

# THE KLEIN-GORDON EQUATION AND STATIONARY PHASE.

Geir Pedersen

Department of Mathematics, UiO.

February 11, 2019

# The Klein-Gordon equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + q\eta = 0 \quad (1)$$

Initial condition:

$$\eta(x, 0) = e^{-(\frac{x}{L})^2}, \quad \frac{\partial}{\partial t} \eta(x, 0) = 0 \quad (2)$$

The solution will depend on  $q$  and  $L$  in the combination  $qL^2$  (Easily demonstrated by rescaling). However, we keep the equation on the given form. In the plots we always have  $L = 10$ .

## Solution methods

- 1 Finite differences.
- 2 Stationary phase
- 3 FFT (presented elsewhere)

# Fourier transform + stationary phase

## 1 *Fourier transform*

Linear equations, constant coefficients, spatially confined initial condition  $\Rightarrow$  Fourier integral for large positive  $x$ :

$$\eta(x, t) = \frac{1}{2\pi} \Re \int_0^{\infty} \tilde{\eta}_0(k) e^{i(kx - \omega(k)t)} dk$$

where  $\omega$  and  $k$  fulfill the dispersion relation.

## 2 *Stationary phase*

For large  $x$  and  $t$  dominant contributions to the Fourier integral comes from the vicinity of the stationary point. An approximate integrand  $\Rightarrow$  explicit asymptotic solution for large  $x$  and  $t$ .

# The Fourier transform

Relations between  $f(x)$  and  $\tilde{f}(k)$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \quad (3)$$

Essential property

$$\frac{d\tilde{f}}{dk} = ik\tilde{f},$$

as shown by integration by parts.

Details in the definition (3) may vary in literature and software.

# The transformed Klein-Gordon equation

Fourier transform applied to (1); (spatial differentiation replaced by power of  $ik$ )

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + (k^2 + q)\tilde{\eta} = 0$$

second order ODE for  $\tilde{\eta}$  with general solution

$$\tilde{\eta}(k, t) = A(k)e^{-i\omega(k)t} + B(k)e^{i\omega(k)t}$$

where  $\omega(k) = \sqrt{q + k^2}$  (dispersion relation)

Transformed initial conditions

$$\tilde{\eta}(k, 0) = \tilde{\eta}_0(k) = L\sqrt{\pi}e^{-(\frac{kL}{2})^2}, \quad \frac{\partial \tilde{\eta}(k, 0)}{\partial t} = 0,$$

yield

$$A = B = \frac{1}{2}\tilde{\eta}_0$$

# The inverse transformation

$$\eta(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \tilde{\eta}_0(k) e^{i(kx - \omega(k)t)} + \tilde{\eta}_0(k) e^{i(kx + \omega(k)t)} \right) dk$$

Initial condition  $\Rightarrow \tilde{\eta}$  is real and symmetric with relation to  $k = 0$   
 $\Rightarrow$  terms in inversion are complex conjugates and

$$\eta(x, t) = \frac{1}{2\pi} \Re \int_{-\infty}^{\infty} \tilde{\eta}_0(k) e^{i(kx - \omega(k)t)} dk,$$

further rewritten to

$$\eta(x, t) = \frac{1}{2\pi} \Re \left\{ \int_0^{\infty} \tilde{\eta}_0(k) e^{i(kx - \omega(k)t)} dk + \int_0^{\infty} \tilde{\eta}_0(k) e^{-i(kx + \omega(k)t)} dk \right\} \quad (4)$$

Waves propagating to left and right are separated.

# Stationary phase

A general Fourier integral

$$I(t) = \int_a^b F(k) e^{it\chi(k)} dk,$$

where  $a$  and  $b$  may be finite or infinite.

Large  $t \Rightarrow$  rapid oscillations of integrand  $\Rightarrow$  cancellation of adjacent contributions  $\Rightarrow I \rightarrow 0$  as  $t \rightarrow \infty$ .

If there is a stationary point,  $k = k_0$ , where

$$\frac{d\chi(k_0)}{dk} = 0$$

the oscillations will be slowest around  $k_0$  and the dominant contribution to the Fourier integral comes from the vicinity of this point.

Approximation of phase function and amplitude factor close to  $k_0$

$$\chi(k) \approx \chi(k_0) + \frac{1}{2}\chi''(k_0)(k - k_0)^2, \quad F(k) \approx F(k_0).$$

This gives

$$\begin{aligned} I(t) &\approx \int_{k_0-\epsilon}^{k_0+\epsilon} F(k_0) e^{it\{\chi(k_0) + \frac{1}{2}\chi''(k_0)(k-k_0)^2\}} dk \\ &\approx F(k_0) e^{it\chi(k_0)} \int_{-\infty}^{\infty} e^{\frac{1}{2}it\chi''(k_0)(k-k_0)^2} dk. \end{aligned}$$

The latter integral is found in mathematical handbooks etc., and

$$I(t) \sim \frac{\sqrt{2\pi}F(k_0)}{\sqrt{t|\chi''(k_0)|}} e^{i(\chi(k_0)t \pm \frac{\pi}{4})}$$



# Stationary phase applied to the Klein Gordon solution

We regard the first term in (4) and recognize  $F$  above as  $\tilde{\eta}_0/(2\pi)$  and the phase as

$$\chi = k \frac{x}{t} - \omega(k).$$

The stationary point is then given according to

$$c_g(k_0) \equiv \frac{d\omega(k_0)}{dk} = \frac{x}{t},$$

$c_g$  is the group velocity! Since  $c_g < 1$ , for all  $k$ , stationary phase is only applicable for  $x < t$ . We then find

$$k_0 = q^{\frac{1}{2}} \frac{x}{t} (1 - (\frac{x}{t})^2)^{-\frac{1}{2}}, \quad \chi(k_0) = -q^{\frac{1}{2}} (1 - (\frac{x}{t})^2)^{\frac{1}{2}},$$
$$\chi''(k_0) = -q^{-\frac{1}{2}} (1 - (\frac{x}{t})^2)^{\frac{3}{2}}.$$

The final asymptotic solution may be written as

$$\eta \sim a(x, t) \cos \theta(x, t),$$

where

$$a = \frac{L^{\frac{1}{2}} q^{\frac{1}{4}}}{(2\frac{t}{L})^{\frac{1}{2}}} \frac{e^{-\frac{L^2 q}{4((\frac{t}{x})^2 - 1)}}}{(1 - (\frac{x}{t})^2)^{\frac{3}{4}}}, \quad \theta = \frac{\pi}{4} - q^{\frac{1}{2}} \sqrt{t^2 - x^2}$$

A local wave number may be defined as  $k_l(x, t) = \frac{\partial \theta}{\partial x}$ . Then  $k_l(x, t) = k_0(x, t)$  follows.

- 1 How does the amplitude ( $a$ ) vary in  $x$  and  $t$  ?
- 2 How does the local wave length vary in  $x$  and  $t$  ?
- 3 How does individual crests propagate relative to the wave system as a whole?

# Numerical method.

Straightforward, explicit, finite difference method:

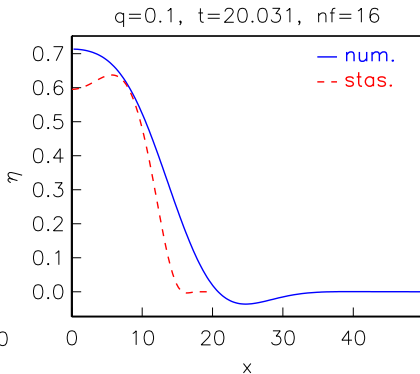
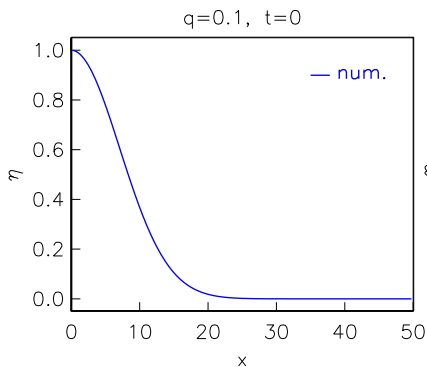
Grid:  $x_i = i\Delta x$ ,  $t_n = n\Delta t$ , nodal values:  $\eta(x_i, t_n) \approx \eta_i^{(n)}$ .

Difference equations:

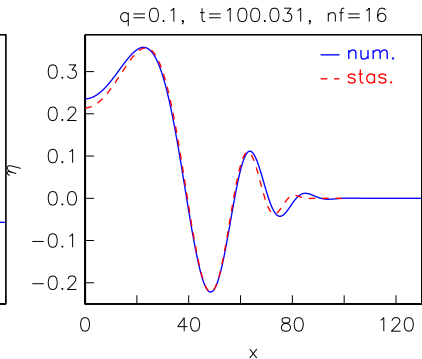
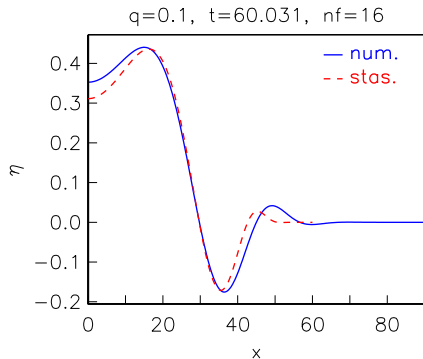
$$\left. \begin{aligned} & \frac{1}{\Delta t^2} \left( \eta_i^{(n+1)} - 2\eta_i^{(n)} + \eta_i^{(n-1)} \right) \\ & - \frac{1}{\Delta x^2} \left( \eta_{i+1}^{(n)} - 2\eta_i^{(n)} + \eta_{i-1}^{(n)} \right) + q\eta_i^{(n)} = 0 \end{aligned} \right\} \quad (5)$$

- 1  $\eta_i^{(0)}$  and  $\eta_i^{(-1)}$  given by initial conditions.
- 2 For each step ( $n = 1, 2, \dots$ ) we compute  $\eta_i^{(n+1)}$  from  $\eta_i^{(n)}$  and  $\eta_i^{(n-1)}$ .
- 3 Grid-refinement tests ( $\Delta x, \Delta t \rightarrow 0$ ).

# Comparison of numeric and asymptotic solutions



# Comparison of numeric and asymptotic solutions



# Comparison of numeric and asymptotic solutions

