Ex. 1 . *Outer solution.*
The outer solution is obtained from

$$\frac{\mathrm{d}y_o}{\mathrm{d}x} + y_o^2 = 0.$$

This is a nonlinear first order equation which is separable and readily solved

$$\frac{1}{y_o^2}\frac{\mathrm{d}y_o}{\mathrm{d}x} = -1 \Rightarrow \frac{1}{y_o} = x + C \Rightarrow y_o = \frac{1}{x+C}.$$

Available is also the solution $y_o = 0$, corresponding to $C = \infty$. Hence y_o may fulfill either of the boundary conditions, but not both. As usual we first assume a boundary layer to the left, at $x = 0$. If this works we are good. Then $C = 1$ and

$$y_o = \frac{1}{1+x}.$$

Inner solution.
We introduce the stretched variable $\xi = x/\delta$:

$$\begin{array}{cccc} \frac{\epsilon}{\delta^2}\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} & +\frac{1}{\delta}\frac{\mathrm{d}y}{\mathrm{d}\xi} & +y^2 & =0 \\ (1) & (2) & (3) & \end{array}$$

Since (3) remains finite as $\epsilon, \delta \to 0$ and (1) must be contained not to reproduce the outer solution the dominant balance is (1) & (2) which gives $\delta \sim \epsilon$. Then (1), (2) $\sim \epsilon^{-1} \gg$ (3) ~ 1. Hence, we choose $\delta = \epsilon$ and find Y, the leading order boundary layer solution, from

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\xi^2} + \frac{\mathrm{d}Y}{\mathrm{d}\xi} = 0,$$

which is integrated to

$$\frac{\mathrm{d}Y}{\mathrm{d}\xi} + Y = A,$$

which, in turn, has the solution

$$Y = A + B^e - \xi.$$

The condition $Y(0) = 0$ then yields $B = -A$.
Matching.
The matching condition

$$\lim_{\xi\to\infty} Y(\xi) = \lim_{x\to 0} y_o(x) \equiv y_{\text{match}},$$

implies $A = 1$ and $y_{\text{match}} = 1$. Then the unified solution becomes

$$y_{\text{unif.}} = Y + y_o - y_{\text{match}} = \frac{1}{x+1} - e^{-\frac{x}{\epsilon}}.$$

Ex. 2 .

a)

x	m	t	C	k	x_0
L	M	T	$\frac{M}{TL^2}$	$\frac{M}{T^2}$	L

Π theorem, part 1: 6 parameters, 3 dimensions $\Rightarrow$ 3 dimensionless numbers. Simplest options:
Make the first, only, with x. The simplest option: $\pi_1 = \frac{x}{x_0}$

Make the next,only, with t. Use also m and k (not x'es): $\pi_2 = \frac{kt^2}{m}$

The last is the only made with C. Avoid x and t. Then the theorem tells us that there is on π between C, m, x_0 and k: $\pi_3 = \frac{Cx_0^2}{\sqrt{mk}}$.

b) We scale the problem according to

$$x = x_c u, \quad t = t_c s.$$

From the transformation of the initial condition from $x(0) = x_0$ to $u(0) = 1$ we find $x_c = x_0$. Then both initial conditions are fulfilled regardless of t_c. Using $x_c = x_0$, inserting the transformation in the ODE and reorganize the result such that the coefficient of the second derivative becomes unity we obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + \frac{Cx_0^2 t_c}{m} u^2 \frac{\mathrm{d}u}{\mathrm{d}s} + \frac{t_c^2 k}{m} u = 0.$$

To reproduce the dimensionless ODE, given in the problem text, we must have

$$\frac{Cx_0^2 t_c}{m} = \epsilon, \quad \frac{t_c^2 k}{m} = 1,$$

which imply

$$t_c = \sqrt{\frac{m}{k}}, \quad \epsilon = \frac{Cx_0^2}{\sqrt{mk}}.$$

t_c expresses the period of the undamped oscillator. ϵ increases with C and x_0 and decreases with m and k, as can be expected.
u, s and ϵ are dimensionless numbers which may expressed by π_1, π_2 and π_3. In fact:

$\epsilon = \pi_3$, $u = \pi_1$ and $\tau = \sqrt{\pi_2}$.

If we had obtained different π's in sub-problem a, the above relations would also be different.

c) Introduction of τ, such that $u = u(s, \tau)$, yields the set

$$\frac{\partial^2 u}{\partial s^2} + 2\epsilon \frac{\partial^2 u}{\partial t \partial \tau} + \epsilon u^2 \frac{\mathrm{d}u}{\mathrm{d}s} + u = O(\epsilon^2),$$

$$u(0,0) = 1, \quad \frac{\partial u(0,0)}{\partial s} + \epsilon \frac{\partial u(0,0)}{\partial \tau} = 0.$$

The expansion $u = u_0(s,\tau) + \epsilon u_1(s,\tau)$ is then inserted.

Order ϵ^0

$$\frac{\partial^2 u_0}{\partial s^2} + u_0 = O,$$

$$u_0(0,0) = 1, \quad \frac{\partial u_0(0,0)}{\partial s} = 0.$$

The solution is

$$u_0 = A(\tau)e^{is} + A^*(\tau)e^{-is} = A(\tau)e^{is} + c.c., \quad A(0) = \frac{1}{2}.$$

Order ϵ^1

$$\frac{\partial^2 u_1}{\partial s^2} + u_1 = -2\frac{\partial^2 u_0}{\partial t \partial \tau} - u_0^2 \frac{\mathrm{d}u_0}{\mathrm{d}s}.$$

$$u_1(0,0) = 0, \quad \frac{\partial u_1(0,0)}{\partial s} = -\frac{\partial u_0(0,0)}{\partial \tau}.$$

Inserting the expression for u_0 on the right hand side of the ODE we arrive at

$$\frac{\partial^2 u_1}{\partial s^2} + u_1 = iA^3 e^{3is} - i\left(2\frac{\mathrm{d}A}{\mathrm{d}\tau} + A^2 A^*\right) e^{is} + c.c.$$

The solution must be damped. Hence, we do not allow u_1 to grow linearly in s. Then, the e^{is} and e^{-is} parts of the right hand side must vanish, which implies

$$2\frac{\mathrm{d}A}{\mathrm{d}\tau} + A^2 A^* = 0.$$

Substituting the polar form $A = ae^{i\psi}$, where a and ψ are real, into this we obtain

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} + \frac{1}{2}a^3 = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\tau} = 0.$$

The first one is a separable equation

$$-2a^{-3}\frac{\mathrm{d}a}{\mathrm{d}\tau} = 1 \quad \Rightarrow \quad a^{-2} = \tau + B \quad \Rightarrow \quad a = (B+\tau)^{-\frac{1}{2}}.$$

The second immediately yields $\psi = D$ =const. The initial condition for A now implies $B = 4$ and $D = 0$. Then

$$A = (4+\tau)^{-\frac{1}{2}}.$$

Assembling the leading order solution we then find

$$A = \frac{1}{\sqrt{1+\frac{\epsilon s}{4}}}\cos(s).$$

Ex. 3 . The velocity of the particle is $\vec{v} = \dot{x}\vec{\imath}$. Then $T = \frac{1}{2}m\dot{x}^2$. Furthermore:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \Rightarrow \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

Then $\dot{x} = p/m$ and

$$H = \dot{x}p - L = \dot{x}p - \frac{1}{2}\dot{x}^2 + \frac{1}{2}k^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Hamilton's canonical equations then become

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial p} = \tfrac{p}{m} \\ \dot{p} &= -\tfrac{\partial H}{\partial x} = -kx \end{aligned}$$

Ex. 4 . Zero derivative implies

$$0 = f'(x) = -\frac{2\sinh(x-2)}{\cosh^3(x-2)} - \epsilon\frac{2x\sinh(x-2)}{\cosh^3(x-2)} + \epsilon\frac{2}{\cosh^2(x-2)}. \tag{1}$$

For the unperturbed problem this gives

$$0 = -\frac{2\sinh(x_0-2)}{\cosh^3(x_0-2)},$$

With the solution $x_0 = 2$.
We set $x = x_0 + x_1$, $x_1 \ll x_0$, meaning that $x_1 \ll 1$. For the leading part of (1) we may use Taylor series expansion

$$\frac{2\sinh(x-2)}{\cosh^3(x-2)} = \frac{2\sinh(x_1)}{\cosh^3(x_1)} = \left(\frac{2\sinh(x)}{\cosh^3(x)}\right)' |_{x=0}\, x_1 + O(x_1^2).$$

Moreover

$$\left(\frac{2\sinh(x)}{\cosh^3(x)}\right)' |_{x=0} = \left(\frac{2}{\cosh^2(x)} - \frac{6\sinh^2(x)}{\cosh^4(x)}\right) |_{x=0} = 2.$$

In the terms of (1) that are linear in ϵ it will suffice to replace x by x_0 alone, which gives

$$0 = f'(x) = -2x_1 + 2\epsilon + O(x_1^2, \epsilon x_1),$$

and $x_1 = \epsilon$. It is fully acceptable to assume that x_1 is of order ϵ in the first place. Hence,

$$x_{\max} = 2 + \epsilon + O(\epsilon^2).$$