Mek4100
The WKB method

Geir Pedersen

Department of Mathematics, UiO

October 16, 2017

Demonstration problem

Oscillation problem with variable coefficient

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + W(\epsilon t)y = 0; \quad y(t_a) = a, \quad y(t_b) = b, \tag{1}$$

where $W > 0$.

ϵ – small parameter $\Rightarrow$ coefficient W is slowly varying.

Note 1: Equation solved by two-scale technique in problem 24b.

Note 2: The problem (1) does not always inherit a solution. Will be demonstrated in specific example.

Note 3: $W = \text{const.} \Rightarrow$ Exact solution $y = A_+ e^{\mathrm{i}\sqrt{W}t} + A_- e^{-\mathrm{i}\sqrt{W}t}$.

Preparation for WKB; rescaling

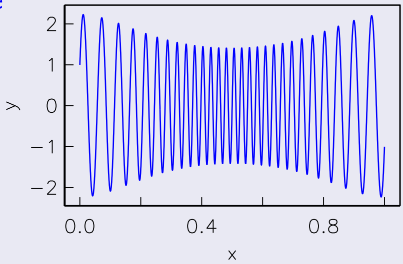
Choose slow scale as free variable $x = \epsilon t$

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + W(x)y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \tag{2}$$

Looks like a boundary layer problem, but solution oscillates rapidly everywhere

(2) not well scaled in the usual sense since $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ becomes unbounded as $\epsilon \to 0$. Scaling in (2) is common; convenient but not necessary.

Solution example



Use of exponential form

Write solution in terms of new unknown

$$y = e^{S(x)}.$$

Substitution into 2 yields equation for $k(x) \equiv S'$:

$$\epsilon^2(k' + k^2) + W = 0, \tag{3}$$

First order nonlinear ODE, called a Ricatti equation.
So far no real approximation or progress are made; (3) is still not solvable in formula for general W. But;

(3) makes a good starting point for dominant balance analysis.
The transformation by means of the exponential form only feasible for linear, homogeneous equations

Dominant balance

$$\begin{array}{ccccccc} \epsilon^2 k' & + & \epsilon^2 k^2 & + & W & = & 0 \\ (1) & + & (2) & + & (3) & = & 0 \end{array} \qquad (4)$$

(1) & (3): $k \sim -\epsilon^{-2} \int W dx \Rightarrow (2) \sim \epsilon^{-2} \gg (1), (3)$. Invalid!

(1) & (2): $k \sim (C+x)^{-1} \Rightarrow (3)$ dominates as $\epsilon \to 0$, $y \sim x + C$. Invalid!

(2) & (3): $k \sim k_0 = \pm \mathrm{i}\epsilon^{-1} W^{\frac{1}{2}}$. $\Rightarrow (1) \sim \epsilon \ll (2), (3)$. Two solutions. OK!*

$$y \sim e^{\pm \mathrm{i}\epsilon^{-1} \int W^{\frac{1}{2}} dx}$$

Describes rapid oscillations. This is the full solution if W is constant.

*: Since $|k| \to \infty$ as $\epsilon \to 0$ we must expect $k^2 \gg k'$

Second balance

$k = k_0 + k_1$, $k_1 \ll k_0$. Substitution in (4)

$$\epsilon^2(k_0' + k_1' + k_0^2 + 2k_0k_1 + k_1^2) + W = 0$$

Canceling of leading order $\epsilon^2 k_0^2 + W = 0$ and $k_1 \ll k_0 \Rightarrow$

$$\epsilon^2(k_0' + 2k_0k_1) = 0,$$

with solution $k_1 = -\frac{1}{2}k_0'/k_0 = -\frac{1}{4}W'/W = O(1)$.

Third balance

$k = k_0 + k_1 + k_2$, $k_2 \ll k_1 \ll k_0$, canceling etc.$\Rightarrow$

$$\epsilon^2(k_1' + k_1^2 + 2k_0k_2) = 0,$$

$$k_2 = -\frac{k_1' + k_1^2}{2k_0} = \mp \mathrm{i}\epsilon\left(\frac{W''}{8W^{\frac{3}{2}}} - \frac{5(W')^2}{32W^2}\right).$$

Assembling the solutions

$$S_\pm = \int k dx = C_\pm \pm \frac{\mathrm{i}}{\epsilon} \int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x} + \ln\left(W^{-\frac{1}{4}}\right) \mp \mathrm{i}\epsilon\alpha,$$

where $\alpha = \int\limits_{x_a}^{x} \{\frac{1}{8} W'' W^{-\frac{3}{2}} - \frac{5}{32}(W')^2 W^{-2}\} d\hat{x}$ and $C_\pm$ is a constant of integration.

The two solutions for y are written $(\epsilon\alpha \ll 1)$

$$y_\pm = A_\pm W^{-\frac{1}{4}} e^{\pm\frac{\mathrm{i}}{\epsilon}\int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} e^{\mp \mathrm{i}\epsilon\alpha} = A_\pm W^{-\frac{1}{4}} e^{\pm\frac{\mathrm{i}}{\epsilon}\int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} (1 \mp \mathrm{i}\epsilon\alpha + O(\epsilon^2))$$

Choosing $A_- = A_+^*$ makes $y_- + y_+$ real. Moreover, real and imaginary parts of A_- may be found as to make the boundary conditions fulfilled.

From k_1 we obtain an amplitude modulation.

Digression: The formal WKB expansion

The end results were expansions of type

$$y = e^{\mathrm{i}\frac{\phi(x)}{\epsilon}} \left(A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \ldots\right). \qquad (5)$$

1. (5) often used as an anzats.
2. Reduces the expansion to unimaginative book-keeping.
3. (5) sometimes not appropriate.
4. The higher A_j seldom significant.

The form (5) akin to solution for constant W, namely
$y = Ae^{\frac{\mathrm{i}}{\epsilon} W^{\frac{1}{2}} x} = Ae^{\frac{\mathrm{i}}{\epsilon} \int W^{\frac{1}{2}} dx}$

The explicit real solution

When α is ignored the sum $y_- + y_+$ may be recast into the form

$$y = W^{-\frac{1}{4}} \left(B \cos\psi + C \sin\psi \right), \qquad (6)$$

where $\psi = \epsilon^{-1} \int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}$ and $A_+ = \frac{1}{2}(B - \mathrm{i}C)$.

Boundary conditions $\Rightarrow$

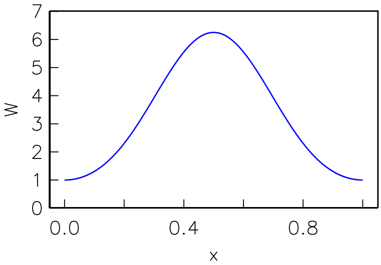
$$B = aW(x_a)^{\frac{1}{4}}, \quad C = \left(bW(x_b)^{\frac{1}{4}} - aW(x_a)^{\frac{1}{4}} \cos\psi(xb) \right) \frac{1}{\sin\psi(x_b)}.$$

No solution if $\sin\psi(x_b) = 0$, meaning $\epsilon^{-1} \int\limits_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x} = n\pi$.

A specific case

Selected parameters (ϵ is not fixed!): $x_a = 0$, $x_b = 1$, $a = 1$, $b = -1$,
and function

$$W^{\frac{1}{2}} = Q + R\cos^2(x - \frac{1}{2})\pi, \quad \psi = \frac{1}{\epsilon}\left[(Q + \frac{1}{2}R)x + \frac{R}{4\pi} sin(2x-1)\pi\right],$$



Coefficient for $Q = 1$ and $R = \frac{3}{2}$.

Numerical method

Define $y_j \approx y(j\Delta x)$ for $j = 0, ...n$ and $\Delta x = \frac{1}{n}$.
Tri-diagonal set of equations, solved by Gaussian elimination

$$y_0 = a,$$
$$\frac{1}{\Delta x^2}(y_{j+1} - 2y_j + y_{j-1}) + \frac{1}{\epsilon^2}W_j y_j + s_j = 0, \quad j = 1, ..., n-1,$$
$$y_n = b,$$

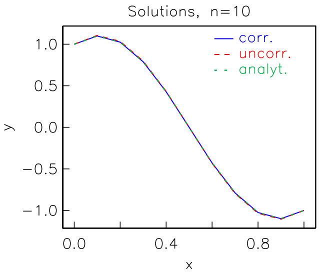
where correction terms

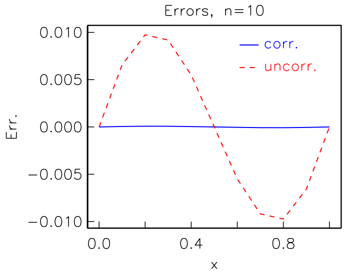
$$s_j = \frac{1}{12\epsilon^2}\left(W_{j+1}y_{j+1} - 2W_j y_j + W_{j-1}y_{j-1}\right),$$

reduces error to $O(\Delta x^4)$.

Test of numerical method

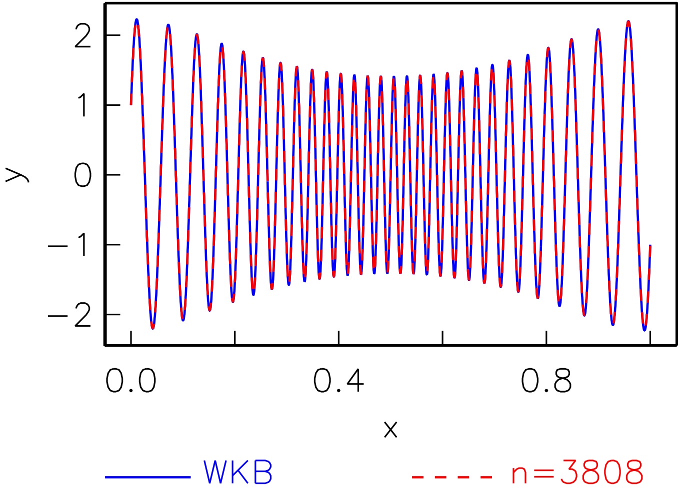
Constant coefficients, large ϵ: $Q = 4$, $R = 0$, $\epsilon = 1$.
WKB formula is exact.





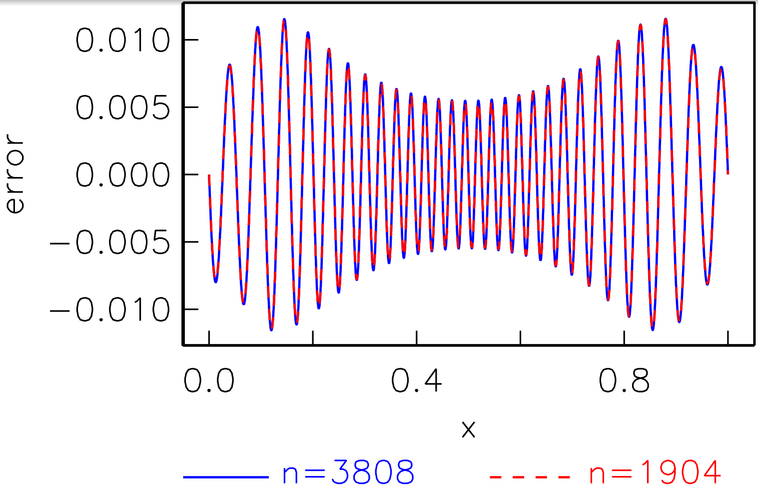
More test runs are performed to assure convergence, but not shown here.
Such verification is tedious, but mandatory!
Resolution must still be checked for small ϵ runs, which are much more demanding.

Solutions for $\epsilon = 0.01$; $Q = 1$ and $R = \frac{3}{2}$



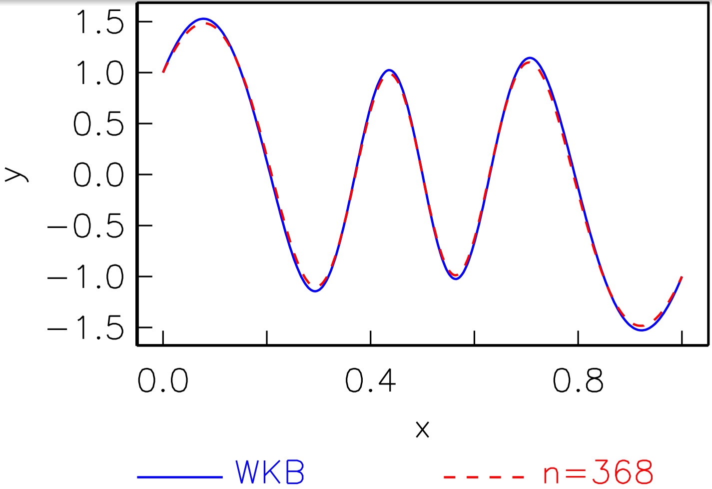
Numerical solution marked by value of n.

Errors for $\epsilon = 0.01$; $Q = 1$ and $R = \frac{3}{2}$



Numerical solution not noticeably dependent on n.
Error is 0.5%, say, of typical value of y.

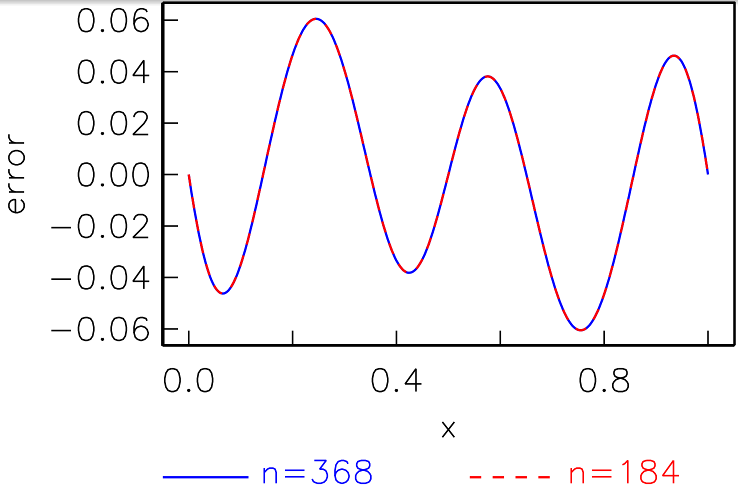
Results $\epsilon = 0.1$



Numerical solution marked by value of n.

For this larger ϵ: WKB still quite good, but error visible

Errors for $\epsilon = 0.1$



Again, numerical solution is not noticeably dependent on n.
Error is 4%, say, of typical value of y.

Convergence in ϵ

$< f > = \int_0^1 f\,dx$, evaluated by trapezoidal integration

$$L_2 = \sqrt{< (y_{\text{num}} - y_{\text{WKB}})^2 >}, \quad E_r = L_2 / \left(\epsilon \sqrt{< (y_{\text{WKB}})^2 >} \right)$$

ϵ	L_2	E_r
0.10	$0.36 \cdot 10^{-1}$	0.38
$0.50 \cdot 10^{-1}$	$0.11 \cdot 10^{-1}$	0.39
$0.25 \cdot 10^{-1}$	$0.14 \cdot 10^{-1}$	0.42
$0.10 \cdot 10^{-1}$	$0.60 \cdot 10^{-2}$	0.48
$0.50 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$	0.33
$0.25 \cdot 10^{-2}$	$0.44 \cdot 10^{-3}$	0.30

nf=32 (measure of resolution)

Solution changes qualitatively with $\epsilon \Rightarrow E_r$ remains of same size, but does not approach a constant.

WKB and a boundary layer problem

Change: sign on coefficient in the equation

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - W(x)y = 0; \quad y(x_a) = a, \quad y(x_b) = b, \tag{7}$$

where $W > 0$. Solutions are now of rapidly growing/decaying nature instead of oscillating.

The boundary layer method

The problem is virtually contained in problem 64 in leaflet. The unified solution becomes

$$y \approx a e^{-\sqrt{W(x_a)}\frac{(x-x_a)}{\epsilon}} + b e^{\sqrt{W(x_b)}\frac{(x-x_b)}{\epsilon}}. \tag{8}$$

Boundary layers at both ends, zero as outer solution.

All the algebra of the first examples repeats itself, except for the occurrence of i, the imaginary unit. Using k_0 and k_1:

$$y \approx A_+ W^{-\frac{1}{4}} e^{\frac{1}{\epsilon}\int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}} + A_- W^{-\frac{1}{4}} e^{-\frac{1}{\epsilon}\int\limits_{x_a}^{x} W^{\frac{1}{2}} d\hat{x}}.$$

Boundary conditions

$$W(x_a)^{-\frac{1}{4}}(A_+ + A_-) = a, \quad W(x_b)^{-\frac{1}{4}}(\gamma A_+ + \gamma^{-1} A_-) = b,$$

where $\gamma = e^{\frac{1}{\epsilon}\int\limits_{x_a}^{x_b} W^{\frac{1}{2}} d\hat{x}} \gg 1$. Hence,

$$A_+ = \frac{a\gamma^{-1} W(x_a)^{\frac{1}{4}} - bW(x_b)^{\frac{1}{4}}}{\gamma^{-1} - \gamma}, \quad A_- = \frac{bW(x_b)^{\frac{1}{4}} - a\gamma W(x_a)^{\frac{1}{4}}}{\gamma^{-1} - \gamma}$$

How to reconcile this with (8) ?

First $\gamma \gg 1 \Rightarrow A_+ \approx b\gamma^{-1}W(x_b)^{\frac{1}{4}}$ and $A_- = aW(x_a)^{\frac{1}{4}}$; thus

$$y \approx \frac{b}{\gamma}\left(\frac{W(x_b)}{W(x)}\right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon}\int\limits_{x}^{x_b} W^{\frac{1}{2}}d\hat{x}} + a\left(\frac{W(x_a)}{W(x)}\right)^{\frac{1}{4}} e^{-\frac{1}{\epsilon}\int\limits_{x_a}^{x} W^{\frac{1}{2}}d\hat{x}}. \quad (9)$$

First term grows rapidly toward x_b: boundary layer at right end. Second term decays rapidly from x_a: boundary layer at left end. Right term significant only when $x - x_a$ is small. Taylor expansion

$$\int\limits_{x_a}^{x} \frac{W^{\frac{1}{2}}}{\epsilon}d\hat{x} = \left(\frac{W(x_a)^{\frac{1}{2}}(x-x_a)}{\epsilon} + \frac{(W(x_a)^{\frac{1}{2}})'(x-x_a)^2}{2\epsilon} + \dots\right),$$

For a region $1 \gg x - x_a \gg \epsilon$ the first term $\gg 1$ while the second $\ll 1$. Example: $x - x_a = \epsilon^{\frac{2}{3}}$; first term $\sim \epsilon^{-\frac{1}{3}}$, second term $\sim \epsilon^{\frac{1}{3}}$. Consequence: second term in (9) vanishes before second term in Taylor expansion becomes important. We may then also put $W(x)/W(x_a) \approx 1$, meaning that k_1 is ignored.

Similar treatment of first term in (9) gives

$$y \approx b e^{-\frac{1}{\epsilon} W(x_b)^{\frac{1}{2}}(x_b - x)} + a e^{-\frac{1}{\epsilon} W(x_a)^{\frac{1}{2}}(x - x_a)}. \tag{10}$$

Which is the boundary layer solution (8) retrieved.

- Homogeneous, linear boundary layer problems may be solved with WKB techniques
- Boundary layer solutions consistent with leading order WKB solution
- Quite some simplification needed to reveal the full relationship

Relation to theorem 3.12 in Logan

Boundary value problem

$$\epsilon y'' + p(x)y' + q(x) = 0, \quad y(0) = a, \quad y(1) = b, \qquad (11)$$

where $\epsilon \to 0$, $p(x) > 0$, $p, q \sim 1$
Again

$$y = e^{S(x)} = e^{\int k d\hat{x}}.$$

It is important that we do not assume (5).

$$
\begin{array}{ccccccccc}
\epsilon k' & + & \epsilon k^2 & + & pk & + & q & = & 0 \\
(1) & + & (2) & + & (3) & + & (4) & = & 0
\end{array}
\tag{12}
$$

(1) & (4): $k \sim -\epsilon^{-2} \int q dx \Rightarrow (2) \sim \epsilon^{-1} \gg (1), (3)$. Invalid!

(1) & (2): $k \sim (C + x)^{-1} \Rightarrow (4)$ dominates as $\epsilon \to 0$, $y \sim x + C$. Invalid!

(2) & (4): $k \sim \pm \mathrm{i} \epsilon^{-\frac{1}{2}} q^{\frac{1}{2}}$. $\Rightarrow (3) \sim \epsilon^{-\frac{1}{2}} \gg (2), (4)$. Invalid!

(2) & (3): $k \sim -\frac{p}{\epsilon}$. One valid solution.

(3) & (4): $k \sim -\frac{q}{p}$. One valid solution. (Not on form (5)!)

(1) & (3): $k \sim C e^{-\epsilon^{-1} \int_0^x p d\hat{x}}$. $(3) \ll (4)$ when $x \gg \epsilon$. Discarded!

Then

$$
y = A e^{-\int \frac{q}{p} dx} + B e^{-\int \frac{p}{\epsilon} dx}
$$

Outer and boundary layer approximations inherited. Theorem 3.12 from Logan may be reproduced; details omitted.