

# The WKBJ method and optics

## MEK4320

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# The WKBJ technique

# WKBJ; example: LSW

Linear shallow water theory (indices mark differentiation)

$$\eta_{tt} - \nabla \cdot (c_0^2 \nabla \eta) = 0 \quad (1)$$

where  $c_0^2 = gh(x, y)$ .

In ray theory

$$\eta(x, y, t) = A(x, y, t) e^{i\chi(x, y, t)}, \quad (2)$$

where  $\vec{k} \equiv \nabla \chi$ ,  $\omega \equiv -\frac{\partial \chi}{\partial t}$ .

Slow variations of  $\vec{k}$  and  $\omega \Rightarrow$  ray equations.

We now insert (2) in (1):

$$\begin{aligned} A_{tt} - i(2\omega A_t + \omega_t A) - \omega^2 A = \\ \nabla \cdot (c_0^2 \nabla A) + i \left( c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}) \right) - c_0^2 k^2 A \end{aligned} \quad (3)$$

Exact, but nothing is achieved either – so far.

## Scales

Typical wavelength:  $\lambda_c$  (fast scale)

Typical length scale for medium change  $L_c$  (slow scale)

Small parameter  $\frac{\lambda_c}{L_c} = \epsilon \ll 1$ .

Typical wave speed:  $c_c = \sqrt{gh_c}$

Typical amplitude:  $A_c$  – no significance as long as in linear regime

Typical phase:  $\chi \sim L_c/\lambda_c = \epsilon^{-1}$

## Rescaling

All derivatives explicit in (3) are with respect to slow variation.

Fast variation inherent in definitions of  $\vec{k}$  and  $\omega$ , only.

$\vec{\kappa} = \lambda_c \vec{k}$ ,  $\hat{\omega} = \lambda_c \omega / c_c$ ,  $\hat{A} = A / A_c$ ,  $\hat{c} = c_0 / c_c$ ,  $\hat{x} = x / L_c$ ,  
 $\hat{t} = c_c t / L_c$ .

## Rescaling of (3)

$$\begin{aligned}\epsilon^2 \hat{A}_{\hat{t}\hat{t}} - i\epsilon(2\hat{\omega}\hat{A}_{\hat{t}} + \hat{\omega}_{\hat{t}}\hat{A}) - \hat{\omega}^2 \hat{A} = \\ \epsilon^2 \hat{\nabla} \cdot (c^2 \hat{\nabla} \hat{A}) + i\epsilon \left( c^2 \hat{\nabla} \hat{A} \cdot \vec{\kappa} + \hat{\nabla} \cdot (c^2 \hat{A} \vec{\kappa}) \right) - c^2 \kappa^2 \hat{A}\end{aligned}\quad (4)$$

Leading terms:  $O(1)$  no slow differentiations

Next order terms:  $O(\epsilon)$  one slow differentiation

Second order terms:  $O(\epsilon^2)$  two slow differentiations

### Leading order

$$\hat{\omega}^2 = c^2 \kappa^2 \Rightarrow \text{restored scales } \omega^2 = c_0^2 k^2 = W^2.$$

With  $\vec{k}_t + \nabla \omega = 0$ ,  $\partial k_i / \partial x_j = \partial k_j / \partial x_i$ : **ray theory retrieved**

$$\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1, 2 \quad (5)$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega = 0 \quad (6)$$

with  $\vec{c}_g = c_0 \vec{k} / k$ . Now  $\vec{k}$  and  $\omega$  are settled.

## Next order; $O(\epsilon)$ relative size

Original scales restored

$$-2\omega A_t - \omega_t A = c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}).$$

With  $\omega = c_0 k$  and  $\vec{c}_g = c_0 \vec{k}/k$  (then  $c_0^2 \vec{k} = \omega \vec{c}_g$ ):

$$-2\omega A_t - \omega_t A = 2\omega \vec{c}_g \cdot \nabla A + A \vec{c}_g \cdot \nabla \omega + \omega A \nabla \cdot \vec{c}_g.$$

Next step multiply with  $A/\omega$  and regroup

$$-(A^2)_t - \frac{A^2}{\omega} (\omega_t + \vec{c}_g \cdot \nabla \omega) = \nabla \cdot (\vec{c}_g A^2).$$

Due to (6) terms within last parentheses on l.h.s cancel out:

$$(A^2)_t = -\nabla \cdot (\vec{c}_g A^2). \quad (7)$$

With  $E = \frac{1}{2}\rho g A^2$  (energy density) and  $\vec{F} = \vec{c}_g E$  (energy flux) equation (7) reads

$$E_t + \nabla \cdot \vec{F} = 0, \quad (8)$$

Averaged energy conservation, as in uniform medium.

# Remarks

## Remark 1

Similar WKBJ approaches apply to most linear wave equations  $\Rightarrow$  result with interpretation as energy conservation is general.

## Remark 2

By means of (6) we have

$$(G(\omega)E)_t + \nabla \cdot (G(\omega)\vec{F}) = 0,$$

for any  $G(\omega)$ .

## Remark 3

If we have coupling with a background current it is the wave action ( $E$  over some frequency) which is conserved.

## Remark 4

To include higher order in  $\epsilon$  we must expand  $A = A_0 + \epsilon A_1 + ..$

# Brief review of ray theory and optics.

# Optics; summary

## Ray theory (geometrical optics)

Harmonic wave, uniform medium  $\Rightarrow$  dispersion relation

$$\omega = W(\vec{k}; H\dots)$$

Slow variation of medium and wave train  $\Rightarrow$  local  $k$  and  $\omega$  fulfill the dispersion relation (approximately) as in uniform medium.

## Physical optics

Harmonic wave, uniform medium  $\Rightarrow$  averaged energy relations

$$\vec{F} = \vec{c}_g E, \quad E = E(A^2, \dots)$$

Slow variation  $\Rightarrow \vec{F} = \vec{c}_g E$  (approximately) as in uniform medium.

# Equation of geometrical optics

## Ray equations

From  $\omega = -\frac{\partial \chi}{\partial t}$ ,  $\vec{k} = \nabla \chi$  and  $\omega = W(\vec{k}, x_i, t)$

$$\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1, 2..$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega = \frac{\partial W}{\partial t}$$

## Recasted to Hamilton's canonical equations

$$\frac{dk_i}{dt} = -\frac{\partial W}{\partial x_i}, \quad i = 1, 2..$$

$$\frac{dx_i}{dt} = \frac{\partial W}{\partial k_i} = (c_g)_i, \quad i = 1, 2..$$

$$\frac{d\omega}{dt} = \frac{\partial W}{\partial t}$$

# Equations of physical optics

## The transport equation

Energy conservation, in general

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Invoking the approximation  $F = c_g E$ :

$$\frac{\partial E}{\partial t} + \frac{\partial(c_g E)}{\partial x} = 0$$

where  $E = E(A^2, \dots)$

More dimensions

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{c}_g E) = 0 \quad (9)$$

...

## calculation of wave field

- ①  $\vec{k}$  and  $\omega$  are obtained from ray theory
- ② The transport equation (9) is solved for  $A$

# Optics in uniform media; two examples

# Uniform medium

$$\omega = W(\vec{k}) \Rightarrow \vec{c}_g = c_g(\vec{k})$$

Ray equations

$$\frac{dk_i}{dt} = 0, \quad i = 1, 2..$$

$$\frac{dx_i}{dt} = (c_g)_i, \quad i = 1, 2..$$

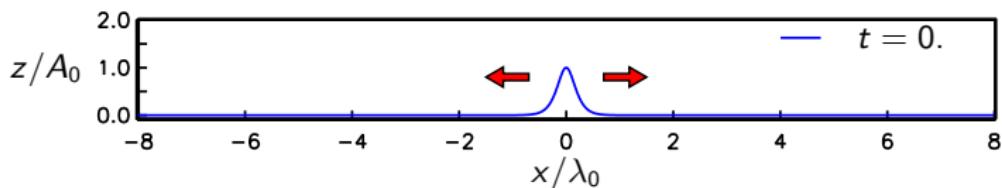
$$\frac{d\omega}{dt} = 0$$

$\vec{k}$  and  $\omega$  conserved along characteristics  $\mathcal{C}$ :  $\frac{d\vec{r}}{dt} = \vec{c}_g$

$\Rightarrow \vec{c}_g$  conserved  $\Rightarrow \mathcal{C}$  are straight lines.

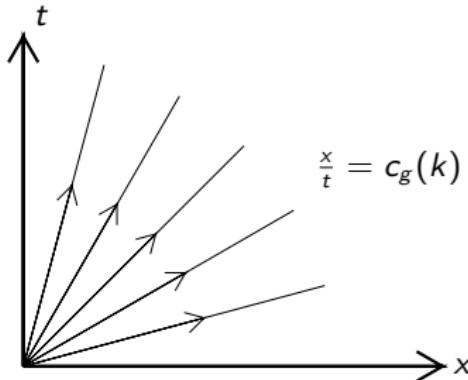
Characteristics are also straight lines viewed in the generalized  $x_i, t$  space.

# Evolution of wave train from confined disturbance



- Previously: Fourier transform  $\Rightarrow$  formal solution as integral.
- When propagation distances are much larger than initial length ( $\lambda_0$ ): application of stationary phase, main spectral contribution from vicinity of  $k_s$  such that  $c_g(k_s) = x/t$ .
- Result: a slowly varying wave train.
- The slowly varying wave train should be within the realm of optics.
- Hence, problem revisited with optics. Approximation: initial disturbance is at  $x = 0$ .

# In 1D: waves from point disturbance, at $x = 0$ at $t = 0$



All  $\mathcal{C}$  intersects at  $x = t = 0 \Rightarrow c_g = x/t$   
 $c_g(k) = x/t \Rightarrow k = k(x/t) \Rightarrow \chi = \int k dx$

Gravity waves in infinite depth; ray theory

$$W = \sqrt{gk} \Rightarrow x/t = c_g = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{1}{4}g\frac{t^2}{x^2} \Rightarrow \\ \chi = -\frac{1}{4}g\frac{t^2}{x} + f(t)$$

$$\text{Furthermore } \frac{\partial \chi}{\partial t} = -\omega = -\sqrt{gk} \Rightarrow \chi = -\frac{1}{4}g\frac{t^2}{x} + \text{const}$$

Phase function as from the stationary phase solution.

## Still infinite depth; the transport equation

$$\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}(c_g A^2) = 0$$

inserted  $c_g = x/t$  and rewritten as

$$\frac{\partial xA^2}{\partial t} + \frac{x}{t} \frac{\partial(xA^2)}{\partial x} = 0$$

General solution:  $xA^2$  is constant along  $\mathcal{C}$   $\Rightarrow$

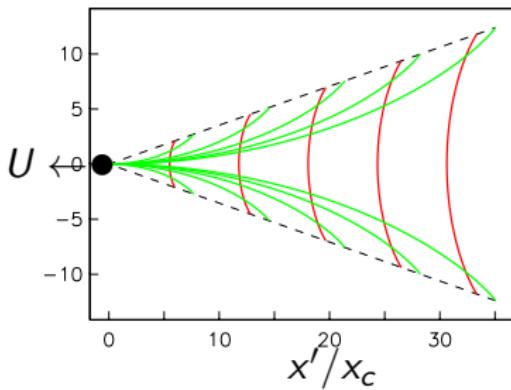
$$A = x^{-\frac{1}{2}} G\left(\frac{x}{t}\right) = t^{-\frac{1}{2}} \hat{G}\left(\frac{x}{t}\right)$$

Consistent with stationary phase (spectrum  $\Rightarrow G$ )

Interpretation: Energy between two characteristics ( $\mathcal{C}$ ) is conserved.

# The Kelvin ship wave pattern.

Point source at the surface



Constant source velocity:  $\vec{U} = -U\vec{i} \Rightarrow$  stationary and slowly varying wave pattern. Frame with fluid at rest  $\Rightarrow$  isotropic dispersion relation

$$\vec{c}' = c_0(k) \frac{\vec{k}}{k} \quad (10)$$

## Change of coordinate system

From fixed coordinate system ( $\vec{r}'$ ) to a moving one ( $\vec{r}$ )

Frame follows the source  $\vec{r} = \vec{r}' - \vec{U}t$ .

New system: stationary source on a uniform current.

A harmonic mode then becomes:

$$A \cos \chi = A \cos(\vec{k} \cdot \vec{r}' - \omega' t) = A \cos(\vec{k} \cdot \vec{r} - (\omega' + \vec{k} \cdot \vec{U})t)$$

$$\omega = c_0(k)k + \vec{U} \cdot \vec{k} \equiv W(k_x, k_y) \quad (11)$$

$$\vec{c} = \left( c_0(k) + \vec{U} \cdot \frac{\vec{k}}{k} \right) \frac{\vec{k}}{k} \quad (12)$$

where  $\vec{k} = k_x \vec{i} + k_y \vec{j}$

Doppler shift  $\Rightarrow$  Anisotropic dispersion.

# Optics

New frame: stationary pattern implies  $\omega = 0$ . (11) yields:

$$W(k_x, k_y) = 0 \quad (13)$$

The group velocity

$$\vec{c}_g = \frac{\partial W}{\partial k_x} \vec{i} + \frac{\partial W}{\partial k_y} \vec{j} \quad (14)$$

Hamilton's equations

$$\frac{d\vec{r}}{dt} = \vec{c}_g, \quad \frac{d\vec{k}}{dt} = 0 \quad (15)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \quad (16)$$

Uniform medium  $\Rightarrow$  Characteristics are straight lines

To carry energy characteristics must pass through the source (the origin)

Straight characteristics through the origin:

$$x = c_{gx} t, \quad y = c_{gy} t.$$

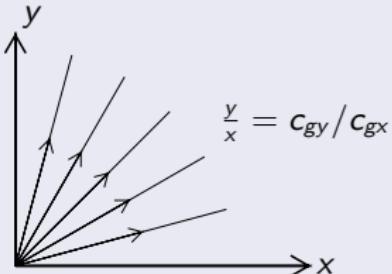
(Even for a stationary pattern characteristics can be parameterized by time; following the transport of energy)

Elimination of  $t$ :

$$\frac{y}{x} = \frac{\frac{\partial W}{\partial k_y}}{\frac{\partial W}{\partial k_x}} \quad (17)$$

combined with  $W(k_x, k_y) = 0$  (13)  $\Rightarrow$  two equations for  $k_x$  and  $k_y$ .

## Characteristics



## The Phase function

$$\chi(\vec{r}) = \chi_0 + \int_{C(\vec{r})} \vec{k} \cdot d\vec{r} \quad (18)$$

where  $\chi_0$  is the phase in the origin and  $C(\vec{r})$  is some integration path.

Choosing  $C$  as a characteristic: Integration trivial because  $\vec{k}$  is constant.

$$\chi(\vec{r}) = \chi_0 + k_x x + k_y y \quad (19)$$

Phase lines  $\chi = -A$ . Two options for visualization/interpretation

- A quick overview of the phase line readily obtained by contour plot of  $\chi$  (Matlab or Python etc.).
- Parameterization of phase lines. Some uncanny trigonometry, but the presence of two families of solutions for  $\vec{k}$  from (17) and (13) is demonstrated.

## Parameterization of phase lines (infinite depth)

$\theta$ : angle between  $\vec{k}$  and negative  $x$ -axis

$$k_x = -k \cos \theta, \quad k_y = k \sin \theta. \quad (20)$$

(13) is rewritten

$$c_0 = U \cos \theta,$$

and  $c_0(k) = \sqrt{g/k}$  then yields

$$k = \frac{g}{U^2 \cos^2 \theta} \quad (21)$$

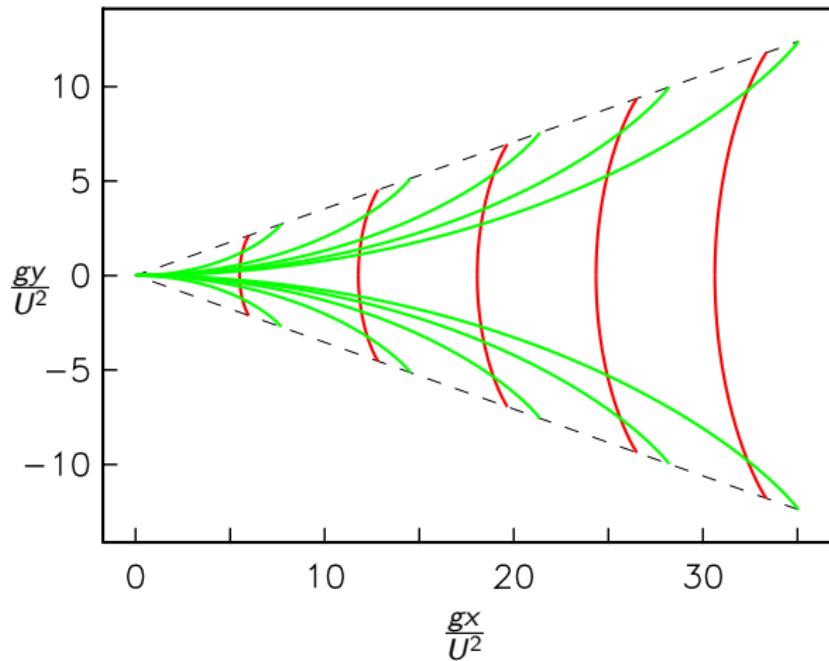
(17) and (19) are solved for  $x$  and  $y$

$$x = \frac{(A - \chi_0)g}{U^2} \cos \theta (1 + \sin^2 \theta) \quad (22)$$

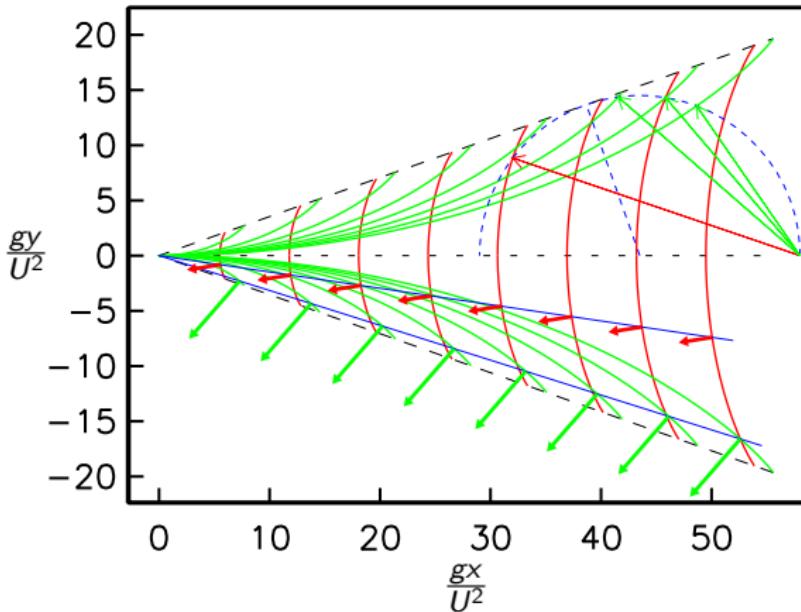
$$y = \frac{(A - \chi_0)g}{U^2} \cos^2 \theta \sin \theta \quad (23)$$

Note:  $y(\theta)/x(\theta)$  extreme for  $\cos \theta = \sqrt{2/3}$  ( $\theta = \theta_c = 35.3^\circ$ )  $\Rightarrow$  jump in direction of phase lines  $\Rightarrow$  independent solutions

Point source: other techniques show  $\chi_0 = \frac{1}{4}\pi, -\frac{1}{4}\pi$  for transverse and diverging waves, respectively.



# The Kelvin pattern, more details



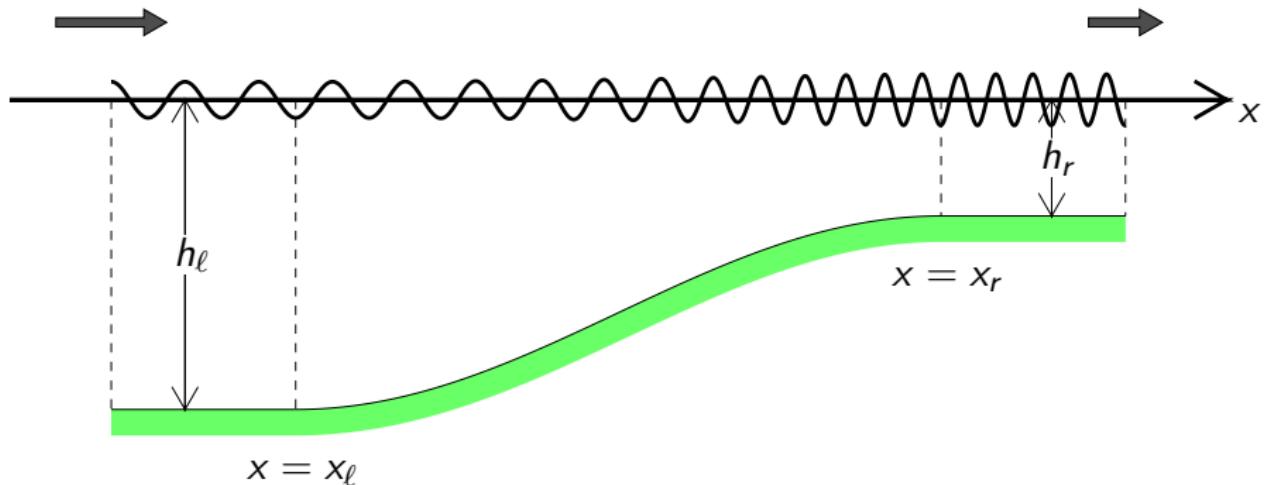
Fat arrows: wave number vectors.

Dashed half circle: propagation with  $\vec{c}_g$  from intersection with  $x$ -axis, subject to  $c_0 = U \cos \theta$ . Thin arrows: corresponding rays.

# Amplification and refraction in bathymetry; Caustics

# Example; inhomogeneous medium, shallow water theory

## GEOMETRY AND WAVE FIELD.



Plane waves. Normal incidence on a sloping bottom.

# Green's law

Plane waves, normal incidence ,  $h = h(x)$ ,  $\vec{k} = k\hat{i}$

Ray theory  $\Rightarrow \omega = \text{const.}$ ,  $k = \omega/c_0$ ,  $\chi = \int kdx$

Transport equation for plane waves  $\Rightarrow$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow F = \text{const.} \Rightarrow c_0 A^2 = \text{const.}$$

Constant energy flux

Substituting  $c_0 = \sqrt{gh}$  we find **Green's law**:

$$A = A_0 \left( \frac{h}{h_0} \right)^{-\frac{1}{4}}, \quad (24)$$

where  $A_0$  and  $h_0$  correspond to a reference location.

## Comparison with accurate numerical solution

$\eta = \hat{\eta}(x)e^{-i\omega t} \Rightarrow$  ODE:

$$\frac{d}{dx} \left( gh(x) \frac{d\hat{\eta}}{dx} \right) + \omega^2 \hat{\eta} = 0. \quad (25)$$

For any  $x_a$  such that  $x_a < x_\ell$  (incident + reflected wave)

$$\hat{\eta} = A_0 e^{ik_\ell x} + R e^{-ik_\ell x}, \quad \text{where } \omega = \sqrt{gh_\ell} k_\ell,$$

where  $R$  is unknown. Annihilation of reflected part

$$\frac{d\hat{\eta}}{dx} + ik_\ell \hat{\eta} = 2iA_0 k_\ell e^{ik_\ell x}, \quad \text{for } x = x_a. \quad (26)$$

At  $x = x_b > x_r$  only transmitted wave:  $\hat{\eta} = T e^{ik_r x}$ , and

$$\frac{d\hat{\eta}}{dx} - ik_r \hat{\eta} = 0, \quad \text{where } \omega = \sqrt{gh_r} k_r. \quad (27)$$

(25,26,27) ODE and boundary conditions for  $\hat{\eta}$ . Moreover:  $A = |\hat{\eta}|$

Numerical approximation :  $\hat{\eta}_j \approx \hat{\eta}(j\Delta x)$ ,  $h_{j+\frac{1}{2}} = h((j + \frac{1}{2})\Delta x)$ ,  
Discrete version of (25)

$$\frac{gh_{j+\frac{1}{2}}(\hat{\eta}_{j+1} - \hat{\eta}_j) - gh_{j-\frac{1}{2}}(\hat{\eta}_j - \hat{\eta}_{j-1})}{\Delta x^2} + \omega^2 \hat{\eta}_j = 0.$$

Boundary conditions (grid from  $i = 0$  to  $i = N$ )

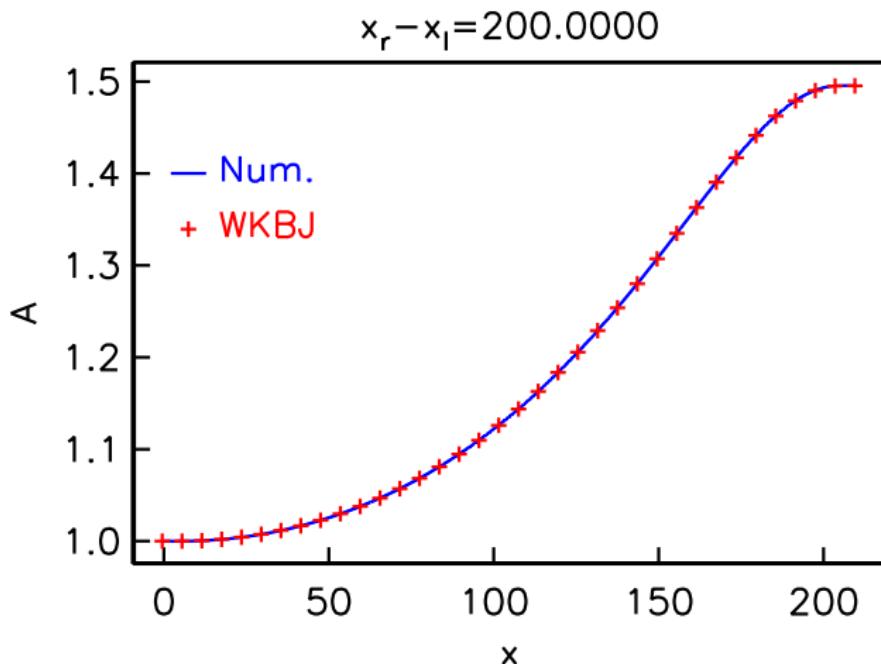
$$\frac{\hat{\eta}_1 - \hat{\eta}_0}{\Delta x} + \frac{i}{2} k_\ell (\hat{\eta}_1 + \hat{\eta}_0) = 2iA_0 k_\ell e^{i\frac{1}{2}k_\ell \Delta x},$$

$$\frac{\hat{\eta}_N - \hat{\eta}_{N-1}}{\Delta x} - \frac{i}{2} k_r (\hat{\eta}_N + \hat{\eta}_{N-1}) = 0.$$

Tri-diagonal set with closure from boundary conditions.

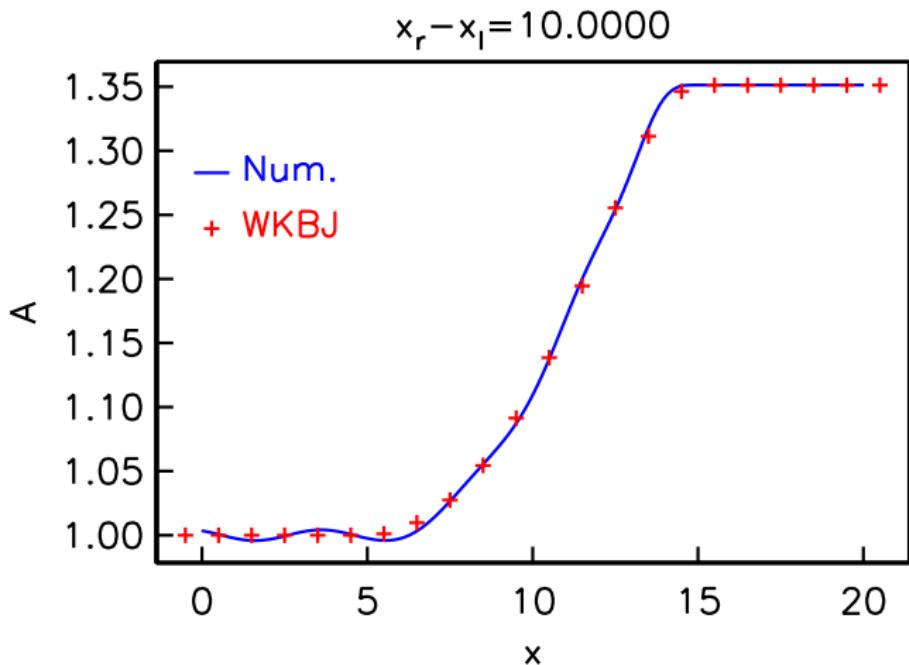
NB: Boundary conditions may be amended by numerical dispersion relation to become exact for the discrete case.

# Gentle slope



$A$  normalized by  $A_\ell$ .  $x$  by  $\hat{x}$ , arbitrary horizontal unit.  
 $x_\ell = 5\hat{x}$ ,  $\lambda_\ell = 8\hat{x}$ ,  $h_r = 0.2h_\ell$ ,  $x_r - x_\ell = 200\hat{x} = 25\lambda_\ell$

# Steep slope



$$x_\ell = 5\hat{x}, \lambda_\ell = 8\hat{x}, h_r = 0.3h_\ell, x_r - x_l = 10\hat{x} = \frac{5}{4}\lambda_\ell$$

# Remarks on optics for shoaling

- Optics good even when  $L = x_r - x_\ell$  and  $\lambda$  are comparable.
- Major discrepancy between optics and accurate numerical solution: Reflections.
- Optics do not incorporate reflections.

# Oblique incidence, plane geometry

We still have  $h = h(x)$ , but  $\vec{k} = k_x \vec{i} + k_y \vec{j}$ .

$\theta$ : angle of incidence;  $k_x = |\vec{k}| \cos \theta$ ,  $k_y = |\vec{k}| \sin \theta$ .

Local wave celerity  $c = \sqrt{gh(x)}$

## Geometrical optics

Hamilton's equations,  $\frac{d\omega}{dt} = 0$  and  $\frac{dk_y}{dt} = 0 \Rightarrow \omega = \text{const.}, k_y = \text{const.}$

Then  $\omega = W(|\vec{k}|, x) = \sqrt{gh}|\vec{k}| = ck_y / \sin \theta$  and

$$\frac{c}{\sin \theta} = \frac{\omega}{k_y} = \text{const.}$$

## Snell's law

## Physical optics; transport equation for $A$

$$\nabla \cdot \vec{F} = 0, \quad \Rightarrow \quad \nabla \cdot (\vec{c}_g A^2) = 0$$

Since  $A = A(x)$  (uniformity in  $y$ ), this yields constant energy flux density in  $x$ -direction

$$c_{gx} A^2 = \text{const.}$$

Now  $c_{gx} = c \cos \theta$  and by Snell's law  $c_{gx} = \omega \cos \theta \sin \theta / k_y$ .

Constant energy flux density then yields

$$\sin(2\theta) A^2 = \text{const.}$$

$\theta < \frac{\pi}{4}$ :  $c_{gx}$  decreases with shoaling  $\Rightarrow$  amplification

$\theta > \frac{\pi}{4}$ :  $c_{gx}$  increases with shoaling due to refraction  $\Rightarrow$  attenuation

# Deep water caustic

Snell's law and amplitude relation

$$\frac{c}{\sin \theta} = \frac{\omega}{k_y} = \text{const.}, \quad c_{gx} A^2 = \text{const.}$$

When  $h \rightarrow h_c = \omega^2/(gk_y^2)$  ( $c \rightarrow \omega/k_y$ )

$$\theta \rightarrow \frac{\pi}{2}, \quad k_x \rightarrow 0, \quad A \rightarrow \infty.$$

Ray parallel to  $y$ -axis; breakdown of optics.

Natural assumption: waves is reflected and propagates into shallower water again.

A mechanism for trapping of waves.

Some more analysis is required.

## Composite solution; Three domains\*

Throughout  $\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)}$   $\Rightarrow$  ODE:

$$\frac{d}{dx} \left( gh(x) \frac{d\hat{\eta}}{dx} \right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0. \quad (28)$$

(i):  $h$  well below  $h_c$

Optics as described previously yield solution. This corresponds to a WKBJ expansion.  $\hat{\eta}(x)$  is wavy.

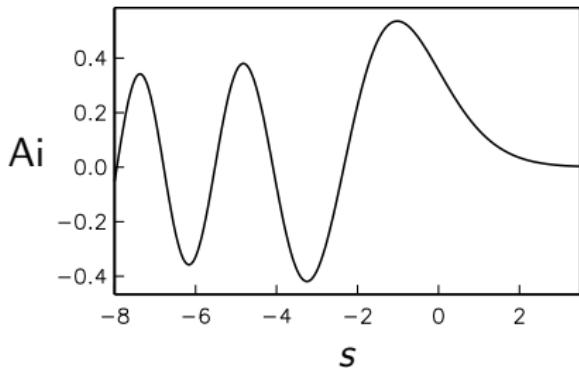
(ii):  $h$  around  $h_c$

Local expansion:  $h \approx h(x_c) + h'(x_c)(x - x_c) + \dots$ . Keeping leading order terms of (28), rescaling

$$\frac{d^2\hat{\eta}}{ds^2} = s\hat{\eta}, \quad \text{where} \quad s = \left( \frac{\omega^2}{h'(x_c)k_y^4} \right)^{\frac{1}{3}} (x - x_c)$$

Solution that remains finite for large  $x - x_c$ :  $\hat{\eta} = BAi(s)$

Combines wavy  $\hat{\eta}$  for  $h < h_c$  with exponential decay for  $h > h_c$ .



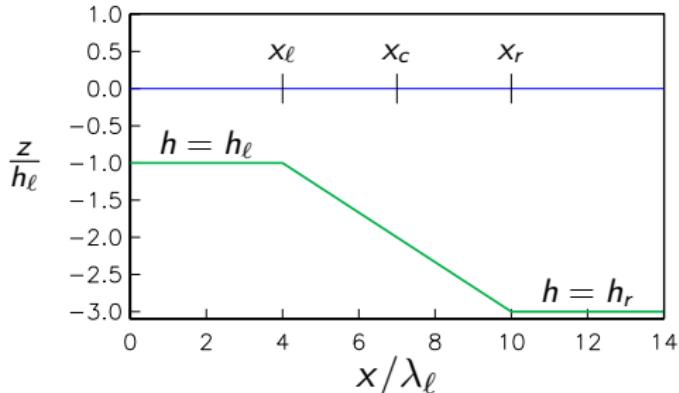
(iii):  $h$  well above  $h_c$

Use WKBJ with  $\hat{\eta} = e^{-S(x)}$ , where  $S$  is real and  $S \rightarrow \infty$  as  $x - x_c \rightarrow \infty$ . Exponential decay.

## Combination

“Asymptotic match”: (i) with (ii), and (ii) with (iii).  
Further details omitted.

# Simplified geometry reflection/transmission problem



$x < x_\ell$ : incident and reflected waves.

If  $h_r < h_c$

Refraction and partial reflection on slope.

$x > x_r$ : uniform transmitted wave.

If  $h_r > h_c$

Total reflection from slope.

$x > x_r$ : decaying  $\eta$ .

Possible caustic

Angle of incidence  $\theta_\ell$ .

$$h_c = \frac{h_\ell}{\sin^2 \theta_\ell}$$

# Formulation of reflection/transmission boundary value problem

$\eta = \hat{\eta}(x)e^{i(k_y y - \omega t)}$   $\Rightarrow$  ODE:

$$\frac{d}{dx} \left( gh(x) \frac{d\hat{\eta}}{dx} \right) + (\omega^2 - gh(x)k_y^2)\hat{\eta} = 0.$$

For  $x_a < x_\ell$  we have, as for normal incidence,

$$\frac{d\hat{\eta}}{dx} + ik_x \hat{\eta} = 2iA_0 k_x e^{ik_x x}.$$

where  $k_x$  is found from  $\omega = W(k_x, k_y, x_a) = \sqrt{gh_\ell} |\vec{k}|$ .

If case  $h_r < h_c$  and  $x_b > x_r$  we again have

$$\frac{d\hat{\eta}}{dx} - ik_x \hat{\eta} = 0,$$

where  $k_x$  is found from  $\omega = W(k_x, k_y, x_b) = \sqrt{gh_r} |\vec{k}|$ .

Reflection,  $h_r > h_c$

Since (28) has constant coefficients when  $x > x_r$  two solutions are readily found

$$\hat{\eta} = B_1 e^{-\alpha x}, \quad \hat{\eta} = B_2 e^{\alpha x}, \quad \text{where} \quad \alpha = \sqrt{k_y^2 - \frac{\omega^2}{gh_r}}.$$

To obtain only the decaying solution we employ

$$\frac{d\hat{\eta}}{dx} + \alpha \hat{\eta} = 0, \quad \text{at} \quad x = x_b.$$

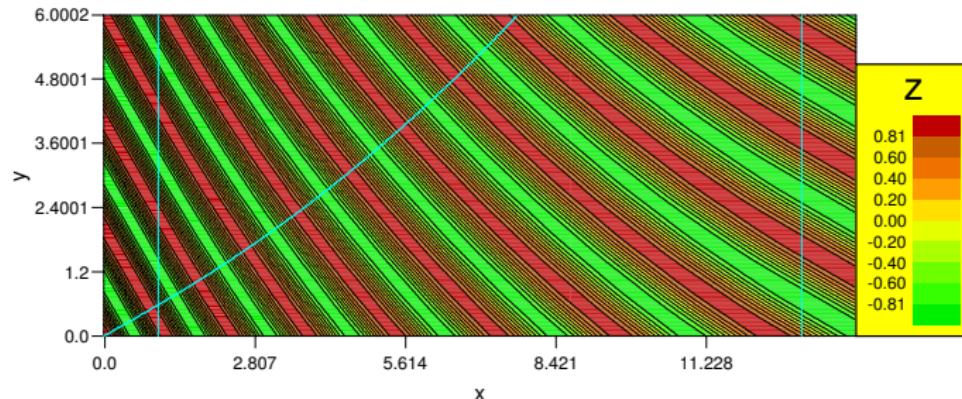
Here  $x_b$  may have any value larger than  $x_r$ .

If  $x_b - x_c$  is sufficiently large we may instead employ

$$\hat{\eta}(x_b) = 0.$$

Numerics as for normal incidence.

Full transmission;  $\theta_\ell = 30^\circ$ ,  $h_r = 3h_\ell$ ,  $h_c = 4h_\ell$

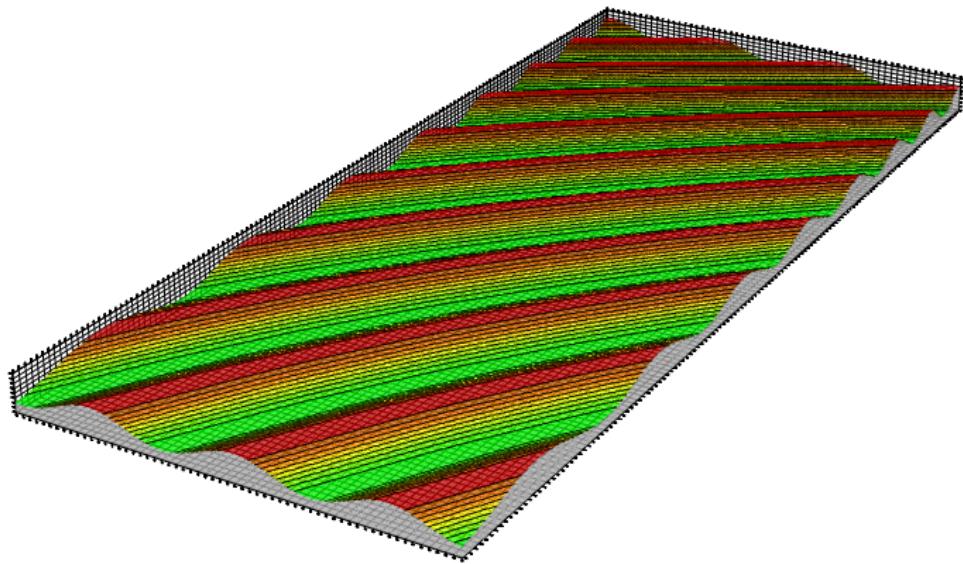


Plot:  $x$  and  $y$  normalized by  $\lambda_\ell = 2\pi c_\ell/\omega$ .

$x = x_\ell$ ,  $x = x_r$  and one ray is marked by lines.

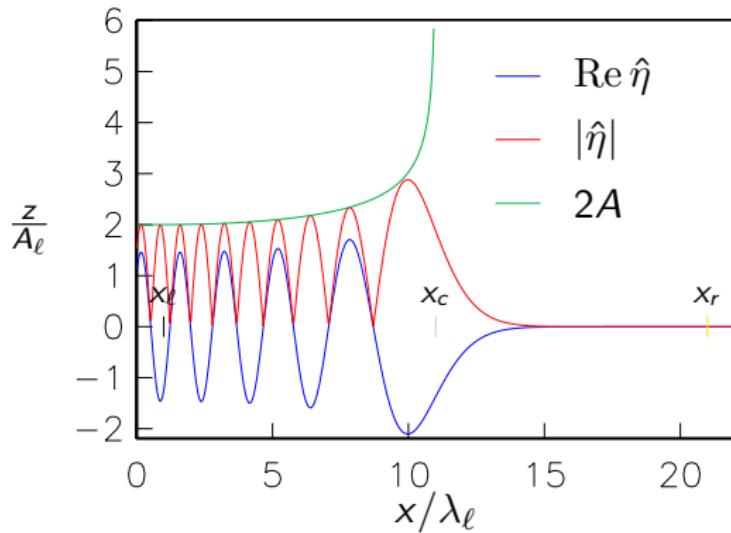
$$x_r - x_\ell = 12\lambda_\ell$$

Virtually no reflection, only refraction.



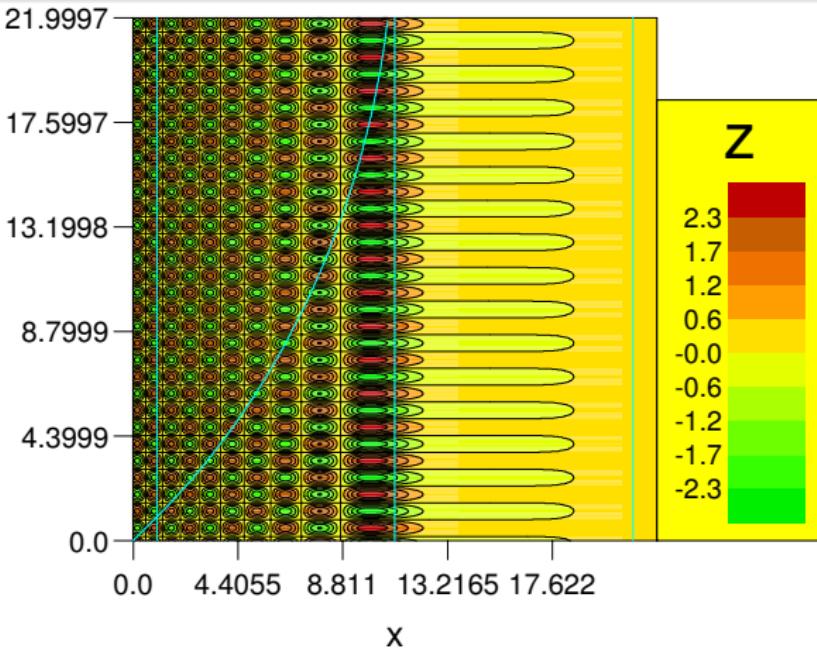
Surface seen from the deep region.

Full reflection;  $\theta_\ell = 45^\circ$ ,  $h_r = 3h_\ell$ ,  $h_c = 2h_\ell$



$A_\ell$  amplitude of incident wave for  $x < x_\ell$ .

Optics for both incident and reflected solution: curve marked with  $2A$ .



Horizontal axes normalized by  $\lambda_\ell$ .

Impermeable walls may be introduced at any  $x_w$  where  $\frac{d\hat{\eta}(x_w)}{dx} = 0$ .  
 Then we have a wave mode, trapped to the shallow region, that propagates in the  $y$  direction. **Edge wave**.

First define the position of the wall  $\Rightarrow$  eigenvalue problem for  $\hat{\eta}$ .