A Boussinesq model for educational purposes

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1 Scaling and equations

Marking dimensional quantities by a star we introduce a coordinate system with the horizontal axis, $ox^\star$, in the undisturbed water level and $oz^\star$ pointing vertically upward. Moreover, we assume a bottom at $z^\star = -h^\star$ and denote the surface elevation and averaged horizontal particle velocity by and $\eta^\star$ and $u^\star$ respectively. Applying the maximum depth, h_0, and a characteristic wavelength, L, as "vertical" and "horizontal" length scales we are then led to the following definition of non-dimensional variables

$$\left.\begin{array}{l} x^\star = L^\star x, \quad t^\star = L^\star (gh_0^\star)^{-\frac{1}{2}} t, \quad \eta^\star = \alpha h_0^\star \eta, \\ z^\star = h_0^\star z, \quad u^\star = \alpha (gh_0^\star)^{\frac{1}{2}} u, \end{array}\right\} \tag{1}$$

where g is the acceleration of gravity and α is an amplitude measure. In computer programs and graphics the above scaling is generally inconvenient. Hence, a different scaling is used in the FORTRAN program that corresponds to replacing α by 1 and L by h_0 in (1).

Long wave approximations may be developed as expansions in the parameters α and $\beta \equiv (h_0/L)^2$. Keeping all terms in α and retaining leading orders in β [11, 12] we arrive at

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left((h + \alpha\eta)\overline{u}\right) \tag{2}$$

$$\frac{\partial \overline{u}}{\partial t} + \frac{1}{2}\alpha\frac{\partial \overline{u}^2}{\partial x} = -\frac{\partial \eta}{\partial x} + \frac{1}{2}\beta h \frac{\partial^2}{\partial x^2}\left(h\frac{\partial \overline{u}}{\partial t}\right) - \frac{1}{6}\beta h^2 \frac{\partial^3 \overline{u}}{\partial^2 x \partial t}, \tag{3}$$

where $\overline{u}$ is the vertical average of u. This set is often referred to as the standard Boussinesq equations. When we delete the $O(\beta)$ terms the Boussinesq equations simplify to the nonlinear shallow water equations. On the other hand, improvements over the standard Boussinesq equations are also available, where more terms are kept or the $O(\beta)$ terms are rewritten in a favourable fashion[7, 10, 2, 8, 9, 5, 3, 6, 4]. A newer review on Boussinesq type equations is given in [1].

2 The solitary wave solution

Solitary waves often play an important role in nonlinear and dispersive long wave theory. For $h \equiv 1$ the simplest solution reads

$$\eta = \alpha \mathrm{Sec}^2(k(x - ct)), \quad k = \sqrt{\frac{3\alpha}{4}}, \quad c\left(1 + \frac{1}{2}\alpha\right), \tag{4}$$

where α is chosen as to give a scaled maximum for η equal to unity. However, this solution is *exact only for the KdV equation* and the employment of (4) as initial condition in the Boussinesq equations will lead to shape adjustments and evolution of (small) residual wave trains that may be confused with numerical errors. Thus, an exact numerical solitary wave solution of the Boussinesq equations is used in the program. The details are not particularly interesting and are thus omitted.

3 Numerical method

When $\alpha, \beta \to 0$ the equations (2,3) simplify to

$$\frac{\partial \eta}{\partial t} = -\frac{\partial (hu)}{\partial x}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x}$$

To solve this set numerically we employ a staggered grid (figure 1) and define discrete unknowns according to

$$\eta^{(n)}_{j-\frac{1}{2}} \approx \eta((j-\frac{1}{2})\Delta x, n\Delta t), \quad u^{(n+\frac{1}{2})}_j \approx u(j\Delta x, (n+\frac{1}{2})\Delta t),$$

where Δx and Δt are the grid increments. Replacing derivatives by differences we then obtain

$$\frac{\eta^{(n)}_{j-\frac{1}{2}} - \eta^{(n-1)}_{j-\frac{1}{2}}}{\Delta t} = -\frac{h_j u^{(n-\frac{1}{2})}_j - h_{j-1} u^{(n-\frac{1}{2})}_{j-1}}{\Delta x} \qquad \text{(i)}$$

$$\frac{u^{(n+\frac{1}{2})}_j - u^{(n-\frac{1}{2})}_j}{\Delta t} = -\frac{\eta^{(n)}_{j+\frac{1}{2}} - \eta^{(n)}_{j-\frac{1}{2}}}{\Delta x} \qquad \text{(ii)}$$

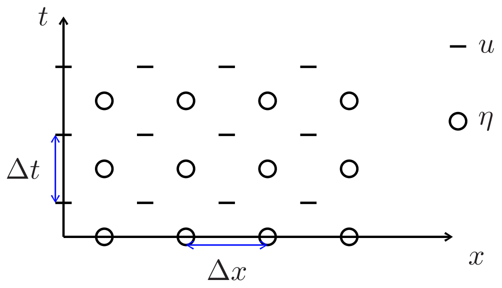
In the advancement of the solution from $t = (n-\frac{1}{2})\Delta t$ to $t = (n+\frac{1}{2})\Delta t$ (i) yields η at $t = n\Delta t$, point by point. Then (ii) yields u at $t = (n+1)\Delta t$. The result is an extremely simple, explicit method.

If the discrete versions of the Boussinesq equations are spelled out in a corresponding manner we obtain lengthy expressions that are hard to read. Hence we employ a little formalism. The numerical approximation to a quantity f at a grid-point with coordinates $(\beta\Delta x, \kappa\Delta t)$ is denoted by $f^{(\kappa)}_\beta$. We build the discrete Boussinesq equations primarily by replacing the unknowns by their discrete counterparts, or averages thereof, and derivatives by finite differences. To make the difference equations more compact and legible we introduce the symmetric difference and average operators, δ_x and $^{-x}$ by

$$\delta_x f^{(\kappa)}_{\beta,\gamma} = \frac{1}{\Delta x}(f^{(\kappa)}_{\beta+\frac{1}{2},\gamma} - f^{(\kappa)}_{\beta-\frac{1}{2},\gamma}), \quad (\overline{f}^x)^{(\kappa)}_{\beta,\gamma} = \frac{1}{2}(f^{(\kappa)}_{\beta+\frac{1}{2},\gamma} + f^{(\kappa)}_{\beta-\frac{1}{2},\gamma}). \qquad (5)$$

We note that the differences and averages are defined at intermediate grid locations as compared to f. Difference and average operators with respect t are defined correspondingly. It is easily shown that these operators are commutative in all linear combinations. To abbreviate the expressions further we also group terms of identical indices inside square brackets, leaving the super- and subscripts outside the right bracket. In nonlinear terms it is also convenient to introduce a special notation for the squared temporal geometrical mean

$$[f^{(*2)}]^{(n)} \equiv f^{(n+\frac{1}{2})} f^{(n-\frac{1}{2})}. \qquad (6)$$



Figur 1: A staggered grid for solution of the Boussinesq equations

The above method for the linearized shallow water equations may then be generalized to the Boussinesq equations according to

$$[\delta_t \eta = -\delta_x (h + \alpha \overline{\eta}^{xt}) u)]_{j+\frac{1}{2}}^{(n-\frac{1}{2})}, \tag{7}$$

$$\left[\delta_t u + \frac{1}{2}\alpha \delta_x \overline{u^{(*2)}}^x = -\delta_x \eta + \frac{1}{2}\beta h \delta_x^2 (h\delta_t u) - \frac{1}{6}\beta h^2 \delta_x^2 \delta_t u + C\right]_j^{(n)}, \tag{8}$$

where we have omitted the bar over u and C is a correction term that is described below. Now the method has become implicit. The discrete continuity and momentum equations give tri-diagonal linear sets of equations to be solved for the new η and u values respectively. The purpose of the geometric mean is to avoid nonlinearities in implicit terms. This representation of the convective term works well for the Boussinesq equations, but is probably less well suited for the non-linear shallow water equations. The method (7,8) is of second order accuracy in the grid increments. The correction term

$$C = \frac{1}{12}\left((\Delta t^2 - \frac{\Delta x^2}{2h})\delta_x^2(h\delta_t u) - \frac{\Delta x^2}{2}\delta_x^2 \delta_t u\right),$$

essentially removes the second order error terms from the "linear shallow water part" of the equations. Hence, it will give improved dispersion characteristics when Δx (and $\sqrt{h}\Delta t$) are comparable to h. We observe that C vanish when h is constant and the Courant number, $C_r = \sqrt{h}\Delta t/\Delta x$, is unity.

When initial conditions for η and u are available at $t = 0$ a first order error is introduced if the latter is used directly for $u_j^{\frac{1}{2}}$. Thus (8) is first used to advance the velocity half a time step forward, before the first normal computational cycle is started. At lateral boundaries, at say $j = 0, N$, no-flux conditions are used: $u_0 = u_N = 0$. Scrutinizing the discrete equations we then realize that no fictitious values for η are needed. For instance, the quantity $\eta_{-\frac{1}{2}}$ that formally appears in (7) at $j = \frac{1}{2}$ is multiplied by u_0 that is zero.

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