The WKBJ method and optics MEK4320

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The WKBJ technique

WKBJ; example: LSW

Linear shallow water theory(indices mark differentiation)

$$\eta_{tt} - \nabla \cdot (c_0^2 \nabla \eta) = 0 \tag{1}$$

where $c_0^2 = gh(x, y)$.
In ray theory

$$\eta(x, y, t) = A(x, y, t)e^{i\chi(x,y,t)}, \tag{2}$$

where $\vec{k} \equiv \nabla\chi$, $\omega \equiv -\frac{\partial\chi}{\partial t}$.
Slow variations of $\vec{k}$ and $\omega \Rightarrow$ ray equations.
We now insert (2) in (1):

$$\begin{aligned} & A_{tt} - i(2\omega A_t + \omega_t A) - \omega^2 A = \\ & \nabla \cdot (c_0^2 \nabla A) + i\left(c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A\vec{k})\right) - c_0^2 k^2 A \end{aligned} \tag{3}$$

Exact, but nothing is achieved either – so far.

Scaling; transformation to non-dimensional form

Scales

Typical wavelength: λ_c (fast scale)

Typical length scale for medium change L_c (slow scale)

Small parameter $\frac{\lambda_c}{L_c} = \epsilon \ll 1$.

Typical wave speed: $c_c = \sqrt{gh_c}$

Typical amplitude: A_c – no significance as long as in linear regime

Typical phase: $\chi \sim L_c/\lambda_c = \epsilon^{-1}$

Rescaling

All derivatives explicit in (3) are with respect to slow variation.

Fast variation inherent in definitions of $\vec{k}$ and ω, only.

$\vec{\kappa} = \lambda_c \vec{k}$, $\hat{\omega} = \lambda_c \omega / c_c$, $\hat{A} = A/A_c$, $\hat{c} = c_0/c_c$, $\hat{x} = x/L_c$,
$\hat{t} = c_c t/L_c$.

Rescaling of (3)

$$\begin{aligned}&\epsilon^2 \hat{A}_{\hat{t}\hat{t}} - i\epsilon(2\hat{\omega}\hat{A}_{\hat{t}} + \hat{\omega}_{\hat{t}}\hat{A}) - \hat{\omega}^2\hat{A} = \\ &\epsilon^2\hat{\nabla}\cdot(c^2\hat{\nabla}\hat{A}) + i\epsilon\left(c^2\hat{\nabla}\hat{A}\cdot\vec{\kappa} + \hat{\nabla}\cdot(c^2\hat{A}\vec{\kappa})\right) - c^2\kappa^2\hat{A}\end{aligned} \tag{4}$$

Leading terms: $O(1)$ no slow differentiations
Next order terms: $O(\epsilon)$ one slow differentiation
Second order terms: $O(\epsilon^2)$ two slow differentiations

Leading order

$\hat{\omega}^2 = c^2\kappa^2 \Rightarrow$ restored scales $\omega^2 = c_0^2 k^2 = W^2$.
With $\vec{k}_t + \nabla\omega = 0$, $\partial k_i/\partial x_j = \partial k_j/\partial x_i$: ray theory retrieved

$$\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1, 2 \tag{5}$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla\omega = 0 \tag{6}$$

with $\vec{c}_g = c_0\vec{k}/k$. Now $\vec{k}$ and ω are settled.

Next order; $O(\epsilon)$ relative size

Original scales restored

$$-2\omega A_t - \omega_t A = c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}).$$

With $\omega = c_0 k$ and $\vec{c}_g = c_0 \vec{k}/k$ (then $c_0^2 \vec{k} = \omega \vec{c}_g$):

$$-2\omega A_t - \omega_t A = 2\omega \vec{c}_g \cdot \nabla A + A\vec{c}_g \cdot \nabla \omega + \omega A \nabla \cdot \vec{c}_g.$$

Next step multiply with A/ω and regroup

$$-(A^2)_t - \frac{A^2}{\omega}(\omega_t + \vec{c}_g \cdot \nabla \omega) = \nabla \cdot (\vec{c}_g A^2).$$

Due to (6) terms within last parentheses on l.h.s cancel out:

$$(A^2)_t = -\nabla \cdot (\vec{c}_g A^2). \tag{7}$$

With $E = \frac{1}{2}\rho g A^2$ (energy density) and $\vec{F} = \vec{c}_g E$ (energy flux) equation (7) reads

$$E_t + \nabla \cdot \vec{F} = 0, \tag{8}$$

Averaged energy conservation, as in uniform medium.

Remarks

Remark 1

Similar WKBJ approaches apply to most linear wave equations $\Rightarrow$ result with interpretation as energy conservation is general.

Remark 2

By means of (6) we have

$$(G(\omega)E)_t + \nabla \cdot \left(G(\omega)\vec{F} \right) = 0,$$

for any $G(\omega)$.

Remark 3

If we have coupling with a background current it is the wave action (E over some frequency) which is conserved.

Remark 4

To include higher order in ϵ we must expand $A = A_0 + \epsilon A_1 + ..$

Brief review of ray theory and optics.

Optics; summary

Ray theory (geometrical optics)

Harmonic wave, uniform medium$\Rightarrow$ dispersion relation

$$\omega = W(\vec{k}; H...)$$

Slow variation of medium and wave train $\Rightarrow$ local k and ω fulfill the dispersion relation (approximately) as in uniform medium.

Physical optics

Harmonic wave, uniform medium $\Rightarrow$ averaged energy relations

$$\vec{F} = \vec{c}_g E, \quad E = E(A^2, ..)$$

Slow variation $\Rightarrow$ $\vec{F} = \vec{c}_g E$ (approximately) as in uniform medium.

Equation of geometrical optics

Ray equations

From $\omega = -\frac{\partial \chi}{\partial t}$, $\vec{k} = \nabla \chi$ and $\omega = W(\vec{k}, x_i, t)$

$$\begin{aligned}\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\ \frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega &= \frac{\partial W}{\partial t}\end{aligned}$$

Recasted to Hamilton's canonical equations

$$\begin{aligned}\frac{\mathrm{d}k_i}{\mathrm{d}t} &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\ \frac{\mathrm{d}x_i}{\mathrm{d}t} &= \frac{\partial W}{\partial k_i} = (c_g)_i\,, \quad i = 1, 2.. \\ \frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\partial W}{\partial t}\end{aligned}$$

Equations of physical optics

The transport equation

Energy conservation, in general

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Invoking the approximation $F = c_g E$:

$$\frac{\partial E}{\partial t} + \frac{\partial (c_g E)}{\partial x} = 0$$

where $E = E(A^2, ..)$

More dimensions

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{c}_g E) = 0 \qquad (9)$$

calculation of wave field

1. $\vec{k}$ and ω are obtained from ray theory
2. The transport equation (9) is solved for A

Optics in uniform media; two examples

Uniform medium

$\omega = W(\vec{k}) \Rightarrow \vec{c}_g = c_g(\vec{k})$

Ray equations

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = 0, \qquad i = 1, 2..$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = (c_g)_i, \quad i = 1, 2..$$

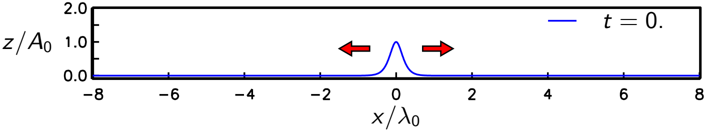
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

$\vec{k}$ and ω conserved along characteristics $\mathcal{C}$: $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{c}_g$

$\Rightarrow \vec{c}_g$ conserved $\Rightarrow \mathcal{C}$ are straight lines.

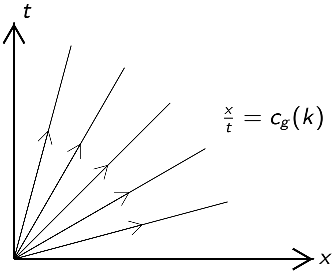
Characteristics are also straight lines viewed in the generalized x_i, t space.

Evolution of wave train from confined disturbance



- Previously: Fourier transform $\Rightarrow$ formal solution as integral.
- When propagation distances are much larger than initial length (λ_0): application of stationary phase, main spectral contribution from vicinity of k_s such that $c_g(k_s) = x/t$.
- Result: a slowly varying wave train.
- The slowly varying wave train should be within the realm of optics.
- Hence, problem revisited with optics. Approximation: initial disturbance is at $x = 0$.

In 1D: waves from point disturbance, at $x = 0$ at $t = 0$



All $\mathcal{C}$ intersects at $x = t = 0 \Rightarrow c_g = x/t$
$c_g(k) = x/t \Rightarrow k = k(x/t) \Rightarrow \chi = \int kdx$

Gravity waves in infinite depth; ray theory
$W = \sqrt{gk} \Rightarrow x/t = c_g = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{1}{4}g\frac{t^2}{x^2} \Rightarrow$
$\chi = -\frac{1}{4}g\frac{t^2}{x} + f(t)$
Furthermore $\frac{\partial \chi}{\partial t} = -\omega = -\sqrt{gk} \Rightarrow \chi = -\frac{1}{4}g\frac{t^2}{x} + \mathrm{const}$
Phase function as from the stationary phase solution.

Still infinite depth; the transport equation

$$\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}(c_g A^2) = 0$$

inserted $c_g = x/t$ and rewritten as

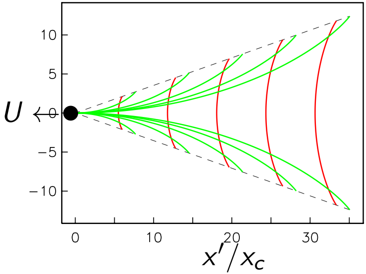
$$\frac{\partial x A^2}{\partial t} + \frac{x}{t}\frac{\partial (x A^2)}{\partial x} = 0$$

General solution: xA^2 is constant along $\mathcal{C} \Rightarrow$

$$A = x^{-\frac{1}{2}} G\left(\frac{x}{t}\right) = t^{-\frac{1}{2}} \hat{G}\left(\frac{x}{t}\right)$$

Consistent with stationary phase (spectrum$\Rightarrow$ G)
Interpretation: Energy between two characteristics ($\mathcal{C}$) is conserved.

Point source at the surface



Constant source velocity: $\vec{U} = -U\vec{\imath} \Rightarrow$ stationary and slowly varying wave pattern. Frame with fluid at rest $\Rightarrow$ isotropic dispersion relation

$$\vec{c}' = c_0(k)\frac{\vec{k}}{k} \quad (10)$$

Change of coordinate system

From fixed coordinate system ($\vec{r}'$) to a moving one ($\vec{r}$)

Frame follows the source $\vec{r} = \vec{r}' - \vec{U}t$.

New system: stationary source on a uniform current.

A harmonic mode then becomes:

$A\cos\chi = A\cos(\vec{k}\cdot\vec{r}' - \omega' t) = A\cos(\vec{k}\cdot\vec{r} - (\omega' + \vec{k}\cdot\vec{U})t)$

$$\omega = c_0(k)k + \vec{U}\cdot\vec{k} \equiv W(k_x, k_y) \tag{11}$$

$$\vec{c} = \left(c_0(k) + \vec{U}\cdot\frac{\vec{k}}{k}\right)\frac{\vec{k}}{k} \tag{12}$$

where $\vec{k} = k_x\vec{\imath} + k_y\vec{\jmath}$

Doppler shift $\Rightarrow$ Anisotropic dispersion.

Optics

New frame: stationary pattern implies $\omega = 0$. (11) yields:

$$W(k_x, k_y) = 0 \tag{13}$$

The group velocity

$$\vec{c}_g = \frac{\partial W}{\partial k_x}\vec{\imath} + \frac{\partial W}{\partial k_y}\vec{\jmath} \tag{14}$$

Hamilton's equations

$$\frac{d\vec{r}}{dt} = \vec{c}_g, \quad \frac{d\vec{k}}{dt} = 0 \tag{15}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \tag{16}$$

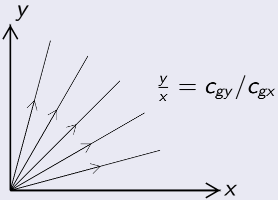
Uniform medium $\Rightarrow$ Characteristics are straight lines
To carry energy characteristics must pass through the source (the origin)

Straight characteristics through the origin:
$x = c_{gx} t$, $y = c_{gy} t$.
(Even for a stationary pattern characteristics can be parameterized by time; following the transport of energy)
Elimination of t:

$$\frac{y}{x} = \frac{\frac{\partial W}{\partial k_y}}{\frac{\partial W}{\partial k_x}} \tag{17}$$

combined with $W(k_x, k_y) = 0$ (13)$\Rightarrow$ two equations for k_x and k_y.

Characteristics



The Phase function

$$\chi(\vec{r}) = \chi_0 + \int_{C(\vec{r})} \vec{k} \cdot d\vec{r} \tag{18}$$

where χ_0 is the phase in the origin and $C(\vec{r})$ is some integration path.
Choosing C as a characteristic: Integration trivial because $\vec{k}$ is constant.

$$\chi(\vec{r}) = \chi_0 + k_x x + k_y y \tag{19}$$

Phase lines $\chi = -A$. Two options for visualization/interpretation

- A quick overview of the phase line readily obtained by contour plot of χ (Matlab or Python etc.).
- Parameterization of phase lines. Some uncanny trigonometry, but the presence of two families of solutions for $\vec{k}$ from (17) and (13) is demonstrated.

Parameterization of phase lines (infinite depth)

θ: angle between $\vec{k}$ and negative x-axis

$$k_x = -k\cos\theta, \quad k_y = k\sin\theta. \tag{20}$$

(13) is rewritten

$$c_0 = U\cos\theta,$$

and $c_0(k) = \sqrt{g/k}$ then yields

$$k = \frac{g}{U^2\cos^2\theta} \tag{21}$$

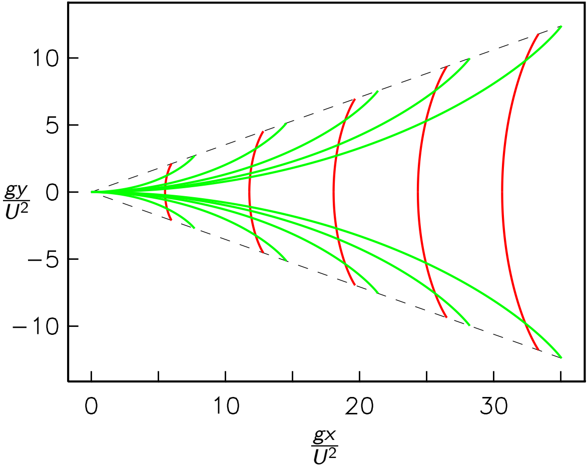
(17) and (19) are solved for x and y

$$x = \frac{(A-\chi_0)g}{U^2}\cos\theta(1+\sin^2\theta) \tag{22}$$

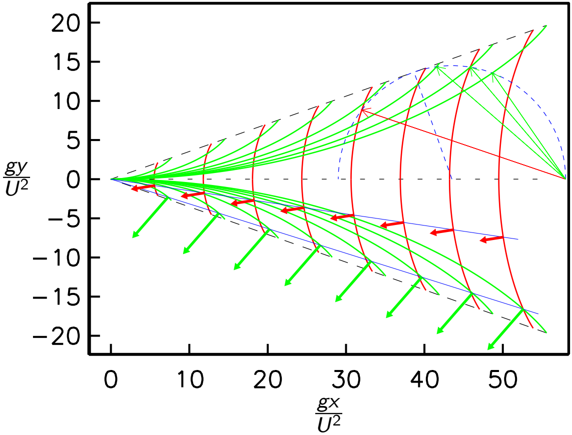
$$y = \frac{(A-\chi_0)g}{U^2}\cos^2\theta\sin\theta \tag{23}$$

Note: $y(\theta)/x(\theta)$ extreme for $\cos\theta = \sqrt{2/3}$ $(\theta = \theta_c = 35.3^\circ)\Rightarrow$
jump in direction of phase lines $\Rightarrow$ independent solutions

Point source: other techniques show $\chi_0 = \frac{1}{4}\pi, -\frac{1}{4}\pi$ for transverse and diverging waves, respectively.



The Kelvin pattern, more details

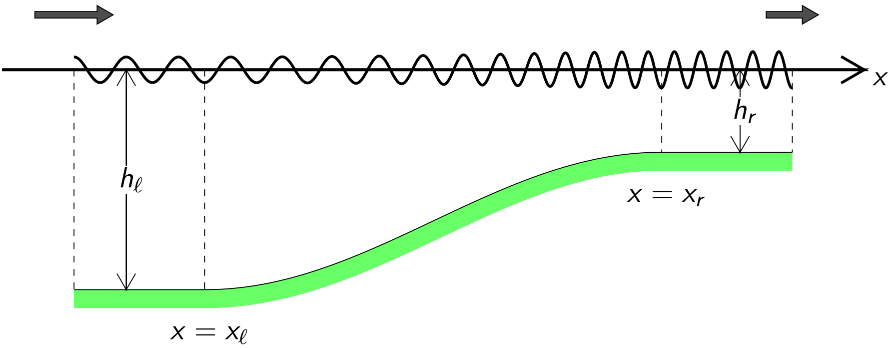


Fat arrows: wave number vectors.

Dashed half circle: propagation with $\vec{c}_g$ from intersection with x-axis, subject to $c_0 = U\cos\theta$. Thin arrows: corresponding rays.

Amplification and refraction in bathymetry; Caustics

GEOMETRY AND WAVE FIELD.



Plane waves. Normal incidence on a sloping bottom.

Green's law

Plane waves, normal incidence , $h = h(x)$, $\vec{k} = k\vec{\imath}$
Ray theory $\Rightarrow \omega = \mathrm{const.}$, $k = \omega/c_0$, $\chi = \int k dx$
Transport equation for plane waves $\Rightarrow$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow F = \mathrm{const.} \Rightarrow c_0 A^2 = \mathrm{const.}$$

Constant energy flux

Substituting $c_0 = \sqrt{gh}$ we find Green's law:

$$A = A_0 \left(\frac{h}{h_0}\right)^{-\frac{1}{4}}, \tag{24}$$

where A_0 and h_0 correspond to a reference location.

Comparison with accurate numerical solution

$\eta = \hat{\eta}(x)e^{-i\omega t} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \omega^2\hat{\eta} = 0. \tag{25}$$

For any x_a such that $x_a < x_\ell$ (incident + reflected wave)

$$\hat{\eta} = A_0 e^{ik_\ell x} + Re^{-ik_\ell x}, \quad \text{where} \quad \omega = \sqrt{gh_\ell}k_\ell,$$

where R is unknown. Annihilation of reflected part

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_\ell\hat{\eta} = 2iA_0k_\ell e^{ik_\ell x}, \quad \text{for} \quad x = x_a. \tag{26}$$

At $x = x_b > x_r$ only transmitted wave: $\hat{\eta} = Te^{ik_r x}$, and

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_r\hat{\eta} = 0, \quad \text{where} \quad \omega = \sqrt{gh_r}k_r. \tag{27}$$

(25,26,27) ODE and boundary conditions for $\hat{\eta}$. Moreover: $A = |\hat{\eta}|$

FDM

Numerical approximation : $\hat{\eta}_j \approx \hat{\eta}(j\Delta x)$, $h_{j+\frac{1}{2}} = h((j+\frac{1}{2})\Delta x)$,
Discrete version of (25)

$$\frac{gh_{j+\frac{1}{2}}(\hat{\eta}_{j+1} - \hat{\eta}_j) - gh_{j-\frac{1}{2}}(\hat{\eta}_j - \hat{\eta}_{j-1})}{\Delta x^2} + \omega^2 \hat{\eta}_j = 0.$$

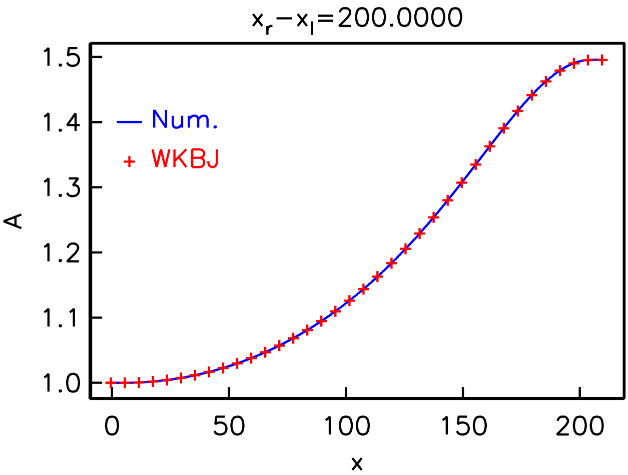
Boundary conditions (grid from $i = 0$ to $i = N$)

$$\frac{\hat{\eta}_1 - \hat{\eta}_0}{\Delta x} + \frac{i}{2} k_\ell (\hat{\eta}_1 + \hat{\eta}_0) = 2iA_0 k_\ell e^{i\frac{1}{2}k_\ell \Delta x},$$

$$\frac{\hat{\eta}_N - \hat{\eta}_{N-1}}{\Delta x} - \frac{i}{2} k_r (\hat{\eta}_N + \hat{\eta}_{N-1}) = 0.$$

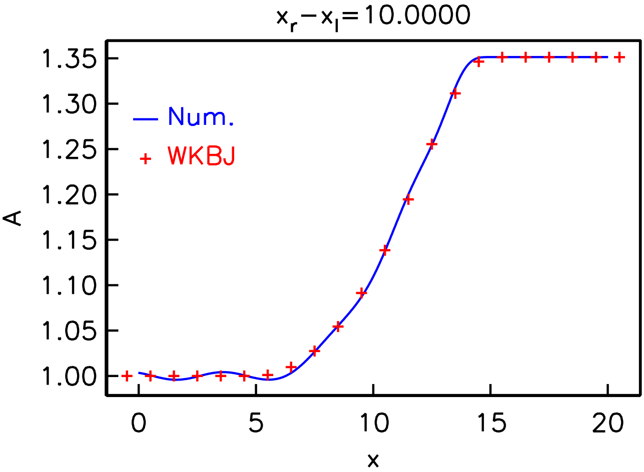
Tri-diagonal set with closure from boundary conditions.
NB: Boundary conditions may be amended by numerical dispersion relation to become exact for the discrete case.

Gentle slope



A normalized by A_ℓ. x by $\hat{x}$, arbitrary horizontal unit.
$x_\ell = 5\hat{x}$, $\lambda_\ell = 8\hat{x}$, $h_r = 0.2h_\ell$, $x_r - x_\ell = 200\hat{x} = 25\lambda_\ell$

Steep slope



$x_\ell = 5\hat{x},\ \lambda_\ell = 8\hat{x},\ h_r = 0.3h_\ell,\ x_r - x_l = 10\hat{x} = \frac{5}{4}\lambda_\ell$

Remarks on optics for shoaling

- Optics good even when $L = x_r - x_\ell$ and λ are comparable.
- Major discrepancy between optics and accurate numerical solution: Reflections.
- Optics do not incorporate reflections.

Oblique incidence, plane geometry

We still have $h = h(x)$, but $\vec{k} = k_x\vec{\imath} + k_y\vec{\jmath}$.
θ: angle of incidence; $k_x = |\vec{k}|\cos\theta$, $k_y = |\vec{k}|\sin\theta$.
Local wave celerity $c = \sqrt{gh(x)}$

Geometrical optics

Hamilton's equations, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}k_y}{\mathrm{d}t} = 0 \Rightarrow \omega = \mathrm{const.}$, $k_y = \mathrm{const.}$.
Then $\omega = W(|\vec{k}|, x) = \sqrt{gh}|\vec{k}| = ck_y/\sin\theta$ and

$$\frac{c}{\sin\theta} = \frac{\omega}{k_y} = \mathrm{const.}$$

Snell's law

Physical optics; transport equation for A

$$\nabla \cdot \vec{F} = 0, \quad \Rightarrow \quad \nabla \cdot \left(\vec{c}_g A^2\right) = 0$$

Since $A = A(x)$ (uniformity in y), this yields constant energy flux density in x-direction

$$c_{gx} A^2 = \text{const.}$$

Now $c_{gx} = c\cos\theta$ and by Snell's law $c_{gx} = \omega \cos\theta \sin\theta / k_y$. Constant energy flux density then yields

$$\sin(2\theta) A^2 = \text{const.}$$

$\theta < \frac{\pi}{4}$: c_{gx} decreases with shoaling $\Rightarrow$ amplification
$\theta > \frac{\pi}{4}$: c_{gx} increases with shoaling due to refraction $\Rightarrow$ attenuation

Deep water caustic

Snell's law and amplitude relation

$$\frac{c}{\sin\theta} = \frac{\omega}{k_y} = \text{const.}, \quad c_{gx}A^2 = \text{const.}$$

When $h \to h_c = \omega^2/(gk_y^2)$ $(c \to \omega/k_y)$

$$\theta \to \frac{\pi}{2}, \quad k_x \to 0, \quad A \to \infty.$$

Ray parallel to y-axis; breakdown of optics.
Natural assumption: waves are reflected and propagate into shallower water again.
A mechanism for trapping of waves.
Some more analysis is required.

Composite solution; Three domains*

Throughout $\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0. \qquad (28)$$

(i): h well below h_c

Optics as described previously yield solution. This corresponds to a WKBJ expansion. $\hat{\eta}(x)$ is wavy.

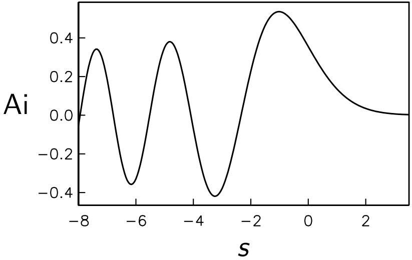
(ii): h around h_c

Local expansion: $h \approx h(x_c) + h'(x_c)(x - x_c) + \ldots$. Keeping leading order terms of (28), rescaling

$$\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}s^2} = s\hat{\eta}, \quad \text{where} \quad s = \left(\frac{\omega^2}{h'(x_c)k_y^4}\right)^{\frac{1}{3}}(x - x_c)$$

Solution that remains finite for large $x - x_c$: $\hat{\eta} = B\mathrm{Ai}(s)$

Combines wavy $\hat{\eta}$ for $h < h_c$ with exponential decay for $h > h_c$.



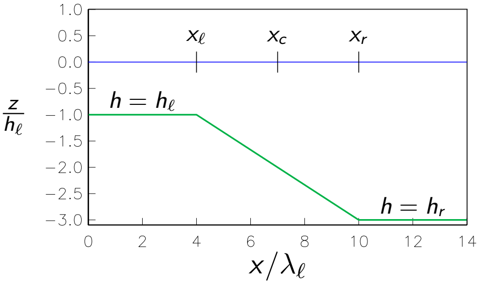
(iii): h well above h_c

Use WKBJ with $\hat{\eta} = e^{-S(x)}$, where S is real and $S \to \infty$ as $x - x_c \to \infty$. Exponential decay.

Combination

"Asymptotic match": (i) with (ii), and (ii) with (iii).
Further details omitted.

Simplified geometry reflection/transmission problem



Possible caustic

Angle of incidence θ_ℓ.

$$h_c = \frac{h_\ell}{\sin^2 \theta_\ell}$$

$x < x_\ell$: incident and reflected waves.

If $h_r < h_c$

Refraction and partial reflection on slope.

$x > x_r$: uniform transmitted wave.

If $h_r > h_c$

Total reflection from slope.

$x > x_r$: decaying η.

Formulation of reflection/transmission boundary value problem

$\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0.$$

For $x_a < x_\ell$ we have, as for normal incidence,

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_x\hat{\eta} = 2iA_0k_x e^{ik_x x}.$$

where k_x is found from $\omega = W(k_x, k_y, x_a) = \sqrt{gh_\ell}|\vec{k}|$.
If case $h_r < h_c$ and $x_b > x_r$ we again have

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_x\hat{\eta} = 0,$$

where k_x is found from $\omega = W(k_x, k_y, x_b) = \sqrt{gh_r}|\vec{k}|$.

Reflection, $h_r > h_c$

Since (28) has constant coefficients when $x > x_r$ two solutions are readily found

$$\hat{\eta} = B_1 e^{-\alpha x}, \quad \hat{\eta} = B_2 e^{\alpha x}, \quad \text{where} \quad \alpha = \sqrt{k_y^2 - \frac{\omega^2}{gh_r}}.$$

To obtain only the decaying solution we employ

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + \alpha\hat{\eta} = 0, \quad \text{at} \quad x = x_b.$$

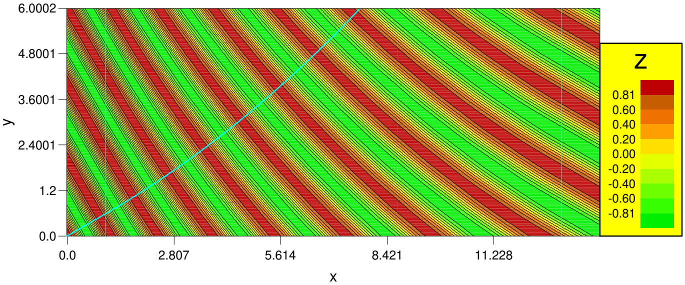
Here x_b may have any value larger than x_r.

If $x_b - x_c$ is sufficiently large we may instead employ

$$\hat{\eta}(x_b) = 0.$$

Numerics as for normal incidence.

Transmission; $\theta_\ell = 30^\circ,\ h_r = 3h_\ell,\ h_c = 4h_\ell$

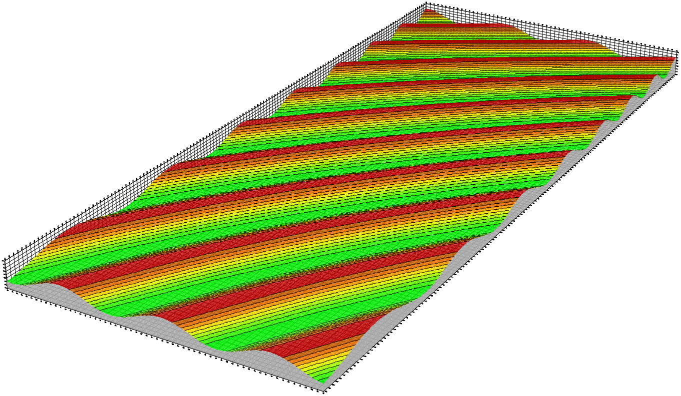


Plot: x and y normalized by $\lambda_\ell = 2\pi c_\ell/\omega$.

$x = x_\ell$, $x = x_r$ and one ray are marked by lines.

$x_r - x_\ell = 12\lambda_\ell$.

Virtually no reflection, only refraction.



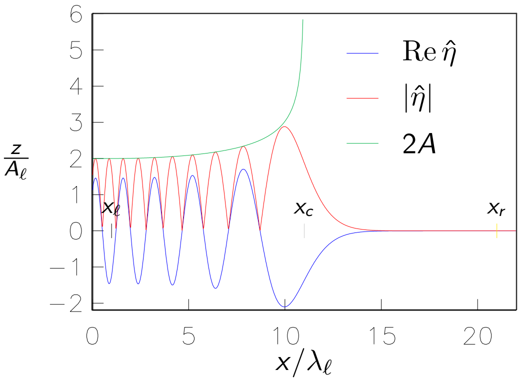
Surface seen from the deep region.

Case with full reflection

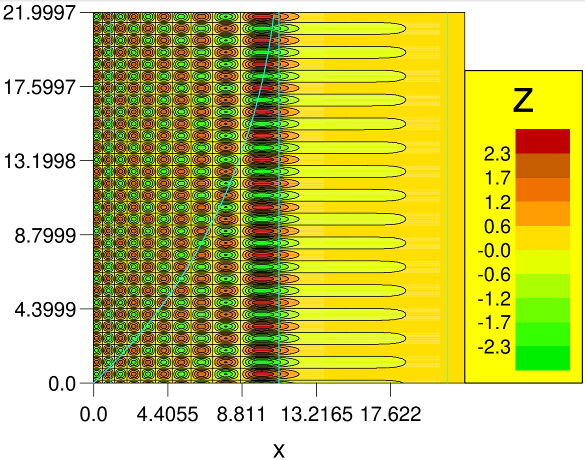
Same geometry and wavelength
$h_r = 3h_\ell,\ x_r - x_\ell = 12\lambda_\ell$

Increased angle of incidence
$\theta_\ell = 45^\circ \Rightarrow h_c = 2h_\ell < h_r$
Full reflection

First a transect $y = \mathrm{const.}$



A_ℓ amplitude of incident wave for $x < x_\ell$.
Optics for both incident and reflected solution: curve marked with $2A$.



Horizontal axes normalized by λ_ℓ.

Impermeable walls may be introduced at any x_w where $\frac{\mathrm{d}\hat{\eta}(x_w)}{\mathrm{d}x} = 0$.

Then we have a wave mode, trapped to the shallow region, that propagates in the y direction. Edge wave.

First define the position of the wall $\Rightarrow$ eigenvalue problem for $\hat{\eta}$.