Mek4100
The π theorem and scaling

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The Buckingam π theorem

Problem contains

- Basic units L_j, $j = 1, n$
- Parameters q_k, $k = 1, m$

$$q_k = p_k \prod_{j=1}^{n} L_j^{a_{jk}} = p_k L_1^{a_{1k}} \cdot \ldots L_n^{a_{nk}}.$$

Seek dimensionless combinations on the form

$$\begin{aligned}
\pi &= \textstyle\prod_{k=1}^{m} q_k^{\alpha_k} \\
&\sim L_1^{a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1m}\alpha_m} \\
&\quad \cdot L_2^{a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{2m}\alpha_m} \\
&\quad \cdot \ldots \\
&\quad \cdot L_n^{a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + a_{nm}\alpha_m}
\end{aligned}$$

π dimensionless when

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1m}\alpha_m &= 0 \\ \ldots. &= 0 \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + a_{nm}\alpha_m &= 0 \end{aligned}$$

Matrix form

$$AX = 0.$$

n equations with m unknowns, where

- $A = \{a_{jk}\}$ is the *Dimension matrix*
- $X = \{\alpha_k\}$

Linear algebra $\Rightarrow$ solution space of dimension $m - r$, where $r = \mathrm{rank}(A) \leq \min(n, m)$.

In other words: We have $m - r$ independent solutions for π.
Infinite number of derived solutions (as π_1^2, $\pi_1\pi_2$) makes choice of π's ambiguous.

π theorem, part i

There are exactly $m - r$ *independent* dimensionless numbers.

Important consequence:
We obtain the numbers by inspection, selection etc. As long as we find $m - r$ independent ones we are good!
Gaussian elimination on $AX = 0$ should be used as last resort, only.

Rescaling

$L_j = \lambda_j \hat{L}_j$.

$$q_k = p_k \prod_{j=1}^{n} L_j^{a_{jk}} = \hat{p}_k \prod_{j=1}^{n} \hat{L}_j^{a_{jk}}, \quad \hat{p}_k = p_k \prod_{j=1}^{n} \lambda_j^{a_{jk}}$$

Logan leaves out p_k and treatise of second part if π theorem is flawed.

Unit-free relation

No good definition in Logan. No good counter-examples.

Attempted definition

Any relation $F(q_1, ..., q_m) = 0$ implies a relation between the numeral values $f(p_1, ..., p_m) = 0$. If f is independent of the choice of units the relation is unit-free.
All useful relations are like this.

π theorem, part ii

Any unit-free relation $F(q_1, ..., q_m) = 0$ can be expressed in terms of dimensionless numbers; $G(\pi_1, ..., \pi_{m-r}) = 0$

We skip the proof.

Example; mathematical pendulum revisited

Parameters

m	ℓ	g	ω
M	L	LT^{-2}	T^{-1}

Use of π theorem, part i

Units on m, ℓ and ω independent $\Rightarrow r = 3$.
Number of π: 4-3=1.
m cannot enter; only quantity with mass.
Remove time between ω and g: $\omega^2/g \sim L^{-1}$.
Remove length $\pi = \ell\omega^2/g$.

$$\pi = \frac{\ell\omega^2}{g}$$

Use of π theorem, part ii

We seek ω expressed by the other parameters:

$$F(\omega, m, \ell, g) = 0$$

Can be expressed as

$$G(\pi) = 0 \quad \Rightarrow \quad \pi = \mathrm{const.}$$

Then

$$\pi = \frac{\ell\omega^2}{g} = \mathrm{const.} \quad \Rightarrow \quad \omega = \mathrm{const.}\sqrt{\frac{g}{\ell}}.$$

Where const. may be a different constant each time.

Mathematical pendulum with finite excursion

New parameter: maximum horizontal excursion x

m	ℓ	g	ω	x
M	L	LT^{-2}	T^{-1}	L

Use of π theorem, part i

Still we have $r = 3$.

Number of π: 5-3=2.

The one from previous example is still applicable

$$\pi_1 = \frac{\ell\omega^2}{g}$$

Need one more. Obvious choice

$$\pi_2 = \frac{x}{\ell}$$

Since x is part of π_2, but not π_1, the two π's are independent.

Use of π theorem, part ii

We seek ω expressed by the other parameters:

$$F(\omega, m, \ell, g, x) = 0$$

Can be expressed as

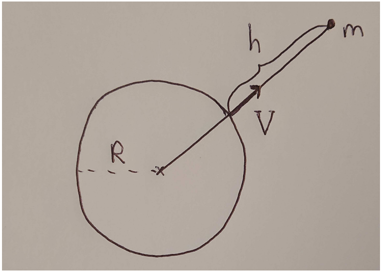
$$G(\pi_1, \pi_2) = 0 \quad \Rightarrow \quad \pi_1 = h(\pi_2),$$

where h is an unknown function.
Then

$$\pi_1 = \frac{\ell\omega^2}{g} = h(\pi_2) \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{\ell}\hat{h}\left(\frac{x}{\ell}\right)}.$$

Where $\hat{h} = h^{\frac{1}{2}}$.

Scaling; The projectile example.



$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -\frac{GmM}{(h+R)^2}$$

$$h(0) = 0, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}t} = V$$

From Newton's 2 law.

At school: h very small (meaning what ?) $\Rightarrow h = Vt - \frac{1}{2}gt^2$
Goal:

- Identify small parameter that tells when h is small
- Make problem dimensionless such that small h limit is OK.

Reshuffling equation ($g = MG/R^2$)

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -\frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

$$h(0) = 0, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}t} = V$$

Solution $h = h(V, g, R, t)$

Observations:

- m cancels out
- M and G always appeared as single entity MG
- Often useful to do some simplifications at the very beginning.

Scaling

$$\overline{t} = \frac{t}{t_c}, \quad \overline{h} = \frac{h}{h_c}.$$

Choice of characteristic time (t_c) and height (h_c) ambiguous, but dimensionless time and height should be of order unity.

$$\frac{h_c}{t_c^2}\frac{\mathrm{d}^2\overline{h}}{\mathrm{d}\overline{t}^2} = -\frac{g}{\left(1 + \frac{h_c}{R}\overline{h}\right)^2}$$

$$\overline{h}(0) = 0, \quad \frac{h_c}{t_c}\frac{\mathrm{d}\overline{h}(0)}{\mathrm{d}\overline{t}} = V$$

Observation: Both $\overline{t}$ and $\overline{h}$ must be π's.

Dimension analysis

h	t	g	R	V
L	T	LT^{-2}	L	LT^{-1}

Number of π: 5-2=3. Choice

1. Obvious: $\pi_1 = \frac{h}{R}$.
2. Now, one with t and not h: $\pi_2 = \frac{Vt}{R}$.
3. Finally, neither h nor t. Then, the subset g, R, V provide a single number (use π theorem on subset!) $\pi_3 = \frac{V}{\sqrt{gR}}$

Feasible scalings:

$$\bar{t} = p(\pi_3)\pi_2, \quad \bar{h} = P(\pi_3)\pi_1.$$

where p and P are functions to be selected.
Low orbit; requirement cannot contain t or h

$$\pi_3 \ll 1$$

$\ll$ means “a magnitude smaller”.

Scaling; attempt 1

Simply put $p = P = 1$

$$\bar{t} = \pi_2 = \frac{Vt}{R}, \quad \bar{h} = \pi_1 = \frac{h}{R}.$$

$$t_c = \frac{R}{V}, \quad h_c = R.$$

Scaled eqs:

$$\pi_3^2 \frac{\mathrm{d}^2\bar{h}}{\mathrm{d}\bar{t}^2} = -\frac{1}{\left(1+\bar{h}\right)^2}, \quad \bar{h}(0) = 0, \quad \frac{\mathrm{d}\bar{h}(0)}{\mathrm{d}\bar{t}} = 1.$$

Limit $\pi_3 \to 0$ ill behaved. Must have $|\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}\bar{t}^2}| \to \infty$ as $\pi_3 \to 0$.
Rubbish scaling.

Scaling; attempt 2. Use g instead of V in $\bar{t}$.

Make $\bar{t}$ from t, g and R (unique, why ?)

$$\bar{t} = \frac{\pi_2}{\pi_3} = \sqrt{\frac{g}{R}}\,t, \quad \bar{h} = \frac{h}{R}.$$

$$t_c = \sqrt{\frac{R}{g}}, \quad h_c = R, \quad p(\pi_3) = \frac{1}{\pi_3}.$$

Scaled eqs:

$$\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}\bar{t}^2} = -\frac{1}{\left(1+\bar{h}\right)^2}, \quad \bar{h}(0) = 0, \quad \frac{\mathrm{d}\bar{h}(0)}{\mathrm{d}\bar{t}} = \pi_3.$$

Limit $\pi_3 \to 0$: "start from rest"; $\bar{h}$ becomes immediately negative; no upward motion

Rubbish again.

Why failure?

Scaling 1

- $h_c = R$. Low orbit: characteristic h not radius of Earth.
- $t_c = \frac{R}{V}$. Time spent by traveling to center of Earth with speed V. Too large for t_c.

h_c and t_c not characteristic at all!

Scaling 2

- $h_c = R$. Still bad.
- $t_c = \sqrt{\frac{R}{g}}$. Like time spent to center of Earth from rest with acceleration g. Again too large for t_c.

Equally stupid as 1.

Proper attempt; leave R out of scaling

express t_c and h_c in terms of V and g, only

$$t_c = \frac{V}{g}, \quad h_c = \frac{V^2}{g}.$$

Observe: $p = P = \pi_3^{-2}$.
t_c is time for retardation from V to 0 by g.
h_c is such that potential energy gh_c is comparable to kinetic energy at $t = 0$. And, $h_c = Vt_c$.
Scaled eqs:

$$\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}\bar{t}^2} = -\frac{1}{\left(1+\pi_3^2\bar{h}\right)^2}, \quad \bar{h}(0) = 0, \quad \frac{\mathrm{d}\bar{h}(0)}{\mathrm{d}\bar{t}} = 1.$$

Limit $\pi_3 \to 0$: $\bar{h} = \bar{t} - \frac{1}{2}\bar{t}^2$. "School result" reproduced.

Lessons learned

- π theorem alone is not sufficient.
- Correct scaling guided by sound interpretations of h_c and t_c.
- Dimensionless variables and coefficients of dimensionless equations are π's.

Finally. Interpretation of low-orbit requirement

$$\pi_3 = \frac{V}{\sqrt{gR}} \ll 1.$$

$\sqrt{gR}$ describes "free fall velocity to center of Earth". That V is much less than this is a reasonable requirement.

But, honestly

Instructive as it may be, that was also a lot of fuzz. Funny how a little theory may make you dance. Here is another approach.

The equation set, once more

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -\frac{g}{\left(1+\frac{h}{R}\right)^2}, \quad h(0)=0, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}t} = V.$$

Fairly clear that the red term should be small for a low orbit. Deletion gives the trivial set

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -g, \quad h(0)=0, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}t} = V,$$

which gives a position

$$h = Vt - \frac{1}{2}gt^2.$$

Ah well, identifying a simplified problem that was easily solved gave an approximate solution

$$h = Vt - \frac{1}{2}gt^2.$$

The peak position then becomes

$$h_{\max} = h(t_{\max}) = V^2/(2g), \quad t_{\max} = V/g.$$

Choosing h_c and t_c accordingly, and claiming $h/R \ll 1$ we find

$$t_c = \frac{V}{g}, \quad h_c = \frac{V^2}{g} = Vt_c, \quad \epsilon \equiv \frac{V^2}{gR} \ll 1$$

The defined ϵ (standard name for small parameter) equals π_3^2.
Next, the full set is scaled accordingly, ϵ will appear and we are ready to invoke a perturbation scheme.
In a more complex case we would often combine simplified solutions, or even heuristic arguments, with dimension analysis to get the equation set into shape and prepare for solution – numerical or analytical.