Problems for Mek 4100

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1 Dimension analysis

Ex. 1 ***Viscous boundary layer.*** A boundary layer evolves from the end ($x = 0$) of a semi-infinite plate. The velocity in the background current is U and the kinematic viscosity coefficient is ν ($[\nu] = L^2/T$). Show that the boundary layer thickness can be expressed as

$$\delta = \frac{\nu}{U} f\left(\frac{Ux}{\nu}\right), \quad x \geq 0 \tag{1}$$

where f is an undetermined function.

Ex. 2 ***Resistance on a ship.*** A ship of length ℓ moves with velocity U in a fluid of density ρ and kinematic viscosity ν. The resistance (force) on the ship (from the fluid) is denoted as F, while the acceleration of gravity is g. Show that

$$F = \rho \ell^2 U^2 f(Fr, Re) \tag{2}$$

where f is an undetermined function. The dimensionless numbers, Fr and Re, that are named the Froude and Reynolds number, respectively, are defined as

$$Fr = \frac{U}{\sqrt{g\ell}}, \quad Re = \frac{U\ell}{\nu} \tag{3}$$

Ex. 3 ***Surface waves.*** We seek the wave celerity (propagation speed), c, of surface waves on an incompressible fluid. The waves are due to gravitation, represented by g, and surface tension (capillary effects), quantified by σ, which has dimension force per length. The density of the fluid is ρ and the wave is characterized by a length λ and height a. The depth is assumed to be infinity (much larger than λ) and will hence not appear in the parameter list.

a) Find a complete set of non-dimensional numbers.

b) We now assume that $a \to 0$. Find a quantity that expresses the importance of surface tension in relation to gravity concerning the wave celerity.

Ex. 4 ***Ship model.*** We focus on the heave motion of a ship in regular seas (periodic waves), and assume 8 relevant parameters

$$g \quad \rho \quad \nu \quad \ell \quad M \quad X_3 \quad a \quad \omega \tag{4}$$

The parameters are briefly explained in table 1, while the depth of the fluid is assumed infinite (comparable to the wave-length or longer).

par.	description	dimension
g	Acceleration of gravity	L/T^2
ρ	Density of the fluid	M/L^3
ν	Kinematic viscosity	L^2/T
ℓ	Length of the ship	L
M	Mass of the ship	M
X_3	The vertical excursion (motion) of the ship	L
a	Amplitude of the incident waves	L
ω	Frequency of the incident waves	$1/T$

Table 1: Parameters for ship model.

a) Find a complete set of non-dimensional numbers.

b) To find the ratio x_3/a a model test is performed with the same fluid as the prototype case (full scale). Discuss the selection of parameters in the model study. Can you foresee any problems ?

Ex. 5 ***Oscillating beam.*** An oscillating beam has length ℓ, stiffness EI and mass per length ratio ρ. The beam is subjected to an axial load P at the two ends. The motion of the beam is then governed by the partial differential equation with boundary conditions

$$\rho w_{tt} + EI w_{xxxx} + P w_{xx} = 0, \quad w = w_{xx} = 0 \ \text{ for } \ x = 0, \ell \tag{5}$$

where $w(x)$ is the transverse displacement distribution of the beam.

a) Employ dimensional analysis to show that the eigenfrequencies must fulfill

$$\omega = \left(\frac{EI}{\rho \ell^4}\right)^{\frac{1}{2}} f\left(\frac{P\ell^2}{EI}\right) \tag{6}$$

where f is an undetermined function.

b) Find the frequencies from the equation (5) and compare with results from the preceding point.

Ex. 6 ***Pipe flow.*** The steady flow of a homogeneous fluid in a straight pipe, with uniform cross-sections, is governed by the parameters of table 2

par.	description
R	Hydraulic radius = area/circumference
ν	Kinematic viscosity
$\beta = -\rho^{-1}\mathrm{d}p/\mathrm{d}x$	Normalized pressure gradient

Table 2: Parameters for the pipe flow problem.

a) The net flux of volume through the pipe is denoted by Q. Show that

$$Q = \Pi \beta \frac{R^4}{\nu}, \tag{7}$$

where Π is a dimensionless quantity. Explain why the density of the fluid cannot enter this relation explicitly.

b) Find Π for a circular pipe, for which the motion is governed by

$$\beta + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0, \quad u(a) = 0, \tag{8}$$

where a is the radius of the pipe.

Ex. 7 ***Scaling of a wave equation.*** Unidirectional, long gravity waves in shallow water are described by the KdV equation. On dimensional form the equation reads

$$\begin{array}{ccccc} \frac{\partial \eta}{\partial t} & +\sqrt{gh}\frac{\partial \eta}{\partial x} & +\frac{3}{2}\sqrt{\frac{g}{h}}\eta\frac{\partial \eta}{\partial x} & +\frac{1}{6}h^2\sqrt{gh}\frac{\partial^3 \eta}{\partial x^3} & = 0 \\ (1) & (2) & (3) & (4) & \end{array} \tag{9}$$

where η, h and g are the surface elevation, equilibrium depth and acceleration of gravity, respectively. We must require that the last two terms,marked by (3) and (4), of the equation are small.

At $t = 0$ we have an initial wave shape of length λ and height A. We assume a scaling

$$\eta = A\overline{\eta}, \quad x = \lambda\overline{x}, \quad t = t_c\overline{t} \tag{10}$$

where a proper t_c (time scale) must be found.

a) Discuss possible choices for t_c by means of dimensional analysis.

b) Determine t_c by assuming that the terms (1) and (2) are dominant (largest by an order of magnitude) in equation (9). Check your result by solving the equation when the presumably small terms (3) and (4) are deleted.

c) Find the condition(s), expressed in the given parameters, for the terms (3) and (4) being much smaller than (1) and (2).

d) Find a criterion for (3) and (4) being of the same order of magnitude.

2 Perturbation methods

Ex. 8 ***Intersection 1.*** x is determined by the equation

$$F(x) = \epsilon G(x), \tag{11}$$

where ϵ is small and an x_0 is given such that $F(x_0) = 0$. Find an approximate solution to the above equation by a regular perturbation series for x and discuss the applicability. Modify the perturbation solution for the case $F'(x_0) = 0$.

Ex. 9 ***Intersection 2.*** Find approximations to the large roots of the equation

$$\tan x = \frac{1}{x} \tag{12}$$

Suggestion: Assume that the roots are close to $n\pi$ for large integers n and a perturbation series of the form

$$x = n\pi + \sum_{j=1}^{\infty} \alpha_j \epsilon^j, \tag{13}$$

where the small parameter, ϵ, must be suitably defined.

Ex. 10 *Jacobi iteration.* A linear set of equations is written

$$\begin{array}{ccccc} x & + & \epsilon a y & = & c \\ \epsilon b x & + & y & = & d \end{array} \tag{14}$$

where $\epsilon \ll 1$.

a) Find solutions on the form

$$x = \sum_{n=0}^{\infty} \epsilon^n x_n \qquad y = \sum_{n=0}^{\infty} \epsilon^n y_n \tag{15}$$

by a regular perturbation expansion. This procedure will correspond to an iteration scheme referred to as Jacobi iteration.

b) Find the interval of convergence for the series in point a).

Ex. 11 *Equating Taylor-series.* An equation is written

$$f(x) = g(\epsilon) \tag{16}$$

where f and g are given as Taylor series

$$f = \sum_{n=1}^{\infty} f_n x^n \qquad g = \sum_{n=1}^{\infty} g_n \epsilon^n \tag{17}$$

Find the first few terms in a perturbation solution for $x(\epsilon)$ in form of a power series in ϵ.

Ex. 12 *Algebraic equation 1.* An equation is written

$$x^2 - 2x + 1 = \epsilon(1 + 2x) \tag{18}$$

where ϵ is small.

a) Try the straigtforward, naive, perturbation scheme. Discuss why this approach is unsuccessful.

b) Employ a general type expansion $x = x_0 + x_1 + x_2 + ...$, where $x_0 \gg x_1 \gg x_2 ...$ when $\epsilon \to 0^+$. The terms must be determined sucessively by dominant balance consideration, at least until a clear pattern emerges.

c) Solve

$$(x - 1)^n = \epsilon x \tag{19}$$

where n is a positive integer.

Ex. 13 *Algebraic equation 2.* An algebraic equation of third order is given as

$$x^3 - x^2 + \epsilon = 0 \tag{20}$$

Find approximations to all solutions for small ϵ.

Ex. 14 ***Regular perturbation of differential equation of first order.*** An equation reads

$$y' + y + \epsilon x y^2 = 0 \quad ; \quad y(0) = 1, \tag{21}$$

where ϵ is small. Is this differential equation separable ? Find the two leading terms in regular perturbation expansion and discuss the uniformity of this solution. Find the exact solution (divide the differential equation by y^2). Then, explain the relation between this and the perturbation solution and use this to assess the latter.

Ex. 15 ***Regular perturbation of differential equation of second order.*** We are given the problem

$$y'' + \epsilon e^{-x} y = 0 \quad ; \quad y(0) = 1,\ y'(0) = 1, \tag{22}$$

where ϵ is small.

a) Show that a regular perturbation expansion leads to

$$y''_{n+1} = -e^{-x} y_n, \tag{23}$$

and that y_n then takes on the form

$$y_n = \sum_{m=0}^{n} (a_{mn} + b_{mn} x) e^{-mx}. \tag{24}$$

Find recursion formulas for a_{mn} and b_{mn}.

b) Invoke the substitution $t = 2\sqrt{\epsilon} e^{-\frac{1}{2}x}$ and show that the exact solution of the differential equation can be expressed in terms of Bessel functions (confer with, for instance, the mathematical handbook of Rottmann). Discuss the prospects for a uniform validity of the perturbation expansion.

Ex. 16 ***Helmholtz method.*** An second order differential equation is written in the form

$$(p(x)y')' + q(x)y = 0 \tag{25}$$

where p and q are periodic, with period L. Morover, p is positive everywhere. We assume that there exists (at least) one periodic solution, y_0, of (25) with period L. In this exercise we will outline a technique for investigation of inhomogeneous equations with respect to existence of solutions that fulfill either periodic or homogeneous boundary conditions. This technique is often denoted as Helmholtz method.

a) Show that there may be periodic solution (period L) of

$$(p(x)y')' + q(x)y = F(x) \tag{26}$$

only if

$$\int_0^L F y_0 \mathrm{d}x = 0 \tag{27}$$

b) We now abandon the requirement of periodic p and q. Instead we assume that there is a solution of the homogeneous equation (25), defined for $0 < x < L$, which fulfill the homogeneous boundary conditions $y_0(0) = y_0(L) = 0$. In that case, show that (26) with boundary conditions $y(0) = y(L) = 0$ have no solutions unless (27) is satisfied.

Ex. 17 ***The Poincaree Lindstedt method.*** We seek a periodic solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y - y^2 = 0 \quad ; \quad y(0) = \epsilon,\ \dot{y}(0) = 0 \tag{28}$$

where ϵ is a small parameter. Introduce $y = \epsilon x$ and determine the first three terms in a perturbation series for x.

Ex. 18 ***Flow between two planes.*** In a dimensionless description a stationary two-dimensional potential-flow is confined between two planes according to

$$1 \geq y \geq b(x),$$

where the lower plane is slightly corrugated

$$b(x) = \epsilon \cos x,$$

where $\epsilon \ll 1$. For the unperturbed problem, identified with $\epsilon = 0$, we assume a uniform flow parallel to the planes. Employing the stream function, ψ, we obtain a simple mathematical description of the problem

$$\nabla^2 \psi = 0 \quad \text{for} \quad b(x) < y < 1, \tag{29}$$

in the fluid, and

$$\psi = 0 \quad \text{at} \quad y = b(x) \quad ; \quad \psi = 1 \quad \text{at} \quad y = 1, \tag{30}$$

as boundary conditions. Inherent in the boundary conditions is a unitary integrated volume flux between the planes.

a) Find the first three terms in a straightforward perturbation expansion for ψ. The leading order term then corresponds to a uniform current associated with $\psi_0 = y$ (the x component of the velocity is $\partial\psi/\partial y$). Observe that the boundary condition at $y = b$ must be expanded as

$$\psi(x, b) = \psi(x, 0) + \psi_y(x, 0)b + \frac{1}{2}\psi_{yy}(x, 0)b^2 + ...$$

where the indices denote partial differentiation.

b) Streamlines are defined by ψ =const. Determine the streamline corresponding to $\psi = c$ as an expression on the form $y = s(c, x)$ by a perturbation expansion.

Ex. 19 ***A generalized Poincaree-Lindstedt method.*** We are given the Van der Pols equation

$$u'' + u = \epsilon u'(1 - u^2) \tag{31}$$

where ϵ is a small parameter.

a) Explain why the term on the right hand side of (31) yields amplitude growth when u is small and damping when u is large. Furthermore, explain why this suggests that there exists only a single possible amplitude for periodic solutions of (31). What, roughly, is the magnitude of this amplitude ?

b) We seek a periodic solution of (31). A convenient procedure is to reformulate the problem according to

$$u = A(\epsilon)x(\tau), \quad \tau = \omega t,$$

and require that x has period 2π in τ. Since $t = 0$ may be chosen arbitrarily and the amplitude is defined through A we may select the initial conditions

$$x(0) = 1, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}\tau} = 0 \quad \text{for} \quad \tau = 0.$$

Find the first two terms in the power series for x, ω and the first term for A.

Ex. 20 ***The Mathieu equation.*** A second order equation inherits a periodic coefficient

$$x'' - \epsilon(\cos 2t)x' + \delta x = 0, \tag{32}$$

where ϵ is small. We seek a δ that enables a periodic solution (period 2π). Find an approximation to this, as well as the associated x, by a perturbation expansion.

Ex. 21 ***Periodic solutions of coupled equations.*** A set of equations for $u(t)$ og $z(t)$ is given by

$$u'' + u = \epsilon(1 - z)u', \tag{33}$$

$$\theta z' + z = u^2, \tag{34}$$

where θ is a real constant and ϵ is small. Calculate the leading first two terms in a perturbation expansion for real, periodic solutions for u og z. Explain why a requirement must be put on the amplitude of u_0 to avoid secular terms.

Ex. 22 ***Nonlinear damping.*** An equation for oscillations includes a nonlinear damping term

$$x'' + \epsilon x^2 x' + x = 0 \quad ; \quad x(0) = 1,\ x'(0) = 0, \tag{35}$$

where $\epsilon \ll 1$ and the free variable is t. Introduce the slow time variable $\tau = \epsilon t$ and insert the perturbation series $x = x_0(t,\tau) + \epsilon x_1(t,\tau) + ...$ in the differential equation. Show that the lowest orders of the perturbation hierarchy become

$$\epsilon^0: \quad \frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \qquad x_0 = 1,\ \frac{\partial x_0}{\partial t} = 0 \ \text{ for } \ t = \tau = 0 \tag{36}$$

$$\epsilon^1: \quad \frac{\partial^2 x_1}{\partial t^2} + x_1 = -x_0^2\frac{\partial x_0}{\partial t} - 2\frac{\partial^2 x_0}{\partial t \partial \tau}; \tag{37}$$

$$x_1 = 0,\ \frac{\partial x_1}{\partial t} + \frac{\partial x_0}{\partial \tau} = 0 \ \text{ for } \ t = \tau = 0$$

Find x_0 and point to which parts of x_1 that can also be determined by the two lowest orders.

Ex. 23 ***Weak resonance and quadratic resistance.*** An oscillator with weak resonant forcing and damping is governed by the equation

$$x'' + \epsilon x'|x'| + x = \epsilon F \cos t \quad ; \quad x(0) = 0,\ x'(0) = 0 \tag{38}$$

This kind of resistance term may, for instance, correspond to a form drag on a pendulum for high Reynolds numbers. Employ the two-scale technique from the preceeding exercise.

a) Find and discuss x_0. (Hint: We may assume that $x_0 = A_0(\tau)\sin t$ and invoke Helmholtz method from problem 16.)

b) For large time the solution from sub-problem approaches a periodic state. Apply this state as the initial state, set $F = 0$ and discuss the damping of free oscillations.

Ex. 24 ***Coefficient with slow variation.***

a) Employ two time scales to find the leading order solution of

$$y'' + (1 + \epsilon f(\epsilon t))y = 0 \quad ; \quad y(0) = 1,\ y'(0) = 0. \tag{39}$$

In particular, work out the solution for $f(x) = \cos(x)$.

b) We have now changed the equation to

$$y'' + f(\epsilon t)y = 0 \quad ; \quad y(0) = 1,\ y'(0) = 0, \tag{40}$$

where f is of order 1. Show that a straightforward application of two-scale expansions break down and employ a modified fast scale to find the leading order behaviour.

Ex. 25 ***Van der Pol's equation.*** The equation reads

$$\ddot{y} + y - \epsilon(1 - y^2)\dot{y} = 0, \tag{41}$$

where $\epsilon \ll 1$.

a) Find equations for the leading order approximation by a two scale expansion

b) Show the existence of a limit cycle, in the sense of a period solution that is approached asymtotically as $t \to \infty$ for a large number of inital conditions.

Ex. 26 ***Mathieu's equation.*** A special case of the Mathieu equation reads

$$\ddot{y} + [\frac{1}{4} + \epsilon(k + 2\cos t)]y = 0 \tag{42}$$

Employ the two time-scale approach to detect the k values for which stable periodic solutions do exist. In this case stability means that amplitude of the stable solution remains finite in the slow time scale.

Ex. 27 ***Dominant balance in algebraic equations.*** Find the two leading terms in the expansions for *all* the roots of

a) $\epsilon x^3 + x^2 - 2x + 1 = 0$.

b) $\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0$.

c) $\epsilon^2 x^8 - \epsilon x^6 + x - 2 = 0$.

Ex. 28 ***Boundary layer, equation of first order.*** We are given the equation set

$$\epsilon y' + x^n y = x^m \quad ; y(0) = 1, \tag{43}$$

where $0 < n < m$. Find the inner and outer solution. Do they match ?

Ex. 29 ***Boundary layer, equation of second order.*** A boundary value problem is defined through

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1, \tag{44}$$

where $\epsilon \to 0^+$.

a) Find the outer solution and explain why there must be a boundary layer at $x = 0$.

b) Find the boundary layer thickness, the inner solution and the unified solution. Sketch the unified solution.

Ex. 30 ***Boundary layer; nonlinear equation.*** An equation set reads

$$\epsilon y'' + 2yy' - 4y^{\frac{3}{2}} = 0, \quad y(0) = 0,\ y'(0) = 1/\epsilon \tag{45}$$

where $\epsilon \to 0^+$. Find outer and inner solution and perform an asymtotic match.

Ex. 31 ***Boundary layer; equation of third order.*** A boundary value problem is specified according to

$$\epsilon^2 y''' - y' + y = 0 \quad ; \quad y(0) = \alpha,\ y(1) = \beta,\ y'(1) = \gamma/\epsilon. \tag{46}$$

In this case we must expect a boundary layer at both $x = 0$ and $x = 1$. Find a unified approximation.

Ex. 32 ***Boundary layer; an eigenvalue problem.*** A beam is latched at one end and fixed at the other. Eigen oscillations are defined through

$$\epsilon u'''' - u'' - \lambda^2 u = 0 \quad u(0) = u'(0) = 0,\ u(1) = u''(1) = 0, \tag{47}$$

where λ, that gives the frequency, is unknown. Finn leading approximations (in ϵ) to u and λ.

Ex. 33 ***Double boundary layer.*** Our boundary value problem reads

$$\epsilon y'' - x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1, \tag{48}$$

where $\epsilon \to 0^+$.

a) Find the outer solution and explain why there must be a boundary layer at $x = 0$.

b) Determine the boundary layer thickness and find the inner solution. Explain why we need a boundary layer at $x = 1$ as well.

c) Find the unified solution by matched asymtotics.

Ex. 34 ***Asymptotic series.*** A Stieltje's integral is defined as

$$y(\epsilon) = \int_0^\infty \frac{\mathrm{e}^{-t}}{1 + \epsilon t} \mathrm{d}t \tag{49}$$

where $\epsilon \geq 0$. We will study the behaviour of the integral in the limit $\epsilon \to 0$. It is obvious that $y(0) = 1$, but we also seek corrections for small ϵ.

a) Show the identity

$$y(\epsilon) = \sum_{j=0}^{n} (-1)^j j! \epsilon^j + r_n(\epsilon) \equiv \sum_{j=0}^{n} a_j + r_n(\epsilon) \equiv s_n(\epsilon) + r_n(\epsilon), \tag{50}$$

where

$$r_n(\epsilon) = (-1)^{n+1}(n+1)!\epsilon^{n+1} \int_0^\infty (1+\epsilon t)^{-(n+2)} \mathrm{e}^{-t} \mathrm{d}t \tag{51}$$

b) Prove that the series $\sum_{j=0}^{\infty} a_j$ *diverge* for all $\epsilon > 0$.

c) Demonstrate that $r_n/a_n \to 0$ when $\epsilon \to 0$ for all n. Explain why this imply that s_n is a close approximation to y, provided ϵ is sufficiently small. Moreover, explain also why there is an optimal choice for n for each ϵ. Such a series, for which the partial sums provide good asymptotic approximations, whereas the series itself may diverge, is denoted as an *asymptotic series*. Singular perturabtions will generally lead to asymptotic series.

The functions y and s_n are depicted in figure 1 for $n = 0, 1, 2, 4, 8$.

3 Calculus of variation

Ex. 35 ***Oscillating water column.*** A non-dimensional model equation for an oscillating column reads

$$(1+\Delta)\ddot{\Delta} + \frac{1}{2}\dot{\Delta}^2 + \Delta = 0, \tag{52}$$

where Δ is the excursion from the mean level.

a) The equation (52) can be derived from Hamilton's principle with the Lagrangian

$$L = \frac{1}{2}(1+\Delta)\dot{\Delta}^2 - \frac{1}{2}\Delta^2. \tag{53}$$

Prove this and find the first integral for (52).

b) We seek periodic solutions of (52). Introduce $\Delta = \epsilon Y$, where $0 < \epsilon \ll 1$ and $Y = O(1)$. You may further assume that $Y(0) = 1, \quad Y'(0) = 0$. Find Y and the eigen-frequency through order ϵ.

c) Bring the solution for Y of the preceding point to to order ϵ^2.

Ex. 36 ***The Hamiltonian.*** A set of material particles is defined through

$$m_i \quad \vec{r}_i(q_1, ...q_N) \quad \text{for} \quad i = 1..n \tag{54}$$

where q_i are the generalized coordinates. Furthermore, the total potential energy is given as $V(q_1, ..., q_N)$.

a) Prove that

$$H \equiv \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \quad - \quad L = \text{const.} \tag{55}$$

is a first integral for the Lagrange's equations. The quantity H is denoted as the *Hamiltonian*.

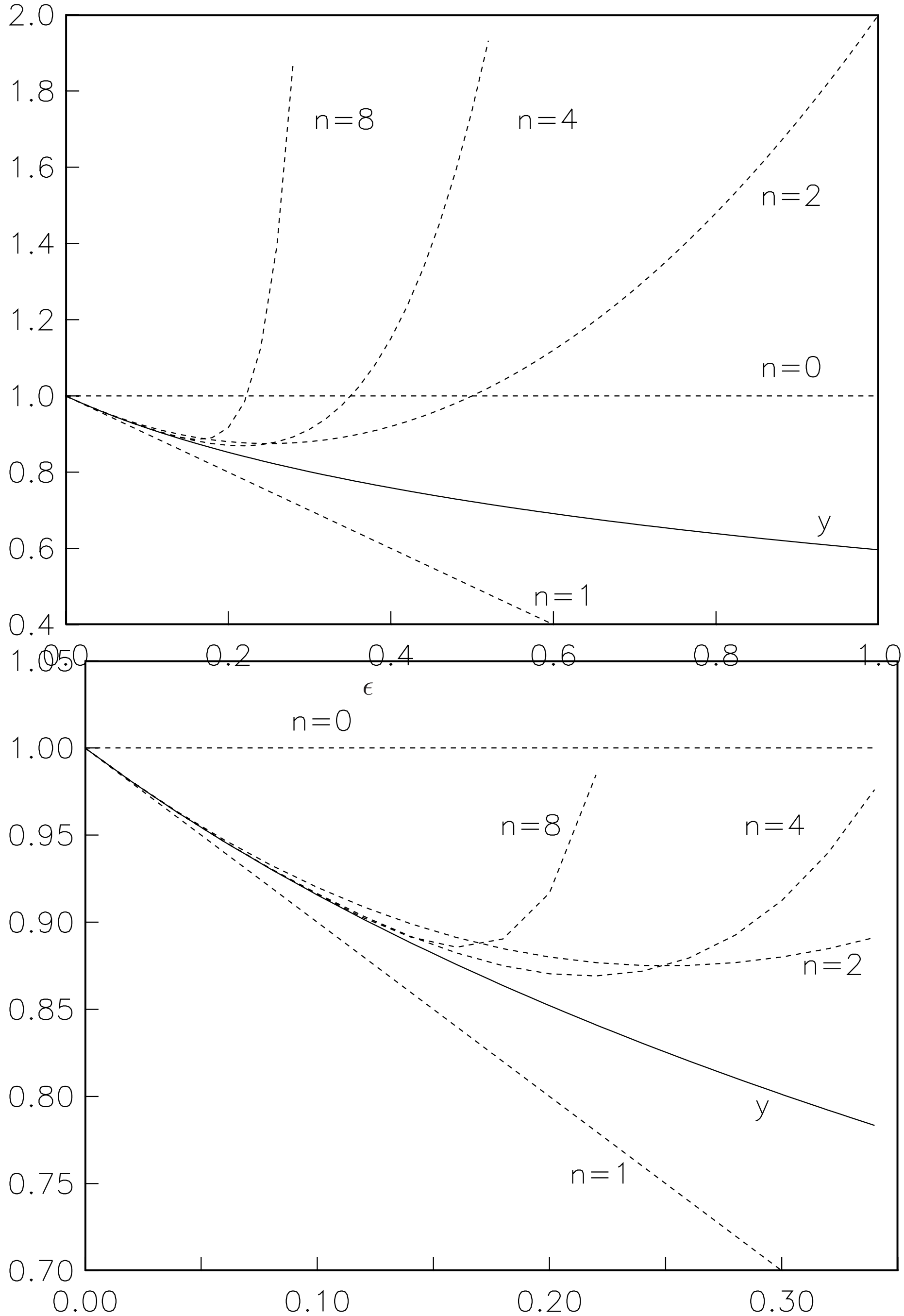


Figure 1: The Stieltje's integral and its asymptotic approximations as functions of ϵ. $n = 1$ refers to the partial sum s_n etc. and y is computed by numerical integration. Lower panel is a close-up of the behaviour for small ϵ.

b) Show that the existence of the first integral corresponds to conservation of mechanical energy.

Ex. 37 ***Boundary conditions for elastic beam..*** The deflection of a beam is governed by

$$EIw'''' + Pw'' - K(x)w + q(x) = 0 \quad \text{for} \quad 0 < x < L, \tag{56}$$

where EI and P are constants. Find variation principles for the equation combined successively with the boundary conditions

a) $w = 0,\ EIw'' = M$ for $x = 0, L$.

b) $EIw''' + Pw' = kw,\ EIw'' = -\kappa w'$ for $x = 0, L$.

Ex. 38 ***System analysis.*** A mathematical pendulum with length L and mass m moves vertical plane. The frictionless support, P, is subjected to a prescribed motion $(X(t), Y(t))$.

a) Employ Hamilton's principle to obtain the differential equation that governs the motion.

b) Recast the equation from the previous point into dimensionless form by the transformation

$$\xi = \frac{X}{L}, \quad \eta = \frac{Y}{L}, \quad \tau = \omega t. \tag{57}$$

Show that the result can be expressed as

$$\Phi'' + \xi'' \cos\Phi + \eta'' \sin\Phi + \Lambda \sin\Phi = 0, \tag{58}$$

where $\Lambda = g/(\omega^2 L)$.

c) Introduce $\xi = \delta\cos\tau$ and $\eta \equiv 0$. Assume that $\Phi = \epsilon\theta$ and $\delta = \epsilon^\nu a$, where ν is an integer, and a, $\theta = O(1)$. Find a ν compatible with finite periodic response for θ in a domain containing $\Lambda = 1$. Sketch the amplitude as function of Λ in the vicinity of $\Lambda = 1$.

d) Set $\xi \equiv 0$ and $\eta = \Delta\cos(2\tau)$. This time we assume $\Phi = \epsilon\theta$ and $\Delta = \epsilon^\mu\alpha$ where μ is an integer and α, $\theta = O(1)$. As in previous point μ must be determined as to yield finite periodic response close to $\Lambda = 1$. Find an equation for the leading approximation to the amplitude.

Ex. 39 ***Hamilton's principle.*** A frictionless constraint rotates in vertical plane with a prescribed, constant rate such that the angle between the horizontal plane and the constraint is ωt. A particle of mass m moves on the constraint under the action of gravity.

a) Write down the Lagrangian and employ the Lagrange-equations to determine the motion of the particle. Assume an initial condition where the distance from the rotation center of the constraint to the particle is r_0 and the particle is at rest relative to the constraint.

b) Find the Hamiltonian. Is it constant? Does the Hamiltonian equal the energy?

c) Next assume that there is no gravity and answer the questions from the preceding point.

d) g is still zero. Enter the relative coordinate system that follows the rotation of the constraint. Calculate both the energy and the Hamiltonian.

4 Miscellaneous

Ex. 40 ***Zeroes and iterations.*** An equation for x is given as

$$\frac{\epsilon}{1+x} = \sin x \tag{59}$$

where ϵ is a positive parameter. Equation (59) inherits an infinite number of solutions. We will seek an approximation for the smallest positive solution for small ϵ.

a) Explain why $x \to 0$ when $\epsilon \to 0$ and find the leading approximation to x.

b) Find the next approximation in a perturbation expansion for x.

Ex. 41 ***Forced pendulum .*** A pendulum, of length ℓ and mass m, is subjected to gravity and a forcing. The motion is governed by (*do not prove this!*)

$$m\ell\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g\sin\theta = F\cos\omega t, \tag{60}$$

where g is the acceleration of gravity and θ is the angle of the the pendulum, measured from the vertical position.

a

We assume a forced periodic response with the frequency of the forcing, ω. Apply dimensional analysis to find a relation between the specified parameters m, ℓ, g, F, ω and the maximum value (amplitude) for θ. Present all the steps in the derivation carefully.

b

Next, we assume a weak resonant forcing ($\omega = \sqrt{g/\ell}$) and a periodic response with small amplitude. This may be realized when the leading non-linear term on the left hand side of (74) is of the same order of magnitude as the forcing. Exploit this to rescale the equation (74) to obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + z - \epsilon z^3 + O(\epsilon^2) = \epsilon\cos\tau, \tag{61}$$

where z is the rescaled angle.

c

Employ a perturbation expansion to find the leading order periodic solution (period 2π in τ) of (61). Tip: the expansion must be partly carried one step beyond the leading order.

Ex. 42 ***The elastic string and Hamilton's principle.***

An elastic string of length ℓ is fixed in the endpoints. The density (mass per length) is ρ and the tension in the string is σ. When η denotes a small transverse displacement from equilibrium the density per length of kinetic and potential energy , respectively, reads

$$T = \frac{1}{2}\rho\left(\frac{\partial\eta}{\partial t}\right)^2, \quad V = \frac{1}{2}\sigma\left(\frac{\partial\eta}{\partial x}\right)^2,$$

where x measures the length along the string. Hamilton's principle then reads

$$\delta\int_0^{t_1}\int_0^{\ell}(T-V)\mathrm{d}x\mathrm{d}t = 0. \tag{62}$$

Find the equation of Euler-Lagrange's. A general formula is not to be used. You must start with the principle and work through all the steps that lead from (62) to the differential equation for this particular case.

Ex. 43 ***Boundary layer.*** An equation set is specified as

$$\epsilon\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)-q(x)y=0,\quad y(0)=y(1)=1, \tag{63}$$

where p and q are analytic functions of x (Taylor series do exist) and are positive in the interval $0\leq x\leq 1$. ϵ is a small parameter.

a Find a solution of (63) by boundary layer theory $\epsilon\to 0$.

b We now violate one of the conditions given above and assume that $q<0$ in the interval $0\leq x\leq 1$. Explain why boundary layer theory is no longer applicable.

Ex. 44 ***Multiple scale, revisited.*** We keep the equation set from previous exercise, point b. Defining $q\equiv -r^2$ and changing the boundary conditions we find

$$\epsilon\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+r^2(x)y=0,\quad y(0)=1,\quad \frac{\mathrm{d}y(0)}{\mathrm{d}x}=0, \tag{64}$$

where p and r are positive in $0\leq x\leq 1$.

a Rescale (64) to obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\mu\,p(\mu z)\frac{\mathrm{d}y}{\mathrm{d}z}+r^2(\mu z)y=0,\quad y(0)=1,\quad \frac{\mathrm{d}y(0)}{\mathrm{d}z}=0, \tag{65}$$

where μ is a small parameter.

b First we assume $r=r_0$ =constant. Find the leading order solution of (65) by a multiple scale expansion.

c We now return to the case with a variable r. Why does the straightforward multiple scale method break down ? Determine the leading order solution by a modified expansion.

Ex. 45 ***The method of Poincaree-Lindstedt.*** Find a periodic solution for

$$(1+\epsilon u)u''+\frac{1}{2}\epsilon(u')^2+u=0$$

Ex. 46 ***First order equation.*** For $x\geq 0$ we have defined the set

$$\epsilon\frac{\mathrm{d}y}{\mathrm{d}x}+p(x)y^2=q(x),\quad y(0)=0, \tag{66}$$

where p and q are positive for all $x\geq 0$. Find an approximate solution when $\epsilon\to 0$.

Ex. 47 ***Bessels equation.*** Bessel's equation of order 0 reads

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \tag{67}$$

For large x we assume that the second term on the left hand side yields a mild damping and that the solution will be an oscillation with a slowly decreasing amplitude in x. Why ?

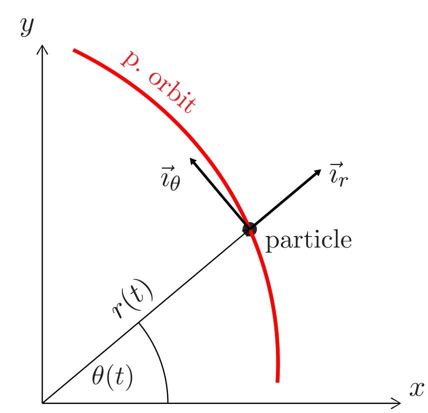
We seek approximations to the solution for large x. This imply that we will not employ specific initial or boundary conditions in this exercise.

By means of the transformation

$$x = \frac{1}{\epsilon} + t,$$

where ϵ is small, can Bessel's equations be recast into a form suitable for a two-scale expansion, with t as the fast scale. Find the leading order approximation for $0 < t < \infty$.

Ex. 48 ***Particle in gravity field.***

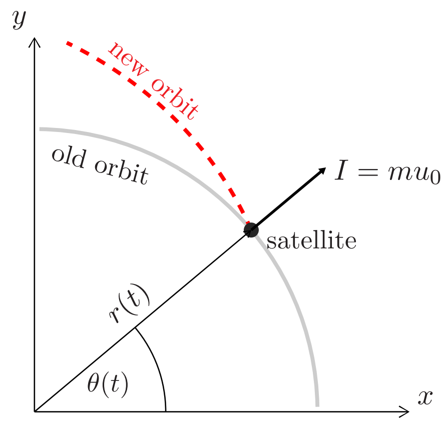


A particle of mass m is orbiting a center of gravity. The potential energy is $V(r) = Gm/r$, where r is the distance from the center. Employ r and θ as generalized coordinates. (See figure). The particle velocity may be expressed as

$$\vec{v} = \dot{r}\vec{\imath}_r + r\dot{\theta}\vec{\imath}_\theta.$$

(Do not show this!)
Find the Lagrangian for the particle and show that there are two first integrals (equations with no higher than first derivatives). What is the physical interpretation of these first integrals ?

Ex. 49 ***Satellite orbit; Dimension analysis, scaling.***



A satellite of mass m is orbiting the Earth. The force from the Earth on the satellite is $F(r) = -Gm/r^2$ (potential $V(r) = Gm/r$), where r is the distance from the centre of the Earth. When r and the angle θ (see figure) are used as coordinates the motion of the satellite is governed by

$$mr^2\dot{\theta} = S = \text{constant}, \tag{68}$$

$$m\ddot{r} - mr(\dot{\theta})^2 = -\frac{Gm}{r^2}, \tag{69}$$

that correspond to preservation of angular momentum and the radial component of Newtons second law, respectively. The dot denotes temporal differentiation. These equations shall *not* be derived.

a) A circular orbit corresponds to $r = r_0$ =constant and $\dot{\theta} = \omega_0$ =constant. Show that the relation $G = r_0^3\omega_0^2$ must be fulfilled. The orbit is then shifted by a short blast from a rocket engine that yields an radial impulse (change of linear momentum) $I = mu_0$, while the angular momentum remains unaltered. Moreover, we ignore the change of the satellite position during the impulse. Presumably, the new orbit will be closed, but not circular. The characteristics of the motion depend on the parameters m, r_0, ω_0, u_0 and t (time). Find a complete set of non-dimensional numbers.

b) The new orbit may be found by solving (68) and (69) with the initial conditions

$$r(0) = r_0, \quad \dot{r}(0) = u_0$$

In the following we assume that I is a weak impulse, implying that the orbit becomes only mildly perturbed, and wish to scale and make equations dimensionless accordingly. Start with eliminating $\dot{\theta}$ between (68) and (69) and remove dimensions to obtain the differential equation with initial conditions

$$(1+\epsilon z)^3\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + z = 0, \quad z(0) = 0, \quad \frac{\mathrm{d}z(0)}{\mathrm{d}\tau} = 1, \tag{70}$$

where $r = r_0(1+\epsilon z)$, τ is a time variable and ϵ is a small dimensionless number. ϵ and τ must be chosen as part of this process and must be related to the findings in point a).

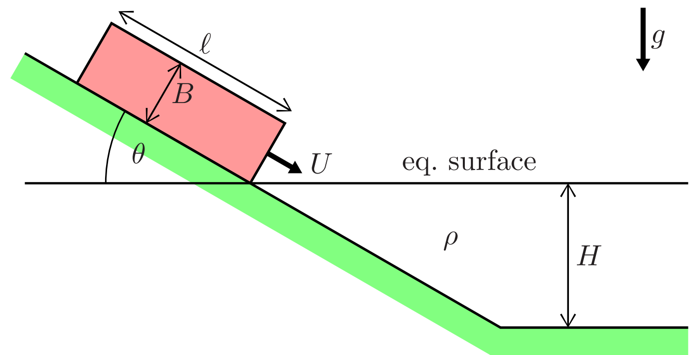


Figure 2: Definition sketch of slide in wavetank.

It is wise to assume that the dimensionless time is given by the non-disturbed state, as $\omega_0 t$, and to continue next with a scrutiny of the initial conditions rather than the differential equation. After all, it is the initial condition that defines the perturbation.

Ex. 50 ***The perturbed satellite orbit..***
We seek an approximate solution to (70) by a straightforward perturbation expansion of z in ϵ.

a) Find the first two terms in a series for z.

b) Explain why the expansion breaks down at order ϵ^2 (for z). What is the remedy ? Work out the full solutions to order ϵ^2.

Ex. 51 ***A boundary layer problem.***
A boundary value problem is defined through

$$\epsilon y'' + y' + \frac{1}{2}y^2 = 0 \quad ; \quad y(0) = 0, \quad y(1) = 1, \tag{71}$$

where $\epsilon \to 0^+$.

Find the leading order approximate solution valid in $0 \leq x \leq 1$ and sketch it.

Ex. 52 ***Dimension analysis of slide into water.***
A rigid, rectangular slide body is forced to move with constant velocity, U, along an inclined plane in the end of a wave tank, see figure. The equilibrium depth in the wave tank is H, the slide has length ℓ and thickness B, the inclination angle is θ, the constant of gravity is g and the density of the fluid in the tank is ρ. The slide spans the total width of the tank, allowing us to assume two-dimensional motion when viscosity is neglected. Since the slide motion is forced the weight of the slide does not affect the fluid response.

a) Find a complete set of dimensionless numbers from the parameter set U, ℓ, B, H, ρ, g and θ.

b) We now introduce the force per width, F, acting on the slide from the liquid and set $t = 0$ as the time of first contact between slide and liquid. Moreover, we study times that are smaller than both ℓ/U and the time needed for any disturbance in the liquid to reach the non-sloping region of the tank. Explain why ℓ and H do not influence F for such times and show

$$F = \rho B U^2 G\left(\frac{U}{\sqrt{gB}}, \frac{Ut}{B}, \theta\right),$$

where G is a function of three real variables.

Ex. 53 ***Forced oscillation.***
A dimensionless and scaled equation with weak resonant forcing and weak damping is given according to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \epsilon u^2 \frac{\mathrm{d}u}{\mathrm{d}t} + u = \epsilon \cos t,$$

where ϵ is a small parameter. We assume that the motion starts from rest, implying $\frac{\mathrm{d}u(0)}{\mathrm{d}t} = u(0) = 0$. Use a two scale expansion to demonstrate that the solution approaches a limit cycle when t becomes large.
You are allowed to assume, or rather to guess, that the zeroth order solution has the form $A \sin t$, where A is slowly varying. Moreover, it is not required that you work out the full solution for A.

Ex. 54 ***Forced pendulum, revisited.*** This is an extension of problem 41. A pendulum, of length ℓ and mass m, is subjected to gravity, a forcing and a linear resistance force

$$m\ell \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \alpha\ell \frac{\mathrm{d}\theta}{\mathrm{d}t} + mg\sin\theta = F\cos\omega t, \tag{72}$$

where g is the acceleration of gravity and θ is the angle of the pendulum, measured from the vertical position.

a) In this case we assume a near resonant forcing

$$\sqrt{g/\ell} = \omega + \Delta\omega,$$

where $\Delta\omega \ll \omega$. and a periodic response with small amplitude. We now assume the leading non-linear term, the forcing the damping and the term due to $\Delta\omega$ are of the same order of magnitude. Exploit this to rescale the equation (72) to obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + (1+\epsilon\gamma)z + \epsilon\beta\frac{\mathrm{d}z}{\mathrm{d}\tau} - \epsilon\kappa z^3 + O(\epsilon^2) = \epsilon\cos\tau, \tag{73}$$

where z is the rescaled angle and γ, β, κ are constants of order unity. Equation (73) is a model equation that combines weak forcing, weak damping and weak linear de-tuning ($\gamma \neq 0$)

b) Employ a perturbation expansion to find the leading order periodic solution (period 2π in τ) of (73).

c) Apply a numerical ODE solver to (73). Start from rest, choose $\epsilon = 0.1$, and run the cases

1. $\gamma = 0$, $\beta = 0$, $\kappa = 1$.
2. $\gamma = 1$, $\beta = 0$, $\kappa = 0$.
3. $\gamma = 0$, $\beta = 1$, $\kappa = 1$.
4. $\gamma = 0$, $\beta = 0.1$, $\kappa = 1$.

Discuss the results

d) Keep $\gamma = 0$ and $\kappa = \beta = 1$, but vary ϵ to obtain maximum excursions close to $\theta_{\max} = 10°$ and $\theta_{\max} = 30°$. Compare with the perturbation solution.

Ex. 55 ***Higher order boundary layer theory.*** A first order problem is given as

$$\epsilon y' + y = 1 + x, \quad y(0) = 0.$$

a) Find the exact solution

b) Find the leading order outer and inner solutions. Design a unified solution and compare with the exact solution.

c) Find all the higher order inner and outer solutions. Are they significant ?

Ex. 56 ***Free pendulum.*** A nonlinear pendulum is governed by

$$\ell\frac{\mathrm{d}^2\theta^\star}{\mathrm{d}^2t^\star} + g\sin\theta^\star = 0, \quad \theta^\star(0) = \theta_0, \quad \frac{\mathrm{d}\theta^\star(0)}{\mathrm{d}t^\star} = 0 \tag{74}$$

where g is the acceleration of gravity, θ_0 is the maximum angle and the star marks dimensional quantities.

a) Scale the problem (74) to obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \epsilon^{-1}\sin(\epsilon x) = 0, \quad x(0) = 1, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = 0, \tag{75}$$

and identify the small parameter ϵ.

b) Show that a straightforward perturbation expansion yields

$$x = \cos t + \epsilon^2\left(\frac{1}{192}(\cos t - \cos(3t)) + \frac{t}{16}\sin t\right) + O(\epsilon^4), \tag{76}$$

and explain the weaknesses of this solution.

c) Employ the Poincare-Lindstedt method to find

$$x = \cos(\omega t) + \frac{\epsilon^2}{192}(\cos(\omega t) - \cos(3\omega t)) + O(\epsilon^4), \quad \omega = 1 - \frac{\epsilon^2}{16} + O(\epsilon^4). \tag{77}$$

and explain the weaknesses of this solution.

d) Use Taylor expansion of the trigonometric functions in (77) to retrieve (76).

e) Now we wish to reproduce ω_1 as given in (77) from (76). To this end we observe that the period is $T = 2\pi + \epsilon^2 T_1$. Determine T_1 from (76) and the requirement that $\frac{\mathrm{d}x(T)}{\mathrm{d}t} = 0$. Then find ω_1.

f) Show that the energy equation from (75) is

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \epsilon^{-2}\left(\cos\epsilon - \cos(\epsilon x)\right) = 0, \tag{78}$$

and that this implies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\frac{\sqrt{2}}{\epsilon}\left(\cos(\epsilon x) - \cos\epsilon\right)^{\frac{1}{2}}, \tag{79}$$

and that the period is given by

$$T = \int_0^1 \frac{2\sqrt{2}\epsilon dx}{\left(\cos\epsilon x - \cos\epsilon\right)^{\frac{1}{2}}}. \tag{80}$$

This integral may then be subjected to numerical integration (beware of the singularity in the integrand at $x = 1$) or series expansion.

Ex. 57 ***Numerical solutions for free oscillations.*** In this exercise the oscillation from the previous problem is investigated by numerical analysis. We will focus on two ifferent values of ϵ, corresponding to $\theta_0 = 10°$ and $\theta_0 = 30°$, respectively.

a) Make a phase diagram where (77) and (79) is compared for both values of ϵ. Discuss the results. In this context phase diagram means a $\frac{\mathrm{d}x}{\mathrm{d}t}$ versus x plot.

b) Solve the initial value problem (75) with a numerical ODE tool. Verify your soluton by comparing with (79).

c) Compare the numerical solution to (77) for both values of ϵ. Discuss the results.

d) Find ω from the numerical solution and compare with the ω from (77).

Ex. 58 ***Numerical testing of two-scale expansions.*** We will test the solutions from problem 25 and the example in section 4.2 of the leaflet "Multiple scale methods" as specified by msm:25 (meaning equation 25 of leaflet). The solution is then msm:32,34,35.

a) Apply an ODE solver to the case from problem 25. Test the effect of resolution.

b) Compare with the perturbation solution, concerning both the transient evolution and the limit cycle. Employ two different values for ϵ, one small and one larger. Discuss.

c) To design a test example we choose

$$T = t + \frac{1}{2\epsilon}\sin(\epsilon t),$$

in sms:35. Find γ and discuss how the the length of the pendulum changes.

d) Solve sms:25 numerically and compare with msm:32,34,35 for one large and one smal ϵ.

Ex. 59 ***Eigen-oscillation in basin..*** In a shallow basin with vertical, impermeable sidewalls the motion for small surface elevations is governed by the PDE

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gh(x)\frac{\partial \eta}{\partial x}\right) = 0 \quad \text{for} \quad 0 < x < L, \tag{81}$$

with the boundary conditions

$$\frac{\partial \eta}{\partial x} = 0 \quad \text{for} \quad x = 0, L. \tag{82}$$

η and h denote the surface elevaton and the equilibrium depth, respectively.

a) We will find eigen-oscillations on the form

$$\eta = y(x)\sin(\omega t), \tag{83}$$

where y and ω are unknowns. We assume a nearly constant depth $h = h_0(1 + \epsilon r(x))$. Rescale the problem to obtain

$$y'' + \lambda y + \epsilon (r y')' = 0 \quad \text{for} \quad 0 < x < 1, \tag{84}$$

and

$$y'(0) = y'(1) = 0.$$

In the last two equations y and x have been redefined.

b) Find y and λ for the non-perturbed problem.

c) Employ Poincare-Lindstedts method

$$y = y_0 + \epsilon y_1 + ..., \quad \lambda = \lambda_0 + \epsilon \lambda_1 + ...,$$

and find the equation for λ_1.

d) Employ the Helmholtz method (problem 16) to find an expression for λ_1 in terms of r.

e) Find λ_1 and y_1 for

(i) $g(x) = \sin(kx)$

(ii) $g(x) = x$

f) Find the exact solution of (84) for case (ii) and indicate how the eigenvalues may be obtained from this solution. (Hint: the transformation $\xi = a\sqrt{x+b}$ yields a standard equation.)

Ex. 60 ***Extreme for a simple functional.*** Consider the functional

$$J(y) = \int_a^b L(x, y, y')\,dx,$$

where

$$L = \frac{1}{2}p(x)(y')^2 + \frac{1}{2}q(x)y^2 + f(x)y. \tag{85}$$

The function y fulfills the boundary conditions

$$y(a) = y_1, \quad y(b) = y_2. \tag{86}$$

a) Find the Euler-Lagranges equation.

b) Define

$$g(\epsilon; h) = J(y + \epsilon h),$$

where y is the solution of the Euler-Lagranges equation with the boundary conditions (86). Determine the expressions L_1 and L_2, both independent of ϵ, such that

$$g(\epsilon; h) = J(y) + \epsilon \int_a^b L_1\,dx + \epsilon^2 \int_a^b L_2\,dx. \tag{87}$$

c) Use the derivation of the Euler-Lagranges equations in Logan to explain why $\int_a^b L_1\,dx$ in (87) is zero.

d) Show $\int_a^b L_1\,dx = 0$ by directly by integration by parts etc.

e) Now we instead insert $y + \delta y$ into J. In analogy to (87) show that

$$J(y + \delta y) = J(y) + \int_a^b \hat{L}_1\,dx + \int_a^b \hat{L}_2\,dx, \tag{88}$$

where L_1 contains only linear terms in δy and $\delta y'$. Correspondingly, L_2 shall contain only quadratic terms. Explain why $\int_a^b \hat{L}_1\,dx$ is zero.

f) Show that the solution of the Euler-Lagranges equations yields a minimum for J when p and q are positive in $[a, b]$.

Ex. 61 ***Euler-Lagranges equations with periodic boundary conditions.*** We define $\mathcal{A}$ as the set of periodic functions on $[a, b]$, such that $y(a) = y(b), \quad y'(a) = y'(b)$. A functional is then defined on $\mathcal{A}$

$$J(y) = \int_a^b L(x, y, y')\,dx,$$

where we rquire that L is periodic in x on $[a, b]$.

a) Show that we obtain the Euler-Lagranges equation on the usual form:

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

b) For $\kappa = 1,\ \beta = \gamma = 0$ the equation for weak resonant forcing (73) takes on the form

$$z'' + z - \epsilon z^3 = \epsilon \cos t, \tag{89}$$

where the free variable now is t, instead of x. Find a Lagrangian density, L, such that (89) becomes the Euler-Lagranges equation for $\delta J = 0$ on the interval $[a, b] = [0, 2\pi]$. Hint: Try $L = \frac{1}{2}(z')^2 - G(z, t)$ and determine G.

c) Let's try to make the best of $a \cos t$ as approximation to the solution of (89). One way to seek a value for a is to define

$$f(a) = J(a \cos t),$$

and then require $\frac{\mathrm{d}f}{\mathrm{d}a} = 0$ (instead of seeking an extremum on $\mathcal{A}$ we seek the extremum over a reduced set constructed by the variation of a, if an extremum it is). Work out the expression for $f(a)$ and find the a such that $\frac{\mathrm{d}f}{\mathrm{d}a} = 0$. Compare with the perturbation solution from problem 54.

d) Does the a, that you have found, correspond to an extremum for f ?

Ex. 62 ***From the exam 2014; algebraic equation.*** An algebraic equation reads

$$x^2 - 3x + 2 = e^{-kx}, \tag{90}$$

and we seek approximate solutions for large values of k.

a) Explain why the solutions correspond roughly to $x \approx 1$ and $x \approx 2$ when $k \to \infty$. It may be fruitful to depict the two sides of equation (90), in the same diagram, for a large k. We proceed with the solution that is close to 1.
Invoke the perturbation series

$$x = 1 + x_1 + x_2 + ... \quad \text{where} \quad 1 \gg x_1 \gg x_2,$$

and find x_1 through dominant balance analysis.

b) Find x_2.

Ex. 63 ***From the exam 2014; Lagrange's equations and perturbations.*** A particle, with mass m, moves in the gravity field on the constraint $y = \frac{1}{2}\alpha x^2$ (see figure 3). The motion takes place in the xy-plane, where the y axis is vertical, whereas the x axis is horizontal. At $t = 0$ the particle is at rest at $x = x_0$.

a) The position of the particle, x, depends on the other parameters x_0, α, g, m and t. Show that x can be expressed as

$$x = x_0 F(\pi_1, \pi_2),$$

where π_1 and π_2 are dimensionless numbers. Give expressions for π_1 and π_2.

b) We use x as position coordinate for the particle. Find the Lagrange's function and the Lagrange's equation. This equation do possess a first integral in this case. Find this and explain its physical significance.

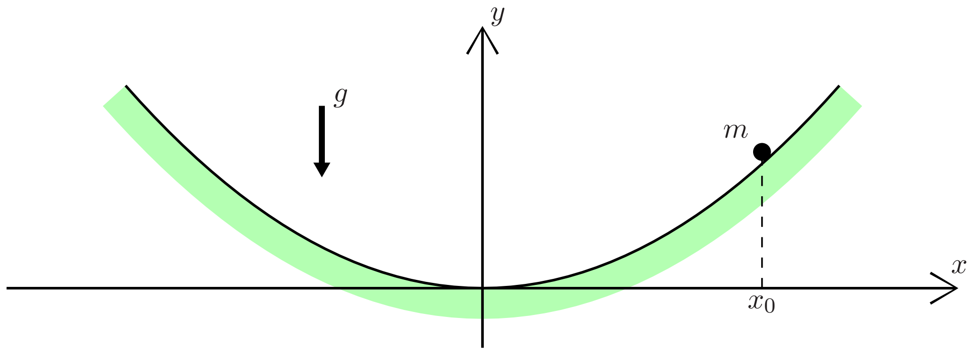


Figure 3: The constraint with the particle in its initial position.

c) We now assume that the particle performs oscillations which are small in the sense that x_0 is small. Rescale the Lagrange's equation from the preceding sub-problem to obtain

$$\left((1+\epsilon z^2\right)\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2}+\epsilon z\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2+z=0, \tag{91}$$

and the initial condition

$$z(0)=1, \quad \frac{\mathrm{d}z(0)}{\mathrm{d}\tau}=0, \tag{92}$$

where τ and z are dimensionless time and horizontal position, respectively, while ϵ is a small parameter.

d) Find a periodic approximation to the solution of (91) and (92), valid to, and including, order ϵ.

Ex. 64 ***From the exam 2014; boundary layers..*** A boundary value problem is defined as

$$\epsilon y''-f(x)y=g(x), \quad y(0)=1, \quad y(1)=2,$$

where f and g are given, positive, functions with convergent Taylor series for all points in the interval $[0,1]$, and ϵ is a small parameter. We seek an approximate solution by means of boundary layer theory.

a) Find the outer solution and explain why there must be boundary layers at both boundaries.

b) Find a unified approximation.

Ex. 65 ***Structure of boundary layer solutions.*** A boundary value problem is defined as

$$\epsilon y''+y'-y=0, \quad y(0)=0, \quad y(1)=1,$$

where ϵ is a small parameter.

a) Find an approximate solution by means of boundary value theory.

b) Find the exact solution.

c) Discuss the differences between the exact and the approximate solution. Does the exact solution inherit any features that may never be represented as power series in ϵ, even when the problem is rescaled ?

Ex. 66 ***Use of dominant balance on Ricatti's equation.*** A general second order equation is written

$$p(x)y'' + q(x)y' + r(x)y = 0. \tag{93}$$

a) Substitute

$$y = e^{\int P(x)dx},$$

and show that P must fullfill the Ricatti equation

$$p(x)(P' + P^2) + q(x)P + r(x) = 0, \tag{94}$$

which is a first order nonlinear equation for P.

b) Bessel's equation of zeroth order reads

$$y'' + \frac{1}{x}y' + y = 0. \tag{95}$$

We will investigate the solutions of this equation for large x, meaning $\frac{1}{x} \ll 1$. Apply dominant balance analysis to the corresponding Ricatti equation to find leading order approximations, P_0, for P.

c) Write $P = P_0 + P_1 + P_2 + ...$, where $P_2 \ll P_1 \ll P_0$ and find P_1 and P_2 by higher order balances.

d) Show that the corresponding approximation to y can be written

$$y \approx x^{-\frac{1}{2}} e^{\pm ix} \left(a_0 + \frac{a_1}{x} + O\left(\frac{1}{x^2}\right)\right),$$

where a_0 and a_1 are constants.

Ex. 67 ***Hamiltons principle does not yield a maximum.*** A harmonic oscillator consists of a spring and a particle of mass m, moving along the x-axis. The stiffness of the spring is k, meaning that the particle is subjected to a force $F(x) = -kx$, where we have assumed that $x = 0$ is the equilibrium position.

a) Find the Lagranges function for this system and work ut the Lagranges equation.

b) Discuss first integrals and work out Hamiltons canonical equations.

c) We assume that the particle starts at $x = x_0$ for $t = 0$. Make the system dimensionless such that the motion for the first cycle is governed by Hamilton's principle in the form

$$J(y) = \int_0^{2\pi} \left\{ \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - y^2 \right\} d\tau, \quad y(0) = 1, \quad y(2\pi) = 1.$$

d) When y solves the Lagranges equation show that $J(y) = 0$ and that

$$J(y + \epsilon h) = \epsilon^2 J(h),$$

for all h fulfilling $h(0) = h(2\pi) = 0$.

e) Explain why the y that gives $\delta J = 0$ is neither a maximum or a minimum. Hint: try $h = \sin(\alpha x)$ for different possible values of α.

Ex. 68 ***Mixed, natural boundary conditions.*** A functional includes end-point contributons according to

$$J(y) = \int_a^b \left\{ \frac{1}{2} p(x)(y')^2 + \frac{1}{2} q(x) y^2 - h(x) y \right\} dx \, + \, F(y(a), y'(a)) \, + \, G(y(b), y'(b)),$$

where F and G are (yet) unknown functions, while $p > 0$ in $[a, b]$.
There is no restriction on y at the end-points. Determine F and G in order to obtain

$$y'(a) = \alpha_1 y(a) + \beta_1, \quad y'(b) = \alpha_2 y(b) + \beta_2,$$

as natural boundary conditons.

Ex. 69 ***From the exam 2008 (MEK3100); an eigenvalueproblem.***
A boundary value problem on the interval $[0, 1]$ is given by

$$y'' + \lambda y + \epsilon y^3 = 0, \quad y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = \frac{1}{2}, \tag{96}$$

where $\epsilon \ll 1$ and λ is undetermined. The last requirement in (96) is a normalisation condition that removes ambiguity in the solutons.
We seek eigenvalue solutions as pairs of λ and $y(x)$ such that (96) is fulfilled.

Show that the unperturbed problem has

$$y^{(n)}(x) = \sin(n\pi x), \quad \lambda^{(n)} = n^2 \pi^2,$$

as an eigensolution for any positive integer n. Use the method of Poincare-Lindstedt on (96) to find the solution with smallest eigenvalue ($n = 1$) through order ϵ.

Ex. 70 ***From the exam 2015.*** An equation is specified according to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \gamma(\epsilon t) \frac{\mathrm{d}y}{\mathrm{d}t} + y = \epsilon f(\epsilon t) \cos(t), \tag{97}$$

where ϵ is a small parameter and γ and f are functions. The initial conditions are

$$y(0) = 1, \quad \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0. \tag{98}$$

a) Apply the multiple scale technique to (97) and (98). Find equations for how the two amplitudes of the leading order approximations evolve with time, but do not solve them in the general form.

b) Find the explicit solution for $\gamma = 0$ and $f(\tau) = e^{-\alpha \tau}$.

Ex. 71 ***From the exam 2015.*** A particle of mass m moves freely in three dimensions in a force field with potential $V = \alpha_x x^2 + \alpha_y y^2$, where x and y are two Cartesian coordinates, with z as the third one.

a) Work out the Lagranges equations in Cartesian coordinates.

b) Show that the Lagranges equations are equivalent to Newtons second law in this case. Are there first intgrals in this case ? If so, what do they correspond to physically?

Ex. 72 ***From the exam 2015.*** A second order equation is written

$$\epsilon y'' + W(x)y = 0, \tag{99}$$

where $\epsilon \to 0$, x is of order 1, and W is positive everywhere and of order 1. The prime denotes differentiaton with respect to x.

a) Substitute

$$y = e^{\int P(x)dx},$$

and show that P must fullfill the equation

$$\epsilon(P' + P^2) + W(x) = 0. \tag{100}$$

Apply the method of dominant balance to find the leading approximation, P_0, for P.

b) Write $P = P_0 + P_1$, where $P_1 \ll P_0$ and find an approximation for P_1 by a secondary dominant balance. Find y corresponding to $P_0 + P_1$.