Forced Pendulum
Numerical approach

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Physical problem and equation

We have a pendulum of length l, with mass m. The pendulum is subject to gravitation as well as both a forcing and linear resistance force. We denote the acceleration of gravity as g, and the angle of the pendulum as θ. The forcing has magnitude F and angular frequency ω, while α is a measure of the resistance force. This gives us the following equation of motion:

$$ml\frac{d^2\theta}{dt^2} + \alpha l\frac{d\theta}{dt} + mg\sin(\theta) = F\cos(\omega t)$$

Scaling

$$\frac{d^2\theta}{dt^2} + \frac{\alpha}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = \frac{F}{ml}\cos(\omega t)$$

We assume that the forcing is near the resonance frequency:

$$\sqrt{g/l} = \omega + \Delta\omega \qquad \Rightarrow \qquad \frac{g}{l} = \omega_0^2 = 1 + 2\omega\Delta\omega + \Delta\omega^2$$

We also assume that the system has an periodic response with small amplitude. When we rescale the equation we assume that the leading non-linear term, the forcing, the damping, and the term associated with $\Delta\omega$ are of the same order of magnitude.

Scaling

First we change variable to $\tau = \omega t$, and scale the angle $\theta = Bz$. We also taylor expand the sine term.

$$z'' + \frac{\alpha}{m\omega} z' + \frac{\omega_0^2}{\omega^2}(z - \frac{B^2}{6} z^3) = \frac{F}{Bml\omega^2} \cos(\tau)$$

We want the non-linear term to be of the same order as the forcing, and we obtain this be choosing:

$$B^2 = \frac{F}{Bml\omega^2} \quad \Rightarrow \quad B = \left(\frac{F}{ml\omega^2}\right)^{\frac{1}{3}}$$

We use our assumptions and introduce:

$$\epsilon = B^2, \quad \beta\epsilon = \frac{\alpha}{m\omega}, \quad \kappa = \frac{1}{6}, \quad \gamma\epsilon = \frac{2\Delta\omega}{\omega} + \left(\frac{\Delta\omega^2}{\omega^2}\right),$$

which implies that the damping and linear detuning are both weak.

Scaling

There is ambiguity, in form of a constant factor, concerning the identification of κ and ϵ.

The equation becomes:

$$z'' + \epsilon\beta z' + (1 + \gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau) + O(\epsilon^2)$$

where the term $\beta z'$ represents linear damping, $(1 + \gamma\epsilon)z$ represents linear detuning, $\kappa\epsilon z^3$ represents nonlinear detuning and $\epsilon\cos(\tau)$ represents the forcing.

Asymptotic solution

We use: $z = z_0 + \epsilon z_1 + O(\epsilon^2)$

This gives the leading order terms:
$\epsilon^0 : \quad z_0'' + z_0 = 0 \quad \Rightarrow \quad z_0 = ae^{i\tau} + c.c$

To decide a we need to use the terms linear in ϵ:
$\epsilon^1 : \quad z_1'' + z_1 = -\gamma z_0 - \beta z_0' + \kappa z_0^3 + \cos(\tau) =$
$e^{i\tau}(-\gamma a - i\beta a + \frac{1}{2} + 3\kappa a|a|^2) + \kappa a^3 e^{3i\tau} + c.c$

Since $e^{i\tau}$ and its complex conjugated solve the homogeneous equation for z_1 we need to exclude them from the RHS in order to avoid secular terms in the particular solution.

Asymptotic solution

$$-\gamma a - i\beta a + \frac{1}{2} + 3\kappa a|a|^2 = 0$$

We write $a = |a|e^{i\delta}$

$$-\gamma|a| - i\beta|a| + 3\kappa|a|^3 = -\frac{e^{-i\delta}}{2} \tag{1}$$

$$|-\gamma|a| - i\beta|a| + 3\kappa|a|^3|^2 = |-\frac{e^{-i\delta}}{2}|^2$$

$$(3\kappa|a|^2 - \gamma)^2|a|^2 + \beta^2|a|^2 = \frac{1}{4}$$

This is a third order equation for $|a|^2 = \rho$

$$9\kappa^2\rho^3 - 6\kappa\gamma\rho^2 + (\gamma^2 + \beta^2)\rho - \frac{1}{4} = 0$$

Asymptotic solution

When we have found $|a|^2$ numerically we find δ from eq. (1) ($\sin(\delta) = -2\beta|a|$ etc.)

With δ and $|a|$ we can write the leading order solution as:

$$z_0(\tau) = 2|a|\cos(\tau + \delta)$$

Numerical solution

In order to solve the equation with a numerical ODE solver we rewrite it as two coupled equations.
We do this by introducing $x = \frac{dz}{d\tau}$:

$$z' = x$$

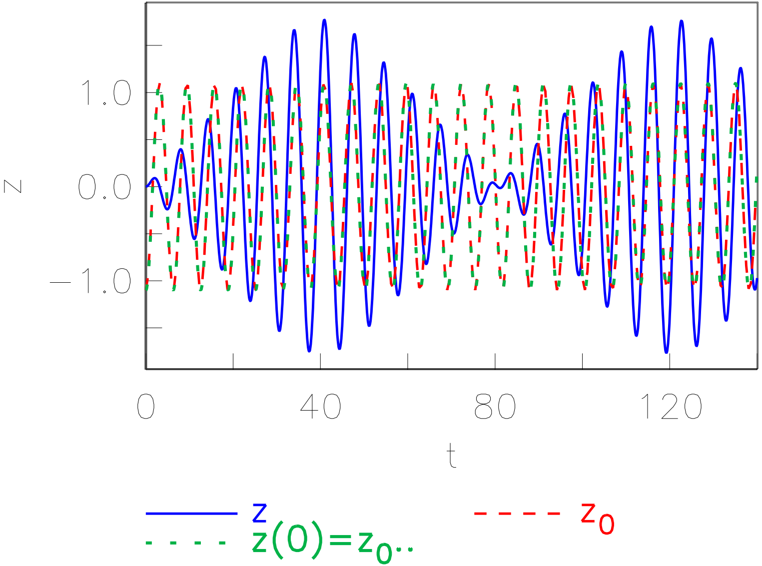
$$x' = -\epsilon\beta x - (1+\epsilon\gamma)z + \epsilon\beta z^3 + \epsilon\cos(\tau)$$

The initial conditions are often $x = z = 0$ or adopted from an analytic solution.

Here we have $\gamma = 0$, $\beta = 0$ and $\kappa = 1$.

This gives the asymptotic solution: $z_0 = -2\left(\frac{1}{6}\right)^{\frac{1}{3}}\cos(\tau)$

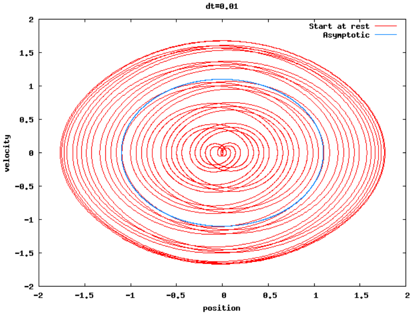
Numerical solution, case 1 $\quad z'' + \epsilon\beta z' + (1+\gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau)$

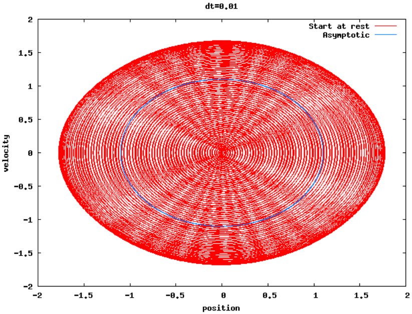


Comments:

- Start from equilibrium $z(0) = z'(0) = 0$. No damping, the solution will not stabilize with a fixed amplitude.
- Start with $z(0) = z_0(0),\ z'(0) = z_0'(0)$: Almost a periodic solution.

$\gamma = 0, \beta = 0$ and $\kappa = 1$

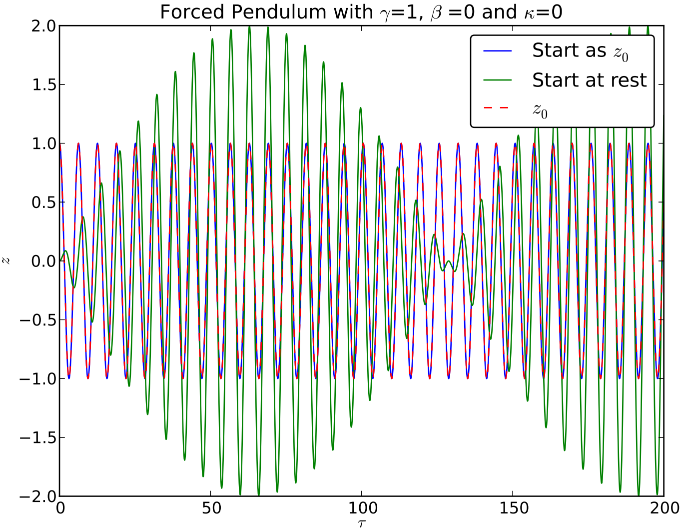




z' versus z: limit cycle is not approached.

Here we have $\gamma = 1$, $\beta = 0$ and $\kappa = 0$.
This gives the asymptotic solution: $z_0 = \cos(\tau)$
We also have an exact solution: $z = \frac{1}{\gamma}\cos(\tau) + A\cos(\sqrt{(1+\gamma\epsilon)}\tau)$
where $A = -\frac{1}{\gamma} = -1$ if we start from equilibrium,
and $A = -\frac{1}{\gamma} + 1 = 0$ if we start like z_0

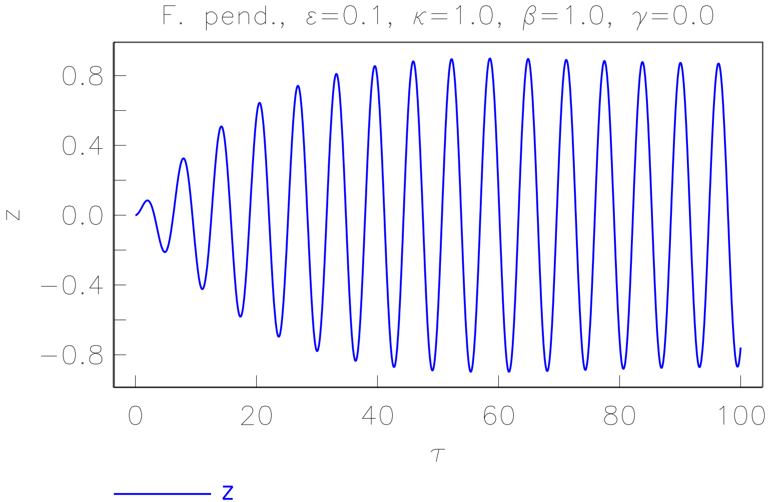
Numerical solution, case 2 $z'' + \epsilon\beta z' + (1+\gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau)$



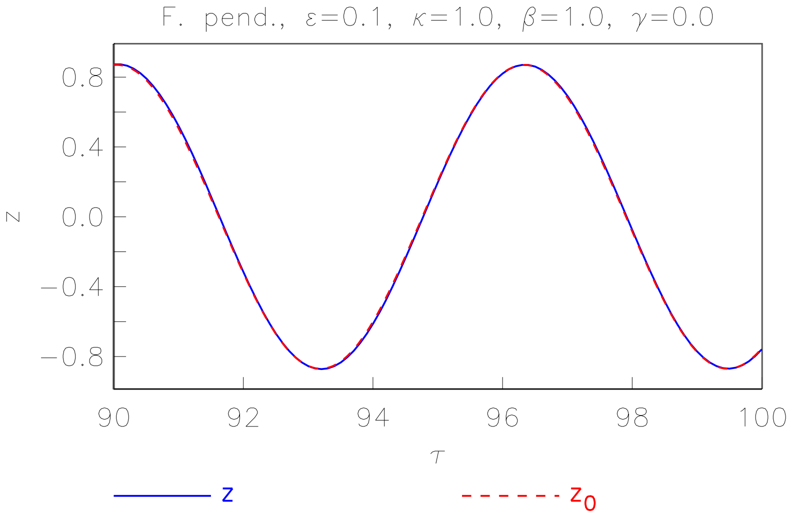
Comments:
- The solution that starts at rest consist of two solution with slightly different angular frequency, hence convolution. Max amplitude is two.
- Solution that starts as z_0 stays as z_0 for a long time. Eventually it will be convoluted as well.

Here we have $\gamma = 0$, $\beta = 1$ and $\kappa = 1$.
This gives the asymptotic solution: $z_0 = 0.8698 \cdot \cos(\tau - 2.08687)$

Numerical solution, case 3 $z'' + \epsilon\beta z' + (1+\gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau)$



Numerical solution, start from equilibrium

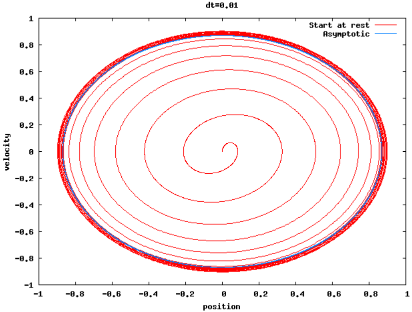


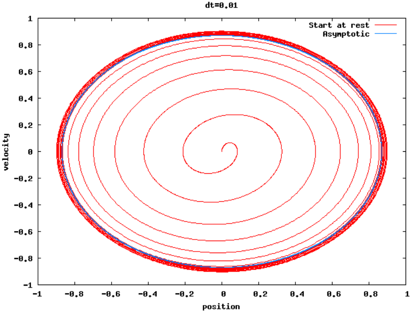
Comparsion with z_0 for large times.

Comments:

- Because of the damping term the solution for all initial conditions stabilizes at a fixed amplitude.
- For large times z_0 and z agree wery well.

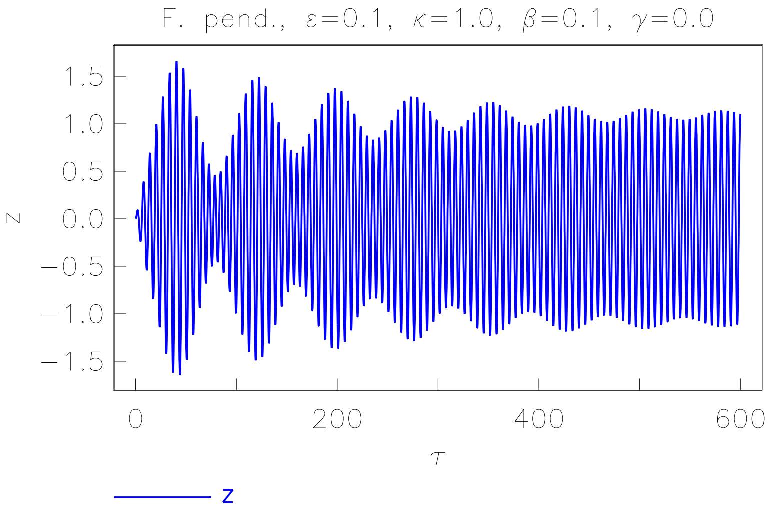
$\gamma = 0, \beta = 1$ and $\kappa = 1$





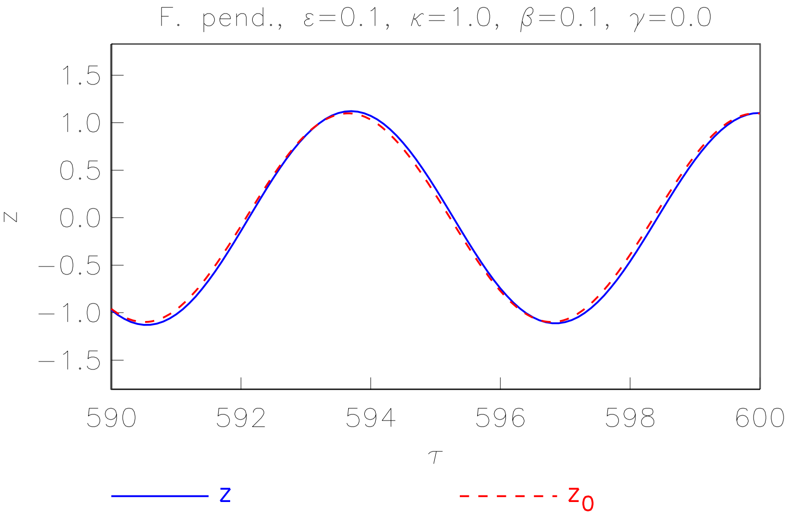
Left: $\tau \leq 200$, right $\tau \leq 1000$.

Here we have $\gamma = 0$, $\beta = 0.1$ and $\kappa = 1$.
This gives the asymptotic solution: $z_0 = 1.0984 \cdot \cos(\tau - 3.03153)$



Numerical solution, start from equilibrium

Numerical solution, case 4 $z'' + \epsilon\beta z' + (1+\gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau)$



Comparsion with z_0 for large times. Slight deviations due to finite τ still noticeable.

Numerical solution, case 4 $z'' + \epsilon\beta z' + (1 + \gamma\epsilon)z - \kappa\epsilon z^3 = \epsilon\cos(\tau)$

Comments:
- Because of the damping term the solution for all initial conditions stabilizes at a fixed amplitude, but it takes a long time.
- Compared to case3 it takes a lot longer for the amplitude to stabilize at a fixed value

For the case $\gamma = 0$ and $\kappa = \beta = 1$ we want to find the value for ϵ that gives an maximum excursion close to $10°$ and $30°$.

$\theta_{1,\max} = 10° \quad \Rightarrow \quad z_{1,\max} = \frac{\pi}{18\sqrt{\epsilon}}$

$\theta_{2,\max} = 30° \quad \Rightarrow \quad z_{2,\max} = \frac{\pi}{6\sqrt{\epsilon}}$

Iterating over $\epsilon \in (0,1]$ while testing for

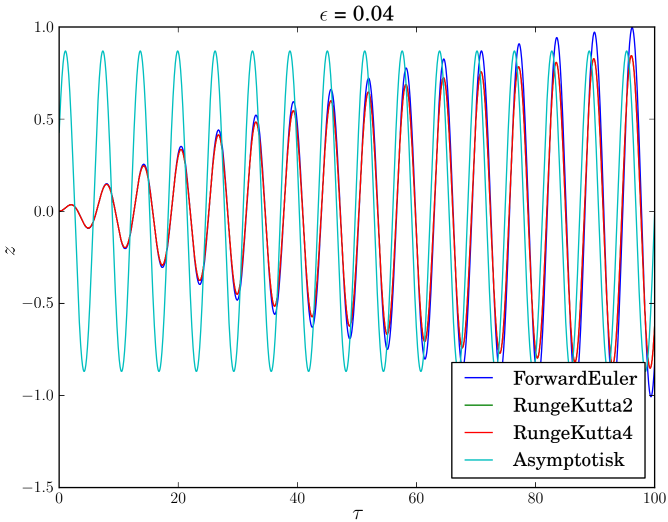
$$\left| z_{numeric,max} - z_{i,max} \right| < \chi$$

where $\chi \ll 1$ and $i = 1, 2$ gives this values for ϵ

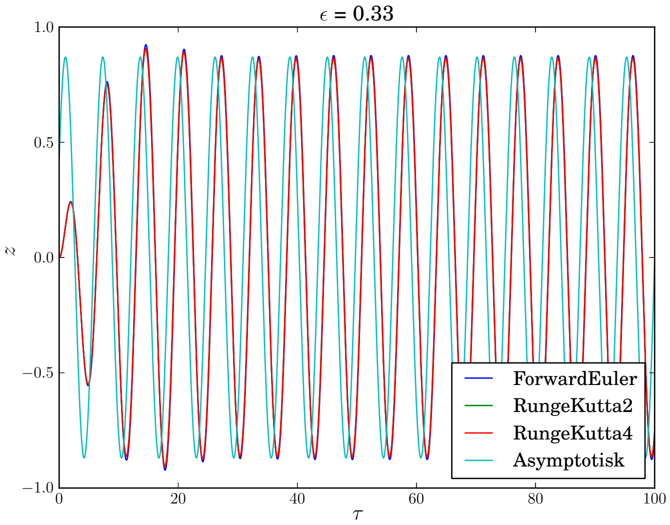
$\theta_{1,max} \quad \Rightarrow \quad \epsilon = 0.04$

$\theta_{2,max} \quad \Rightarrow \quad \epsilon = 0.33$

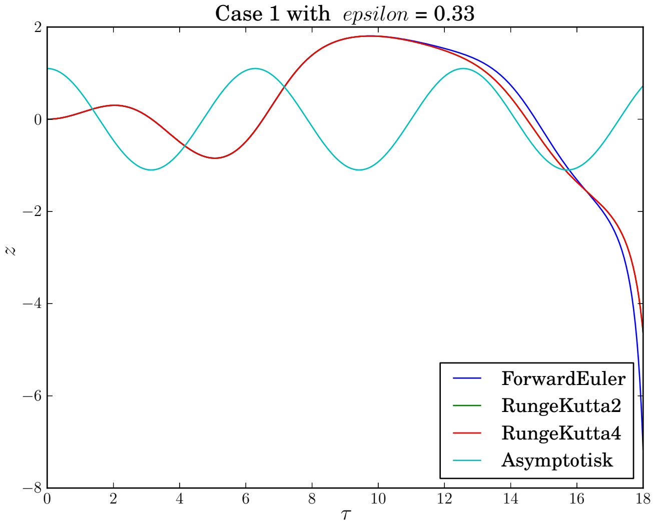
$\gamma = 0, \beta = 1$ and $\kappa = 1$



$\gamma = 0, \beta = 1$ and $\kappa = 1$



$\gamma = 0, \beta = 0$ and $\kappa = 1$



Case 1 with $\epsilon = 0.33 \qquad \Rightarrow \qquad \theta_{2,\max} = 30^\circ$
Comments:
Convolution makes restoring term change its sign ($\kappa\epsilon z^3 > z$); solution becomes unstable.