The WKBJ method and optics MEK4320

Geir Pedersen

Department of Mathematics, UiO

November 4, 2015

WKBJ; example: LSW

Linear shallow water theory(indices mark differentiation)

$$\eta_{tt} - \nabla \cdot (c_0^2 \nabla \eta) = 0 \tag{1}$$

where $c_0^2 = gh(x, y)$.
In ray theory

$$\eta(x, y, t) = A(x, y, t) e^{i\chi(x,y,t)}, \tag{2}$$

where $\vec{k} \equiv \nabla \chi$, $\omega \equiv -\frac{\partial \chi}{\partial t}$.
Slow varations of $\vec{k}$ and $\omega \Rightarrow$ ray equations.
We now insert (2) in (1):

$$\begin{aligned} &A_{tt} - i(2\omega A_t + \omega_t A) - \omega^2 A = \\ &\nabla \cdot (c_0^2 \nabla A) + i\left(c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k})\right) - c_0^2 k^2 A \end{aligned} \tag{3}$$

Exact, but nothing is achieved either – so far.

Scaling; transformation to non-dimensional form

Scales

Typical wavelength: λ_c (fast scale)
Typical length scale for medium change L_c (slow scale)
Small parameter $\frac{\lambda_c}{L_c} = \epsilon \ll 1$.
Typical wave speed: $c_c = \sqrt{gh_c}$
Typical amplitude: A_c – no significance as long as in linear regime
Typical phase: $\chi \sim L_c/\lambda_c = \epsilon^{-1}$

Rescaling

All derivatives explicit in (3) are with respect to slow variation.
Fast variation inherent in definitions of $\vec{k}$ and ω, only.
$\vec{\kappa} = \lambda_c \vec{k}$, $\hat{\omega} = \lambda_c \omega / c_c$, $\hat{A} = A/A_c$, $\hat{c} = c_0/c_c$, $\hat{x} = x/L_c$,
$\hat{t} = c_c t / L_c$.

Rescaling of (3)

$$\begin{aligned} &\epsilon^2 \hat{A}_{\hat{t}\hat{t}} - i\epsilon(2\hat{\omega}\hat{A}_{\hat{t}} + \hat{\omega}_{\hat{t}}\hat{A}) - \hat{\omega}^2\hat{A} = \\ &\epsilon^2\hat{\nabla}\cdot(c^2\hat{\nabla}\hat{A}) + i\epsilon\left(c^2\hat{\nabla}\hat{A}\cdot\vec{\kappa} + \hat{\nabla}\cdot(c^2\hat{A}\vec{\kappa})\right) - c^2\kappa^2\hat{A} \end{aligned} \tag{4}$$

Leading terms: $O(1)$ no slow differentiations
Next order: $O(\epsilon)$ one slow differentiation
Second order: $O(\epsilon^2)$ two slow differentiations

Leading order

$\hat{\omega}^2 = c^2\kappa^2 \Rightarrow$ restored scales $\omega^2 = c_0^2k^2 = W^2$.
With $\vec{k}_t + \nabla\omega = 0$, $\partial k_i/\partial x_j = \partial k_j/\partial x_i$: ray theory retrieved

$$\frac{\partial k_i}{\partial t} + \vec{c}_g\cdot\nabla k_i = -\frac{\partial W}{\partial x_i}, \quad i = 1, 2 \tag{5}$$

$$\frac{\partial\omega}{\partial t} + \vec{c}_g\cdot\nabla\omega = 0 \tag{6}$$

with $\vec{c}_g = c_0\vec{k}/k$. Now $\vec{k}$ and ω are settled.

Next order; $O(\epsilon)$ relative size

Original scales restored

$$-2\omega A_t - \omega_t A = c_0^2 \nabla A \cdot \vec{k} + \nabla \cdot (c_0^2 A \vec{k}).$$

With $\omega = c_0 k$ and $\vec{c}_g = c_0 \vec{k}/k$ (then $c_0^2 \vec{k} = \omega \vec{c}_g$):

$$-2\omega A_t - \omega_t A = 2\omega \vec{c}_g \cdot \nabla A + A\vec{c}_g \cdot \nabla \omega + \omega A \nabla \cdot \vec{c}_g.$$

Next step multiply with A/ω and regroup

$$-(A^2)_t - \frac{A^2}{\omega}(\omega_t + \vec{c}_g \cdot \nabla \omega) = \nabla \cdot (\vec{c}_g A^2).$$

Due to (6) terms within last parentheses on l.h.s cancel out:

$$(A^2)_t = -\nabla \cdot (\vec{c}_g A^2). \tag{7}$$

With $E = \frac{1}{2}\rho g A^2$ (energy density) and $\vec{F} = \vec{c}_g E$ (energy flux) equation (7) reads

$$E_t + \nabla \cdot \vec{F} = 0, \tag{8}$$

Averaged energy conservation, as in uniform medium.

Remarks

Remark 1

Similar WKBJ approaches apply to most linear wave equations $\Rightarrow$ result with interpretation as energy conservation is general.

Remark 2

By means of (6) we have

$$(G(\omega)E)_t + \nabla \cdot \left(G(\omega)\vec{F}\right) = 0,$$

for any $G(\omega)$.

Remark 3

If we have coupling with a background current it is the wave action (E over some frequency) which is conserved.

Remark 4

To include higher order in ϵ we must expand $A = A_0 + \epsilon A_1 + ..$

Stråleteori (geometrisk optikk)
Harmonisk bølge, uniformt medium$\Rightarrow$ dispersjonsrelasjon
$\omega = W(\vec{k}; H...)$

Langsom variasjon av medium og bølgetog$\Rightarrow$ lokale k og ω oppfyller dispersjonsrelasjonen (tilnærmet)

Fysisk optikk
Harmonisk bølge, uniformt medium$\Rightarrow$
$\vec{F} = \vec{c}_g E$, $E = E(A^2, ..)$

Langsom variasjon etc. $\Rightarrow \vec{F} = \vec{c}_g E$ (tilnærmet)

GEOMETRISK OPTIKK; likninger

Strålelikninger

Fra $\omega = -\frac{\partial \chi}{\partial t}$, $\vec{k} = \nabla \chi$ og $\omega = W(\vec{k}, x_i, t)$

$$
\begin{aligned}
\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\partial \omega}{\partial t} + \vec{c}_g \cdot \nabla \omega &= \frac{\partial W}{\partial t}
\end{aligned}
$$

Skrevet som Hamiltons kanoniske likninger

$$
\begin{aligned}
\frac{\mathrm{d}k_i}{\mathrm{d}t} &= -\frac{\partial W}{\partial x_i}, \quad i = 1, 2.. \\
\frac{\mathrm{d}x_i}{\mathrm{d}t} &= \frac{\partial W}{\partial k_i} = (c_g)_i \,, \quad i = 1, 2.. \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\partial W}{\partial t}
\end{aligned}
$$

Transportlikning

Generell energilikning

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Innsatt tilnærmelsen $F = c_g E$:

$$\frac{\partial E}{\partial t} + \frac{\partial (c_g E)}{\partial x} = 0$$

der $E = E(A^2, ..)$

Flere dimensjoner

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{c}_g E) = 0 \tag{9}$$

Utregning av bølgefelt

1. $\vec{k}$ og ω finnes fra stråleteori
2. Transportlikning (9) løses for A

Uniformt medium

$\omega = W(\vec{k}) \Rightarrow \vec{c}_g = c_g(\vec{k})$

Strålelikninger

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = 0, \qquad i = 1, 2..$$

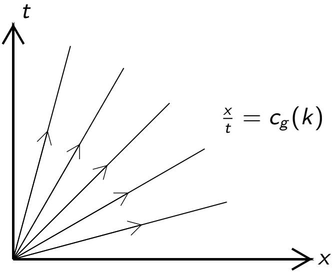
$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = (c_g)_i, \quad i = 1, 2..$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

$\vec{k}$ og ω bevart langs karakteristikk $\mathcal{C}$: $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{c}_g$

$\Rightarrow \vec{c}_g$ bevart $\Rightarrow \mathcal{C}$ er rette linjer

Uniformt medium: bølger fra konsentrert forstyrrelse



Alle $\mathcal{C}$ gjennom $x = t = 0 \Rightarrow c_g = x/t$
$c_g(k) = x/t \Rightarrow k = k(x/t) \Rightarrow \chi = \int k dx$

Uendelig dyp; stråleteori
$W = \sqrt{gk} \Rightarrow x/t = c_g = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{1}{4}g\frac{t^2}{x^2} \Rightarrow$
$\chi = -\frac{1}{4}g\frac{t^2}{x} + f(t)$
Videre $\frac{\partial \chi}{\partial t} = -\omega = -\sqrt{gk} \Rightarrow \chi = -\frac{1}{4}g\frac{t^2}{x} + \text{const}$
Fasefunksjon som fra stasjonær fase.

Uendelig dyp; transportlikning

$$\frac{\partial A^2}{\partial t} + \nabla \cdot (\vec{c}_g A^2) = 0$$

med $c_g = x/t$ omskrives til

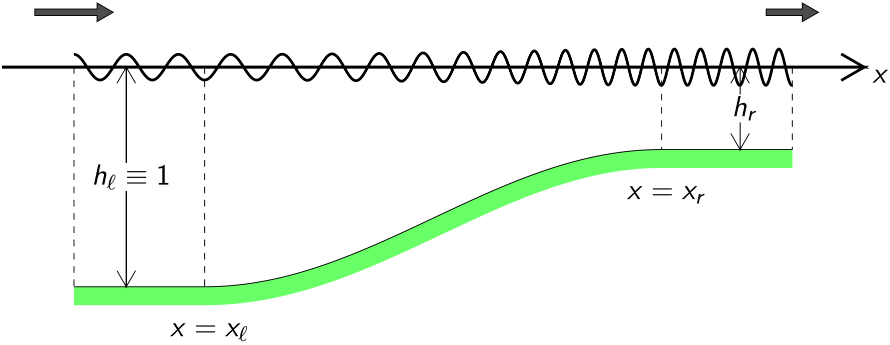
$$\frac{\partial x A^2}{\partial t} + \frac{x}{t}\frac{\partial (x A^2)}{\partial x} = 0$$

Generell løsning xA^2 konstant langs $\mathcal{C} \Rightarrow$

$$A = x^{-\frac{1}{2}} G\left(\frac{x}{t}\right) = t^{-\frac{1}{2}} \hat{G}\left(\frac{x}{t}\right)$$

Konsistent med stasjonær fase (spektrum$\Rightarrow$ G)
Tolkning: Energi mellom to $\mathcal{C}$ bevart.

GEOMETRY AND WAVEFIELD.



Plane waves incident on a sloping bottom.

Eksempel continues: Green's lov

Plant, normalt infall, $h = h(x)$, $\vec{k} = k\vec{\imath}$
Geometrisk optikk (stråleteori)$\Rightarrow$ $\omega = \mathrm{konst.}$, $k = \omega/c_0$,
$\chi = \int k dx$
Transportlikning for plane bølger $\Rightarrow$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow F = \mathrm{konst.} \Rightarrow c_0 A^2 = \mathrm{konst.}$$

Konstant energifluks

Insatt $c_0 = \sqrt{gh}$, Greens lov:

$$A = A_0 \left(\frac{h}{h_0}\right)^{-\frac{1}{4}}, \tag{10}$$

der A_0 og h_0 beskriver en referansetilstand.

Sammenlikning med velkontrollert numerisk løsning

$\eta = \hat{\eta}(x)e^{-i\omega t} \Rightarrow$ ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(gh(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \omega^2\hat{\eta} = 0. \tag{11}$$

For en x_a slik at $x_a < x_\ell$ (inkommende + reflektert bølge)

$$\hat{\eta} = A_0 e^{ik_\ell x} + Re^{-ik_\ell x}, \quad \text{der} \quad \omega = \sqrt{gh_\ell}k_\ell.$$

Gir randbetingelse

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + ik_\ell\hat{\eta} = 2iA_0 k_\ell e^{ik_\ell x}, \quad \text{for} \quad x = x_a. \tag{12}$$

Ved $x = xb > x_r$ bare transmittert bølge: $\hat{\eta} = Te^{ik_r x}$

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} - ik_r\hat{\eta} = 0, \quad \text{der} \quad \omega = \sqrt{gh_r}k_r. \tag{13}$$

2 ordens ODE og 2 randbetingelser for kompleks $\hat{\eta}$. Tolker: $A = |\hat{\eta}|$

FDM

Numerical approximation : $\hat{\eta}_j \approx \hat{\eta}(j\Delta x)$, $\hat{h}_{j+\frac{1}{2}} = h((j+\frac{1}{2})\Delta x)$,
Discrete version of (11)

$$\frac{gh_{j+\frac{1}{2}}(\hat{\eta}_{j+1} - \hat{\eta}_j) - gh_{j-\frac{1}{2}}(\hat{\eta}_j - \hat{\eta}_{j-1})}{\Delta x^2} + \omega^2 \hat{\eta}_j = 0.$$

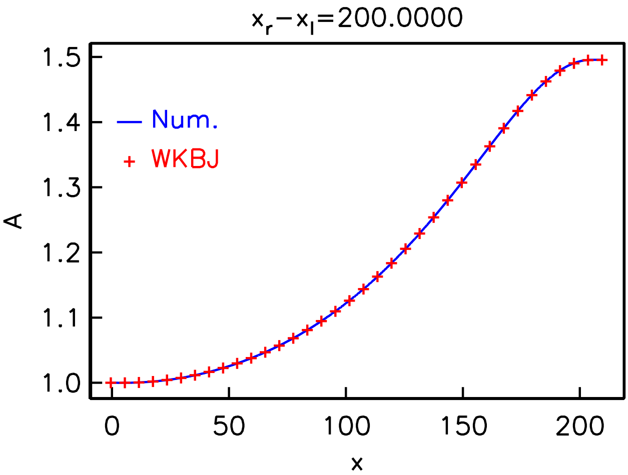
Boundary conditions (grid from $i = 0$ to $i = N$)

$$\frac{\hat{\eta}_1 - \hat{\eta}_0}{\Delta x} + \frac{i}{2}(\hat{\eta}_1 + \hat{\eta}_0) = 2iA_0 k_\ell e^{i\frac{1}{2}k_\ell \Delta x},$$

$$\frac{\hat{\eta}_N - \hat{\eta}_{N-1}}{\Delta x} - \frac{i}{2}(\hat{\eta}_N + \hat{\eta}_{N-1}) = 0.$$

Tri-diagonal set with closure from boundary conditions.
NB: Boundary conditions may be amended by numerical dispersion relation to become exact for the discrete case.

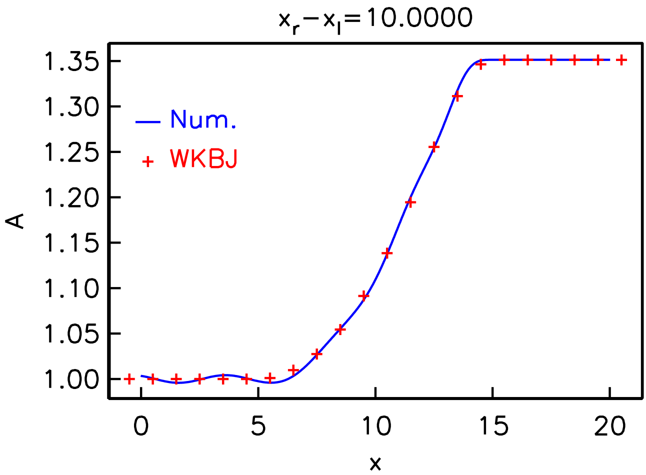
Gentle slope



$x = x_l = 5$: amplitude 1, periode 8.

$h_r = 0.2$, $x_r - x_l = 200$

Steep slope



$x = x_l = 5$: amplitude 1, periode 8.

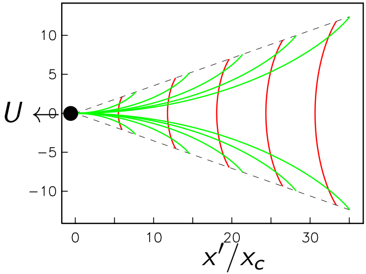
$h_r = 0.3$, $x_r - x_l = 10$

Remarks on optics for shoaling

- Optics good even when $L = x_r - x_\ell$ and λ are comparable.
- Major discrepancy between optics and accurate numrical solution: Reflections.
- Optics do not incorporate reflections.

Skipsbølgemønster.

Punktforstyrrelse i overflaten



Konstant hastighet: $\vec{U} = -U\vec{\imath} \Rightarrow$ stasjonært og langsomt varierende bølgesystem. Isotrop dispersjonsrelasjon

$$\vec{c}' = c_0(k)\frac{\vec{k}}{k} \qquad (14)$$

Bytte av koordinatsystem

Vi følger forstyrrelsen $\vec{r} = \vec{r}' - \vec{U}t$.

Harmonisk mode:

$A \cos \chi_H = A \cos(\vec{k} \cdot \vec{r}' - \omega' t) = A \cos(\vec{k} \cdot \vec{r} - (\omega' + \vec{k} \cdot \vec{U})t)$

$$\omega = c_0(k)k + \vec{U} \cdot \vec{k} \equiv W(k_x, k_y) \tag{15}$$

$$\vec{c} = \left(c_0(k) + \vec{U} \cdot \frac{\vec{k}}{k} \right) \frac{\vec{k}}{k} \tag{16}$$

der $\vec{k} = k_x \vec{\imath} + k_y \vec{\jmath}$

Dopplerskift$\Rightarrow$ Anisotrop dispersjon

Stråleteori

Stasjonært mønster gir $\omega = 0$. (15) gir da:

$$W(k_x, k_y) = 0 \tag{17}$$

Gruppehastighet

$$\vec{c}_g = \frac{\partial W}{\partial k_x}\vec{\imath} + \frac{\partial W}{\partial k_y}\vec{\jmath} \tag{18}$$

Hamiltons likninger

$$\frac{d\vec{r}}{dt} = \vec{c}_g, \qquad \frac{d\vec{k}}{dt} = 0 \tag{19}$$

der

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \tag{20}$$

Uniformt medium $\Rightarrow$ Karakteristikkene blir rette linjer
Bare de karakteristikker kan bære energi som går gjennom forstyrrelsen (origo)

Må ha

$x = c_{gx}t$, $y = c_{gy}t$.

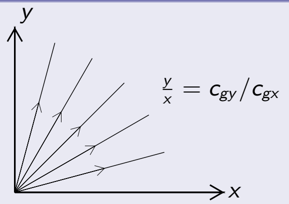
(Selv om mønstret er stajonært kan karakteristikker parameteriseres vha. tiden)

Eliminasjon av t:

$$\frac{y}{x} = \frac{\frac{\partial W}{\partial k_y}}{\frac{\partial W}{\partial k_x}} \tag{21}$$

kombinert med $W(k_x, k_y) = 0$ (17)$\Rightarrow$ to likninger for k_x og k_y.

Karakteristikker



Fasefunksjonen

$$\chi(\vec{r}) = \chi_0 + \int\limits_{C(\vec{r})} \vec{k} \cdot d\vec{r} \tag{22}$$

der χ_0 er fasen i origo og $C(\vec{r})$ integrasjonsvei
Integrasjon langs karakteristikkene er triviell fordi $\vec{k}$ er konstant på hver karakteristikk.

$$\chi(\vec{r}) = \chi_0 + k_x x + k_y y \tag{23}$$

Faselinjer $\chi = -A$. To muligheter for visualisering/tolkning

- En raskt danne seg et bilde av faselinjene ved å plotte nivålinjene til χ i feks. Matlab.
- En kan parameterisere faselinjer. En del trigonometri, men en demonstrerer to uavhengige løsninger for $\vec{k}$ fra (21) og (17)

Parameterisering av faselinjer for uendelig dyp

θ: vinklen mellom $\vec{k}$ og negativ x-akse

$$k_x = -k\cos\theta, \quad k_y = k\sin\theta. \tag{24}$$

(17) kan skrives om til

$$c_0 = U\cos\theta,$$

og $c_0(k) = \sqrt{g/k}$ gir da

$$k = \frac{g}{U^2\cos^2\theta} \tag{25}$$

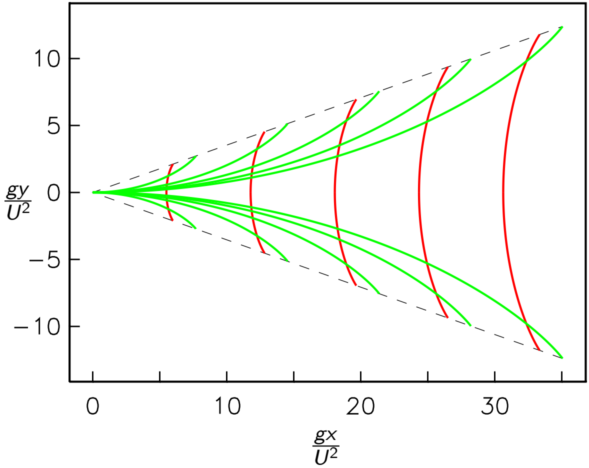
(21) og (23) løses for x og y

$$x = \frac{(A-\chi_0)g}{U^2}\cos\theta(1+\sin^2\theta) \tag{26}$$

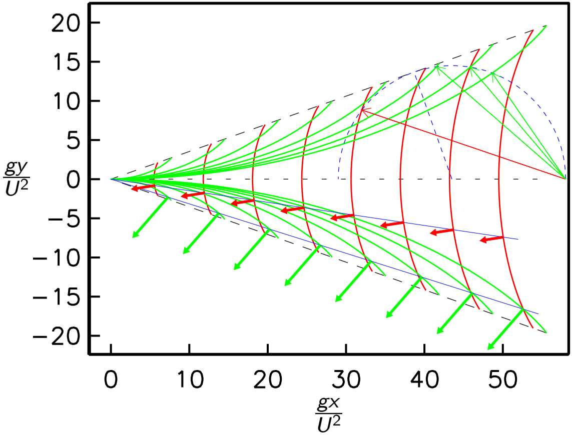
$$y = \frac{(A-\chi_0)g}{U^2}\cos^2\theta\sin\theta \tag{27}$$

NB: $y(\theta)/x(\theta)$ ekstremum for $\cos\theta = \sqrt{2/3}$ $(\theta = \theta_c = 35.3^\circ) \Rightarrow$ faselinjene har knekker $\Rightarrow$ uavhengige løsninger

Punktforstyrrelse: kan vises, vha. andre teknikker, at $\chi_0 = \frac{1}{4}\pi, -\frac{1}{4}\pi$ for hhv. hekk og baugbølger.



The Kelvin pattern, more details



Fat arrows: wave number vectors.

Dashed half circle: propagation with $\vec{c}_g$ from intersection with x-axis, subject to $c_0 = U\cos\theta$. Thin arrows: corresponding rays.