

# Mek4100

## Two-scale perturbation methods

Geir Pedersen

Department of Mathematics, UiO

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# Motivation

- A number of problems inherit several temporal or spatial scales
- Example: Boundary layer problem; albeit here the rapid scale is only present in the boundary layer
- Several **global** scales  $\Rightarrow$  a new method is required
- Linear, homogeneous equations: WKB(J) is an alternative

## Example 1: damped oscillation

ODE with initial conditions

$$\frac{d^2y}{dt^2} + \epsilon \frac{dy}{dt} + y = 0; \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 0, \quad (1)$$

$\epsilon$  – small parameter.

Physical interpretation: weak resistance force proportional to the velocity

# Direct (naive) perturbation

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \Rightarrow$$

$O(\epsilon^0)$

$$\frac{d^2 y_0}{dt^2} + y_0 = 0,$$

$$y_0(0) = 1, \quad \frac{dy_0(0)}{dt} = 0.$$

Solution:

$$y_0 = \cos t$$

$O(\epsilon^1)$

$$\frac{d^2 y_1}{dt^2} + y_1 = -\frac{dy_0}{dt} = \sin t,$$

$$y_1(0) = \frac{dy_1(0)}{dt} = 0.$$

Resonance (secular terms)  $\Rightarrow$

$$y_1 = \frac{1}{2}(\sin t - t \cos t).$$

Breakdown due to

Effect of small resistance accumulates. Exact solution (presented later) implies  $y \rightarrow 0$  as  $t \rightarrow \infty$

Hence,  $\epsilon y_1 \approx y - y_0$  must be comparable to  $y_0$

Poincare-Lindsted: not applicable, why ?

# Introduction of a slow time variable

New time

$$\tau = \epsilon t,$$

is introduced **in addition** to the fast time  $t$ . Hence

$$y = y(t, \tau),$$

which is defined in the quadrant  $[t \geq 0] \times [\tau \geq 0]$  as if  $t$  and  $\tau$  were independent.

Much redundancy: only the line  $\tau = \epsilon t$  has direct significance.

Temporal derivatives transform

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

# The transformed problem

Damped oscillation equation in terms of  $t$  and  $\tau$  yields PDE

$$\frac{\partial^2 y}{\partial t^2} + y + \epsilon \left( 2 \frac{\partial^2 y}{\partial t \partial \tau} + \frac{\partial y}{\partial t} \right) + \epsilon^2 \left( \frac{\partial^2 y}{\partial \tau^2} + \frac{\partial y}{\partial \tau} \right) = 0;$$

$$y(0, 0) = 1, \quad \frac{\partial y(0, 0)}{\partial t} + \epsilon \frac{\partial y(0, 0)}{\partial \tau} = 0.$$

## Considerations

- 1  $t$  and  $\tau$  **are** not “really” independent, but  
Solution of the PDE provides solution for ODE  
Physical effects behind scales may sometimes be conceived as independent
- 2 Anyway, an ODE for a PDE; good bargain?  
Yes, as long as we can solve the PDE

# Two-scale perturbation

The series

$$y = y_0(t, \tau) + \epsilon y_1(t, \tau) + \dots,$$

$O(\epsilon^0)$

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0; \quad y_0(0, 0) = 1, \quad \frac{\partial y_0(0, 0)}{\partial t} = 0.$$

The solution for  $y_0$  becomes

$$y_0 = A_0(\tau) \cos t + B_0(\tau) \sin t, \quad A_0(0) = 1, \quad B_0(0) = 0$$

$A_0, B_0$  must be determined to the next order.



$(\epsilon^1)$

$$\begin{aligned}\frac{\partial^2 y_1}{\partial t^2} + y_1 &= -\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} \\ &= (A_0 + 2\frac{dA_0}{d\tau}) \sin t - (B_0 + 2\frac{dB_0}{d\tau}) \cos t;\end{aligned}$$

$$y_1(0,0) = 0, \quad \frac{\partial y_1(0,0)}{\partial t} = -\frac{\partial y_0(0,0)}{\partial \tau}.$$

Avoid secular terms  $\Rightarrow$

$$A_0 + 2\frac{dA_0}{d\tau} = B_0 + 2\frac{dB_0}{d\tau} = 0.$$

## $O(\epsilon^1)$ , cont.

Initial conditions for  $A_0, B_0 \Rightarrow$

$$A_0 = e^{-\frac{1}{2}\tau}, \quad B_0 = 0.$$

No particular solution to  $O(\epsilon)$ :

$$y_1 = A_1(\tau) \cos t + B_1(\tau) \sin t, \quad A_1(0) = 0, \quad B_1(0) = \frac{1}{2}$$

## Complete solution

$$y = e^{-\frac{1}{2}\epsilon t} \cos t + \epsilon(A_1(\epsilon t) \cos t + B_1(\epsilon t) \sin t) + O(\epsilon^2).$$

!  $\epsilon^2$ : secular terms appear; can be eliminated by introducing  $\tau_1 = \epsilon^2 t$ .

# Comparing with exact solution

Exact

$$y = e^{-\frac{1}{2}\epsilon t} \left( \cos \omega t + \frac{\epsilon}{2\omega} \sin \omega t \right),$$

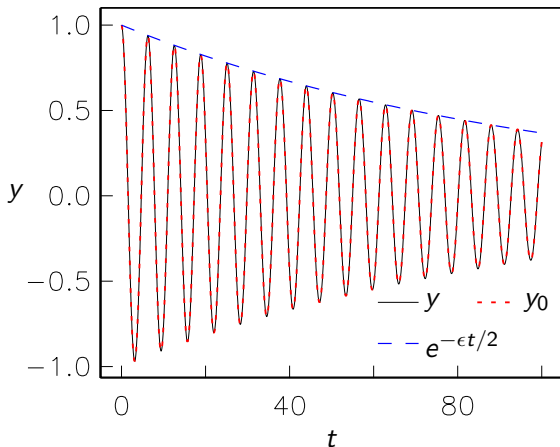
where  $\omega = \sqrt{1 - \frac{1}{4}\epsilon^2} = 1 + O(\epsilon)^2$ .

Two-scale approximation

$$y = e^{-\frac{1}{2}\epsilon t} \cos t + \epsilon(A_1(\epsilon t) \cos t + B_1(\epsilon t) \sin t) + O(\epsilon^2).$$

**The two solutions agree, including the initial condition  $B_1(0) = \frac{1}{2}$ .**

## Graphical comparison, $\epsilon = 0.02$ .



## Example 2: nonlinear oscillations

Scaled equation

$$\frac{d^2x}{dt^2} + x - \frac{\epsilon}{6}x^3 = 0, \quad x(0) = 1, \quad \frac{dx(0)}{dt} = 0.$$

Poincare-Lindsted

We seek a periodic solution with frequency  $\omega = \omega_0 + \epsilon\omega_1 + \dots$

Two-scale method

We regard  $\epsilon t$  a slow time scale that modulates the phase.

# Two-scale expansion, nonlinear pendulum

Invoke  $\tau = \epsilon t$

$$\frac{\partial^2 x}{\partial t^2} + x + \epsilon \left( 2 \frac{\partial^2 x}{\partial t \partial \tau} - \frac{1}{6} x^3 \right) + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} = 0;$$

$$x(0, 0) = 1, \quad \frac{\partial x(0, 0)}{\partial t} + \epsilon \frac{\partial x(0, 0)}{\partial \tau} = 0.$$

Perturbation series

$$x = x_0(t, \tau) + \epsilon x_1(t, \tau) + \dots,$$

$\epsilon^0$

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \quad x_0(0, 0) = 1, \quad \frac{\partial x_0(0, 0)}{\partial t} = 0.$$

Exponential form  $\Rightarrow$

$$x_0 = A_0(\tau)e^{it} + \bar{A}_0(\tau)e^{-it}, \quad A_0(0) = \frac{1}{2},$$

where  $\bar{A}_0$  is the complex conjugate of  $A_0$ .

$\epsilon^1$ 

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t^2} + x_1 &= \frac{1}{6}x_0^3 - 2\frac{\partial^2 x_0}{\partial t \partial \tau} \\ &= \frac{1}{6}A_0^3 e^{3it} + \left(-2i\frac{dA_0}{d\tau} + \frac{1}{2}\bar{A}_0 A_0^2\right) e^{it} + \text{c.c.},\end{aligned}$$

$$x_1(0,0) = 0, \quad \frac{\partial x_1(0,0)}{\partial t} = -\frac{\partial x_0(0,0)}{\partial \tau}, \quad (2)$$

where c.c. indicates the addition of the complex conjugate.

Annihilation of secular terms  $\Rightarrow$

$$i\frac{dA_0}{d\tau} - \frac{1}{4}\bar{A}_0 A_0^2 = 0.$$



$\epsilon^1$ , cont.

From previous slide

$$i \frac{dA_0}{d\tau} - \frac{1}{4} \bar{A}_0 A_0^2 = 0.$$

Insertion of  $A_0 = |A_0|e^{i\psi} \Rightarrow$

$$\frac{d|A_0|}{d\tau} = 0, \quad \frac{d\psi}{d\tau} = -\frac{1}{4}|A_0|^2,$$

Initial condition  $A_0(0) = \frac{1}{2} \Rightarrow$

$$A_0 = \frac{1}{2} e^{-\frac{i}{16}\tau}$$

Then, initial conditions  $\Rightarrow$

$$x_1 = -\frac{A_0^3}{48} e^{3it} + A_1(\tau) e^{it} + \text{c.c.}, \quad A_1(0) = \frac{1}{384}$$

## The two leading orders combined

$$\begin{aligned}x &= \frac{1}{2}e^{i(1-\frac{\epsilon}{16})t} - \frac{\epsilon}{384}e^{3i(1-\frac{\epsilon}{16})t} + \epsilon A_1(\tau)e^{it} + \text{c.c.} + O(\epsilon^2) \\&= \cos(1 - \frac{\epsilon}{16})t - \frac{\epsilon}{192} \cos 3(1 - \frac{\epsilon}{16})t \\&\quad + \epsilon a_1(\epsilon t) \cos t - \epsilon b_1(\epsilon t) \sin t + O(\epsilon^2),\end{aligned}\tag{3}$$

where  $A_1 = a_1 + ib_1$ .

Can be verified by Poincare-Lindsted's method.

## Example 3: Pendulum with prescribed length variation

Conservation of angular momentum (around support)

$$\ell \ddot{\phi} + 2\dot{\ell} \dot{\phi} + g\phi = 0,$$

$\phi$ =excursion,  $\ell$ =length and the dot indicates derivation with respect to time.

Scaling

$$t = \sqrt{\frac{g}{\ell(0)}} t^*, \quad \gamma = \frac{\ell}{\ell(0)}, \quad \theta = \frac{\phi}{\phi_c}.$$

Slow scale,  $\tau = \epsilon t$ , describes change of  $\gamma$  (dimensionless length).

$$\gamma(\tau) \frac{d^2\theta}{d\tau^2} + 2\epsilon \frac{d\gamma}{d\tau} \frac{d\theta}{d\tau} + \theta = 0.$$

# Attempt: direct application of two-scale method

PDE

$$\gamma(\tau) \frac{\partial^2 \theta}{\partial t^2} + \theta + 2\epsilon \left( \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial t} + \gamma \frac{\partial^2 \theta}{\partial t \partial \tau} \right) + \epsilon^2 \left( 2 \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial \tau} + \gamma \frac{\partial^2 \theta}{\partial \tau^2} \right) = 0.$$

Initial conditions (choice of  $\phi_c$ )

$$\theta(0, 0) = 1, \quad \frac{\partial \theta(0, 0)}{\partial t} + \epsilon \frac{\partial \theta(0, 0)}{\partial \tau} = 0.$$

Expansion  $\theta = \theta_0(t, \tau) + \epsilon \theta_1(t, \tau) + \dots$

# Direct application..

$O(\epsilon^0)$

$$\gamma \frac{\partial^2 \theta_0}{\partial t^2} + \theta_0 = 0; \quad \theta_0(0) = 1, \quad \frac{\partial \theta_0(0)}{\partial t} = 0.$$

solution

$$\theta_0 = A_0(\tau) e^{i\gamma^{-\frac{1}{2}} t} + \bar{A}_0 e^{-i\gamma^{-\frac{1}{2}} t}, \quad A_0(0) = \frac{1}{2},$$

## Direct application..

$O(\epsilon^1)$

$$\gamma \frac{\partial^2 \theta_1}{\partial t^2} + \theta_1 = h_s; \quad \theta_1(0,0) = 0, \quad \frac{\partial \theta_1(0,0)}{\partial t} = -\frac{\partial \theta_0}{\partial \tau},$$

where

$$\begin{aligned} h_s &= -2\gamma \frac{\partial^2 \theta_0}{\partial t \partial \tau} - 2 \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial t} \\ &= - \left( 2i\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + i\gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} + 2tA_0\gamma^{-1} \frac{\partial \gamma}{\partial \tau} \right) e^{i\gamma^{-\frac{1}{2}} t} + \text{c.c.} \end{aligned}$$

Appearance of  $t \Rightarrow$  secular terms in  $\theta_1$ .

Reason for failure

**The fast scale (period) is non-constant; it varies with  $\tau$ .**

# Modified two-scale method; variable fast scale

Variable time scale ( scale is fast, but it's variation is slow)

$$\frac{dT}{dt} = \sigma(\tau).$$

Transformation

$$\frac{d}{dt} = \sigma \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau},$$

$$\frac{d^2}{dt^2} = \sigma^2 \frac{\partial^2}{\partial T^2} + \epsilon \left( 2\sigma \frac{\partial^2}{\partial T \partial \tau} + \frac{d\sigma}{d\tau} \frac{\partial}{\partial T} \right) + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

We must choose (determine)  $\sigma$  as to avoid secular terms.

Perturbation series

$$\theta = \theta(T, \tau) + \epsilon \theta_1(T, \tau) + \dots$$

PDE

$$\gamma\sigma^2\frac{\partial^2\theta}{\partial T^2} + \theta + \epsilon\left(2\sigma\frac{d\gamma}{d\tau}\frac{\partial\theta}{\partial T} + 2\sigma\gamma\frac{\partial^2\theta}{\partial T\partial\tau} + \gamma\frac{d\sigma}{d\tau}\frac{\partial\theta}{\partial T}\right) + \epsilon^2\left(2\frac{d\gamma}{d\tau}\frac{\partial\theta}{\partial\tau} + \gamma\frac{\partial^2\theta}{\partial\tau^2}\right) = 0.$$

$O(\epsilon^0)$

$$\gamma\sigma^2\frac{\partial^2\theta_0}{\partial T^2} + \theta_0 = 0; \quad \theta_0(0,0) = 1, \quad \frac{\partial\theta_0(0,0)}{\partial T} = 0.$$

In previous attempt  $\tau$  appears explicitly in the exponent.

This can be avoided by  $\sigma = \gamma^{-\frac{1}{2}} \Rightarrow$

$$\theta_0 = A_0(\tau)e^{iT} + \bar{A}_0(\tau)e^{-iT}, \quad A_0(0) = \frac{1}{2},$$



$O(\epsilon^1)$

$$\frac{\partial^2 \theta_1}{\partial T^2} + \theta_1 \equiv h_s,$$

where

$$\begin{aligned} h_s &= -2\sigma\gamma \frac{\partial^2 \theta_0}{\partial T \partial \tau} - 2\sigma \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial T} - \gamma \frac{d\sigma}{d\tau} \frac{\partial \theta_0}{\partial T} \\ &= -i \left( 2\sigma\gamma \frac{dA_0}{d\tau} + 2\sigma A_0 \frac{d\gamma}{d\tau} + \gamma \frac{d\sigma}{d\tau} A_0 \right) e^{iT} + \text{c.c.} \\ &= -i \left( 2\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + \frac{3}{2} \gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} \right) e^{iT} + \text{c.c.} \end{aligned}$$

Annihilation of secular terms  $\Rightarrow$  coefficient of  $e^{iT}$  is zero  $\Rightarrow$   
Separable equation (ODE) for  $A_0$  and  $A_0(0) = \frac{1}{2}$ ,  $\gamma(0) = 1 \Rightarrow$

$$A_0 = \frac{1}{2} \gamma^{-\frac{3}{4}}$$

## Physical note: wave action

Energy in pendulum motion :  $E = E_s + E_\ell$

$E_\ell$ : potential energy due to change in  $\ell$

$E_s$ : Energy due to the oscillations

$$E_s = \frac{1}{2} m \ell^2 \dot{\phi}^2 + m g \ell (1 - \cos \phi),$$

Small amplitude, scaling, invocation of two-scale solution  $\theta_0 \Rightarrow$

$$E_s = 2 m g \ell_0 \phi_c^2 \gamma A_0^2 (1 + O(\epsilon)).$$

$E_s$  is not constant, but

### Wave action

$$\frac{E_s}{\omega} \approx \text{const.}, \quad \omega = \sqrt{\frac{g}{\ell}}$$

**is constant; general result for oscillations and waves in time dependent medium**