

Mek4100

Two-scale perturbation methods

Geir Pedersen

Department of Mathematics, UiO

October 6, 2020

Motivation

- A number of problems inherit several temporal or spatial scales.
- Example: Boundary layer problem; albeit here the rapid scale is only present in the boundary layer.
Allows different expansions in boundary layer and outer region, followed by matching.
- Several **global** scales \Rightarrow a new method is required.
- Linear, homogeneous equations: WKB(J) is an efficient alternative.
- Most general method: multiple scale expansions.

Example 1: damped oscillation

ODE with initial conditions

$$\frac{d^2y}{dt^2} + \epsilon \frac{dy}{dt} + y = 0; \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 0. \quad (1)$$

ϵ – small parameter.

Unperturbed problem: The linear, harmonic oscillator.

Physical interpretation of the ϵ term: weak resistance force proportional to the velocity.

Direct (naive) perturbation

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \Rightarrow$$

$O(\epsilon^0)$

$$\frac{d^2 y_0}{dt^2} + y_0 = 0,$$

$$y_0(0) = 1, \quad \frac{dy_0(0)}{dt} = 0.$$

Solution:

$$y_0 = \cos t$$

$O(\epsilon^1)$

$$\frac{d^2 y_1}{dt^2} + y_1 = -\frac{dy_0}{dt} = \sin t,$$

$$y_1(0) = \frac{dy_1(0)}{dt} = 0.$$

Mathematical resonance (secular terms) \Rightarrow

$$y_1 = \frac{1}{2}(\sin t - t \cos t).$$

Then $\epsilon t \sim 1 \Rightarrow \epsilon y_1 \sim y_0$; breakdown.

Breakdown due to

Effect of small resistance accumulates. Exact solution (presented later) implies $y \rightarrow 0$ as $t \rightarrow \infty$, while y_0 is periodic.

Hence, $\epsilon y_1 \approx y - y_0$ must be comparable to y_0

Poincare-Lindsted: not applicable, why ?

Introduction of a slow time variable

New time

$$\tau = \epsilon t,$$

is introduced **in addition** to the fast time t . Hence

$$y = y(t, \tau),$$

which is defined in the quadrant $[t \geq 0] \times [\tau \geq 0]$ as if t and τ were independent.

Much redundancy: only the line $\tau = \epsilon t$ has direct significance.

Temporal derivatives transform

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

The transformed problem

Damped oscillation equation in terms of t and τ yields PDE

$$\frac{\partial^2 y}{\partial t^2} + y + \epsilon \left(2 \frac{\partial^2 y}{\partial t \partial \tau} + \frac{\partial y}{\partial t} \right) + \epsilon^2 \left(\frac{\partial^2 y}{\partial \tau^2} + \frac{\partial y}{\partial \tau} \right) = 0;$$

$$y(0, 0) = 1, \quad \frac{\partial y(0, 0)}{\partial t} + \epsilon \frac{\partial y(0, 0)}{\partial \tau} = 0.$$

Considerations

- 1 t and τ **are** not “really” independent, but solution of the PDE provides solution for ODE.
Physical effects behind scales may sometimes be conceived as independent.
- 2 Anyway, an ODE for a PDE; good bargain?
Yes, as long as we can solve the PDE.

Two-scale perturbation

The series

$$y = y_0(t, \tau) + \epsilon y_1(t, \tau) + \dots,$$

All terms must remain finite or, rather, vanish in time.

$O(\epsilon^0)$

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0; \quad y_0(0, 0) = 1, \quad \frac{\partial y_0(0, 0)}{\partial t} = 0.$$

The solution for y_0 becomes

$$y_0 = A_0(\tau) \cos t + B_0(\tau) \sin t, \quad A_0(0) = 1, \quad B_0(0) = 0$$

A_0, B_0 must be determined to the next order.

(ϵ^1)

$$\begin{aligned}\frac{\partial^2 y_1}{\partial t^2} + y_1 &= -\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} \\ &= (A_0 + 2\frac{dA_0}{d\tau}) \sin t - (B_0 + 2\frac{dB_0}{d\tau}) \cos t;\end{aligned}$$

$$y_1(0,0) = 0, \quad \frac{\partial y_1(0,0)}{\partial t} = -\frac{\partial y_0(0,0)}{\partial \tau}.$$

Avoid (secular) terms that grow in $t \Rightarrow$

$$A_0 + 2\frac{dA_0}{d\tau} = B_0 + 2\frac{dB_0}{d\tau} = 0.$$

$O(\epsilon^1)$, cont.

Initial conditions for $A_0, B_0 \Rightarrow$

$$A_0 = e^{-\frac{1}{2}\tau}, \quad B_0 = 0.$$

No particular solution to $O(\epsilon)$:

$$y_1 = A_1(\tau) \cos t + B_1(\tau) \sin t, \quad A_1(0) = 0, \quad B_1(0) = \frac{1}{2}$$

Complete solution

$$y = e^{-\frac{1}{2}\epsilon t} \cos t + \epsilon(A_1(\epsilon t) \cos t + B_1(\epsilon t) \sin t) + O(\epsilon^2).$$

! ϵ^2 : secular terms may appear; can be eliminated by introducing $\tau_1 = \epsilon^2 t$. We will not pursue this in MEK4100.

Comparing with exact solution

Exact

Original second order ODE is linear and has constant coefficients.
Solution readily found:

$$y = e^{-\frac{1}{2}\epsilon t} \left(\cos \omega t + \frac{\epsilon}{2\omega} \sin \omega t \right),$$

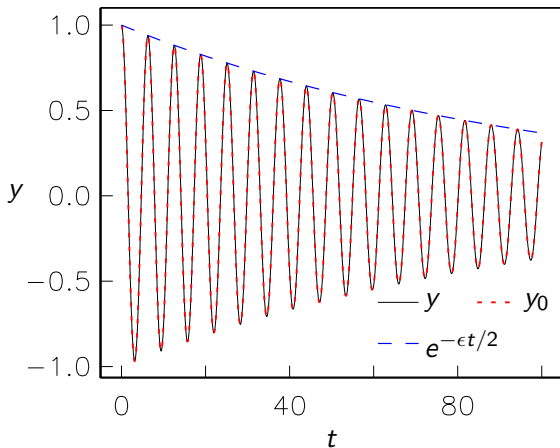
where $\omega = \sqrt{1 - \frac{1}{4}\epsilon^2} = 1 + O(\epsilon)^2$.

Two-scale approximation

$$y = e^{-\frac{1}{2}\epsilon t} \cos t + \epsilon(A_1(\epsilon t) \cos t + B_1(\epsilon t) \sin t) + O(\epsilon^2).$$

The two solutions agree, including the initial condition $B_1(0) = \frac{1}{2}$.

Graphical comparison, $\epsilon = 0.02$.



Example 2: nonlinear oscillations

Scaled equation

$$\frac{d^2x}{dt^2} + x - \frac{\epsilon}{6}x^3 = 0, \quad x(0) = 1, \quad \frac{dx(0)}{dt} = 0.$$

Poincare-Lindstedt's method

We seek a periodic solution with frequency $\omega = \omega_0 + \epsilon\omega_1 + \dots$.
 $\epsilon\omega_1 t$ may be regarded as a slow time compared to $\omega_0 t$.

Two-scale method

We regard ϵt a slow time scale that modulates the phase.
(The phase is, say, ωt .)

Two-scale expansion, nonlinear pendulum

$$\tau = \epsilon t, \quad x = x(t, \tau)$$

Transformation: ODE \Rightarrow PDE

$$\frac{\partial^2 x}{\partial t^2} + x + \epsilon \left(2 \frac{\partial^2 x}{\partial t \partial \tau} - \frac{1}{6} x^3 \right) + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} = 0;$$

$$x(0, 0) = 1, \quad \frac{\partial x(0, 0)}{\partial t} + \epsilon \frac{\partial x(0, 0)}{\partial \tau} = 0.$$

Perturbation series

$$x = x_0(t, \tau) + \epsilon x_1(t, \tau) + \dots,$$

ϵ^0

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \quad x_0(0, 0) = 1, \quad \frac{\partial x_0(0, 0)}{\partial t} = 0.$$

Exponential form \Rightarrow

$$x_0 = A_0(\tau)e^{it} + \bar{A}_0(\tau)e^{-it}, \quad A_0(0) = \frac{1}{2},$$

where \bar{A}_0 is the complex conjugate of A_0 .

ϵ^1

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t^2} + x_1 &= \frac{1}{6}x_0^3 - 2\frac{\partial^2 x_0}{\partial t \partial \tau} \\ &= \frac{1}{6}A_0^3 e^{3it} + \left(-2i\frac{dA_0}{d\tau} + \frac{1}{2}\bar{A}_0 A_0^2\right) e^{it} + \text{c.c.},\end{aligned}$$

$$x_1(0,0) = 0, \quad \frac{\partial x_1(0,0)}{\partial t} = -\frac{\partial x_0(0,0)}{\partial \tau}, \quad (2)$$

where c.c. indicates the addition of the complex conjugate.

Annihilation of secular terms \Rightarrow

$$i\frac{dA_0}{d\tau} - \frac{1}{4}\bar{A}_0 A_0^2 = 0.$$

ϵ^1 , cont.

From previous slide

$$i \frac{dA_0}{d\tau} - \frac{1}{4} \bar{A}_0 A_0^2 = 0.$$

Insertion of $A_0 = |A_0|e^{i\psi} \Rightarrow$

$$\frac{d|A_0|}{d\tau} = 0, \quad \frac{d\psi}{d\tau} = -\frac{1}{4}|A_0|^2,$$

Initial condition $A_0(0) = \frac{1}{2} \Rightarrow |A_0|(0) = \frac{1}{2}, \quad \psi(0) = 0 \Rightarrow$

$$A_0 = \frac{1}{2} e^{-\frac{i}{16}\tau}$$

Then, initial conditions \Rightarrow

$$x_1 = -\frac{A_0^3}{48} e^{3it} + A_1(\tau) e^{it} + \text{c.c.}, \quad A_1(0) = \frac{1}{384}$$

The two leading orders combined

$$\begin{aligned}x &= \frac{1}{2}e^{i(1-\frac{\epsilon}{16})t} - \frac{\epsilon}{384}e^{3i(1-\frac{\epsilon}{16})t} + \epsilon A_1(\tau)e^{it} + \text{c.c.} + O(\epsilon^2) \\&= \cos(1 - \frac{\epsilon}{16})t - \frac{\epsilon}{192} \cos 3(1 - \frac{\epsilon}{16})t \\&\quad + \epsilon a_1(\epsilon t) \cos t - \epsilon b_1(\epsilon t) \sin t + O(\epsilon^2),\end{aligned}\tag{3}$$

where $A_1 = a_1 + ib_1$.

Can be verified by Poincare-Lindsted's method.

Example 3: Pendulum with prescribed length variation

Conservation of angular momentum (around support)

$$\ell \ddot{\phi} + 2\dot{\ell}\dot{\phi} + g\phi = 0, \quad \phi(0) = \phi_0, \quad \dot{\phi}(0) = 0$$

ϕ =excursion, ℓ =length and the dot indicates derivation with respect to time.

Scaling

$$t = \sqrt{\frac{g}{\ell(0)}} t^*, \quad \gamma = \frac{\ell}{\ell(0)}, \quad \theta = \frac{\phi}{\phi_0}.$$

Slow scale, $\tau = \epsilon t$, describes change of γ (dimensionless length).

$$\gamma(\tau) \frac{d^2\theta}{dt^2} + 2\epsilon \frac{d\gamma}{d\tau} \frac{d\theta}{dt} + \theta = 0.$$

Attempt: direct application of two-scale method

Excursion is function of both times $\theta = \theta(t, \tau)$.

Transformation \Rightarrow PDE

$$\gamma(\tau) \frac{\partial^2 \theta}{\partial t^2} + \theta + 2\epsilon \left(\frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial t} + \gamma \frac{\partial^2 \theta}{\partial t \partial \tau} \right) + \epsilon^2 \left(2 \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial \tau} + \gamma \frac{\partial^2 \theta}{\partial \tau^2} \right) = 0.$$

Initial conditions

$$\theta(0, 0) = 1, \quad \frac{\partial \theta(0, 0)}{\partial t} + \epsilon \frac{\partial \theta(0, 0)}{\partial \tau} = 0.$$

Expansion $\theta = \theta_0(t, \tau) + \epsilon \theta_1(t, \tau) + \dots$

Direct application..

$O(\epsilon^0)$

$$\gamma \frac{\partial^2 \theta_0}{\partial t^2} + \theta_0 = 0; \quad \theta_0(0) = 1, \quad \frac{\partial \theta_0(0)}{\partial t} = 0.$$

solution

$$\theta_0 = A_0(\tau) e^{i\gamma^{-\frac{1}{2}}t} + \bar{A}_0 e^{-i\gamma^{-\frac{1}{2}}t}, \quad A_0(0) = \frac{1}{2},$$

Direct application..

$O(\epsilon^1)$

$$\gamma \frac{\partial^2 \theta_1}{\partial t^2} + \theta_1 = h_s; \quad \theta_1(0,0) = 0, \quad \frac{\partial \theta_1(0,0)}{\partial t} = -\frac{\partial \theta_0}{\partial \tau},$$

where

$$\begin{aligned} h_s &= -2\gamma \frac{\partial^2 \theta_0}{\partial t \partial \tau} - 2 \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial t} \\ &= - \left(2i\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + i\gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} + 2\textcolor{red}{t} A_0 \gamma^{-1} \frac{\partial \gamma}{\partial \tau} \right) e^{i\gamma^{-\frac{1}{2}} t} + \text{c.c.} \end{aligned}$$

Linear appearance of $\textcolor{red}{t} \Rightarrow$ secular terms in θ_1 . Cannot be removed since γ is function of τ , only.

Reason for failure

The fast scale (period) is non-constant; it varies with τ .

Modified two-scale method; variable fast scale

Variable time scale (scale is fast, but it's variation is slow)

$$\frac{dT}{dt} = \sigma(\tau).$$

Transformation

$$\frac{d}{dt} = \sigma \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau},$$

$$\frac{d^2}{dt^2} = \sigma^2 \frac{\partial^2}{\partial T^2} + \epsilon \left(2\sigma \frac{\partial^2}{\partial T \partial \tau} + \frac{d\sigma}{d\tau} \frac{\partial}{\partial T} \right) + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

We must choose (determine) σ as to avoid secular terms.

Perturbation series

$$\theta = \theta(T, \tau) + \epsilon \theta_1(T, \tau) + \dots$$

PDE

$$\gamma\sigma^2 \frac{\partial^2 \theta}{\partial T^2} + \theta + \epsilon \left(2\sigma \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial T} + 2\sigma\gamma \frac{\partial^2 \theta}{\partial T \partial \tau} + \gamma \frac{d\sigma}{d\tau} \frac{\partial \theta}{\partial T} \right) + \epsilon^2 \left(2 \frac{d\gamma}{d\tau} \frac{\partial \theta}{\partial \tau} + \gamma \frac{\partial^2 \theta}{\partial \tau^2} \right) = 0.$$

$O(\epsilon^0)$

$$\gamma\sigma^2 \frac{\partial^2 \theta_0}{\partial T^2} + \theta_0 = 0; \quad \theta_0(0,0) = 1, \quad \frac{\partial \theta_0(0,0)}{\partial T} = 0.$$

In previous attempt τ appeared explicitly in the exponent, which led to secular term to the next order.

This can be avoided by $\sigma = \gamma^{-\frac{1}{2}} \Rightarrow$

$$\theta_0 = A_0(\tau)e^{iT} + \bar{A}_0(\tau)e^{-iT}, \quad A_0(0) = \frac{1}{2},$$

$O(\epsilon^1)$

$$\frac{\partial^2 \theta_1}{\partial T^2} + \theta_1 \equiv h_s,$$

where

$$\begin{aligned} h_s &= -2\sigma\gamma \frac{\partial^2 \theta_0}{\partial T \partial \tau} - 2\sigma \frac{d\gamma}{d\tau} \frac{\partial \theta_0}{\partial T} - \gamma \frac{d\sigma}{d\tau} \frac{\partial \theta_0}{\partial T} \\ &= -i \left(2\sigma\gamma \frac{dA_0}{d\tau} + 2\sigma A_0 \frac{d\gamma}{d\tau} + \gamma \frac{d\sigma}{d\tau} A_0 \right) e^{iT} + \text{c.c.} \\ &= -i \left(2\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + \frac{3}{2} \gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} \right) e^{iT} + \text{c.c.} \end{aligned}$$

Annihilation of secular terms \Rightarrow coefficient of e^{iT} is zero \Rightarrow (ODE) for A_0 .

$$2\gamma^{\frac{1}{2}} \frac{dA_0}{d\tau} + \frac{3}{2} \gamma^{-\frac{1}{2}} A_0 \frac{d\gamma}{d\tau} = 0$$

Separable equation (ODE) for A_0

$$\frac{1}{A_0} \frac{dA_0}{d\tau} = -\frac{3}{4\gamma} \frac{d\gamma}{d\tau}$$

Integration and $A_0(0) = \frac{1}{2}$, $\gamma(0) = 1 \Rightarrow$

$$A_0 = \frac{1}{2} \gamma^{-\frac{3}{4}}$$

Physical note: wave action

Energy in pendulum motion : $E = E_s + E_\ell$

E_ℓ : potential energy due to change in ℓ

E_s : Energy due to the oscillations

$$E_s = \frac{1}{2} m \ell^2 \dot{\phi}^2 + m g \ell (1 - \cos \phi),$$

Small amplitude, scaling, invocation of two-scale solution $\theta_0 \Rightarrow$

$$E_s = 2 m g \ell_0 \phi_0^2 \gamma A_0^2 (1 + O(\epsilon)).$$

E_s is not constant, but

Wave action

$$\frac{E_s}{\omega} \approx \text{const.}, \quad \omega = \sqrt{\frac{g}{\ell}}$$

is constant; general result for oscillations and waves in a time dependent medium or on a current.