Simplifying the ikelihood Estimation of Two Variance Components in a Random Effects Model

Model

$$y_i = u_i + \varepsilon_i \quad (i = 1, \dots, n)$$
$$u = (u_1, \dots, u_n)' \sim N\left(0, G\sigma_g^2\right)$$
$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)' \sim N(0, I\sigma_\varepsilon^2)$$

Likelihood Function

$$L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) = \left\|G\sigma_g^2 + I\sigma_\varepsilon^2\right\|^{-\frac{1}{2}} Exp\left\{-\tfrac{1}{2} y'\left[\sigma_g^2 G + I\sigma_\varepsilon^2\right]^{-1} y\right\}$$

Let $G = VDV'$ be the eigenvalue decomposition of G, where V is a matrix with eigenvectors, satisfying $V'V = VV' = I$ and D is a diagonal matrix with the eigenvalues of G in the diagonal. Using $G = VDV'$ and $VV' = I$ we have

$$\begin{aligned} L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) &= \left\|VDV'\sigma_g^2 + VV'\sigma_\varepsilon^2\right\|^{-\frac{1}{2}} Exp\left\{-\tfrac{1}{2} y'\left[\sigma_g^2 VDV' + VV'^{\sigma_\varepsilon^2}\right]^{-1} y\right\} \\ &= \left\|V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V'\right\|^{-\frac{1}{2}} Exp\left\{-\frac{1}{2} y'\left[V\left[D\sigma_g^2 + I\right]\sigma_\varepsilon^2 V'\right]^{-1} y\right\} \end{aligned}$$

Further, using $[AB]^{-1} = B^{-1}A^{-1}$ and $V^{-1} = V'$

$$\left[V\left[D\sigma_g^2 + I\right]\sigma_\varepsilon^2 V'\right]^{-1} = \sigma_\varepsilon^{-2} V[D\lambda + I]^{-1}V'$$

where: $\lambda = \frac{\sigma_g^2}{\sigma_\varepsilon^2}$.

Furthermore, the determinant of $V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V'$ is simply the product of it's eigenvalues, therefore,

$$\left\|V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V'\right\| = \prod_{i=1}^{n}\left(d_i\sigma_g^2 + \sigma_\varepsilon^2\right) = \prod_{i=1}^{n}\left(d_i\sigma_g^2\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} + \sigma_\varepsilon^2\right) = \sigma_\varepsilon^{2n}\prod_{i=1}^{n}(d_i\lambda + 1)$$

Combining the above results we have

$$L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) = \left[\sigma_\varepsilon^{2n}\prod_{i=1}^{n}(d_i\lambda + 1)\right]^{-\frac{1}{2}} Exp\left\{-\frac{y'V[D\lambda + I]^{-1}V'y}{2\sigma_\varepsilon^2}\right\}$$

Therefore, $f\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) = -2\times\log\left\{L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right)\right\}$ becomes,

$$\begin{aligned} l\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) &= log\left\{\sigma_\varepsilon^{2n}\prod_{i=1}^{n}(d_i\lambda + 1)\right\} + \sigma_\varepsilon^{-2}\sum_{i=1}^{n}\frac{\tilde{y}_i^2}{d_i\lambda + 1} \\ &= n\times log\left(\sigma_\varepsilon^2\right) + \sum_{i=1}^{n}\log(d_i\lambda + 1) + \sigma_\varepsilon^{-2}\sum_{i=1}^{n}\frac{\tilde{y}_i^2}{d_i\lambda + 1} \end{aligned}$$

where: $\tilde{y} = V'y$.