Mek4100
Two-scale perturbation methods

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Motivation

- A number of problems inherit several temporal or spatial scales.
- Example: Boundary layer problem; albeit here the rapid scale is only present in the boundary layer.
  Allows different expansions in boundary layer and outer region, followed by matching.
- Several global scales $\Rightarrow$ a new method is required.
- Linear, homogeneous equations: WKB(J) is an efficient alternative.
- Most general method: multiple scale expansions.

Example 1: damped oscillation

ODE with initial conditions

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0; \quad y(0) = 1, \quad \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0. \tag{1}$$

ϵ – small parameter.

Unpertubed problem: The linear, harmonic oscillator.
Physical interpretation of the ϵ term: weak resistance force proportional to the velocity.

Direct (naive) perturbation

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \Rightarrow$$

$O(\epsilon^0)$

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0,$$

$$y_0(0) = 1, \quad \frac{\mathrm{d}y_0(0)}{\mathrm{d}t} = 0.$$

Solution:

$$y_0 = \cos t$$

$O(\epsilon^1)$

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -\frac{\mathrm{d}y_0}{\mathrm{d}t} = \sin t,$$

$$y_1(0) = \frac{\mathrm{d}y_1(0)}{\mathrm{d}t} = 0.$$

Mathematical resonance (secular terms) $\Rightarrow$

$$y_1 = \frac{1}{2}(\sin t - t\cos t).$$

Then $\epsilon t \sim \Rightarrow \epsilon y_1 \sim y_0$; breakdown.

Breakdown due to

Effect of small resistance accumulates. Exact solution (presented later) implies $y \to 0$ as $t \to \infty$, while y_0 is periodic.
Hence, $\epsilon y_1 \approx y - y_0$ must be comparable to y_0
Poincare-Lindsted: not applicable, why ?

Introduction of a slow time variable

New time

$$\tau = \epsilon t,$$

is introduced in addition to the fast time t. Hence

$$y = y(t, \tau),$$

which is defined in in the quadrant $[t \geq 0] \times [\tau \geq 0]$ as if t and τ were independent.
Much redundancy: only the line $\tau = \epsilon t$ has direct significance.

Temporal derivatives transform

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

The transformed problem

Damped oscillation equation in terms of t and τ yields PDE

$$\frac{\partial^2 y}{\partial t^2} + y + \epsilon(2\frac{\partial^2 y}{\partial t \partial \tau} + \frac{\partial y}{\partial t}) + \epsilon^2(\frac{\partial^2 y}{\partial \tau^2} + \frac{\partial y}{\partial \tau}) = 0;$$

$$y(0,0) = 1, \quad \frac{\partial y(0,0)}{\partial t} + \epsilon\frac{\partial y(0,0)}{\partial \tau} = 0.$$

Considerations

1. t and τ **are** not “really” independent, but
   solution of the PDE provides solution for ODE.
   Physical effects behind scales may sometimes be conceived as independent.
2. Anyway, an ODE for a PDE; good bargain?
   Yes, as long as we can solve the PDE.

Two-scale perturbation

The series

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \ldots,$$

All terms must remain finite or, rather, vanish in time.

$O(\epsilon^0)$

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0; \quad y_0(0,0) = 1, \quad \frac{\partial y_0(0,0)}{\partial t} = 0.$$

The solution for y_0 becomes

$$y_0 = A_0(\tau)\cos t + B_0(\tau)\sin t, \quad A_0(0) = 1, \quad B_0(0) = 0$$

A_0, B_0 must be determined to the next order.

(ϵ^1)

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= -\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} \\
&= (A_0 + 2\frac{\mathrm{d}A_0}{\mathrm{d}\tau}) \sin t - (B_0 + 2\frac{\mathrm{d}B_0}{\mathrm{d}\tau}) \cos t;
\end{aligned}$$

$$y_1(0,0) = 0, \quad \frac{\partial y_1(0,0)}{\partial t} = -\frac{\partial y_0(0,0)}{\partial \tau}.$$

Avoid (secular) terms that grow in $t \Rightarrow$

$$A_0 + 2\frac{\mathrm{d}A_0}{\mathrm{d}\tau} = B_0 + 2\frac{\mathrm{d}B_0}{\mathrm{d}\tau} = 0.$$

$O(\epsilon^1)$, cont.

Initial conditions for A_0, B_0 $\Rightarrow$

$$A_0 = e^{-\frac{1}{2}\tau}, \quad B_0 = 0.$$

No particular solution to $O(\epsilon)$:

$$y_1 = A_1(\tau)\cos t + B_1(\tau)\sin t, \quad A_1(0) = 0,\ B_1(0) = \frac{1}{2}$$

Complete solution

$$y = e^{-\frac{1}{2}\epsilon t}\cos t + \epsilon(A_1(\epsilon t)\cos t + B_1(\epsilon t)\sin t) + O(\epsilon^2).$$

! $\boldsymbol{\epsilon^2}$: secular terms may appear; can be eliminated by in troducing $\tau_1 = \epsilon^2 t$. We will not pursue this in MEK4100.

Comparing with exact solution

Exact

Original second order ODE is linear and has constant coefficients. Solution readily found:

$$y = e^{-\frac{1}{2}\epsilon t}\left(\cos \omega t + \frac{\epsilon}{2\omega}\sin \omega t\right),$$

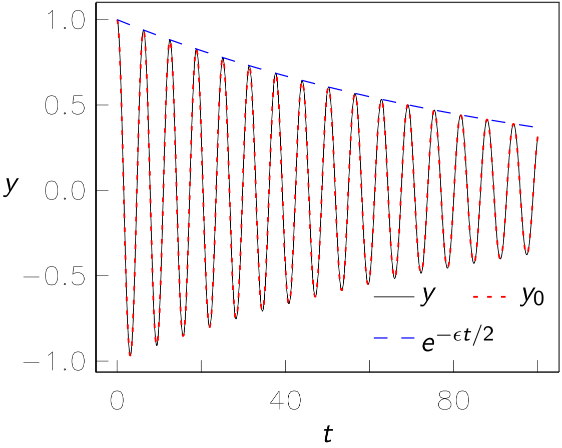
where $\omega = \sqrt{1 - \frac{1}{4}\epsilon^2} = 1 + O(\epsilon)^2$.

Two-scale approximation

$$y = e^{-\frac{1}{2}\epsilon t}\cos t + \epsilon(A_1(\epsilon t)\cos t + B_1(\epsilon t)\sin t) + O(\epsilon^2).$$

The two solutions agree, including the initial condition $B_1(0) = \frac{1}{2}$.

Graphical comparison, $\epsilon = 0.02$.



Example 2: nonlinear oscillations

Scaled equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x - \frac{\epsilon}{6}x^3 = 0, \quad x(0) = 1, \frac{\mathrm{d}x(0)}{\mathrm{d}t} = 0.$$

Poincare-Lindstedt's method

We seek a periodic solution with frequency $\omega = \omega_0 + \epsilon\omega_1 + \dots$.
$\epsilon\omega_1 t$ may be regarded as a slow time compared to $\omega_0 t$.

Two-scale method

We regard ϵt a slow time scale that modulates the phase.
(The phase is, say, ωt.)

Two-scale expansion, nonlinear pendulum

$\tau = \epsilon t$, $x = x(t, \tau)$

Transformation: ODE $\Rightarrow$ PDE

$$\frac{\partial^2 x}{\partial t^2} + x + \epsilon \left(2 \frac{\partial^2 x}{\partial t \partial \tau} - \frac{1}{6} x^3 \right) + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} = 0;$$

$$x(0,0) = 1, \quad \frac{\partial x(0,0)}{\partial t} + \epsilon \frac{\partial x(0,0)}{\partial \tau} = 0.$$

Perturbation series

$$x = x_0(t,\tau) + \epsilon x_1(t,\tau) + \ldots,$$

ϵ^0

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0; \quad x_0(0,0) = 1, \quad \frac{\partial x_0(0,0)}{\partial t} = 0.$$

Exponential form $\Rightarrow$

$$x_0 = A_0(\tau)e^{it} + \overline{A}_0(\tau)e^{-it}, \quad A_0(0) = \frac{1}{2},$$

where $\overline{A_0}$ is the complex conjugate of A_0.

ϵ^1

$$
\begin{aligned}
\frac{\partial^2 x_1}{\partial t^2} + x_1 &= \frac{1}{6}x_0^3 - 2\frac{\partial^2 x_0}{\partial t \partial \tau} \\
&= \frac{1}{6}A_0^3 e^{3it} + \left(-2i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{1}{2}\overline{A}_0 A_0^2\right) e^{it} + \mathrm{c.c.},
\end{aligned}
$$

$$
x_1(0,0) = 0, \qquad \frac{\partial x_1(0,0)}{\partial t} = -\frac{\partial x_0(0,0)}{\partial \tau}, \tag{2}
$$

where $\mathrm{c.c.}$ indicates the addition of the complex conjugate.
Annihilation of secular terms$\Rightarrow$

$$
i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} - \frac{1}{4}\overline{A}_0 A_0^2 = 0.
$$

ϵ^1, cont.

From previous slide

$$i\frac{\mathrm{d}A_0}{\mathrm{d}\tau} - \frac{1}{4}\overline{A}_0 A_0^2 = 0.$$

Insertion of $A_0 = |A_0|e^{i\psi} \Rightarrow \Rightarrow$

$$\frac{\mathrm{d}|A_0|}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\frac{1}{4}|A_0|^2,$$

Initial condition $A_0(0) = \frac{1}{2} \Rightarrow |A_0|(0) = \frac{1}{2}, \quad \psi(0) = 0 \Rightarrow$

$$A_0 = \frac{1}{2}e^{-\frac{i}{16}\tau}$$

Then, initial conditions $\Rightarrow$

$$x_1 = -\frac{A_0^3}{48}e^{3it} + A_1(\tau)e^{it} + \mathrm{c.c.}, \quad A_1(0) = \frac{1}{384}$$

The two leading orders combined

$$
\begin{aligned}
x &= \frac{1}{2}e^{i(1-\frac{\epsilon}{16})t} - \frac{\epsilon}{384}e^{3i(1-\frac{\epsilon}{16})t} + \epsilon A_1(\tau)e^{it} + \mathrm{c.c.} + O(\epsilon^2) \\
&= \cos(1-\frac{\epsilon}{16})t - \frac{\epsilon}{192}\cos 3(1-\frac{\epsilon}{16})t \qquad (3) \\
&\quad +\epsilon a_1(\epsilon t)\cos t - \epsilon b_1(\epsilon t)\sin t + O(\epsilon^2),
\end{aligned}
$$

where $A_1 = a_1 + ib_1$.
Can be verified by Poincare-Lindsted's method.

Example 3: Pendulum with prescribed length variation

Conservation of angular momentum (around support)

$$\ell\ddot{\phi} + 2\dot{\ell}\dot{\phi} + g\phi = 0, \quad \phi(0) = \phi_0, \quad \dot{\phi}(0) = 0$$

ϕ=excursion, ℓ=length and the dot indicates derivation with respect to time.

Scaling

$$t = \sqrt{\frac{g}{\ell(0)}}t^*, \quad \gamma = \frac{\ell}{\ell(0)}, \quad \theta = \frac{\phi}{\phi_0}.$$

Slow scale, $\tau = \epsilon t$, describes change of γ (dimensionless length).

$$\gamma(\tau)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\epsilon\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta = 0.$$

Attempt: direct application of two-scale method

Excursion is function of both times $\theta = \theta(t, \tau)$.

Transformation $\Rightarrow$ PDE

$$\gamma(\tau)\frac{\partial^2\theta}{\partial t^2} + \theta + 2\epsilon\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial t} + \gamma\frac{\partial^2\theta}{\partial t\partial\tau}\right) + \epsilon^2\left(2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial\tau} + \gamma\frac{\partial^2\theta}{\partial\tau^2}\right) = 0.$$

Initial conditions

$$\theta(0,0) = 1, \quad \frac{\partial\theta(0,0)}{\partial t} + \epsilon\frac{\partial\theta(0,0)}{\partial\tau} = 0.$$

Expansion $\theta = \theta_0(t,\tau) + \epsilon\theta_1(t,\tau) + \ldots$

Direct application..

$O(\epsilon^0)$

$$\gamma\frac{\partial^2\theta_0}{\partial t^2}+\theta_0=0;\quad \theta_0(0)=1,\quad \frac{\partial\theta_0(0)}{\partial t}=0.$$

solution

$$\theta_0=A_0(\tau)e^{i\gamma^{-\frac{1}{2}}t}+\overline{A}_0e^{-i\gamma^{-\frac{1}{2}}t},\quad A_0(0)=\frac{1}{2},$$

Direct application..

$O(\epsilon^1)$

$$\gamma\frac{\partial^2\theta_1}{\partial t^2}+\theta_1=h_s;\quad \theta_1(0,0)=0,\quad \frac{\partial\theta_1(0,0)}{\partial t}=-\frac{\partial\theta_0}{\partial\tau},$$

where

$$\begin{aligned} h_s &= -2\gamma\frac{\partial^2\theta_0}{\partial t\partial\tau}-2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial t}\\ &= -\left(2i\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau}+i\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}+2tA_0\gamma^{-1}\frac{\partial\gamma}{\partial\tau}\right)e^{i\gamma^{-\frac{1}{2}}t}+\mathrm{c.c.}\end{aligned}$$

Linear appearance of t $\Rightarrow$ secular terms in θ_1. Cannot be removed since γ is function of τ, only.

Reason for failure

The fast scale (period) is non-constant; it varies with τ.

Modified two-scale method; variable fast scale

Variable time scale (scale is fast, but it's variation is slow)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \sigma(\tau).$$

Transformation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sigma\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \sigma^2\frac{\partial^2}{\partial T^2} + \epsilon(2\sigma\frac{\partial^2}{\partial T\partial \tau} + \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial}{\partial T}) + \epsilon^2\frac{\partial^2}{\partial \tau^2}.$$

We must choose (determine) σ as to avoid secular terms.
Perturbation series

$$\theta = \theta(T,\tau) + \epsilon\theta_1(T,\tau) + ...$$

PDE

$$\gamma\sigma^2\frac{\partial^2\theta}{\partial T^2}+\theta+\epsilon\left(2\sigma\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial T}+2\sigma\gamma\frac{\partial^2\theta}{\partial T\partial\tau}+\gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial T}\right)$$
$$+\epsilon^2\left(2\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial\tau}+\gamma\frac{\partial^2\theta}{\partial\tau^2}\right)\quad=0.$$

$O(\epsilon^0)$

$$\gamma\sigma^2\frac{\partial^2\theta_0}{\partial T^2}+\theta_0=0;\quad \theta_0(0,0)=1,\quad \frac{\partial\theta_0(0,0)}{\partial T}=0.$$

In previous attempt τ appeared explicitly in the exponent, which led to secular term to the next order.
This can be avoided by $\sigma=\gamma^{-\frac{1}{2}}\Rightarrow$

$$\theta_0=A_0(\tau)e^{iT}+\overline{A}_0(\tau)e^{-iT},\quad A_0(0)=\frac{1}{2},$$

$O(\epsilon^1)$

$$\frac{\partial^2\theta_1}{\partial T^2} + \theta_1 \equiv h_s,$$

where

$$\begin{aligned} h_s &= -2\sigma\gamma\frac{\partial^2\theta_0}{\partial T\partial\tau} - 2\sigma\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial T} - \gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial T} \\ &= -i\left(2\sigma\gamma\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + 2\sigma A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + \gamma\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}A_0\right)e^{iT} + \mathrm{c.c.} \\ &= -i\left(2\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{3}{2}\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)e^{iT} + \mathrm{c.c.} \end{aligned}$$

Annihilation of secular terms $\Rightarrow$ coefficient of e^{iT} is zero $\Rightarrow$ (ODE) for A_0.

$$2\gamma^{\frac{1}{2}}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} + \frac{3}{2}\gamma^{-\frac{1}{2}}A_0\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = 0$$

Separable equation (ODE) for A_0

$$\frac{1}{A_0}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} = -\frac{3}{4\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}$$

Integration and $A_0(0) = \frac{1}{2}$, $\gamma(0) = 1 \Rightarrow$

$$A_0 = \frac{1}{2}\gamma^{-\frac{3}{4}}$$

Physical note: wave action

Energy in pendulum motion :$E = E_s + E_\ell$
E_ℓ: potential energy due to change in ℓ
E_s: Energy due to the oscillations

$$E_s = \frac{1}{2} m\ell^2 \dot{\phi}^2 + mg\ell(1 - \cos\phi),$$

Small amplitude, scaling, invocation of two-scale solution $\theta_0 \Rightarrow$

$$E_s = 2mg\ell_0\phi_0^2\gamma A_0^2(1 + O(\epsilon)).$$

E_s is not constant, but

Wave action

$$\frac{E_s}{\omega} \approx \text{const.}, \quad \omega = \sqrt{\frac{g}{\ell}}$$

is constant; general result for oscillations and waves in a time dependent medium or on a current.