Long wave modeling.

NLSW and Boussinesq equations; motivation and derivation

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Primitive and general hydrodynamic equations

The Navier-Stokes *(NS)* equation, primitive form

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} \equiv \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \tau_f - g\vec{k}$$

$$\nabla \cdot \vec{v} = 0$$

where $\vec{v}$ = velocity, $\mathrm{D}/\mathrm{D}t$ = material derivative, p = pressure and τ_f is viscous/turbulent part of stress tensor. In words:

acceleration = - pressure gradient + friction + gravity

net outflow from any fluid volume = 0

Boundary conditions: impermeable, no-slip, free (surface), artificial

Key problems: turbulence model, free surface tracking, under-resolved boundary layers, etc.

Generalization/alternatives to NS include multi-phase, multi-material...

Applicability of primitive models

- General, but inaccurate, free surface techniques may be imbedded (VOF, level-set, SPH ...)
- Industrial (CFX, Fluent...) and open-source (OpenFoam) solvers
- Computations readily become very heavy $\Rightarrow$ numerical solutions are under-resolved or unattainable
- Thin wall boundary layers; cannot be resolved
- Feasible only in local and idealized studies
- The burden of the computations often lead to wavering of the physics ?
- Analytic solutions are sparse, circumstantial and cumbersome

Surprisingly (?) little insight (?) in hydrodynamic wave theory yet stem from "full computational models".

Simplified theories are still crucial, but general models become increasingly important

Full potential theory

Non-rotational motion $\Rightarrow$ potential ϕ: $\nabla\phi \equiv \vec{v}$

$$\nabla^2\phi = 0 \quad \text{for} \quad -h < z < \eta$$

Free surface $(z = \eta)$ $\left(\frac{\mathrm{D}}{\mathrm{D}t} = \partial/\partial t + \vec{v}\cdot\nabla\right)$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0, \qquad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial\phi}{\partial z}$$

Bottom $(z = -h)$

$$\frac{\mathrm{D}h}{\mathrm{D}t} = 0 \Rightarrow \frac{\partial\phi}{\partial n} = 0$$

Surface waves often well described.

Potential flow models

1. Integral equation discretized by panels. Boundary location updated in time as part of the method.
2. FFT techniques; approximations at free surface
3. Solution of the Euler's equaton of motion by finite volumes etc. (related to NS solvers)
4. Somewhat in fashion (!?): Higher order expansions of vertical structures by layers, polynomials ....

Not incorporated: viscous effects, turbulence, overturning waves, Coriolis force...
Computation still heavy. Useful for local simulations and for assessing validity of simpler models.
Still true: More efficient and robust models must be employed for large scale modeling

Approximate theories

Basis of approximations; Scales for surface gravity waves

Acceleration scale

g acceleration of gravity
Remark: scale for particle acceleration in gravity waves is always the same, regardless of size of problem

Length scales

λ wavelength	h depth	A amplitude (η)
L_h depth variations	L_λ variation of λ, A...	

Often $L_h \sim L_\lambda$

Velocity and time scales

May be built from length and acceleration scales

Approximations; Regimes

- $\frac{A}{h}, \frac{A}{\lambda} \ll 1 \Rightarrow$ linear and weakly non-linear theories
- $\frac{\lambda}{h} \ll 1 \Rightarrow$ deep water
- $\frac{h}{\lambda} \ll 1 \Rightarrow$ **shallow water; long wave theory**
- $\frac{h}{L_h}, \frac{\lambda}{L_\lambda} \ll 1 \Rightarrow$ multiple scale methods: ray theory (sec. 3 in Comp.); narrow band (nearly uniform waves)

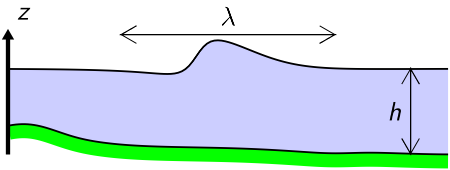
Different requirements may be combined; long wave theory is often combined with weak non-linearity.

Definition of characteristic scales may be vague or ambiguous

Ocean modeling

Tools of the trade

- Depth integrated models for long waves
- Ray tracing, wave kinematics
- Efficient and robust numerical techniques



Long waves $\lambda/h \gg 1$ (2D case for simplicity)

U, W – characteristic horizontal and vertical velocities

$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{W}{h} \sim \frac{U}{\lambda} \Rightarrow \frac{W}{U} \sim \frac{h}{\lambda} \ll 1$

$\Rightarrow$ vertical motion is small $\Rightarrow$ pressure nearly hydrostatic

Lessons learned from the wave mode as $kH \to 0$

The sinusiodal wave mode for gravity surface has been investigated.

- In the limit $kH \to 0$:
  - Particle trajectories becomes horizontal lines.
  - Vertical velocities much smaller than horizontal ones.
  - The horizontal velocity is uniform in z.
  - The pressure is hydrostatic.
- Corrections for small kH.
  - Is of order $(kH)^2$.
  - Potential, pressure and horizontal velocity are approximated by quadratic polynomials in z.

The mode is a solution of the linear equations in uniform depth.
May the above properties carry over to nonlinear equations and variable depth as well ?
May this then be exploited to design approximate equations with a simpler structure than full potential theory ?

The hydrostatic approximation; The NLSW equations

A depth integrated continuity equation

Split horizontal and vertical velocity

$$\mathbf{v} = \mathbf{v}_h + w\mathbf{k}, \quad \nabla = \nabla_h + \mathbf{k}\frac{\partial}{\partial z},$$

where

$$\mathbf{v}_h = u\mathbf{i} + v\mathbf{j}, \quad \nabla_h = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y},$$

We rename the equilibrium depth $h(x, y, t)$, whereas H is the typical value of h.

Mass balance in a vertical cylindrical volume, S

The footprint in the xy-plane is Ω

The volume is confined by the bottom, $z = -h$, and reach beyond the free surface η.

Volume of fluid within S

$$\iint\limits_{\Omega} (\eta + h) dxdy,$$

Flux of fluid volume into S

$$-\int\limits_{\Gamma} \int\limits_{-h}^{\eta} \mathbf{v}_h \cdot \mathbf{n} dzds,$$

where Γ is the circumference of Ω and ds is arc length along Γ.

We define the depth integrated velocity

$$\mathbf{U} = \int_{-h}^{\eta} \mathbf{v}_h dz,$$

and express volume conservation in S ($\mathbf{n}$ independent of z)

$$\iint_{\Omega} \frac{\partial}{\partial t}(\eta + h) dxdy \; = \; - \int_{\Gamma} \mathbf{U} \cdot \mathbf{n} ds$$

Use of Gauss' theorem then yield

$$\iint_{\Omega} \left\{ \frac{\partial}{\partial t}(\eta + h) + \nabla_h \cdot \mathbf{U} \right\} dxdy \; = \; 0$$

The general depth integrated continuity equation

Mass balance for any Ω requires

$$\frac{\partial \eta}{\partial t} + \frac{\partial h}{\partial t} = -\nabla_h \cdot \mathbf{U}.$$

This is an exact depth integrated continuity equation, but it is useful only if we "know what to insert" for $\mathbf{U}$.

Generally the depth is time independent; $\frac{\partial h}{\partial t} = 0$.

Alternative derivation

The continuity equation is written

$$\frac{\partial w}{\partial z} = -\nabla_h \cdot \mathbf{v}_h.$$

Integrated from bottom to surface

$$w(x,y,\eta,t) - w(x,y,-h,t) = \int_{-h}^{\eta} \frac{\partial w}{\partial z} dz = -\int_{-h}^{\eta} \nabla_h \cdot \mathbf{v}_h dz. \quad (1)$$

Kinematic condition at the free surface

$$w(x,y,\eta,t) = \frac{\partial \eta}{\partial t} + \mathbf{v}_h(x,y,\eta,t) \cdot \nabla_h \eta.$$

Kinematic condition at the bottom

$$w(x,y,-h,t) = -\left(\frac{\partial h}{\partial t} + \mathbf{v}_h(x,y,-h,t) \cdot \nabla_h h\right).$$

Rewriting rightmost integral in (1)

$$\int_{-h}^{\eta} \nabla_h \cdot \mathbf{v}_h dz = \nabla_h \cdot \int_{-h}^{\eta} \mathbf{v}_h dz - \nabla_h \eta \cdot \mathbf{v}_h(x, y, \eta, t) - \nabla_h h \cdot \mathbf{v}_h(x, y, -h, t).$$

Recognition of the integral on the right hand side as $\mathbf{U}$ and insertion in (1) $\Rightarrow$ some terms cancel out and:

$$\frac{\partial \eta}{\partial t} + \frac{\partial h}{\partial t} = -\nabla_h \cdot \mathbf{U},$$

is obtained, again.

The momentum equation

Euler's equation of motion (horizontal and vertical)

$$
\begin{aligned}
\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} &= -\frac{1}{\rho}\nabla_h p, \\
\frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,
\end{aligned}
$$

where $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}_h \cdot \nabla_h + w\frac{\partial}{\partial z}$. Integration of z component from surface ($p_a = 0$)

$$
p = \rho g(\eta - z) - \rho \int\limits_{\eta}^{z} \frac{\mathrm{D}w}{\mathrm{D}t} dz.
$$

$g\eta$ balances the extra weight from wave through elevation.
Last term comes from vertical accelerations.
Experience with modes $\Rightarrow$ last term $\ll \rho g \eta$.

Assessment of vertical acceleration term in linear case

$$\rho \int\limits_{\eta}^{z} \frac{\mathrm{D}w}{\mathrm{D}t} dz \approx \rho h \frac{\partial w(x,y,0,t)}{\partial t} \approx \rho h \frac{\partial^2 \eta}{\partial t^2}.$$

For a wave mode we then have

$$h \frac{\partial^2 \eta}{\partial t^2} = -h\omega^2 \eta.$$

Acceleration term much less than $\rho g \eta$ implies

$$h\omega^2 \ll g \Rightarrow (kh)^2 \ll \frac{gh}{c^2}$$

For long waves $c \approx \sqrt{gh}$ and this correpond to $(kh)^2 \ll 1$.

We assume

$$\rho g \eta \gg \rho \int_{\eta}^{z} \frac{\mathrm{D}w}{\mathrm{D}t} dz,$$

also for variable depth and nonlinear waves. Then

$$p = \rho g(\eta - z),$$

and the horizontal part of the momentum equation becomes

$$\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} = -g\nabla_h \eta.$$

Hence, horizontal particle acceleration independent of $z \Rightarrow \mathbf{v}_h$ remains independent of z if initially so; for instance for waves propagating into quiescent water. Consequences

1. $\mathbf{v}_h = \mathbf{v}_h(x, y, t) \Rightarrow \frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} = \frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h$
2. $\mathbf{U} = \int_{-h}^{\eta} \mathbf{v}_h dz = (h + \eta)\mathbf{v}_h$

Nonlinear Shallow Water Equations

Summary of derivation on preceding slides

Vertical acceleration neglected $\Rightarrow$ hydrostatic pressure $\Rightarrow$ no vertical variation in horizontal velocity $\Rightarrow$ 3D physics, 2D maths.

NLSW

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h = -g\nabla_h \eta$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h+\eta)\mathbf{v}_h)$$

η: surface elevation, $\mathbf{v}_h$: velocity (horizontal), ∇_h: horizontal gradient operator

Efficient and simple numerical solution; hyperbolic equations.

A number of analytical solutions.

May include bores (simple representation of breaking wave), Coriolis effects and bottom drag, *but wave dispersion is lost*.

Use: Ocean modeling; tides, tsunamis, storm surges.

Simplifications of hydrostatic theory

Plane waves, $\mathbf{v}_h = u\mathbf{i}$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left((h+\eta)u\right)$$

Linearized as well

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial t} = -\frac{\partial (hu)}{\partial x},$$

or, by elimination

$$\frac{\partial^2 u}{\partial t^2} - g\frac{\partial^2 (hu)}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gh\frac{\partial \eta}{\partial x}\right) = 0,$$

Constant depth $h = H$; $c_0 = \sqrt{gH}$

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} = 0.$$

Standard wave equations with solutions

$$\eta = F(x - c_0 t) + G(x + c_0 t), \quad u = \frac{c_0}{H}\{F(x - c_0 t) - G(x + c_0 t)\}.$$

Derivation of solution; substitution $\xi_1 = x - c_0 t$, $\xi_2 = x + c_0 t \Rightarrow$

$$0 = \frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} = -4c_0^2 \frac{\partial^2 \eta}{\partial \xi_1 \partial \xi_2},$$

$$\frac{\partial^2 \eta}{\partial \xi_1 \partial \xi_2} = 0 \Rightarrow \frac{\partial \eta}{\partial \xi_1} = f(\xi_1) \Rightarrow \eta = F(\xi_1) + G(\xi_2),$$

where $F' = f$.

First glance at topographic effects on waves

Linear shallow water waves over topography

Linear plane wave, non-constant depth

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(g h(x) \frac{\partial \eta}{\partial x} \right) = 0,$$

Assume standing waves (separation of variables)

$$\eta(x,t) = \hat{\eta}(x)\cos(\omega t + \Delta),$$

where ω is given and Δ may chosen freely. Substitution into wave equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x) \frac{\mathrm{d}\eta}{\mathrm{d}x} \right) + \frac{\omega^2}{g}\hat{\eta} = 0.$$

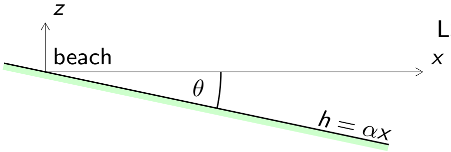
Still, cannot be solved, in general.

Particular depth profiles

- Linear profile: $h = \alpha x$. The classic case; much studied.
- Simpler solutions for $h = \alpha x^q$, with $q = \frac{4}{3}$, 2, 4.

Linear profile $h = \alpha x$

LSW dubious ? $\rightarrow$



$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \kappa\hat{\eta} = 0,$$

where $\kappa = \frac{\omega^2}{g\alpha}$.
Regular singularity at $x = 0$ (zero of order ≤ 2 for coeff. of η_{xx})

Transformation to Bessel's equation

Second order ODEs with low order polynomials as coefficients often have special names, or may be transformed to named equations.
In this case

$$s = 2\sqrt{\kappa x} \quad \Rightarrow \quad s\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}s^2} + \frac{\mathrm{d}\hat{\eta}}{\mathrm{d}s} + s\hat{\eta} = 0,$$

The Bessel equation of order 0.

Bessel functions

Not given by elementary functions, as such. Properties are still well known and described in handbooks, at WEB sites etc.
Two independent solutions of Bessel equation of order 0 are
$\mathrm{J}_0(s)$ - Bessel function, $\mathrm{Y}_0(s)$ - Neumann function.

Approximations for the Bessel functions

Small s; Fröbenius series

$$\mathrm{J}_0(s) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}s^2\right)^n}{(n!)^2}, \quad \mathrm{Y}_0(s) = \frac{2}{\pi}(\log(\frac{1}{2}s)+\gamma)\mathrm{J}_0(s)+\frac{1}{\pi}\sum_{n=0}^{\infty} a_n s^{2n},$$

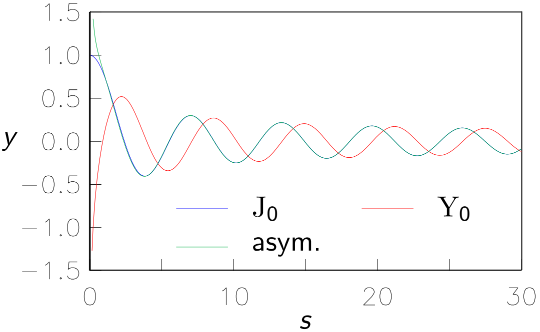
(γ, a_n constants).

J_0 is analytic at $s = 0$.
Y_0 is singular at $s = 0$.

Large s; Asymptotic approximations

$$\mathrm{J}_0(s) \sim \sqrt{\frac{2}{\pi s}}\cos(s - \frac{\pi}{4}), \quad \mathrm{Y}_0(s) \sim \sqrt{\frac{2}{\pi s}}\sin(s - \frac{\pi}{4}).$$

First term of asymptotic series.



Bessel functions J_0, Y_0 plotted with the asymptotic approximation of J_0.

Asymptotic approximation good even for quite moderate arguments.

Values for J_0, Y_0 in this and following figures are obtained by use of intrinsic functions in GNU-fortran.

Our solution

If $x = 0$ is excluded from the domain

$$\hat{\eta} = A\mathrm{J}_0(s) + B\mathrm{Y}_0(s) \quad \text{where} \quad s = 2\sqrt{\kappa x},$$

For large s we may invoke the asymptotic expressions, and rewrite result as

$$\hat{\eta} \sim a_0 \, (\kappa x)^{-\frac{1}{4}} \cos(2\sqrt{\kappa x} + \delta) \equiv a(x) \cos \chi,$$

where A, B, a_0 and δ are related coefficients.
Amplitude part: a is proportional to $h^{-\frac{1}{4}}$; Green's law.
Phase part (χ): wave-length increases with h.
Large x (large s) solutions nearly periodic in x with slow variations of amplitude and wave length. We will return to these results in the more general context of optics (ray theory).

Free waves; restoration of temporal dependence

On the previous slides a_0, δ and Δ may be chosen freely.
Each choice gives a standing wave.
By trigonomerc formulas such standing waves may be combined to propagating waves.
The a general expression for propagating waves may be found as

$$\eta = (\kappa x)^{-\frac{1}{4}} \left(a_1 \cos(2\sqrt{\kappa x} - \omega t + \delta_1) + a_2 \cos(2\sqrt{\kappa x} + \omega t + \delta_2) \right) .$$

It is stressed that this still is an approximation for large $\sqrt{\kappa x}$.

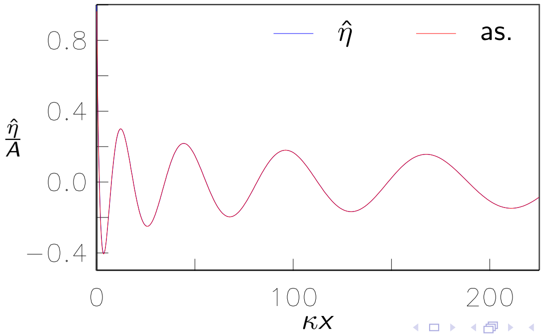
When $x = 0$ (the beach) is *included*; standing wave

Singularity at the beach

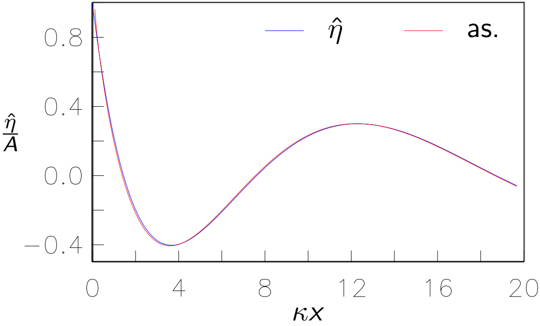
$x \to 0$ and $B \neq 0 \Rightarrow \hat{\eta} \sim \log(x)$; singular.
Corresponds to $u \sim \frac{1}{x}$; volume flux, $h\hat{u}$, finite as $x \to 0$.

Proper solution with $\hat{\eta} < \infty$ and $hu = 0$ at $x = 0$.

$$\hat{\eta} = A\mathrm{J}_0(2\sqrt{\kappa x}), \quad \eta = A\mathrm{J}_0(2\sqrt{\kappa x})\cos(\omega t + \Delta).$$



Blow-up of near-shore region



Derivation of Boussinesq equations

Dispersive long wave models

Models developed by perturbation/iteration/series expansion, assuming small
$\epsilon \equiv (H/\lambda)^2$ and $\alpha \equiv A/H$, where H is typical depth.
Leading pressure modifications by vertical accelerations included.
Huge diversity in in formulations and accuracy

Boussinesq type models

- 1880$\rightarrow$ Theoretical applications (KdV...)
- 1966 first numerical Boussinesq models put to use
- 1990$\rightarrow$ new formulations, increased validity range
- Increase of computer power $\Rightarrow$ large scale models feasible
- Important for some tsunami features
- Important model for coastal engineering
- A step toward more general models from (N)LSW; assessment of dispersion effects

Specifications.

- Long waves in shallow water.
- Non-linear waves.
- 2-D motion (vertical $+$ 1 horizontal direction).
- Slow depth variations ($\frac{\partial h}{\partial x}$ not large).

Scaling

1. Characteristic depth, H, used for "vertical" quantities (z, h, w, η).
2. Characteristic wave length , ℓ, used for "horizontal" quantities (u, x, t).
3. Velocities and surface elevation are, in addition, scaled by an amplitude factor, α.

The long wave expansion requires

$$\epsilon \equiv \frac{H^2}{\ell^2} \ll 1 \tag{2}$$

Non-dimensional quantities

$$z^\star = Hz, \qquad x^\star = \ell x, \qquad t^\star = \ell (gH)^{-\frac{1}{2}} t,$$
$$h^\star = Hh(x), \qquad \eta^\star = \alpha H \eta, \qquad u^\star = \alpha (gH)^{\frac{1}{2}} u,$$
$$w^\star = \epsilon^{\frac{1}{2}} \alpha (gH)^{\frac{1}{2}} w, \quad p^\star = \rho g H p,$$

Scaled basic equations.

Boundary conditions

$$\begin{array}{llll} p = 0, & \eta_t + \alpha u \eta_x = w, & \mathrm{at} & z = \alpha\eta \\ & w = -h_x u & \mathrm{at} & z = -h \end{array} \tag{3}$$

Euler's equation of motion:

$$u_t + \alpha u u_x + \alpha w u_z = -\alpha^{-1} p_x \tag{4}$$
$$\epsilon (w_t + \alpha u w_x + \alpha w w_z) = -\alpha^{-1} (p_z - 1) \tag{5}$$

and the continuity equation

$$u_x + w_z = 0 \tag{6}$$

Strategy

1. Shallow water equations $\Rightarrow$ pressure nearly hydrostatic, u nearly uniform in z.
2. Definition of depth averaged velocity $\overline{u}$. Exact depth averaged continuity equation in terms of $\overline{u}$.
3. Utilizing point 1: Continuity equation and boundary conditions $\Rightarrow$ w expressed by z, η and $\overline{u}$.
4. Expression for w inserted in equation for z-momentum $\Rightarrow$ p expressed by z, η and $\overline{u}$. Gives a correction to hydrostatic pressure.
5. Insertion in horizontal momentum equation, depth averaging $\Rightarrow$ depth-averaged momentum equation.

Alternative techniques: Direct expansion in z, Luke's principle...

Step 1. Hydrostatic theory revisited.

All terms of order ϵ omitted.

From z-component of Euler's equation of motion:

$$p = \alpha\eta - z + O(\alpha\epsilon) \tag{7}$$

x-component:

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\eta_x + O(\epsilon) = \mathrm{function}(x) + O(\epsilon)$$

Then

$$u_z = O(\epsilon) \tag{8}$$

NLSW equations are reproduced

$$u_t + \alpha u u_x = -\eta_x + O(\epsilon)$$

$$\eta_t = -\{(h + \alpha\eta)u\}_x$$

Step 2: depth averaged velocity and continuity equation

Definition of depth average

$$\overline{u} = (h + \alpha\eta)^{-1} \int_{-h}^{\alpha\eta} u dz \tag{9}$$

Then (8)$\Rightarrow$ $u - \overline{u} = O(\epsilon)$
Depth averaged continuity equation:

$$\eta_t = -\{(h + \alpha\eta)\overline{u}\}_x \tag{10}$$

This continuity equation is exact to all orders in α and ϵ

Step 3: Express w in terms of η and $\overline{u}$

From the kinematic surface condition it follows

$$w = w|_{z=\alpha\eta} + \int_{\alpha\eta}^{z} w_z dz = \eta_t + \int_{0}^{z} w_z dz + O(\alpha) \qquad (11)$$

Then equation (6), zero divergence, yields

$$w = \eta_t - \int_{0}^{z} u_x dz + O(\alpha) = \eta_t - \int_{0}^{z} \overline{u}_x dz + O(\alpha, \epsilon) \qquad (12)$$

Since $\overline{u}$ is independent of z we obtain

$$w = \eta_t - z\overline{u}_x + O(\alpha, \epsilon) \qquad (13)$$

The z-component of the momentum equation, (5), is integrated

$$p = p|_{z=\alpha\eta} + \int_{\alpha\eta}^{z} p_z dz = \int_{\alpha\eta}^{z} \{-1 - \alpha\epsilon(w_t + \alpha ...)\} dz \qquad (14)$$

Omitting small terms $\Rightarrow$

$$p = \alpha\eta - z - \alpha\epsilon \int_{0}^{z} w_t dz + O(\epsilon\alpha^2) \qquad (15)$$

Insertion of expression (13) for w then implies

$$p = \alpha\eta - z - \epsilon\alpha(z\eta_{tt} - \frac{1}{2} z^2 \overline{u}_{xt}) + O(\alpha\epsilon^2, \alpha^2\epsilon) \qquad (16)$$

Step 5: averaging the x-component of the mom. eq.

Use of $u - \overline{u} = O(\epsilon)$ in momentum equation ($\overline{(u_t)} = \overline{u}_t + O(\alpha\epsilon)$ etc., see next slide)

$$\overline{u}_t + \alpha \overline{u}\,\overline{u}_x + O(\alpha\epsilon) = -\alpha^{-1}\overline{p_x}. \qquad (17)$$

Invoking (16) on the r.h.s.

$$-\alpha^{-1}p_x = -\eta_x + \epsilon(z\eta_{ttx} - \frac{1}{2}z^2\overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon).$$

Averaging ($(h+\alpha\eta)^{-1}\int_{-h}^{\alpha\eta} \cdot dz$) $\Rightarrow$

$$\begin{aligned} -\alpha^{-1}\overline{p_x} &= -\eta_x - \epsilon(\tfrac{1}{2}h\eta_{ttx} + \tfrac{1}{6}h^2\overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon) \\ &= -\eta_x + \epsilon(\tfrac{1}{2}h(h\overline{u}_t)_{xx} - \tfrac{1}{6}h^2\overline{u}_{xxt}) + O(\epsilon^2, \alpha\epsilon) \end{aligned}$$

where $\eta_t = -(h\overline{u})_x + O(\alpha)$ from (10) is used to obtain a better structure of the final equation.

Step 5: continues....

The convective term $\alpha\overline{u}\overline{u}_x$ is, by itself, $O(\alpha)$. Replacing u by $\overline{u}$ thus yields error $O(\alpha,\epsilon)$. The term $\alpha w u_z$ is $O(\alpha,\epsilon)$ since $u_z = O(\epsilon)$.

For the temporal derivative of the acceleration

$$\begin{aligned}
\int_{-h}^{\alpha\eta} u_t dz &= \frac{\partial}{\partial t}\left(\int_{-h}^{\alpha\eta} u dz\right) - \alpha u|_{z=\alpha\eta}\frac{\partial\eta}{\partial t} \\
&= \frac{\partial}{\partial t}\left((h+\alpha\eta)\overline{u}\right) - \alpha\overline{u}\frac{\partial\eta}{\partial t} + O(\alpha\epsilon) = (h+\alpha\eta)\overline{u}_t + O(\alpha\epsilon).
\end{aligned}$$

Observe the cancellation of $O(\alpha)$ terms in last line.
Hence, $\overline{(u_t)} = \overline{u}_t + O(\alpha\epsilon)$, and we may collect all terms of the averaged equation of motion.

Standard Boussinesq equations
(sometimes named Peregrine's Boussinesq equation)

$$\begin{aligned} \overline{u}_t + \alpha \overline{u}\, \overline{u}_x &= -\eta_x + \epsilon\{\frac{1}{2}h(h\overline{u}_t)_{xx} - \frac{1}{6}h^2\overline{u}_{xxt}\} \\ &+ O(\epsilon^2, \alpha\epsilon) \qquad (18) \end{aligned}$$

$$\eta_t = -\{(h + \alpha\eta)\overline{u}\}_x \qquad (19)$$

Constitute two equations for the unknowns η and $\overline{u}$.

New in relation to NLSW: dispersion.
Same nonlinear terms as NLSW.

More on Boussinesq type equations

Depth integrated theory; Boussinesq

Boussinesq equations with 2 horizontal dimensions

$$\frac{\partial \bar{\mathbf{v}}_h}{\partial t} + \alpha \bar{\mathbf{v}}_h \cdot \nabla_h \bar{\mathbf{v}}_h = -\nabla_h \eta + \epsilon \left(\frac{1}{2} h \nabla_h \nabla_h \cdot (h \frac{\partial \bar{\mathbf{v}}_h}{\partial t}) - \frac{1}{6} h^2 \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t} \right)$$
$$-\epsilon \kappa h^2 (\nabla_h^3 \eta + \nabla_h \nabla_h \cdot \frac{\partial \bar{\mathbf{v}}_h}{\partial t}) + O(\epsilon^2, \alpha\epsilon)$$

$$\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h + \alpha\eta)\bar{\mathbf{v}}_h)$$

Derivation as with 1 horizontal dimension.
Blue term $=O(\epsilon^2, \epsilon\alpha)$: optimization of dispersion properties (never mind detail now; see later slides)
Numerical solution much heavier than for shallow water eq. – implicit solution strategy needed. Still, much faster to solve than primitive equations. Wave dispersion included.

The z_α formulation

Popular formulation from Nwogu, later extended by others (Kennedy, Kirby, Wu, Liu, Lynett..)
Velocity profile

$$\vec{v} = \vec{v}_s + \epsilon(z_\alpha \nabla_h \frac{\partial \eta}{\partial t} - \frac{1}{2} z_\alpha^2 \nabla_h \nabla_h \cdot \vec{v}_*) + O(\epsilon^2),$$

$\vec{v}_s$ = surface velocity, $\vec{v}_*$ velocity at any depth.
Velocity at $z_\alpha(x, y)$
$\mathbf{v}(x, y, t) \equiv \vec{v}(x, y, z_\alpha(x, y), t)$,
used as unknown. Optimization of dispersion on flat bottom $\Rightarrow$
$z_\alpha = -0.531h$
Extra nonlinearities, $O(\epsilon\alpha)$, may be kept in derivation.

z_α not related to nonlinear parameter α

Generalized Boussinesq equations

Hsiao et al. (2002):

$$\eta_t = -\nabla_h \cdot [(h + \alpha\eta)(\mathbf{v} + \epsilon\mathbf{M})] + O(\epsilon^2),$$

$$\mathbf{v}_t + \frac{\alpha}{2}\nabla_h(\mathbf{v}^2) = -\nabla_h\eta - \epsilon\left[\frac{1}{2}z_\alpha^2\nabla_h\nabla_h\cdot\mathbf{v}_t + z_\alpha\nabla_h\nabla_h\cdot(h\mathbf{v}_t)\right]$$
$$+\alpha\epsilon\nabla_h(D_1 + \alpha D_2 + \alpha^2 D_3) + O(\epsilon^2) + \mathbf{N} + \mathbf{E},$$

where index t denotes temporal differentiation and

$$\mathbf{M} = [\tfrac{1}{2}z_\alpha^2 - \tfrac{1}{6}(h^2 - \alpha h\eta + \alpha^2\eta^2)]\nabla_h\nabla_h\cdot\mathbf{v}$$
$$+[z_\alpha + \tfrac{1}{2}(h - \alpha\eta)\nabla_h\nabla_h\cdot(h\mathbf{v})].$$

Extra nonlinearities marked with blue.

Furthermore...

$$
\begin{aligned}
D_1 &= \eta\nabla\cdot(h\mathbf{v}_t) - \tfrac{1}{2}z_\alpha^2\mathbf{v}\cdot\nabla\nabla\mathbf{v} - z_\alpha\mathbf{v}\cdot\nabla\nabla\cdot(h\mathbf{v}) - \tfrac{1}{2}(\nabla\cdot(h\mathbf{v}))^2,\\
D_2 &= \tfrac{1}{2}\eta^2\nabla\cdot\mathbf{v}_t + \eta\mathbf{v}\nabla\nabla\cdot(h\mathbf{v}) - \eta\nabla\cdot(h\mathbf{v})\nabla\cdot\mathbf{v},\\
D_3 &= \tfrac{1}{2}\eta^2\left[\mathbf{v}\cdot\nabla\nabla\cdot\mathbf{v} - (\nabla\cdot\mathbf{v})^2\right],\\
\mathbf{E} &= H^{-1}\nabla_h(\nu(x,y,t)\nabla_h(H\mathbf{v}),\\
\mathbf{N} &= -\tfrac{\alpha}{\mu}\tfrac{K}{H}|\mathbf{v}|\mathbf{v}.
\end{aligned}
$$

Unsystematic terms:

E is dissipation term for capturing of breaking waves

N is bottom drag.

Programs freely available on WEB (Funwave and Coulwave).

There are some issues with these models

Dispersion relation for single harmonic mode

Mode

$$\eta = A\cos(kx - \omega t)$$

Full potential theory in present scaling ($h = 1$)

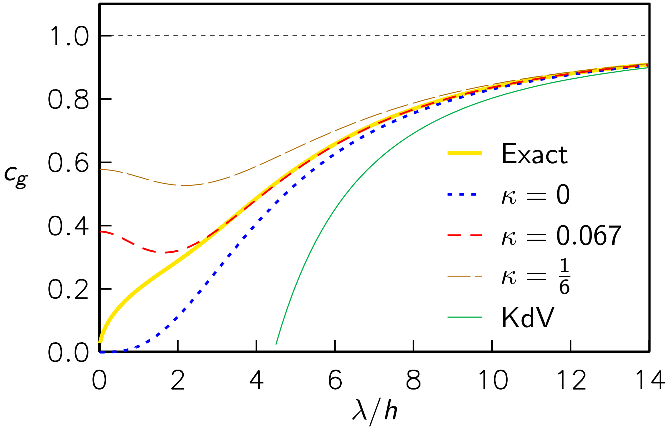
$$c^2 = \frac{1}{k\epsilon^{\frac{1}{2}}}\tanh(\epsilon^{\frac{1}{2}}k) = 1 - \frac{1}{3}\epsilon k^2 + \frac{2}{15}\epsilon^2 k^4 + \ldots$$

Presented Boussinesq models fulfill

$$c^2 = \frac{1 + \kappa\epsilon k^2}{1 + (\frac{1}{3} + \kappa)\epsilon k^2} = 1 - \frac{1}{3}\epsilon k^2 + \left(\frac{1}{9} + \frac{1}{3}\kappa + \frac{2}{3}\kappa^2\right)\epsilon^2 k^4 + \ldots,$$

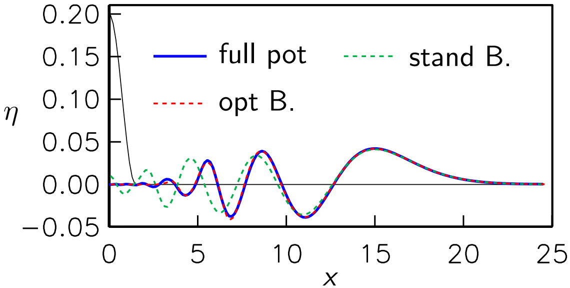
Standard Boussinesq with averaged velocity	$\kappa = 0$
Optimized Boussinesq	$\kappa = 0.067$
Optimized $z_\alpha = -0.531h$	$\kappa = 0.067$
$z_\alpha = -h$ ($\vec{v}$ at bottom)	$\kappa = \frac{1}{6}$

Dispersion properties



$\kappa = \frac{1}{6} \rightarrow u$ at bottom, $\kappa = 0 \rightarrow$ averaged u,
$\kappa = 0.067 \rightarrow$ optimal choice
KdV defined later.
$c_g = \mathrm{d}\omega/\mathrm{d}k$ – group velocity

Effect of dispersion



Linear evolution from short initial elevation

Half of a symmetric solution shown.

Black line: half the initial condition, width $\sim$ 3 depths

Front: Good agreement for all Boussinesq formulations

Rear: Improved model superior, standard B. too dispersive

Observe: No corresponding improvement for steep bottom gradients is published.

Derivation of the KdV equation.

Assumptions

- Long waves – as for the Boussinesq equations
- Weakly non-linear waves. $\alpha \sim \epsilon$. We set $\alpha = \epsilon$.
- Unidirectional waves; propagation in the positive x-direction

Strategies

1. Ad-hoc combination of non-linear term from characteristic formulation (later section) and term inspired from the linear dispersion relation.
2. Transform to frame moving with linear shallow water speed $\Rightarrow$ Unidirectional waves will display slow temporal variations only.
3. Correct the equations for the Riemann-invariantens in the framework of the Boussinesq equations.

We employ alternative 2

Boussinesq equations for $h = 1$.

$$\eta_t = -\{(1 + \alpha\eta)\overline{u}\}_x \tag{20}$$

$$\overline{u}_t + \alpha\overline{u}\,\overline{u}_x = -\eta_x + \epsilon\frac{1}{3}\overline{u}_{xxt} \tag{21}$$

$\alpha, \epsilon \to 0 \Rightarrow \eta_{tt} - \eta_{xx} = 0 \Rightarrow$

$$\eta = F(x - t) + G(x + t),$$
$$u = F(x - t) - G(x + t)$$

Waves in positive x-direction $\Rightarrow G \equiv 0$ and

$$\eta_t + \eta_x = 0, \quad u_t + u_x = 0 \tag{22}$$

Standard transport equation.

Is there a generalization of (22) that include corrections of order (α, ϵ) ?

Starting point

Small $\epsilon \Rightarrow$ non-linearity and dispersion are both weak.
Wave shape vary only slowly, celerity near 1.
The solution changes only slowly in a frame moving with unitary speed.

Coordinate transformation

$$\xi = x - t \qquad \tau = \epsilon t \tag{23}$$

τ is a slow time variable.

Transformation of equation of continuity, (20):

$$\epsilon\eta_\tau - \eta_\xi = -(1 + \epsilon\eta)\overline{u}_\xi - \epsilon\overline{u}\eta_\xi \tag{24}$$

Elimination of $\overline{u}$ in h.o. terms

Leading order: $\eta_\xi = \overline{u}_\xi + O(\epsilon)$. Hence:

$$\eta = \overline{u} + O(\epsilon) \tag{25}$$

provided, for instance, $\eta = \overline{u} = 0$ at $x = \infty$.

Small $\overline{u}$ terms in (24) replaced using (25):

$$\epsilon\eta_\tau - \eta_\xi = -\overline{u}_\xi - 2\epsilon\eta\eta_\xi + O(\epsilon^2) \tag{26}$$

Momentum eq. (21) with $\overline{u} = \eta$ in small terms

$$\overline{u}_\xi = \eta_\xi + \epsilon\eta_\tau + \epsilon\eta\eta_\xi + \frac{1}{3}\epsilon\eta_{\xi\xi\xi} + O(\epsilon^2) \tag{27}$$

Elimination of $\overline{u}_\xi$ between (26) and (27) $\Rightarrow$ The KdV equation:

$$\epsilon\eta_\tau + \frac{3}{2}\epsilon\eta\eta_\xi + \frac{1}{6}\epsilon\eta_{\xi\xi\xi} = O(\epsilon^2) \tag{28}$$

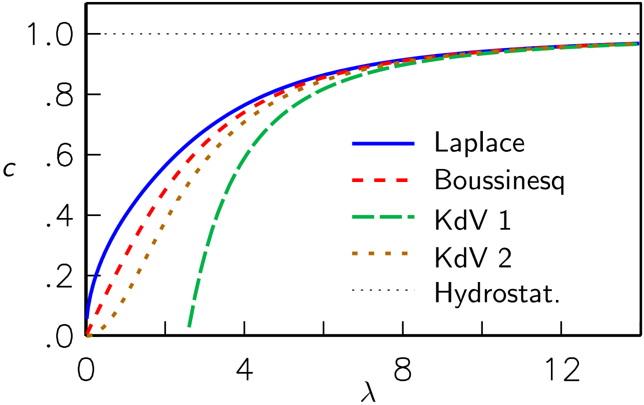
Transformation back to x and t

$$\eta_t + (1 + \frac{3}{2}\epsilon\eta)\eta_x + \frac{1}{6}\epsilon\eta_{xxx} = O(\epsilon^2) \tag{29}$$

Or; $\eta_t = -\eta_x + O(\epsilon)\Rightarrow$

$$\eta_t + (1 + \frac{3}{2}\epsilon\eta)\eta_x - \frac{1}{6}\epsilon\eta_{xxt} = O(\epsilon^2) \tag{30}$$

Dispersion relations for KdV



Phase velocity as function som funksjon of wave length

KdV 1: $\eta_t + (1 + \frac{3}{2}\epsilon\eta)\eta_x + \frac{\epsilon}{6}\eta_{xxx} = 0$

KdV 2: $\eta_t + (1 + \frac{3}{2}\epsilon\eta)\eta_x - \frac{\epsilon}{6}\eta_{xxt} = 0$

What is gained by long wave theory

1. Physical contents more transparent
2. Important closed form solutions of NLSW and the KdV equations
3. NLSW equations are hyperbolic with characteristics and shocks
4. Unknown upper bound of fluid replaced by coefficients in η
5. The number of dimensions reduced by 1 (depth integration)

Last two points crucial for numerical solution