ASYMPTOTIC WAVE-FRONT

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The Fourier transform

Transform and inversion (may be defined in slightly different ways)

$$\tilde{f} = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-kx}\mathrm{d}x, \quad f = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\mathrm{e}^{kx}\mathrm{d}k.$$

Initial surface elevation: $\eta(x,0) = \eta_0(x)$; no initial velocities.
Fourier transform on linear inviscid set $\Rightarrow$ The right-going wave system:

$$\eta(x,t) = \tfrac{1}{2\pi}\Re\int_0^\infty \tilde{\eta}_0 \mathrm{e}^{\imath\chi}\mathrm{d}k$$
$$\chi \equiv kx - \omega(k)t,$$

where η_0 is assumed symmetric around $x = 0$.

Stationary phase

Stationary point
$$\frac{\mathrm{d}\chi(k_s)}{\mathrm{d}k} = 0$$
corresponding to
$$c_g(k_s) = \frac{x}{t}$$

Neighbourhood of k_s $\Rightarrow$ dominant contribution to Fourier integral
Limitation in finite depth
Depth: H
$c_g \le c_0 \equiv \sqrt{gH}$ $\Rightarrow$ Stationary phase for $\frac{x}{t} \le c_0$
Neighbourhood $\frac{x}{t} \sim c_0$ $\Rightarrow$ different approach

Wave front: close to $x = c_0 t$

Main contributions in Fourier integral from $k \to 0$.

$$\omega = \sqrt{gk\tanh(kH)} = c_0 k(1 - \frac{1}{6}(kH)^2 + O((kH)^4)),$$

where $c_0 = \sqrt{gh}$.

$$\chi \approx kx - c_0\left(k - \frac{H^2}{6}k^3\right)t$$

and

$$\begin{aligned}\eta(x,t) &\sim \frac{1}{2\pi}\Re\int_0^\infty \tilde{\eta}_0(0)\mathrm{e}^{\imath\left(kx-(c_0k-\frac{H^2}{6}c_0k^3)t\right)}\mathrm{d}k \\ &= \frac{\tilde{\eta}_0(0)}{2\pi}\int_0^\infty \cos\left(k(x-c_0t) - \frac{H^2}{6}c_0tk^3\right)\mathrm{d}k\end{aligned} \qquad (1)$$

Transformation to the Airy function

The substitution $k = s/m$ and $z = (x - c_0t)/m$, where

$$m = \left(\frac{1}{2}H^2c_0t\right)^{\frac{1}{3}},$$

in (1) $\Rightarrow$

$$\eta(x,t) \sim \frac{\eta_0(0)}{2m}I(z),$$

with

$$I(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}s^3 + zs\right)\mathrm{d}s$$

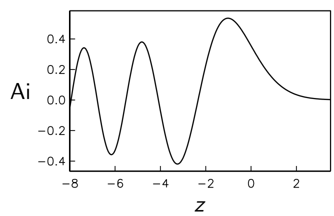
Integral formula for the Airy function, Ai.

The Airy function

$F = \mathrm{Ai}$ is the solution of

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} - zF = 0, \qquad (2)$$

which vanish as $z \to \infty$ and fulfills $\int_{-\infty}^{\infty} F\mathrm{d}z = 1$.



Ai links an exponential behaviour to an oscillating one.

The wave front

close to $x = c_0 t$

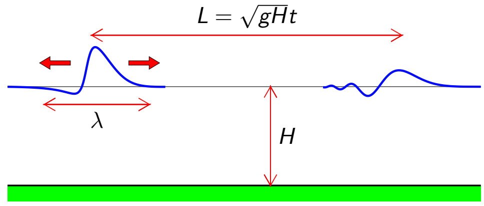
$$\eta \sim \frac{V}{2m}\mathrm{Ai}(z) = \frac{\frac{1}{2}V}{(\frac{1}{2}c_0H^2t)^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{x-c_0t}{(\frac{1}{2}c_0H^2t)^{\frac{1}{3}}}\right), \qquad (3)$$

where $V = \tilde{\eta}(0)$ is the volume (per width) under $\eta(x,0)$

- Leading crest close to $x = c_0 t$
- Shape independent of shape of $\eta(x,0)$
- Maximum wave height decays as $\sim t^{-\frac{1}{3}}$
  Will eventually dominate trailing waves ($\sim t^{-\frac{1}{2}}$).
- Length of leading crest increases as $\sim t^{\frac{1}{3}}$

Analysis may be extended to $V = 0$. Then front decays faster than trailing waves.

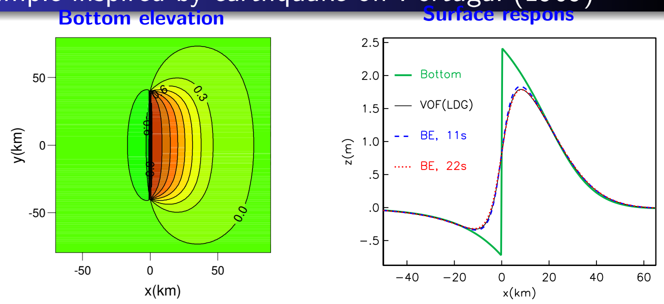
Example: Wave front and dispersion in tsunami propagation



Dispersion often neglected for tsunamis due to large wavelength ($kH \ll 1 \Rightarrow c \approx c_g \approx \sqrt{gH}$).
Its significance depends on:

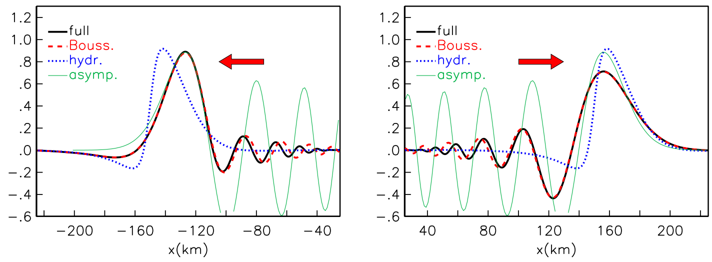
1. Extent of source relative to depth.
2. Propagation distance.

Example inspired by earthquake off Portugal (1969)



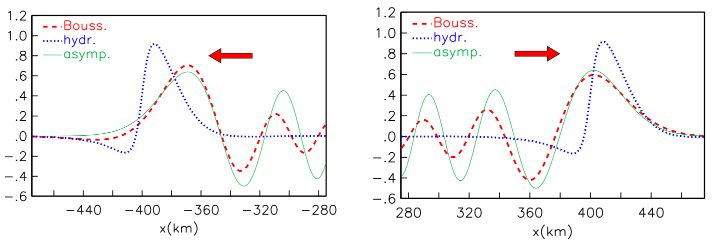
Magnitude: $M_s = 7.9$, $H = 5000\,\mathrm{m}$, inverse thrust fault, large dip angle $\approx 50°$, fault length $\approx 70\,\mathrm{km} \Rightarrow$ rather confined bottom uplift
Left panel: co-seismic bottom-uplift from Okada's formula
Right panel: hydrodynamic response for center line from 2D theories
2D response used as initial condition.

After $t = 11.3 \Rightarrow L = c_0 t = 150\,\mathrm{km}$



Curves:
'full': full potential theory (numerical)
'Bouss.': Boussinesq equation (sec. 2.11 in Comp.) (numerical)
'hydr.': Shallow water solution (sec. 2.10); half the initial shape
'asymp.': Asymptotic solution for wave front

After $t = 30$ min $\Rightarrow L = 400\mathrm{km}$



Now asymptote and numerical solution are becoming close.