# WKBJ METODEN
# OG
# FYSISK OPTIKK
# MEK4320

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Linear shallow water theory(indices mark differentiation)

$$\eta_{tt} - \nabla \cdot (c_0^2 \nabla \eta) = 0 \tag{1}$$

where $c_0^2 = gh(x, y)$.
In ray theory

$$\eta(x, y, t) = A(x, y, t) e^{i\chi(x,y,t)}, \tag{2}$$

where $\vec{k} \equiv \nabla \chi$, $\omega \equiv -\frac{\partial \chi}{\partial t}$.
Slow varations of $\vec{k}$ and $\omega \Rightarrow$ ray equations.
We now insert (2) in (1):

$$\begin{aligned} &A_{tt} - i(2\omega A_t + \omega_t A) - \omega^2 A = \\ &\nabla \cdot (c_0^2 \nabla A) + i \left( c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}) \right) - c_0^2 k^2 A \end{aligned} \tag{3}$$

So far: exact, but nothing is achieved either – so far.

# Scaling; transformation to non-dimensional form

## Scales

Typical wavelength: $\lambda_c$

Typical length scale for medium change $L_c$

Small parameter $\frac{\lambda_c}{L_c} = \epsilon \ll 1$.

Typical wave speed: $c_c = \sqrt{gh_c}$

Typical amplitude: $A_c$ – no significance as long as in linear regime

Typical phase: $\chi \sim L_c/\lambda_c = \epsilon^{-1}$

## Rescaling

All derivatives explicit in (3) are with respect to slow variation.

Fast variation inherent in definitions of $\vec{k}$ and $\omega$, only.

$\vec{\kappa} = \lambda_c \vec{k}$, $\hat{\omega} = \lambda_c \omega / c_c$, $\hat{A} = A/A_c$, $\hat{c} = c_0/c_c$, $\hat{x} = x/L_c$,
$\hat{t} = c_c t/L_c$.

## Rescaling of (3)

$$\epsilon^2 \hat{A}_{\hat{t}\hat{t}} - i\epsilon(2\hat{\omega}\hat{A}_{\hat{t}} + \hat{\omega}_{\hat{t}}\hat{A}) - \hat{\omega}^2\hat{A} = \\ \epsilon^2\hat{\nabla}\cdot(c^2\hat{\nabla}\hat{A}) + i\epsilon\left(c^2\hat{\nabla}\hat{A}\cdot\vec{\kappa} + \hat{\nabla}\cdot(c^2\hat{A}\vec{\kappa})\right) - c^2\kappa^2\hat{A} \tag{4}$$

Leading terms: $O(1)$ no slow differentiations
Next order: $O(\epsilon)$ one slow differentiation
Second order: $O(\epsilon^2)$ two slow differentiations

### Leading order

$\hat{\omega}^2 = c^2\kappa^2 \Rightarrow$ restored scales $\omega^2 = c_0^2 k^2 = W^2$.

With $\vec{k}_t + \nabla\omega = 0$, $\partial k_i/\partial x_j = \partial k_j/\partial x_i$: ray theory retrieved

$$\frac{\partial k_i}{\partial t} + \vec{c}_g\cdot\nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1,2 \tag{5}$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g\cdot\nabla\omega = 0 \tag{6}$$

with $\vec{c}_g = c_0\vec{k}/k$. Now $\vec{k}$ and $\omega$ is settled.

# Next order; $O(\epsilon)$ relative size

Original scales restored

$$-2\omega A_t - \omega_t A = c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}).$$

With $\omega = c_0 k$ and $\vec{c}_g = c_0 \vec{k}/k$ (then $c_0^2 \vec{k} = \omega \vec{c}_g$):

$$-2\omega A_t - \omega_t A = 2\omega \vec{c}_g \cdot \nabla A + A \vec{c}_g \cdot \nabla \omega + \omega A \nabla \cdot \vec{c}_g .$$

Next step multiply with $A/\omega$ and regroup

$$-(A^2)_t - \frac{A^2}{\omega} (\omega_t + \vec{c}_g \cdot \nabla \omega) = \nabla \cdot (\vec{c}_g A^2).$$

Due to (6) terms within last parentheses on l.h.s cancel out:

$$(A^2)_t = -\nabla \cdot (\vec{c}_g A^2). \tag{7}$$

With $E = \frac{1}{2}\rho g A^2$ (energy density) and $\vec{F} = \vec{c}_g E$ (energy flux) equation (7) reads

$$E_t + \nabla \cdot \vec{F} = 0, \tag{8}$$

Averaged energy conservation, as in uniform medium.

# Remarks

## Remark 1

Similar WKBJ approaches apply to most linear wave equations $\Rightarrow$ result with interpretation as energy conservation is general.

## Remark 2

By means of (6) we have

$$(G(\omega)E)_t + \nabla \cdot \left( G(\omega)\vec{F} \right) = 0,$$

for any $G(\omega)$.

## Remark 3

If we have coupling with a background current it is the wave action ($E$ over some frequency) which is conserved.

## Remark 4

To include higher order in $\epsilon$ we must expand $A = A_0 + \epsilon A_1 + ..$

# OPTIKK;OPPSUMMERING

Stråleteori (geometrisk optikk)

Harmonisk bølge, uniformt medium$\Rightarrow$ dispersjonsrelasjon

$\omega = W(\vec{k}; H...)$

Langsom variasjon av medium og bølgetog$\Rightarrow$ lokale $k$ og $\omega$ oppfyller dispersjonsrelasjonen (tilnærmet)

Fysisk optikk

Harmonisk bølge, uniformt medium$\Rightarrow$

$\vec{F} = c_g E, \; E = E(A^2, ..)$

Langsom variasjon etc. $\Rightarrow \vec{F} = \vec{c}_g E$ (tilnærmet)

# GEOMETRISK OPTIKK; likninger

## Strålelikninger

Fra $\omega = -\frac{\partial \chi}{\partial t}$, $\vec{k} = \nabla \chi$ og $\omega = W(\vec{k}, x_i, t)$

$$\begin{aligned}
\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega &= \frac{\partial W}{\partial t}
\end{aligned}$$

## Skrevet som Hamiltons kanoniske likninger

$$\begin{aligned}
\frac{\mathrm{d} k_i}{\mathrm{d} t} &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\mathrm{d} x_i}{\mathrm{d} t} &= \frac{\partial W}{\partial k_i} = (c_g)_i \,, \quad i = 1, 2.. \\
\frac{\mathrm{d} \omega}{\mathrm{d} t} &= \frac{\partial W}{\partial t}
\end{aligned}$$

## Transportlikning

Generell energilikning

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Innsatt tilnærmelsen $F = c_g E$:

$$\frac{\partial E}{\partial t} + \frac{\partial (c_g E)}{\partial x} = 0$$

der $E = E(A^2, ..)$

Flere dimensjoner

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{c}_g E) = 0 \tag{9}$$

### Utregning av bølgefelt

1. $\vec{k}$ og $\omega$ finnes fra stråleteori
2. Transportlikning (9) løses for $A$

# Uniformt medium

$\omega = W(\vec{k}) \Rightarrow \vec{c}_g = c_g(\vec{k})$

Strålelikninger

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = 0, \quad i = 1, 2..$$

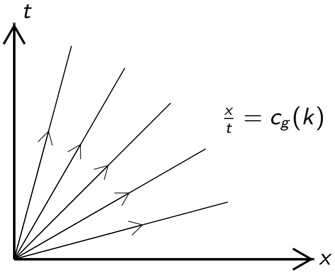
$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = 0, \quad i = 1, 2..$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

$\vec{k}$ og $\omega$ bevart langs karakteristikk $\mathcal{C}$: $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{c}_g$

$\Rightarrow$ $\vec{c}_g$ bevart $\Rightarrow$ $\mathcal{C}$ er rette linjer

# Uniformt medium: bølger fra konsentrert forstyrrelse




Alle $\mathcal{C}$ gjennom $x = t = 0 \Rightarrow c_g = x/t$
$c_g(k) = x/t \Rightarrow k = k(x/t) \Rightarrow \chi = \int k dx$

Uendelig dyp; stråleteori
$W = \sqrt{gk} \Rightarrow x/t = c_g = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{1}{4}g\frac{t^2}{x^2} \Rightarrow$
$\chi = -\frac{1}{4}g\frac{t^2}{x} + f(t)$
Videre $\frac{\partial \chi}{\partial t} = -\omega = -\sqrt{gk} \Rightarrow \chi = -\frac{1}{4}g\frac{t^2}{x} + \mathrm{const}$
Fasefunksjon som fra stasjonær fase.

# Uendelig dyp; transportlikning

$$\frac{\partial A^2}{\partial t} + \nabla \cdot (\vec{c}_g A^2) = 0$$

med $c_g = x/t$ omskrives til

$$\frac{\partial x A^2}{\partial t} + \frac{x}{t}\frac{\partial (x A^2)}{\partial x} = 0$$

Generell løsning $xA^2$ konstant langs $\mathcal{C}$ $\Rightarrow$

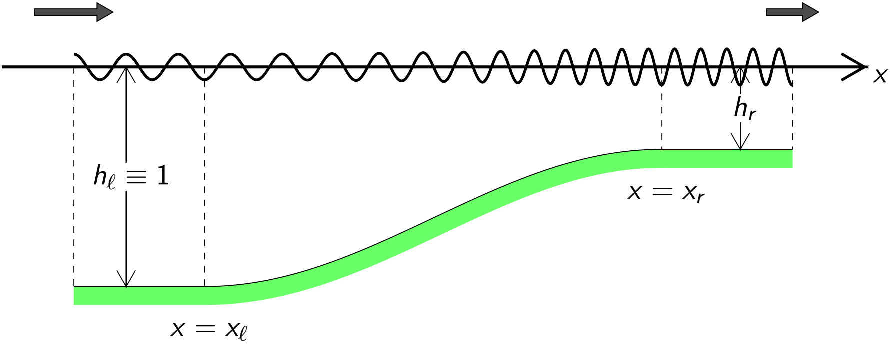
$$A = x^{-\frac{1}{2}} G\left(\frac{x}{t}\right) = t^{-\frac{1}{2}} \hat{G}\left(\frac{x}{t}\right)$$

Konsistent med stasjonær fase (spektrum$\Rightarrow$ $G$)

Tolkning: Energi mellom to $\mathcal{C}$ bevart.

# Example; inhomogeneous medium

**GEOMETRY AND WAVEFIELD.**




Plane waves incident on sloping bottom.

# Eksempel continues: Green's lov

Plant, normalt infall, $h = h(x)$, $\vec{k} = k\vec{\imath}$

Geometrisk optikk (stråleteori)$\Rightarrow$ $\omega = \mathrm{konst.}$, $k = \omega / c_0$,

$\chi = \int k dx$

Transportlikning for plane bølger $\Rightarrow$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow F = \mathrm{konst.} \Rightarrow c_0 A^2 = \mathrm{konst.}$$

Konstant energifluks

Stråleteori + transportlikning $\Rightarrow$ Greens lov:

$$A = A_0 \left( \frac{h}{h_0} \right)^{-\frac{1}{4}}, \tag{10}$$

der $A_0$ og $h_0$ beskriver en referansetilstand.

## Sammenlikning med nøyaktig numerisk løsning

$\eta = \hat{\eta}(x)e^{-i\omega t} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \omega^2\hat{\eta} = 0.$$

For en $x_a$ slik at $x_a < x_\ell$ (ikom. +refl. bølge)

$$\hat{\eta} = A_0 e^{ik_\ell x} + Re^{-ik_\ell x}, \quad \text{der} \quad \omega = \sqrt{gh_\ell}k_\ell.$$

Gir randbetingelse

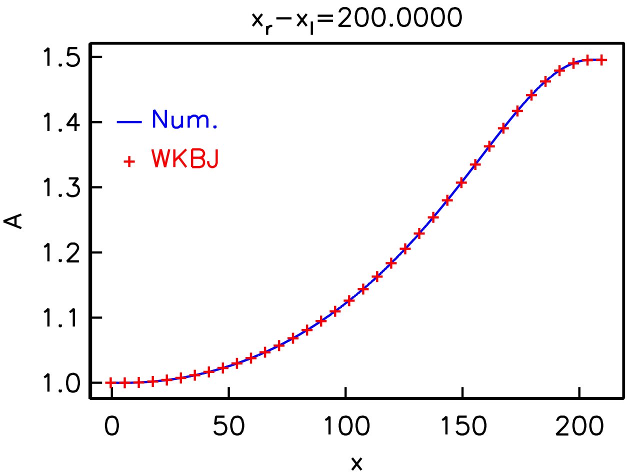
$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_\ell\hat{\eta} = 2iA_0 k_\ell e^{ik_\ell x}, \quad \text{for} \quad x = x_a.$$

Ved $x = xb > x_r$ bare transmittert bølge $\hat{\eta} = Te^{ik_r x}$

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_r\hat{\eta} = 0, \quad \text{der} \quad \omega = \sqrt{gh_r}k_r.$$

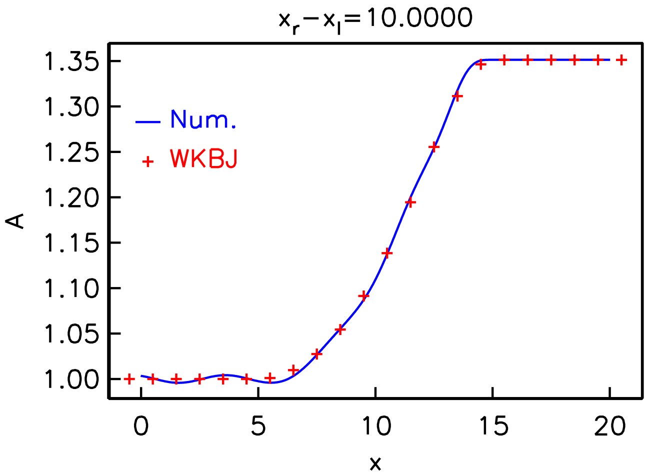
2 ordens ODE og 2 randbet. for kompleks $\hat{\eta}$. Tolker: $A = |\hat{\eta}|$

# Gentle slope




$x = x_l = 5$: amplitude 1, periode 8.

$h_r = 0.2$, $x_r - x_l = 200$




$x = x_l = 5$: amplitude 1, periode 8.
$h_r = 0.3$, $x_r - x_l = 10$