The WKBJ method and optics MEK4320

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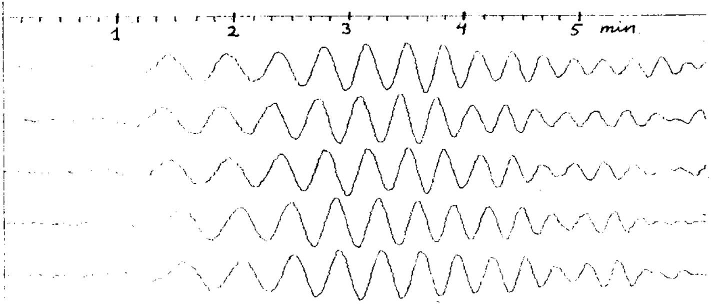
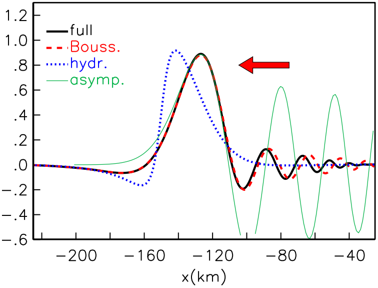
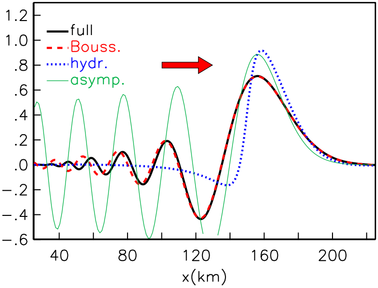
Ray theory.



Note: Taken from the compendium; too little information on the graph. *Don't make this error in your assignment.*

Surface-elevation from tsunami computation





From the slides on asymptotic wave front. (elevation in m)
Different curves corresponds to different theories.
Slowly varying trains start developing behind the wave front.
Larger times: longer trains, slower variations.
Typical for dispersive initial value problems.

Slowly varying wave trains; mathematical form

A solution is assumed on the form

$$\eta(x,t) = A(x,t)e^{i\chi(x,t)}.$$

Concept: local wave number and frequency

$$\vec{k} \equiv \frac{\partial \chi}{\partial x}, \quad \omega \equiv -\frac{\partial \chi}{\partial t}.$$

Justification of concept

Taylor expansion around x_0, t_0

$$\chi(x,t) = \chi(x_0,t_0) + k(x-x_0) - \omega(t-t_0) + T_2 + \ldots$$

Where the second order terms are

$$T_2 = \frac{1}{2}k_x(x-x_0)^2 + k_t(x-x_0)(t-t_0) - \frac{1}{2}\omega_t(t-t_0)^2$$

Define $\lambda = 2\pi/k$. Then

$$\chi(x_0 + \lambda, t_0) - \chi(x_0, t_0) = k\lambda + ... = 2\pi + ...$$

If the implicit terms (from T_2 etc.) are much smaller than 2π then λ approximates the wavelength of a nearly harmonic wave.
For the first term in T_2 this require $k_x\lambda^2 \ll k\lambda \Rightarrow$

$$\frac{k_x}{k}\lambda \ll 1.$$

Interpretation: the relative variation of k over a wavelength is small.
Correpondingly; from similar requirements also for period it follows $Tk_t/k \ll 1$ etc.
Generally, slow variations of k and ω require slow variation of the medium.

Local invocation of dispersion relation

At a location x, t the medium has given properties depending on x and t. If the medium was uniform with these properties we may find a corresponding dispersion relation.
When the wave train nearly is a harmonic mode, locally, we assume that k and ω approximately fulfill this dispersion relation

$$\omega = W(k, x, t),$$

where the explicit dependence on x and t comes from the variation of the medium.

Examples on W

Shallow water $W(k, x, t) = \sqrt{gH(x, t)}k$
Gravity waves; general depth $W(k, x, t) = \sqrt{gk\tanh(kH(x, t))}$
Here $H(x, t)$ is a depth that vary in space and time.

Sometimes $\omega = W(k, x, t)$ is used as is. On other occasions we need the *ray equations*.

The ray equations

From the definition of $k = \chi_x$ and $\omega = -\chi_t$ it follows

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0.$$

Insertion of $\omega(x,t) = W(k(x,t), x, t)$ and use of the chain rule

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega}{\partial x} = -\frac{\partial W}{\partial k}\frac{\partial k}{\partial x} - \frac{\partial W}{\partial x}.$$

Recognizing $c_g = \frac{\partial W}{\partial k}$ we write

$$\frac{\partial k}{\partial t} + c_g\frac{\partial k}{\partial x} = -\frac{\partial W}{\partial x}.$$

Describes evolution of k along the characteristic $\frac{\mathrm{d}x}{\mathrm{d}t} = c_g$.
The group velocity, c_g, appears again.

Differentiating $\omega = W$:

$$\frac{\partial \omega}{\partial t} = \frac{\partial W}{\partial k}\frac{\partial k}{\partial t} + \frac{\partial W}{\partial t}.$$

Then $k_t + \omega_x = 0$ yields

$$\frac{\partial \omega}{\partial t} + c_g \frac{\partial \omega}{\partial x} = \frac{\partial W}{\partial t}.$$

See compendium for

- More dimensions
- Rephrasing the ray equations as Hamilton's canonical equations.

The relation for A, the transport equation

Energy conservation

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where E and F are depth integrated density and flux, respectively. Applies also to averaged quantities.

Invocation of relation for uniform medium

Harmonic wave, uniform medium $\Rightarrow$ averaged energy relations

$$\overline{F} = \vec{c}_g \overline{E}, \quad \overline{E} = CA^2,$$

(Gravity waves: $C = \frac{1}{2}\rho g$)
Slow variation $\Rightarrow \overline{F} = c_g \overline{E}$ (approximately) as in uniform medium.

Energy conservation

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{F}}{\partial x} = 0,$$

then gives

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial (c_g \overline{E})}{\partial x} = 0,$$

or, for gravity waves

$$\frac{\partial A^2}{\partial t} + \frac{\partial (c_g A^2)}{\partial x} = 0$$

The WKBJ technique

WKBJ; example: LSW

Linear shallow water theory(indices mark differentiation)

$$\eta_{tt} - \nabla\cdot(c_0^2\nabla\eta) = 0 \tag{1}$$

where $c_0^2 = gh(x,y)$.
In ray theory

$$\eta(x,y,t) = A(x,y,t)e^{i\chi(x,y,t)}, \tag{2}$$

where $\vec{k} \equiv \nabla\chi$, $\omega \equiv -\frac{\partial\chi}{\partial t}$.
Slow variations of $\vec{k}$ and $\omega \Rightarrow$ ray equations.
We now insert (2) in (1):

$$\begin{aligned} &A_{tt} - i(2\omega A_t + \omega_t A) - \omega^2 A = \\ &\nabla\cdot(c_0^2\nabla A) + i\left(c_0^2\nabla A\cdot\vec{k} + \nabla\cdot(c_0^2 A\vec{k})\right) - c_0^2k^2A \end{aligned} \tag{3}$$

Exact, but nothing is achieved either – so far.

Scaling; transformation to non-dimensional form

Scales

Typical wavelength: λ_c (fast scale)

Typical length scale for medium change L_c (slow scale)

Small parameter $\frac{\lambda_c}{L_c} = \epsilon \ll 1$.

Typical wave speed: $c_c = \sqrt{gh_c}$

Typical amplitude: A_c – no significance as long as in linear regime

Typical phase: $\chi \sim L_c/\lambda_c = \epsilon^{-1}$

Rescaling

All derivatives explicit in (3) are with respect to slow variation.

Fast variation inherent in definitions of $\vec{k}$ and ω, only.

$\vec{\kappa} = \lambda_c \vec{k}$, $\hat{\omega} = \lambda_c \omega / c_c$, $\hat{A} = A/A_c$, $\hat{c} = c_0/c_c$, $\hat{x} = x/L_c$, $\hat{t} = c_c t/L_c$.

Rescaling of (3)

$$\epsilon^2 \hat{A}_{\hat{t}\hat{t}} - i\epsilon(2\hat{\omega}\hat{A}_{\hat{t}} + \hat{\omega}_{\hat{t}}\hat{A}) - \hat{\omega}^2\hat{A} = \\ \epsilon^2\hat{\nabla}\cdot(c^2\hat{\nabla}\hat{A}) + i\epsilon\left(c^2\hat{\nabla}\hat{A}\cdot\vec{\kappa} + \hat{\nabla}\cdot(c^2\hat{A}\vec{\kappa})\right) - c^2\kappa^2\hat{A} \tag{4}$$

Leading terms: $O(1)$ no slow differentiations
Next order terms: $O(\epsilon)$ one slow differentiation
Second order terms: $O(\epsilon^2)$ two slow differentiations

Leading order

$\hat{\omega}^2 = c^2\kappa^2 \Rightarrow$ restored scales $\omega^2 = c_0^2 k^2 = W^2$.
With $\vec{k}_t + \nabla\omega = 0$, $\partial k_i/\partial x_j = \partial k_j/\partial x_i$: ray theory retrieved

$$\frac{\partial k_i}{\partial t} + \vec{c}_g\cdot\nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1,2 \tag{5}$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g\cdot\nabla\omega = 0 \tag{6}$$

with $\vec{c}_g = c_0\vec{k}/k$. Now $\vec{k}$ and ω are settled.

Next order; $O(\epsilon)$ relative size

Original scales restored

$$-2\omega A_t - \omega_t A = c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}).$$

With $\omega = c_0 k$ and $\vec{c}_g = c_0 \vec{k}/k$ (then $c_0^2 \vec{k} = \omega \vec{c}_g$):

$$-2\omega A_t - \omega_t A = 2\omega \vec{c}_g \cdot \nabla A + A \vec{c}_g \cdot \nabla \omega + \omega A \nabla \cdot \vec{c}_g.$$

Next step multiply with A/ω and regroup

$$-(A^2)_t - \frac{A^2}{\omega}(\omega_t + \vec{c}_g \cdot \nabla \omega) = \nabla \cdot (\vec{c}_g A^2).$$

Due to (6) terms within last parentheses on l.h.s cancel out:

$$(A^2)_t = -\nabla \cdot (\vec{c}_g A^2). \qquad (7)$$

With $E = \frac{1}{2}\rho g A^2$ (energy density) and $\vec{F} = \vec{c}_g E$ (energy flux) equation (7) reads

$$E_t + \nabla \cdot \vec{F} = 0, \qquad (8)$$

Averaged energy conservation, as in uniform medium.

Remarks

Remark 1

Similar WKBJ approaches apply to most linear wave equations $\Rightarrow$ result with interpretation as energy conservation is general.

Remark 2

By means of (6) we have

$$(G(\omega)E)_t + \nabla \cdot \left(G(\omega)\vec{F} \right) = 0,$$

for any $G(\omega)$.

Remark 3

If we have coupling with a background current it is the wave action (E over some frequency) which is conserved.

Remark 4

To include higher order in ϵ we must expand $A = A_0 + \epsilon A_1 + ..$

Brief review of ray theory and optics.

Starting point

Assume solution on form

$$\eta(\vec{r}, t) = A(\vec{r}, t)e^{i\chi(\vec{r}, t)}.$$

Define local wave number and frequency

$$\vec{k} \equiv \nabla\chi, \quad \omega \equiv -\frac{\partial\chi}{\partial t}.$$

From definition it follows

$$\frac{\partial\vec{k}}{\partial t} + \nabla\omega = 0, \quad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}.$$

Optics; summary

Ray theory (geometrical optics)

Harmonic wave, uniform medium$\Rightarrow$ dispersion relation

$$\omega = W(\vec{k}; H...)$$

Slow variation of medium and wave train $\Rightarrow$ local k and ω fulfill the dispersion relation (approximately) as in uniform medium.

Physical optics

Harmonic wave, uniform medium $\Rightarrow$ averaged energy relations

$$\vec{F} = \vec{c}_g E, \quad E = E(A^2, ..)$$

Slow variation $\Rightarrow$ $\vec{F} = \vec{c}_g E$ (approximately) as in uniform medium.

Equation of geometrical optics

Ray equations

From $\omega = -\frac{\partial \chi}{\partial t}$, $\vec{k} = \nabla \chi$ and $\omega = W(\vec{k}, x_i, t)$

$$\begin{aligned}
\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega &= \frac{\partial W}{\partial t}
\end{aligned}$$

Recasted to Hamilton's canonical equations

$$\begin{aligned}
\frac{\mathrm{d} k_i}{\mathrm{d} t} &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\mathrm{d} x_i}{\mathrm{d} t} &= \frac{\partial W}{\partial k_i} = (c_g)_i\,, \quad i = 1, 2.. \\
\frac{\mathrm{d} \omega}{\mathrm{d} t} &= \frac{\partial W}{\partial t}
\end{aligned}$$

The transport equation

Energy conservation, in general

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Invoking the approximation $F = c_g E$:

$$\frac{\partial E}{\partial t} + \frac{\partial (c_g E)}{\partial x} = 0$$

where $E = E(A^2, ..)$

More dimensions

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{c}_g E) = 0 \qquad (9)$$

calculation of wave field

1. $\vec{k}$ and ω are obtained from ray theory
2. The transport equation (9) is solved for A

Optics in uniform media; two examples

Uniform medium

$\omega = W(\vec{k}) \Rightarrow \vec{c}_g = c_g(\vec{k})$

Ray equations

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = 0, \qquad i = 1, 2..$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = (c_g)_i, \quad i = 1, 2..$$

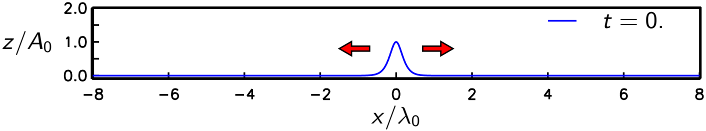
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

$\vec{k}$ and ω conserved along characteristics $\mathcal{C}$: $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{c}_g$

$\Rightarrow$ $\vec{c}_g$ conserved $\Rightarrow$ $\mathcal{C}$ are straight lines.

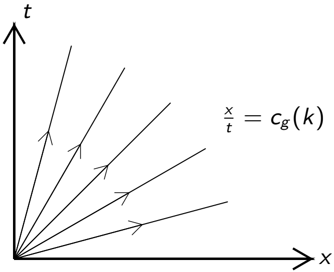
Characteristics are also straight lines viewed in the generalized x_i, t space.

Evolution of wave train from confined disturbance



- Previously: Fourier transform $\Rightarrow$ formal solution as integral.
- When propagation distances are much larger than initial length (λ_0): application of stationary phase, main spectral contribution from vicinity of k_s such that $c_g(k_s) = x/t$.
- Result: a slowly varying wave train.
- The slowly varying wave train should be within the realm of optics.
- Hence, problem revisited with optics. Approximation: initial disturbance is at $x = 0$.

In 1D: waves from point disturbance, at $x = 0$ at $t = 0$



All $\mathcal{C}$ intersects at $x = t = 0 \Rightarrow c_g = x/t$
$c_g(k) = x/t \Rightarrow k = k(x/t) \Rightarrow \chi = \int k dx$

Gravity waves in infinite depth; ray theory
$W = \sqrt{gk} \Rightarrow x/t = c_g = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{1}{4}g\frac{t^2}{x^2} \Rightarrow$
$\chi = -\frac{1}{4}g\frac{t^2}{x} + f(t)$
Furthermore $\frac{\partial\chi}{\partial t} = -\omega = -\sqrt{gk} \Rightarrow \chi = -\frac{1}{4}g\frac{t^2}{x} + \mathrm{const}$
Phase function as from the stationary phase solution.

Still infinite depth; the transport equation

$$\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}(c_g A^2) = 0$$

inserted $c_g = x/t$ and rewritten as

$$\frac{\partial x A^2}{\partial t} + \frac{x}{t}\frac{\partial (x A^2)}{\partial x} = 0$$

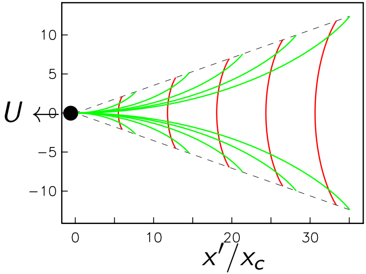
General solution: xA^2 is constant along $\mathcal{C}$ $\Rightarrow$

$$A = x^{-\frac{1}{2}} G\left(\frac{x}{t}\right) = t^{-\frac{1}{2}} \hat{G}\left(\frac{x}{t}\right)$$

Consistent with stationary phase (spectrum$\Rightarrow$ G)
Interpretation: Energy between two characteristics ($\mathcal{C}$) is conserved.

The Kelvin ship wave pattern.

Point source at the surface



Constant source velocity: $\vec{U} = -U\vec{\imath} \Rightarrow$ stationary and slowly varying wave pattern. Frame with fluid at rest $\Rightarrow$ isotropic dispersion relation

$$\vec{c}' = c_0(k)\frac{\vec{k}}{k} \tag{10}$$

Change of coordinate system

From fixed coordinate system ($\vec{r}'$) to a moving one ($\vec{r}$)

Frame follows the source $\vec{r} = \vec{r}' - \vec{U}t$.

New system: stationary source on a uniform current.

A harmonic mode then becomes:

$A\cos\chi = A\cos(\vec{k}\cdot\vec{r}' - \omega' t) = A\cos(\vec{k}\cdot\vec{r} - (\omega' + \vec{k}\cdot\vec{U})t)$

$$\omega = c_0(k)k + \vec{U}\cdot\vec{k} \equiv W(k_x, k_y) \tag{11}$$

$$\vec{c} = \left(c_0(k) + \vec{U}\cdot\frac{\vec{k}}{k}\right)\frac{\vec{k}}{k} \tag{12}$$

where $\vec{k} = k_x\vec{\imath} + k_y\vec{\jmath}$

Doppler shift $\Rightarrow$ Anisotropic dispersion.

Optics

New frame: stationary pattern implies $\omega = 0$. (11) yields:

$$W(k_x, k_y) = 0 \tag{13}$$

The group velocity

$$\vec{c}_g = \frac{\partial W}{\partial k_x}\vec{\imath} + \frac{\partial W}{\partial k_y}\vec{\jmath} \tag{14}$$

Hamilton's equations

$$\frac{d\vec{r}}{dt} = \vec{c}_g, \quad \frac{d\vec{k}}{dt} = 0 \tag{15}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \tag{16}$$

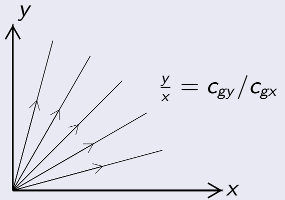
Uniform medium $\Rightarrow$ Characteristics are straight lines
To carry energy characteristics must pass through the source (the origin)

Straight characteristics through the origin:
$x = c_{gx}t$, $y = c_{gy}t$.
(Even for a stationary pattern characteristics can be parameterized by time; following the transport of energy)
Elimination of t:

$$\frac{y}{x} = \frac{\frac{\partial W}{\partial k_y}}{\frac{\partial W}{\partial k_x}} \tag{17}$$

combined with $W(k_x, k_y) = 0$ (13)$\Rightarrow$ two equations for k_x and k_y.

Characteristics



The Phase function

$$\chi(\vec{r}) = \chi_0 + \int_{C(\vec{r})} \vec{k} \cdot d\vec{r} \tag{18}$$

where χ_0 is the phase in the origin and $C(\vec{r})$ is some integration path.
Choosing C as a characteristic: Integration trivial because $\vec{k}$ is constant.

$$\chi(\vec{r}) = \chi_0 + k_x x + k_y y \tag{19}$$

Phase lines $\chi = -A$. Two options for visualization/interpretation

- A quick overview of the phase line readily obtained by contour plot of χ (Matlab or Python etc.).
- Parameterization of phase lines. Some uncanny trigonometry, but the presence of two families of solutions for $\vec{k}$ from (17) and (13) is demonstrated.

Parameterization of phase lines (infinite depth)

θ: angle between $\vec{k}$ and negative x-axis

$$k_x = -k\cos\theta, \quad k_y = k\sin\theta. \tag{20}$$

(13) is rewritten

$$c_0 = U\cos\theta,$$

and $c_0(k) = \sqrt{g/k}$ then yields

$$k = \frac{g}{U^2\cos^2\theta} \tag{21}$$

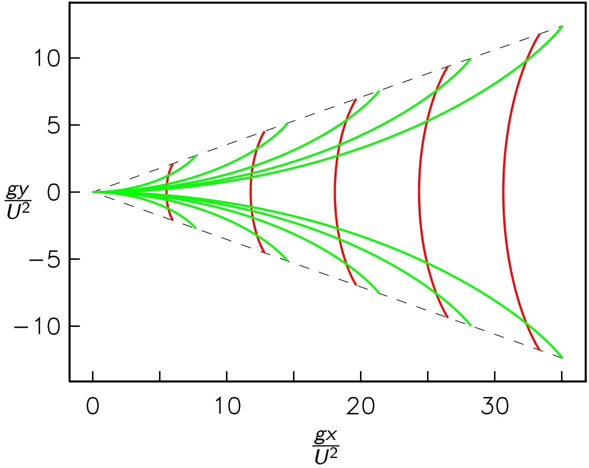
(17) and (19) are solved for x and y

$$x = \frac{(A-\chi_0)g}{U^2}\cos\theta(1+\sin^2\theta) \tag{22}$$

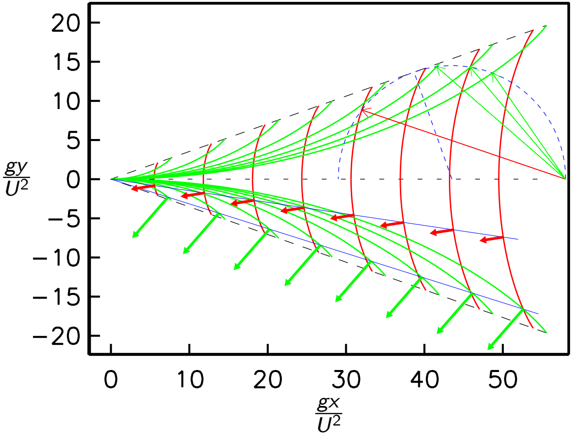
$$y = \frac{(A-\chi_0)g}{U^2}\cos^2\theta\sin\theta \tag{23}$$

Note: $y(\theta)/x(\theta)$ extreme for $\cos\theta = \sqrt{2/3}$ ($\theta = \theta_c = 35.3^\circ$)$\Rightarrow$ jump in direction of phase lines $\Rightarrow$ independent solutions

Point source: other techniques show $\chi_0 = \frac{1}{4}\pi, -\frac{1}{4}\pi$ for transverse and diverging waves, respectively.



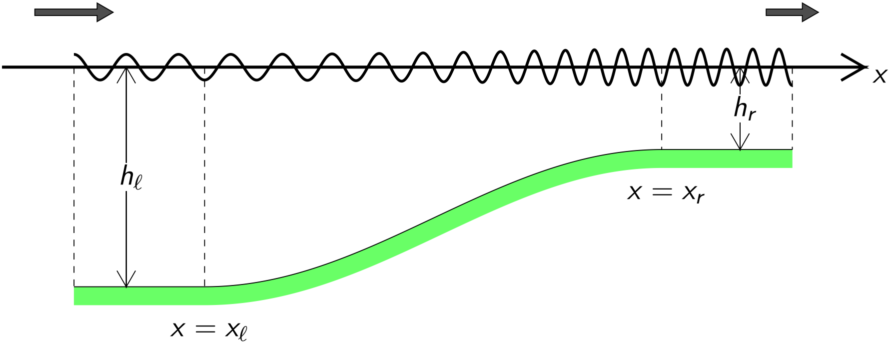
The Kelvin pattern, more details



Fat arrows: wave number vectors.
Dashed half circle: propagation with $\vec{c}_g$ from intersection with x-axis, subject to $c_0 = U\cos\theta$. Thin arrows: corresponding rays.

Amplification and refraction in bathymetry; Caustics

GEOMETRY AND WAVE FIELD.



Wave with frequency ω incident on a slope.

Plane waves, normal incidence , $h = h(x)$, $\vec{k} = k\vec{\imath}$
Dispersion relation $\omega = W(k, x) = \sqrt{gh(x)}k$
Characteristic equation for ω

$$\frac{\partial \omega}{\partial t} + c_g \frac{\partial \omega}{\partial x} = \frac{\partial W}{\partial t} = 0,$$

ω constant along characteristic.
Frequency is fixed for $x < x_\ell \Rightarrow \omega$ constant everywhere.
$\omega = W \Rightarrow k = \omega / c_0$, where $c_0 = \sqrt{gh}$

The phase follows from $\chi_t = -\omega$ and $\chi_x = k$

$$\chi = \int k dx - \omega t = \omega \int \frac{1}{\sqrt{gh}} dx - \omega t.$$

The transport equation and Green's law

Energy conservation

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{F}}{\partial x} = 0.$$

No time variation of averaged quantities $\Rightarrow$

$$\frac{\partial \overline{F}}{\partial x} = 0 \Rightarrow \overline{F} = \mathrm{const.} \Rightarrow c_0 A^2 = \mathrm{const.}$$

Constant energy flux

Substituting $c_0 = \sqrt{gh}$ we find Green's law:

$$A = A_0 \left(\frac{h}{h_0}\right)^{-\frac{1}{4}}, \tag{24}$$

where A_0 and h_0 correspond to a reference location.

Comparison with an accurate numerical solution

$\eta = \hat{\eta}(x)e^{-i\omega t}$ in $\frac{\partial \eta^2}{\partial t^2} - \frac{\partial}{\partial x}\left(gh\frac{\partial \eta}{\partial x}\right) = 0 \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \omega^2\hat{\eta} = 0. \tag{25}$$

For any x_a such that $x_a < x_\ell$ (incident + reflected wave)

$$\hat{\eta} = A_0 e^{ik_\ell x} + Re^{-ik_\ell x}, \quad \text{where} \quad \omega = \sqrt{gh_\ell}k_\ell,$$

where R is unknown. Annihilation of reflected part

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_\ell\hat{\eta} = 2iA_0k_\ell e^{ik_\ell x}, \quad \text{for} \quad x = x_a. \tag{26}$$

At $x = x_b > x_r$ only transmitted wave: $\hat{\eta} = Te^{ik_r x}$, and

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_r\hat{\eta} = 0, \quad \text{where} \quad \omega = \sqrt{gh_r}k_r. \tag{27}$$

(25,26,27) ODE and boundary conditions for $\hat{\eta}$. Moreover: $A = |\hat{\eta}|$

FDM

Numerical approximation : $\hat{\eta}_j \approx \hat{\eta}(j\Delta x)$, $h_{j+\frac{1}{2}} = h((j+\frac{1}{2})\Delta x)$,
Discrete version of (25)

$$\frac{gh_{j+\frac{1}{2}}(\hat{\eta}_{j+1} - \hat{\eta}_j) - gh_{j-\frac{1}{2}}(\hat{\eta}_j - \hat{\eta}_{j-1})}{\Delta x^2} + \omega^2 \hat{\eta}_j = 0.$$

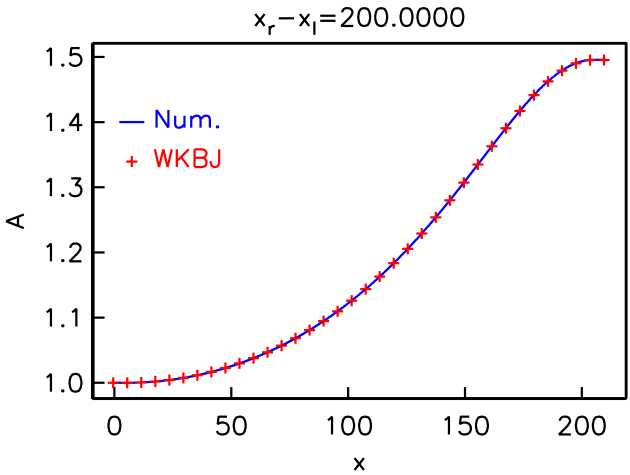
Boundary conditions (grid from $i = 0$ to $i = N$)

$$\frac{\hat{\eta}_1 - \hat{\eta}_0}{\Delta x} + \frac{i}{2} k_\ell (\hat{\eta}_1 + \hat{\eta}_0) = 2iA_0 k_\ell e^{i\frac{1}{2}k_\ell \Delta x},$$

$$\frac{\hat{\eta}_N - \hat{\eta}_{N-1}}{\Delta x} - \frac{i}{2} k_r (\hat{\eta}_N + \hat{\eta}_{N-1}) = 0.$$

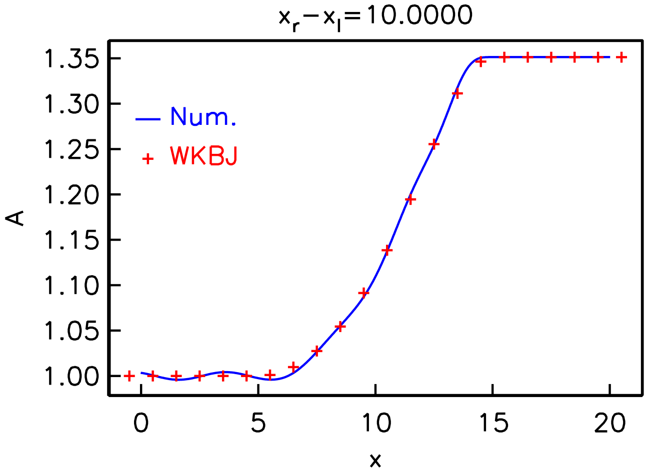
Tri-diagonal set with closure from boundary conditions.
NB: Boundary conditions may be amended by numerical dispersion relation to become exact for the discrete case.

Gentle slope



A normalized by A_ℓ. x by $\hat{x}$, arbitrary horizontal unit.
$x_\ell = 5\hat{x}$, $\lambda_\ell = 8\hat{x}$, $h_r = 0.2h_\ell$, $x_r - x_\ell = 200\hat{x} = 25\lambda_\ell$

Steep slope



$x_\ell = 5\hat{x},\ \lambda_\ell = 8\hat{x},\ h_r = 0.3h_\ell,\ x_r - x_l = 10\hat{x} = \frac{5}{4}\lambda_\ell$

Remarks on optics for shoaling

- Optics good even when $L = x_r - x_\ell$ and λ are comparable.
- Major discrepancy between optics and accurate numerical solution: Reflections.
- Optics do not incorporate reflections.

Oblique incidence, plane geometry

We still have $h = h(x)$, but $\vec{k} = k_x\vec{\imath} + k_y\vec{\jmath}$.
θ: angle of incidence; $k_x = |\vec{k}|\cos\theta$, $k_y = |\vec{k}|\sin\theta$.
Local wave celerity $c = \sqrt{gh(x)}$

Geometrical optics

Hamilton's equations, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}k_y}{\mathrm{d}t} = 0 \Rightarrow \omega = \mathrm{const.}$, $k_y = \mathrm{const.}$.
Then $\omega = W(|\vec{k}|, x) = \sqrt{gh}|\vec{k}| = ck_y/\sin\theta$ and

$$\frac{c}{\sin\theta} = \frac{\omega}{k_y} = \mathrm{const.}$$

Snell's law

Physical optics; transport equation for A

$$\nabla \cdot \vec{F} = 0, \quad \Rightarrow \quad \nabla \cdot \left(\vec{c}_g A^2\right) = 0$$

Since $A = A(x)$ (uniformity in y), this yields constant energy flux density in x-direction

$$c_{gx} A^2 = \text{const.}$$

Now $c_{gx} = c\cos\theta$ and by Snell's law $c_{gx} = \omega\cos\theta\sin\theta/k_y$. Constant energy flux density then yields

$$\sin(2\theta)A^2 = \text{const.}$$

$\theta < \frac{\pi}{4}$: c_{gx} decreases with shoaling $\Rightarrow$ amplification
$\theta > \frac{\pi}{4}$: c_{gx} increases with shoaling due to refraction $\Rightarrow$ attenuation

Deep water caustic

Snell's law and amplitude relation

$$\frac{\sqrt{gh}}{\sin\theta} = \frac{c}{\sin\theta} = \frac{\omega}{k_y} = \text{const.}, \quad c_{gx}A^2 = \text{const.}$$

When $h \to h_c = \omega^2/(gk_y^2)$ $(c \to \omega/k_y)$, we have $\sin\theta \to 1$ and

$$\theta \to \frac{\pi}{2}, \quad k_x,\ c_{gx} \to 0, \quad A \to \infty.$$

Ray parallel to y-axis; breakdown of optics.
Natural assumption: waves are reflected and propagate into shallower water again.
A mechanism for trapping of waves.
Some more analysis is required.

Composite solution; Three domains*

Throughout $\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0. \tag{28}$$

(i): h well below h_c

Optics as described previously yield solution. This corresponds to a WKBJ expansion. $\hat{\eta}(x)$ is wavy.

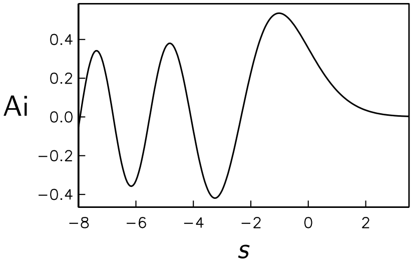
(ii): h around h_c

Local expansion: $h \approx h(x_c) + h'(x_c)(x - x_c) + \ldots$. Keeping leading order terms of (28), rescaling

$$\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}s^2} = s\hat{\eta}, \quad \text{where} \quad s = \left(\frac{\omega^2}{h'(x_c)k_y^4}\right)^{\frac{1}{3}}(x - x_c)$$

Solution that remains finite for large $x - x_c$: $\hat{\eta} = B\mathrm{Ai}(s)$

Combines wavy $\hat{\eta}$ for $h < h_c$ with exponential decay for $h > h_c$.



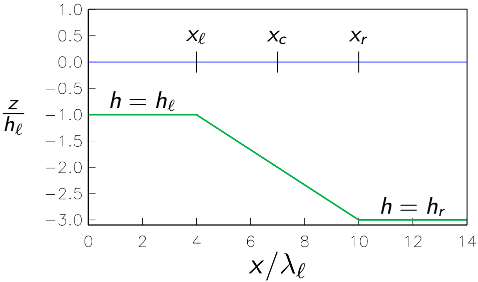
(iii): h well above h_c

Use WKBJ with $\hat{\eta} = e^{-S(x)}$, where S is real and $S \to \infty$ as $x - x_c \to \infty$. Exponential decay.

Combination

"Asymptotic match": (i) with (ii), and (ii) with (iii).
Further details omitted.

Simplified geometry reflection/transmission problem



Possible caustic

Angle of incidence θ_ℓ.

$$h_c = \frac{h_\ell}{\sin^2 \theta_\ell}$$

$x < x_\ell$: incident and reflected waves.

If $h_r < h_c$

Refraction and partial reflection on slope.

$x > x_r$: uniform transmitted wave.

If $h_r > h_c$

Total reflection from slope.

$x > x_r$: decaying η.

Formulation of reflection/transmission boundary value problem

$\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0.$$

For $x_a < x_\ell$ we have, as for normal incidence,

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_x\hat{\eta} = 2iA_0 k_x e^{ik_x x}.$$

where k_x is found from $\omega = W(k_x, k_y, x_a) = \sqrt{gh_\ell}|\vec{k}|$.
If case $h_r < h_c$ and $x_b > x_r$ we again have

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_x\hat{\eta} = 0,$$

where k_x is found from $\omega = W(k_x, k_y, x_b) = \sqrt{gh_r}|\vec{k}|$.

Reflection, $h_r > h_c$

Since (28) has constant coefficients when $x > x_r$ two solutions are readily found

$$\hat{\eta} = B_1 e^{-\alpha x}, \quad \hat{\eta} = B_2 e^{\alpha x}, \quad \text{where} \quad \alpha = \sqrt{k_y^2 - \frac{\omega^2}{g h_r}}.$$

To obtain only the decaying solution we employ

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + \alpha\hat{\eta} = 0, \quad \text{at} \quad x = x_b.$$

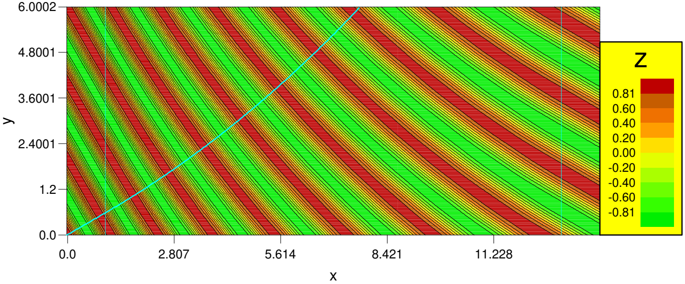
Here x_b may have any value larger than x_r.

If $x_b - x_c$ is sufficiently large we may instead employ

$$\hat{\eta}(x_b) = 0.$$

Numerics as for normal incidence.

Transmission; $\theta_\ell = 30^\circ,\ h_r = 3h_\ell,\ h_c = 4h_\ell$

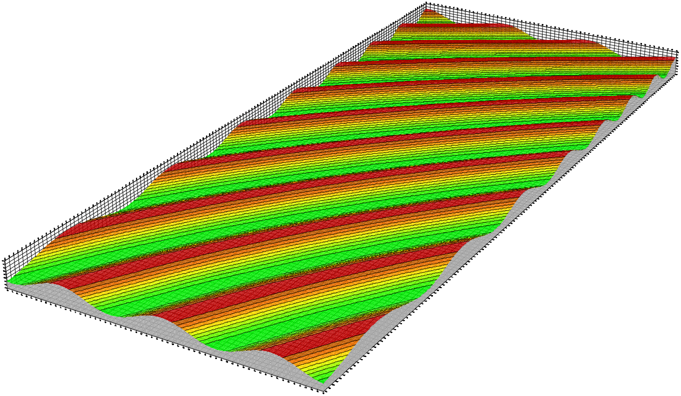


Plot: x and y normalized by $\lambda_\ell = 2\pi c_\ell/\omega$.

$x = x_\ell$, $x = x_r$ and one ray are marked by lines.

$x_r - x_\ell = 12\lambda_\ell$.

Virtually no reflection, only refraction.



Surface seen from the deep region.

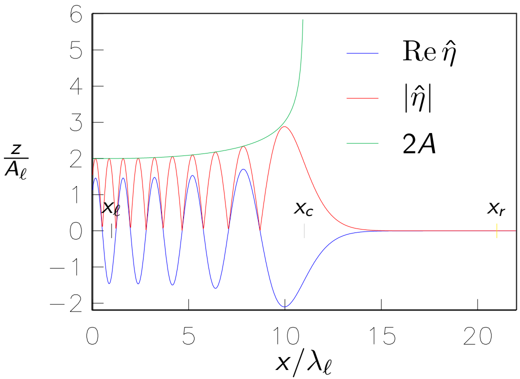
Case with full reflection

Same geometry and wavelength
$h_r = 3h_\ell,\ x_r - x_\ell = 12\lambda_\ell$

Increased angle of incidence
$\theta_\ell = 45^\circ \Rightarrow h_c = 2h_\ell < h_r$
Full reflection

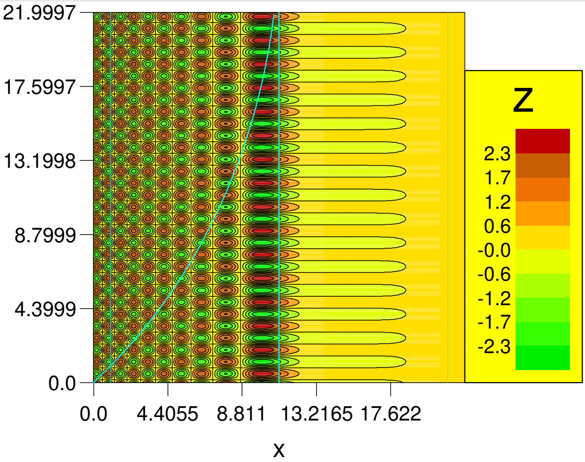
First a transect $y = \mathrm{const.}$

Full reflection; $\theta_\ell = 45°$, $h_r = 3h_\ell$, $h_c = 2h_\ell$



A_ℓ amplitude of incident wave for $x < x_\ell$.

Optics for both incident and reflected solution: curve marked with $2A$.



Horizontal axes normalized by λ_ℓ.

Impermeable walls may be introduced at any x_w where $\frac{\mathrm{d}\hat{\eta}(x_w)}{\mathrm{d}x} = 0$.

Then we have a wave mode, trapped to the shallow region, that propagates in the y direction. Edge wave.

First define the position of the wall $\Rightarrow$ eigenvalue problem for $\hat{\eta}$.