A note on a BIM model made for runup

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0.0.1 Introduction

This note is a brief description of a full potential flow model with particular adaptations to moving shorelines. The model is closely related to the higher order technique of [6, 1], while the formulation differs from the standard references [2, 3, 4]. However, while these references employ high order polynomials for interpolation along the contour we use cubic splines. This makes inclusion of boundary conditions simpler and does allow for the inclusion of a moving shoreline in particular.

At the shoreline point we assume analyticity which in principle excludes cases with contact angles larger than 90°. Also some other features, such as the numerical integration procedure along the contours, differ from the references. Boundary integral methods are not efficient for the computation of very thin swash tongues that evolve during runup/withdrawal of higher waves. This is ironical since the flow may be so simple that even the shallow water equations are unnecessarily complicated (see [5]). The proximity of the surface and the bottom parts of the contour requires a smaller time step relative to the grid spacing and a higher order numerical integration than in deeper water. A runup value should then only be accepted if a systematic refinement sequence of at least three grids, with the the same integration rule, produce consistent results and point to at least three correct digits.

1 A boundary integral method

In the fluid the motion is governed by the Laplace equation

$$\nabla^2\phi = 0 \quad \text{for} \quad -h < z < \eta.$$

At the free surface ($z = \eta$) the Bernoulli equation is expressed as

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} - \frac{1}{2}(\nabla\phi)^2 + \eta = 0 \tag{1}$$

The kinematic condition at the surface is written in the Lagrangian form

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial\phi}{\partial z} \quad \frac{\mathrm{D}\xi}{\mathrm{D}t} = \frac{\partial\phi}{\partial x},$$

where (η, ξ) is the position of a surface particle. At rigid boundaries (bottom or sidewalls) we have

$$\frac{\partial \phi}{\partial n} = 0,$$

where n denotes the direction normal to the boundary.

This model is related to the high order technique of [1]. However, to allow more flexible boundary conditions, as sloping beaches, the high order polynomials are replaced by cubic splines for the spatial interpolation between nodes. Accordingly the order of the temporal scheme is reduced to third order accuracy. The key features then become

- Lagrangian particles are used along the free surface. At other boundaries both fixed and moving nodes may be employed

- Cauchy's formula for complex velocity ($q = u - \mathrm{i}v$) is used to produce an implicit relation between the velocity components along the surface

$$\alpha \mathrm{i} q(z_p) = \mathrm{PV} \oint_C \frac{q(z)}{z_p - z} dz \tag{2}$$

  where α is the interior angle. Following [1] the integral equation (2) is rephrased in terms of the velocity components tangential and normal to the contour, denoted by $u^{(s)}$ and $v^{(s)}$, respectively. Invoking the relation

$$u^{(s)} - \mathrm{i}v^{(s)} = \mathrm{e}^{\mathrm{i}\theta}(u - iv),$$

  where θ is the angle between the tangent and the x-axis, we then obtain

$$\alpha \mathrm{i}(u_p^{(s)} - v_p^{(s)}) = \mathrm{e}^{\mathrm{i}\theta_p} \mathrm{PV} \oint_C \frac{u^{(s)} - iv^{(s)}}{z_p - z} ds. \tag{3}$$

  For rigid boundaries, where the normal velocity is known, the real component of this equation is imposed, while the imaginary component is used at free surfaces, where u_s is known from the integration of the Bernoulli equation. The equation set is established by collocation in the sense that z_p runs through all nodes to produce as many equations as unknowns (see figure 1).

- Cubic splines for field variables – solution is twice continuously differentiable

- Combination of Taylor series expansion and multi-step technique used for time integration. This allows for variable time stepping. For instance, ϕ_p is advanced one time step n to $n+1$ according to

$$\phi_{(n+1)} = \phi_{(n)} + \Delta t_{(n)} \left(\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)_{(n)} + \frac{1}{2}\Delta t_{(n)}^2 \left(\frac{\mathrm{D}^2\phi}{\mathrm{D}t^2}\right)_{(n)} + \frac{1}{6}\frac{\Delta t_{(n)}^3}{\Delta t_{(n-1)}} \left(\frac{\mathrm{D}^2\phi}{\mathrm{D}t^2}_{(n)} - \frac{\mathrm{D}^2\phi}{\mathrm{D}t^2}_{(n-1)}\right).$$

  The last, backward difference both increases the accuracy and stabilize the scheme. The first temporal derivative of ϕ is obtained from the Bernoulli equation. We then also obtain the local derivative, $\frac{\partial\phi}{\partial t}$, that is used to define a boundary value problem for the temporal derivatives of the velocity. This is identical to the problem for the velocities themselves, given by (2) or (3) with u and v replaced by their local time derivatives. Formulas like the one above are applied to the other principal unknowns ξ and η.

- Special treatment of corner points; invocation of analyticity.
- Like most models of this kind some filtering is required in the nonlinear case to avoid growth of noise. A five point smoothing formula is applied to this end.

In sum we have a "moderately high order" method that is lower order compared to the method [1], but at the same time less restricted at the boundaries.

The computational cycle consists of the following main steps

1. We know velocities (and more) at t. Time stepping by discrete surface condition give ϕ (potential) and particle positions at the surface for $t+\Delta t$
2. ϕ at surface yields the tangential velocity at the surface
3. Crucial step: Tangential velocity at surface and bottom condition (normal velocities) yield equations for the other velocity component through the integral equation (3) that is equivalent to the Laplace equation
4. $\partial\phi/\partial t$ is obtained from the Bernoulli equation (1). The tangential component of the temporal derivative of the velocity is then obtained, in analogy to step 2, and the linear equation set from step 3 is solved with a new right hand side to obtain the remaining component of $\partial\mathbf{v}/\partial t$.
5. The Bernoulli equation is differentiated, materially, with respect to t and $\frac{\mathrm{D}^2\phi}{\mathrm{D}t^2}$ is computed.
6. Now the first and second order Lagrangian derivatives of ϕ, ξ and η are computed at $t+\Delta t$ and the cycle may repeat itself.

The whole problem is then posed in terms of the position of the fluid boundary and the velocity potential there. Values of velocities within the fluid may be obtained by choosing z_p as an interior point and put $\alpha = 2\pi$ in the Cauchy relation (2), which then provide explicit expressions for u_p and v_p. In the linear case the procedure is substantially simplified since the geometry is constant and matrices involved in the third step may be computed and factorized only once.

References

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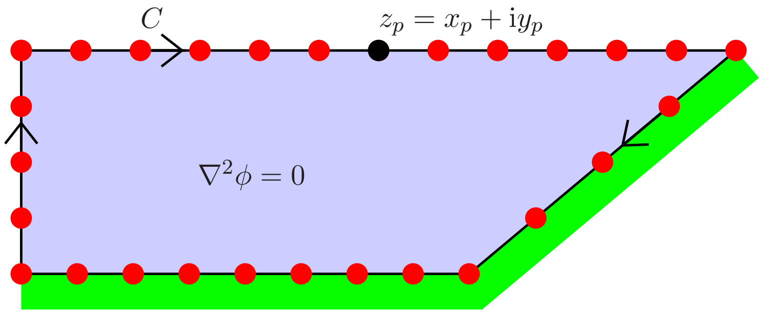

Figure 1: Definition sketch of computational domain in BIM method

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