The Van der Pol equation

We wish to study the unforced Van der Pol oscillator described by:

$$y + y - \epsilon(1 - y^2)\dot{y} = 0 \qquad (1)$$

Applying a multiple scale method with the slowly varying time $\tau = \epsilon t$ and assuming $y(t, \tau) = y_0 + \epsilon y_1 + O(\epsilon \check{s})$ we get the following equations:

$$\begin{aligned} \epsilon^0: &\quad \ddot{y}_0 + y_0 = 0 \\ \epsilon^1: &\quad \ddot{y}_1 + y_1 = -2y_{t\tau} - +y_0 - y_0^2 y_{0t} \end{aligned}$$

This yields a leading order approximation:

$$y = R(\tau)\cos(t + \theta(\tau)) + O(\epsilon)$$

with

$$\theta = constant\,,\; R = \frac{2}{\sqrt{1 + \mathrm{D}e^{-\tau}}}$$

where D and θ are both constants to be determined by initial conidtions which we can choose.

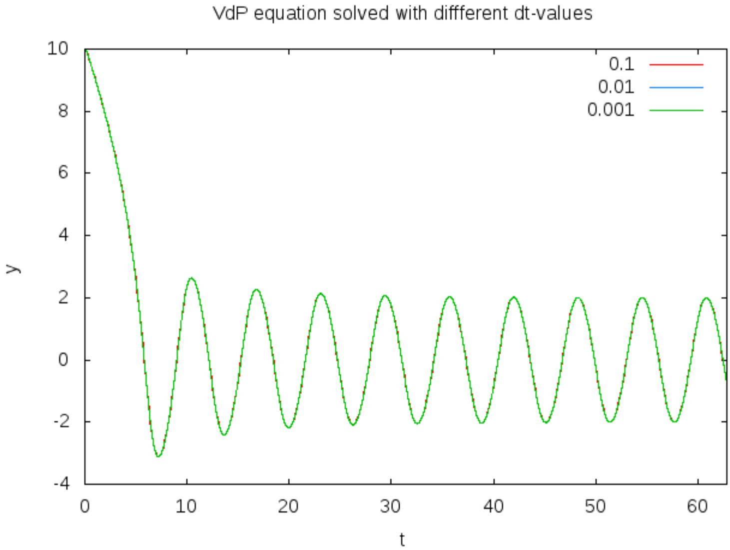

Figure 1: Eq.(1) with initial conditions $y(0) = 10$ and $\dot{y}(0) = 0$ solved by Runge-Kutta method of 4th order.

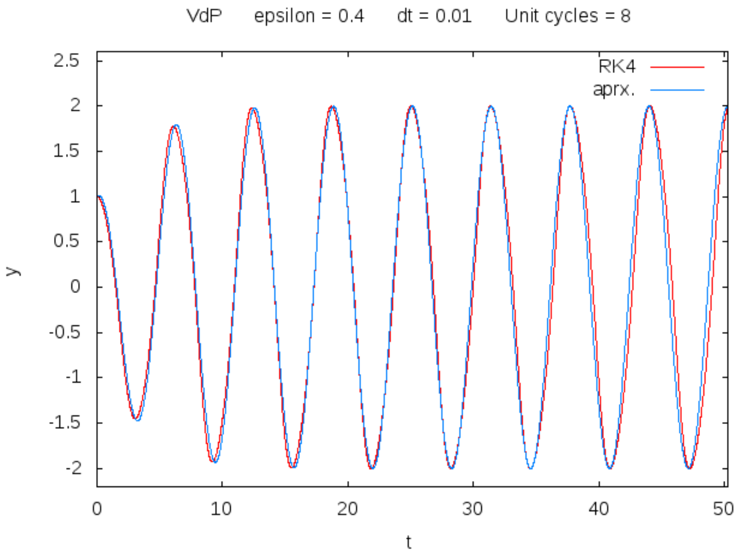

Figure 2: Comparing numerical solution and approximate solution when $y(0) = 1$, $\dot{y}(0) = 0$.

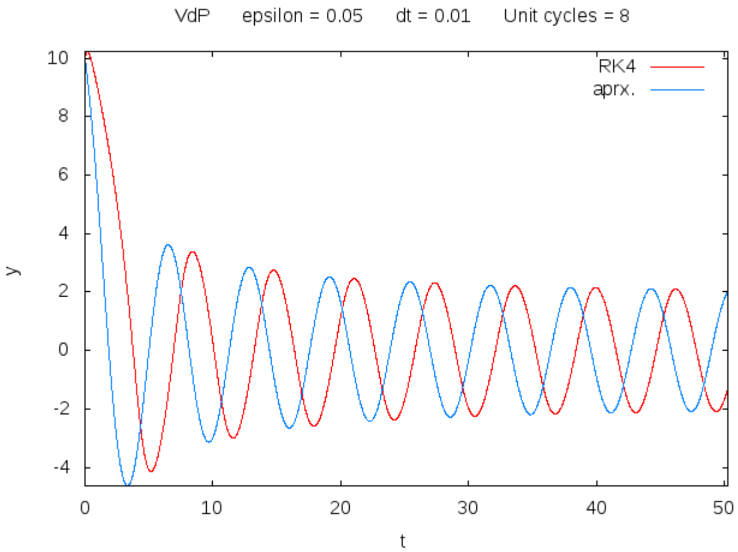

Figure 3: Initial position outside limit circle gives less accuracy. $y(0) = 10$, $\dot{y}(0) = 3$.

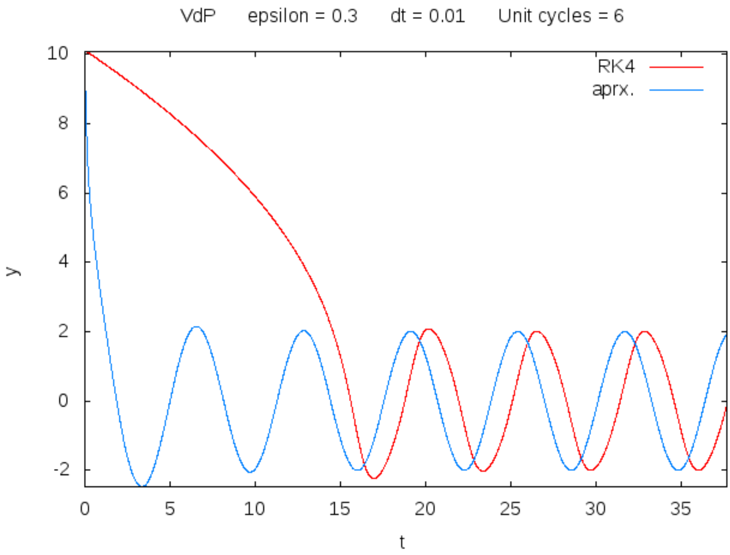

Figure 4: Increasing epsilon results in large errors in the beginning of the interval.

Section 4 in leaflet

We want to solve the scaled equation:

$$\gamma(\tau)\theta_{tt} + 2\epsilon\gamma_T\theta_t + \theta = 0 \qquad \theta(0) = 1\,,\ \theta_t(0) = 0 \qquad (23)$$

Where γ represents the scaled length of the rod. The leading order beheavior is obtained by introducing both a slow and a fast time scale which gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sigma\frac{\mathrm{d}}{\mathrm{d}T} + \epsilon\frac{\mathrm{d}}{\mathrm{d}\tau}$$

With $\tau = \epsilon t$ and $\frac{\mathrm{d}T}{\mathrm{d}t} = \sigma$. In order to get rid of secular terms in the equation for the 0th order we must choose $\sigma = \gamma^{-\frac{1}{2}}$. The 0th order solution is then:

$$A_0(\tau)e^{iT} + \bar{A}_0(\tau)e^{-iT} \quad , \quad A_0(0) = \frac{1}{2}$$

Now, to avoid secular terms in the first order equation A_0 is found to be:

$$A_0 = Constant \cdot \gamma^{-\frac{3}{4}}$$

With $\frac{\mathrm{d}T}{\mathrm{d}t} = \gamma^{-\frac{1}{2}}$ and $T = t + \frac{1}{2\epsilon} sin(\epsilon t)$ we have:

$$\gamma = (1 + \frac{1}{2}\cos(\epsilon t))^{-2}$$

All we need to describe the leading order behaviour is now acquired.

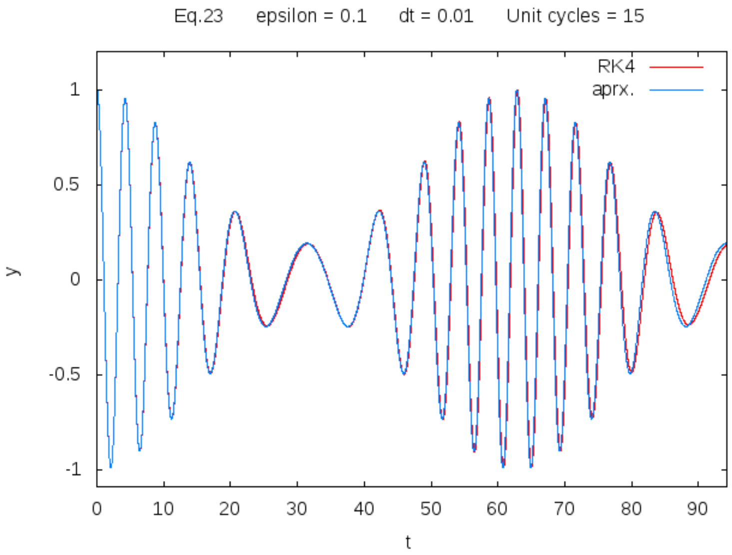

Figure 5: Envelope and cycles well approximated for small epsilon.

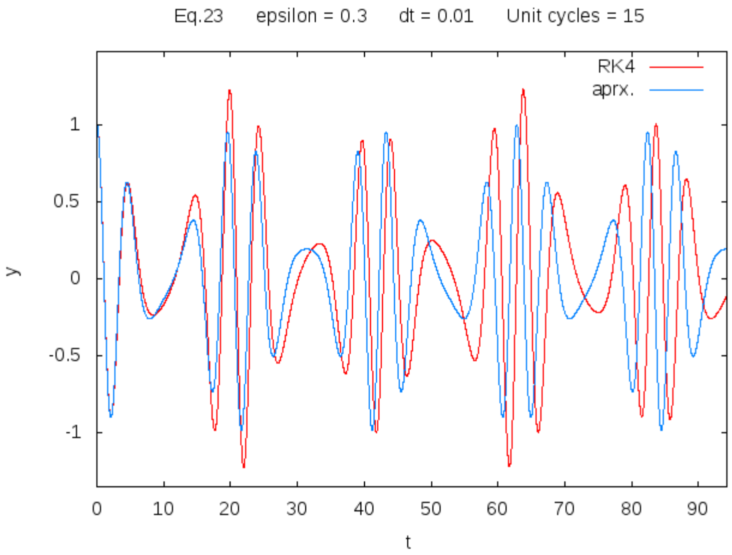

Figure 6: Still an O.K. estimate but signs of trouble ahead.

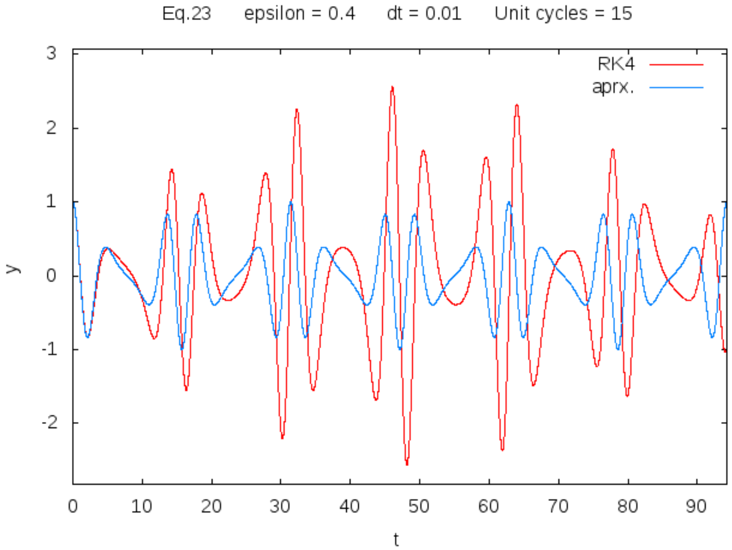

Figure 7: At $\epsilon \approx 0.4$ our method pretty much fails.