The Stokes wave solution.
MEK4320

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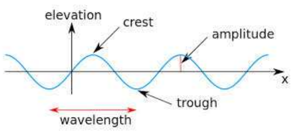
Background

A first step from this...

Typical figure from the net.



Surface gravity waves in elementary coarses and popular dissemination

- Linear periodic solution
- Smooth, rounded, regular.
- Real waves seldom look like this.

to this...

and this...

Stokes waves

- Nonlinear periodic surface gravity waves of permanent shape
- First presented in 1847
- Stability and modulation still an active field of research
- Here: basic solution in infinite depth.

Described by two parameters:

- Amplitude a^*
- Wavelength $\lambda^* = 2\pi/k^*$

Wave celerity, c^*, surface elevation, ζ^* and velocity potential, ϕ^*, must be found.
* marks dimensional coordinates.

Basic equations

Full potential theory in deep water

Surface elevation ζ^*, potential ϕ^*, infinite depth.
* marks dimensional quantities.

$$\zeta^*{}_{t^*} + \phi^*{}_{x^*}\zeta^*{}_{x^*} = \zeta^*{}_{z^*}, \quad z^* = \zeta^*; \tag{1}$$

$$\phi^*{}_{t^*} + \frac{1}{2}(\nabla^*\phi^*)^2 + g\zeta^* = 0, \quad z^* = \zeta^*; \tag{2}$$

$$(\nabla^*)^2\phi^* = 0, \quad z^* < \zeta^*; \tag{3}$$

$$\nabla^*\phi^* \to 0, \quad z^* \to -\infty; \tag{4}$$

Surface conditions (1,2), Laplace equation (3), vanishing motion in depth (4).

Linear harmonic mode

Deletion of nonlinear terms

$$\zeta^*{}_{t^*} = \zeta^*{}_{z^*}, \quad z^* = 0;$$
$$\phi^*{}_{t^*} + g\zeta^* = 0, \quad z^* = 0;$$
$$(\nabla^*)^2\phi^* = 0, \quad z^* < 0;$$
$$\nabla^*\phi^* \to 0, \quad z^* \to -\infty;$$

Solution

$$\zeta^* = a^*\cos(k^*(x^* - c_f^* t^*)),$$
$$\phi^* = a^* c_f^* e^{k^* z^*}\sin(k^*(x^* - c_f^* t^*)) \quad c_f^* = \sqrt{\frac{g}{k^*}}$$

Goal: find nonlinear correction to this

Preparation for perturbation method:

Parameters, scaling, expansion parameter (ϵ) and eigenvalue (c)

Dimension analysis

Parameters

1. λ^* wavelength. Represented by $k^* = 2\pi/\lambda^*$.
2. a^* a measure of amplitude. will be made unique later.
3. g. These are gravity waves.
4. c^* The wave celerity. Must be found. In the linear approximation $c^* = c_f^* = \sqrt{\frac{g}{k^*}}$

The Pi theorem

Two dimensionless numbers

- $\epsilon = a^*k^*$; wave steepness. Assumed small.
- The other is c^*/c_f^*.

$$c^* = c_f^* f(\epsilon), \tag{5}$$

where the function f must be found.

Scaling

Dimensionless variables

$$z = k^*z^*, \qquad x = k^*x^*,$$
$$t = k^*c_f^*t^*, \quad \zeta^* = a^*\zeta,$$
$$\phi^* = c_f^*a^*\phi, \quad c^* = c_f^*c.$$

Waves of permanent form

composite phase variable: $\theta = x - ct$
Fields : $\zeta = \zeta(\theta)$, $\phi = \phi(\theta, z)$
Only two free variables instead of three.
Wavelength fixated by requiring periode$= 2\pi$ in θ.
Introduction of wave celerity, c, as independent unknown is crucial.

Dimensionless equations

Boundary conditions at $z = \epsilon\zeta$:

$$-c\phi_{,\theta} + \frac{1}{2}\epsilon(\phi_{,\theta}^2 + \phi_{,z}^2) + \zeta = 0 \qquad (6)$$
$$-c\zeta_{,\theta} + \epsilon\zeta_{,\theta}\phi_{,\theta} = \phi_{,z} \qquad (7)$$

Indices after comma: partial derivation.
In the bulk of the fluid $z < \epsilon\zeta$:

$$\phi_{,\theta\theta} + \phi_{,zz} = 0 \qquad (8)$$

At infinite depth $z \to -\infty$:

$$\phi_{,\theta},\ \phi_{,z} \to 0 \qquad (9)$$

c is an explicit unknown (eigenvalue)

Form of solutions; normalization condition

Form of periodic solution

Periodic solution (period 2π due to scaling with k^*)$\Rightarrow$ Fourier series for ζ

$$\zeta = b_0 + \sum_{j=1}^{\infty} b_j \cos(j\theta) + \sum_{j=1}^{\infty} e_j \sin(j\theta). \qquad (10)$$

Averaged surface $\Rightarrow$ $b_0 = 0$. Crest of the lowest harmonic at $\theta = 0$ $\Rightarrow$ $e_1 = 0$.
Unique definitions of a^* and ϵ $\Rightarrow$ *normalization* condition

$$b_1 = 1$$

It will turn out that all e_j are zero.

Series for ϕ

$$\phi = U_0\theta + \sum_{j=1}^{\infty} F_j(z)\sin(j\theta) + \sum_{j=1}^{\infty} G_j(z)\cos(j\theta).$$

Each term must fulfill the Laplace equations independently $\Rightarrow$ F and G functions are exponentials.
Vanishing velocity for $z \to \infty$ $\Rightarrow$ $U_0 = 0$.

$$\phi = \sum_{j=1}^{\infty} e^{jz}\{B_j \sin(j\theta) + E_j \cos(j\theta)\}. \qquad (11)$$

It will turn out that all E_j are zero.

The perturbation solution

Series expansion

Simultaneous expansion of ζ, ϕ and c; The Poincare-Lindstedt method.

$$\zeta = \zeta_0(\theta) + \epsilon\zeta_1(\theta) + \epsilon^2\zeta_2(\theta) + \cdots \qquad (12)$$
$$\phi = \phi_0(\theta, z) + \epsilon\phi_1(\theta, z) + \epsilon^2\phi_2(\theta, z) + \cdots \qquad (13)$$
$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots \qquad (14)$$

Requirement: ϕ_i and ζ_i periodic in θ. Period must equal 2π.
Each ζ_j must conform to (10)
Since the equations for $z < 0$ are linear

$$\phi_{j,\theta\theta} + \phi_{j,zz} = 0$$
$$\phi_{j,\theta}, \phi_{j,z} \to 0 \qquad \text{for} \quad z \to -\infty$$

For all j. Taken care of by each ϕ_j conforming to (11)

The expansion and geometrical nonlinearity

Challenges

1. With expansion of c and nonlinearities: quite some book keeping.
2. Surface conditions apply at unknown position $z = \epsilon\zeta$.

Point 2: terms in surface conditions must be expanded to express the surface conditions in terms of field variables at $z = 0$. Example

$$\begin{aligned}\phi_{,\theta}(\theta, \epsilon\zeta) = \;& \phi_{,\theta}(\theta, 0) + \epsilon\phi_{,\theta z}(\theta, 0)\zeta + \tfrac{1}{2}\epsilon^2\phi_{,\theta zz}(\theta, 0)\zeta^2 + \dots \\ = \;& \phi_{0,\theta} + \epsilon\left(\phi_{0,\theta z}\zeta_0 + \phi_{1,\theta}\right) \\ & + \epsilon^2\left(\tfrac{1}{2}\phi_{,\theta zz}\zeta_0^2 + \phi_{1,\theta z}\zeta_0 + \phi_{0,\theta z}\zeta_1 + \phi_{2,\theta}\right) + \dots\end{aligned}$$

where the argument $(\theta, 0)$ is implicit in the lower two lines.

Leading order; linear solution

Keeping $O(1)$ terms:

$$\begin{aligned} -c_0\phi_{0,\theta} + \zeta_0 &= 0, && \text{for } z = 0, \\ c_0\zeta_{0,\theta} + \phi_{0,z} &= 0, && \text{for } z = 0, \\ \phi_{0,\theta\theta} + \phi_{0,zz} &= 0, && \text{for } z < 0, \\ \phi_{0,\theta}, \phi_{0,z} &\to 0, && \text{for } z \to -\infty. \end{aligned} \tag{15}$$

Linear harmonic wave mode is reproduced in dimensionless form

$$\zeta_0 = \cos\theta, \quad \phi_0 = e^z \sin\theta, \quad c_0 = 1, \tag{16}$$

where the normalization condition has been applied.

Order ϵ^1

At $z = 0$:

$$\left.\begin{aligned} -c_0\phi_{1,\theta} + \zeta_1 &= c_1\phi_{0,\theta} - \tfrac{1}{2}(\phi_{0,\theta}^2 + \phi_{0,z}^2) + c_0\phi_{0,\theta z}\zeta_0 \equiv R_a, \\ c_0\zeta_{1,\theta} + \phi_{1,z} &= -c_1\zeta_{0,\theta} + \zeta_{0,\theta}\phi_{0,\theta} - \phi_{0,\theta zz}\zeta_0 \equiv R_b. \end{aligned}\right\} \tag{17}$$

Elimination of ζ_1 and $c_0 = 1 \Rightarrow$

$$\phi_{1,\theta\theta} + \phi_{1,z} = R_b - R_{a,\theta} \quad \text{for} \quad z = 0.$$

Insert expressions for ζ_0, ϕ_0 in right hand side

$$\phi_{1,\theta\theta} + \phi_{1,z} = 2c_1 \sin\theta \quad \text{for} \quad z = 0 \tag{18}$$

Solution of type (11) for ϕ_1

$$\phi_1 = \sum_{j=1}^{\infty} e^{jz}\{B_j^{(1)}\sin(j\theta) + E_j^{(1)}\cos(j\theta)\},$$

is then inserted into (18).

Result

$$\sum_{j=1}^{\infty}(j - j^2)\{B_j^{(1)}\sin(j\theta) + E_j^{(1)}\cos(j\theta)\} = 2c_1\sin\theta.$$

Uniqueness of Fourier series

- $c_1 = 0$ (terms with $j = 1$ vanish)
- $B_j^{(1)} = E_j^{(1)} = 0$ for $j > 1$.
- Normalization and crest at $\theta = 0 \Rightarrow B_1^{(1)} = E_1^{(1)} = 0$

Hence

$$c_1 = 0, \quad \phi_1 = 0$$

Then (17) $\Rightarrow$

$$\zeta_1 = \frac{1}{2}\cos 2\theta.$$

Order ϵ^2

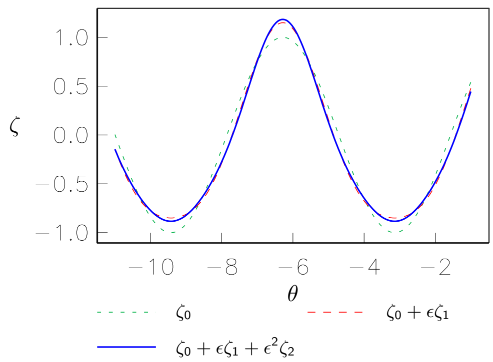
Same structure as for $O(\epsilon^1)$. Longer expressions, but $c_1 = 0$ and $\phi_1 = 0$ help.
Details omitted.

$$\zeta_2 = \frac{3}{8}\cos 3\theta, \quad \phi_2 = 0, \quad c_2 = \frac{1}{2}.$$

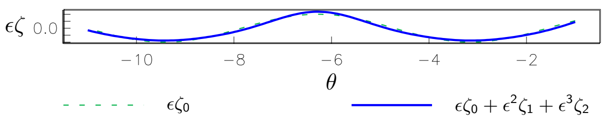
To this order: modification of celerity !
With dimensions restored

$$c^* = \sqrt{\frac{g}{k^*}}\left(1 + \frac{1}{2}(a^*k^*)^2 + \cdots\right)$$

Wave shape; $\epsilon = 0.3$



True aspect ratio; $\epsilon = 0.3$



- Narrower crests, wider troughs.
- Total wave height increased by ζ_2.
- Quite large ϵ. Still looks far from extreme.

Comments

Some properties and remarks

Stokes-drift:

$$Q^* = \frac{1}{T^*}\int_0^{T^*}\int_{-\infty}^{\zeta^*} u^* dz^* dt^* = \frac{1}{2k^*}\sqrt{\frac{g}{k^*}}\epsilon^2 + O(\epsilon^3) > 0.$$

There is net volume transport in the direction of wave advance.

Instabilities

- Benjamin & Feir (1967): The Stokes wave is always unstable in infinite depth.
- Combination of shorter and longer wavelength grow $\Rightarrow$ modulations.
- Instability slow for small amplitudes.
- $\epsilon > 0.2$, say, span-wise instability $\Rightarrow$ three dimensional motion.
- Stokes waves may be stable on finite depth.
- Much, much more.

Narrow band approximation. Modulation equations*

$$2\zeta = A(x,t)e^{i(kx-\omega t)} + \epsilon A_2(x,t)e^{2i(kx-\omega t)} + \cdots + c.c.$$

where A, A_1 etc. display only slow variation with respect to x and t.
Some algebra$\Rightarrow$

$$i\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}|A^2|A = 0 \qquad (19)$$

where $\tau = \epsilon t$ and $\xi = \epsilon(x - c_g t)$. Here c_g is the linear group velocity for wave number k.
The cubic Schrödinger equation.