**Ex. 1 .** The method of Poincaree-Lindstedt.
Redefine time $\tau = \omega t$ and **require that $y$ and all its approximations inherit period $2\pi$ in $\tau$.**

$$\omega^2 y'' + \left(1 + \epsilon\omega^2 (y')^2\right) y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}\tau}$ etc.

$$y = y_0 + \epsilon y_1 + ...,$$
$$\omega = \omega_0 + \epsilon\omega_1 + ...$$

$O(\epsilon^0)$:

$$\omega_0^2 y_0'' + y_0 = 0, \quad y_0(0) = 1, \quad y_0'(0) = 0,$$

solution

$$y_0 = \cos(\omega_0^{-1}\tau).$$

Period $2\pi$ in $\tau \Rightarrow \omega_0 = 1$.

$O(\epsilon^1)$:

$$y_1'' + y_1 = -2\omega_1 y_0'' - (y_0')^2 y_0 = 2\omega_1 \cos\tau - \sin^2\tau\cos\tau = (2\omega_1 + \frac{1}{4})\cos\tau - \frac{1}{4}\cos 3\tau,$$
$$y_1(0) = 0, \quad y_1'(0) = 0,$$

Period $2\pi$ in $\tau$ for $y_1 \Rightarrow \omega_1 = -\frac{1}{8}$.

$$y_1 = \frac{1}{32}(\cos 3\tau - \cos\tau)$$

**Ex. 2 .** $f = \epsilon F$ yields

$$g = \epsilon F(x - g)$$

Must have $g = \epsilon G$, where $G$ is of order 1. Then

$$G = F(x - \epsilon G) \tag{1}$$

Leading order solution $G \approx G_0 = F(x)$. Taylor series expansion of $F$ in (1)

$$G = F(x) - \epsilon F'(x)G + \frac{1}{2}(\epsilon)^2 F''(x)G^2 + O(\epsilon^3) \tag{2}$$

We now try $G = G_0 + \epsilon G_1 + ..$

$$\begin{aligned}
O(\epsilon^0): &\quad G_0 = F(x) & \\
O(\epsilon^1): &\quad G_1 = -F'(x)G_0 & = -F'(x)F(x) \\
O(\epsilon^2): &\quad G_2 = -F'(x)G_1 + \tfrac{1}{2}F''(x)G_0^2 & = \{F(F')^2 + \tfrac{1}{2}F^2F''\}
\end{aligned}$$

Only $G_0$ and $G_1$ are required.

**Ex. 3 .**

**a)** Using $\Theta$ for dimension temperature:

| Parameter | $T$ | $t$ | $\kappa$ | $y$ | $T_0$ |
|---|---|---|---|---|---|
| Dimension | $\Theta$ | $T$ | $\frac{L^2}{T}$ | $L$ | $\Theta$ |

Number of $\pi$'s: $5 - 3 = 2$

$$\pi_1 = \frac{T}{T_0}, \quad \pi_2 = \frac{y}{\sqrt{\kappa t}}$$

According the the $\pi$ theorem a relation between $T$ and the other parameters can be expressed

$$\pi_1 = F(\pi_2) \Rightarrow T = T_0 F(\pi_2)$$

.

**b)** $F$ depends on only one composite variable, namely $\pi_2$, then an ODE should suffice. Now

$$\frac{\partial T}{\partial t} = T_0 \frac{\partial}{\partial t} F\left(\frac{y}{\sqrt{\kappa t}}\right) = -\frac{1}{2} T_0 \frac{y}{\sqrt{\kappa}} t^{-\frac{3}{2}} F'$$

$$\frac{\partial^2 T}{\partial y^2} = T_0 \frac{\partial^2}{\partial y^2} F\left(\frac{y}{\sqrt{\kappa t}}\right) = \frac{T_0}{\kappa t} F''$$

Then multiplying $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2}$ with $\kappa t / T_0$ gives

$$-\frac{1}{2} \hat{\pi} \frac{\mathrm{d}F}{\mathrm{d}\hat{\pi}} = \frac{\mathrm{d}^2 F}{\mathrm{d}\hat{\pi}^2}$$

The boundary conditions become

$$\begin{aligned} T(0,t) &= 0 \quad &\Rightarrow \quad F(0) &= 0 \\ T(y,0) &= T_0 \quad &\Rightarrow \quad F(\infty) &= 1 \end{aligned}$$

**Ex. 4 .**

**a)**

$$T = \frac{1}{2} m \vec{v}_1^2 + \frac{1}{2} m \vec{v}_2^2 = m(\dot{x}_G^2 + \dot{y}_G^2 + \ell^2 \dot{\theta}^2),$$

(with some canceling).

$$V = mgy_1 + mgy_2 = 2mgy_G$$

and $L = T - V$.

**b)** First integrals

1. $\dfrac{\partial L}{\partial x_G} = 0 \Rightarrow \text{const.} = \dfrac{\partial L}{\partial \dot{x}_G} = 2m\dot{x}_G$
   Conservation of horizontal momentum.
2. $\dfrac{\partial L}{\partial \theta} = 0 \Rightarrow \text{const.} = \dfrac{\partial L}{\partial \dot{\theta}} = 2m\ell^2\dot{\theta}$
   Conservation of angular momentum with respect to the center of gravity.
3. $\dfrac{\partial L}{\partial t} = 0 \Rightarrow \text{const.} = L - \dot{x}_G\dfrac{\partial L}{\partial \dot{x}_G} - \dot{y}_G\dfrac{\partial L}{\partial \dot{y}_G} - \dot{\theta}\dfrac{\partial L}{\partial \dot{\theta}} = -(T+V)$
   Conservation of energy.

**Ex. 5 .** This is taken from slides presented in the course.
Write solution in terms of new unknown

$$y = e^{S(x)}.$$

Substitution into differential equation yields equation for $k(x) \equiv S'$:

$$\epsilon^2(k' + k^2) - W = 0,$$

called a Ricatti equation.
Dominant balance

$$\begin{array}{ccccccc} \epsilon^2 k' & + & \epsilon^2 k^2 & - & W & = & 0 \\ (1) & + & (2) & + & (3) & = & 0 \end{array} \tag{3}$$

(1) & (3): $k \sim \epsilon^{-2}\int W dx \Rightarrow$(2)$\sim \epsilon^{-2} \gg$ (1), (3). Invalid!

(1) & (2): $k \sim (C+x)^{-1} \Rightarrow$(3) dominates as $\epsilon \to 0$, $y \sim x + C$. Invalid!

(2) & (3): $k \sim k_0 = \pm\epsilon^{-1}W^{\frac{1}{2}}$. $\Rightarrow$(1)$\sim \epsilon \ll$ (2), (3). Two solutions. OK!*

$$y \sim e^{\pm\epsilon^{-1}\int W^{\frac{1}{2}} dx}$$

Second balance
$k = k_0 + k_1$, $k_1 \ll k_0$. Substitution in (3)

$$\epsilon^2(k_0' + k_1' + k_0^2 + 2k_0k_1 + k_1^2) + W = 0$$

Canceling of leading order $\epsilon^2 k_0^2 + W = 0$ and $k_1 \ll k_0 \Rightarrow$

$$\epsilon^2(k_0' + 2k_0k_1) = 0,$$

with solution $k_1 = -\frac{1}{2}k_0'/k_0 = -\frac{1}{4}W'/W = O(1)$.

$$S_\pm = \int k dx = C_\pm \pm \frac{1}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x} + \ln\left(W^{-\frac{1}{4}}\right),$$

which with $C_+ = A$ and $C_- = B$ yields the desired solution.