Long wave modeling.

Boussinesq equations; motivation and derivation

MEK4320

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August 8, 2018

Primitive and general hydrodynamic equations

The Navier-Stokes *(NS)* equation, primitive form

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} \equiv \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \mathcal{D} - g\vec{k}$$

$$\nabla\cdot\vec{v} = 0$$

where $\vec{v}$ = velocity, $\mathrm{D}/\mathrm{D}t$ = material derivative, p = pressure and $\mathcal{D}$ is viscous/turbulent term. In words:

acceleration = - pressure gradient + friction + gravity

net outflux from any fluid volume = 0

Boundary conditions: impermeable, no-slip, free (surface), artificial

Key problems: turbulence model, free surface tracking, under-resolved boundary layers, etc.

Generalization/alternatives to NS include multi-phase, multi-material...

Applicability of primitive models

- General, but inaccurate, free surface techniques (VOF, SPH ...)
- Industrial (CFX, Fluent...) and open-source (OpenFoam) solvers
- Computations readily become very heavy $\Rightarrow$ numerical solutions are under-resolved or unattainable
- Thin wall boundary layers; cannot be resolved
- Feasible only in local and idealized studies
- The burden of the computations often lead to wavering of the physics ?
- Analytic solutions are sparse, circumstantial and cumbersome

Surprisingly (?) little insight in hydrodynamic wave theory yet stem from "full computational models".

Simplified theories are still crucial, but general models become increasingly important

Full potential theory

Non-rotational motion $\Rightarrow$ potential ϕ: $\nabla\phi \equiv \vec{v}$

$$\nabla^2\phi = 0 \quad \text{for} \quad -h < z < \eta$$

Free surface $(z = \eta)$ $(\frac{\mathrm{D}}{\mathrm{D}t} = \partial/\partial t + \vec{v}\cdot\nabla)$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0, \qquad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial\phi}{\partial z}$$

Bottom $(z = -h)$

$$\frac{\mathrm{D}h}{\mathrm{D}t} = 0 \Rightarrow \frac{\partial\phi}{\partial n} = 0$$

Surface waves often well described.

Potential flow models

1. Integral equation discretized by panels. Boundary location updated in time as part of the method.
2. FFT techniques; approximations at free surface

Not incorporable: viscous effects, turbulence, overturning waves, Coriolis force...

Computation still heavy. Useful for local simulations and for assessing validity of simpler models.

More efficient and robust models must be employed for large scale modeling

Approximative theories

Basis of approximations; Scales for surface gravity waves

Acceleration scale

g acceleration of gravity
Remark: scale for particle acceleration in gravity waves is always the same, regardless of size of problem

Length scales

λ wavelength	h depth	A amplitude (η)
L_h depth variations	L_λ variation of λ, A...	

Often $L_h \sim L_\lambda$

Velocity and time scales

May be built from length and acceleration scales

Approximations; Regimes

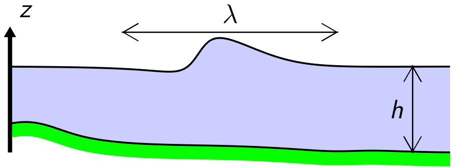
- $\frac{A}{h}, \frac{A}{\lambda} \ll 1 \Rightarrow$ linear and weakly non-linear theories
- $\frac{\lambda}{h} \ll 1 \Rightarrow$ deep water
- $\frac{h}{\lambda} \ll 1 \Rightarrow$ **shallow water; long wave theory**
- $\frac{h}{L_h}, \frac{\lambda}{L_\lambda} \ll 1 \Rightarrow$ multiple scale methods: ray theory; narrow band (nearly uniform waves)

Different requirements may be combined; long wave theory is often combined with weak non-linearity.

Definition of characteristic scales may be vague or ambiguous

Tools of the trade

- Depth integrated models for long waves
- Ray tracing, wave kinematics
- Efficient and robust numerical techniques



Long waves $\lambda/h \gg 1$ (2D case for simplicity)

U, W – characteristic horizontal and vertical velocities

$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{W}{h} \sim \frac{U}{\lambda} \Rightarrow \frac{W}{U} \sim \frac{h}{\lambda} \ll 1$

$\Rightarrow$ vertical motion is small $\Rightarrow$ pressure nearly hydrostatic

Nonlinear Shallow Water Equations

Vertical acceleration neglected $\Rightarrow$ hydrostatic pressure $\Rightarrow$ no vertical variation in horizontal velocity $\Rightarrow$ 3D physics, 2D maths.

NLSW

$$\frac{\partial \bar{\mathbf{v}}_h}{\partial t} + \bar{\mathbf{v}}_h \cdot \nabla_h \bar{\mathbf{v}}_h = -g\nabla_h \eta$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h+\eta)\bar{\mathbf{v}}_h)$$

η: surface elevation, $\bar{\mathbf{v}}_h$: velocity (horizontal), ∇_h: horizontal gradient operator

Efficient and simple numerical solution; hyperbolic equations.
May include bores, Coriolis effects and bottom drag, *but wave dispersion is lost.*
Use: Ocean modeling; tides, tsunamis, storm surges.

Dispersive long wave models

Models developed by perturbation/iteration/series expansion, assuming small
$\epsilon \equiv (H/\lambda)^2$ and $\alpha \equiv A/H$, where H is typical depth.
Leading pressure modifications by vertical accelerations included.
Huge diversity in in formulations and accuracy

Boussinesq type models

- 1880$\rightarrow$ Theoretical applications (KdV...)
- 1966 first numerical Boussinesq models put to use
- 1990$\rightarrow$ new formulations, increased validity range
- Increase of computer power $\Rightarrow$ large scale models feasible
- Important for some tsunami features
- Important model for coastal engineering
- A step toward more general models from (N)LSW; assessment of dispersion effects

Derivation of Boussinesq equations

Antagelser/krav.

- Lange bølger på grunt vann.
- Ikke-lineære bølger.
- 2-D bevegelse (vertikalen + 1 horisontal retning).
- Langsom variasjon av dyp.

Skalering

1. Et typisk dyp, H, brukes for skalering av "vertikale" størrelser (z, h, w, η).
2. En typisk bølgelengde, ℓ, brukes for skalering av horisontale størrelser (u, x , t).
3. Feltstørrelser skaleres i tillegg med en faktor, α, som er et mål for amplituden.

Langbølgeutvikling forutsetter

$$\epsilon \equiv \frac{H^2}{\ell^2} \ll 1 \tag{1}$$

Dimensjonering

$$z^\star = Hz, \qquad x^\star = \ell x, \qquad t^\star = \ell(gH)^{-\frac{1}{2}}t,$$

$$h^\star = Hh(x), \qquad \eta^\star = \alpha H\eta, \qquad u^\star = \alpha(gH)^{\frac{1}{2}}u,$$

$$w^\star = \epsilon^{\frac{1}{2}}\alpha(gH)^{\frac{1}{2}}w, \quad p^\star = \rho gHp,$$

Skalerte likninger.

Randbetingelser

$$\begin{array}{lll} p = 0, \quad \eta_t + \alpha u\eta_x = w, & \text{ved} & z = \alpha\eta \\ w = -h_x u & \text{ved} & z = -h \end{array} \tag{2}$$

Eulers bevegelseslikninger:

$$u_t + \alpha u u_x + \alpha w u_z = -\alpha^{-1}p_x \tag{3}$$

$$\epsilon(w_t + \alpha u w_x + \alpha w w_z) = -\alpha^{-1}(p_z - 1) \tag{4}$$

og kontinuitetslikningen:

$$u_x + w_z = 0 \tag{5}$$

Strategi

1. Gruntvannslikninger utledes $\Rightarrow$ trykk tilnærmet hydrostatisk, u nesten uniform i vertikalen.
2. Definisjon av $\overline{u}$ = dybdemiddel. Eksakt kontinuitetslikning uttrykt ved $\overline{u}$.
3. Gitt punkt 1: Kontlikning + randbet. $\Rightarrow$ w uttrykt ved z, η og $\overline{u}$.
4. Uttrykk for w innsatt likning for z-momentum $\Rightarrow$ p uttrykt ved z, η og $\overline{u}$. Korreksjon til hydrostatikk.
5. Insetting i horisontal momentumlikning og integrasjon over væskedyp $\Rightarrow$ Dybde-integrert momentumlikning.

Andre teknikker: Direkte rekke-utviklinger i z...

Steg 1. Hydrostatisk teori.

Alle ledd av orden ϵ sløyfes.

Fra z-komponent av bev.likn:

$$p = \alpha\eta - z + O(\alpha\epsilon) \tag{6}$$

x-komponent av bevegelseslikningen:

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\eta_x + O(\epsilon) = \mathrm{funksjon}(x) + O(\epsilon)$$

(6)$\Rightarrow$ $\frac{\partial p}{\partial x} = \alpha O(\epsilon) \Rightarrow$

$$u_z = O(\epsilon) \tag{7}$$

Likninger (NLSW) i kompendium gjenskapes

$$u_t + \alpha u u_x = -\eta_x + O(\epsilon)$$

$$\eta_t = -\{(h + \alpha\eta)u\}_x$$

Step 2: depth averaged velocity and continuity equation

Definition of depth average

$$\overline{u} = (h + \alpha\eta)^{-1} \int\limits_{-h}^{\alpha\eta} u dz \tag{8}$$

Then (7)$\Rightarrow$ $u - \overline{u} = O(\epsilon)$
Depth averaged continuity equation:

$$\eta_t = -\{(h + \alpha\eta)\overline{u}\}_x \tag{9}$$

This continuity equation is exact to all orders in α and ϵ

Step 3: Express w in terms of η and $\overline{u}$

From the kinematic surface condition it follows

$$w = w|_{z=\alpha\eta} + \int_{\alpha\eta}^{z} w_z dz = \eta_t + \int_{0}^{z} w_z dz + O(\alpha) \qquad (10)$$

Then equation (5), zero divergence, yields

$$w = \eta_t - \int_{0}^{z} u_x dz + O(\alpha) = \eta_t - \int_{0}^{z} \overline{u}_x dz + O(\alpha, \epsilon) \qquad (11)$$

Since $\overline{u}$ is independent of z we obtain

$$w = \eta_t - z\overline{u}_x + O(\alpha, \epsilon) \qquad (12)$$

Step 4: The corrected pressure expression

The z-component of the momentum equation, (4), is integrated

$$p = p|_{z=\alpha\eta} + \int_{\alpha\eta}^{z} p_z dz = \int_{\alpha\eta}^{z} \{-1 - \alpha\epsilon(w_t + \alpha ...)\} dz \qquad (13)$$

Omitting small terms $\Rightarrow$

$$p = \alpha\eta - z - \alpha\epsilon \int_{0}^{z} w_t dz + O(\epsilon\alpha^2) \qquad (14)$$

Insertion of expression (12) for w then implies

$$p = \alpha\eta - z - \epsilon\alpha(z\eta_{tt} - \frac{1}{2}z^2\overline{u}_{xt}) + O(\alpha\epsilon^2, \alpha^2\epsilon) \qquad (15)$$

Step 5: averaging the x-component of the mom. eq.

Use of $u - \overline{u} = O(\epsilon)$ in momentum equation

$$u_t + \alpha \overline{u}\,\overline{u}_x + O(\alpha\epsilon) = -\alpha^{-1} p_x. \tag{16}$$

Invoking (15) on the r.h.s.

$$-\alpha^{-1} p_x = -\eta_x + \epsilon(z\eta_{ttx} - \frac{1}{2} z^2 \overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon).$$

Averaging ($(h+\alpha\eta)^{-1} \int_{-h}^{\alpha\eta} \cdot dz$) $\Rightarrow$

$$\begin{aligned} -\overline{\alpha^{-1} p_x} &= -\eta_x - \epsilon(\tfrac{1}{2} h\eta_{ttx} + \tfrac{1}{6} h^2 \overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon) \\ &= -\eta_x + \epsilon(\tfrac{1}{2} h(h\overline{u}_t)_{xx} - \tfrac{1}{6} h^2 \overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon) \end{aligned}$$

where $\eta_t = -(h\overline{u})_x + O(\alpha)$ from (9) is used to obtain a better structure of the final equation.

Step 5: continues....

The convective term of (16) is independent of z and unaltered by averaging.
For the temporal derivative of the acceleration

$$\begin{aligned}\int_{-h}^{\alpha\eta} u_t dz &= \frac{\partial}{\partial t}\left(\int_{-h}^{\alpha\eta} u dz\right) - \alpha u|_{z=\alpha\eta}\frac{\partial\eta}{\partial t} \\ &= \frac{\partial}{\partial t}((h+\alpha\eta)\overline{u}) - \alpha\overline{u}\frac{\partial\eta}{\partial t} + O(\alpha\epsilon) = (h+\alpha\eta)\overline{u}_t + O(\alpha\epsilon).\end{aligned}$$

Hence, $\overline{(u_t)} = \overline{u}_t + O(\alpha\epsilon)$, and we may collect all terms of the averaged equation of motion.

The Boussinesq equations

$$\begin{aligned} \overline{u}_t + \alpha \overline{u}\,\overline{u}_x &= -\eta_x + \epsilon\{\frac{1}{2}h(h\overline{u}_t)_{xx} - \frac{1}{6}h^2\overline{u}_{xxt}\} \\ &+ O(\epsilon^2, \alpha\epsilon) \qquad (17) \end{aligned}$$

which combined with (9):

$$\eta_t = -\{(h + \alpha\eta)\overline{u}\}_x \qquad (18)$$

constitutes two equations for the unknowns η and $\overline{u}$.

New in relation to NLSW: dispersion.

More on Boussinesq type equations

Depth integrated theory; Boussinesq

Boussinesq equations with 2 horizontal dimensions

$$\frac{\partial \bar{\mathbf{v}}_h}{\partial t} + \alpha \bar{\mathbf{v}}_h \cdot \nabla_h \bar{\mathbf{v}}_h = -\nabla_h \eta + \epsilon \left(\frac{1}{2} h \nabla_h \nabla_h \cdot (h \frac{\partial \bar{\mathbf{v}}_h}{\partial t}) - \frac{1}{6} h^2 \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t} \right)$$
$$+ \epsilon \kappa h^2 (\nabla_h^3 \eta + \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t}) + O(\epsilon^2, \alpha\epsilon)$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h + \alpha \eta) \bar{\mathbf{v}}_h)$$

Derivation as with 1 horizontal dimension.
Blue term $= O(\epsilon^2, \epsilon\alpha)$: optimization of dispersion properties
Numerical solution much heavier than for shallow water eq. – implicit solution strategy needed. Still, much faster to solv than primitive equations. Wave dispersion included.

The z_α formulation

Popular formulation from Nwogu, later extended by others (Kennedy, Kirby, Wu, Liu, Lynett..)
Velocity profile

$$\vec{v} = \vec{v}_s + \epsilon(z_\alpha \nabla_h \frac{\partial \eta}{\partial t} - \frac{1}{2} z_\alpha^2 \nabla_h \nabla_h \cdot \vec{v}_*) + O(\epsilon^2),$$

$\vec{v}_s$ = surface velocity, $\vec{v}_*$ velocity at any depth.
Velocity at $z_\alpha(x, y)$
$\mathbf{v}(x, y, t) \equiv \vec{v}(x, y, z_\alpha(x, y), t)$,
used as unknown. Optimization of disperson on flat bottom $\Rightarrow$
$z_\alpha = -0.531h$
Extra nonlinearities, $O(\epsilon\alpha)$, may be kept in derivation.

z_α **not related to nonlinear parameter** α

Generalized Boussinesq equations

Hsiao et al. (2002):

$$\eta_t = -\nabla_h \cdot [(h + \alpha\eta)(\mathbf{v} + \epsilon\mathbf{M})] + O(\epsilon^2),$$
$$\mathbf{v}_t + \frac{\alpha}{2}\nabla_h(\mathbf{v}^2) = -\nabla_h\eta - \epsilon\left[\frac{1}{2}z_\alpha^2\nabla_h\nabla_h\cdot\mathbf{v}_t + z_\alpha\nabla_h\nabla_h\cdot(h\mathbf{v}_t)\right]$$
$$+\alpha\epsilon\nabla_h(D_1 + \alpha D_2 + \alpha^2 D_3) + O(\epsilon^2) + \mathbf{N} + \mathbf{E},$$

where index t denotes temporal differentiation and

$$\mathbf{M} = [\tfrac{1}{2}z_\alpha^2 - \tfrac{1}{6}(h^2 - \alpha h\eta + \alpha^2\eta^2)]\nabla_h\nabla_h\cdot\mathbf{v}$$
$$+[z_\alpha + \tfrac{1}{2}(h - \alpha\eta)\nabla_h\nabla_h\cdot(h\mathbf{v})].$$

Extra nonlinearities marked with blue.

Furthermore...

$$
\begin{aligned}
D_1 &= \eta\nabla\cdot(h\mathbf{v}_t) - \tfrac{1}{2}z_\alpha^2\mathbf{v}\cdot\nabla\nabla\mathbf{v} - z_\alpha\mathbf{v}\cdot\nabla\nabla\cdot(h\mathbf{v}) - \tfrac{1}{2}(\nabla\cdot(h\mathbf{v}))^2,\\
D_2 &= \tfrac{1}{2}\eta^2\nabla\cdot\mathbf{v}_t + \eta\mathbf{v}\nabla\nabla\cdot(h\mathbf{v}) - \eta\nabla\cdot(h\mathbf{v})\nabla\cdot\mathbf{v},\\
D_3 &= \tfrac{1}{2}\eta^2\left[\mathbf{v}\cdot\nabla\nabla\cdot\mathbf{v} - (\nabla\cdot\mathbf{v})^2\right],\\
\mathbf{E} &= H^{-1}\nabla_h(\nu(x,y,t)\nabla_h(H\mathbf{v}),\\
\mathbf{N} &= -\tfrac{\alpha}{\mu}\tfrac{K}{H}|\mathbf{v}|\mathbf{v}.
\end{aligned}
$$

Unsystematic terms:
E is dissipation term for capturing of breaking waves
N is bottom drag.
Programs freely available on WEB (Funwave and Coulwave).
There are issues with these equations

Dispersion relation for single harmonic mode

Mode

$$\eta = A\cos(kx - \omega t)$$

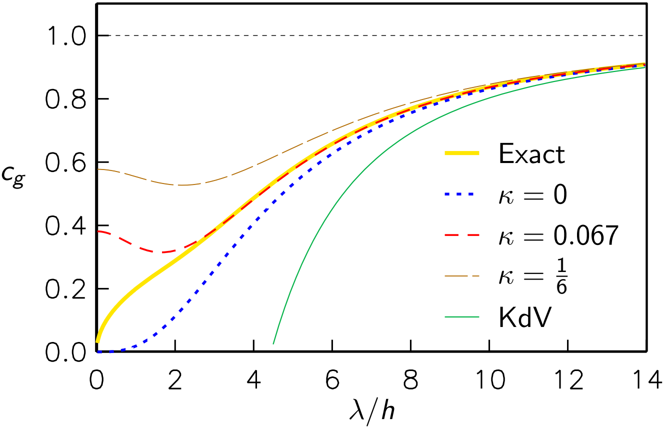
Full potential theory in present scaling ($h = 1$)

$$c^2 = \frac{1}{k\epsilon^{\frac{1}{2}}}\tanh(\epsilon^{\frac{1}{2}}k) = 1 - \frac{1}{3}\epsilon k^2 + \frac{2}{15}\epsilon^2 k^4 + \ldots$$

Many Boussinesq models fulfill

$$c^2 = \frac{1 + \kappa\epsilon k^2}{1 + (\frac{1}{3} + \kappa)\epsilon k^2} = 1 - \frac{1}{3}\epsilon k^2 + \ldots,$$

Standard Boussinesq with averaged velocity	$\kappa = 0$
Optimized Boussinesq	$\kappa = 0.067$
Optimized $z_\alpha = -0.531h$	$\kappa = 0.067$
$z_\alpha = -h$ ($\vec{v}$ at bottom)	$\kappa = \frac{1}{6}$

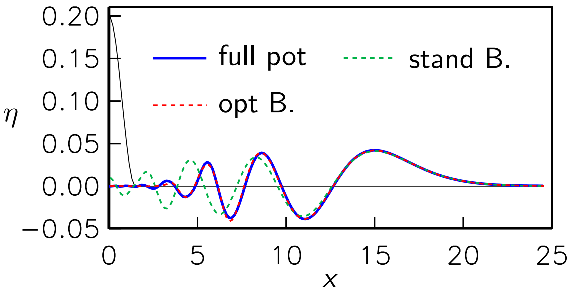


$\kappa = \frac{1}{6} \rightarrow u$ at bottom, $\kappa = 0 \rightarrow$ averaged u,

$\kappa = 0.067 \rightarrow$ optimal choice

$c_g = \mathrm{d}\omega/\mathrm{d}k$ – group velocity

Effect of dispersion



Linear evolution from short initial elevation

Black: half the initial condition, width $\sim$ 3 depths

Front: Good agreement for all Boussinesq formulations

Rear: Improved model superior, standard B. too dispersive

Observe: No corresponding improvement for steep bottom gradients

Utledning av KdV likninger.

Antagelser

- Lange bølger – som for Boussinesq-likninger
- Svakt ikke-lineære bølger. $(\alpha \sim \epsilon)$
- Ensretting: bølgeforplantning bare i positiv x-retning

Strategier

1. Klatter sammen ikke-lineært ledd fra hydrostatisk teori og ledd avledet fra den lineære dispersjonsrelasjonen.
2. Innfører koordinatsystem som beveger seg med lineær gruntvanns-hastighet $\Rightarrow$ For ensrettede bølger vil alt endre seg langsomt i tiden.
3. Finne korreksjoner på likningene for Riemann-invariantene innenfor rammen av Boussinesq-likningene.

Velger alternativ 2

Boussinesq-likninger for $h = 1$.

$$\eta_t = -\{(1 + \alpha\eta)\overline{u}\}_x \tag{19}$$

$$\overline{u}_t + \alpha\overline{u}\,\overline{u}_x = -\eta_x + \epsilon\frac{1}{3}\overline{u}_{xxt} \tag{20}$$

$\alpha, \epsilon \to 0 \Rightarrow$

$$\eta = F(x - t) + G(x + t),$$
$$u = F(x - t) - G(x + t)$$

Bølger i positiv x-retning $\Rightarrow$ $G \equiv 0$ og

$$\eta_t + \eta_x = 0, \quad u_t + u_x = 0 \tag{21}$$

Dvs. standard transportlikning.

Finnes det generaliseringer av (21) som inneholder korreksjoner til $O(\alpha, \epsilon)$?

Utgangspunkt

For små α, ϵ er ikke-linearitet og dispersjon svake.
Form varierer svakt, fasehastighet nær 1.
Løsning endrer seg lite når vi beveger oss med hastighet 1.

Koordinatskifte

$$\xi = x - t \qquad \tau = \epsilon t \tag{22}$$

τ er en langsom variabel.

Innsetting i kontinuitetslikning, (19):

$$\epsilon\eta_\tau - \eta_\xi = -(1 + \alpha\eta)\overline{u}_\xi - \alpha\overline{u}\eta_\xi \tag{23}$$

Eliminasjon av $\overline{u}$

Ledende orden: $\eta_\xi = \overline{u}_\xi + O(\alpha, \epsilon)$. Dvs:

$$\eta = \overline{u} + O(\alpha, \epsilon) \tag{24}$$

hvis feks. $\eta = \overline{u} = 0$ for $x = \infty$.

Vi kvitter oss med mest mulig $\overline{u}$ i (23) vha. (24):

$$\epsilon\eta_\tau - \eta_\xi = -\overline{u}_\xi - 2\alpha\eta\eta_\xi + O(\epsilon^2, \alpha\epsilon) \qquad (25)$$

Innsetting i bev.likn. (20) med $\overline{u} = \eta$ i små ledd $\Rightarrow$

$$\overline{u}_\xi = \eta_\xi + \epsilon\eta_\tau + \alpha\eta\eta_\xi + \frac{1}{3}\epsilon\eta_{\xi\xi\xi} + O(\epsilon^2, \alpha\epsilon) \qquad (26)$$

Innsetting i kont.likn. (25) $\Rightarrow$ KdV likningen:

$$\epsilon\eta_\tau + \frac{3}{2}\alpha\eta\eta_\xi + \frac{1}{6}\epsilon\eta_{\xi\xi\xi} = O(\epsilon^2, \alpha\epsilon) \qquad (27)$$

Tilbaketransformasjon

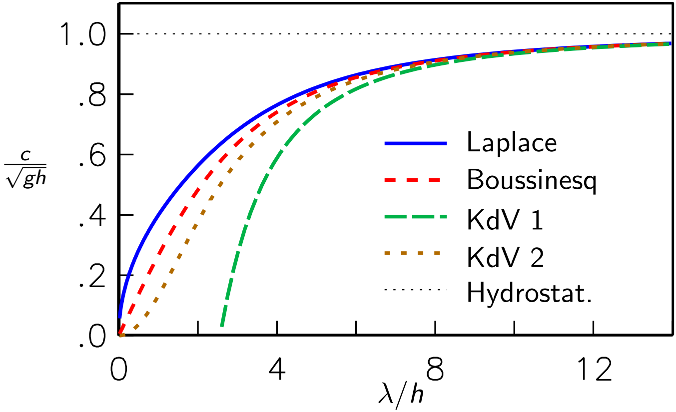
Gjeninnføres x og t :

$$\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x + \frac{1}{6}\epsilon\eta_{xxx} = O(\epsilon^2, \alpha\epsilon) \qquad (28)$$

Eller; $\eta_t = \eta_x + O(\alpha, \epsilon) \Rightarrow$

$$\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x - \frac{1}{6}\epsilon\eta_{xxt} = O(\epsilon^2, \alpha\epsilon) \qquad (29)$$

Dispersjonsrelasjoner for KdV



Fasehastighet som funksjon av bølgelengde

KdV 1: $\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x + \frac{\epsilon}{6}\eta_{xxx} = 0$

KdV 2: $\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x - \frac{\epsilon}{6}\eta_{xxt} = 0$

What is gained by long wave theory

1. Physical contents more transparent
2. Important closed form solutions of NLSW and the KdV equations
3. NLSW equations are hyperbolic with characteristics and shocks
4. Unknown upper bound of fluid replaced by coefficients in η
5. The number of dimensions reduced by 1 (depth integration)

Last two points crucial for numerical solution