UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MEK 3100/4100 — Mathematical methods in mechanics.

Day of examination: Monday 9. June 2008.

Examination hours: 14:30 – 17:30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: Mathematical handbook, by K. Rottmann.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This set includes three problems that will first be given in English and then in Norwegian. A formula sheet is included at the end.

Oppgavesettet inneholder 3 oppgaver som først gis på engelsk, deretter på norsk. Et formelark er inkludert til slutt.

Problem 1.
A boundary value problem is defined as

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1,$$

where $\epsilon \to 0^+$. Find the leading order approximate solution for $0 \le x \le 1$.

Problem 2.
A functional is defined according to

$$I(y) = \int_a^b L(x, y, y')dx$$

where y is twice differentiable, $y(a) = A$ and $y(b) = B$. Derive the Euler-Lagrange equation from the requirement of zero variation, $\delta I = 0$.

Problem 3.
A boundary value problem on the interval $[0, 1]$ is given by

$$y'' + \lambda y + \epsilon y^3 = 0, \quad y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = \frac{1}{2}, \tag{1}$$

where $\epsilon \ll 1$ and λ is undetermined. The last requirement in (1) is a normalisation condition that removes ambiguity in the solutons.
We seek eigenvalue solutions as pairs of λ and $y(x)$ such that (1) is fulfilled.

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Show that the unperturbed problem has

$$y^{(n)}(x) = \sin(n\pi x), \quad \lambda^{(n)} = n^2\pi^2,$$

as an eigensolution for any positive integer n. Use the method of Poincare-Lindstedt on (1) to find the solution with smallest eigenvalue ($n = 1$) through order ϵ.

— End of problems in English —

Oppgave 1.
Et randverdiproblem er definert ved

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1,$$

der $\epsilon \to 0^+$. Finn den ledende ordens tilnærmete løsning for $0 \le x \le 1$.

Oppgave 2.
En funksjonal er definert som

$$I(y) = \int_a^b L(x, y, y')dx$$

der y er to ganger kontinuerlig deriverbar, $y(a) = A$ og $y(b) = B$. Utled Euler-Lagrange likningen fra krav om null variasjon, $\delta I = 0$.

Oppgave 3.
Et randverdiproblem på intervallet $[0, 1]$ er gitt ved

$$y'' + \lambda y + \epsilon y^3 = 0, \quad y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = \frac{1}{2}, \tag{2}$$

der $\epsilon \ll 1$ og λ er ubestemt. Det siste kravet i (2) er en normaliseringsbetingelse som fjerner flertydighet i løsningene.
Vi søker egenverdiløsninger som par av λ og $y(x)$ slik at (2) er oppfylt.
Vis at det uperturberte problemet har

$$y^{(n)}(x) = \sin(n\pi x), \quad \lambda^{(n)} = n^2\pi^2,$$

som egenløsning for alle positive heltall n. Bruk Poincare-Lindstedt's metode på (2) for å finne løsningen med den minste egenverdien ($n = 1$) til orden ϵ.

— Slutt på norske oppgaver —

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Formulas for Mek3100/4100

Trigonometric formulas

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$,
$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$
$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$,
$\cos^2\theta\sin\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin 3\theta$, $\cos\theta\sin^2\theta = \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta$,
$\cos\theta\cos 2\theta = \frac{1}{2}(\cos\theta + cos 3\theta)$, $\sin\theta\sin 2\theta = \frac{1}{2}(\cos\theta - \cos 3\theta)$,
$\cos\theta\sin 2\theta = \frac{1}{2}(\sin\theta + \sin 3\theta)$, $\sin\theta\cos 2\theta = \frac{1}{2}(-\sin\theta + 3\sin\theta)$,
$\cos\theta\cos 3\theta = \frac{1}{2}(\cos 2\theta + \cos 4\theta)$, $\sin\theta\sin 3\theta = \frac{1}{2}(\cos 2\theta - \cos 4\theta)$
$\cos\theta\sin 3\theta = \frac{1}{2}(\sin 4\theta + \sin 2\theta)$, $\sin\theta\cos 3\theta = \frac{1}{2}(\sin 4\theta - \sin 2\theta)$
$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$,
$\sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $\cosh(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$,
$\sin\theta = \theta - \frac{1}{6}\theta^3 + ... = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\theta^{(2j+1)}$, $\cos\theta = \theta - \frac{1}{2}\theta^2 + ... = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\theta^{2j}$.

Taylors formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_n,$$

where the residual is $R_n = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{(n+1)}$ for some c between a and x.

First order differential equations

The equation set

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x), \quad y(0) = a, \tag{3}$$

has the solution

$$y(x) = e^{-\int_0^x f(t)\mathrm{d}t}\left(a + \int_0^x g(t)e^{\int_0^t f(s)\mathrm{d}s}\mathrm{d}t\right).$$

Particular solutions

Equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = F(t) \tag{4}$$

where ω is a nonzero constant. Selected inhomogeneous solutions

$F(t)$	Part. løsning
$\cos\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2 - \sigma^2)^{-1}\cos\sigma t$
$\sin\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2 - \sigma^2)^{-1}\sin\sigma t$
$\cos\omega t$	$\frac{1}{2\omega}t\sin\omega t$
$\sin\omega t$	$-\frac{1}{2\omega}t\cos\omega t$