MEK4320, Problems for 6/5 2020

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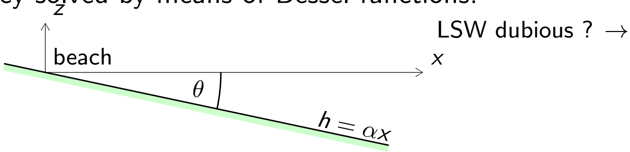
Compendium, section 2.10: LSW equation for $h = \alpha x$ and constant frequency solved by means of Bessel functions.



a: Use of ray theory.

Reproduce the result ($\kappa = \omega^2/\alpha g$)

$$\eta = a_1(\kappa x)^{-\frac{1}{4}}\cos(2\sqrt{\kappa x} - \omega t + \delta_1) + a_2(\kappa x)^{-\frac{1}{4}}\cos(2\sqrt{\kappa x} + \omega t + \delta_2),$$

from section 2.10 by means of ray theory and the transport equation.

Solution to a

Must reproduce $\eta = A(x)\cos(\chi)$ with $A = a_1(\kappa x)^{-\frac{1}{4}}$ and $\chi = 2\sqrt{\kappa x} - \omega t + \delta_1$ (other sign is similar)

Dispersion relation $\omega = W(k,x) = \sqrt{g\alpha x}k$
Characteristic equation for ω

$$\frac{\partial \omega}{\partial t} + c_g \frac{\partial \omega}{\partial x} = \frac{\partial W}{\partial t} = 0,$$

ω may be constant and $k = \omega/\sqrt{g\alpha x}$.
From $\chi_x = k$

$$\chi = \int k dx + F(t) = \frac{\omega}{\sqrt{g\alpha}} \int x^{-\frac{1}{2}} dx + F(t) = \frac{2\omega\sqrt{x}}{\sqrt{g\alpha}} + F(t)$$

Then $\chi_t = \pm\omega \Rightarrow F' = -\omega \Rightarrow \chi = 2\sqrt{\kappa x} \pm \omega t + \text{const.}$.

Transport equation

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{F}}{\partial x} = 0.$$

No local time variation of averaged quantities $\Rightarrow$
$\overline{F} = c_g \overline{E} = \mathrm{const} \Rightarrow \sqrt{gh}A^2 = \mathrm{const} \Rightarrow A = \mathrm{const}\, x^{-\frac{1}{4}}$.
Both A and χ are reproduced with ray theory.

b: Requirement for asymptotics.

For the use of ray theory we must require $|\lambda k_x/k| \ll 1$. Explain this. On the other hand, the use of asymptotic approximations for the Bessel functions in section 2.10 requires $\sqrt{\kappa x} \gg 1$. Show that these two requirements are the same.

Solution

$k = \omega/\sqrt{g\alpha x}$, $k_x = -\frac{1}{2}\omega/(x\sqrt{g\alpha x})$, $\lambda = 2\pi/k \Rightarrow$

$$1 \gg \left|\frac{\lambda k_x}{k}\right| = \pi\sqrt{\frac{g\alpha}{\omega^2 x}} = \frac{\pi}{\sqrt{\kappa x}}.$$

Problem 4, chapter 3.5

Same set-up as preceding problem except for $h(x) = h_0(\frac{x}{l})^2$.
Same scheme for application of ray theory.

a: Use ray theory to determine the phase function χ.

Solution

Again $k = \pm\omega/\sqrt{gh}$ and integration of $\chi_x = k$ and $\chi_t = \omega \Rightarrow$

$$\chi = \pm\beta \log(x/x_0) + \omega t + \mathrm{const.}$$

where

$$\beta = \frac{l\omega}{\sqrt{gh_0}},$$

and x_0 is some reference value.

Constant energy flux now implies

$$\sqrt{gh}A^2 = \text{const.} \quad \Rightarrow \quad A = \text{const}\, x^{-\frac{1}{2}}$$

Combining the two solutions

$$\eta = A_+ x^{-\frac{1}{2}} e^{i(\beta \log(x/x_0) + \omega t)} + A_- x^{-\frac{1}{2}} e^{i(-\beta \log(x/x_0) + \omega t)}.$$

b

Give an interpretation of the expression $\frac{1}{c}\frac{\partial c}{\partial x}\lambda$. Which requirement must we impose on $\beta = \frac{\omega l}{\sqrt{gh_0}}$ for ray theory to be valid?

Solution

Expression has interpretation as change of medium over an wavelength.

Calculation with $c = \sqrt{gh}$ and $\lambda = 2\pi/k$

$$\frac{1}{c}\frac{\partial c}{\partial x}\lambda = \frac{2\pi}{\beta}.$$

Ray theory requires slowly varying medium $\Rightarrow \beta \gg 1$.

c

Find a solution of the shallow water equation of the form $\eta = e^{i\omega t}x^q$. Compare this with the results we found with the help of ray theory.

Solution

Prefer to do it step by step.

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gh(x)\frac{\partial\eta}{\partial x}\right) = 0,$$

Inserted $\eta = e^{i\omega t}\hat{\eta}(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \frac{\omega^2}{g}\hat{\eta} = 0.$$

Next, insert expression for $h(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right)+\beta^2\hat{\eta}=0.$$

β does appear again! Equation may be rewritten

$$x^2\hat{\eta}''+2x\hat{\eta}'+\beta^2\hat{\eta}=0.$$

If we insert a power for $\hat{\eta}$ all terms will have the same power $\Rightarrow$ solution on form $\hat{\eta}=x^q$ possible for some q.
Insertion of $\hat{\eta}=x^q$

$$\left(q^2+q+\beta^2\right)x^q=0.$$

Solution when

$$q_\pm=\frac{1}{2}\left(-1\pm\sqrt{1-4\beta^2}\right).$$

For $\beta > \frac{1}{2}$

$$q_{\pm} = -\frac{1}{2} \pm iq_i, \quad q_i = \beta\sqrt{1 - \frac{1}{4}\beta^{-2}} = \beta - \frac{1}{8}\beta^{-1} + O(\beta^{-3}).$$

Full solution $\eta = A_+ x^{q_+} e^{i\omega t} + A_- x^{q_-} e^{i\omega t}$.
Recasting first part to same form as ray solution

$$\eta = A_+ x^{q_+} e^{i\omega t} + \ldots = \overline{A}_+ x^{-\frac{1}{2}} e^{i(q_i \log(x/x_0) + \omega t)} + \ldots$$

where $\overline{A}_+ = A_+ x_0^{iq_i}$. $q_i = \beta + O(1/\beta)$: close to ray solution when β large.

Problem 1; section 3.1

In one end of a wave tank, a wave is generated with length $\lambda = 1\,\mathrm{m}$. The wave generator has been on for $t = 100\,\mathrm{s}$. Estimate the number of wave crests in the channel in the three cases where the water depth is $h = 4\,\mathrm{m}$, $h = 0.5\,\mathrm{m}$ and $h = 0.1\,\mathrm{m}$. Assume that the wave channel is long enough that reflection from the opposite end does not happen.

Extra question: An observer stands still while the generated wave train passes. How many crest will she count ?

Solution

First: $h = 4\,\mathrm{m}$ deep water , $h = 0.5\,\mathrm{m}$ and $h = 0.1\,\mathrm{m}$ (nearly) shallow water (problem 9 from assignment).
Dispersion relation: λ and h give ω, period T etc.
Number of crests produced at paddle

$$N_p = \frac{t}{T} = \frac{ct}{\lambda}.$$

Length of wave train after t: energy moves with $c_g \Rightarrow$ front has reached a distance $L = c_g t$ from paddle.
Number of crests in wave train

$$N_t = \frac{L}{\lambda} = \frac{c_g t}{\lambda}.$$

Shallow water: $N_t = N_p$, deep water $N_t = \frac{1}{2} N_p$.

Solution

Extra question

Observer at distance L from paddle. End of wave train moves with c_g.

It passes in

$$t_p = \frac{L}{c_g} = t.$$

Crests observed

$$\frac{t}{T} = N_p,$$

for both deep and shallow water.