

Multiple scale methods

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1 Background

Many physical problems involve more than one temporal or spatial scale. One important example is the boundary layer problem, where we recognize one fast and one slow scale. In that case the two scales are separated, in the sense that fast scale only affects the boundary layer, while the outer solution is governed by the slow scale alone. In fact, the slow scale is also present in the boundary layer, but influences only the higher order approximations that we do not address in Mek 3100. In boundary layer theory we exploit the dominance of the fast scale in the boundary layer to find an approximate inner solution. The inner fast solution and the outer slow solution is then matched to provide a complete approximation.

In other problems both the slow and fast scales may present everywhere. The scales may have different physical origins, such as in the first example below where a fast oscillation is subjected to a slow damping. Another example is propagation of short waves in a slowly varying medium, where the fast scale corresponds to the wavelength while the slow scale is associated with the variation of the medium. In other cases the slow scale may be due to gentle nonlinear effects that modify periods or rates of change for the system. Subsequently we will retrieve the periodic solution of a nonlinear pendulum, that has previously been found by Poincare-Lindsted's method, by a two scale perturbation expansion.

In a multiple scale technique we introduce several time, or space, variables, that are scaled differently and regarded as being independent. As for other perturbation methods, this one is best outlined through examples.

For linear problems we have a particularly powerful and simple version of multiple scale expansions called the WKB(J) method. This is a topic in other courses and will not be described herein.

2 Example: oscillation with mild damping

We consider a scaled equation that governs oscillations

$$\frac{d^2y}{dt^2} + \epsilon \frac{dy}{dt} + y = 0; \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 0, \quad (1)$$

where ϵ is a small parameter. The second term on the left hand side can be interpreted as a resistance term, proportional to the velocity, that acts on a mathematical pendulum.

2.1 Failure of the direct approach

We substitute the naive expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots, \quad (2)$$

into and (1) and collect powers of ϵ .

$O(\epsilon^0)$

$$\frac{d^2 y_0}{dt^2} + y_0 = 0; \quad y_0(0) = 1, \quad \frac{dy_0(0)}{dt} = 0. \quad (3)$$

The solution is

$$y_0 = \cos t$$

$O(\epsilon^1)$

$$\frac{d^2 y_1}{dt^2} + y_1 = -\frac{dy_0}{dt} = \sin t; \quad y_1(0) = \frac{dy_1(0)}{dt} = 0. \quad (4)$$

We observe that the right hand side in the differential equation corresponds to resonance and yields secular terms. The solution for y_1 is

$$y_1 = \frac{1}{2}(\sin t - t \cos t).$$

Since the last term in y_1 grows linearly in time, ϵy_1 will become comparable with y_0 for large $t \sim 1/\epsilon$. Hence, the first terms of the expansion (2) provide a local (small t) approximation, at most. The shortcoming of (2) is related to the breakdown of the straightforward approach on nonlinear perturbation problems, but is more transparent to explanation. The small resistance term in (1) will slowly, but accumulative, absorb energy and damp the motion. Hence, even though the term itself is small the long term effect is crucial and the solution cannot be described as being periodic plus a small correction. The consequence for a naive expansion, like (2), must be that the ordering requirement ($y_0 \gg \epsilon y_1 \gg \dots$) is violated.

It is quite obvious that Poincare-Lindsted's method is of no avail in this case. However, it may be instructive to try and fail. If we introduce $\bar{t} = \omega t$, where $\omega = \omega_0 + \epsilon \omega_1 + \dots$, there will be no value of ω_1 for which the secular term vanishes. Details are left to the reader.

2.2 The two-scale expansion

In the present problem we may recognize a fast time scale for the oscillation and a slow time scale for the damping. We introduce the latter scale according to the new temporal variable¹

$$\tau = \epsilon t. \quad (5)$$

We will now consider the fast scale, t , and the slow scale, τ , as independent variables. The total time derivative is then transformed according to

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}. \quad (6)$$

¹A priori it is far from obvious that the slow scale is ϵt , but it is a reasonable first attempt.

Substituting these identities into (1) we obtain

$$\frac{\partial^2 y}{\partial t^2} + y + \epsilon(2\frac{\partial^2 y}{\partial t \partial \tau} + \frac{\partial y}{\partial t}) + \epsilon^2(\frac{\partial^2 y}{\partial \tau^2} + \frac{\partial y}{\partial \tau}) = 0; \quad (7)$$

$$y(0, 0) = 1, \quad \frac{\partial y(0, 0)}{\partial t} + \epsilon \frac{\partial y(0, 0)}{\partial \tau} = 0. \quad (8)$$

At this state no progress is apparent. On the contrary, the splitting of the time variable has turned an ordinary differential equation into a partial one that also looks more messy. However, the point is that (7) is better suited for a perturbation approach, in spite of its more complex form and that the solution of (7) contains all the solutions of (1). The perturbation series

$$y = y_0(t, \tau) + \epsilon y_1(t, \tau) + \dots,$$

now yields the perturbation hierarchy

$O(\epsilon^0)$

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0; \quad y_0(0, 0) = 1, \quad \frac{\partial y_0(0, 0)}{\partial t} = 0. \quad (9)$$

To this order only derivatives with respect to the fast variable appears. The slow variable τ is implicit in the constants of integration

$$y_0 = A_0(\tau) \cos t + B_0(\tau) \sin t, \quad A_0(0) = 1, \quad B_0(0) = 0 \quad (10)$$

$O(\epsilon^1)$

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} = (A_0 + 2\frac{dA_0}{d\tau}) \sin t - (B_0 + 2\frac{dB_0}{d\tau}) \cos t; \quad (11)$$

$$y_1(0, 0) = 0, \quad \frac{\partial y_1(0, 0)}{\partial t} = -\frac{\partial y_0(0, 0)}{\partial \tau}. \quad (12)$$

To avoid secular terms we must require

$$A_0 + 2\frac{dA_0}{d\tau} = B_0 + 2\frac{dB_0}{d\tau} = 0.$$

Combination with the last two relations of (10) yields

$$A_0 = e^{-\frac{1}{2}\tau}, \quad B_0 = 0.$$

There is no particular solution to $O(\epsilon)$, while the homogeneous solution becomes

$$y_1 = A_1(\tau) \cos t + B_1(\tau) \sin t, \quad A_1(0) = 0, \quad B_1(0) = \frac{1}{2}$$

After collection of all contributions and insertion of $\tau = \epsilon t$ the solution reads

$$y = e^{-\frac{1}{2}\epsilon t} \cos t + \epsilon(A_1(\epsilon t) \cos t + B_1(\epsilon t) \sin t) + O(\epsilon^2). \quad (13)$$

To next order we find $B_1 \sim \tau e^{-\frac{1}{2}\tau} + \dots$. Even though y_1 is finite it will not remain small in relation to y_0 . We *may* interpret this as the presence of a kind of secular terms. Anyway, the perturbation expansion to $O(\epsilon)$ must be expected to be local ($\epsilon t \ll 1$). For this particular problem we may contemplate on combining the two-scale technique with a Poincare-Lindsted approach. However, in general we must introduce a second order slow scale $\tau_1 = \epsilon^2 t$ and express the solution as $y(t, \tau, \tau_1)$. In principle, this may be continued to any order, but multiple scale schemes are generally not brought beyond the leading order.

2.3 The exact solution

The solution of (1) reads

$$y = e^{-\frac{1}{2}\epsilon t} \left(\cos \omega t + \frac{\epsilon}{2\omega} \sin \omega t \right),$$

where $\omega = \sqrt{1 - \frac{1}{4}\epsilon^2} = 1 + O(\epsilon)^2$. We observe that (13) is correct, including $B_1(0) = \frac{1}{2}$.

3 Example: nonlinear oscillation

In dimensionless form the equation for a pendulum, with the leading nonlinearity, can be written

$$\frac{d^2 x}{dt^2} + x - \frac{\epsilon}{6} x^3 = 0, \quad x(0) = 1, \quad \frac{dx(0)}{dt} = 0. \quad (14)$$

Naturally, this is a problem that is typically treated with the Poincare-Lindsted method, where a periodic solution with frequency $\omega = \omega_0 + \epsilon \omega_1 + \dots$ is anticipated. However, we may also regard $\epsilon \omega_1 t$ as a slow scale associated with the mild nonlinear modulation of the phase. In fact, the Poincare-Lindsted method may be regarded as a particularly simple version of a multiple scale method, for which all the slow scales are absorbed in the phase through the expansion of a single frequency.

3.1 The two-scale perturbation

As in the previous example we introduce $\tau = \epsilon t$ and (14) yields

$$\begin{aligned} \frac{\partial^2 x}{\partial t^2} + x + \epsilon \left(2 \frac{\partial^2 x}{\partial t \partial \tau} - \frac{1}{6} x^3 \right) + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} &= 0; \\ x(0, 0) &= 1, \quad \frac{\partial x(0, 0)}{\partial t} + \epsilon \frac{\partial x(0, 0)}{\partial \tau} = 0. \end{aligned} \quad (15)$$

Inserting the series

$$x = x_0(t, \tau) + \epsilon x_1(t, \tau) + \dots,$$

we then find

$$O(\epsilon^0)$$

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \quad x_0(0, 0) = 1, \quad \frac{\partial x_0(0, 0)}{\partial t} = 0. \quad (16)$$

It is now convenient to work with the solution in exponential form

$$x_0 = A_0(\tau)e^{it} + \bar{A}_0(\tau)e^{-it}, \quad A_0(0) = \frac{1}{2}, \quad (17)$$

where \bar{A}_0 is the complex conjugate of A_0 .

$O(\epsilon^1)$

$$\begin{aligned} \frac{\partial^2 x_1}{\partial t^2} + x_1 &= \frac{1}{6}x_0^3 - 2\frac{\partial^2 x_0}{\partial t \partial \tau} \\ &= \frac{1}{6}A_0^3 e^{3it} + \left(-2i\frac{dA_0}{d\tau} + \frac{1}{2}\bar{A}_0 A_0^2\right) e^{it} + \text{c.c.}, \end{aligned} \quad (18)$$

$$x_1(0, 0) = 0, \quad \frac{\partial x_1(0, 0)}{\partial t} = -\frac{\partial x_0(0, 0)}{\partial \tau}, \quad (19)$$

where c.c. indicates that the part of the right hand side that is explicit is complex conjugated and then added. Elimination of secular terms implies For

$$i\frac{dA_0}{d\tau} - \frac{1}{4}\bar{A}_0 A_0^2 = 0.$$

This differential equation in τ may, for instance, be solved by invoking the polar form $A_0 = |A_0|e^{i\psi}$. The real and the imaginary parts of the differential equation then imply

$$\frac{d|A_0|}{d\tau} = 0, \quad \frac{d\psi}{d\tau} = -\frac{1}{4}|A_0|^2.$$

Invoking the initial condition for A_0 we obtain

$$A_0 = \frac{1}{2}e^{-\frac{i}{16}\tau}.$$

The solution for x_1 becomes

$$x_1 = -\frac{A_0^3}{48}e^{3it} + A_1(\tau)e^{it} + \text{c.c.}.$$

The initial conditions to this order then yield $A_1(0) = 1/384$.

Putting everything together we arrive at

$$x = \frac{1}{2}e^{i(1-\frac{\epsilon}{16})t} - \frac{\epsilon}{384}e^{3i(1-\frac{\epsilon}{16})t} + \epsilon A_1(\tau)e^{it} + \text{c.c.} + O(\epsilon^2) \quad (20)$$

$$= \cos(1 - \frac{\epsilon}{16})t - \frac{\epsilon}{192} \cos 3(1 - \frac{\epsilon}{16})t + \epsilon a_1 \cos t - \epsilon b_1 \sin t + O(\epsilon^2), \quad (21)$$

where a_1, b_1 are the real and imaginary parts of A_1 , respectively. It is left to reader to prove that this is consistent with the result from Poincare-Lindsted's method.

4 Pendulum of non-constant length

There are numerous other relevant examples to two-scale expansions. One is boundary layer problems, where the application of the technique is ambiguous. A favourable specification may then produce a global solution without any explicit matching. On the other hand, the mathematical structure of the boundary layer solution becomes less transparent than with the standard boundary layer expansion. Hence, we put that example aside. Instead we present another problem that presents new mathematical as well as physical aspects.

A mathematical pendulum is attached to its support by a flexible rod with negligible mass. The length of the rod is assumed to be a prescribed function of time. The equation for angular momentum, around the support, yields²

$$\ell\ddot{\phi} + 2\dot{\ell}\dot{\phi} + g\phi = 0, \quad (22)$$

where ϕ is the angle of the pendulum, ℓ is the length and the dot denotes differentiation with respect to time t^* . Dimensions are removed by the transformation

$$t = \sqrt{\frac{g}{\ell(0)}}t^*, \quad \gamma = \frac{\ell}{\ell(0)}, \quad \theta = \frac{\phi}{\phi_c}.$$

Moreover, we introduce a slow scale $\tau = \epsilon t$ owing to rate of temporal change of ℓ . The equation of motion then becomes

$$\gamma(\tau)\frac{d^2\theta}{d\tau^2} + 2\epsilon\frac{d\gamma}{d\tau}\frac{d\theta}{d\tau} + \theta = 0. \quad (23)$$

As initial conditions we choose

$$\theta(0) = 1, \quad \frac{d\theta}{d\tau} = 0.$$

4.1 Attempt: direct application of two scales

When the slow scale is invoked also in θ we obtain

$$\gamma(\tau)\frac{\partial^2\theta}{\partial\tau^2} + \theta + 2\epsilon\left(\frac{d\gamma}{d\tau}\frac{\partial\theta}{\partial\tau} + \gamma\frac{\partial^2\theta}{\partial\tau\partial t}\right) + \epsilon^2\left(2\frac{d\gamma}{d\tau}\frac{\partial\theta}{\partial\tau} + \gamma\frac{\partial^2\theta}{\partial\tau^2}\right) = 0. \quad (24)$$

The perturbation expansion $\theta = \theta_0(t, \tau) + \epsilon\theta_1(t, \tau) + \dots$ yields

$\mathcal{O}(\epsilon^0)$

$$\gamma\frac{\partial^2\theta_0}{\partial\tau^2} + \theta_0 = 0; \quad \theta_0(0, 0) = 1, \quad \frac{\partial\theta_0(0, 0)}{\partial\tau} = 0. \quad (25)$$

The solution reads

$$\theta_0 = A_0(\tau)e^{i\gamma^{-\frac{1}{2}}t} + \overline{A_0}e^{-i\gamma^{-\frac{1}{2}}t}, \quad A_0(0) = \frac{1}{2}, \quad (26)$$

²Mechanical energy is not conserved due to the action of whatever device causing the variation of the length of the rod.

$O(\epsilon^1)$

$$\gamma \frac{\partial^2 \theta_1}{\partial t^2} + \theta_1 = h_s; \quad \theta_1(0) = 0, \quad \frac{\partial \theta_1(0)}{\partial t} = -\frac{\partial \theta_0}{\partial \tau}, \quad (27)$$

where the right hand side is

$$h_s = -2\gamma \frac{\partial^2 \theta_0}{\partial t \partial \tau} - 2 \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial t} = - \left(2i\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + i\gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} + 2tA_0\gamma^{-1} \frac{\partial \gamma}{\partial \tau} \right) e^{i\gamma^{-\frac{1}{2}}t} + \text{c.c.} \quad (28)$$

In the last term the fast variable t appears. We cannot remove this term by means of selecting any particular $A_0(\tau)$ and, hence, the solution for θ_1 contains a secular term proportional to t^2 .

This attempt clearly failed. The solution found is locally valid for small times, at best.

4.2 Modified two-scale method

The reason for the failure of the previous approach is quite obvious. Provided $\gamma(\ell)$ varies sufficiently slowly the solution will become nearly periodic at all times. However, the period is dependent on the length of the pendulum and is thus non-constant. This reflects in the appearance of τ in the exponents of the expression (26) for θ_0 .

We need a fast time scale with slow variations and introduce a new fast scale, T , through

$$\frac{dT}{dt} = \sigma(\tau).$$

This relation implies a nearly linear relation between T and t . The transformation of the temporal derivatives now becomes

$$\frac{d}{dt} = \sigma \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{d^2}{dt^2} = \sigma^2 \frac{\partial^2}{\partial T^2} + \epsilon(2\sigma \frac{\partial^2}{\partial T \partial \tau} + \frac{d\sigma}{d\tau} \frac{\partial}{\partial T}) + \epsilon^2 \frac{\partial^2}{\partial \tau^2}. \quad (29)$$

The objective is to choose, or rather determine, the factor σ as to avoid secular terms. The transformed equation reads

$$\gamma \sigma^2 \frac{\partial^2 \theta}{\partial T^2} + \theta + \epsilon \left(2\sigma \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial T} + 2\sigma \gamma \frac{\partial^2 \theta}{\partial T \partial \tau} + \gamma \frac{d\sigma}{d\tau} \frac{\partial \theta}{\partial T} \right) + \epsilon^2 \left(2 \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial \tau} + \gamma \frac{\partial^2 \theta}{\partial \tau^2} \right) = 0. \quad (30)$$

Substitution of the perturbation series then yields

$O(\epsilon^0)$

$$\gamma \sigma^2 \frac{\partial^2 \theta_0}{\partial T^2} + \theta_0 = 0; \quad \theta_0(0,0) = 1, \quad \frac{\partial \theta_0(0,0)}{\partial T} = 0. \quad (31)$$

To avoid linear terms in T in the right hand side to the next order we must avoid explicit presence of the slow scale in the exponents (phases) of the solution for θ_0 . This is achieved by setting $\sigma = \gamma^{-\frac{1}{2}}$. The solution then becomes

$$\theta_0 = A_0(\tau)e^{iT} + \bar{A}_0(\tau)e^{-iT}, \quad A_0(0) = \frac{1}{2}, \quad (32)$$

$O(\epsilon^1)$

Now we obtain the modified equation

$$\frac{\partial^2 \theta_1}{\partial T^2} + \theta_1 = -2\sigma\gamma \frac{\partial^2 \theta_0}{\partial T \partial \tau} - 2\sigma \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial T} - \gamma \frac{d\sigma}{d\tau} \frac{\partial \theta_0}{\partial T} \equiv h_s, \quad (33)$$

where

$$\begin{aligned} h_s &= -i \left(2\sigma\gamma \frac{dA_0}{d\tau} + 2\sigma A_0 \frac{d\gamma}{d\tau} + \gamma \frac{d\sigma}{d\tau} A_0 \right) e^{iT} + \text{c.c.} \\ &= -i \left(2\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + \frac{3}{2} \gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} \right) e^{iT} + \text{c.c.} \end{aligned}$$

Secular terms are avoided when the contents of the parentheses is zero. This requires that A_0 solves a separable differential equation that is readily integrated

$$A_0 \gamma^{\frac{3}{4}} = \text{const.} \quad (34)$$

Since γ is prescribed we have now obtained A_0 as a function of τ and the leading behaviour of the solution is established.

Reinsertion of t , instead of T , is obtained by substituting

$$T = \int_0^t \gamma^{-\frac{1}{2}} dt = \frac{1}{\epsilon} \int_0^{\epsilon t} \gamma^{-\frac{1}{2}} d\tau,$$

in the expression for θ_0 .

4.3 Important note

A pendulum of variable length is a simple representative for oscillation/wave systems with properties that vary in time and space. We shall thus make a physical interpretation of our two-scale solution.

The energy of the pendulum may be divided in one part, E_s , that is due to oscillatory motion and another, E_ℓ , that is the change in energy due to the varying length of the pendulum

$$E = E_s + E_\ell, \quad (35)$$

$$E_\ell = \frac{1}{2} m \dot{\ell}^2 - mg\ell, \quad (36)$$

$$E_s = \frac{1}{2} m \ell^2 \dot{\phi}^2 + mg\ell(1 - \cos \phi), \quad (37)$$

Since we assume small amplitude motion it is consistent with (22) to do the approximation $1 - \cos \phi = \frac{1}{2} \phi^2$. By introducing non-dimensional quantities and replacing θ by θ_0 we obtain

$$E_s = 2mg\ell_0 \phi_c^2 \gamma A_0^2 (1 + O(\epsilon)). \quad (38)$$

It is obvious that (34) do not imply a constant E_s . However, if we introduce the local (in time) frequency $\omega = \sqrt{g/\ell}$ the relations (34) and (38) imply

$$\frac{E_s}{\omega} = \text{const.} \quad (39)$$

The quantity E_s/ω is named *wave action* and its conservation is a general property for oscillatory systems with slow variation of the governing characteristics.