THE KLEIN-GORDON EQUATION AND STATIONARY PHASE.

Geir Pedersen

Department of Mathematics, UiO.

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The Klein-Gordon equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + q\eta = 0 \tag{1}$$

Initial condition:

$$\eta(x,0) = e^{-(\frac{x}{L})^2}, \quad \frac{\partial}{\partial t}\eta(x,0) = 0 \tag{2}$$

The solution will depend on q and L in the combination qL^2 (Easily demonstrated by rescaling). However, we keep the equation on the given form. In the plots we always have $L = 10$.

Solution methods

1. Finite differences.
2. Stationary phase
3. FFT (presented elsewhere)

Fourier transform + stationary phase

1. *Fourier transform*
   Linear equations, constant coefficients, spatially confined initial condition $\Rightarrow$ Fourier integral

$$\eta(x,t) = \frac{1}{2\pi}\Re\int_0^\infty \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k$$

   where ω and k fulfill the dispersion relation.

2. *Stationary phase*
   For large x and t dominant contributions to the Fourier integral comes from the vicinity of the stationary point. An approximate integrand $\Rightarrow$ explicit asymptotic solution for large x and t.

The Fourier transform

Relations between $f(x)$ and $\tilde{f}(k)$

$$\tilde{f}(k) = \int\limits_{-\infty}^{\infty} f(x) e^{-\mathrm{i}kx} \mathrm{d}x, \quad f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \tilde{f}(k) e^{\mathrm{i}kx} \mathrm{d}k \qquad (3)$$

Essential property

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}x} = \mathrm{i}k\tilde{f},$$

as shown by integration by parts.

Details in the definition (3) may vary in literature and software.

The transformed Klein-Gordon equation

Fourier transform applied to (1); (spatial differentiation replaced by power of $\mathrm{i}k$)

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + (k^2 + q)\tilde{\eta} = 0$$

second order ODE for $\tilde{\eta}$ with general solution

$$\tilde{\eta}(k,t) = A(k)e^{-\mathrm{i}\omega(k)t} + B(k)e^{\mathrm{i}\omega(k)t}$$

where $\omega(k) = \sqrt{q + k^2}$ (dispersion relation)
Transformed initial conditions

$$\tilde{\eta}(k,0) = \tilde{\eta}_0(k) = L\sqrt{\pi}e^{-\left(\frac{kL}{2}\right)^2}, \quad \frac{\partial \tilde{\eta}(k,0)}{\partial t} = 0,$$

yield

$$A = B = \frac{1}{2}\tilde{\eta}_0$$

The inverse transformation

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)} + \tilde{\eta}_0(k)e^{\mathrm{i}(kx+\omega(k)t)}\right)\mathrm{d}k$$

Initial condition and dispersion relaton $\Rightarrow$ $\tilde{\eta}_0$ is real, $\tilde{\eta}_0(-k) = \tilde{\eta}_0(k)$ and $\tilde{\omega}(-k) = \omega(k)$. and symmetric with relation to $k = 0$.

Exploiting properties of η_0 and ω; substitute $\ell = -k$

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\tilde{\eta}_0(k)e^{\mathrm{i}(kx+\omega(k)t)}\right)\mathrm{d}k &= -\int_{\infty}^{-\infty}\left(\tilde{\eta}_0(\ell)e^{\mathrm{i}(-\ell x+\omega(\ell)t)}\right)\mathrm{d}\ell \\ &= \int_{-\infty}^{\infty}\left(\tilde{\eta}_0(\ell)e^{-\mathrm{i}(\ell x-\omega(\ell)t)}\right)\mathrm{d}\ell.\end{aligned}$$

Terms in inversion are complex conjugates

Terms in inversion are complex conjugates $\Rightarrow$

$$\eta(x,t) = \frac{1}{2\pi}\Re\int\limits_{-\infty}^{\infty} \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k,$$

Further rewriting

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty} \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k \\
&= \int\limits_{0}^{\infty} \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + \int\limits_{-\infty}^{0} \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k
\end{aligned}$$

$$\begin{aligned}
\Re\int\limits_{-\infty}^{0} \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k &= \Re\int\limits_{0}^{\infty} \tilde{\eta}_0(\ell)e^{-\mathrm{i}(\ell x+\omega(\ell)t)}\mathrm{d}\ell \\
&= \Re\int\limits_{0}^{\infty} \tilde{\eta}_0(\ell)e^{\mathrm{i}(\ell x+\omega(\ell)t)}\mathrm{d}\ell
\end{aligned}$$

$$\eta(x,t) = \frac{1}{2\pi}\Re\left\{\int\limits_0^\infty \tilde{\eta}_0(k)e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + \int\limits_0^\infty \tilde{\eta}_0(k)e^{-\mathrm{i}(kx+\omega(k)t)}\mathrm{d}k\right\} \tag{4}$$

Waves propagating to left and right are separated.

Dispersion properties

$$\omega = \sqrt{q+k^2}, \quad c = \sqrt{1+\frac{q}{k^2}}, \quad c_g = \frac{1}{c}.$$

c decreases with k, increases with λ.
c_g increases with k, decreases with λ.
$c,\ c_g \to 1$ as $k \to \infty$.

Stationary phase

A general Fourier integral

$$I(t) = \int_a^b F(k) e^{\mathrm{i}t\chi(k)} \mathrm{d}k,$$

where a and b may be finite or infinite.
Large $t \Rightarrow$ rapid oscillations of integrand $\Rightarrow$ cancellation of adjacent contributions $\Rightarrow I \to 0$ as $t \to \infty$.
If there is a stationary point, $k = k_0$, where

$$\frac{\mathrm{d}\chi(k_0)}{\mathrm{d}k} = 0$$

the oscillations will be slowest around k_0 and the dominant contribution to the Fourier integral comes from the vicinity of this point.

Approximation of phase function and amplitude factor close to k_0

$$\chi(k) \approx \chi(k_0) + \frac{1}{2}\chi''(k_0)(k-k_0)^2, \quad F(k) \approx F(k_0).$$

This gives

$$\begin{aligned} I(t) &\approx \int_{k_0-\epsilon}^{k_0+\epsilon} F(k_0) e^{\mathrm{i}t\{\chi(k_0)+\frac{1}{2}\chi''(k_0)(k-k_0)^2\}} \mathrm{d}k \\ &\approx F(k_0) e^{\mathrm{i}t\chi(k_0)} \int_{-\infty}^{\infty} e^{\frac{1}{2}\mathrm{i}t\chi''(k_0)(k-k_0)^2} \mathrm{d}k. \end{aligned}$$

The latter integral is found in mathematical handbooks etc., and

$$I(t) \sim \frac{\sqrt{2\pi}F(k_0)}{\sqrt{t|\chi''(k_0)|}} e^{\mathrm{i}\left(\chi(k_0)t \pm \frac{\pi}{4}\right)}$$

Stationary phase applied to the Klein Gordon solution

We regard the first term in (4) and recognize F above as $\tilde{\eta}_0/(2\pi)$ and

$$\chi = k\frac{x}{t} - \omega(k).$$

The stationary point is then given according to

$$c_g(k_0) \equiv \frac{\mathrm{d}\omega(k_0)}{\mathrm{d}k} = \frac{x}{t},$$

c_g is the group velocity! Since $c_g < 1$, for all k, stationary phase is only applicable for $x < t$. We then find

$$k_0 = q^{\frac{1}{2}} \tfrac{x}{t} \left(1 - \left(\tfrac{x}{t}\right)^2\right)^{-\frac{1}{2}}, \qquad \chi(k_0) = -q^{\frac{1}{2}} \left(1 - \left(\tfrac{x}{t}\right)^2\right)^{\frac{1}{2}},$$

$$\chi''(k_0) = -q^{-\frac{1}{2}} \left(1 - \left(\tfrac{x}{t}\right)^2\right)^{\frac{3}{2}}.$$

The final asymptotic solution may be written as

$$\eta \sim a(x,t)\cos\theta(x,t),$$

where

$$a = \frac{L^{\frac{1}{2}} q^{\frac{1}{4}}}{(2\frac{t}{L})^{\frac{1}{2}}} \frac{e^{-\frac{L^2 q}{4((\frac{t}{x})^2-1)}}}{(1-(\frac{x}{t})^2)^{\frac{3}{4}}}, \quad \theta = \frac{\pi}{4} - q^{\frac{1}{2}}\sqrt{t^2-x^2}$$

Local wave number $k_l(x,t) = \frac{\partial\theta}{\partial x}$. Then $k_l(x,t) = k_0(x,t)$ follows.

How does the amplitude (a) vary in x and t ?

$x \to 0 \Rightarrow a \sim t^{-\frac{1}{2}}$; a decreases with x; $a \to 0$ as $x \to t$.

How does the local wave length vary in x and t ?

k_0 increases with x and hence λ decreases with x. (c_g decreases with wavelength).

How does individual crests propagate relative to the wave system as a whole?

Moves foreward and gains on front ($x = t$) since $c > 1 > c_g$.

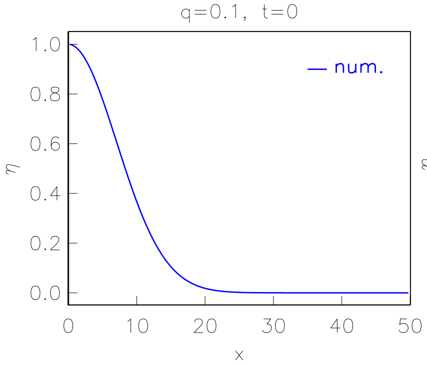
Numerical method.

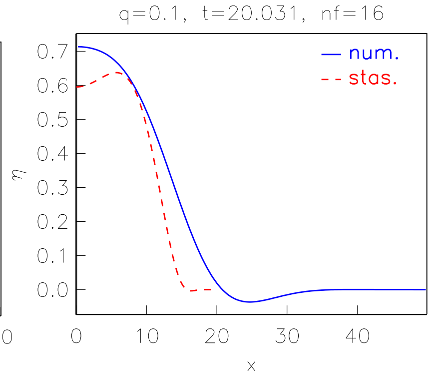
Straightforward, explicit, finite difference method:
Grid: $x_i = i\Delta x$, $t_n = n\Delta t$, nodal values:$\eta(x_i, t_n) \approx \eta_i^{(n)}$.
Difference equations:

$$\left.\begin{array}{l} \frac{1}{\Delta t^2}\left(\eta_i^{(n+1)} - 2\eta_i^{(n)} + \eta_i^{(n-1)}\right) \\ \quad - \frac{1}{\Delta x^2}\left(\eta_{i+1}^{(n)} - 2\eta_i^{(n)} + \eta_{i-1}^{(n)}\right) + q\eta_i^{(n)} = 0 \end{array}\right\} \qquad (5)$$

1. $\eta_i^{(0)}$ and $\eta_i^{(-1)}$ given by initial conditions.
2. For each step $(n = 1, 2...)$ we compute $\eta_i^{(n+1)}$ from $\eta_i^{(n)}$ and $\eta_i^{(n-1)}$.
3. Grid-refinement tests $(\Delta x, \Delta t \to 0)$.

Comparison of numeric and asymptotic solutions





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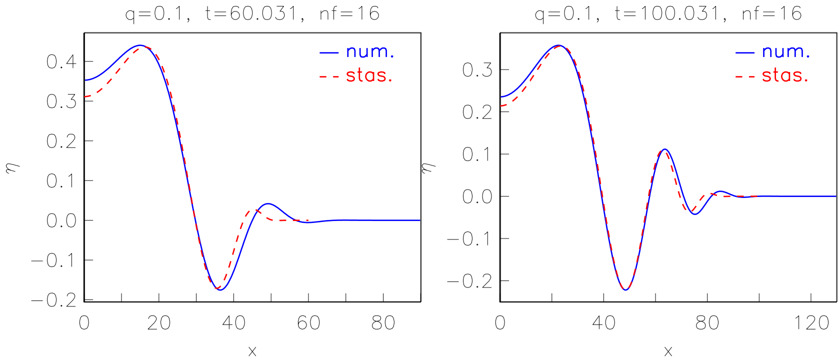

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