THE KLEIN-GORDON EQUATION AND STATIONARY PHASE.

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The Klein-Gordon equation

$$\frac{\partial^2\eta}{\partial t^2}-\frac{\partial^2\eta}{\partial x^2}+q\eta=0 \qquad (1)$$

Initial condition:

$$\eta(x,0)=e^{-(\frac{x}{L})^2},\quad \frac{\partial}{\partial t}\eta(x,0)=0 \qquad (2)$$

The solution will depend on q and L in the combination qL^2 (Easily demonstrated by rescaling). However, we keep the equation on the given form. In the plots we always have $L=10$.

Solution methods

1. Finite differences.
2. Stationary phase
3. FFT (presented elsewhere)

Fourier transform + stationary phase

1. *Fourier transform*
   Linear equations, constant coefficients, spatially confined initial condition $\Rightarrow$ Fourier integral
   $$\eta(x,t)=\frac{1}{2\pi}\Re\int_0^\infty \tilde\eta_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k$$
   where ω and k fulfill the dispersion relation.
2. *Stationary phase*
   For large x and t dominant contributions to the Fourier integral comes from the vicinity of the stationary point. An approximate integrand $\Rightarrow$ explicit asymptotic solution for large x and t.

The Fourier transform

Relations between $f(x)$ and $\tilde f(k)$

$$\tilde f(k)=\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x,\quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde f(k)e^{\mathrm{i}kx}\mathrm{d}k \qquad (3)$$

Essential property

$$\frac{\mathrm{d}\tilde f}{\mathrm{d}x}=\mathrm{i}k\tilde f,$$

as shown by integration by parts.
Details in the definition (3) may vary in literature and software.

The transformed Klein-Gordon equation

Fourier transform applied to (1); (spatial differentiation replaced by power of ik)

$$\frac{\partial^2\tilde\eta}{\partial t^2}+(k^2+q)\tilde\eta=0$$

second order ODE for $\tilde\eta$ with general solution

$$\tilde\eta(k,t)=A(k)e^{-\mathrm{i}\omega(k)t}+B(k)e^{\mathrm{i}\omega(k)t}$$

where $\omega(k)=\sqrt{q+k^2}$ (dispersion relation)
Transformed initial conditions

$$\tilde\eta(k,0)=\tilde\eta_0(k)=L\sqrt{\pi}e^{-(\frac{kL}{2})^2},\quad \frac{\partial\tilde\eta(k,0)}{\partial t}=0,$$

yield

$$A=B=\frac{1}{2}\tilde\eta_0$$

The inverse transformation

$$\eta(x,t)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\tilde\eta_0(k)e^{\mathrm{i}(kx-\omega(k)t)}+\tilde\eta_0(k)e^{\mathrm{i}(kx+\omega(k)t)}\right)\mathrm{d}k$$

Initial condition and dispersion relaton $\Rightarrow$ $\tilde\eta_0$ is real, $\tilde\eta_0(-k)=\tilde\eta_0(k)$ and $\tilde\omega(-k)=\omega(k)$. and symmetric with relation to $k=0$.

Exploiting properties of η_0 and ω; substitute $\ell=-k$

$$\int_{-\infty}^{\infty}\left(\tilde\eta_0(k)e^{\mathrm{i}(kx+\omega(k)t)}\right)\mathrm{d}k = -\int_{\infty}^{-\infty}\left(\tilde\eta_0(\ell)e^{\mathrm{i}(-\ell x+\omega(\ell)t)}\right)\mathrm{d}\ell = \int_{-\infty}^{\infty}\left(\tilde\eta_0(\ell)e^{-\mathrm{i}(\ell x-\omega(\ell)t)}\right)\mathrm{d}\ell.$$

Terms in inversion are complex conjugates

Terms in inversion are complex conjugates $\Rightarrow$

$$\eta(x,t) = \frac{1}{2\pi}\Re \int_{-\infty}^{\infty} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k,$$

Further rewriting

$$\begin{aligned}
&\int_{-\infty}^{\infty} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k \\
&= \int_{0}^{\infty} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k + \int_{-\infty}^{0} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k \\
\Re \int_{-\infty}^{0} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k \quad &= \Re \int_{0}^{\infty} \tilde{\eta}_0(\ell) e^{-\mathrm{i}(\ell x+\omega(\ell)t)} \mathrm{d}\ell \\
&= \Re \int_{0}^{\infty} \tilde{\eta}_0(\ell) e^{\mathrm{i}(\ell x+\omega(\ell)t)} \mathrm{d}\ell
\end{aligned}$$

$$\eta(x,t) = \frac{1}{2\pi}\Re \left\{ \int_{0}^{\infty} \tilde{\eta}_0(k) e^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k + \int_{0}^{\infty} \tilde{\eta}_0(k) e^{-\mathrm{i}(kx+\omega(k)t)} \mathrm{d}k \right\} \tag{4}$$

Waves propagating to left and right are separated.

Dispersion properties

$$\omega = \sqrt{q+k^2}, \quad c = \sqrt{1+\frac{q}{k^2}}, \quad c_g = \frac{1}{c}.$$

c decreases with k, increases with λ.
c_g increases with k, decreases with λ.
$c,\ c_g \to 1$ as $k \to \infty$.

Stationary phase

A general Fourier integral

$$I(t) = \int_{a}^{b} F(k) e^{\mathrm{i}t\chi(k)} \mathrm{d}k,$$

where a and b may be finite or infinite.
Large $t \Rightarrow$ rapid oscillations of integrand $\Rightarrow$ cancellation of adjacent contributions $\Rightarrow I \to 0$ as $t \to \infty$.
If there is a stationary point, $k = k_0$, where

$$\frac{\mathrm{d}\chi(k_0)}{\mathrm{d}k} = 0$$

the oscillations will be slowest around k_0 and the dominant contribution to the Fourier integral comes from the vicinity of this point.

Approximation of phase function and amplitude factor close to k_0

$$\chi(k) \approx \chi(k_0) + \frac{1}{2}\chi''(k_0)(k-k_0)^2, \quad F(k) \approx F(k_0).$$

This gives

$$\begin{aligned}
I(t) \quad &\approx \int_{k_0-\epsilon}^{k_0+\epsilon} F(k_0) e^{\mathrm{i}t\{\chi(k_0)+\frac{1}{2}\chi''(k_0)(k-k_0)^2\}} \mathrm{d}k \\
&\approx F(k_0) e^{\mathrm{i}t\chi(k_0)} \int_{-\infty}^{\infty} e^{\frac{1}{2}\mathrm{i}t\chi''(k_0)(k-k_0)^2} \mathrm{d}k.
\end{aligned}$$

The latter integral is found in mathematical handbooks etc., and

$$I(t) \sim \frac{\sqrt{2\pi}F(k_0)}{\sqrt{t|\chi''(k_0)|}} e^{\mathrm{i}(\chi(k_0)t \pm \frac{\pi}{4})}$$

Stationary phase applied to the Klein Gordon solution

We regard the first term in (4) and recognize F above as $\tilde{\eta}_0/(2\pi)$ and

$$\chi = k\frac{x}{t} - \omega(k).$$

The stationary point is then given according to

$$c_g(k_0) \equiv \frac{\mathrm{d}\omega(k_0)}{\mathrm{d}k} = \frac{x}{t},$$

c_g is the group velocity! Since $c_g < 1$, for all k, stationary phase is only applicable for $x < t$. We then find

$$k_0 = q^{\frac{1}{2}}\frac{x}{t}\left(1-\left(\tfrac{x}{t}\right)^2\right)^{-\frac{1}{2}}, \qquad \chi(k_0) = -q^{\frac{1}{2}}\left(1-\left(\tfrac{x}{t}\right)^2\right)^{\frac{1}{2}},$$

$$\chi''(k_0) = -q^{-\frac{1}{2}}\left(1-\left(\tfrac{x}{t}\right)^2\right)^{\frac{3}{2}}.$$

The final asymptotic solution may be written as

$$\eta \sim a(x,t)\cos\theta(x,t),$$

where

$$a = \frac{L^{\frac{1}{2}}q^{\frac{1}{4}}}{(2\frac{t}{L})^{\frac{1}{2}}} \frac{e^{-\frac{L^2 q}{4((\frac{t}{x})^2-1)}}}{\left(1-(\frac{x}{t})^2\right)^{\frac{3}{4}}}, \quad \theta = \frac{\pi}{4} - q^{\frac{1}{2}}\sqrt{t^2-x^2}$$

Local wave number $k_l(x,t) = \frac{\partial\theta}{\partial x}$. Then $k_l(x,t) = k_0(x,t)$ follows.

How does the amplitude (a) vary in x and t ?
$x \to 0 \Rightarrow a \sim t^{-\frac{1}{2}}$; a decreases with x; $a \to 0$ as $x \to t$.

How does the local wave length vary in x and t ?
k_0 increases with x and hence λ decreases with x. (c_g decreases with wavelength).

How does individual crests propagate relative to the wave system as a whole?
Moves foreward and gains on front ($x = t$) since $c > 1 > c_g$.

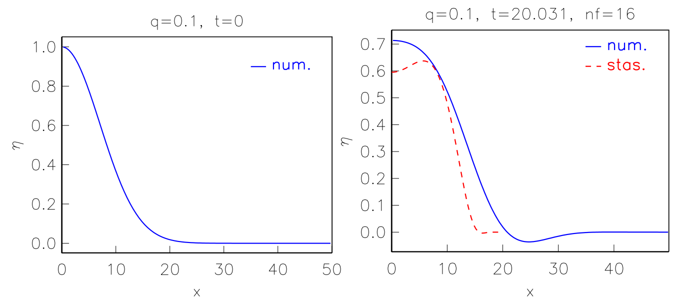
Numerical method.

Straightforward, explicit, finite difference method:
Grid: $x_i = i\Delta x$, $t_n = n\Delta t$, nodal values:$\eta(x_i, t_n) \approx \eta_i^{(n)}$.
Difference equations:

$$\left.\begin{array}{l} \frac{1}{\Delta t^2}\left(\eta_i^{(n+1)} - 2\eta_i^{(n)} + \eta_i^{(n-1)}\right) \\ -\frac{1}{\Delta x^2}\left(\eta_{i+1}^{(n)} - 2\eta_i^{(n)} + \eta_{i-1}^{(n)}\right) + q\eta_i^{(n)} = 0 \end{array}\right\} \quad (5)$$

1. $\eta_i^{(0)}$ and $\eta_i^{(-1)}$ given by initial conditions.
2. For each step $(n = 1, 2 \ldots)$ we compute $\eta_i^{(n+1)}$ from $\eta_i^{(n)}$ and $\eta_i^{(n-1)}$.
3. Grid-refinement tests $(\Delta x, \Delta t \to 0)$.

Comparison of numeric and asymptotic solutions



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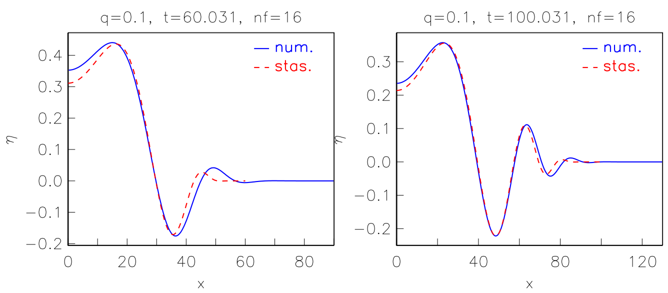

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