Maximum Likelihood Estimation of Two Variance Components in a Random Effects Model

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The code presented below contains functions for evaluating the likelihood of a random effects model using the eigenvalue decomposition of the co-variance matrix associated to a random effect. Several authors have explited the equivalence between Gaussian processes and reandom regressions on eigenvectors, a few examples of these are: de los Campos et al. (2010) , Zhou and Stephens (2012) and Janss et al., (2012).

Model

$$y_i = u_i + \varepsilon_i \quad (i = 1, \dots, n)$$
$$u = (u_1, \dots, u_n)' \sim N\left(0, G\sigma_g^2\right)$$
$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)' \sim N(0, I\sigma_\varepsilon^2)$$

Likelihood Function

$$L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) = \left\| G\sigma_g^2 + I\sigma_\varepsilon^2 \right\|^{-\frac{1}{2}} Exp\left\{-\tfrac{1}{2} y'\left[\sigma_g^2 G + I\sigma_\varepsilon^2\right]^{-1} y\right\}$$

Let $G = VDV'$ be the eigenvalue decomposition of G, where V is a matrix with eigenvectors, satisfying $V'V = VV' = I$ and D is a diagonal matrix with the eigenvalues of G in the diagonal. Using $G = VDV'$ and $VV' = I$ we have

$$L\left(\sigma_g^2, \sigma_\varepsilon^2; y\right) = \left\| VDV'\sigma_g^2 + VV'\sigma_\varepsilon^2 \right\|^{-\frac{1}{2}} Exp\left\{-\tfrac{1}{2} y'\left[\sigma_g^2 VDV' + VV'^{\sigma_\varepsilon^2}\right]^{-1} y\right\}$$
$$= \left\| V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V' \right\|^{-\frac{1}{2}} Exp\left\{-\frac{1}{2} y'\left[V\left[D\sigma_g^2 + I\right]\sigma_\varepsilon^2 V'\right]^{-1} y\right\}$$

Further, using $[AB]^{-1} = B^{-1}A^{-1}$ and $V^{-1} = V'$

$$\left[V\left[D\sigma_g^2 + I\right]\sigma_\varepsilon^2 V'\right]^{-1} = \sigma_\varepsilon^{-2} V[D\lambda + I]^{-1} V'$$

where: $\lambda = \frac{\sigma_g^2}{\sigma_\varepsilon^2}$.

Furthermore, the determinant of $V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V'$ is simply the product of it's eigenvalues, therefore,

$$\left\| V\left[D\sigma_g^2 + I\sigma_\varepsilon^2\right]V' \right\| = \prod_{i=1}^{n} \left(d_i\sigma_g^2 + \sigma_\varepsilon^2\right) = \prod_{i=1}^{n} \left(d_i\sigma_g^2 \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} + \sigma_\varepsilon^2\right) = \sigma_\varepsilon^{2n} \prod_{i=1}^{n} (d_i\lambda + 1)$$

Combining the above results we have

$$L(\sigma_g^2, \sigma_\varepsilon^2; y) = [\sigma_\varepsilon^{2n} \prod_{i=1}^{n}(d_i\lambda + 1)]^{-\frac{1}{2}} Exp\left\{-\frac{y'V[D\lambda+I]^{-1}V'y}{2\sigma_\varepsilon^2}\right\}$$

Therefore, $f(\sigma_g^2, \sigma_\varepsilon^2; y) = -2\times \log\{L(\sigma_g^2, \sigma_\varepsilon^2; y)\}$ becomes,

$$l(\sigma_g^2, \sigma_\varepsilon^2; y) = log\{\sigma_\varepsilon^{2n} \prod_{i=1}^{n}(d_i\lambda + 1)\} + \sigma_\varepsilon^{-2} \sum_{i=1}^{n} \frac{\tilde{y}_i^2}{d_i\lambda+1}$$
$$= n\times log(\sigma_\varepsilon^2) + \sum_{i=1}^{n} \log(d_i\lambda + 1) + \sigma_\varepsilon^{-2} \sum_{i=1}^{n} \frac{\tilde{y}_i^2}{d_i\lambda+1}$$

where: $\tilde{y} = V'y$.