

# Mek4100 The $\pi$ theorem and scaling

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## The Buckingham $\pi$ theorem

Problem contains

- Basic units  $L_j, j = 1, n$
- Parameters  $q_k, k = 1, m$

$$q_k = p_k \prod_{j=1}^n L_j^{a_{jk}} = p_k L_1^{a_{1k}} \cdot \dots \cdot L_n^{a_{nk}}.$$

Seek dimensionless combinations on the form

$$\begin{aligned} \pi &= \prod_{k=1}^m q_k^{\alpha_k} \\ &\sim L_1^{a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1m}\alpha_m} \\ &\quad \cdot L_2^{a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2m}\alpha_m} \\ &\quad \cdot \dots \\ &\quad \cdot L_n^{a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nm}\alpha_m} \end{aligned}$$

$\pi$  dimensionless when

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\alpha_2 + \dots + a_{1m}\alpha_m &= 0 \\ \dots &= 0 \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots + a_{nm}\alpha_m &= 0 \end{aligned}$$

Matrix form

$$AX = 0.$$

$n$  equations with  $m$  unknowns, where

- $A = \{a_{jk}\}$  is the *Dimension matrix*
- $X = \{\alpha_k\}$

Linear algebra  $\Rightarrow$  solution space of dimension  $m - r$ , where  $r = \text{rank}(A) \leq \min(n, m)$ .

In other words: We have  $m - r$  independent solutions for  $\pi$ . Infinite number of derived solutions (as  $\pi_1^2, \pi_1\pi_2$ ) makes choice of  $\pi$ 's ambiguous.

### $\pi$ theorem, part i

There are exactly  $m - r$  independent dimensionless numbers.

**Important consequence:**

We obtain the numbers by inspection, selection etc. As long as we find  $m - r$  independent ones we are good!

Gaussian elimination on  $AX = 0$  should be used as last resort, only.

### Rescaling

$$\hat{L}_j = \lambda_j L_j.$$

$$q_k = p_k \prod_{j=1}^n L_j^{a_{jk}} = \hat{p}_k \prod_{j=1}^n \hat{L}_j^{a_{jk}}, \quad \hat{p}_k = p_k \prod_{j=1}^n \lambda_j^{a_{jk}}$$

Logan leaves out  $p_k$  and treatise of second part if  $\pi$  theorem is flawed.

### Unit-free relation

No good definition in Logan. No good counter-examples.

### Attempted definition

Any relation  $F(q_1, \dots, q_m) = 0$  implies a relation between the numeral values  $f(p_1, \dots, p_m) = 0$ . If  $f$  is independent of the choice of units the relation is unit-free.

All useful relations are like this.

### $\pi$ theorem, part ii

Any unit-free relation  $F(q_1, \dots, q_m) = 0$  can be expressed in terms of dimensionless numbers;  $G(\pi_1, \dots, \pi_{m-r}) = 0$

We skip the proof.

## Example; mathematical pendulum revisited

Parameters

$$\frac{m}{M} \quad \frac{\ell}{L} \quad \frac{g}{LT^{-2}} \quad \frac{\omega}{T^{-1}}$$

### Use of $\pi$ theorem, part i

Units on  $m, \ell$  and  $\omega$  independent  $\Rightarrow r = 3$ .

Number of  $\pi$ :  $4 - 3 = 1$ .

$m$  cannot enter; only quantity with mass.

Remove time between  $\omega$  and  $g$ :  $\omega^2/g \sim L^{-1}$ .

Remove length  $\pi = \ell \omega^2/g$ .

$$\pi = \frac{\ell \omega^2}{g}$$

### Use of $\pi$ theorem, part ii

We seek  $\omega$  expressed by the other parameters:

$$F(\omega, m, \ell, g) = 0$$

Can be expressed as

$$G(\pi) = 0 \Rightarrow \pi = \text{const.}$$

Then

$$\pi = \frac{\ell \omega^2}{g} = \text{const.} \Rightarrow \omega = \text{const.} \sqrt{\frac{g}{\ell}}.$$

Where const. may be a different constant each time.

## Mathematical pendulum with finite excursion

New parameter: maximum horizontal excursion  $x$

$$\frac{m}{M} \frac{\ell}{L} \frac{g}{LT^{-2}} \frac{\omega}{T^{-1}} \frac{x}{L}$$

### Use of $\pi$ theorem, part i

Still we have  $r = 3$ .

Number of  $\pi$ :  $5-3=2$ .

The one from previous example is still applicable

$$\pi_1 = \frac{\ell \omega^2}{g}$$

Need one more. Obvious choice

$$\pi_2 = \frac{x}{\ell}$$

Since  $x$  is part of  $\pi_2$ , but not  $\pi_1$ , the two  $\pi$ 's are independent.

### Use of $\pi$ theorem, part ii

We seek  $\omega$  expressed by the other parameters:

$$F(\omega, m, \ell, g, x) = 0$$

Can be expressed as

$$G(\pi_1, \pi_2) = 0 \Rightarrow \pi_1 = h(\pi_2),$$

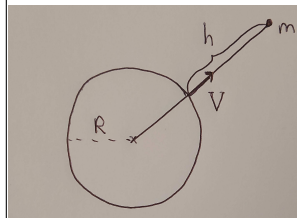
where  $h$  is an unknown function.

Then

$$\pi_1 = \frac{\ell \omega^2}{g} = h(\pi_2) \Rightarrow \omega = \sqrt{\frac{g}{\ell}} \hat{h} \left( \frac{x}{\ell} \right).$$

Where  $\hat{h} = h^{\frac{1}{2}}$ .

## Scaling: The projectile example.



$$m \frac{d^2 h}{dt^2} = - \frac{GmM}{(h+R)^2}$$

$$h(0) = 0, \quad \frac{dh(0)}{dt} = V$$

From Newton's 2 law.

At school:  $h$  very small (meaning what?)  $\Rightarrow h = Vt - \frac{1}{2}gt^2$

Goal:

- Identify small parameter that tells when  $h$  is small
- Make problem dimensionless such that small  $h$  limit is OK.

Reshuffling equation ( $g = MG/R^2$ )

$$\frac{d^2 h}{dt^2} = - \frac{g}{(1 + \frac{h}{R})^2}$$

$$h(0) = 0, \quad \frac{dh(0)}{dt} = V$$

Solution  $h = h(V, g, R, t)$

Observations:

- $m$  cancels out
- $M$  and  $G$  always appeared as single entity  $MG$
- Often useful to do some simplifications at the very beginning.

### Scaling

$$\bar{t} = \frac{t}{t_c}, \quad \bar{h} = \frac{h}{h_c}.$$

Choice of characteristic time ( $t_c$ ) and height ( $h_c$ ) ambiguous, but dimensionless time and height should be of order unity.

$$\frac{h_c}{t_c^2} \frac{d^2 \bar{h}}{d\bar{t}^2} = - \frac{g}{(1 + \frac{h_c}{R} \bar{h})^2}$$

$$\bar{h}(0) = 0, \quad \frac{h_c}{t_c} \frac{d\bar{h}(0)}{d\bar{t}} = V$$

Observation: Both  $\bar{t}$  and  $\bar{h}$  must be  $\pi$ 's.

### Dimension analysis

$$\frac{h}{L} \quad \frac{t}{T} \quad \frac{g}{LT^{-2}} \quad \frac{R}{L} \quad \frac{V}{LT^{-1}}$$

Number of  $\pi$ :  $5-2=3$ . Choice

- 1 Obvious:  $\pi_1 = \frac{h}{R}$ .
- 2 Now, one with  $t$  and not  $h$ :  $\pi_2 = \frac{Vt}{R}$ .
- 3 Finally, neither  $h$  nor  $t$ . Then, the subset  $g$ ,  $R$ ,  $V$  provide a single number (use  $\pi$  theorem on subset!)  $\pi_3 = \frac{V}{\sqrt{gR}}$

Feasible scalings:

$$\bar{t} = p(\pi_3)\pi_2, \quad \bar{h} = P(\pi_3)\pi_1.$$

where  $p$  and  $P$  are functions to be selected.

Low orbit; requirement cannot contain  $t$  or  $h$

$$\pi_3 \ll 1$$

$\ll$  means "a magnitude smaller".

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### Scaling; attempt 1

Simply put  $p = P = 1$

$$\bar{t} = \pi_2 = \frac{Vt}{R}, \quad \bar{h} = \pi_1 = \frac{h}{R}.$$

$$t_c = \frac{R}{V}, \quad h_c = R.$$

Scaled eqs:

$$\pi_3 \frac{d^2 \bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \quad \bar{h}(0) = 0, \quad \frac{d\bar{h}(0)}{d\bar{t}} = 1.$$

Limit  $\pi_3 \rightarrow 0$  ill behaved. Must have  $|\frac{d^2 \bar{h}}{d\bar{t}^2}| \rightarrow \infty$  as  $\pi_3 \rightarrow 0$ .  
**Rubbish scaling.**

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### Scaling; attempt 2. Use $g$ instead of $V$ in $\bar{t}$ .

Make  $\bar{t}$  from  $t$ ,  $g$  and  $R$  (unique, why ?)

$$\bar{t} = \pi_2 = \sqrt{\frac{g}{R}}t, \quad \bar{h} = \frac{h}{R}.$$

$$t_c = \sqrt{\frac{R}{g}}, \quad h_c = R, \quad p(\pi_3) = \frac{1}{\pi_3}.$$

Scaled eqs:

$$\frac{d^2 \bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \quad \bar{h}(0) = 0, \quad \frac{d\bar{h}(0)}{d\bar{t}} = \pi_3.$$

Limit  $\pi_3 \rightarrow 0$ : "start from rest";  $\bar{h}$  becomes immediately negative; no upward motion

**Rubbish again.**

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### Why failure?

#### Scaling 1

- $h_c = R$ . Low orbit: characteristic  $h$  not radius of Earth.
- $t_c = \frac{R}{V}$ . Time spent by traveling to center of Earth with speed  $V$ . Too large for  $t_c$ .

$h_c$  and  $t_c$  not characteristic at all!

#### Scaling 2

- $h_c = R$ . Still bad.
- $t_c = \sqrt{\frac{R}{g}}$ . Like time spent to center of Earth from rest with acceleration  $g$ . Again too large for  $t_c$ .

Equally stupid as 1.

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### Proper attempt; leave $R$ out of scaling

express  $t_c$  and  $h_c$  in terms of  $V$  and  $g$ , only

$$t_c = \frac{V}{g}, \quad h_c = \frac{V^2}{g}.$$

Observe:  $p = P = \pi_3^{-2}$ .

$t_c$  is time for retardation from  $V$  to 0 by  $g$ .

$h_c$  is such that potential energy  $gh_c$  is comparable to kinetic energy at  $t = 0$ . And,  $h_c = Vt_c$ .

Scaled eqs:

$$\frac{d^2 \bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\pi_3^2 \bar{h})^2}, \quad \bar{h}(0) = 0, \quad \frac{d\bar{h}(0)}{d\bar{t}} = 1.$$

Limit  $\pi_3 \rightarrow 0$ :  $\bar{h} = \bar{t} - \frac{1}{2}\bar{t}^2$ . "School result" reproduced.

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### Lessons learned

- $\pi$  theorem alone is not sufficient.
- Correct scaling guided by sound interpretations of  $h_c$  and  $t_c$ .
- Dimensionless variables and coefficients of dimensionless equations are  $\pi$ 's.

Finally. Interpretation of low-orbit requirement

$$\pi_3 = \frac{V}{\sqrt{gR}} \ll 1.$$

$\sqrt{gR}$  describes "free fall velocity to center of Earth". That  $V$  is much less than this is a reasonable requirement.

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## But, honestly

Instructive as it may be, that was also a lot of fuzz. Funny how a little theory may make you dance. Here is another approach.

The equation set, once more

$$\frac{d^2 h}{dt^2} = -\frac{g}{\left(1 + \frac{h}{R}\right)^2}, \quad h(0) = 0, \quad \frac{dh(0)}{dt} = V.$$

Fairly clear that the **red term** should be small for a low orbit.

Deletion gives the trivial set

$$\frac{d^2 h}{dt^2} = -g, \quad h(0) = 0, \quad \frac{dh(0)}{dt} = V,$$

which gives a position

$$h = Vt - \frac{1}{2}gt^2.$$

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Ah well, identifying a simplified problem that was easily solved gave an approximate solution

$$h = Vt - \frac{1}{2}gt^2.$$

The peak position then becomes

$$h_{\max} = h(t_{\max}) = V^2/(2g), \quad t_{\max} = V/g.$$

Choosing  $h_c$  and  $t_c$  accordingly, and claiming  $h/R \ll 1$  we find

$$t_c = \frac{V}{g}, \quad h_c = \frac{V^2}{g} = Vt_c, \quad \epsilon \equiv \frac{V^2}{gR} \ll 1$$

The defined  $\epsilon$  (standard name for small parameter) equals  $\pi_3^2$ .

Next, the full set is scaled accordingly,  $\epsilon$  will appear and we are ready to invoke a perturbation scheme.

In a more complex case we would often combine simplified solutions, or even heuristic arguments, with dimension analysis to get the equation set into shape and prepare for solution – numerical or analytical.

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