THE KLEIN-GORDON EQUATION AND STATIONARY PHASE.

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The Klein-Gordon equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + q\eta = 0 \tag{1}$$

Initial condition:

$$\eta(x,0) = e^{-\left(\frac{x}{L}\right)^2}, \quad \frac{\partial}{\partial t}\eta(x,0) = 0 \tag{2}$$

The solution will depend on q og L in the combination qL^2 (Easily demonstrated by rescaling). However, we keep the equation on the given form. In the plots we always have $L = 10$.

Solution methods

1. Finite differences.
2. Stationary phase
3. FFT (presented elsewhere)

Fourier transform + stationary phase

1. *Fourier transform*
   Linear equations, constant coefficients, spatially confined initial condition ⇒ Fourier integral for large positive x:

   $$\eta(x,t) = \frac{1}{2\pi}\Re\int_0^\infty \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k$$

   where ω and k fulfill the dispersion relation.
2. *Stationary phase*
   For large x and t dominante contributions to the Fourier integral comes from the vicinity of the stationary point. An approximate integrand ⇒ explicit asymptotic solution for large x and t.

The Fourier transform

Relations between $f(x)$ and $\tilde{f}(k)$

$$\tilde{f}(k) = \int\limits_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x, \quad f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k \qquad (3)$$

Essential property

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}x} = \mathrm{i}k\tilde{f},$$

as shown by integration by parts.

Details in the definition (3) may vary in literature and software.

The transformed Klein-Gordon equation

Fourier transform applied to (4); (spatial differentiation replaced by power of $\mathrm{i}k$)

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + (k^2 + q)\tilde{\eta} = 0$$

second order ODE for $\tilde{\eta}$ with general solution

$$\tilde{\eta}(k,t) = A(k)e^{-\mathrm{i}\omega(k)t} + B(k)e^{\mathrm{i}\omega(k)t}$$

where $\omega(k) = \sqrt{q + k^2}$ (dispersion relation)
Transformed initial conditions

$$\tilde{\eta}(k,0) = \tilde{\eta}_0(k) = L\sqrt{\pi}e^{\left(\frac{kL}{2}\right)^2}, \quad \frac{\partial \tilde{\eta}(k,0)}{\partial t} = 0,$$

yield

$$A = B = \frac{1}{2}\tilde{\eta}_0$$

The inverse transformation

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\tilde{\eta_0}(k)e^{\mathrm{i}(kx-\omega(k)t)} + \tilde{\eta_0}(k)e^{\mathrm{i}(kx+\omega(k)t)}\right)\mathrm{d}k$$

Initial condition $\Rightarrow$ $\tilde{\eta}$ is real and symmetric with relation to $k = 0 \Rightarrow$ two terms in inversion are complex conjugates and

$$\eta(x,t) = \frac{1}{2\pi}\Re\int_{-\infty}^{\infty}\tilde{\eta_0}(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k,$$

further rewritten to

$$\eta(x,t) = \frac{1}{2\pi}\Re\left\{\int_{0}^{\infty}\tilde{\eta_0}(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + \int_{0}^{\infty}\tilde{\eta_0}(k)e^{-\mathrm{i}(kx+\omega(k)t)}\mathrm{d}k\right\} \tag{4}$$

Waves propagating to left and right are separated.

Stationary phase

A general Fourier integral

$$I(t) = \int_a^b F(k) e^{\mathrm{i}t\chi(k)} \mathrm{d}k,$$

where a and b may be finite or infinite.
Large $t \Rightarrow$ rapid oscillations of integrand $\Rightarrow$ cancellation of adjacent contributions $\Rightarrow I \to 0$ as $t \to \infty$.
If there is a stationary point, $k = k_0$, where

$$\frac{\mathrm{d}\chi(k_0)}{\mathrm{d}k} = 0$$

the oscillations will be slowest around k_0 and the dominanat contribution to the Fourier integral comes from the vicinity of this point.

Approximation of phase function and amplitude factor close to k_0

$$\chi(k) \approx \chi(k_0) + \frac{1}{2}\chi''(k_0)(k-k_0)^2, \quad F(k) \approx F(k_0).$$

This gives

$$\begin{aligned} I(t) &\approx \int\limits_{k_0-\epsilon}^{k_0+\epsilon} F(k_0) e^{\mathrm{i}t\{\chi(k_0)+\frac{1}{2}\chi''(k_0)(k-k_0)^2\}} \mathrm{d}k \\ &\approx F(k_0) e^{\mathrm{i}t\chi(k_0)} \int\limits_{-\infty}^{\infty} e^{\frac{1}{2}\mathrm{i}t\chi''(k_0)(k-k_0)^2} \mathrm{d}k. \end{aligned}$$

The latter integral is found in mathematical handbooks etc., and

$$I(t) \sim \frac{\sqrt{2\pi} F(k_0)}{\sqrt{t|\chi''(k_0)|}} e^{\mathrm{i}\left(\chi(k_0)t \pm \frac{\pi}{4}\right)}$$

Stationary phase applied to the Klein Gordon solution

We regard the first term in (6) and recognize F above as $\tilde{\eta}_0/(2\pi)$ and the phase as

$$\chi = k\frac{x}{t} - \omega(k).$$

The stationary point is then given according to

$$c_g(k_0) \equiv \frac{\mathrm{d}\omega(k_0)}{\mathrm{d}k} = \frac{x}{t},$$

where $c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ is the group velocity! Since $c_g < 1$, for all k, stationary phase is only applicable for $x < t$. We then find

$$k_0 = q^{\frac{1}{2}}\frac{x}{t}(1-(\tfrac{x}{t})^2)^{-\frac{1}{2}}, \qquad \chi(k_0) = -q^{\frac{1}{2}}(1-(\tfrac{x}{t})^2)^{\frac{1}{2}},$$
$$\chi''(k_0) = -q^{-\frac{1}{2}}(1-(\tfrac{x}{t})^2)^{\frac{3}{2}}.$$

The final asymptotic solution may be written as

$$\eta \sim a(x,t)\cos\theta(x,t),$$

where

$$a = \frac{L^{\frac{1}{2}} q^{\frac{1}{4}}}{(2\frac{t}{L})^{\frac{1}{2}}} \frac{e^{-\frac{L^2 q}{4((\frac{t}{x})^2 - 1)}}}{(1-(\frac{x}{t})^2)^{\frac{3}{4}}}, \quad \theta = \frac{\pi}{4} - q^{\frac{1}{2}}\sqrt{t^2 - x^2}$$

A local wave number may defined as $k_l(x,t) = \frac{\partial\theta}{\partial x}$. Then viser $k_l(x,t) = k_0(x,t)$.

1. How does the amplitude (a) vary in x and t ?
2. How does the local wave length vary in x and t ?
3. How does individual crests propagate relative to the wave system as a whole?

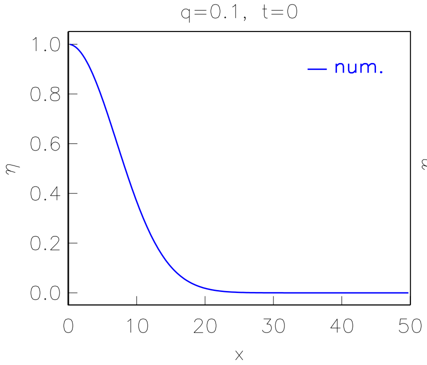
Numerical method.

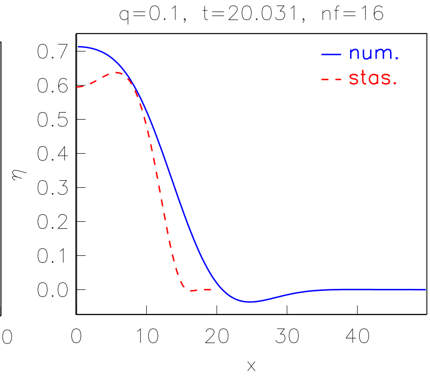
Straightforward, explicit, finite difference method:
Grid: $x_i = i\Delta x$, $t_n = n\Delta t$, nodal values:$\eta(x_i, t_n) \approx \eta_i^{(n)}$.
Difference equations:

$$\left.\begin{array}{l} \frac{1}{\Delta t^2}\left(\eta_i^{(n+1)} - 2\eta_i^{(n)} + \eta_i^{(n-1)}\right) \\ \quad - \frac{1}{\Delta x^2}\left(\eta_{i+1}^{(n)} - 2\eta_i^{(n)} + \eta_{i-1}^{(n)}\right) + q\eta_i^{(n)} = 0 \end{array}\right\} \tag{5}$$

1. $\eta_i^{(0)}$ and $\eta_i^{(-1)}$ given by initial conditions.
2. For each step $(n = 1, 2...)$ we compute $\eta_i^{(n+1)}$ from $\eta_i^{(n)}$ and $\eta_i^{(n-1)}$.
3. Grid-refinement tests $(\Delta x, \Delta t \to 0)$.

Comparison of numeric and asymptotic solutions





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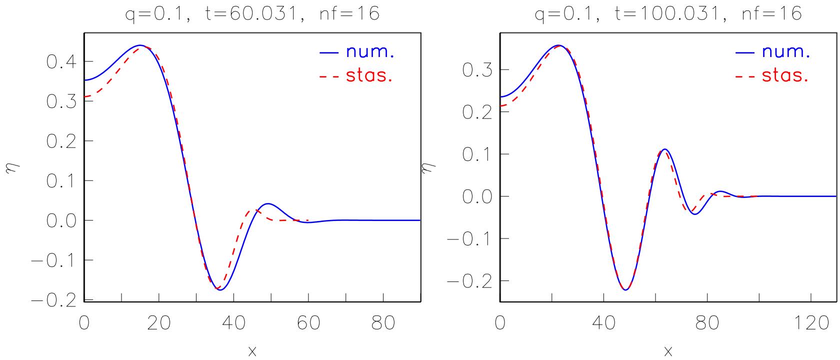

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