UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MEK 3100/4100 — Mathematical methods in mechanics

Day of examination: Friday 8. June 2007.

Examination hours: 14:30 – 17:30.

This problem set consists of 4 pages.

Appendices:

Permitted aids: Mathematical handbook, by K. Rottmann.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This set includes three exercises that will first be given in English and then in Norwegian. However, the figure is given only once. In addition a formula sheet is included at the end.

Exercise 1. A boundary layer problem

A boundary value problem is defined through

$$\epsilon y'' + y' + \frac{1}{2}y^2 = 0 \quad ; \quad y(0) = 0, \quad y(1) = 1, \tag{1}$$

where $\epsilon \to 0^+$.

Find the leading order approximate solution valid in $0 \leq x \leq 1$ and sketch it.

Exercise 2. Dimension analysis

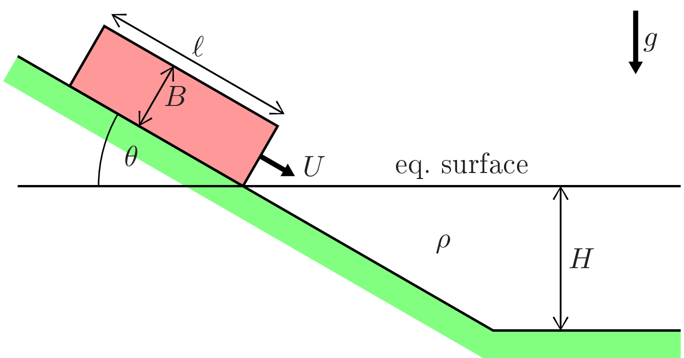

Figure: Definisjonskisse av skred i bølgetank. (Definition sketch of slide in wavetank).

A rigid, rectangular slide body is forced to move with constant velocity, U, along an inclined plane in the end of a wave tank, see figure. The equilibrium

(Continued on page 2.)

depth in the wavetank is H, the slide has length ℓ and thickness B, the inclination angle is θ, the constant of gravity is g and the density of the fluid in the tank is ρ. The slide spans the total width of the tank, allowing us to assume two-dimensional motion when viscosity is neglected. Since the slide motion is forced the weight of the slide does not affect the fluid response.

a) Find a complete set of dimensionless numbers from the parameter set U, ℓ, B, H, ρ, g and θ.

b) We now introduce the force per width, F, acting on the slide from the liquid and set $t = 0$ as the time of first contact between slide and liquid. Moreover, we study times that are smaller than both ℓ/U and the time needed for any disturbance in the liquid to reach the non-sloping region of the tank. Explain why ℓ and H do not influence F for such times and show

$$F = \rho B U^2 G\left(\frac{U}{\sqrt{gB}}, \frac{Ut}{B}\right),$$

where G is a function of two real variables.

Exercise 3. Forced oscillation
A dimensionless and scaled equation with weak resonant forcing and weak damping is given according to

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \epsilon u^2\frac{\mathrm{d}u}{\mathrm{d}t} + u = \epsilon\cos t,$$

where ϵ is a small parameter. We assume that the motion starts from rest, implying $\frac{\mathrm{d}u(0)}{\mathrm{d}t} = u(0) = 0$. Use a two scale expansion to demonstrate that the solution approaches a limit cycle when t becomes large.
You are allowed to assume, or rather to guess, that the zeroth order solution has the form $A\sin t$, where A is slowly varying. Moreover, it is not required that you work out the full solution for A.

— *End of exercises in English* —

Oppgave 1. Et grensesjiktsproblem
Et grenseverdiproblem er gitt ved

$$\epsilon y'' + y' + \frac{1}{2}y^2 = 0 \quad ; \quad y(0) = 0, \quad y(1) = 1, \tag{2}$$

der $\epsilon \to 0^+$.
Finn den ledende ordens tilnærmelsen som er gyldig for $0 \le x \le 1$ og tegn en skisse av den.

Oppgave 2. Dimensjonsanalyse
Et stivt rektangulært skred beveger seg på et skråplan med en tvungen hastighet , U, ned i en bølgetank (se figur foran). Likevektsdypet i tanken er H, skredet har lengde ℓ og tykkelse B, helningsvinkelen er θ, tyngdens akselerasjon er g og tettheten av væsken er ρ. Skredet fyller hele tankens bredde slik at vi kan anta todimensjonal bevegelse når viskøse effekter neglisjeres. Fordi skredet har tvungen bevegelse vil dets tyngde ikke påvirke væskebevegelsen.

(Continued on page 3.)

a) Finn et komplett sett dimensjonsløse tall fra parameterene U, ℓ, B, H, ρ, g og θ.

b) Vi introduserer nå den kraft per bredde, F, som virker fra væsken på skredet og setter $t = 0$ som det tidspunkt der skredet først kommer i kontakt med væsken. Vi betrakter tider som er mindre enn både ℓ/U og den tiden som er nødvendig for at noen forstyrrelse i væsken skal nå området med horisontal bunn. Forklar hvorfor F nå ikke avhenger av ℓ og H og vis

$$F = \rho B U^2 G\left(\frac{U}{\sqrt{gB}}, \frac{Ut}{B}\right),$$

der G er en funksjon av to reelle variable.

Oppgave 3. Tvungen svingning
En dimensjonsløs og skalert likning er gitt ved

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \epsilon u^2 \frac{\mathrm{d}u}{\mathrm{d}t} + u = \epsilon \cos t,$$

der ϵ er en liten parameter. Vi antar at bevegelsen starter fra ro, dvs. $\frac{\mathrm{d}u(0)}{\mathrm{d}t} = u(0) = 0$. Bruk en toskalautvikling til å demonstrere at løsningen nærmer seg en grensesykel for store t.
Du kan anta, eller skal vi si gjette, at nullteordensløsningen har formen $A \sin t$, der A varierer langsomt. Dessuten, det kreves ikke at du finner den fulle løsningen for A.

— *Slutt på norske oppgaver* —

(Continued on page 4.)

Formulas for Mek3100/4100

Trigonometric formulas

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$,

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$,

$\cos^2\theta\sin\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin 3\theta$, $\cos\theta\sin^2\theta = \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta$,

$\cos\theta\cos 2\theta = \frac{1}{2}(\cos\theta + cos3\theta)$, $\sin\theta\sin 2\theta = \frac{1}{2}(\cos\theta - \cos 3\theta)$,

$\cos\theta\sin 2\theta = \frac{1}{2}(\sin\theta + \sin 3\theta)$, $\sin\theta\cos 2\theta = \frac{1}{2}(-\sin\theta + 3\sin\theta)$,

$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$,

$\sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $\cosh(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$,

$\sin\theta = \theta - \frac{1}{6}\theta^3 + ... = \sum\limits_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\theta^{(2j+1)}$, $\cos\theta = \theta - \frac{1}{2}\theta^2 + ... = \sum\limits_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\theta^{2j}$.

Taylors formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_n,$$

where the residual is $R_n = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{(n+1)}$ for some c between a and x.

First order differential equations

The equation set

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x), \quad y(0) = a, \tag{3}$$

has the solution

$$y(x) = e^{-\int\limits_0^x f(t)\mathrm{d}t}\left(a + \int\limits_0^x g(t)e^{\int\limits_0^t f(s)\mathrm{d}s}\mathrm{d}t\right).$$

Particular solutions

Equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = F(t) \tag{4}$$

where ω is a nonzero constant. Selected inhomogeneous solutions

$F(t)$	Part. løsning
$\cos\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2-\sigma^2)^{-1}\cos\sigma t$
$\sin\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2-\sigma^2)^{-1}\sin\sigma t$
$\cos\omega t$	$\frac{1}{2\omega}t\sin\omega t$
$\sin\omega t$	$-\frac{1}{2\omega}t\cos\omega t$