MEK4320
Energy in surface waves.

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Two approaches

1. Averaged fluxes and densities are calculated directly.
   This approach is given, albeit imprecisely, in leaflet sec. 2.4.
   - Important for physical insight. Should be done first.
2. An energy equation is derived from the basic equations for momentum and mass conservation, together with the boundary conditions.
   - Gain: correct, general expressions for densities and fluxes are found, also including nonlinearity
   - Gain: existence and expressions for capillary energy follows from the basic equation.
   - Disadvantage: much algebra. In particular related to the surface tension.

We will only do one horizontal dimension.

Application of a uniform surface pressure will lead to a constant addition of p_a to the pressure throughout the fluid, while the velocities and the surface elevation are unaffected.

In the energy balance p_a will enter only through the pressure work at surfaces. Looking at any volume, Ω, with surface Σ, the effect of the work exerted by p_a is:

$$\int\limits_{\Sigma} -p_a \vec{v} \cdot \vec{n} dA = \int\limits_{\Omega} -p_a \nabla \cdot \vec{v} d\Omega = 0,$$

where Gauss' theorem is employed and we have invoked $\nabla \cdot \vec{v} = 0$. We exempt p_a in the subsequent calculations by putting it equal to zero

Direct evaluation of densities and fluxes

The wave solution

The harmonic solution reads

$$\phi = -\frac{a\omega \cosh k(z + H)}{k \sinh kH} \cos(kx - \omega t), \quad \eta = a\sin(kx - \omega t),$$

where ω and k fulfill the dispersion relation.

In the integration we mainly need simple integrals akin to

$$\int_{x_1}^{x_1+\lambda} \sin^2(kx - \omega t)dx = \frac{1}{2}\lambda,$$

and

$$\int_{-H}^{0} \cosh^2(k(z + H))dz = \frac{1}{2}(\frac{1}{k}\cosh kH \sinh kH + H).$$

Calculation of energy densities.

Accuracy of basic solution
Our wave mode solution is linear in the amplitude (a). $\Rightarrow$ Only approximate energy densities may be calculated. Linear modes yield energy densities of order a^2 correctly to leading order.

Energy densities
Densities is averaged over a wavelength and measured per crest length.

(i)   Kinetic energy, E_k.

(ii) Potential energy due to gravity, E_p^t.

(iii) Potential energy due to capillary effects, E_p^k.

Notation: (K2.1), say, is equation 2.1 in leaflet. Indices denote partial derivatives.

Averaged kinetic energy

The full expression for averaged kinetic energy:

$$E_k = \frac{1}{\lambda} \int\limits_{x_1}^{x_1+\lambda} \int\limits_{-H}^{\eta} \frac{1}{2}\rho \vec{v}^2 \mathrm{d}z \, \mathrm{d}x \qquad (1)$$

where the result is independent of x_1 due to periodicity. The integral in z can be split in two. One from the bottom to $z = 0$ and anther from $z = 0$ to the free surface ($z = \eta$). Since the integrand is quadratic in a the latter will be of order $O(a^3)$ which may be neglected in comparison with the first. Insertion of the linear wave mode followed by straightforward integration yields

$$E_k = \frac{1}{4}\rho g a^2 (1 + \frac{\sigma k^2}{\rho g}) + O(a^3) \qquad (2)$$

In contrast to the leaflet[a] we define the zero level at $z = 0$.

$$\int\limits_{x_1}^{x_1+\lambda} \int\limits_{-H}^{\eta} \rho g z \mathrm{d}z \, \mathrm{d}x = -\frac{1}{2}\rho g \lambda H^2 + \int\limits_{x_1}^{x_1+\lambda} \frac{1}{2}\rho g \eta^2 \mathrm{d}x \qquad (3)$$

The first term in the rightmost expression is the equilibrium energy, while the second is the increase due to the wave. Hence

$$E_p^t = \frac{1}{\lambda} \int\limits_{x_1}^{x_1+\lambda} \frac{1}{2}\rho g \eta^2 \mathrm{d}x = \frac{1}{4}\rho g a^2 + O(a^3) \qquad (4)$$

[a] The odd definition below (K2.26) is necessary to mask the erroneous omission of advection of potential energy in the leaflet.

Potential energy due to surface tension

Stretching of surface: work is done against the surface tension.
Shrinking of surface: energy may be retrieved.
Simple flat surface $\Rightarrow$
potential energy$=\sigma A + \mathrm{const.}$, where A is surface area.
Assume this holds for a general surface[c] We study energy per
width; Area per width = arc-length $\Rightarrow$

$$\int\limits_{x_1}^{x_1+\lambda} \sigma\sqrt{1+\eta_x^2}\mathrm{d}x = \sigma\lambda + \int\limits_{x_1}^{x_1+\lambda} \frac{1}{2}\sigma\eta_x^2\mathrm{d}x + O(a^4) \qquad (5)$$

First term is the equilibrium part; the wave part becomes

$$E_p^k = \frac{1}{\lambda}\int\limits_{x_1}^{x_1+\lambda} \frac{1}{2}\sigma\eta_x^2\mathrm{d}x + O(a^4) = \frac{1}{4}\sigma k^2 a^2 + O(a^3) \qquad (6)$$

[c] Shown later and in leaflet through derivation of energy equation.

The total potential energy

$$E_p = E_p^t + E_p^k = \frac{1}{4}\rho g a^2 (1 + \frac{\sigma k^2}{\rho g}) + O(a^3),\qquad(7)$$

which is the same as the average kinetic energy

$$E_k = E_p,$$

In average potential and kinetic energies are equal.
This is a general property for linear waves.
It is also true for linear oscillators, such as a pendulum in the
gravity field.

Energy transport through a vertical cross-section

Advection of kinetic and gravity-potential energy

Transport through interface element $d\sigma$:

$\gamma \vec{v} \cdot \vec{n} d\sigma$, γ is density (i.e. $\rho g z$) per volume, $\vec{n}$ unit normal.

Here $\vec{n} = \vec{i}$, transport per width: $d\sigma \Rightarrow dz$.

Advection of capillary-potential energy

Due to transport of material surface with u at surface.

Effect of pressure work

Pressure times normal velocity, here: $pudz$

Effect of surface tension work

Force due to surface tension $\cdot \vec{v}$ at the surface.

Transport of potential energy from gravitation, kinetic energy and the effect of pressure work is combined.

$$f_1 = \int\limits_{-H}^{\eta} (\frac{1}{2}\rho \vec{v}^2 + \rho g z + p) u \mathrm{d}z \tag{8}$$

The expression simplifies by means of Euler's pressure equation[e] (below (K2.7) with $p_a = 0$)

$$f_1 = -\int\limits_{-H}^{\eta} (\rho \phi_t - p_a) \phi_x \mathrm{d}z = -\int\limits_{-H}^{0} \rho \phi_t \phi_x \mathrm{d}z + O(a^3) \tag{9}$$

The averaged flux becomes

$$F_1 = -\frac{1}{T} \int\limits_{t}^{t+T} \int\limits_{-H}^{0} \rho \phi_t \phi_x \mathrm{d}z \mathrm{d}t + O(a^3) \tag{10}$$

Energy transport due to surface tension

f_2 = advection + effect of surface tension

- advection $= \sigma \frac{\mathrm{d}s}{\mathrm{d}x} u = \sigma \times$ transport of surface length.
- effect of surface tension $= \sigma \vec{s} \cdot \vec{v}$.

Here s is arc-length of surface and $\vec{s}$ is unit tangent.

Combined

$$f_2 = \sigma[(1+\eta_x^2)^{\frac{1}{2}} u - (1+\eta_x^2)^{-\frac{1}{2}}(u+\eta_x w)]_{z=\eta} = -\sigma\eta_x\phi_z|_{z=0} + O(a^3) \tag{11}$$

Averaged:

$$F_2 = -\frac{1}{T} \int\limits_{t}^{t+T} f_2 \mathrm{d}t$$

We find F_1 and F_2 by inserting the wave mode solution and do the straightforward integration.

Recognition of group velocity

With $F = F_1 + F_2$ we may write

$$F = c_g E \tag{12}$$

where $c_g = \mathrm{d}\omega/\mathrm{d}k$ is the group velocity.

The relation (12) is a general property for linear wave modes.

c_g is interpreted as the celerity of wave energy

Conservation laws derived from the PDEs

Conservation of energy for a volume Ω may be expressed as

$$\frac{\partial}{\partial t} \int\limits_{\Omega} \varepsilon \mathrm{d}\Omega = - \int\limits_{\Sigma} \vec{q} \cdot \vec{n} \mathrm{d}A \tag{13}$$

where Σ is the surface that envelopes Ω, ϵ is the energy density per volume and $\vec{q}$ is the energy flux density.
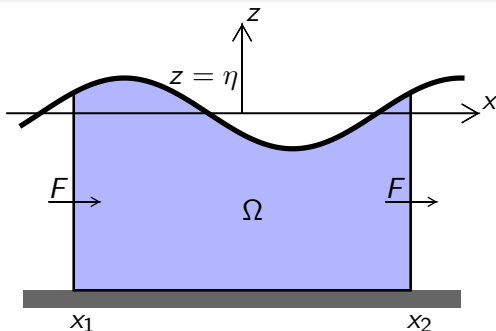
Side track

Ω fixed in time (not true in our case!): Division by the volume, use of Gauss' theorem on the right hand side, followed by $\Omega \to 0$

$$\varepsilon_t = -\nabla \cdot \vec{q}, \tag{14}$$

which is the differential form.

The integrated energy equation for Ω

$$\frac{d}{dt}(e_p^{(t)} + e_k + e_p^{(k)}) = F|_{x=x_1} - F|_{x=x_2} \qquad (15)$$

$e_p^{(t)}$, e_k, $e_p^{(k)}$ are obtained by integrating the energy densities in Ω.
Objective: Produce (15) from basic equations, manipulations and interpretations. Recognize the expressions from the direct approach.

Euler's equation of motion:

$$\rho \vec{v}_t = -\nabla p - \rho \nabla (\frac{1}{2}\vec{v}^2) - \rho \nabla \Phi \qquad (16)$$

where the convective term is simplified due to $\nabla \times \vec{v} = 0$.
We multiply (16) by $\vec{v}$ and integrate in Ω. Gauss' theorem and
$\rho =$ const. $\Rightarrow$

$$\int_{\Omega} (\frac{1}{2}\rho \vec{v}^2)_t d\Omega = -\int_{r} (\quad p + \tfrac{1}{2}\rho \vec{v}^2 + \rho \Phi \quad)\vec{v} \cdot \vec{n} ds \qquad (17)$$

$$(i) \quad (ii) \quad (iii) \qquad (18)$$

where r is the union of the free surface and the vertical cross
sections at x_1 and x_2 (the contribution from the bottom is zero).
On the left hand side of (17) we recognize the kinetic energy.

Temporal derivation may be moved outside integral by Leibniz' rule (only the surface is time dependent):

$$\frac{d}{dt} \int\limits_{\Omega} G d\Omega = \int\limits_{\Omega} \frac{\partial G}{\partial t} d\Omega + \int\limits_{x_1}^{x_2} G|_{z=\eta} \vec{v} \cdot \vec{n} ds \qquad (19)$$

Here

$$\int\limits_{\Omega} (\frac{1}{2}\rho\vec{v}^2)_t d\Omega = \frac{d}{dt} \int\limits_{\Omega} (\frac{1}{2}\rho\vec{v}^2) d\Omega - \int\limits_{x_1}^{x_2} (\frac{1}{2}\rho\vec{v}^2)|_{z=\eta} \vec{v} \cdot \vec{n} ds \qquad (20)$$

and

$$\frac{d}{dt} \int\limits_{\Omega} \Phi d\Omega = \int\limits_{x_1}^{x_2} \Phi|_{z=\eta} \vec{v} \cdot \vec{n} ds, \qquad (21)$$

since $\frac{\partial \Phi}{\partial t} = 0$.

> Equation (17) reorganized
> Last term (20) and (ii) cancel out
> (21) ⇒ (iii) = time derivative of total potential energy in the gravity field.

$$\frac{d}{dt}(e_p^{(t)} + e_k) = -\int_{x_1}^{x_2} p\vec{v}|_{z=\eta} \cdot \vec{n}ds + I_1 - I_2 \qquad (22)$$

where p is the pressure under the surface and

$$e_k = \int_{\Omega} \frac{1}{2}\rho\vec{v}^2 d\Omega \qquad e_p^{(t)} = \int_{\Omega} \rho\Phi d\Omega \qquad (23)$$

$$I_i = \int_{-H}^{\eta} \{(p + \frac{1}{2}\rho\vec{v}^2 + \rho\Phi)u\}|_{x=x_i}dz \qquad (24)$$

All in place, relative to direct approach, save the capillary contributions (from first term on r.h.s of (22))

Expression for I_i contains, in sequence:

1. Effect of pressure in vertical transect
2. Advection of kinetic energy in vertical transect
3. Advection of potential energy in gravity field in vertical transect

Hence, when $\sigma = 0$ we have $I = F$.

Moreover, when $\sigma = 0$ the first term on r.h.s. of (22) vanishes.

Hence, we have shown the simplified version of (15)

$$\frac{d}{dt}(e_p^{(t)} + e_k) = F|_{x=x_1} - F|_{x=x_2}$$

First term on r.h.s. of (22) can be rewritten to yield $e_p^{(k)}$ term as well as contribution to F.

Details follow after "EXTRA" flag, but will not be required.

Definition of vertically integrated energy densities, $E_k^{(i)}$ and $E_p^{(i)}$

$$e_k = \int_\Omega \frac{1}{2}\rho\vec{v}^2 d\Omega = \int_{x_1}^{x_2} E_k^{(i)} dx, \quad \text{where} \quad E_k^{(i)} = \int_{-H}^{\eta} \frac{1}{2}\rho\vec{v}^2 dz \quad (25)$$

$$e_p^{(t)} = \int_\Omega \rho\Phi d\Omega = \int_{x_1}^{x_2} E_p^{(i)} dx, \quad \text{where} \quad E_p^{(i)} = \int_{-H}^{\eta} \rho\Phi dz \quad (26)$$

F (I) on preceding slides are vertically integrated fluxes, now renamed $F^{(i)}$. Dividing (15) by $x_2 - x_1$ and forming the limit $x_2 - x_1 \to 0$ we obtain the depth integrated energy equation

$$\frac{\partial}{\partial t}(E_p^{(i)} + E_k^{(i)}) = -\frac{\partial F^{(i)}}{\partial x} \quad (27)$$

The averaged quantities, E_k etc, may then be obtained by inserting the wave mode solution in $E_k^{(i)}$ etc., removing the equilibrium contribution, make the appropriate approximations and average over t or x.

EXTRAS

Last step: recasting the surface integral that includes p.

Dynamic boundary condition at $z = \eta$

$$p\vec{n} = -\sigma\frac{\partial\vec{s}}{\partial s} \Rightarrow p = -\frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} \tag{28}$$

Integration by parts

$$-\int\limits_{x_1}^{x_2} p\vec{v}\cdot\vec{n}ds = \int\limits_{x_1}^{x_2}\sigma\vec{v}\cdot d\vec{s} = (\sigma\vec{v}\cdot\vec{s})|_{x_1}^{x_2} - \int\limits_{x_1}^{x_2}\sigma\vec{s}\cdot d\vec{v} \tag{29}$$

First term on r.h.s. is the effect the surface tension.

Last term must correspond to the combination of temporal change and advection of potential capillary energy. We must unravel them.

We must identify the following as a part of the last term in (29)

Capillary energy as defined previously

$$e_p^{(k)} \equiv \sigma L = \int\limits_{x_1}^{x_2} \sigma ds = \int\limits_{x_1}^{x_2} \sigma s_x dx, \quad \text{where} \quad s_x = \sqrt{1 + \eta_x^2} \quad (30)$$

Time differentiation

$$\frac{\mathrm{d}e_p^{(k)}}{\mathrm{d}t} = \sigma \frac{\mathrm{d}L}{\mathrm{d}t} = \int\limits_{x_1}^{x_2} \frac{\sigma \eta_x \eta_{xt}}{s_x} dx \quad (31)$$

Manipulations on last term of (29)

Definitions: $\hat{u} = u(x, \eta, t)$, $\hat{w} = w(x, \eta, t)$

Kinematic boundary condition: $\hat{w} = \eta_t + \hat{u}\eta_x \Rightarrow \hat{w}_x = \eta_{xt} + ...$

Substitution $d\vec{v} = (\vec{\imath}\,\hat{u}_x + \vec{k}\,\hat{w}_x)dx$ and rearrangement[f]

$$\int\limits_{x_1}^{x_2} \sigma \vec{s} \cdot d\vec{v} = \int\limits_{x_1}^{x_2} \sigma \left(\frac{\eta_x \eta_{xt}}{s_x} + \frac{\partial}{\partial x}(s_x \hat{u}) \right) dx$$

$$= \frac{\mathrm{d}e_p^{(k)}}{\mathrm{d}t} + (\hat{u}\sigma s_x)|_{x_1}^{x_2}$$

where the last term is advection of capillary energy.

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[f] Alternative: $\vec{s} \cdot d\vec{v}$ can be interpreted as the extension of a material surface segment. Temporal change of L is achieved through taking the advection of arc-length $(s_x \hat{u})$ out of Ω.

If everything is collected we retrieve (15) with the energy flux:

$$F = \int\limits_{-H}^{\eta} \{(p + \frac{1}{2}\rho\vec{v}^2 + \rho\Phi)u\}dz + \sigma u|_{z=\eta}\frac{\partial s}{\partial x} - \sigma\vec{v}\cdot\vec{s} \qquad (32)$$

By means of Euler's pressure equation F is simplified further

$$F = -\int\limits_{-H}^{\eta} \phi_t\phi_x dz + \sigma u|_{z=\eta}\frac{\partial s}{\partial x} - \sigma\vec{v}\cdot\vec{s} \qquad (33)$$

Insertion of linear wave mode reproduces previous results.