Unidirectional waves and velocity profiles
– radially symmetric and plane Boussinesq solutions –

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1 preface

This is an extension of the short note *On radial Boussinesq equations and velocity profiles – short note for the Mjølnir project–*, dated 2004. There may be sections with no explicit motivation and redundancy, as this note is meant to serve only for internal use in projects.

2 Introduction, basic equations

We will use the the phrase unidirectional waves for wave systems that may be described mathematically by a single horizonatal space variable. Two main examples to this are

- Impacts in the shallow ocean often generate a radially symmetric wave system that may give rise to nonlinear and weakly dispersive tsunamis, that can be goverened by the Boussinesq equation.
- A wave paddle in a tank will generate unidirectional waves. In such cases it is important to derive the whole charateritics of the wave, including velocity profiles, from the time series of surface elevations obtained from wave gauges.

Besides there are many circumstances where waves may be regarded as unidirectional as a good approximation.

We make quantities dimensionless with A (surface amplitude), h_0 (characteristic dpth), λ (characteristic wavelength) and $\lambda/\sqrt{gh_0}$ as scales for amplitude, vertical length, horizontal length and time, respectively. With the depth averaged velocity $\overline{\mathbf{v}}_h$ the Boussinesq equations then read

$$\frac{\partial \overline{\mathbf{v}}_h}{\partial t} + \epsilon \overline{\mathbf{v}}_h \cdot \nabla \overline{\mathbf{v}}_h = -\nabla \eta + \frac{\beta}{2} h \nabla \nabla \cdot (h \frac{\partial \overline{\mathbf{v}}_h}{\partial t}) - \frac{\beta}{6} h^2 \nabla^2 \frac{\partial \overline{\mathbf{v}}_h}{\partial t} + O(\beta^2, \beta\epsilon) \quad (1)$$

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot ((h + \epsilon\eta)\overline{\mathbf{v}}_h), \quad (2)$$

With dimensionless parameters $\epsilon = A/h$, $\beta = (h/\lambda)^2$

Employing the averaged velocity potential, ϕ, as primary unknown we obtain the alternative form[1]

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\epsilon(\nabla\phi)^2 + \eta - \frac{\beta}{2} h \nabla \cdot \nabla (h \frac{\partial \phi}{\partial t}) + \frac{\beta}{6} h^2 \nabla^2 \frac{\partial \phi}{\partial t} + O(\beta^2, \beta\epsilon) \quad (3)$$

[1] This formulation may be unstable for extreme bottom gradients

$$\frac{\partial\eta}{\partial t} = -\nabla\cdot\left[(h+\epsilon\eta)\left(\nabla\phi + \beta h\left(\frac{1}{6}\frac{\partial\eta}{\partial t} - \frac{1}{3}\nabla h\cdot\nabla\phi\right)\nabla h\right)\right], \tag{4}$$

3 Radial symmetry

We assume $\eta = \eta(r,t)$, $\overline{\mathbf{v}}_h = \overline{u}\vec{\imath}_r$, $\phi = \phi(r,t)$, where $r = \sqrt{x^2+y^2}$. The transformation relations are

$$\begin{array}{ll} \nabla = \vec{\imath}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{\imath}_\theta\frac{\partial}{\partial\theta} & \nabla\eta = \frac{\partial\eta}{\partial r}\vec{\imath}_r \\ \nabla^2\eta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right) & \nabla\cdot(h\nabla\phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial\phi}{\partial r}\right) \\ \overline{\mathbf{v}}_h\cdot\nabla\overline{\mathbf{v}}_h = \overline{u}\frac{\partial\overline{u}}{\partial r}\vec{\imath}_r & \nabla\cdot\overline{\mathbf{v}}_h = \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u}) \\ \nabla\nabla\cdot\overline{\mathbf{v}}_h = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u})\right) & \end{array} \tag{5}$$

The radial Boussinesq equations now become

$$\frac{\partial\overline{u}}{\partial t} + \epsilon\overline{u}\frac{\partial\overline{u}}{\partial r} = -\frac{\partial\eta}{\partial r} + \frac{\beta}{2}h\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(hr\frac{\partial\overline{u}}{\partial t}\right)\right) - \frac{\beta}{6}h^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{u}}{\partial t}\right)\right) + O(\beta^2,\beta\epsilon) \tag{6}$$

$$\frac{\partial\eta}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left((h+\epsilon\eta)r\overline{u}\right), \tag{7}$$

or

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\epsilon\left(\frac{\partial}{\partial r}\phi\right)^2 + \eta - \frac{\beta}{2}\frac{h}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial^2\phi}{\partial tr}\right) + \frac{\beta}{6}\frac{h^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial^2\phi}{\partial rt}\right) + O(\beta^2,\beta\epsilon) \tag{8}$$

$$\frac{\partial\eta}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r(h+\epsilon\eta)\frac{\partial\phi}{\partial r} + \beta rh\left(\frac{1}{6}\frac{\partial\eta}{\partial t} - \frac{1}{3}\frac{\partial h}{\partial r}\frac{\partial\phi}{\partial r}\right)\frac{\partial h}{\partial r}\right], \tag{9}$$

4 Unidirectional waves

Next we assume constant depth, $h\equiv 1$ and that all waves propagate from the center of symmetry. This imply that the radial propagation speed is $c = 1 + O(\epsilon,\beta)$. Then we employ the transformation and assumption:

$$\xi = r - t, \quad \tau = \beta t, \quad \frac{1}{r} = O(\beta). \tag{10}$$

Substitution into (7,6) yields

$$\beta\frac{\partial\eta}{\partial\tau} - \frac{\partial\eta}{\partial\xi} = -\frac{\partial\overline{u}}{\partial\xi} - \epsilon\frac{\partial}{\partial\xi}(\eta\overline{u}) - \frac{\overline{u}}{r} + O(\beta^2,\beta\epsilon), \tag{11}$$

$$\beta\frac{\partial\overline{u}}{\partial\tau} - \frac{\partial\overline{u}}{\partial\xi} + \epsilon u\frac{\partial\overline{u}}{\partial\xi} = -\frac{\partial\eta}{\partial\xi} - \frac{\beta}{3}\frac{\partial^3\overline{u}}{\partial\xi^3} + O(\beta^2,\beta\epsilon). \tag{12}$$

The leading order balance then becomes $\overline{u} = \eta + O(\beta,\epsilon)$ and the above equations imply

$$\beta\frac{\partial\eta}{\partial\tau} = \frac{\partial\eta}{\partial\xi} - \frac{\partial\overline{u}}{\partial\xi} - \epsilon\frac{\partial}{\partial\xi}(\eta^2) - \frac{\overline{u}}{r} + O(\beta^2,\beta\epsilon), \tag{13}$$

$$\beta\frac{\partial\eta}{\partial\tau} = \frac{\partial\overline{u}}{\partial\xi} - \frac{\partial\eta}{\partial\xi} - \frac{1}{2}\epsilon\frac{\partial}{\partial\xi}(\eta^2) - \frac{\beta}{3}\frac{\partial^3\overline{u}}{\partial\xi^3} + O(\beta^2,\beta\epsilon). \tag{14}$$

Elimination of temporal derivatives then gives

$$0 = \frac{\partial \eta}{\partial \xi} - \frac{\partial \overline{u}}{\partial \xi} - \frac{\epsilon}{4}\frac{\partial}{\partial \xi}(\eta^2) - \frac{1}{2}\frac{\overline{u}}{r} + \frac{\beta}{6}\frac{\partial^3 \overline{u}}{\partial \xi^3} + O(\beta^2, \beta\epsilon). \tag{15}$$

This may be used to eliminate $\partial\overline{u}/\partial\xi$ from either (11) or (12). When $\overline{u}$ is removed from the higher order terms by means of the dominant balance $\overline{u} \approx \eta$ we then arrive at the radial KdV equation

$$\beta\frac{\partial \eta}{\partial \tau} + \frac{3\epsilon}{2}\eta\frac{\partial \eta}{\partial \xi} - \frac{\beta}{3}\frac{\partial^3 \eta}{\partial \xi^3} + \frac{\eta}{r} = 0 \tag{16}$$

Restoring r in (15), integrating and reorganizing the resulting expression we obtain

$$\overline{u} - \frac{\beta}{6}\frac{\partial^2 \overline{u}}{\partial r^2} + O(\beta^2, \beta\epsilon) = \eta - \frac{\epsilon}{4}\eta^2 - \frac{1}{2}\int_{\infty}^{r} \frac{\overline{u}}{\hat{r}} \mathrm{d}\hat{r} \equiv R, \tag{17}$$

where we have assumed equilibrium at infinity, or

$$\overline{u} - \frac{\beta}{6}\frac{\partial^2 \overline{u}}{\partial r^2} = \eta - \frac{\epsilon}{4}\eta^2 - \frac{1}{2}\int_{\infty}^{r} \frac{\eta}{\hat{r}} \mathrm{d}\hat{r} + O(\beta^2, \beta\epsilon). \tag{18}$$

Corresponding expressions for plane waves may be obtained by omitting the term with the integral and replacing r by x. The equations (17) and (18) may be used to assign initial values to $\overline{u}$ when η is specified. The advantage of (17) is that its regular at $r = 0$ since $\overline{u}$ is zero at the symmetry point, while (18) has a simpler form when η is known. To solve (17) for $\overline{u}$ we have to employ an iteration scheme, which on the other hand is quite straightforward. The second derivative of $\overline{u}$ may be replaced by a second derivative of η, but this will produce an increased noise level in $\overline{u}$.

5 Velocity profiles

To leading order the requirement of zero divergence imply

$$\frac{\partial w}{\partial z} = -\Theta + O(\beta), \tag{19}$$

where w is the vertical velocity component and Θ is independent of z. For plane and radially symmetric waves we have

$$\Theta = \frac{\partial \overline{u}}{\partial x}, \quad \Theta = \frac{1}{r}\frac{\partial (ru)}{\partial r}, \tag{20}$$

respectively. Invoking the kinematic condition at the free surface we then find

$$w = \frac{\mathrm{D}\eta}{\mathrm{D}t} + (\epsilon\eta - z)\Theta + O(\beta), \tag{21}$$

where all nonlinearities are retained for completeness. Irrotational fluid motion imply

$$\frac{\partial u}{\partial z} = \beta\frac{\partial w}{\partial \xi}, \tag{22}$$

where ξ is either x or r. Integrating this equation from the surface ($z = \eta$) we obtain

$$u = u_s - \beta(\epsilon\eta - z)\left[\frac{\partial}{\partial\xi}\frac{\mathrm{D}\eta}{\mathrm{D}t} + \epsilon\Theta\frac{\partial\eta}{\partial\xi}\right] - \frac{\beta}{2}(\epsilon\eta - z)^2\frac{\partial\Theta}{\partial\xi} + O(\beta^2), \tag{23}$$

where u_s is the velocity at the surface. We observe that the occurences of $\overline{u}$ in the $O(\beta)$ terms may be replaced by u_s, or u, within an error of β^2. Averaging (23) we find

$$\overline{u} = u_s - \frac{\beta}{2}(\epsilon\eta + 1)\left[\frac{\partial}{\partial\xi}\frac{\mathrm{D}\eta}{\mathrm{D}t} + \epsilon\Theta\frac{\partial\eta}{\partial\xi}\right] - \frac{\beta}{6}(\epsilon\eta - 1)^2\frac{\partial\Theta}{\partial\xi} + O(\beta^2). \tag{24}$$

Invoking once again the assumption of radially spreading waves and large r, (10), the relation first simplifies to

$$\overline{u} = u_s - \frac{\beta}{2}(\epsilon\eta + 1)\left[\frac{\partial}{\partial\xi}\frac{\mathrm{D}\eta}{\mathrm{D}t} + \epsilon\frac{\partial\eta}{\partial\xi}\frac{\partial\overline{u}}{\partial\xi}\right] - \frac{\beta}{6}(\epsilon\eta - 1)^2\frac{\partial^2\overline{u}}{\partial\xi^2} + O(\beta^2), \tag{25}$$

and by inserting $\overline{u} = \eta + O(\epsilon, \beta)$ and $\frac{\mathrm{D}\eta}{\mathrm{D}t} = -\frac{\partial}{\partial\xi}\eta + O(\epsilon, \beta)$ to

$$\overline{u} = u_s + \frac{\beta}{3}\frac{\partial^2\overline{u}}{\partial\xi^2} + O(\beta^2, \beta\epsilon). \tag{26}$$

We observe that the last two expressions are equal for plane waves and radial symmetry. Replacing $\overline{u}$ on the left hand side of (26) by u_s gives an equation for u_s. However, this is a standard Helmholz equation that should not be used. Neither is it useful to involve an explit differentiation of $\overline{u}$ (growth of noise). When $\overline{u}$ and η is known we may instead use (17) to find the explicit relation

$$u_s = 2R - \overline{u} + O(\beta^2, \beta\epsilon) \tag{27}$$

where R is as in (17). It then remains to compute the surface potential $\psi \equiv \Phi(r, \eta, t)$, where Φ is the full 3D potential. It follows

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\Phi}{\partial\xi} + \epsilon\frac{\partial\Phi}{\partial y}\frac{\partial\eta}{\partial\xi} = u_s + \epsilon\frac{\mathrm{D}\eta}{\mathrm{D}t}\frac{\partial\eta}{\partial\xi} = u_s - \epsilon\left(\frac{\partial\eta}{\partial\xi}\right)^2 + O(\epsilon^2, \beta\epsilon). \tag{28}$$

Integration then yields ψ.

For the velocity at the bottom (23) yields

$$u_b = u_s - \beta(\epsilon\eta - 1)\left[\frac{\partial}{\partial\xi}\frac{\mathrm{D}\eta}{\mathrm{D}t} + \epsilon\Theta\frac{\partial\eta}{\partial\xi}\right] - \frac{\beta}{2}(\epsilon\eta - 1)^2\frac{\partial\Theta}{\partial\xi} + O(\beta^2). \tag{29}$$

To leading order this implies

$$u_b = u_s + \frac{\beta}{2}\frac{\partial^2\overline{u}}{\partial\xi^2} + O(\epsilon^2, \beta\epsilon). \tag{30}$$

With an implicit error $O(\epsilon^2, \beta\epsilon)$ we may rewrite this expression in several ways

$$u_b - \frac{\beta}{6}\frac{\partial^2 u_b}{\partial\xi^2} = \overline{u}, \quad u_b = \frac{3}{2}\overline{u} - \frac{1}{2}u_s. \tag{31}$$

The former is a modified Helmholtz equation with favourable properties. The latter expression is explicit and handy when u_s is already computed.

6 Plane waves and time series

For plane waves we have

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = O(\beta, \epsilon), \quad \frac{\partial^2 \overline{u}}{\partial t^2} - \frac{\partial^2 \overline{u}}{\partial x^2} = O(\beta, \epsilon).$$

If we know a time series for η a time series for $\overline{u}$ may then be obtained from a modified version of (18)

$$\overline{u} - \frac{\beta}{6}\frac{\partial^2 \overline{u}}{\partial t^2} = \eta - \frac{\epsilon}{4}\eta^2 + O(\beta^2, \beta\epsilon). \tag{32}$$

This equation must be solved implicitly with boundary conditions at t_a and t_b, say. The homogenous solution of (32) is

$$\overline{u}_H = Ae^{\sqrt{\frac{\beta}{6}}t} + Be^{-\sqrt{\frac{\beta}{6}}t},$$

where A and B are constants. The boundary value problem clearly has a boundary value nature. In particular, if the interval length, $t_b - t_a$, is large compared to $\sqrt{\beta}$, the boundary values will not influence the solution in the major, inner part of the interval. Or, to put it differently, any choice of reasonable boundary conditions will give the same inner solution.

When $\overline{u}$ is found from (32) we use the same equation to obtain the double derivative of $\overline{u}$. This is a much more robust procedure than numerical derivation of $\overline{u}$. Taking into account only the leading order terms (24) and (23) simplify to

$$\begin{aligned} u_s &= \overline{u} - \tfrac{\beta}{3}\overline{u}_{tt}, \\ u &= u_s - \beta\{(\epsilon\eta - z) - \tfrac{1}{2}(\epsilon\eta - z)^2\}\overline{u}_{tt}, \end{aligned}$$

where indices indicate derivations. *this formula must be checked with Jensen et al (2003), JFM.*