1 a)

ε^0: $x_0'' + x_0 = 0$, $x_0'(0) = 0$, $x_0(0) = 1$

$x_0 = \frac{1}{2}(e^{it} + e^{-it}) = \cos t$

ε^1: $x_1'' + x_1 = x_0^3 = \frac{1}{8}(e^{3it} + 3e^{it} + 3e^{-it} + e^{-3it})$

$x_1'(0) = x_1(0) = 0$

$x_1 = \frac{1}{64}\{e^{it} + e^{-it} - (e^{3it} + e^{-3it})\}$

$- \frac{3}{16} it \{e^{it} - e^{-it}\}$ - non periodic

$|x_1/x_0| \to \infty$ as $t \to \infty$

b) $\tau = \omega t$, $\omega = \omega_0 + \varepsilon\omega_1 + \ldots$

ε^0: $\omega_0^2 \ddot{x}_0 + x_0 = 0$; $x_0 = \frac{1}{2}(e^{it} + e^{-it})$

ε^1: $\ddot{x}_1 + x_1 = x_0^3 + 2\omega_0\omega_1 x_0$

$= \frac{1}{8}(e^{3i\tau} + e^{-3i\tau})$

$+ (\omega_1 + \frac{3}{8})(e^{i\tau} + e^{-i\tau})$

$\omega_1 = -\frac{3}{8}$

2 a) $\tau = \varepsilon t$

ε^0: $\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$, $\frac{\partial y_0}{\partial t}(0,0) = 0$, $y_0(0,0) = 1$

$y_0 = A(\tau)\cos t + B(\tau)\sin t$, $A(0) = 1$, $B(0) = 0$

ε^1: $\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2\frac{\partial^2 y_0}{\partial t \partial \tau} - \gamma(\tau) y_0 + f(\tau)\cos t$

$= \{-2B' - \gamma B + f\}\cos t + \{2A' + \gamma A\}\sin t$

$2B' + \gamma B = f$, $2A' + \gamma A = 0$

b) $\gamma = 0$ $\quad A = A(0) = 1$

$B = \int_0^\tau \frac{1}{2} e^{-\alpha\hat{\tau}}\, d\hat{\tau} = \frac{1}{2\alpha}\left(1 - e^{-\alpha\tau}\right)$

3 $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V$

a) $0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2\alpha_x x - m\ddot{x}$ etc.

b) $m\ddot{x} = -2\alpha_x x = -\frac{\partial V}{\partial x} = F^{(x)}$ etc.

$\frac{\partial L}{\partial t} = 0 \Rightarrow \dot{x}\frac{\partial L}{\partial \dot{x}} + \dots - L = T + V = \text{const}$ (energy)

$\frac{\partial L}{\partial z} = 0 \Rightarrow m\dot{z} = \text{const}$ (cons. momentum)

4 a) $y = e^{\int P\,dx} \Rightarrow y' = yP,\ y'' = yP' + yP^2$

$0 = \varepsilon y'' + w y = \{\varepsilon(P' + P^2) + w\} y$

$\varepsilon P' + \varepsilon P^2 + w = 0$

$\quad (1) \quad (2) \quad (3)$

(2) & (3) : $P_0 = \pm \frac{i}{\varepsilon^{\frac{1}{2}}} w^{\frac{1}{2}}$, $(1) \sim \varepsilon^{\frac{1}{2}} \ll (2), (3)$ OK

(1) & (3) $P_0 = \frac{1}{\varepsilon} \int w\,dx$, $(2) \sim \varepsilon^{-1} \gg (1), (3)$ NO

(1) & (2) $P_0 = \frac{1}{x}$, $(1) \sim \varepsilon \ll (3)$ No

b) $P = P_0 + P_1$, $P_1 \ll P_0$

$\varepsilon(P_0' + P_1' + P_0^2 + 2P_0P_1 + P_1^2) + w = 0$

$\quad (1) \quad (2) \quad (0) \quad (3) \quad (4) \quad (0)$

(0) : Cancel due to leading order l.

(4) $\ll$ (3), Presumably $P_1' \ll P_0'$

$P_1 = -\frac{P_0'}{2P_0} = \frac{w'}{4w}$, $P_1' \ll P_0'$ OK.

$\int P\,dx = \frac{+i}{-\varepsilon^{\frac{1}{2}}}\int w\,dx + \ln w^{\frac{1}{4}} + \hat{C} \Rightarrow y = C_1 w^{\frac{1}{4}} e^{\pm \frac{i}{\varepsilon^{\frac{1}{2}}} \int_0^x w\,dx}$