Ex. 22 .

a)

x	m	t	C	k	x_0
L	M	T	$\frac{M}{TL^2}$	$\frac{M}{T^2}$	L

Π theorem, part 1: 6 parameters, 3 dimensions $\Rightarrow$ 3 dimensionless numbers. Simplest options:
Make the first, only, with x_*. The simplest option: $\pi_1 = \frac{x_*}{x_0}$

Make the next,only, with t_*. Use also m and k (not x'es): $\pi_2 = \frac{kt_*^2}{m}$

The last is the only made with C. Avoid x_* and t_*. Then the theorem tells us that there is on π between C, m, x_0 and k: $\pi_3 = \frac{Cx_0^2}{\sqrt{mk}}$.

b) We scale the problem according to

$$x_* = x_c x, \quad t_* = t_c t.$$

From the transformation of the initial condition from $x_*(0) = x_0$ to $x(0) = 1$ we find $x_c = x_0$. Then both initial conditions are fulfilled regardless of t_c. Using $x_c = x_0$, inserting the transformation in the ODE and reorganize the result such that the coefficient of the second derivative becomes unity we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{Cx_0^2 t_c}{m} x^2 \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{t_c^2 k}{m} x = 0.$$

To reproduce the dimensionless ODE, given in the problem text, we must have

$$\frac{Cx_0^2 t_c}{m} = \epsilon, \quad \frac{t_c^2 k}{m} = 1,$$

which imply

$$t_c = \sqrt{\frac{m}{k}}, \quad \epsilon = \frac{Cx_0^2}{\sqrt{mk}}.$$

t_c expresses the period of the undamped oscillator. ϵ increases with C and x_0 and decreases with m and k, as can be expected.
x, s and ϵ are dimensionless numbers which may expressed by π_1, π_2 and π_3. In fact:
$\epsilon = \pi_3$, $x = \pi_1$ and $t = \sqrt{\pi_2}$.

If we had obtained different π's in sub-problem a, the above relations would also be different.

c) Introduction of τ, such that $x = x(t, \tau)$, yields the set

$$\frac{\partial^2 x}{\partial t^2} + 2\epsilon \frac{\partial^2 x}{\partial t \partial \tau} + \epsilon x^2 \frac{\mathrm{d}x}{\mathrm{d}t} + x = O(\epsilon^2),$$

$$x(0,0) = 1, \quad \frac{\partial x(0,0)}{\partial t} + \epsilon \frac{\partial x(0,0)}{\partial \tau} = 0.$$

The expansion $x = x_0(s, \tau) + \epsilon x_1(t, \tau)$ is then inserted.

Order ϵ^0

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = O,$$

$$x_0(0,0) = 1, \quad \frac{\partial x_0(0,0)}{\partial t} = 0.$$

The solution is

$$x_0 = A(\tau)e^{it} + A^*(\tau)e^{-it} = A(\tau)e^{it} + c.c., \quad A(0) = \frac{1}{2}.$$

Order ϵ^1

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -2\frac{\partial^2 x_0}{\partial t \partial \tau} - x_0^2 \frac{\mathrm{d}x_0}{\mathrm{d}t}.$$

$$x_1(0,0) = 0, \quad \frac{\partial x_1(0,0)}{\partial t} = -\frac{\partial x_0(0,0)}{\partial \tau}.$$

Inserting the expression for x_0 on the right hand side of the ODE we arrive at

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = iA^3 e^{3it} - i\left(2\frac{\mathrm{d}A}{\mathrm{d}\tau} + A^2 A^*\right) e^{it} + c.c.$$

The solution must be damped. Hence, we do not allow x_1 to grow linearly in t. Then, the e^{it} and e^{-it} parts of the right hand side must vanish, which implies

$$2\frac{\mathrm{d}A}{\mathrm{d}\tau} + A^2 A^* = 0.$$

Substituting the polar form $A = ae^{i\psi}$, where a and ψ are real, into this we obtain

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} + \frac{1}{2}a^3 = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\tau} = 0.$$

The first one is a separable equation

$$-2a^{-3}\frac{\mathrm{d}a}{\mathrm{d}\tau} = 1 \quad \Rightarrow \quad a^{-2} = \tau + B \quad \Rightarrow \quad a = (B+\tau)^{-\frac{1}{2}}.$$

The second immediately yields $\psi = D$ =const. The initial condition for A now implies $B = 4$ and $D = 0$. Then

$$A = (4+\tau)^{-\frac{1}{2}}.$$

Assembling the leading order solution we then find

$$A = \frac{1}{\sqrt{1+\frac{\epsilon t}{4}}}\cos(t).$$