MEK4320, solution to extra problem 13.

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Introduction

When indices are used for differentiation the LSW equations read

$$\eta_t = -(hu)_x, \quad u_t = -g\eta_x. \tag{1}$$

In this problem we will study energy within the framework of linear shallow water theory. Hence, we shall use properties like hydrostatic pressure and vertically uniform u whenever appropriate.

a Energy denity and flux.

Question

Show, by direct calculation, that the energy density and flux become, respectively,

$$E = \frac{1}{2}\rho h u^2 + \frac{1}{2}\rho g \eta^2, \quad F = \rho g h \eta u.$$

Solution

Density per volume of kinetic energy: $\frac{1}{2}\rho u^2$.
Here we have neglected the contribution from the vertical component in accordance with LSW theory.
Density per horizontal area (utilization of $u = u(x,t)$).

$$E_k = \int_{-h}^{\eta} \frac{1}{2}\rho u^2 dz = \frac{1}{2}\rho u^2 \int_{-h}^{\eta} dz = \frac{1}{2}\rho u^2 (h+\eta) \approx \frac{1}{2}\rho h u^2,$$

where the cubic term is deleted.

Corresponding integral for potential energy

$$E_p = \int_{-h}^{\eta} \rho g z dz = \frac{1}{2}\rho g \eta^2 - \frac{1}{2}\rho g h^2,$$

where the last term is the equilibrium contribution. Omitting this we find

$$E = \frac{1}{2}\rho h u^2 + \frac{1}{2}\rho g \eta^2.$$

The flux

The flux of kinetic energy is ignored since it is cubic in the field variables.
Integrating advection of potential energy and effect of pressure work, using $p = \rho g(\eta - z)$ in accordance with LSW theory

$$F = \int_{-h}^{\eta} (pu + \rho g z u) dz = \int_{-h}^{\eta} \rho g \eta u dz = \rho g \eta u (h+\eta) \approx \rho g h \eta u.$$

b The wave mode.

Problem text

Assume constant depth and find a wave mode solution to (1). To this end assume a form

$$\eta = A\cos(kx - \omega t), \quad u = U\cos(kx - \omega t).$$

Solution

With constant depth (1) gives

$$\eta_t = -hu_x, \quad u_t = -g\eta_x.$$

Substitution of mode into this

$$\omega A \sin(kx - \omega t) = hkU \sin(kx - \omega t), \quad \omega U \sin(kx - \omega t) = gkA \sin(kx - \omega t).$$

Deletion of common factors and re-organizing give

$$\frac{A}{U} = \frac{hk}{\omega}, \quad \frac{A}{U} = \frac{\omega}{gk}. \quad \Rightarrow \quad \omega^2 = ghk^2, \quad \frac{U}{A} = \frac{c}{h}.$$

c Averaged densities for mode.

Problem text

Use the solution of the previous point to show that

$$\overline{E} = \frac{1}{2}\rho g A^2, \quad \overline{F} = \sqrt{gh}\,\overline{E}.$$

Solution

By inserting the mode we find

$$\begin{aligned} E &= \tfrac{1}{2}\rho h u^2 + \tfrac{1}{2}\rho g \eta^2 &&= \rho g A^2 \cos^2(kx - \omega t) \\ F &= \rho g h \eta u &&= \rho g c A^2 \cos^2(kx - \omega t) \end{aligned}$$

The average of $\cos^2$ over a wavelength or period is $\frac{1}{2}$. Hence

$$\begin{aligned} \overline{E} &= \tfrac{1}{2}\rho g A^2 \\ \overline{F} &= \tfrac{1}{2}\rho g c A^2 = c\overline{E} \end{aligned}$$

where $c = c_g = \sqrt{gh}$

d Fulfillment of energy equation.

Problem text

The energy equation reads

$$E_t + F_x = 0. \qquad (2)$$

Show that this is fulfilled by invoking (1).

Solution

Differentiation with careful use of product rule gives

$$\begin{array}{ccccccc} E_t + F_x & = & \rho huu_t & +\rho g\eta\eta_t & +\rho g\eta(hu)_x & +\rho ghu\eta_x \\ & & \text{(i)} & \text{(ii)} & \text{(iii)} & \text{(iv)} \end{array}$$

Continuity eq., $\eta_t = -(hu)_x$, causes (ii) and (iii) to cancel out.
Momentum eq., $u_t = -g\eta_x$, causes (i) and (iv) to cancel out.
Hence $E_t + F_x = 0$.

e Derivation of the energy equation from the PDE's.

Problem text

Start with (1) and derive (2).

Solution

Multiply the momentum equation with ρhu (same as multiply by ρ and integrate over flow depth)

$$\rho huu_t = -\rho ghu\eta_x.$$

Left hand side is time derivative of E_k. Squinting at the expression for F we rewrite the right hand side

$$(\frac{1}{2}\rho hu^2)_t = -(\rho ghu\eta)_x + \rho g\eta(hu)_x.$$

First term on rhs. is $-F_x$. The latter term is rewritten by the continuity equation

$$(\frac{1}{2}\rho hu^2)_t = -F_x - \rho g\eta\eta_t = -F_x - (\frac{1}{2}\rho g\eta^2)_t.$$

From preceding slide

$$(\frac{1}{2}\rho hu^2)_t = -F_x - \rho g\eta\eta_t = -F_x - (\frac{1}{2}\rho\eta^2)_t.$$

The term within the last parantheses is the potential energy. Moving this to the other side we obtain

$$E_t = -F_x.$$