Compulsory assignment 1

MEK 4100, spring 2014

January 23, 2014

Formalities

Deadline and submitting

You must turn your assignment in before 15.00, Friday, the 21th of February, in the mailbox of Geir Pedersen at the Department of Mathematics (7th floor NHAbelshus).

Requirements

The assignment comprises problem 15 and 18 from the leaflet. Your submission will be reviewed much like an ordinary exam, though with more emphasis on a complete and legible text. Acceptance then requires a 70% score. The four parts of the assignment have the weights 15a, 15b, 18a and 18b have the weights 30%, 20%, 30% and 20%, respectively.

Ex. 15 ***Regular perturbation of differential equation of second order.*** We are given the problem

$$y'' + \epsilon e^{-x} y = 0 \quad ; \quad y(0) = 1,\ y'(0) = 1, \tag{1}$$

where ϵ is small.

a) Show that a regular perturbation expansion leads to

$$y''_{n+1} = -e^{-x} y_n, \tag{2}$$

and that y_n then takes on the form

$$y_n = \sum_{m=0}^{n} (a_{mn} + b_{mn} x) e^{-mx}. \tag{3}$$

Find recursion formulas for a_{mn} and b_{mn}.

b) Invoke the substitution $t = 2\sqrt{\epsilon} e^{-\frac{1}{2}x}$ and show that the exact solution of the differential equation can be expressed in terms of Bessel functions (confer with, for instance, the mathematical handbook of Rottmann). Discuss the prospects for a uniform validity of the perturbation expansion.

Ex. 18 ***Flow between two planes.*** In a dimensionless description a stationary two-dimensional potential-flow is confined between two planes according to

$$1 \geq y \geq b(x),$$

where the lower plane is slightly corrugated

$$b(x) = \epsilon \cos x,$$

where $\epsilon \ll 1$. For the unperturbed problem, identified with $\epsilon = 0$, we assume a uniform flow parallel to the planes. Employing the stream function, ψ, we obtain a simple mathematical description of the problem

$$\nabla^2 \psi = 0 \quad \text{for} \quad b(x) < y < 1, \tag{4}$$

in the fluid, and

$$\psi = 0 \quad \text{at} \quad y = b(x) \quad ; \quad \psi = 1 \quad \text{at} \quad y = 1, \tag{5}$$

as boundary conditions. Inherent in the boundary conditions is a unitary integrated volume flux between the planes.

a) Find the first three terms in a straightforward perturbation expansion for ψ. The leading order term then corresponds to a uniform current associated with $\psi_0 = y$ (the x component of the velocity is $\partial\psi/\partial y$). Observe that the boundary condition at $y = b$ must be expanded as

$$\psi(x, b) = \psi(x, 0) + \psi_y(x, 0) b + \frac{1}{2} \psi_{yy}(x, 0) b^2 + ...$$

where the indices denote partial differentiation.

b) Streamlines are defined by ψ =const. Determine the streamline corresponding to $\psi = c$ as an expression on the form $y = s(c, x)$ by a perturbation expansion.