Numerical investigations on nonlinear and dispersive wave propagation. MEK4320

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Introduction

The problems below are all related to the educational Boussinesq model described on the course pages.

In problem 2 through 6 numerical applications shall be performed. Throughout these problems we assume that the computational domain is $0 \leq x \leq L$, where the length of the numerical wave tank, L, is either given or chosen appropriately by the user. Other parameters are also either given or must be chosen by the student. The choice of parameters in numerical simulations is an important task and may often involve both trial and error and simplified analysis. The simulations start with either a solitary wave solution, for which the sbouss program provides initial values for both u and η, or an initial elevation, specified through an input file, combined with zero initial velocity.

Problem 1 ***Numerical dispersion relation.***
We assume constant depth which equals h_0 in dimensionless coordinates. Then there are solutions of the linear shallow water equations on the form

$$\eta = \operatorname{Re} \hat{\eta} e^{\imath(kx-\omega t)}, \quad u = \operatorname{Re} \hat{u} e^{\imath(kx-\omega t)}. \tag{1}$$

a) Find the relations between ω and k and between $\hat{u}$ and $\hat{\eta}$.

b) In the discrete equations (i) and (ii) in the document "A Boussinesq model for educational purposes", which provides a description of the model, we may correspondingly insert

$$\eta^{(n)}_{j-\frac{1}{2}} = \operatorname{Re} \hat{\eta} e^{\imath(k(j-\frac{1}{2})\Delta x - \omega_N n \Delta t)}, \quad u^{(n+\frac{1}{2})}_{j} = \operatorname{Re} \hat{u} e^{\imath(kj\Delta x - \omega_N (n+\frac{1}{2})\Delta t)}, \tag{2}$$

where the numerical frequency, ω_N, is given an index to distinguish it from the frequency in sub-problem a. Show that the dispersion relation becomes

$$\frac{2}{\Delta t} \sin\left(\frac{\omega_N \Delta t}{2}\right) = \pm\sqrt{h_0}\frac{2}{\Delta x} \sin\left(\frac{k\Delta x}{2}\right). \tag{3}$$

What is the shortest wave for which (3) is meaningful ? Are the waves in the numerical description dispersive ?
Hint: When finding the relation 3 it is helpful to remove the exponential factor corresponding to the center point of each discrete equation. The remaining parts may then immediately be expressed in terms of sine functions.

c) In this sub-problem we will investigate the stability of the numerical method. Stability requires that no mode will grow in time regardless of the value of k. Explain briefly why this is necessary and sufficient for stability. Moreover, instability will occur when (3) yields an ω_N with a positive imaginary part. When ω_N fulfills (3), so does the complex conjugate of ω_N (show this), implying that the existence any complex ω_N implies instability.

Then, show that the numerical solution is unstable (grows exponentially in time) when

$$\mathrm{Co} \equiv \frac{\sqrt{h_0}\Delta t}{\Delta x} > 1. \tag{4}$$

The quantity Co is called the Courant number. (Tip: you may assume that $|\sin(\beta)| \leq 1$ is equivalent to β being real.)

d) Show that the numerical dispersion relation is exact for

$$\mathrm{Co} = 1. \tag{5}$$

e) For the Boussinesq equations we have

$$\omega = \pm h_0^{\frac{1}{2}} k(1 - \frac{1}{6}(kh_0)^2 + O((kh_0)^4)). \tag{6}$$

Show this.

From (3), which is the numerical dispersion relation for the shallow water equations, we may find a similar expression

$$\omega_N = \pm h_0^{\frac{1}{2}} k(1 - \kappa(kh_0)^2 + O((kh_0)^4)). \tag{7}$$

Find κ and a relation between Δx and Δt for which (7) becomes equal to (6), meaning that a numerical solution of the shallow water equations possesses approximately the same dispersion properties as an exact solution of the linear Boussinesq equation.

Tip: expand the left and right hand sides of (3) in Taylor series to obtain $\omega_N + ()\omega_N{}^3 + ... = h_0^{\frac{1}{2}}(k + ()k^3 + ...)$. Leading order then implies $\omega_N = h_0^{\frac{1}{2}} k + O(k)^3$. Insert this into the $\omega_N{}^3$ term and explain why we then obtain the correct relation (7).

Problem 2 ***Solitary wave propagation and grid effects.***

We will test the performance of the model by investigating how well the shape of a solitary wave is preserved etc. What model (LSW, NLSW, linear dispersive or Boussinesq) is the appropriate choice for this task ? Use the sbouss program to simulate the propagation of a solitary wave, with amplitude $\alpha = 0.1$, starting at $x = 20$ until $t = 70$ on constant depth $h = 1$. Make certain that the numerical wave tank is long enough. Use Matlab, or some other tool, to find the maximum η at $t = 70$ when you choose $\Delta x = 2$, $\Delta x = 1$ and $\Delta x = 0.5$, while the time step reduction factor is set to 0.5 and the discrete correction term is turned off. Do also depict all three solutions in the same diagram. Comment on what resolution that seems relevant.

Problem 3 ***Numerical instability.***

Make a LSW simulation with $\Delta x = 0.5$ and $\Delta t = 1$ and a solitary wave, $A = 0.1$, as initial condition. The depth is still $h = 1$. Run the simulation until $t = 40$. What happens and why ?

Problem 4 ***Wave dispersion.***
Use Matlab, for instance, to produce an initial elevation

$$\eta(x,0) = \begin{cases} 2A\cos^2\left(\frac{\pi(x-x_0)}{\lambda}\right) & \text{if} \quad -\frac{1}{2}\lambda < x - x_0 < \frac{1}{2}\lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

which is combined with $u(x,0) = 0$. Explain why this initial condition has continuous first derivative and why it to leading order yields a wave in each direction with amplitude A and length λ.
Choose $A = 0.2$, $\lambda = 20$, $x_0 = 0$, a tank length $L = 100$ and simulation time $t = 80$. Remember that the wave tank starts at $x = 0$. We still have constant depth $h = 1$.

a) Solve the LSW equations. What should the result be ? Adjust the resolution to obtain this result well enough (explain what you mean).

b) Solve the linearized Boussinesq equations. Plot the solution at $t = 80$. Relate the solution to results given in the course leaflet.

c) Solve the full Boussinesq equations. What is the difference from the previous case ? What may the leading wave be ? Depict the solution together with the previous one.

d) Solve the NLSW equations. You will see some short features. Use grid refinement to see if they change dramatically. If they do they are artifacts and the solution is invalid.

e) Solve the NLSW equations again, but focus on the times before the artifacts appear. What happens to the shape ? Illustrate with figures. What may be the physical interpretation of this shape change ?

Problem 5 ***Bores.***
Again we start from rest and use (8) as initial elevation with $x_0 = 0$. However, this time we employ a long initial condition and a long wave tank according to $L = 1000$ and $\lambda = 100$. In the two first subproblems we have constant depth $h = 1$.

a) We focus on $t < 200$. Do simulations with NLSW and Boussinesq equations. How long are the results similar ? Illustrate with graphs.

b) Do the Boussinesq simulation until $t = 800$. What happens? Use Matlab to compare the second crest from the front to the solitary wave solution in the syllabus (slides or in: Lecture Notes Mek 4320: Hydrodynamic Wave theory.) Can you expect perfect agreement ? How can the sbouss program be used to make a more accurate comparison ?

c) A sequence of crests like the ones in the previous sub-problem is called an undular bore. It may also be generated due to shoaling. Make a depth file to sbouss which corresponds to $h = 1$ for $x < 40$, $h = 0.2$ for $150 > x > 50$ and a linear slope in between. Run a solitary wave with amplitude $A = 0.05$ from the deep to the shallow region (you need a fine grid due to the shallow shelf). Show that an undular bore is generated. Explain the relation to the previous sub problem.

Problem 6 ***Numerical dispersion.***
In (8) we choose $\lambda = 16$ and $x_0 = 0$, whereas the choice of A is arbitrary. Moreover, we select a flat bottom ($h \equiv 1$) with $L = 130$. Do two simulations with

(i) The linearized Boussinesq equations and $\Delta x = \Delta t = \frac{1}{2}$. Set the time step explicitly, as explained in the documentation of sbouss.

(ii) LSW equations, $\Delta x = \frac{1}{2}\sqrt{17}$ and $\Delta t = \frac{1}{2}$.

Depict both solutions at a time just before they reach the farther end of the wave tank. How can they be so similar ? (tip: look at the first problem)

Problem 7 ***Relation to standard wave equation.*** We assume linear equations, but non-constant depth.

a) Show that the LSW equations (disperson terms omitted) yields:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 (hu)}{\partial x^2} = 0. \tag{9}$$

b) Find an equation corresponding to (9) when the dispersion term is retained.

c) Ignore the dispersion term again. Elininate all η values from the discrete equations (i) and (ii) in the document "A Boussinesq model for educational purposes" to obtain:

$$\frac{u_j^{(n+\frac{3}{2})} - 2u_j^{(n+\frac{1}{2})} + u_j^{(n-\frac{1}{2})}}{\Delta t^2} = \frac{h_{j+1}u_{j+1}^{(n+\frac{1}{2})} - 2h_j u_j^{(n+\frac{1}{2})} + h_{j-1}u_{j-1}^{(n+\frac{1}{2})}}{\Delta x^2}.$$

Interpret this discrete equation.

Problem 8 ***Numerical stability for the Boussinesq equations.*** We set $\alpha = 0$ and $h = h_0$ =constant in the set (7) and (8) from "A Boussinesq model for educational purposes".

a) Write out the difference equations in the style of (i) and (ii).

b) Insert a mode of type (2) and show that the numerical dispersion relation becomes:

$$\hat{\omega}^2 = \frac{h_0 \hat{k}^2}{1 + \frac{\beta}{3} h_0^2 \hat{k}^2},$$

where $\hat{k} = \frac{2}{\Delta x}\sin\left(\frac{k\Delta x}{2}\right)$ and $\hat{\omega} = \frac{2}{\Delta t}\sin\left(\frac{\omega_N \Delta t}{2}\right)$.

c) Find the criterion for numerical stability.