A Boussinesq model for educational purposes
MEK4320

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Formulation

Scaling

$$x^\star = L^\star x,\quad t^\star = L^\star (g h_0^\star)^{-\frac{1}{2}} t,\quad \eta^\star = \alpha h_0^\star \eta,$$
$$z^\star = h_0^\star z,\quad u^\star = \alpha (g h_0^\star)^{\frac{1}{2}} u,$$

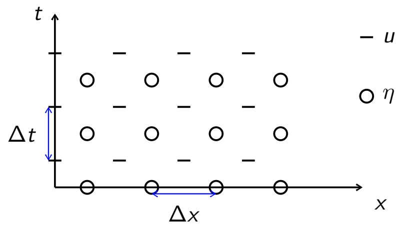
Boussinesq equations

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left((h+\alpha\eta)\overline{u}\right)$$
$$\frac{\partial \overline{u}}{\partial t} + \frac{1}{2}\alpha\frac{\partial \overline{u}^2}{\partial x} = -\frac{\partial \eta}{\partial x} + \frac{1}{2}\beta h \frac{\partial^2}{\partial x^2}\left(h\frac{\partial \overline{u}}{\partial t}\right) - \frac{1}{6}\beta h^2 \frac{\partial^3 \overline{u}}{\partial^2 x \partial t},$$

$\beta = 0 \Rightarrow$ NLSW equations
$\beta = 0,\ \alpha = 0 \Rightarrow$ LSW equations

Finite difference discretization

A staggered grid



The discrete approximation

$$\eta^{(n)}_{j-\frac{1}{2}} \approx \eta((j-\frac{1}{2})\Delta x, n\Delta t),\quad u^{(n+\frac{1}{2})}_j \approx u(j\Delta x, (n+\frac{1}{2})\Delta t),$$

where Δx and Δt are the grid increments.

Discrete LSW equation; constant depth

Differential equations

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x},\quad \frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x}.$$

Difference equations Derivatives $\Rightarrow$ mid-point differences

$$\frac{\eta^{(n)}_{j-\frac{1}{2}} - \eta^{(n-1)}_{j-\frac{1}{2}}}{\Delta t} = -h\frac{u^{(n-\frac{1}{2})}_j - u^{(n-\frac{1}{2})}_{j-1}}{\Delta x} \quad \text{(i)}$$
$$\frac{u^{(n+\frac{1}{2})}_j - u^{(n-\frac{1}{2})}_j}{\Delta t} = -\frac{\eta^{(n)}_{j+\frac{1}{2}} - \eta^{(n)}_{j-\frac{1}{2}}}{\Delta x} \quad \text{(ii)}$$

Simple discretization due to grid structure.
Explicit method for LSW.

LSW dispersion relation (problem 1a); stability (1c)

LSW equation; constant depth

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x},\quad \frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x}.$$

Wave mode

$$\eta = \mathrm{Re}\,\hat{\eta} e^{\imath(kx-\omega t)},\quad u = \mathrm{Re}\,\hat{u} e^{\imath(kx-\omega t)}.$$

Subsitution into LSW equation

$$-\imath\omega\hat{\eta}e^{\imath(kx-\omega t)} = -\imath kh\hat{u}e^{\imath(kx-\omega t)},\quad -\imath\omega\hat{u}e^{\imath(kx-\omega t)} = -\imath k\hat{\eta}e^{\imath(kx-\omega t)}.$$

Deletion of common factors

$$\omega\hat{\eta} = kh\hat{u},\ \omega\hat{u} = k\hat{\eta} \Rightarrow \frac{\hat{\eta}}{\hat{u}} = \frac{kh}{\omega},\ \frac{\hat{\eta}}{\hat{u}} = \frac{\omega}{k} \Rightarrow \omega^2 = hk^2.$$

Dispersion relation.

Numerical dispersion relation (problem 1b)

Mode now reads

$$\eta^{(n)}_{j-\frac{1}{2}} = \mathrm{Re}\,\hat{\eta}e^{\imath(k(j-\frac{1}{2})\Delta x - \omega_N n\Delta t)},\quad u^{(n+\frac{1}{2})}_j = \mathrm{Re}\,\hat{u}e^{\imath(kj\Delta x - \omega_N(n+\frac{1}{2})\Delta t)}.$$

Inserted in (i) (continuity eq.)

$$\hat{\eta}\frac{e^{\imath(k(j-\frac{1}{2})\Delta x - \omega_N n\Delta t)} - e^{\imath(k(j-\frac{1}{2})\Delta x - \omega_N(n-1)\Delta t)}}{\Delta t} =$$
$$-h\hat{u}\frac{e^{\imath(kj\Delta x - \omega_N(n-\frac{1}{2})\Delta t)} - e^{\imath(k(j-1)\Delta x - \omega_N(n-\frac{1}{2})\Delta t)}}{\Delta x}$$

Both differences centered at $x = (j-\frac{1}{2})\Delta x$ and $t = (n-\frac{1}{2})\Delta t$.
Extract corresponding factor $E = e^{\imath(k(j-\frac{1}{2})\Delta x - \omega_N(n-\frac{1}{2})\Delta t)}$ from both sides.

$$\hat{\eta} E \frac{e^{-\frac{1}{2}\imath\omega_N\Delta t} - e^{\frac{1}{2}\imath\omega_N\Delta t}}{\Delta t} = -h\hat{u}E\frac{e^{\imath\frac{1}{2}k\Delta x} - e^{-\frac{1}{2}\imath k\Delta x}}{\Delta x}.$$

Invokation of expontial/trig. relation, deletion of common factors

$$\hat{\eta}\frac{\sin(\frac{1}{2}\omega_N\Delta t)}{\frac{1}{2}\Delta t} = h\hat{u}\frac{\sin(\frac{1}{2}k\Delta x)}{\frac{1}{2}\Delta x}.$$

Correspondingly from (ii)

$$\hat{u}\frac{\sin(\frac{1}{2}\omega_N\Delta t)}{\frac{1}{2}\Delta t} = \hat{\eta}\frac{\sin(\frac{1}{2}k\Delta x)}{\frac{1}{2}\Delta x}.$$

Again we have two expressions for $\hat{\eta}/\hat{u}$. Claiming them to be equal

$$\left(\frac{\sin(\frac{1}{2}\omega_N\Delta t)}{\frac{1}{2}\Delta t}\right)^2 = h\left(\frac{\sin(\frac{1}{2}k\Delta x)}{\frac{1}{2}\Delta x}\right)^2,$$

or

$$\sin(\frac{1}{2}\omega_N\Delta t) = \pm\frac{\sqrt{h}\Delta t}{dx}\sin(\frac{1}{2}k\Delta x).$$

A numerical dispersion relation

Relation between $\hat{u}$ and $\hat{\eta}$ on preceding slide.

Related topic: numerical stability

$$\sin(\frac{1}{2}\omega_N\Delta t) = \pm\frac{\sqrt{h}\Delta t}{dx}\sin(\frac{1}{2}k\Delta x).$$

Real ω_N requires $|\sin(\frac{1}{2}\omega_N\Delta t)| \le 1$.
Otherwise compex conjugate pair of solutions $\omega_N = \omega_r \pm \imath\omega_i$.
One of these yields exponential growth $\Rightarrow$ instability.
Real ω_N for all k requires

$$\mathrm{Co} \equiv \frac{\sqrt{h_0}\Delta t}{\Delta x} \le 1.$$

The CFL criterion

Co is the Courant number. Most ustable mode $\frac{1}{2}k\Delta x = 1$.

Common interpretation of CFL criterion: The signal speed in the grid $(\Delta x/dt)$ cannot be smaller than that from the PDE system.

Numerical dispersion

Right going wave

$$\sin(\frac{1}{2}\omega_N\Delta t) = \frac{\sqrt{h}\Delta t}{dx}\sin(\frac{1}{2}k\Delta x) = \mathrm{Co}\sin(\frac{1}{2}k\Delta x). \quad (*)$$

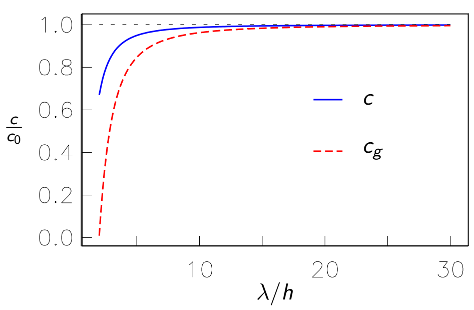
In general: ω_N not linear in k $\Rightarrow$ artificial (numerical) dispersion
Special case: $\mathrm{Co} = 1$ $\Rightarrow$ no numerical dispersion.
Waves shorter than $2\Delta x$ not meaningfully resolved in grid.

Differentiation of '(*)' with repect to k

$$\cos(\frac{1}{2}\omega_N\Delta t)c_g = \cos(\frac{1}{2}k\Delta x).$$

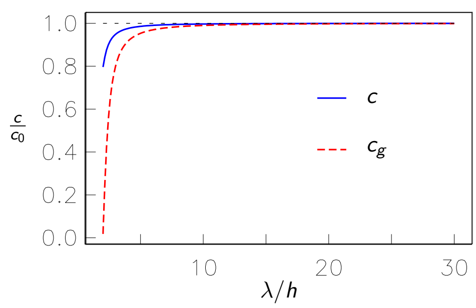
$c_g \to 0$ when $\frac{1}{2}k\Delta x \to 1$.

$\mathrm{Co} = \frac{1}{2}$, $\Delta x = 1$



Strong numerical dispersion when $\lambda < 10\Delta x$, say.

$\mathrm{Co} = 0.9$, $\Delta x = 1$



Weaker numerical dispersion when $\mathrm{Co} = \frac{c_0\Delta t}{\Delta x}$ closer to 1.

Nature of numerical dispersion

The numerical dispersion is normal; $\frac{\mathrm{d}c}{\mathrm{d}k} < 0$.

Expansion for small k (problem 1e)

KdV $\quad \omega = \pm h^{\frac{1}{2}} k \left(1 - \frac{1}{6}(kh)^2\right)$

Boussinesq $\quad \omega = \pm h^{\frac{1}{2}} k \left(1 - \frac{1}{6}(kh)^2 + O((kh)^4)\right)$

LSW, num. $\quad \omega_N = \pm h^{\frac{1}{2}} k \left(1 - \frac{\Delta x^2}{24h^2}(1 - \mathrm{Co}^2)(kh)^2 + O(k^4)\right)$

Rather similar type relations.