1a

$$y_o' - y_o = 0 \quad \Rightarrow \quad y_o = A_o e^x.$$

Fulfilling both $y_o(0) = 0$ and $y_o(1) = 1$ is impossible $\Rightarrow$boundary layer.
Layer at $x = 0$ then

$$y_o = e^{x-1}$$

1b

Transformation $\xi = x/\delta$

$$\begin{array}{ccccc} \frac{\epsilon}{\delta^2}Y'' & +\xi Y' & -\delta\xi Y & = & 0 \\ (1) & (2) & (3) & & \end{array}$$

Clear (3) $\ll$ (2) $\Rightarrow$(1) & (2) must be dominant balance and $\delta = \sqrt{\epsilon}$

$$Y'' + \xi Y' = 0 \quad \Rightarrow \quad Y' = Ae^{-\frac{1}{2}\xi^2}.$$

From the formula

$$Y = Y(0) + \int_0^\xi Y'(\hat{\xi})d\hat{\xi} = A\int_0^\xi e^{-\frac{1}{2}\hat{\xi}^2}d\hat{\xi} = \sqrt{\frac{\pi}{2}}A\,\mathrm{erf}\left(\frac{\xi}{\sqrt{2}}\right)$$

Matching

$$y_{\mathrm{match}} = \lim_{\xi\to\infty} Y = \sqrt{\frac{\pi}{2}}A,$$

$$y_{\mathrm{match}} = \lim_{x\to 0} y_o = e^{-1},$$

Then $A = \sqrt{2}/(e\sqrt{\pi})$ and

$$y_{\mathrm{unif}} = y_o + Y - y_{\mathrm{match}} = e^{-1}\left(e^x + \mathrm{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) - 1\right).$$

2a

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\partial^2 y}{\partial t^2} + 2\epsilon\frac{\partial^2 y}{\partial\tau\partial t} + O(\epsilon^2)$$

$y = y_0 + \epsilon y_1..$

ϵ^0

$$\frac{\partial^2 y_0}{\partial t^2} + \omega^2 y_0 = 0, \quad y_0(0,0) = 1, \quad \frac{\partial y_0(0,0)}{\partial t} = 0.$$

Solution

$$y_0 = A(\tau)e^{i\omega(\tau)t} + c.c., \quad A(0) = \frac{1}{2}.$$

ϵ^1

$$\frac{\partial^2 y_1}{\partial t^2} + \omega^2 y_1 = -2\frac{\partial^2 y_0}{\partial \tau \partial t} = -(2i\omega A' - 2\omega A \omega' t)e^{i\omega(\tau)t} + c.c.$$

where $A' = \frac{\mathrm{d}A}{\mathrm{d}\tau}$ etc.
Only way to avoid terms $\sim t^2 e^{i\omega(\tau)t}$ in y_1 is to require $A \equiv 0$, which yields nothing.

2b

Now $y = y(T, \tau)$.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \sigma^2 \frac{\partial^2 y}{\partial T^2} + \epsilon \left(2\sigma \frac{\partial^2 y}{\partial \tau \partial T} + \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial y}{\partial T} \right) + O(\epsilon^2)$$

$y = y_0 + \epsilon y_1 ..$

ϵ^0

$$\sigma^2 \frac{\partial^2 y_0}{\partial T^2} + \omega^2 y_0 = 0, \quad y_0(0,0) = 1, \quad \frac{\partial y_0(0,0)}{\partial T} = 0.$$

To avoid a slow coefficient in the exponent, which leads to secular terms in next order, we put $\sigma = \omega$ and

$$y_0 = A(\tau)e^{iT} + c.c., \quad A(0) = \frac{1}{2}.$$

ϵ^1

$$\omega^2 \frac{\partial^2 y_1}{\partial T^2} + \omega^2 y_1 = -2\omega \frac{\partial^2 y_0}{\partial \tau \partial T} - \omega' \frac{\partial y_0}{\partial \tau} = -i(2\omega A' + A\omega')e^{iT} + c.c.$$

Avoid secular terms: $2\omega A' + A\omega' = 0 \Rightarrow A^2\omega = \text{const}$.
Use of initial condition

$$A = \frac{1}{2}\sqrt{\frac{\omega(0)}{\omega(\tau)}}.$$

3a

$\pi_1 = h\omega^2/g$, $\pi_2 = kh$.
$\pi_1 = F(\pi_2) \Rightarrow$

$$\omega^2 = \frac{g}{h}F(kh) = gkG(kh).$$

3b

Observe: $F(kh) = kh \tanh kh$.

$$kh \tanh kh = \frac{h\omega^2}{g} \to \infty$$

Since $\tanh kh < 1$ must have $kh \to \infty$.
$\tanh kh \to 1$ as $kh \to \infty \Rightarrow k_0 h = \frac{h\omega^2}{g}$ and $k_0 = \frac{\omega^2}{g}$

3c

Work instead with $\kappa = kh$ (π_2) and $\Omega = h\omega^2/g$ (π_1)

$$\Omega = \kappa \tanh \kappa.$$

Large $\kappa \Rightarrow e^{-2\kappa} \ll 1$

$$\tanh \kappa = \frac{1 - e^{-2\kappa}}{1 + e^{-2\kappa}} = 1 - 2e^{-2\kappa} + O(e^{-4\kappa})$$

$\kappa = \kappa_0 + \kappa_1$ with $\kappa_0 \gg \kappa_1$

$$\Omega = \kappa \tanh \kappa = \kappa \left(1 - 2e^{-2\kappa} + O(e^{-4\kappa})\right)$$

and possible dominant terms

$$\begin{array}{ccccc} \Omega & = & \kappa_0 & +\kappa_1 & -2\kappa_0 e^{-2(\kappa_0+\kappa_1)} \\ (1) & & (2) & (3) & (4) \end{array}$$

Dominant balance: (1) & (2), secondary balance (3) & (4).
Moreover, guess that $e^{-2(\kappa_0+\kappa_1)} \approx e^{-2\kappa_0}$. Must require $\kappa_1 \ll 1$ ($\kappa_1 \ll \kappa_0$ is not sufficient). Then

$$\kappa_1 = 2\kappa_0 e^{-2\kappa_0},$$

and indeed $\kappa_1 \ll 1$.

Some may attempt something like

$$\Omega = (\kappa_0+\kappa_1)\tanh(\kappa_0+\kappa_1) \approx (\kappa_0+\kappa_1)\tanh\kappa_0 = \kappa_0+\kappa_1+(\tanh(\kappa_0)-1)(\kappa_0+\kappa_1) \approx \kappa_0+\kappa_1+(\tanh(\kappa_0)-1)\kappa_0$$

Removing leading order we then obtain

$$\kappa_1 = -(\tanh(\kappa_0) - 1)\kappa_0.$$

This is by no means wrong, but less transparent. Still $\kappa_1 \ll 1$ needs to be adressed.

4a

4b

Integration by parts $\Rightarrow$

$$\delta J = \left(\frac{\partial L}{\partial y'}\delta y\right)_{x=b} + \int_a^b \left\{\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right\}\delta y dx.$$

$\delta J = 0$ for all δy that are zero at the ends yields the Euler equation. Then the integral vanishes and $\delta y(b) \neq 0 \Rightarrow$

$$\left(\frac{\partial L}{\partial y'}\right)_{x=b} = 0$$

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