UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MEK4100 — Mathematical Methods in Mechanics

Day of examination: Thursday 16. June 2016

Examination hours: 9.00 – 13.00

This problem set consists of 3 pages.

Appendices: Formula sheet

Permitted aids: Mathematical handbook, by K. Rottmann.
Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 25%)

A boundary layer problem is specified as

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - xy = 0, \quad y(0) = 0, \quad y(1) = 1, \tag{1}$$

where ϵ is a small and positive number.

1a (weight 10%)

Find the outer solution and explain why there must be a boundary layer. Assume that the boundary layer is situated at $x = 0$.

1b (weight 15%)

Find the unified solution. Here you should express a part of the solution by means of the error function which is defined at the end of the exam set.

Problem 2 (weight 30%)

An initial value problem is given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (\omega(\epsilon t))^2 y = 0, \quad y(0) = 1, \quad \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0, \tag{2}$$

where ϵ is a small constant.

2a (weight 15%)

Introduce the slow scale $\tau = \epsilon t$ and apply a direct two scale method to (2). Show that this methods breaks down and that we cannot find the amplitude of the leading order approximation.

2b (weight 15%)

Redefine the fast scale by defining T, such that $\frac{\mathrm{d}T}{\mathrm{d}t} = \sigma(\tau)$. Find the leading order solution, including the amplitude.

Problem 3 (weight 30%)

For periodic gravity waves on the surface of the ocean we may assume a relation between the angular frequency ω (dimension 1/time), the wave number k (dimension 1/length), the depth h, and the acceleration of gravity g.

3a (weight 10%)

Use the π theorem to find the general form of ω as function of the other parameters.

3b (weight 10%)

By means of hydrodynamic equations we find (do not attempt to show this!)

$$\omega^2 = gk \tanh kh. \tag{3}$$

We wish to solve this equation for k when ω is specified. Explain why $kh \to \infty$ as $h\omega^2/g \to \infty$ and find k_0, the leading order approximation for k, in this limit.

3c (weight 10%)

Find an improved approximation, $k_0 + k_1$, by means of a perturbation expansion. Explain the steps carefully.

Problem 4 (weight 15%)

A functional is defined as

$$J(y) = \int_a^b L(x, y, y')dx,$$

where y is fixed at the end-points $y(a) = y_1$, $y(b) = y_2$.

4a (weight 10%)

Derive the general form of the Euler equation for the y which yields an extreme for J.

4b (weight 5%)

Relax the condition at $x = b$, such that y is only fixed at $x = a$. Then, y must also fulfill a boundary condition to yield an extreme for J. Show this and find the form of the condition.

Specific formulas

Definition of the error function (left), and relation to a class of integrals (right)

$$\operatorname{erf}(s) = \frac{2}{\sqrt{\pi}} \int\limits_0^s e^{-t^2} dt, \quad \int\limits_0^s e^{-\alpha t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \operatorname{erf}(\sqrt{\alpha} s),$$

where α is a constant. It is also given that $\operatorname{erf}(s) \to 1$ as $s \to \infty$.

THE END