On problem 2, page 165, in Logan

We are given the ODE

$$u'' - u = \epsilon t u, \tag{1}$$

with the initial conditions, at $t = 0$,

$$u(0) = 1, \quad u'(0) = -1. \tag{2}$$

A standard perturbation expansion, $u = u_0 + \epsilon u_1 + ...$, yields

$$u = e^{-t} + \frac{\epsilon}{4}\Big(\sinh t - (t^2 + t)e^{-t}\Big) + O(\epsilon^2). \tag{3}$$

Comment: As for large t the requirement $u_0 \gg \epsilon u_1$ is violated. Hence, the approximation ((3)) must be expected to be locally valid for small t, only.

Taylor series

We seek a Taylor series

$$u = \sum_{n=0}^{n=\infty} a_n t^n, \quad a_n = \frac{u^{(n)}(0)}{n!}, \tag{4}$$

where the superscript indicates differentiation. Such a series may be obtained by insertion of the series in the ODE, followed by the requirement that the resulting series, accumulated from all terms in (1), must vanish to every power in t. Here, an alternative approach is outlined. Repeated differentiation on (1) yields

$$\begin{aligned}
u'' &= (1 + \epsilon t)u, \\
u''' &= \epsilon u + (1 + \epsilon t)u', \\
u^{(4)} &= 2\epsilon u' + (1 + \epsilon t)u'', \\
... & .. \; ... \\
u^{(n)} &= (n-2)\epsilon u^{(n-3)} + (1 + \epsilon t)u^{(n-2)},
\end{aligned}$$

where the last, general, line may be proven with induction. Since (2) defines $u(0)$ and $u'(0)$ we now find the derivatives to any order recursively, by substituting 0 for t, and employ them in (4).