# 1. A Lagrangian model applied to benchmarks 1 and 3

G. Pedersen

*Mechanics Division, Department of Mathematics, University of Oslo*
*E-mail: geirkp@math.uio.no*

## 1.1. *The models*

The main runup model of this section is based on Lagrangian Boussinesq equations that are fully nonlinear and possess standard dispersion properties. It is run in both hydrostatic (NLSW) and dispersive mode. For comparison we employ also an Eulerian FDM code for the standard Boussinesq equations and a Boundary Integral Method (BIM) for full potential theory. The first is used for linear computations and is without any particular runup feature; runup heights are found from the vertical elevation at the shore. The BIM is based on Cauchy's formula, Lagrangian surface nodes and spline interpolation. All models allow for a variable spatial resolution, but this feature is employed only for the BIM and the Eulerian models. Systematic grid refinement is employed for all applications reported.

Expressed in terms of the depth averaged velocity, $\overline{u}$, and the total depth, $H$ the Lagrangian long wave equations read

$$\begin{aligned} &H\frac{\partial x}{\partial a} = H_0 \text{ (a)}, \; (1-r_1)\frac{\partial \overline{u}}{\partial t} = -g\frac{H}{H_0}\frac{\partial H}{\partial a} + g\frac{\partial h}{\partial x} - r_2 + S_1 + S_2 \text{ (b)}, \\ &S_1 = \frac{1}{2}H\frac{\partial^3 h}{\partial t^2 \partial x}, \; S_2 = -\frac{1}{g}\left(2\overline{u}\frac{\partial^2 h}{\partial t \partial x} + \frac{\partial^2 h}{\partial t^2}\right)\frac{\partial \overline{u}}{\partial t} - \frac{\partial^2 h}{\partial t \partial x}\frac{\partial H}{\partial t} + H\overline{u}\frac{\partial^3 h}{\partial t \partial x^2}, \end{aligned} \tag{1}$$

where $a$ is the Lagrangian coordinate, $h$ is the equilibrium depth and $H_0 = H(a, 0)$. The position $x$ relates to the velocity according to $\partial x/\partial t = \overline{u}$. Only the shallow water (hydrostatic) terms are spelled out in (1b). A more complete description of the equations in absence of a slide, including the dispersion terms ($r_1$ and $r_2$), is found in Jensen et al. (2003)[2] and references therein. The principal forcing due to a time dependent bottom is the modified source of horizontal momentum from the second term on the right hand side of (1b). Higher order source terms, $S_1$ and $S_2$, have been included particularly for benchmark 3. Replacing $H$ by $h$ in $S_1$ we obtain the linear part that is employed in the Eulerian Boussinesq equation, while $S_2$ is exclusively nonlinear.

A moving shoreline is associated with a fixed $a$, where the condition $H = 0$ is employed. The set (1) is solved by finite differences on a grid that is staggered in space and time. The hydrostatic (NLSW) version is explicit, while dispersion requires implicitness. No smoothing or filtering is employed with the long wave models.

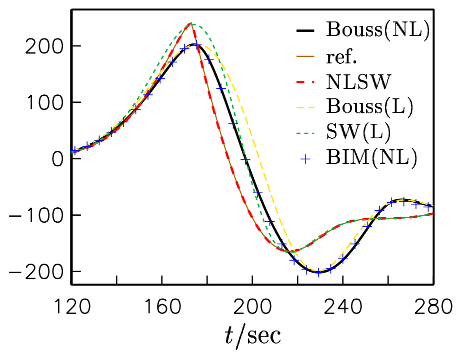


Fig. 1. Left: withdrawal (m). Right: surface (m) at selected times.

### 1.2. *Benchmark problem 1*

The reference solution[1] exists in a semi-infinite domain, but have been made available until 50 km from the coast. We add an deep water region ($x > 50$ km) to avoid that reflections from the seaward boundary reach the shore region for $t < 280$ s. The alternatives are open boundary conditions or sponge layers that may introduce additional errors and uncertainties.

The inundation lengths shown in figure 1 are all close to convergence (increment $\Delta a = 16$ m for the nonlinear long wave models). The NLSW solution agrees very well with the reference solution (ref.). Dispersion has some effect and reduces the draw-down and increases the runup. The maximum runup and drawdown are nearly equal for linear and nonlinear models. Since the initial condition inherits very small elevation/depth ratios this is to be expected in the hydrostatic approximation[4]. That also the linear (L) and nonlinear (NL) Boussinesq models yield nearly identical extrema indicates that dispersion and nonlinearity are important in the deep and shallow regions, respectively, with no or little overlap.

The Boussinesq and full potential (BIM) models yield nearly identical solutions near shore, while there are small discrepancies in the waves propagating into deeper water (not shown).

From a modeling point of view the NLSW solution is the most interesting (challenging) one. At maximum drawdown there is a (near) cusp on the inundation curve corresponding to huge accelerations, a solution that is close to being "weak" and slow convergence of the numerical model. In figure 2 we observe that the accuracy for the velocity is much poorer than for the inundation length. The convergence of the numerical solution is particularly slow close to $t = 173\,\mathrm{s}$ where the reference solution is slightly multi-valued. These features have not been pursued further. Moreover, for the maximum withdrawal the NLSW model displays a linear convergence except for extremely fine resolutions. For the smooth Boussinesq solution we obtain quadratic convergence (dashes in right panel).

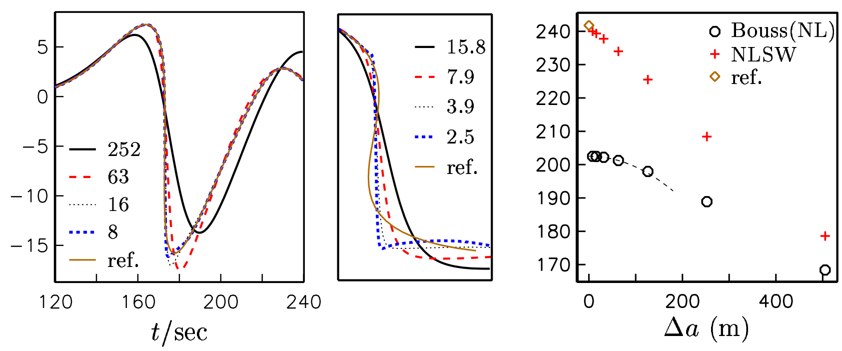


Fig. 2. Left and mid panels: shoreline velocities (m/s). NLSW computations, marked by $\Delta a$ (m), are compared to the reference solution. Mid panel: blow-up of region $(172, 176) \times (-18, 5)$. Right: Convergence of maximum withdrawal (m).

### 1.3. *Benchmark problem 3*

In this problem[3] a slide with time dependent shape and acceleration penetrates the water. All results for this benchmark is given in meters and dimensionless time units, $\sqrt{\delta/g\mu^2}$, as defined in the reference. There is a singularity at the rear of the slide body ($x = 0$), while it is undefined for $x < 0$. Hence, if the fluid reaches $x = 0$ before the height of the slide body becomes negligible at this point, conceptual and practical problems arise in the numerical solution. For case A this occur before the first runup maximum. This is not so for case B for which the NLSW and Boussinesq(*) equations yield:

| $\Delta a$ | 0.16* | 0.16 | 0.31 | 0.63 | 2.51 | 5.04 | 10.17 |
|---|---|---|---|---|---|---|---|
| Runup | 0.495 | 0.495 | 0.495 | 0.496 | 0.491 | 0.458 | 0.468 |
| Drawdown | -0.235 | -0.233 | -0.231 | -0.226 | -0.207 | -0.221 | -0.318 |

The solutions for the different models are shown in figure 3. For case A and $t = 1.5$ the reference solution agree closely with the nonlinear solutions except for a small discrepancy very close to the shore. The differences between the linear and nonlinear solutions are significant for case B at $t = 2.5$, while effects of dispersion are visible only at the front of the offshore propagating wave. The Eulerian model SW(L) is close to the reference solution for both cases (difference up to 0.0025 m with shoreline resolution $\Delta x = 15$ m for case A and $t = 1.5$). For case B inclusion of the dispersive

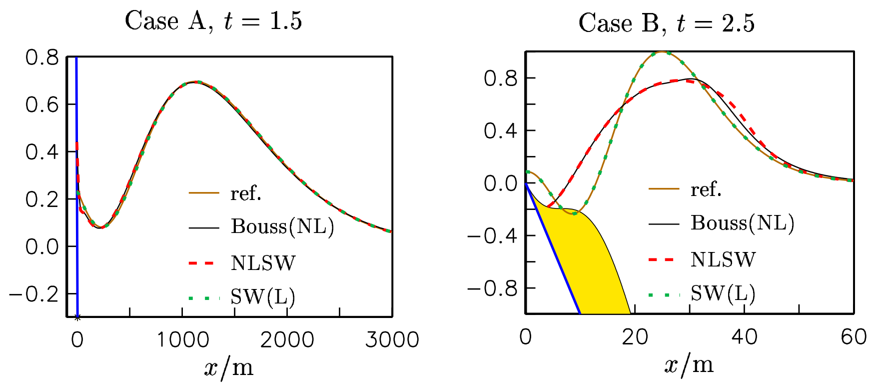


Fig. 3. $\eta$ (m); comparison of models.

source term, $S_1$, reduces the maximum height of the outgoing wave from 1.42 m to 1.33 m at $t = 4.5$. $S_2$ then increases the height by 0.01 m. For case A and $t = 1.5$ the term $S_1$ reduces the wave height from 0.698 m to 0.692 m, while the effect of $S_2$ is negligible (less than $10^{-5}$ m).

## 1.4. *Remarks*

Lagrangian (and ALE) models are undoubtedly useful for idealized studies, as benchmarks 1 and 3, where they produce very accurate results. However, such models have poorer prospects for complex tsunami studies. Further discussion and references on related models are given elsewhere in this volume[4].

## References

1. G. F. Carrier, T. T. Wu, and H. Yeh, *J. Fluid Mech.* **475**, 79 (2003).
2. A. Jensen, G. Pedersen, and D. J. Wood, *J. Fluid. Mech.* **486**, 161 (2003).
3. P. L.-F. Liu, P. Lynett, and C. E. Synolakis, *J. Fluid Mech.* **478**, 101 (2003).
4. G. Pedersen, *This volume.*