Extra problems for Mek 4320

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Ex. 1 ***Shallow water waves on a two-layer fluid.*** A fluid consists of two layers, with different densities. We assume that they do not mix, which imply that there is a sharp interface between the layers with a jump in the density. We ignore capillary effects, compressibility and viscosity. The equilibrium thicknesses of the upper and lower layers are h_1 (constant.) and $h_2(x)$, respectively. We assume motion in one horizontal direction (x) and the vertical (z) only. The lower layer, with density ρ_2, is then confined to $-(h_1 + h_2) \leq z \leq -h_1 + \eta_2$, where $\eta_2(x, t)$ is the displacement of interface between the layers. The upper layer with density ρ_1, is then confined to $-h_1 + \eta_2 \leq z \leq \eta_1$, where $\eta_1(x, t)$ is the surface elevation. Make a sketch of the fluid configuration.

Furthermore, we assume long gravity waves and a hydrostatic pressure distribution. The horizontal velocities of the two layers are u_1 and u_2, respectively, and we assume that they are vertically uniform within each layer at $t = 0$. The acceleration of gravity is g and the relative density difference between the layers is $\epsilon = \frac{\rho_2 - \rho_1}{\rho_2}$. The motion is then governed by the equations

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = -g \frac{\partial \eta_1}{\partial x} \tag{1}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = -g(1-\epsilon) \frac{\partial \eta_1}{\partial x} - g\epsilon \frac{\partial \eta_2}{\partial x}, \tag{2}$$

$$\frac{\partial}{\partial t}(\eta_1 - \eta_2) = -\frac{\partial}{\partial x}\left((h_1 + \eta_1 - \eta_2)u_1\right) \tag{3}$$

$$\frac{\partial \eta_2}{\partial t} = -\frac{\partial}{\partial x}\left((h_2 + \eta_2)u_2\right) \tag{4}$$

a) Derive these equations. You must start with the Eulers equations of motion in each layer and then proceed much as is done in the Compendium for the NLSW equations. It may be a good idea first to repeat the derivation of the Compendium with only x and t.

b) Henceforth, we assume a flat bottom h_2 =constant. Find the harmonic modes for the linearized equation set. Show that there are two classes of modes and discuss their properties.

c) The equation set does also inherit solution in the form of pulses with permanent shape. Again there are two modes, corresponding to the modes in the previous sub-problem.

Ex. 2 ***Wave breaking.*** A pulse is propagating in the positive x-direction. Initially it has the form

$$\eta = \begin{cases} 0 & \text{when} \quad x < -L \\ A(x+L)/L & \text{when} \quad -L < x < 0 \\ A(L-x)/L & \text{when} \quad 0 < x < L \\ 0 & \text{when} \quad L < x \end{cases} \tag{5}$$

The equilibrium depth is h and we assume weakly nonlinear shallow water theory (only leading nonlinearities are included). When does the wave break?

Ex. 3 ***Wave source in shallow water.*** An equilibrium depth is defined by the profile

$$h(x) = \begin{cases} h_0 x/L & \text{for} \quad x < L \\ h_0 & \text{for} \quad x \geq L \end{cases} \tag{6}$$

However, we will have wave propagation in the xy plane. At a location with $h = h_m < h_0$ a wave maker (buoy) produces circular-symmetric waves with frequency ω. We assume that all wave energy that reaches the shore $h = 0$ is absorbed. Employ ray theory for shallow water waves to estimate the fraction of the energy from the wave maker that escapes into deep water ($h = h_0$) Briefly discuss the modifications to this solution when we invoke finite depth. How much of the energy will then escape if $h_0 \to \infty$.

Ex. 4 ***Stokes waves and the Klein-Gordon equation.*** A nonlinear Klein-Gordon equation is written

$$u^*_{t^*t^*} - c_0^2 u^*_{x^*x^*} + qu^* - r(u^*)^3 = 0. \tag{7}$$

Scale the equation to obtain

$$u_{tt} - u_{xx} + u - \epsilon u^3 = 0 \tag{8}$$

Assume a periodic wave of permanent form. Adapt the scaling such that $u = O(1)$ and ϵ is small. Find the first two terms in a “Stokes Wave” solution in this case.

Ex. 5 ***Internal Waves.*** In a stratified fluid we have the density

$$\rho = \begin{cases} \rho_0 - \Delta\rho & \text{when} \quad z < -B \\ \rho_0 + \dfrac{\Delta\rho}{B} z & \text{when} \quad -B \leq z \leq B \\ \rho_0 + \Delta\rho & \text{when} \quad B < z \end{cases} \tag{9}$$

where the z-axis is pointing vertically downwards.

a) We wish to approximate the buoyancy frequency (Brunt-Väisälä), N, by a constant for $-B < z < B$. When is this justified ?

b) Internal modes due to the stratification is governed by

$$\hat{w}'' + k^2 \left(\frac{N^2}{\omega^2} - 1 \right) \hat{w} = 0, \tag{10}$$

where $\hat{w}$ defines the vertical variation of the vertical velocity component, k is the wave number in the horizontal direction and ω is the frequency.

Show that, for the stratification in (9), modes are given by (10) combined with the boundary conditions

$$\left.\begin{array}{ll} \hat{w}' - k\hat{w} = 0 & \text{at} \quad z = -B \\ \hat{w}' + k\hat{w} = 0 & \text{at} \quad z = B \end{array}\right\} \tag{11}$$

c) Solve the eigenvalue problem from the preceding point to obtain implicit expressions for $\beta = \sqrt{\frac{N^2}{\omega^2} - 1}$. Show that there are two groups of modes, with $\hat{w}$ that are symmetric and anti-symmetric with respect to $z = 0$, respectively.

d) The dispersion relations may be represented as the intersections between a functions that are linear and periodic in β. Show this and explain why the symmetric modes have 1, 3, 5 etc. extremes in the interval $-B < z < B$, whereas the anti-symmetric modes have 2, 4, 6 ... Depict $\hat{w}$ for the lowest two modes of each kind.

e) Instead of an unbounded fluid we now assume a semi-unbounded one with a rigid lid at $z = -H$. Outline briefly how to find the modes in this case.

Ex. 6 ***Fourier transform and stationary phase.*** The linearized KdV equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{c_0 H^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{12}$$

is to be solved, subjected to the initial condition

$$\eta(x, 0) \equiv \eta_0(x) = \frac{Q}{2L\sqrt{\pi}} e^{-\left(\frac{x}{2L}\right)^2}. \tag{13}$$

a) Apply the Fourier transform to the equations and show that

$$\eta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\eta_0}(k) e^{i(kx - \omega(k)t)} dk,$$

where $\tilde{\eta_0}$ is the transform of η_0. What is the relation between k and ω?

b) Show that the solution may be recast into

$$\eta(x, t) = \frac{1}{\pi} \mathrm{Re} \left\{ \int_{0}^{\infty} \tilde{\eta_0}(k) e^{i(kx - \omega(k)t)} dk \right\}. \tag{14}$$

This result differs from the corresponding ones from section 2.7 in the Compendium and from the slides on the Klein-Gordon example by a factor 2. Why ?

c) Let $L \to 0$ in (13). Use results from the Compendium to find the exact inversion of (14), meaning find $\eta(x, t)$, in this case.

d) Apply the stationary phase method to (14).

Ex. 7 ***An exact solution of the LSW equation for a trapped mode.*** Elimination of the velocities from the usual LSW equations yields

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla \cdot (gH(x, y) \nabla \eta) = 0, \tag{15}$$

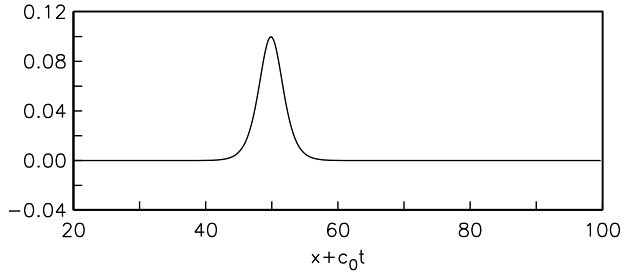
where H is the equilibrium depth. In the start of section 4.1, in the Compendium, a trapped mode, which is an exact solution of linear potential theory is given. In this case $h = \tan \theta x$.

a) Find the corresponding exact solution for (15)

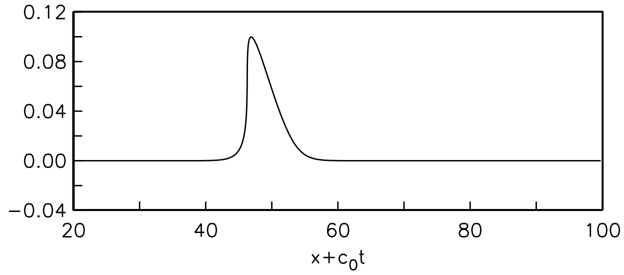
b) When is the solution from the previous subproblem a good approximation to that from the Compendium?

Ex. 8 ***Guess the equation.***

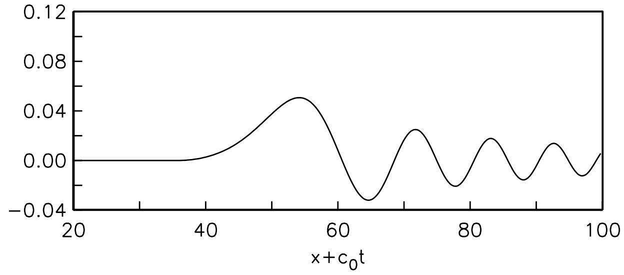
(a): Which equation(s) give(s) this surface?



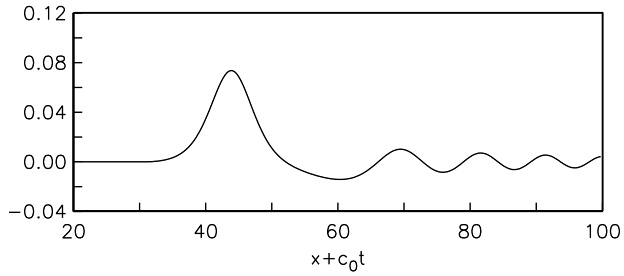
(b): and this ? ($t = 20$)



(c): and this, then ? .



(d): and, finally, this ?



The figure shows the evolution of the surface elevation from a symmetric initial wave, moving towards the left, as obtained by the application of different equations. The plots are normalized with respect to equilibrium depth and the initial height and length are 0.1 and 10, respectively. The coordinate system moves in the negative x direction with the shallow water speed. At $t = 0$ the crest is at $x' = x + c_0 t = 50$. Apart from in diagram (b) the elapsed time equals 200 units, meaning that 200 depth would have been covered when moving with the linear shallow water speed. Match panels and possible equations (there may be more than on set for each panel).

Ex. 9 ***Deep and shallow water.*** Double numbers within parantheses, such as (2.14), refer to equations in the Compendium.

a) Assume that $kH \to \infty$. Show (in detail) how (2.14) and (2.15) simplifies to

$$\phi = -ace^{kz}\cos(kx - \omega t), \quad u = a\omega e^{kz}\sin(kx - \omega t), \quad w = -a\omega e^{kz}\cos(kx - \omega t).$$

b) For the solution of the preceding point: as we move downwards from the surface; at which value of z/λ is the velocity reduced to 1% of the value at the surface?

c) We now assume that kH, and hence $|kz|$ are small. Expand (2.14) and (2.15) in kH and kz, keeping the terms necessary to obtain an expression

$$\phi = -\frac{ac_0}{kH}\left(1 + k^2 q(z,H) + O(k^4)\right)\cos\chi, \quad \omega = c_0 k(1 - \frac{1}{6}(kH)^2 + O(k^4)), \tag{16}$$

where $c_0 = \sqrt{gH}$, $\chi = kx - \omega t$ and q is a second order polynomial of z and H. Find also the corresponding velocities.

d) Employ the z component of the equation of motion to obtain an expression for the pressure

$$p = \rho g(\eta - z) - \rho\int_0^z \frac{\partial w}{\partial t}dz. \tag{17}$$

Subsitute from (16) and show that the pressure is approximately hydrostatic, in the sense that $p \approx \rho g(\eta - z)$, when $kH \to 0$. An important quantity is the horizontal pressure gradient, because this relates to the horizontal acceleration. Find a measure of the relative deviation in $\partial p/\partial x$ from that of the hydrostatic pressure when $\lambda = 4\pi H$ by taking the $k^2 q$ term in (16) into consideration.

e) Correspondingly, show that u is independent of z when $kH \to 0$ and find the total variation of u in a vertical column. Does the result relate to that of the preceding point ?

f) Show that $w \to 0$ as $kH \to 0$. What is the consequence for the particle trajectories?

g) The velocity potential should fulfill

$$\nabla^2\phi = 0.$$

In what sense does the approximation (16) satisfy this equation ?

Ex. 10 ***Nonlinear terms.***

We want to investigate the omission of nonlinear terms in the equations used for the harmonic wave mode given by (2.14) and (2.15). To this end we insert these expressions into the nonlinear versions of the equations and assess the magnitude of the terms we omit. Even though this procedure neither yields qualitative error estimates or a rigorous proof for the consistency of linearization it does give an indication of validity.

a) Assume infinite depth $kH \to \infty$. Insert the harmonic mode in the kinematic boundary condition (2.5) and expand it to find all quadratic terms in a. Compare the maximum size of the nonlinear terms to the maximum linear ones. Identify a parameter that measures the significance of nonlinear terms.

b) Apply the same analysis to (2.7) versus (2.9).

c) We now assume long waves $kH \to 0$. Look at the size of the nonlinear terms of (2.5), as compared to the linear ones, for this case. What is the parameter that measures significance of nonlinearity now ?

Ex. 11 ***Klein-Gordon's equation and stationary phase.***
The equation of Klein-Gordon, expressed in some set of non-dimensional variables, reads:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + q\eta = 0, \tag{18}$$

for $t > 0$ and $-\infty < x < \infty$. Waves are generated from the initial condition

$$\eta(x, 0) = e^{-\left(\frac{x}{L}\right)^2}; \quad \frac{\partial \eta(x, 0)}{\partial t} = 0, \tag{19}$$

a) Explain why the solution may depend on q and L only in the combination qL^2. However, keep the equation in the form (18). Subject the equation and initial conditions to the Fourier transform and show that the solution, for large positive x, may be obtained through the reverse transform

$$\eta(x, t) = \frac{1}{2\pi} \Re \int_0^\infty \tilde{\eta_0}(k) e^{i(kx - \omega(k)t)} dk. \tag{20}$$

b) Invert (20) by means of the stationary phase approximation. Discuss the solution: which waves (short or long) arrives first at a given point ? How does the wave heights evolve in time.

c) (Optional, used as part of mandatory assignment) Set $q = 0.1$ and $L = 10$. On the web cite for notes on MEK4320 you will find numerical solutions for a few selected times. Invoke MatLab, or another suitable tool, to compare your stationary phase solution to the numerical one. Both the plots and authentic snippet must be included in your paper.

Ex. 12 ***Periodic waves on an inclined plane.*** The case of linear variations of depth, $h = \alpha x$, and constant frequency are treated at the end of section 2.10 in the compendium.

a) *Use of ray theory.*
Reproduce the result ($\kappa = \omega^2/\alpha g$)

$$\eta = a_1(\kappa x)^{-\frac{1}{4}} \cos(2\sqrt{\kappa x} - \omega t + \delta_1) + a_2(\kappa x)^{-\frac{1}{4}} \cos(2\sqrt{\kappa x} + \omega t + \delta_2),$$

from section 2.10 by means of ray theory and the transport equation.

b) *Requirement for asymptotics.*
For the use of ray theory we must require $\lambda k_x/k \ll 1$. Explain this. On the other hand, the use of asymptotic approximations for the Bessel functions in section 2.10 requires $\sqrt{\kappa x} \gg 1$. Show that these two requirements are the same.

Ex. 13 ***Energy and the LSW equation.*** When indices are used for differentiation the LSW equations read

$$\eta_t = -(hu)_x, \quad u_t = -g\eta_x. \tag{21}$$

In this problem we will study energy within the framework of linear shallow water theory. Hence, we shall use properties like hydrostatic pressure and vertically uniform u whenever appropriate.

a) Energy denity and flux.
Show, by direct calculation, that the energy density and flux become, respectively,

$$E = \frac{1}{2}\rho h u^2 + \frac{1}{2}\rho g \eta^2, \quad F = \rho g h \eta u.$$

b) The wave mode.
Assume constant depth and find a wave mode solution to (21). To this end assume a form

$$\eta = A\cos(kx - \omega t), \quad u = U\cos(kx - \omega t).$$

c) Averaged densities for mode.
Use the solution of the previous point to show that

$$\overline{E} = \frac{1}{2}\rho g A^2, \quad \overline{F} = \sqrt{gh}\,\overline{E}.$$

d) Fulfillment of energy equation.
The energy equation reads

$$E_t + F_x = 0. \tag{22}$$

Show that is fulfilled by invoking (21).

e) Derivation of the energy equation from the PDE's.
Start with (21) and derive (22).