# Mek4100
# Two-scale perturbation methods

Geir Pedersen

Department of Mathematics, UiO

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# Motivation

- A number of problems inherit several temporal or spatial scales
- Example: Boundary layer problem; albeit here the rapid scale is only present in the boundary layer
- Several global scales $\Rightarrow$ a new method is required
- Linear, homogeneous equations: WKB(J) is an alternative

# Example 1: damped oscillation

ODE with initial conditions

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0; \quad y(0) = 1, \quad \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0, \tag{1}$$

$\epsilon$ – small parameter.

Physical interpretation: weak resistance force proportional to the velocity

## Direct (naive) perturbation

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \Rightarrow$$

### $O(\epsilon^0)$

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0,$$

$$y_0(0) = 1, \quad \frac{\mathrm{d}y_0(0)}{\mathrm{d}t} = 0.$$

Solution:

$$y_0 = \cos t$$

## $O(\epsilon^1)$

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -\frac{\mathrm{d}y_0}{\mathrm{d}t} = \sin t,$$

$$y_1(0) = \frac{\mathrm{d}y_1(0)}{\mathrm{d}t} = 0.$$

Resonance (secular terms) $\Rightarrow$

$$y_1 = \frac{1}{2}(\sin t - t\cos t).$$

## Breakdown due to

Effect of small resistance accumulates. Exact solution (presented later) implies $y \to 0$ as $t \to \infty$

Hence, $\epsilon y_1 \approx y - y_0$ must be comparable to $y_0$

Poincare-Lindsted: not applicable, why ?

# Introduction of a slow time variable

New time

$$\tau = \epsilon t,$$

is introduced in addition to the fast time $t$. Hence

$$y = y(t, \tau),$$

which is defined in in the quadrant $[t \geq 0] \times [\tau \geq 0]$ as if $t$ and $\tau$ were independent.
Much redundancy: only the line $\tau = \epsilon t$ has direct significance.

Temporal derivatives transform

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

# The transformed problem

Damped oscillation equation in terms of $t$ and $\tau$ yields PDE

$$\frac{\partial^2 y}{\partial t^2} + y + \epsilon(2\frac{\partial^2 y}{\partial t \partial \tau} + \frac{\partial y}{\partial t}) + \epsilon^2(\frac{\partial^2 y}{\partial \tau^2} + \frac{\partial y}{\partial \tau}) = 0;$$

$$y(0,0) = 1, \quad \frac{\partial y(0,0)}{\partial t} + \epsilon \frac{\partial y(0,0)}{\partial \tau} = 0.$$

## Considerations

1. $t$ and $\tau$ **are** not "really" independent, but
   solution of the PDE provides solution for ODE
   Physical effects behind scales may sometimes be conceived as independent
2. Anyway, an ODE for a PDE; good bargain?
   Yes, as long as we can solve the PDE

# Two-scale perturbation

The series

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + ...,$$

All terms must remain finite or, rather, vanish in time.

### $O(\epsilon^0)$

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0; \quad y_0(0,0) = 1, \quad \frac{\partial y_0(0,0)}{\partial t} = 0.$$

The solution for $y_0$ becomes

$$y_0 = A_0(\tau)\cos t + B_0(\tau)\sin t, \quad A_0(0) = 1, \quad B_0(0) = 0$$

$A_0$, $B_0$ must be determined to the next order.

## $(\epsilon^1)$

$$
\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= -\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} \\
&= (A_0 + 2\frac{\mathrm{d}A_0}{\mathrm{d}\tau})\sin t - (B_0 + 2\frac{\mathrm{d}B_0}{\mathrm{d}\tau})\cos t;
\end{aligned}
$$

$$
y_1(0,0) = 0, \quad \frac{\partial y_1(0,0)}{\partial t} = -\frac{\partial y_0(0,0)}{\partial \tau}.
$$

Avoid growing (secular) terms $\Rightarrow$

$$
A_0 + 2\frac{\mathrm{d}A_0}{\mathrm{d}\tau} = B_0 + 2\frac{\mathrm{d}B_0}{\mathrm{d}\tau} = 0.
$$

## $O(\epsilon^1)$, cont.

Initial conditions for $A_0$, $B_0$ $\Rightarrow$

$$A_0 = e^{-\frac{1}{2}\tau}, \quad B_0 = 0.$$

No particular solution to $O(\epsilon)$:

$$y_1 = A_1(\tau)\cos t + B_1(\tau)\sin t, \quad A_1(0) = 0,\ B_1(0) = \frac{1}{2}$$

## Complete solution

$$y = e^{-\frac{1}{2}\epsilon t}\cos t + \epsilon(A_1(\epsilon t)\cos t + B_1(\epsilon t)\sin t) + O(\epsilon^2).$$

**!** $\boldsymbol{\epsilon^2}$: secular terms may appear; can be eliminated by in troducing $\tau_1 = \epsilon^2 t$.

# Comparing with exact solution

**Exact**

$$y = e^{-\frac{1}{2}\epsilon t}\left(\cos\omega t + \frac{\epsilon}{2\omega}\sin\omega t\right),$$

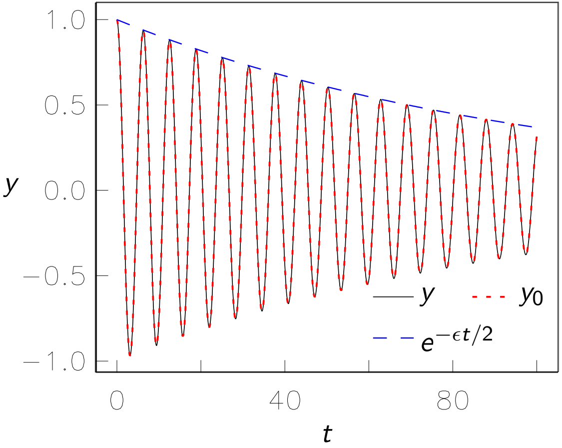
where $\omega = \sqrt{1-\frac{1}{4}\epsilon^2} = 1 + O(\epsilon)^2$.

**Two-scale approximation**

$$y = e^{-\frac{1}{2}\epsilon t}\cos t + \epsilon(A_1(\epsilon t)\cos t + B_1(\epsilon t)\sin t) + O(\epsilon^2).$$

**The two solutions agree, including the initial condition $B_1(0) = \frac{1}{2}$.**

# Graphical comparison, $\epsilon = 0.02$.



# Example 2: nonlinear oscillations

Scaled equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x - \frac{\epsilon}{6}x^3 = 0, \quad x(0) = 1, \frac{\mathrm{d}x(0)}{\mathrm{d}t} = 0.$$

## Poincare-Lindsted

We seek a periodic solution with frequency $\omega = \omega_0 + \epsilon\omega_1 + \ldots$.

## Two-scale method

We regard $\epsilon t$ a slow time scale that modulates the phase.

## Two-scale expansion, nonlinear pendulum

Invoke $\tau = \epsilon t$

$$\frac{\partial^2 x}{\partial t^2} + x + \epsilon\left(2\frac{\partial^2 x}{\partial t \partial \tau} - \frac{1}{6}x^3\right) + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} = 0;$$

$$x(0,0) = 1, \quad \frac{\partial x(0,0)}{\partial t} + \epsilon \frac{\partial x(0,0)}{\partial \tau} = 0.$$

Perturbation series

$$x = x_0(t,\tau) + \epsilon x_1(t,\tau) + ...,$$

## $\epsilon^0$

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \quad x_0(0,0) = 1, \quad \frac{\partial x_0(0,0)}{\partial t} = 0.$$

Exponential form $\Rightarrow$

$$x_0 = A_0(\tau)e^{it} + \overline{A}_0(\tau)e^{-it}, \quad A_0(0) = \frac{1}{2},$$

where $\overline{A_0}$ is the complex conjugate of $A_0$.

## $\epsilon^1$

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t^2} + x_1 &= \frac{1}{6}x_0^3 - 2\frac{\partial^2 x_0}{\partial t \partial \tau}\\ &= \frac{1}{6}A_0^3 e^{3it} + \left(-2i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{1}{2}\overline{A}_0 A_0^2\right) e^{it} + \mathrm{c.c.},\end{aligned}$$

$$x_1(0,0) = 0, \qquad \frac{\partial x_1(0,0)}{\partial t} = -\frac{\partial x_0(0,0)}{\partial \tau}, \tag{2}$$

where $\mathrm{c.c.}$ indicates the addition of the complex conjugate.
Annihilation of secular terms⇒

$$i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} - \frac{1}{4}\overline{A}_0 A_0^2 = 0.$$

## $\epsilon^1$, cont.

From previous slide

$$i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} - \frac{1}{4}\overline{A}_0 A_0^2 = 0.$$

Insertion of $A_0 = |A_0|e^{i\psi} \Rightarrow$

$$\frac{\mathrm{d}|A_0|}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\frac{1}{4}|A_0|^2,$$

Initial condition $A_0(0) = \frac{1}{2} \Rightarrow$

$$A_0 = \frac{1}{2}e^{-\frac{i}{16}\tau}$$

Then, initial conditions $\Rightarrow$

$$x_1 = -\frac{A_0^3}{48}e^{3it} + A_1(\tau)e^{it} + \text{c.c.}, \quad A_1(0) = \frac{1}{384}$$

# The two leading orders combined

$$
\begin{aligned}
x &= \frac{1}{2}e^{i(1-\frac{\epsilon}{16})t} - \frac{\epsilon}{384}e^{3i(1-\frac{\epsilon}{16})t} + \epsilon A_1(\tau)e^{it} + \mathrm{c.c.} + O(\epsilon^2) \\
&= \cos(1-\frac{\epsilon}{16})t - \frac{\epsilon}{192}\cos 3(1-\frac{\epsilon}{16})t \qquad (3) \\
&\quad +\epsilon a_1(\epsilon t)\cos t - \epsilon b_1(\epsilon t)\sin t + O(\epsilon^2),
\end{aligned}
$$

where $A_1 = a_1 + ib_1$.
Can be verified by Poincare-Lindsted's method.

# Example 3: Pendulum with prescribed length variation

Conservation of angular momentum (around support)

$$\ell\ddot{\phi} + 2\dot{\ell}\dot{\phi} + g\phi = 0,$$

$\phi$=excursion, $\ell$=length and the dot indicates derivation with respect to time.
Scaling

$$t = \sqrt{\frac{g}{\ell(0)}}t^*, \quad \gamma = \frac{\ell}{\ell(0)}, \quad \theta = \frac{\phi}{\phi_c}.$$

Slow scale, $\tau = \epsilon t$, describes change of $\gamma$ (dimensionless length).

$$\gamma(\tau)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\epsilon\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta = 0.$$

# Attempt: direct application of two-scale method

PDE

$$\gamma(\tau)\frac{\partial^2\theta}{\partial t^2}+\theta+2\epsilon\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial t}+\gamma\frac{\partial^2\theta}{\partial t\partial\tau}\right)+\epsilon^2\left(2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial\tau}+\gamma\frac{\partial^2\theta}{\partial\tau^2}\right)=0.$$

Initial conditions (choice of $\phi_c$)

$$\theta(0,0)=1,\quad \frac{\partial\theta(0,0)}{\partial t}+\epsilon\frac{\partial\theta(0,0)}{\partial\tau}=0.$$

Expansion $\theta=\theta_0(t,\tau)+\epsilon\theta_1(t,\tau)+\ldots$

# Direct application..

## $O(\epsilon^0)$

$$\gamma\frac{\partial^2\theta_0}{\partial t^2} + \theta_0 = 0; \quad \theta_0(0) = 1, \quad \frac{\partial\theta_0(0)}{\partial t} = 0.$$

solution

$$\theta_0 = A_0(\tau)e^{i\gamma^{-\frac{1}{2}}t} + \overline{A}_0 e^{-i\gamma^{-\frac{1}{2}}t}, \quad A_0(0) = \frac{1}{2},$$

# Direct application..

## $O(\epsilon^1)$

$$\gamma\frac{\partial^2\theta_1}{\partial t^2} + \theta_1 = h_s; \quad \theta_1(0,0) = 0, \quad \frac{\partial\theta_1(0,0)}{\partial t} = -\frac{\partial\theta_0}{\partial\tau},$$

where

$$\begin{aligned} h_s &= -2\gamma\frac{\partial^2\theta_0}{\partial t\partial\tau} - 2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial t} \\ &= -\left(2i\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + i\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + 2tA_0\gamma^{-1}\frac{\partial\gamma}{\partial\tau}\right)e^{i\gamma^{-\frac{1}{2}}t} + \mathrm{c.c.} \end{aligned}$$

Appearance of $t \Rightarrow$ secular terms in $\theta_1$.

## Reason for failure

**The fast scale (period) is non-constant; it varies with $\tau$.**

# Modified two-scale method; variable fast scale

Variable time scale ( scale is fast, but it's variation is slow)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \sigma(\tau).$$

Transformation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sigma\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \sigma^2\frac{\partial^2}{\partial T^2} + \epsilon(2\sigma\frac{\partial^2}{\partial T\partial \tau} + \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial}{\partial T}) + \epsilon^2\frac{\partial^2}{\partial \tau^2}.$$

We must choose (determine) $\sigma$ as to avoid secular terms.
Perturbation series

$$\theta = \theta(T,\tau) + \epsilon\theta_1(T,\tau) + ...$$

PDE

$$\gamma\sigma^2\frac{\partial^2\theta}{\partial T^2}+\theta+\epsilon\left(2\sigma\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial T}+2\sigma\gamma\frac{\partial^2\theta}{\partial T\partial\tau}+\gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial T}\right)+\epsilon^2\left(2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial\tau}+\gamma\frac{\partial^2\theta}{\partial\tau^2}\right)=0.$$

## $O(\epsilon^0)$

$$\gamma\sigma^2\frac{\partial^2\theta_0}{\partial T^2}+\theta_0=0;\quad \theta_0(0,0)=1,\quad \frac{\partial\theta_0(0,0)}{\partial T}=0.$$

In previous attempt $\tau$ appears explicitly in the exponent.
This can be avoided by $\sigma=\gamma^{-\frac{1}{2}}\Rightarrow$

$$\theta_0=A_0(\tau)e^{iT}+\overline{A}_0(\tau)e^{-iT},\quad A_0(0)=\frac{1}{2},$$

## $O(\epsilon^1)$

$$\frac{\partial^2\theta_1}{\partial T^2} + \theta_1 \equiv h_s,$$

where

$$\begin{aligned}
h_s &= -2\sigma\gamma\frac{\partial^2\theta_0}{\partial T\partial\tau} - 2\sigma\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial T} - \gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial T} \\
&= -i\left(2\sigma\gamma\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + 2\sigma A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + \gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}A_0\right)e^{iT} + \mathrm{c.c.} \\
&= -i\left(2\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{3}{2}\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)e^{iT} + \mathrm{c.c.}
\end{aligned}$$

Annihilation of secular terms $\Rightarrow$ coefficient of $e^{iT}$ is zero $\Rightarrow$ (ODE) for $A_0$.

$$2\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{3}{2}\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = 0$$

Separable equation (ODE) for $A_0$

$$\frac{1}{A_0}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} = -\frac{3}{4\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}$$

Integration and $A_0(0) = \frac{1}{2}$, $\gamma(0) = 1 \Rightarrow$

$$A_0 = \frac{1}{2}\gamma^{-\frac{3}{4}}$$

# Physical note: wave action

Energy in pendulum motion :$E = E_s + E_\ell$
$E_\ell$: potential energy due to change in $\ell$
$E_s$: Energy due to the oscillations

$$E_s = \frac{1}{2} m\ell^2 \dot{\phi}^2 + mg\ell(1 - \cos\phi),$$

Small amplitude, scaling, invocation of two-scale solution $\theta_0 \Rightarrow$

$$E_s = 2mg\ell_0 \phi_c^2 \gamma A_0^2 (1 + O(\epsilon)).$$

$E_s$ is not constant, but

**Wave action**

$$\frac{E_s}{\omega} \approx \text{const.}, \quad \omega = \sqrt{\frac{g}{\ell}}$$

**is constant; general result for oscillations and waves in time dependent medium**