 What is the matrix for CCNOT in the standard basis? Prove your answer.

Exercise III.30 Use a single Toffoli gate to implement each of NOT, NAND, and XOR.

Exercise III.31 Use Toffoli gates to implement FAN-OUT. FAN-OUT would seem to violate the No-cloning Theorem, but it doesn't. Explain why.

Exercise III.32 Design a quantum circuit to transform $|000\rangle$ into the entangled state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

Exercise III.33 Show that $|+\rangle, |-\rangle$ is an ON basis.

Exercise III.34 Prove:

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle),\\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).\end{aligned}$$

Exercise III.35 What are the possible outcomes (probabilities and resulting states) of measuring $a|+\rangle + b|-\rangle$ in the *computational basis* (of course, $|a|^2 + |b|^2 = 1$)?

Exercise III.36 Prove that $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

Exercise III.37 Prove:

$$\begin{aligned}H(a|0\rangle + b|1\rangle) &= a|+\rangle + b|-\rangle,\\ H(a|+\rangle + b|-\rangle) &= a|0\rangle + b|1\rangle.\end{aligned}$$

Exercise III.38 Prove $H = (X + Z)/\sqrt{2}$.