Long wave modelling

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Primitive and general hydrodynamic equations

The Navier-Stokes *(NS)* equation, primitive form

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} \equiv \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \mathcal{D} - g\vec{k}$$

$$\nabla\cdot\vec{v} = 0$$

where $\vec{v}$ = velocity, $\mathrm{D}/\mathrm{D}t$ = material derivative, p = pressure and $\mathcal{D}$ is viscous/turbulent term. In words:

acceleration = - pressure gradient + friction + gravity

net outflux from any fluid volume = 0

Boundary conditions: impermeable, no-slip, free (surface), artificial

Key problems: turbulence model, free surface tracking, under-resolved boundary layers, etc.

Generalization/alternatives to NS include multi-phase, multi-material...

Applicability of primitive models

- Computations readily become very heavy $\Rightarrow$ numerical solutions are under-resolved or unattainable
- Feasible only in local and idealized studies
- The burden of the computations often lead to wavering of the physics ?
- Analytic solutions are sparse, circumstantial and cumbersome

Surprisingly (?) little insight in hydrodynamic wave theory yet stem from "full computational models".

Simplified theories are still crucial, but general models become increasingly important

Full potential theory

Non-rotational motion $\Rightarrow$ potential ϕ: $\nabla\phi \equiv \vec{v}$

$$\nabla^2\phi = 0 \quad \text{for} \quad -h < z < \eta$$

Free surface ($z = \eta$) ($\frac{\mathrm{D}}{\mathrm{D}t} = \partial/\partial t + \vec{v}\cdot\nabla$)

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0, \quad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial\phi}{\partial z}$$

Bottom ($z = -h$)

$$\frac{\mathrm{D}h}{\mathrm{D}t} = 0 \Rightarrow \frac{\partial\phi}{\partial n} = 0$$

Not incorporable: viscous effects, turbulence, overturning waves, Coriolis force. Surface waves often well described.

Still heavy models, particularly in 3D

Large scale tsunami or ocean modeling requires approximative theory

Basis of approximations; Scales for surface gravity waves

Acceleration scale

g acceleration of gravity
Remark: scale for particle acceleration in gravity waves is always the same, regardless of size of problem

Length scales

λ wavelength	h depth	A amplitude (η)
L_h depth variations	L_λ variation of λ	

Often $L_h \sim L_\lambda$

Velocity and time scales

May be built from length and acceleration scales

Approximations; Regimes

- $\frac{A}{h}, \frac{A}{\lambda} \ll 1 \Rightarrow$ linear and weakly non-linear theories
- $\frac{\lambda}{h} \ll 1 \Rightarrow$ deep water
- $\frac{h}{\lambda} \ll 1 \Rightarrow$ **shallow water; long wave theory**
- $\frac{h}{L_h}, \frac{\lambda}{L_\lambda} \ll 1 \Rightarrow$ multiple scale methods: ray theory; narrow band (nearly uniform waves)

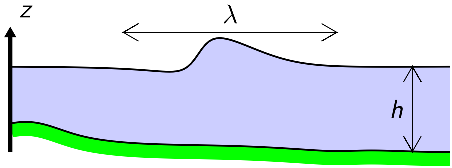
Different requirements may be combined; long wave theory is often combined with weak non-linearity.

Definition of characteristic scales may be vague or ambiguous

Tsunami and ocean modeling

Tools of the trade

- Depth integrated models for long waves
- Efficient and robust numerical techniques
- Ray tracing, wave kinematics



Long waves $\lambda/h \gg 1$ (2D case for simplicity)

U, W – characteristic horizontal and vertical velocities

$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{W}{h} \sim \frac{U}{\lambda} \Rightarrow \frac{W}{U} \sim \frac{h}{\lambda} \ll 1$

$\Rightarrow$ vertical motion is small $\Rightarrow$ pressure nearly hydrostatic

Nonlinear Shallow Water Equations

Vertical acceleration neglected $\Rightarrow$ hydrostatic pressure $\Rightarrow$ no vertical variation in horizontal velocity $\Rightarrow$ 3D physics, 2D maths.

NLSW

$$\frac{\partial \bar{\mathbf{v}}_h}{\partial t} + \bar{\mathbf{v}}_h \cdot \nabla_h \bar{\mathbf{v}}_h = -g \nabla_h \eta$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h + \eta)\bar{\mathbf{v}}_h)$$

η: surface elevation, $\bar{\mathbf{v}}_h$: velocity (horizontal), ∇_h: horizontal gradient operator

Efficient and simple numerical solution; hyperbolic equations. May include bores, Coriolis effects and bottom drag, *but wave dispersion is lost.*

Use: propagation of tsunamis, inundation. The dominant tsunami model (MOST at PTWS/NOAA, TUNAMI, COMCOT, CLAWPACK)

Dispersive long wave models

Models developed by perturbation/iteration/series expansion, assuming small
$\mu \equiv d/\lambda$ and $\epsilon \equiv A/d$, where d is typical depth (h).
Leading pressure modifications by vertical accelerations included.
Huge diversity in in formulations and accuracy

Boussinesq type models

- 1880$\longrightarrow$ Theoretical applications (KdV, KP...)
- 1990$\longrightarrow$ new formulations, increased validity range
- Increase of computer power $\Rightarrow$ large scale models feasible
- Important for some tsunami features
- Important model for coastal engineering
- A step toward more general models from (N)LSW; assessment of dispersion effects

Depth integrated theory; Boussinesq

Vertical acceleration small, but not neglected
$(\lambda/h > 2)$

$$\frac{\partial \bar{\mathbf{v}}_h}{\partial t} + \bar{\mathbf{v}}_h \cdot \nabla_h \bar{\mathbf{v}}_h = -g\nabla_h \eta + \frac{1}{2} h \nabla_h \nabla_h \cdot (h \frac{\partial \bar{\mathbf{v}}_h}{\partial t}) - \frac{1}{6} h^2 \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t}$$
$$-\kappa h^2 (\nabla_h^3 \eta + \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t})$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h + \eta)\bar{\mathbf{v}}_h)$$

Numerical solution much heavier than for shallow water eq. – implicit solution strategy needed. Still, much faster to solv than primitive equations. Wave dispersion included.

Use: propagation of moderately short tsunamis, particularly from non-earthquake sources. Coastal engineering. Wave theory.

Example; Boussinesq equations used at ICG/UiO/NGI

GloBouss: Generalization of standard Boussinesq equations

- Model developed for large scale dispersive tsunami simulations
- Enhanced linear dispersion, like FUNWAVE/COULWAVE
- Less general, but much simpler and efficient than FUNWAVE/COULWAVE
- Geographic coordinates: x -longitude, y -latitude
  u and v are corresponding velocity components
  Scaled and dimensionless equations
- Rotational effects (Coriolis) included

Continuity equation (identical with NLSW model)

$$c_\phi \frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\{(h+\epsilon\eta)u\} - \frac{\partial}{\partial y}\{c_\phi(h+\epsilon\eta)v\},$$

where $c_\phi = \cos\phi$ is a map factor

Momentum equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \epsilon\left(\frac{u}{c_\phi}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{1}{c_\phi}\frac{\partial \eta}{\partial x} + fv - \gamma\mu^2 h^2 \frac{1}{c_\phi}\frac{\partial D_\eta}{\partial x} \\
&\quad + \frac{\mu^2}{2}\frac{h}{c_\phi^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(c_\phi h\frac{\partial v}{\partial t}\right)\right] \\
&\quad - \mu^2\left(\frac{1}{6}+\gamma\right)\frac{h^2}{c_\phi^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(c_\phi\frac{\partial v}{\partial t}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} + \epsilon\left(\frac{u}{c_\phi}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= -\frac{\partial \eta}{\partial y} - fu - \gamma\mu^2 h^2 \frac{\partial D_\eta}{\partial y} \\
&\quad + \frac{\mu^2}{2} h\frac{\partial}{\partial y}\left[\frac{1}{c_\phi}\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial t}\right) + \frac{1}{c_\phi}\frac{\partial}{\partial y}\left(c_\phi h\frac{\partial v}{\partial t}\right)\right] \\
&\quad - \mu^2\left(\frac{1}{6}+\gamma\right)h^2\frac{\partial}{\partial y}\left[\frac{1}{c_\phi}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{1}{c_\phi}\frac{\partial}{\partial y}\left(c_\phi\frac{\partial v}{\partial t}\right)\right]
\end{aligned}
$$

LSW, with Coriolis terms (f), nonlinear terms, dispersion terms,
Dispersion correction terms, D_η is Laplacian of η
Lengthy appearance of equation, but structure well suited for numerical solution

Significant tsunamis after 2004

- 27 Mars, 2005, Sumatra
  $M = 8.7$: 4 m runup, casualties ?
- 17 July, 2006, Java
  $M = 7.8$: slow earthquake, 20 m runup, 700 perished
- 13 January , 2007, Kuril Islands
  $M = 8.1$: small waves
- 1 April, 2007, Solomon Islands
  $M = 8.1$: 52 dead
- 15 August, 2007, Peru
  $M = 8.0$: runup 10 m, destructions, 3 dead
- 13 September, 2008, South Sumatra
  $M = 8.5$: runup 4 m, damages

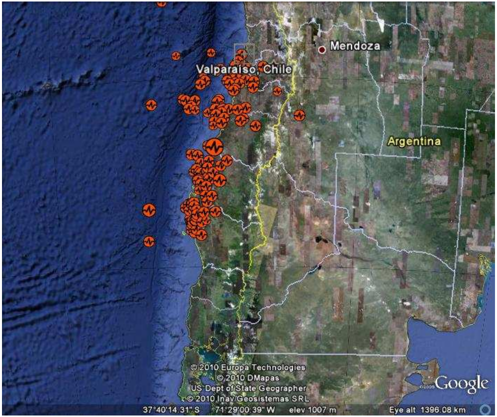
- 29 September, 2009, Samoa
  $M = 8.3$: 111 perished
- 3 January, 2010, Solomon Islands
  $M = 7.1$: 5-7 m runup
- 12 January, 2010, Haiti
  $M = 7.0$: 3 m runup, several casualties
- 27 February, 2010, Chile
  $M = 8.3$: 20 m runup, large destruction locally, 100 ? casualties
- 25 October, 2010, south Sumatra
  $M = 7.7$: 3 m (?) runup, ¿ 300 ? casualties

Chile 2010, 20m runup (USC Tsunami research center)

Tsunami research Dep. Math. UiO

- Experimental and theoretical studys since 1979.
- Participation in European research project 1992-1998.
- Membership in the center of excellence: International Centre for Geohazards (ICG) from 2003.
- Contributions: model development, verification, analysis, lab experiments, projects (Åkneset, EU projects ...)

Chile, February 27, 2010, $M = 8.8$

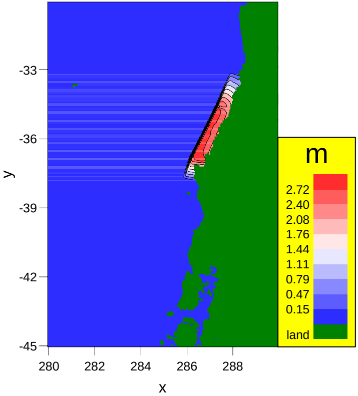


After-shocks

From Google-earth, February 28. Indicates location close to coast.

2010 event north of 1960 earthquake with $M = 9.5$

Source for the Chile 2010 event



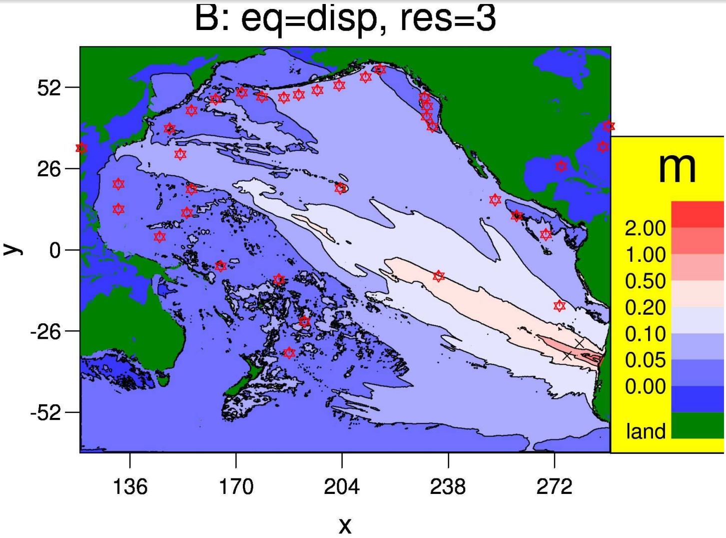
Observations

Destructive tsunami locally.
Most attention given to the earthquake.
Not a global tsunami.

Initial elevation (left)

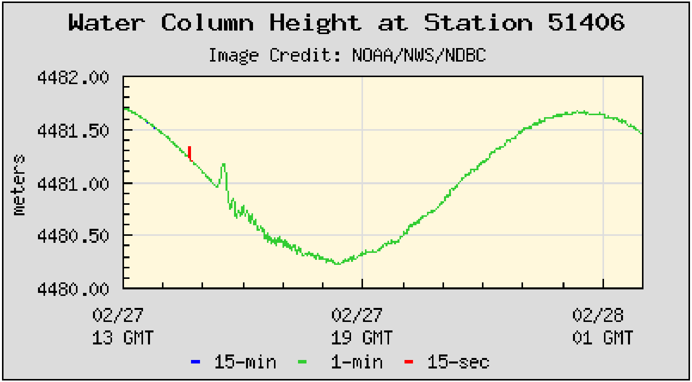
M is set to 8.8.
Source located near shore.
Based on Okada's formula.
Large extension.

Dispersive Globouss simulation, max. elevation



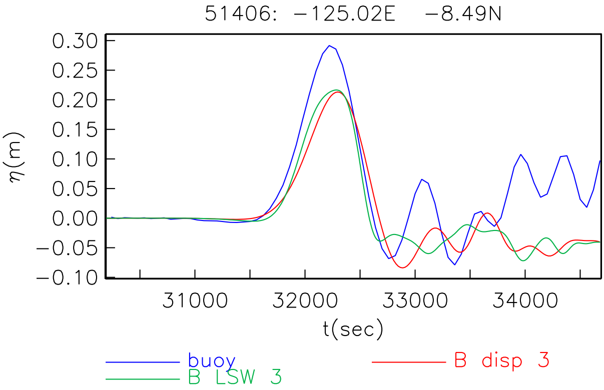
Dart buoy locations are marked with stars

Example of Dart buoy series from NOAA

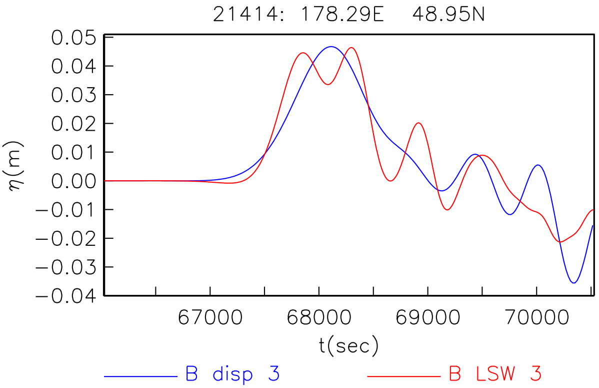


The tide and meteorological sea changes must be removed

Comparison with dart buoy



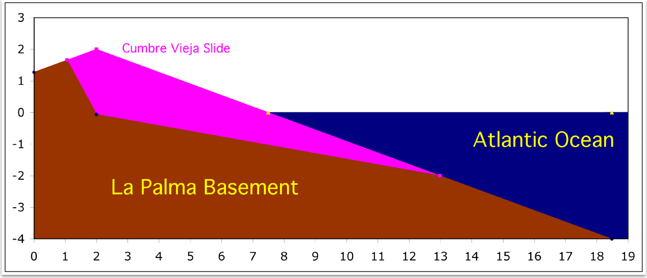
Estimated dispersion parameter $\tau = 7.4 \cdot 10^{-3}$; mild effect only
Good agreement. Difference corresponds to $\Delta M \approx 0.1 - 0.2$

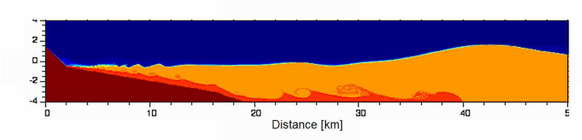


$\tau = 1.5 \cdot 10^{-2}$: effects of dispersion apparent, but still moderate. More complex times series. Tsunami signal from buoy not extracted yet.

Example on slide generated tsunami
Dispersion effect more enhanced

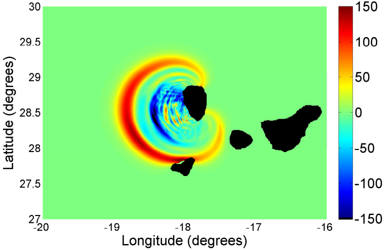
The potential La Palma event

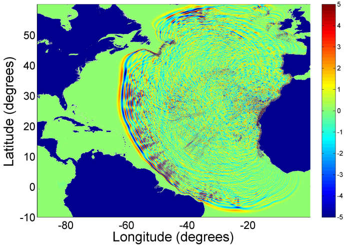




Definition sketch and 2D simulation.
The multi-material Sage model used for slide and generation

Oceanic propagation

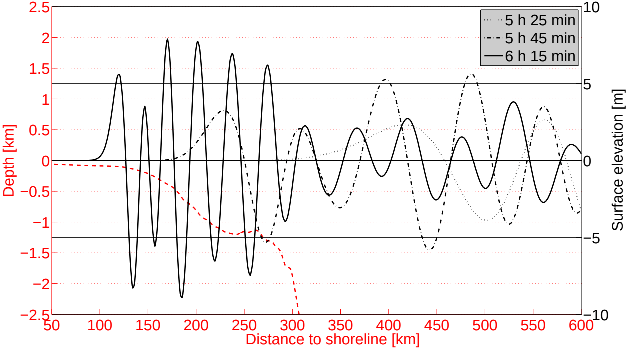




Surfaces after 10 min and 5h 45 min

This scenario would be a major disaster
Stronger effect of dispersion than for the seismic tsunamis
Small net volume $\Rightarrow$ leading crest not dominant

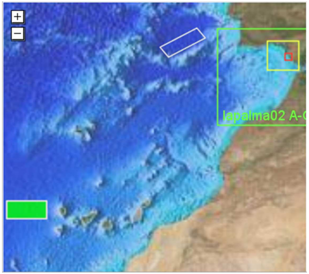
Toward Surinam

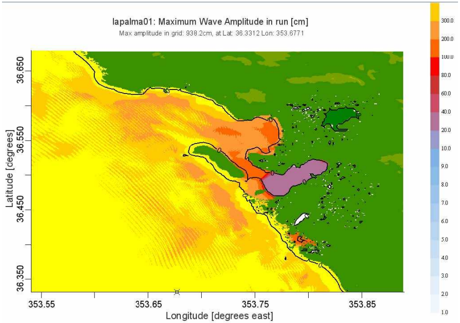


Amplification during shoaling

Small net volume in tsunami ⇒ leading crest not dominant due to dispersion

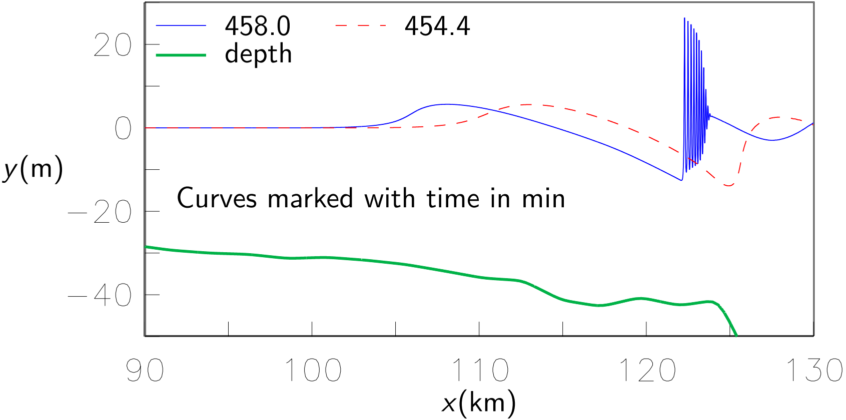
Inundation at Cadiz





GloBouss simulation input to nested 3 level simulation with standard model COMmit (NOAA), using NLSW equations

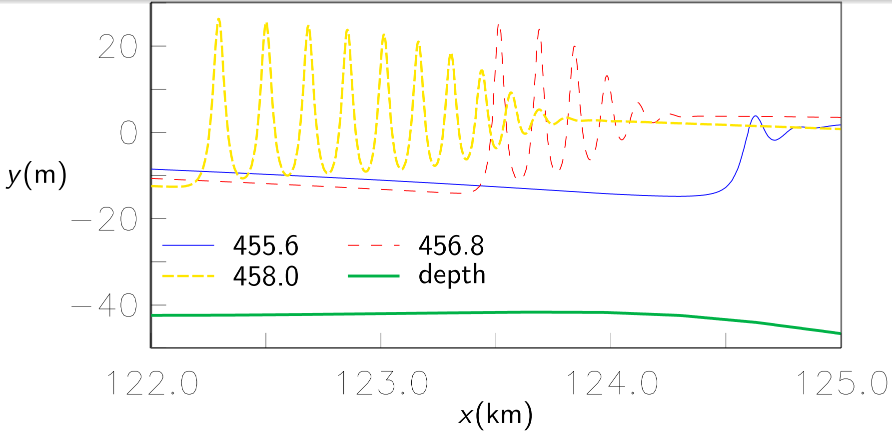
North American continental shelf



Second crest steeper and of higher amplitude than leading one (masked by shoaling)

Second crest at the shelf: some phenomenon at the front

Closeup



Undular bore: fission into solitary waves, doubling of wave height, will break when A/h reach 0.7

Undular bore observed also for seismic tsunamis, not included in NLSW models

Undular bore in river Severn

Tides may produce undular bores in rivers and estuaries (Picture: P. Rash)

Mascaret in Gironde, Bordeaux

Tidal, undular bore used for water sports
(Picture: J. Dassié)