

The Stokes wave solution. MEK4320

Geir Pedersen

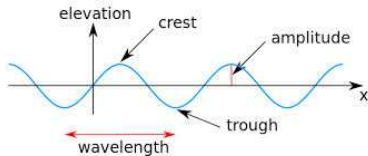
Department of Mathematics, University of Oslo

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Background

A first step from this...

Typical figure from the net.



Surface gravity waves in elementary courses and popular dissemination

- Linear periodic solution
- Smooth, rounded, regular.
- Real waves seldom look like this.

to this...



and this...



Stokes waves

- Nonlinear periodic surface gravity waves of permanent shape
- First presented in 1847
- Stability and modulation still an active field of research
- Here: basic solution in infinite depth.

Described by two parameters:

- Amplitude a^*
- Wavelength $\lambda^* = 2\pi/k^*$

Wave celerity, c^* , surface elevation, ζ^* and velocity potential, ϕ^* , must be found.

* marks dimensional coordinates.

Basic equations

Full potential theory in deep water

Surface elevation ζ^* , potential ϕ^* , infinite depth.

* marks dimensional quantities.

$$\zeta^*_{t^*} + \phi^*_{x^*} \zeta^*_{x^*} = \zeta^*_{z^*}, \quad z^* = \zeta^*; \quad (1)$$

$$\phi^*_{t^*} + \frac{1}{2}(\nabla^* \phi^*)^2 + g \zeta^* = 0, \quad z^* = \zeta^*; \quad (2)$$

$$(\nabla^*)^2 \phi^* = 0, \quad z^* < \zeta^*; \quad (3)$$

$$\nabla^* \phi^* \rightarrow 0, \quad z^* \rightarrow -\infty; \quad (4)$$

Surface conditions (1,2), Laplace equation (3), vanishing motion in depth (4).

Linear harmonic mode

Deletion of nonlinear terms

$$\zeta^*_{t^*} = \zeta^*_{z^*}, \quad z^* = 0;$$

$$\phi^*_{t^*} + g\zeta^* = 0, \quad z^* = 0;$$

$$(\nabla^*)^2 \phi^* = 0, \quad z^* < 0;$$

$$\nabla^* \phi^* \rightarrow 0, \quad z^* \rightarrow -\infty;$$

Solution

$$\zeta^* = a^* \cos(k^*(x^* - c_f^* t^*)),$$

$$\phi^* = a^* c_f^* e^{k^* z^*} \sin(k^*(x^* - c_f^* t^*)) \quad c_f^* = \sqrt{\frac{g}{k^*}}$$

Goal: find nonlinear correction to this

Preparation for perturbation method:

Parameters, scaling, expansion parameter (ϵ)
and eigenvalue (c)

Dimension analysis

Parameters

- 1 λ^* wavelength. Represented by $k^* = 2\pi/\lambda^*$.
- 2 a^* a measure of amplitude. will be made unique later.
- 3 g . These are gravity waves.
- 4 c^* The wave celerity. Must be found. In the linear approximation $c^* = c_f^* = \sqrt{\frac{g}{k^*}}$

The Pi theorem

Two dimensionless numbers

- $\epsilon = a^* k^*$; **wave steepness**. Assumed small.
- The other is c^*/c_f^* .

$$c^* = c_f^* f(\epsilon), \quad (5)$$

where the function f must be found.

Dimensionless variables

$$\begin{aligned}z &= k^* z^*, & x &= k^* x^*, \\t &= k^* c_f^* t^*, & \zeta^* &= a^* \zeta, \\ \phi^* &= c_f^* a^* \phi, & c^* &= c_f^* c.\end{aligned}$$

Waves of permanent form

composite phase variable: $\theta = x - ct$

Fields : $\zeta = \zeta(\theta)$, $\phi = \phi(\theta, z)$

Only two free variables instead of three.

Wavelength fixated by requiring periode = 2π in θ .

Introduction of wave celerity, c , as independent unknown is crucial.

Dimensionless equations

Boundary conditions at $z = \epsilon\zeta$:

$$-c\phi_{,\theta} + \frac{1}{2}\epsilon(\phi_{,\theta}^2 + \phi_{,z}^2) + \zeta = 0 \quad (6)$$

$$-c\zeta_{,\theta} + \epsilon\zeta_{,\theta}\phi_{,\theta} = \phi_{,z} \quad (7)$$

Indices after comma: partial derivation.

In the bulk of the fluid $z < \epsilon\zeta$:

$$\phi_{,\theta\theta} + \phi_{,zz} = 0 \quad (8)$$

At infinite depth $z \rightarrow -\infty$:

$$\phi_{,\theta}, \phi_{,z} \rightarrow 0 \quad (9)$$

c is an explicit unknown (eigenvalue)

Form of solutions; normalization condition

Form of periodic solution

Periodic solution (period 2π due to scaling with k^*) \Rightarrow Fourier series for ζ

$$\zeta = b_0 + \sum_{j=1}^{\infty} b_j \cos(j\theta) + \sum_{j=1}^{\infty} e_j \sin(j\theta). \quad (10)$$

Averaged surface $\Rightarrow b_0 = 0$. Crest of the lowest harmonic at $\theta = 0$
 $\Rightarrow e_1 = 0$.

Unique definitions of a^* and $\epsilon \Rightarrow$ *normalization* condition

$$b_1 = 1$$

It will turn out that all e_j are zero.

Series for ϕ

$$\phi = U_0\theta + \sum_{j=1}^{\infty} F_j(z) \sin(j\theta) + \sum_{j=1}^{\infty} G_j(z) \cos(j\theta).$$

Each term must fulfill the Laplace equations independently $\Rightarrow F$ and G functions are exponentials.

Vanishing velocity for $z \rightarrow \infty \Rightarrow U_0 = 0$.

$$\phi = \sum_{j=1}^{\infty} e^{jz} \{B_j \sin(j\theta) + E_j \cos(j\theta)\}. \quad (11)$$

It will turn out that all E_j are zero.

The perturbation solution

Series expansion

Simultaneous expansion of ζ , ϕ and c ; The Poincare-Lindstedt method.

$$\zeta = \zeta_0(\theta) + \epsilon \zeta_1(\theta) + \epsilon^2 \zeta_2(\theta) + \dots \quad (12)$$

$$\phi = \phi_0(\theta, z) + \epsilon \phi_1(\theta, z) + \epsilon^2 \phi_2(\theta, z) + \dots \quad (13)$$

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \dots \quad (14)$$

Requirement: ϕ_i and ζ_i periodic in θ . Period must equal 2π .

Each ζ_j must conform to (10)

Since the equations for $z < 0$ are linear

$$\begin{aligned} \phi_{j,\theta\theta} + \phi_{j,zz} &= 0 \\ \phi_{j,\theta}, \phi_{j,z} &\rightarrow 0 \quad \text{for } z \rightarrow -\infty \end{aligned}$$

For all j . Taken care of by each ϕ_j conforming to (11)

The expansion and geometrical nonlinearity

Challenges

- 1 With expansion of c and nonlinearities: quite some book keeping.
- 2 Surface conditions apply at unknown position $z = \epsilon\zeta$.

Point 2: terms in surface conditions must be expanded to express the surface conditions in terms of field variables at $z = 0$. Example

$$\begin{aligned}\phi_{,\theta}(\theta, \epsilon\zeta) &= \phi_{,\theta}(\theta, 0) + \epsilon\phi_{,\theta z}(\theta, 0)\zeta + \frac{1}{2}\epsilon^2\phi_{,\theta zz}(\theta, 0)\zeta^2 + \dots \\ &= \phi_{0,\theta} + \epsilon(\phi_{0,\theta z}\zeta_0 + \phi_{1,\theta}) \\ &\quad + \epsilon^2\left(\frac{1}{2}\phi_{,\theta zz}\zeta_0^2 + \phi_{1,\theta z}\zeta_0 + \phi_{0,\theta z}\zeta_1 + \phi_{2,\theta}\right) + \dots\end{aligned}$$

where the argument $(\theta, 0)$ is implicit in the lower two lines.

Leading order; linear solution

Keeping $O(1)$ terms:

$$\begin{aligned} -c_0\phi_{0,\theta} + \zeta_0 &= 0, & \text{for } z = 0, \\ c_0\zeta_{0,\theta} + \phi_{0,z} &= 0, & \text{for } z = 0, \\ \phi_{0,\theta\theta} + \phi_{0,zz} &= 0, & \text{for } z < 0, \\ \phi_{0,\theta}, \phi_{0,z} &\rightarrow 0, & \text{for } z \rightarrow -\infty. \end{aligned} \tag{15}$$

Linear harmonic wave mode is reproduced in dimensionless form

$$\zeta_0 = \cos \theta, \quad \phi_0 = e^z \sin \theta, \quad c_0 = 1, \tag{16}$$

where the normalization condition has been applied.

Order ϵ^1

At $z = 0$:

$$\left. \begin{aligned} -c_0\phi_{1,\theta} + \zeta_1 &= c_1\phi_{0,\theta} - \frac{1}{2}(\phi_{0,\theta}^2 + \phi_{0,z}^2) + c_0\phi_{0,\theta z}\zeta_0 \equiv R_a, \\ c_0\zeta_{1,\theta} + \phi_{1,z} &= -c_1\zeta_{0,\theta} + \zeta_{0,\theta}\phi_{0,\theta} - \phi_{0,\theta z}\zeta_0 \equiv R_b. \end{aligned} \right\} \quad (17)$$

Elimination of ζ_1 and $c_0 = 1 \Rightarrow$

$$\phi_{1,\theta\theta} + \phi_{1,z} = R_b - R_{a,\theta} \quad \text{for } z = 0.$$

Insert expressions for ζ_0 , ϕ_0 in right hand side

$$\phi_{1,\theta\theta} + \phi_{1,z} = 2c_1 \sin \theta \quad \text{for } z = 0 \quad (18)$$

Solution of type (11) for ϕ_1

$$\phi_1 = \sum_{j=1}^{\infty} e^{jz} \{ B_j^{(1)} \sin(j\theta) + E_j^{(1)} \cos(j\theta) \},$$

is then inserted into (18).

Result

$$\sum_{j=1}^{\infty} (j - j^2) \{B_j^{(1)} \sin(j\theta) + E_j^{(1)} \cos(j\theta)\} = 2c_1 \sin \theta.$$

Uniqueness of Fourier series

- $c_1 = 0$ (terms with $j = 1$ vanish)
- $B_j^{(1)} = E_j^{(1)} = 0$ for $j > 1$.
- Normalization and crest at $\theta = 0 \Rightarrow B_1^{(1)} = E_1^{(1)} = 0$

Hence

$$c_1 = 0, \quad \phi_1 = 0$$

Then (17) \Rightarrow

$$\zeta_1 = \frac{1}{2} \cos 2\theta.$$

Order ϵ^2

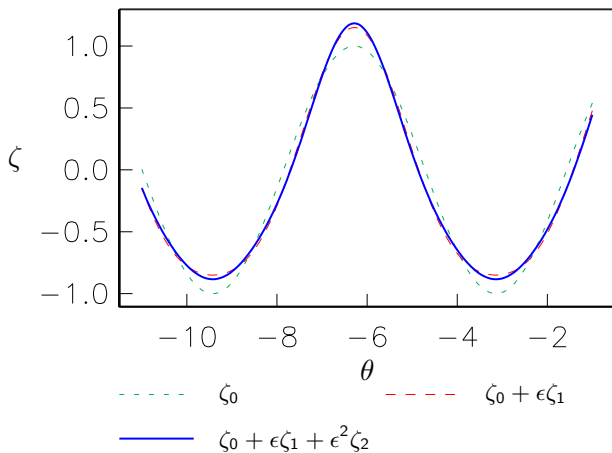
Same structure as for $O(\epsilon^1)$. Longer expressions, but $c_1 = 0$ and $\phi_1 = 0$ help.
Details omitted.

$$\zeta_2 = \frac{3}{8} \cos 3\theta, \quad \phi_2 = 0, \quad c_2 = \frac{1}{2}.$$

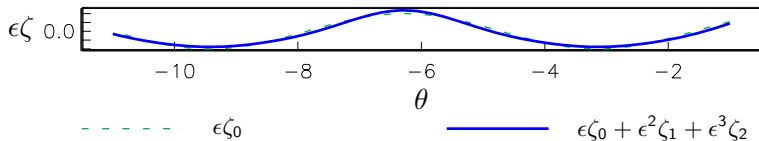
To this order: modification of celerity !
With dimensions restored

$$c^* = \sqrt{\frac{g}{k^*}} \left(1 + \frac{1}{2} (a^* k^*)^2 + \dots \right)$$

Wave shape; $\epsilon = 0.3$



True aspect ratio; $\epsilon = 0.3$



- Narrower crests, wider troughs.
- Total wave height increased by ζ_2 .
- Quite large ϵ . Still looks far from extreme.

Comments

Some properties and remarks

Stokes-drift:

$$Q^* = \frac{1}{T^*} \int_0^{T^*} \int_{-\infty}^{\zeta^*} u^* dz^* dt^* = \frac{1}{2k^*} \sqrt{\frac{g}{k^*}} \epsilon^2 + O(\epsilon^3) > 0.$$

There is net volume transport in the direction of wave advance.

Instabilities

- Benjamin & Feir (1967): The Stokes wave is always unstable in infinite depth.
- Combination of shorter and longer wavelength grow \Rightarrow modulations.
- Instability slow for small amplitudes.
- $\epsilon > 0.2$, say, span-wise instability \Rightarrow three dimensional motion.
- Stokes waves may be stable on finite depth.
- Much, much more.

Narrow band approximation. Modulation equations*

$$2\zeta = A(x, t)e^{i(kx - \omega t)} + \epsilon A_2(x, t)e^{2i(kx - \omega t)} + \dots + c.c.$$

where A , A_1 etc. display only slow variation with respect to x and t .

Some algebra \Rightarrow

$$i\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}|A|^2 A = 0 \quad (19)$$

where $\tau = \epsilon t$ and $\xi = \epsilon(x - c_g t)$. Here c_g is the linear group velocity for wave number k .

The cubic Schrödinger equation.