

# Mek4100

## The WKB method

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# Demonstration problem

Oscillation problem with variable coefficient

$$\frac{d^2 y}{dt^2} + W(\epsilon t)y = 0; \quad y(t_a) = a, \quad y(t_b) = b, \quad (1)$$

where  $W > 0$ .

$\epsilon$  – small parameter  $\Rightarrow$  coefficient  $W$  is slowly varying.

**Note 1:** Equation solved by two-scale technique in problem 24b.

**Note 2:** The problem (1) does not always inherit a solution. Will be demonstrated in specific example.

**Note 3:**  $W = \text{const.} \Rightarrow y = A_+ e^{i\sqrt{W}t} + y = A_- e^{-i\sqrt{W}t}$   
Oscillatory solution.

# Preparation for WKB; rescaling

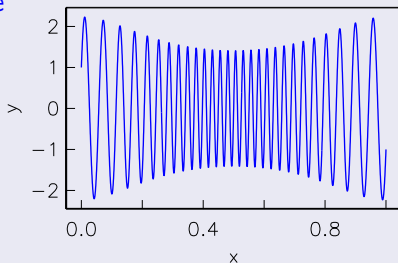
Choose slow scale as free variable  $x = \epsilon t$

$$\epsilon^2 \frac{d^2 y}{dx^2} + W(x)y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \quad (2)$$

Looks like a boundary layer problem, but solution oscillates rapidly everywhere

(2) not well scaled in the usual sense since  $\frac{d^2 y}{dx^2}$  becomes unbounded as  $\epsilon \rightarrow 0$ . Scaling in (2) is convenient, but not necessary.

## Solution example



# Use of exponential form

Write solution in terms of new unknown

$$y = e^{S(x)}.$$

Substitution into 2 yields equation for  $k(x) \equiv S'$ :

$$\epsilon^2(k' + k^2) + W = 0, \quad (3)$$

First order nonlinear ODE, called a Ricatti equation.

So far no real approximation or progress are made; (3) is still not solvable in formula for general  $W$ . But;

(3) makes a good starting point for dominant balance analysis.

The transformation by means of the exponential form only feasible for linear, homogeneous equations

# Dominant balance

$$\begin{aligned}\epsilon^2 k' + \epsilon^2 k^2 + W &= 0 \\ (1) + (2) + (3) &= 0\end{aligned}\tag{4}$$

(1) & (3):  $k \sim -\epsilon^{-2} \int W dx \Rightarrow (2) \sim \epsilon^{-2} \gg (1), (3)$ . Invalid!

(1) & (2):  $k \sim (C+x)^{-1} \Rightarrow (3)$  dominates as  $\epsilon \rightarrow 0$ ,  $y \sim x+C$ . Invalid!

(2) & (3):  $k \sim k_0 = \pm i \epsilon^{-1} W^{\frac{1}{2}} \Rightarrow (1) \sim \epsilon \ll (2), (3)$ . Two solutions.  
OK!\*

$$y \sim e^{\pm i \epsilon^{-1} \int W^{\frac{1}{2}} dx}$$

Describes rapid oscillations. This is the full solution if  $W$  is constant.

\*: Since  $|k| \rightarrow \infty$  as  $\epsilon \rightarrow 0$  we must expect  $k^2 \gg k'$

## Second balance

$k = k_0 + k_1$ ,  $k_1 \ll k_0$ . Substitution in (4)

$$\epsilon^2(k_0' + k_1' + k_0^2 + 2k_0k_1 + k_1^2) + W = 0$$

Canceling of leading order  $\epsilon^2 k_0^2 + W = 0$  and  $k_1 \ll k_0 \Rightarrow$

$$\epsilon^2(k_0' + 2k_0k_1) = 0,$$

with solution  $k_1 = -\frac{1}{2}k_0'/k_0 = -\frac{1}{4}W'/W = O(1)$ .

## Third balance

$k = k_0 + k_1 + k_2$ ,  $k_2 \ll k_1 \ll k_0$ , canceling etc.  $\Rightarrow$

$$\epsilon^2(k_1' + k_1^2 + 2k_0k_2) = 0,$$

$$k_2 = -\frac{k_1' + k_1^2}{2k_0} = \mp i\epsilon \left( \frac{W''}{8W^{\frac{3}{2}}} - \frac{5(W')^2}{32W^2} \right).$$

# Assembling the solutions

$$S_{\pm} = \int k dx = C_{\pm} \pm \frac{i}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x} + \ln \left( W^{-\frac{1}{4}} \right) \mp i\epsilon\alpha,$$

where  $\alpha = \frac{1}{8} \int_{x_a}^x W'' W^{-\frac{3}{2}} d\hat{x}$  and  $C_{\pm}$  is a constant of integration.

The two solutions for  $y$  are written ( $\epsilon\alpha \ll 1$ )

$$y_{\pm} = A_{\pm} W^{-\frac{1}{4}} e^{\pm \frac{i}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}} e^{\mp i\epsilon\alpha} = A_{\pm} W^{-\frac{1}{4}} e^{\pm \frac{i}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}} (1 \mp i\epsilon\alpha + O(\epsilon^2))$$

Choosing  $A_- = A_+^*$  makes  $y_- + y_+$  real. Moreover, real and imaginary parts of  $A_-$  may be found as to make the boundary conditions fulfilled.

From  $k_1$  we obtain an amplitude modulation.

## Digression: The formal WKB expansion

The end results were expansions of type

$$y = e^{i\frac{\phi(x)}{\epsilon}} (A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \dots). \quad (5)$$

- ❶ (5) often used as an ansatz.
- ❷ Reduces the expansion to unimaginative book-keeping.
- ❸ (5) sometimes not appropriate.
- ❹ The higher  $A_j$  seldom significant.

The form (5) akin to solution for constant  $W$ , namely

$$y = Ae^{\frac{i}{\epsilon} W^{\frac{1}{2}} x} = Ae^{\frac{i}{\epsilon} \int W^{\frac{1}{2}} dx}$$



# The explicit real solution

When  $\alpha$  is ignored the sum  $y_- + y_+$  may be recast into the form

$$y = W^{-\frac{1}{4}} (B \cos \psi + C \sin \psi), \quad (6)$$

where  $\psi = \epsilon^{-1} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}$  and  $A_+ = \frac{1}{2}(B - iC)$ .

Boundary conditions  $\Rightarrow$

$$B = aW(x_a)^{\frac{1}{4}}, \quad C = \left( bW(x_b)^{\frac{1}{4}} - aW(x_a)^{\frac{1}{4}} \cos \psi(x_b) \right) \frac{1}{\sin \psi(x_b)}.$$

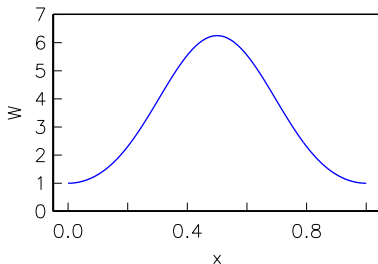
No solution if  $\sin \psi(x_b) = 0$ , meaning  $\epsilon^{-1} \int_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x} = n\pi$ .

## A specific case

Selected parameters ( $\epsilon$  is not fixed!):  $x_a = 0$ ,  $x_b = 1$ ,  $a = 1$ ,  
 $b = -1$ ,  
and function

$$W^{\frac{1}{2}} = Q + R \cos^2(x - \frac{1}{2})\pi, \quad \psi = \frac{1}{\epsilon} \left[ (Q + \frac{1}{2}R)x + \frac{R}{4\pi} \sin(2x - 1)\pi \right],$$

where  $Q = 1$  and  $R = \frac{3}{2}$ .



Define  $y_j \approx y(j\Delta x)$  for  $j = 0, \dots, n$  and  $\Delta x = \frac{1}{n}$ .

Tri-diagonal set of equations, solved by Gaussian elimination

$$y_0 = a,$$

$$\frac{1}{\Delta x^2} (y_{j+1} - 2y_j + y_{j-1}) + \frac{1}{\epsilon^2} W_j y_j + s_j = 0, \quad j = 1, \dots, n-1,$$

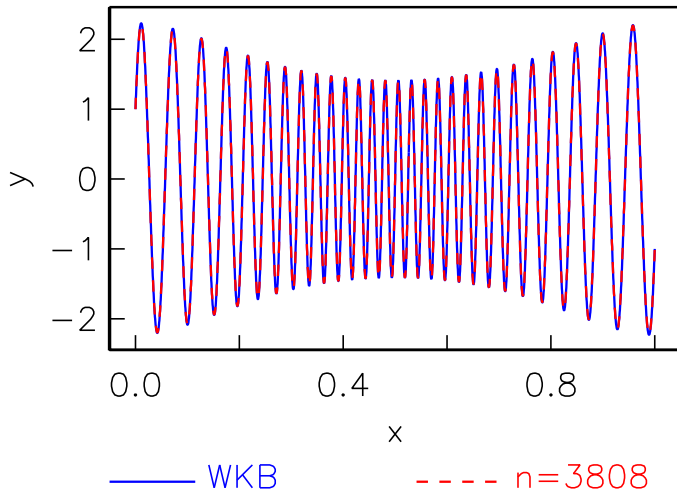
$$y_n = b,$$

where correction terms

$$s_j = \frac{1}{12\epsilon^2} (W_{j+1}y_{j+1} - 2W_jy_j + W_{j-1}y_{j-1}),$$

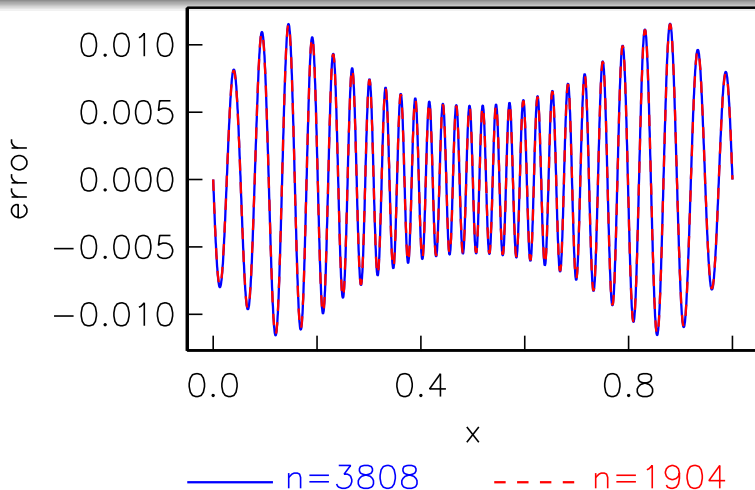
reduces error to  $O(\Delta x^4)$ .

## Solutions for $\epsilon = 0.01$



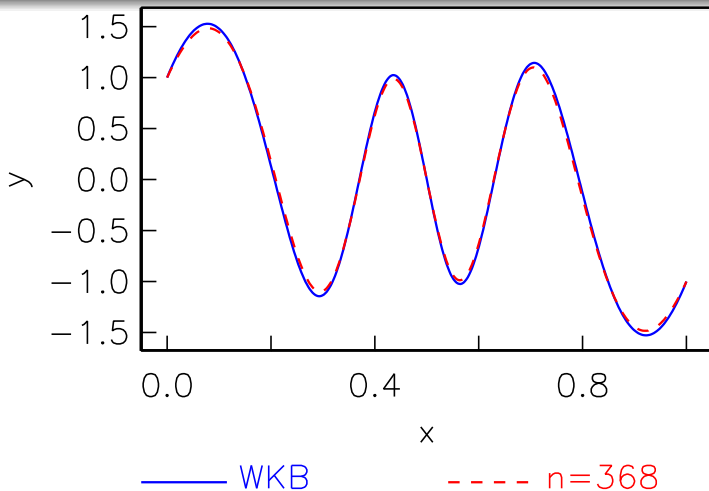
Numerical solution marked by value of  $n$ .

## Errors for $\epsilon = 0.01$



Numerical solution not noticeably dependent on  $n$ .  
Error is 0.5%, say, of typical value of  $y$ .

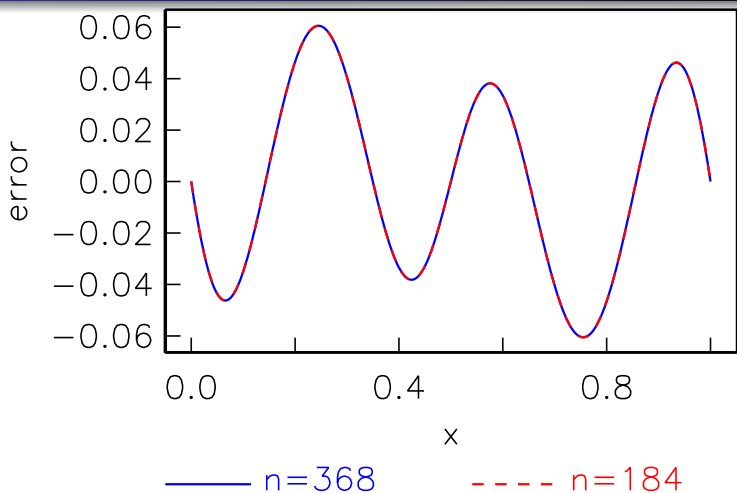
## Results $\epsilon = 0.1$



Numerical solution marked by value of  $n$ .

For this larger  $\epsilon$ : WKB still quite good, but error visible

## Errors for $\epsilon = 0.1$



Again, numerical solution is not noticeably dependent on  $n$ .  
Error is 4%, say, of typical value of  $y$ .

## Convergence in $\epsilon$

$\langle f \rangle = \int_0^1 f dx$ , evaluated by trapezoidal integration

$$L_2 = \sqrt{\langle (y_{\text{num}} - y_{\text{WKB}})^2 \rangle}, \quad E_r = L_2 / \left( \epsilon \sqrt{\langle (y_{\text{WKB}})^2 \rangle} \right)$$

$\epsilon$	$L_2$	$E_r$
0.10	$0.36 \cdot 10^{-1}$	0.38
$0.50 \cdot 10^{-1}$	$0.11 \cdot 10^{-1}$	0.39
$0.25 \cdot 10^{-1}$	$0.14 \cdot 10^{-1}$	0.42
$0.10 \cdot 10^{-1}$	$0.60 \cdot 10^{-2}$	0.48
$0.50 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$	0.33
$0.25 \cdot 10^{-2}$	$0.44 \cdot 10^{-3}$	0.30

$\text{nf}=32$  (measure of resolution)

Solution changes qualitatively with  $\epsilon \Rightarrow E_r$  remains of same size,  
but does not approach a constant.



# WKB and a boundary layer problem

Change: sign on coefficient in the equation

$$\epsilon^2 \frac{d^2 y}{dx^2} - W(x)y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \quad (7)$$

where  $W > 0$ . Solutions are now of rapidly growing/decaying nature instead of oscillating.

## The boundary layer method

The problem is virtually contained in problem 64 in leaflet. The unified solution becomes

$$y \approx ae^{-\sqrt{W(x_a)}\frac{(x-x_a)}{\epsilon}} + be^{\sqrt{W(x_b)}\frac{(x-x_b)}{\epsilon}}. \quad (8)$$

Boundary layers at both ends, zero as outer solution.

# The WKB expansion applied to (7)

All the algebra of the first examples repeats itself, except for the occurrence of  $i$ , the imaginary unit. Using  $k_0$  and  $k_1$ :

$$y \approx A_+ W^{-\frac{1}{4}} e^{\frac{1}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}} + A_- W^{-\frac{1}{4}} e^{-\frac{1}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}}.$$

Boundary conditions

$$W(x_a)^{-\frac{1}{4}}(A_+ + A_-) = a, \quad W(x_b)^{-\frac{1}{4}}(\gamma A_+ + \gamma^{-1} A_-) = b,$$

where  $\gamma = e^{\frac{1}{\epsilon} \int_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x}} \gg 1$ . Hence,

$$A_+ = \frac{a\gamma^{-1}W(x_a)^{\frac{1}{4}} - bW(x_b)^{\frac{1}{4}}}{\gamma^{-1} - \gamma}, \quad A_- = \frac{bW(x_b)^{\frac{1}{4}} - a\gamma W(x_a)^{\frac{1}{4}}}{\gamma^{-1} - \gamma}$$

How to reconcile this with (8) ?

First  $\gamma \gg 1 \Rightarrow A_+ \approx b\gamma^{-1}W(x_b)^{\frac{1}{4}}$  and  $A_- = aW(x_a)^{\frac{1}{4}}$ ; thus

$$y \approx \frac{b}{\gamma} \left( \frac{W(x_b)}{W(x)} \right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon} \int_x^{x_b} W^{\frac{1}{2}} d\hat{x}} + a \left( \frac{W(x_a)}{W(x)} \right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon} \int_{x_a}^x W^{\frac{1}{2}} d\hat{x}}. \quad (9)$$

First term grows rapidly toward  $x_b$ : boundary layer at right end.

Second term decays rapidly from  $x_a$ : boundary layer at left end.

Right term significant only when  $x - x_a$  is small. Taylor expansion

$$\int_{x_a}^x \frac{W^{\frac{1}{2}}}{\epsilon} d\hat{x} = \left( \frac{W(x_a)^{\frac{1}{2}}(x - x_a)}{\epsilon} + \frac{(W(x_a)^{\frac{1}{2}})'(x - x_a)^2}{2\epsilon} + \dots \right),$$

For a region  $1 \gg x - x_a \gg \epsilon$  is the first term  $\gg 1$  while the second is  $\ll 1$ . Example:  $x - x_a = \epsilon^{\frac{2}{3}}$ ; first term  $\sim \epsilon^{-\frac{1}{3}}$ , second term  $\sim \epsilon^{\frac{1}{3}}$ .

Consequence: second term in (9) vanishes before second term in

Taylor expansion becomes important. We may then also put

$W(x)/W(x_a) \approx 1$ , meaning that  $k_1$  is ignored.

Similar treatment of first term in (9) gives

$$y \approx be^{-\frac{1}{\epsilon} W(x_b)^{\frac{1}{2}}(x_b-x)} + ae^{-\frac{1}{\epsilon} W(x_a)^{\frac{1}{2}}(x-x_a)}. \quad (10)$$

Which is the boundary layer solution (8) retrieved.

- Homogeneous, linear boundary layer problems may be solved with WKB techniques
- Boundary layer solutions consistent with leading order WKB solution
- Quite some simplification needed to reveal the full relationship

## Relation to theorem 3.12 in Logan

Boundary value problem

$$\epsilon y'' + p(x)y' + q(x) = 0, \quad y(0) = a, \quad y(1) = b, \quad (11)$$

where  $\epsilon \rightarrow 0$ ,  $p(x) > 0$ ,  $p, q \sim 1$

Again

$$y = e^{S(x)} = e^{\int k d\hat{x}}.$$

It is important that we do not assume (5).

$$\begin{aligned}\epsilon k' + \epsilon k^2 + pk + q &= 0 \\ (1) + (2) + (3) + (4) &= 0\end{aligned}\tag{12}$$

(1) & (4):  $k \sim -\epsilon^{-2} \int q dx \Rightarrow (2) \sim \epsilon^{-1} \gg (1), (3)$ . Invalid!

(1) & (2):  $k \sim (C+x)^{-1} \Rightarrow (4)$  dominates as  $\epsilon \rightarrow 0$ ,  $y \sim x + C$ . Invalid!

(2) & (4):  $k \sim \pm i \epsilon^{-\frac{1}{2}} q^{\frac{1}{2}}$ .  $\Rightarrow (3) \sim \epsilon^{-\frac{1}{2}} \gg (2), (4)$ . Invalid!

(2) & (3):  $k \sim -\frac{p}{\epsilon}$ . One valid solution.

(3) & (4):  $k \sim -\frac{q}{p}$ . One valid solution. (Not on form (5)!)

(1) & (3):  $k \sim C e^{-\epsilon^{-1} \int_0^x p d\hat{x}}$ .  $(3) \ll (4)$  when  $x \gg \epsilon$ . Discarded!

Then

$$y = A e^{-\int \frac{q}{p} dx} + B e^{-\int \frac{p}{\epsilon} dx}$$

Outer and boundary layer approximations inherited. Theorem 3.12 from Logan may be reproduced; details omitted.