21 Internal Waves

In a stratified fluid we have the density

$$\rho = \begin{cases} \rho_0 - \Delta\rho & \text{when} \quad z < -B \\ \rho_0 + \dfrac{\Delta\rho}{B} z & \text{when} \quad -B \le z \le B \\ \rho_0 + \Delta\rho & \text{when} \quad B < z \end{cases} \tag{19}$$

where the z-axis is pointing vertically downwards.

a

We wish to approximate the buoyancy frequency (Brunt-Väisälä), N, by a constant for $-B < z < B$. When is this justified ?

b

Internal modes due to the stratification is governed by

$$\hat{w}'' + k^2 \left(\frac{N^2}{\omega^2} - 1 \right) \hat{w} = 0, \tag{20}$$

where $\hat{w}$ defines the vertical variation of the vertical velocity component, k is the wave number in the horizontal direction and ω is the frequency.
Show that, for the stratification in (19), modes are given by (20) combined with the boundary conditions

$$\left. \begin{array}{ll} \hat{w}' - k\hat{w} = 0 \quad \text{at} & z = -B \\ \hat{w}' + k\hat{w} = 0 \quad \text{at} & z = B \end{array} \right\} \tag{21}$$

c

Solve the eigenvalue problem from the preceding point to obtain implicit expressions for $\beta = \sqrt{\frac{N^2}{\omega^2} - 1}$. Show that there are two groups of modes, with $\hat{w}$ that are symmetric and anti-symmetric with respect to $z = 0$, respectively.

d

The dispersion relations may be represented as the intersections between a functions that are linear and periodic in β. Show this and explain why the symmetric modes have 1, 3, 5 etc. extrema in the interval $-B < z < B$, whereas the antisymmetric modes have 2, 4, 6 ...
Depict $\hat{w}$ for the lowest two modes of each kind.

e

Instead of an unbounded fluid we now assume a semi-unbounded one with a rigid lid at $z = -H$. Outline briefly how to find the modes in this case.