Mek4100
The WKB method

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Demonstration problem

Oscillation problem with variable coefficient

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + W(\epsilon t)y = 0; \quad y(t_a) = a, \quad y(t_b) = b, \qquad (1)$$

where $W > 0$.

ϵ – small parameter $\Rightarrow$ coefficient W is slowly varying.

Note 1: Equation solved by two-scale technique in problem 24b.

Note 2: The problem (1) does not always inherit a solution. Will be demonstrated in specific example.

Note 3: $W = \mathrm{const.} \Rightarrow$ Exact solution $y = A_+ e^{\mathrm{i}\sqrt{W}t} + A_- e^{-\mathrm{i}\sqrt{W}t}$.

Preparation for WKB; rescaling

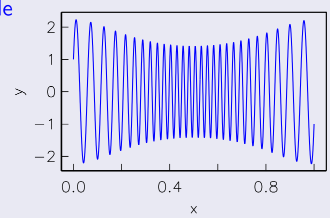
Choose slow scale as free variable $x = \epsilon t$

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + W(x)y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \qquad (2)$$

Looks like a boundary layer problem, but solution oscillates rapidly everywhere.

(2) not well scaled in the usual sense since $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ becomes unbounded as $\epsilon \to 0$. Scaling in (2) is common; convenient but not necessary.

Solution example



Use of exponential form

Write solution in terms of new unknown

$$y = e^{S(x)}.$$

Substitution into 2 yields equation for $k(x) \equiv S'$:

$$\epsilon^2(k' + k^2) + W = 0, \qquad (3)$$

First order nonlinear ODE, called a Ricatti equation.

So far no real approximation or progress are made; (3) is still not solvable in formula for general W. But;

(3) makes a good starting point for dominant balance analysis.

The transformation by means of the exponential form only feasible for linear, homogeneous equations

Dominant balance

$$\begin{array}{ccccccc} \epsilon^2 k' & + & \epsilon^2 k^2 & + & W & = & 0 \\ (1) & + & (2) & + & (3) & = & 0 \end{array} \qquad (4)$$

(1) & (3): $k \sim -\epsilon^{-2} \int W dx \Rightarrow$ (2)$\sim \epsilon^{-2} \gg$ (1), (3). Invalid!

(1) & (2): $k \sim (C + x)^{-1} \Rightarrow$ (3) dominates as $\epsilon \to 0$, $y \sim x + C$. Invalid!

(2) & (3): $k \sim k_0 = \pm \mathrm{i}\epsilon^{-1} W^{\frac{1}{2}}$. $\Rightarrow$ (1)$\sim \epsilon \ll$ (2), (3). Two solutions. OK!*

$$y \sim e^{\pm \mathrm{i}\epsilon^{-1} \int W^{\frac{1}{2}} dx}$$

Describes rapid oscillations. This is the full solution if W is constant.

*: Since $|k| \to \infty$ as $\epsilon \to 0$ we could guess $k^2 \gg k'$ in the first place

Second balance

$k = k_0 + k_1$, $k_1 \ll k_0$. Substitution in (4)

$$\epsilon^2(k_0' + k_1' + k_0^2 + 2k_0 k_1 + k_1^2) + W = 0$$

Canceling of leading order $\epsilon^2 k_0^2 + W = 0$ and $k_1 \ll k_0 \Rightarrow$

$$\epsilon^2(k_0' + 2k_0 k_1) = 0,$$

with solution $k_1 = -\frac{1}{2}k_0'/k_0 = -\frac{1}{4}W'/W = O(1)$.

Third balance

$k = k_0 + k_1 + k_2$, $k_2 \ll k_1 \ll k_0$, canceling etc.$\Rightarrow$

$$\epsilon^2(k_1' + k_1^2 + 2k_0 k_2) = 0,$$

$$k_2 = -\frac{k_1' + k_1^2}{2k_0} = \mp \mathrm{i}\epsilon \left(\frac{W''}{8W^{\frac{3}{2}}} - \frac{5(W')^2}{32W^2} \right).$$

Assembling the solutions

$$S_\pm = \int k dx = C_\pm \pm \frac{\mathrm{i}}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x} + \ln\left(W^{-\frac{1}{4}}\right) \mp \mathrm{i}\epsilon\alpha,$$

where $\alpha = \int_{x_a}^{x} \{\frac{1}{8}W''W^{-\frac{3}{2}} - \frac{5}{32}(W')^2 W^{-2}\} d\hat{x}$ and $C_\pm$ is a constant of integration.
The two solutions for y are written ($\epsilon\alpha \ll 1$)

$$y_\pm = A_\pm W^{-\frac{1}{4}} e^{\pm\frac{\mathrm{i}}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} e^{\mp\mathrm{i}\epsilon\alpha} = A_\pm W^{-\frac{1}{4}} e^{\pm\frac{\mathrm{i}}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} (1 \mp \mathrm{i}\epsilon\alpha + O(\epsilon^2))$$

Choosing $A_- = A_+^*$ makes $y_- + y_+$ real. Moreover, real and imaginary parts of A_- may be found as to make the boundary conditions fulfilled.
From k_1 we obtain an amplitude modulation given by $W^{-\frac{1}{4}}$.

Digression: The formal WKB expansion

The end results were expansions of type

$$y = e^{\mathrm{i}\frac{\phi(x)}{\epsilon}} \left(A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \ldots\right). \tag{5}$$

1. (5) often used as an anzats.
2. Reduces the expansion to unimaginative book-keeping.
3. (5) sometimes not appropriate.
4. The higher A_j seldom significant.

The form (5) is akin to solution for constant W, namely $y = Ae^{\frac{\mathrm{i}}{\epsilon}W^{\frac{1}{2}}x} = Ae^{\frac{\mathrm{i}}{\epsilon}\int W^{\frac{1}{2}}dx}$

The explicit real solution

When α is ignored the sum $y_- + y_+$ may be recast into the form

$$y = W^{-\frac{1}{4}}(B\cos\psi + C\sin\psi), \tag{6}$$

where $\psi = \epsilon^{-1}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}$ and $A_+ = \frac{1}{2}(B - \mathrm{i}C)$.
Boundary conditions $\Rightarrow$

$$B = aW(x_a)^{\frac{1}{4}}, \quad C = \left(bW(x_b)^{\frac{1}{4}} - aW(x_a)^{\frac{1}{4}}\cos\psi(xb)\right)\frac{1}{\sin\psi(x_b)}.$$

No solution if $\sin\psi(x_b) = 0$, meaning $\epsilon^{-1}\int_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x} = n\pi$.

A specific case

Selected parameters (ϵ is not fixed!): $x_a = 0$, $x_b = 1$, $a = 1$, $b = -1$,
and function

$$W^{\frac{1}{2}} = Q + R\cos^2(x - \frac{1}{2})\pi, \quad \psi = \frac{1}{\epsilon}\Big[(Q + \frac{1}{2}R)x + \frac{R}{4\pi}sin(2x-1)\pi\Big],$$

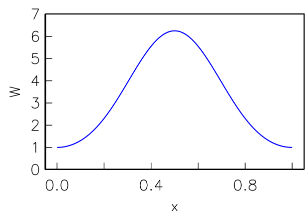


Figure: $W(x)$ for $Q = 1$ and $R = \frac{3}{2}$.

Numerical method

Define $y_j \approx y(j\Delta x)$ for $j = 0, \ldots n$ and $\Delta x = \frac{1}{n}$.
Tri-diagonal set of equations, solved by Gaussian elimination

$$y_0 = a,$$
$$\frac{1}{\Delta x^2}(y_{j+1} - 2y_j + y_{j-1}) + \frac{1}{\epsilon^2}W_j y_j + s_j = 0, \quad j = 1, \ldots, n-1,$$
$$y_n = b,$$

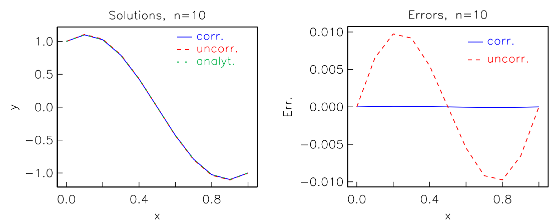
where correction terms

$$s_j = \frac{1}{12\epsilon^2}(W_{j+1}y_{j+1} - 2W_j y_j + W_{j-1}y_{j-1}),$$

reduce errors to $O(\Delta x^4)$.

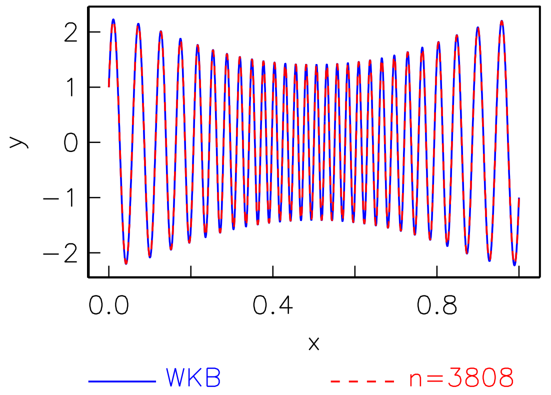
Test of numerical method

Constant coefficients, large ϵ: $Q = 4$, $R = 0$, $\epsilon = 1$.
WKB formula is exact.



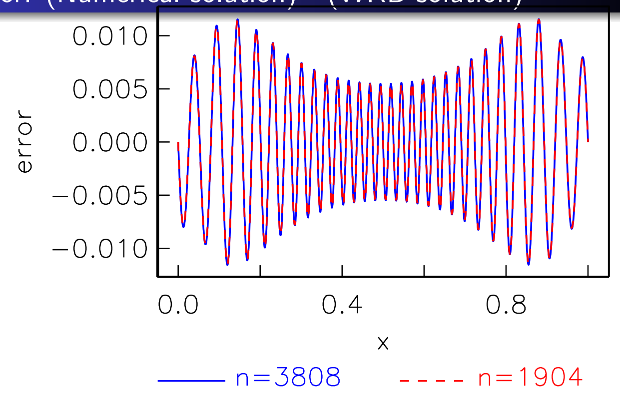
More test runs are performed to assure convergence, but not shown here.
Such verification is tedious, but mandatory!
Resolution must still be checked for runs with small ϵ, which are much more demanding.

Solutions for $\epsilon = 0.01$; $Q = 1$ and $R = \frac{3}{2}$



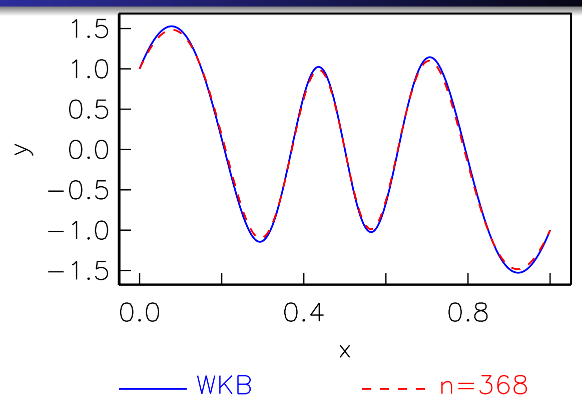
Numerical solution marked by value of n (number of points).

Error: (Numerical solution) - (WKB solution)



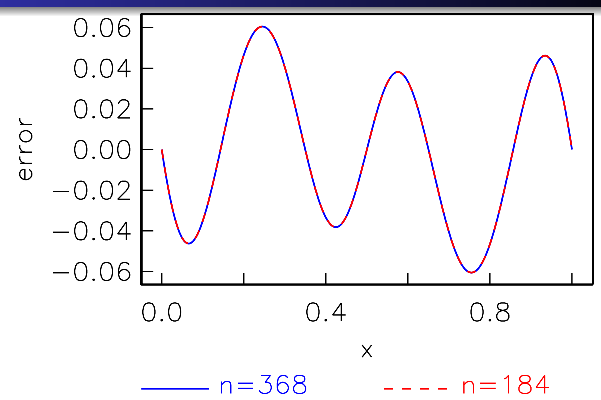
Still, $\epsilon = 0.01$, $Q = 1$ and $R = \frac{3}{2}$.
Difference "numerical - WKB" is not noticeably dependent on n.
Error is 0.5%, say, of typical value of y.

Results $\epsilon = 0.1$



Numerical solution marked by value of n.
For this larger ϵ: WKB still quite good, but error visible

Errors for $\epsilon = 0.1$



Again, numerical solution is not noticeably dependent on n.
Error is 4%, say, of typical value of y.

Convergence in ϵ

$< f >= \int_0^1 f\,dx$, evaluated by trapezoidal integration

$$L_2 = \sqrt{< (y_{\text{num}} - y_{\text{WKB}})^2 >}, \quad E_r = L_2 / \left(\epsilon \sqrt{< (y_{\text{WKB}})^2 >}\right)$$

ϵ	L_2	E_r
0.10	$0.36 \cdot 10^{-1}$	0.38
$0.50 \cdot 10^{-1}$	$0.11 \cdot 10^{-1}$	0.39
$0.25 \cdot 10^{-1}$	$0.14 \cdot 10^{-1}$	0.42
$0.10 \cdot 10^{-1}$	$0.60 \cdot 10^{-2}$	0.48
$0.50 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$	0.33
$0.25 \cdot 10^{-2}$	$0.44 \cdot 10^{-3}$	0.30

nf=32 (measure of resolution)
Solution changes qualitatively with $\epsilon \Rightarrow E_r$ remains of same size, but does not approach a constant in displayed range.

WKB and a boundary layer problem

Change: sign on coefficient in the equation

$$\epsilon^2 \frac{d^2 y}{dx^2} - W(x) y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \tag{7}$$

where $W > 0$. Solutions are now of rapidly growing/decaying nature instead of oscillating.

The boundary layer method

The problem is virtually contained in problem 64 in leaflet. The unified solution becomes

$$y \approx a e^{-\sqrt{W(x_a)}\frac{(x-x_a)}{\epsilon}} + b e^{\sqrt{W(x_b)}\frac{(x-x_b)}{\epsilon}}. \tag{8}$$

Boundary layers at both ends, zero as outer solution.

The WKB expansion applied to (7)

All the algebra of the first examples repeats itself, except for the occurrence of i, the imaginary unit. Using k_0 and k_1:

$$y \approx A_+ W^{-\frac{1}{4}} e^{\frac{1}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} + A_- W^{-\frac{1}{4}} e^{-\frac{1}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}}.$$

Boundary conditions

$$W(x_a)^{-\frac{1}{4}}(A_+ + A_-) = a, \quad W(x_b)^{-\frac{1}{4}}(\gamma A_+ + \gamma^{-1}A_-) = b,$$

where $\gamma = e^{\frac{1}{\epsilon}\int_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x}} \gg 1$. Hence,

$$A_+ = \frac{a\gamma^{-1}W(x_a)^{\frac{1}{4}} - bW(x_b)^{\frac{1}{4}}}{\gamma^{-1}-\gamma}, \quad A_- = \frac{bW(x_b)^{\frac{1}{4}} - a\gamma W(x_a)^{\frac{1}{4}}}{\gamma^{-1}-\gamma}$$

How to reconcile this with (8) ?

First $\gamma \gg 1 \Rightarrow A_+ \approx b\gamma^{-1}W(x_b)^{\frac{1}{4}}$ and $A_- = aW(x_a)^{\frac{1}{4}}$; thus

$$y \approx \frac{b}{\gamma}\left(\frac{W(x_b)}{W(x)}\right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon}\int_{x}^{x_b} W^{\frac{1}{2}} d\hat{x}} + a\left(\frac{W(x_a)}{W(x)}\right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon}\int_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}}. \quad (9)$$

First term grows rapidly toward x_b: boundary layer at right end. Second term decays rapidly from x_a: boundary layer at left end. Right term significant only when $x - x_a$ is small. Taylor expansion

$$\int_{x_a}^{x} \frac{W^{\frac{1}{2}}}{\epsilon} d\hat{x} = \left(\frac{W(x_a)^{\frac{1}{2}}(x-x_a)}{\epsilon} + \frac{(W(x_a)^{\frac{1}{2}})'(x-x_a)^2}{2\epsilon} + \ldots\right),$$

For a region $1 \gg x - x_a \gg \epsilon$ the first term $\gg 1$ while the second $\ll 1$. Example: $x - x_a = \epsilon^{\frac{2}{3}}$; first term $\sim \epsilon^{-\frac{1}{3}}$, second term $\sim \epsilon^{\frac{1}{3}}$. Consequence: second term in (9) vanishes before second term in Taylor expansion becomes important. We may then also put $W(x)/W(x_a) \approx 1$, meaning that k_1 is ignored.

Similar treatment of first term in (9) gives

$$y \approx be^{-\frac{1}{\epsilon}W(x_b)^{\frac{1}{2}}(x_b - x)} + ae^{-\frac{1}{\epsilon}W(x_a)^{\frac{1}{2}}(x-x_a)}. \quad (10)$$

Which is the boundary layer solution (8) retrieved.

- Homogeneous, linear boundary layer problems may be solved with WKB techniques
- Boundary layer solutions consistent with leading order WKB solution
- Quite some simplification needed to reveal the full relationship

Relation to theorem 3.12 in Logan

Boundary value problem

$$\epsilon y'' + p(x)y' + q(x) = 0, \quad y(0) = a, \quad y(1) = b, \quad (11)$$

where $\epsilon \to 0$, $p(x) > 0$, $p, q \sim 1$
Again

$$y = e^{S(x)} = e^{\int k d\hat{x}}.$$

It is important that we do not assume (5).

$$\begin{array}{ccccccccc} \epsilon k' & + & \epsilon k^2 & + & pk & + & q & = & 0 \\ (1) & + & (2) & + & (3) & + & (4) & = & 0 \end{array} \quad (12)$$

(1) & (4): $k \sim -\epsilon^{-2}\int q dx \Rightarrow (2) \sim \epsilon^{-1} \gg (1), (3)$. Invalid!
(1) & (2): $k \sim (C+x)^{-1} \Rightarrow (4)$ dominates as $\epsilon \to 0$, $y \sim x + C$. Invalid!
(2) & (4): $k \sim \pm i\epsilon^{-\frac{1}{2}}q^{\frac{1}{2}}. \Rightarrow (3) \sim \epsilon^{-\frac{1}{2}} \gg (2), (4)$. Invalid!
(2) & (3): $k \sim -\frac{p}{\epsilon}$. One valid solution.
(3) & (4): $k \sim -\frac{q}{p}$. One valid solution. (Not on form (5)!)
(1) & (3): $k \sim Ce^{-\epsilon^{-1}\int_0^x pd\hat{x}}$. $(3) \ll (4)$ when $x \gg \epsilon$. Discarded!

Then

$$y = Ae^{-\int \frac{q}{p}dx} + Be^{-\int \frac{p}{\epsilon}dx}$$

Outer and boundary layer approximations inherited. Theorem 3.12 from Logan may be reproduced; details omitted.