Solitary waves; MEK4320.

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History

Solitary wave; literally wave with a single crest
currently concept also includes permanent shape

J. Scott Russel (1834): Observations in canals, experiments:

$$c = \sqrt{g(h+A)}$$

where g is the acceleration of gravity, h the equilibrium depth A the amplitude.

J. Boussinesq (1871): First theoretical description.

1960-1980: Solitons and their properties are in fashion in physics –
"Collision properties" (particle analogy)
Generation from initial conditions

The KdV solitary wave solution.

The KdV equation (length scale = equilibrium depth)

$$\eta_t + (1 + \frac{3}{2}\eta)\eta_x + \frac{1}{6}\eta_{xxx} = 0 \qquad (1)$$

Solitary wave of permanent form

$$\eta(x,t) = \alpha\, Y(x - ct),$$

where $\max Y = Y(0) = 1$ and $Y(\xi), Y'(\xi)... \to 0$ når $\xi \to \pm\infty$.
α is amplitude/depth; length is limited.
The wave celerity, c, must be determined.
Substitution into (1)

$$-cY' + (1 + \frac{3}{2}\alpha Y)Y' + \frac{1}{6}Y''' = 0,$$

Integration yields

$$(1-c)Y + \frac{3}{4}\alpha Y^2 + \frac{1}{6}Y'' = K_1.$$

$Y(\xi)... \to 0$ when $\xi \to \pm\infty$ implies $K_1 = 0$.
Integrating factor Y'

$$(1-c)YY' + \frac{3}{4}\alpha Y^2 Y' + \frac{1}{6}Y''Y' = 0,$$

and

$$\frac{1}{2}(1-c)Y^2 + \frac{1}{4}\alpha Y^3 + \frac{1}{12}\left(Y'\right)^2 = K_2.$$

vanishing Y, Y' in infinity $\Rightarrow K_2 = 0$. Furthermore, $\max Y = Y(0) = 1 \Rightarrow c = 1 + \frac{1}{2}\alpha$. Since $0 < Y < 1$ we may write

$$Y' = \pm Y\sqrt{3\alpha(1-Y)}.$$

This is a separable, first order ODE.

Discussion of sign

- $\xi \to \infty$: $Y \to 0$; must have $Y' = -Y\sqrt{3\alpha} \Rightarrow$
  $Y = \mathrm{const.} \times e^{-\sqrt{3\alpha}\xi}$.
- $\xi \to -\infty$: $Y \to 0$; must have $Y' = +Y\sqrt{3\alpha} \Rightarrow$
  $Y = \mathrm{const.} \times e^{+\sqrt{3\alpha}\xi}$.
- Only local extreme when $Y = 1 \Rightarrow$ solution must grow from $\xi = -\infty$ to $\xi = 0$, then decrease to $\xi = \infty$.

$+$ in separated equation for $\xi < 0$, $-$ for $\xi > 0$. Solutions readily obtained and combined.

The KdV solitary wave solution

$$\eta = \alpha\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{3\alpha}(x-ct)\right), \quad c = (1+\frac{1}{2}\alpha). \qquad (2)$$

where $\mathrm{sech} \equiv 1/\cosh$.

Properties

- Shape: single positive hump.
- Speed is slightly larger than linear shallow water speed (which is unity in present scaling)
- Speed increases linearly with amplitude (approximation valid within KdV theory)
- Length of hump inversely proportional to $1/\sqrt{\alpha}$.
  Higher solitary waves are shorter; consequence of balance of nonlinearity and dispersion.

More comments

- (2) exact only for the KdV equation
- Solitary wave solutions exist for all consistent Boussinesq-type equations, but generally not in explicit, closed form.
- The solitary wave requires combination of dispersion and nonlinearity.
- Full potential theory: perturbation techniques or numerical solutions
  maximum amplitude $A = 0.83h$, stability limits $A < 0.78h$ (longitudinal), $A < 0.72h$ (span-wise)
- (2) is leading order approximation for all solitary solutions for gravity surface waves.
- Solitary waves do exist also for other physical systems.

Perturbation solutions of full potential theory*

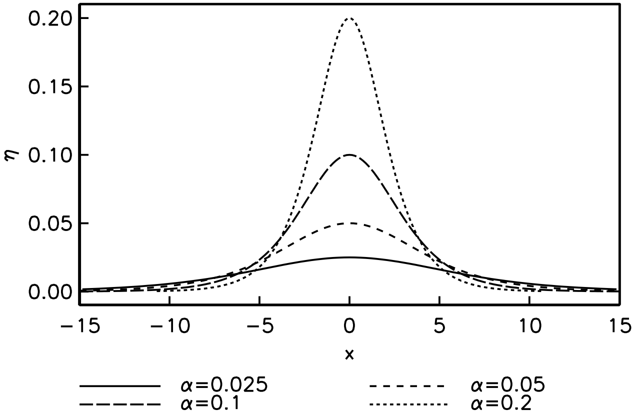
Still: the equilibrium depth is length scale, the shallow water celerity is velocity scale.
After Grimshaw (1971) and Fenton (1972).

$$\eta(x,t) = \alpha\, F(\kappa(x-ct)), \quad u = ...$$

where F is maximum 1. Perturbation series in α

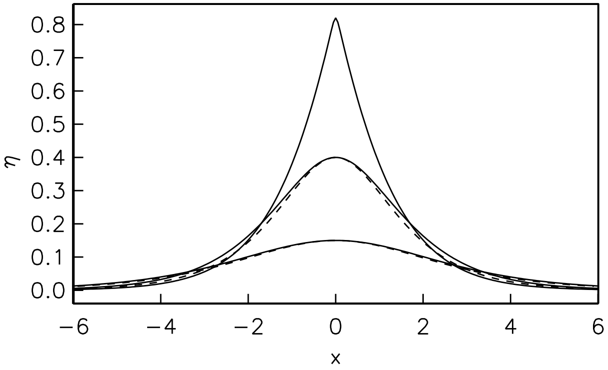
$$\begin{aligned}
\eta = &\quad \alpha s^2 - \alpha^2 \tfrac{3}{4} s^2 p^2 + \alpha^3 (\tfrac{5}{8} - \tfrac{101}{80} s^2) s^2 p^2 + O(\alpha^4) \\
c^2 = &\quad 1 + \alpha + \tfrac{1}{20}\alpha^2 - \tfrac{3}{70}\alpha^3 + O(\alpha^4)) \\
\kappa = &\quad \tfrac{1}{2}(3\alpha)^{\frac{1}{2}} (1 - \tfrac{5}{8}\alpha + \tfrac{71}{128}\alpha^2 + O(\alpha^3))
\end{aligned}$$

where $s = \mathrm{sech}(\kappa(x-ct))$ and $p = \tanh(\kappa(x-ct))$.
Corresponding series for velocities.
Third order solution becomes inaccurate for large α.



Solitary wave profiles from the standard Boussinesq equation

Length decreases with amplitude.



Boussinesq solitary waves (dashed) and full potential theory, numerical, (solid line) for $\alpha = 0.15, 0.4$. Potential solution for $\alpha = 0.82$ near maximum; crest close to cusp with 120° opening.

"Collision" of solitary waves $\Rightarrow$ *soliton*.

Evolution of solitary waves from initial conditions

Collision properties

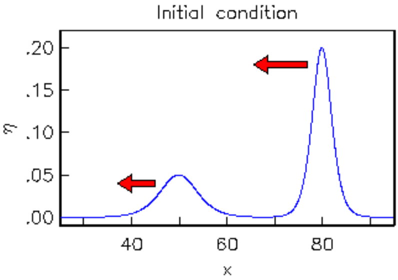
Amplitudes α_1 and α_2; same direction of wave advance.
Definition $f_i = e^{-\sqrt{3\alpha_i}(x - \hat{x}_i - (1+\frac{1}{2}\alpha_i)t)}$ where $\hat{x}_i$ are reference positions

Exact interaction solution of the KdV equation $(\beta_i = \sqrt{\alpha_i})$

$$\frac{\eta}{4} = \frac{\beta_1^2 f_1 + \beta_2^2 f_2 + 2(\beta_1 - \beta_2)^2 f_1 f_2 + \left(\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1}\right)^2 (\beta_2^2 f_1^2 f_2 + \beta_1^2 f_1 f_2^2)}{\left(1 + f_1 + f_2 + \left(\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1}\right)^2 f_1 f_2\right)^2}$$

- Analysis for $t \to \pm\infty$: solitary waves of original amplitudes re-emerges after interaction
- This analogy with percussion of particles motivates the *soliton* denotation.
- Exact collision properties shown only for KdV equation; or leading order behaviours in perturbation expansions.

Interaction; large difference amplitude

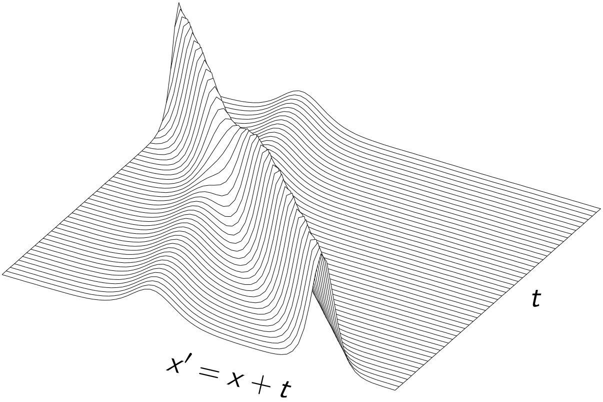


$\alpha_1 = 0.05$, $\alpha_2 = 0.2$

- Propagation to the left. Larger wave overtakes the smaller one.
- Substantial difference in celerity $\Rightarrow$ interaction of moderate duration $\Rightarrow$ limited for nonlinear interactions between waves.

Weak interaction

$\alpha_1 = 0.05$ and $\alpha_2 = 0.2$

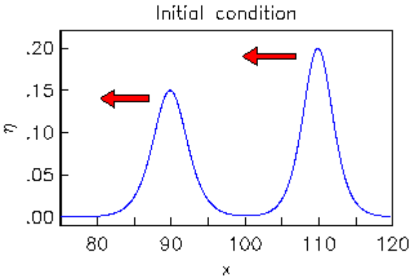


Soliton interactions

Large amplitude: phase shift in direction of wave advance

Small amplitude: phase shift in opposite direction; not as expected from a nonlinear interaction.

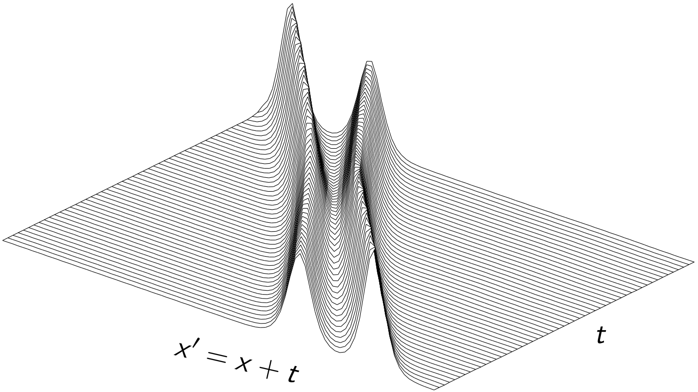
Interaction; small difference in amplitude



$\alpha_1 = 0.15$, $\alpha_2 = 0.2$

- Wave advance to the left. Large wave overtakes the small one.
- Small difference in wave celerity $\Rightarrow$ longer time for nonlinear effects to influence the interaction.

$\alpha_1 = 0.15$ and $\alpha_2 = 0.2$



Soliton interactions

Waves do not pass through each other; they exchange amplitudes

Analogy to particles in elastic collisions$\Rightarrow$ soliton

Generation from initial elevation

Inverse scattering theory $\Rightarrow$ a few exact solutions of the KdV equation.
Early numerical solutions.

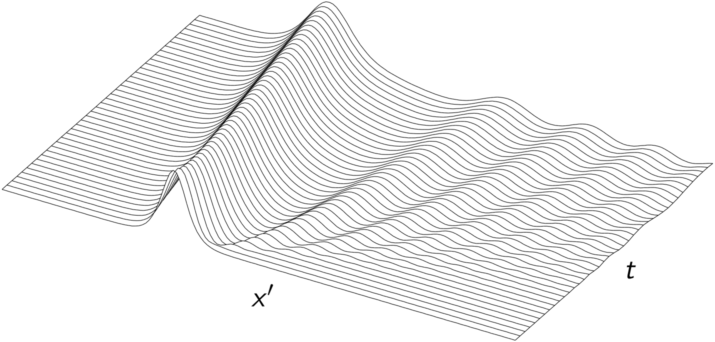
In general: net initial elevation $\Rightarrow$ at least one soliton produced.

Initial conditions

$$\eta(x,0) = \alpha \operatorname{sech}^2\left(\frac{4.15}{L}(x - x_0)\right)$$

NB: not soliton profile for general α and L.
Exact solutions do exist for this particular initial shape, but derivation is involved.
Instead: numerical simulations.

Example; $\alpha = 0.1,\ L = 10$

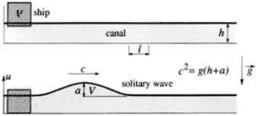


Surface depicted in moving frame $x' = x - ct$.
$\alpha = 0.1,\ L = 10 \Rightarrow 1$ soliton $+$ wave train.

Solitary waves rapidly produced from initial elevations when amplitudes are high.
More numerical experiments in the mandatory assignment.

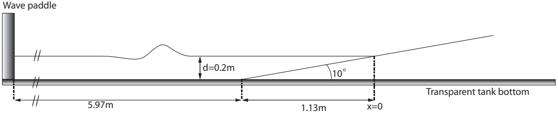
Solitary waves in reality

Russells' experiment (1834)



Rough generation; a solitary wave still emerges.

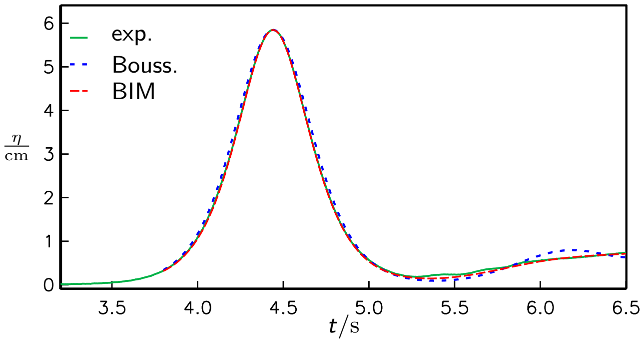
Hydrodynamic Laboratory UiO



Hydraulic piston type paddle. Solitary waves employed in a number of experiments since 1979.

Configuration shown used to investigate effects viscous boundary layers for solitary waves in wave tanks. Generated solitary wave on next slide.

Time series from acoustic probe



(Phys. Fluids **25***, 012102 (2013), Master Thesis by E. Lindstrøm)*
$\alpha = 0.295$, $H = 20\,\mathrm{cm}$, propagation distance $\sim 25H$.

- Full potential theory (BIM) and matches very closely.
- Larger, but still small, deviation for Boussinesq type model (Bouss). Boussinesq model is different from the standard one.
- Later part of time series influenced by reflection from beach in front of gauge, as well as trailing waves from the generation.

River bore; example

The Mascaret

-Tidal phenomenon in the tributaries of the Gironde estuary (1); the Dordogne (2) and the Garonne rivers (3).

-Tidal wave is concentrated due to narrowing bathymetry and steepened due to nonlinearity.

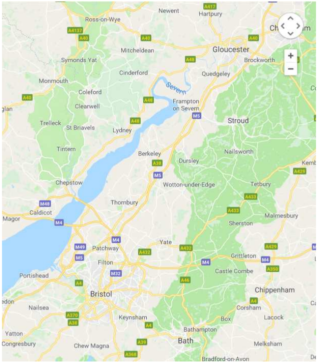
-Strongest around the spring and autumn equinox.

-Also in Severn (Bristol channel), Trent, the Seine, Indus, Bramhmaputra, Qiantang, Amazon, Orinoco and others.

The front of the Mascaret in Dordogne, below Libourne, September 2017.
Undular bore propagating upstream at the turn of the tide.
Series of solitary waves, affected by river-banks, shoals and muddy bottom.

The Mascaret. Side view.

Undular bore in the river Severn

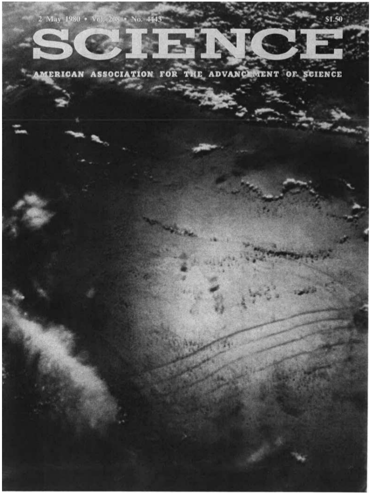


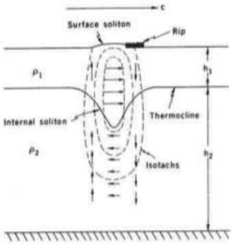
Left: Google maps, right: picture by P. Rash.

Possible tsunami-canal-bore, Japan 2016

Image from the Miyagi police 22 Nov. 2016.
Weak tsunami from 7.4M earthquake outside Japan.

Internal undular bore in the Indian Ocean





Osborne & Burch, 1980

Internal undular bore observed in the field.

Presumably caused by tidal current and bathymetry.

Left: Surface signature on satellite photo.

Summary

- Solitary wave: single crested wave of permanent form and speed.
- Requires both nonlinearity and dispersion. Length decreases with height.
- Simple solution for the KdV equation.
- Solitary wave solutions exist also for Boussinesq eq. and full potential theory. Less simple.
- Collision properties have motivated the name soliton. Not exact beyond KdV theory.
- Solitary waves readily emerges from initial conditions of large large amplitude.
- Consequence of preceding point: solitary waves are easy to generate in a laboratory. Important as a reference wave type in experimental and theoretical undertakings.
- Solitary waves are less abundant in nature. Trains of solitary waves, undular bores, do evolve in relation to tides and tsunamis.