Problem 1

Equation

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1,$$

This problem is identical to problem 31 from the leaflet.

Outer solution

Ignoring the $O(\epsilon)$ term we have

$$x^2 y_o' - y_o = 0,$$

which can be solved by the formula below (2) in the formula sheet

$$y_o = Ce^{-\frac{1}{x}},$$

where C is a constant of integration. We observe that $y_o \to 0$ as $x \to 0$, which implies that y_o cannot satisfy the boundary condition at $x = 0$. Thus, we require $y_o(1) = 1$ and assume a boundary layer at $x = 0$. If this doe not work we will later have to assume boundary layers at both $x = 0$ and $x = 1$. We now find $c = e$ and

$$y_o = e^{1-\frac{1}{x}}.$$

Inner solution

Stretched variable $\xi = x/\delta$:

$$\begin{array}{cccc} \frac{\epsilon}{\delta^2}\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} & +\delta\xi^2\frac{\mathrm{d}y}{\mathrm{d}\xi} & +y & =0 \\ (1) & (2) & (3) & \end{array}$$

(1) must be retained. We observe that (2) approaches zero when $\delta \to 0$, while (3) remains finite. Hence, (2) cannot be part of a dominant balance. For (1) and (3) to be of the same order we must require $\delta \sim \epsilon^{\frac{1}{2}}$. Choosing $\delta = \epsilon^{\frac{1}{2}}$ we find the leading order equation for the inner solution

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\xi^2} - Y = 0,$$

which inherits the solution

$$Y = Ae^{-\xi} + Be^{\xi}.$$

The second part, $Be^{\xi} = Be^{\frac{x}{\sqrt{\epsilon}}}$, grows exponentially as $\xi \to \infty$ and cannot be matched to the finite outer solution. Invoking the boundary condition at $x = 0$ we then obtain

$$Y = e^{-\xi} = e^{-\frac{x}{\sqrt{\epsilon}}}$$

Matching

We match the solutions in the region $\epsilon^{\frac{1}{2}} \sim \delta \ll x \ll 1$

$$Y = e^{-\frac{x}{\sqrt{\epsilon}}} \quad \sim \quad 0,$$

$$y_o = e^{1-\frac{1}{x}} \quad \sim \quad 0.$$

The common solution in the overlap region , called y_{match}, is then zero and we may write

$$y_{\mathrm{unif}} = y_o + Y - y_{\mathrm{match}} = e^{-\frac{x}{\sqrt{\epsilon}}} + e^{1-\frac{1}{x}}$$

Problem 2

The derivation is given in the textbook, but we prefer a slightly different notation. The variation of y is denoted by δy and $\delta y(a) = \delta y(b) = 0$ since both y and $y+\delta y$ must obey the same boundary conditions. Then

$$\delta I = \delta \int_a^b L dx = \int_a^b \left(\frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial y'} \delta y' \right) dx.$$

The last term can be transformed by integration by parts to give an expression containing only δy and not $\delta y'$.

$$\int_a^b \left(\frac{\partial L}{\partial y'} \delta y' \right) dx = - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \delta y \right) dx + \left[\frac{\partial L}{\partial y'} \delta y \right]_a^b.$$

Because δy vanishes at both end points the last term disappears and we find

$$\delta I = \int_a^b \left\{ \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) \right\} \delta y \, dx.$$

Now then, the last crucial step is that δI must vanish for all applicable δy. If we assume that the expression within the $\{\}$ is nonzero at one point, $\hat{x}$, and continuous there is an interval, $\hat{J}$, around $\hat{x}$ where the expression is of one sign. If we choose an δy that is of one sign on a sub-interval of $\hat{J}$ then δI will be nonzero. Therefore, to have $\delta I = 0$ for all admissible δy the expression within $\{\}$ must be zero everywhere and we have derived the Euler-Lagranges equation

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) = 0$$

Problem 3

The unperturbed problem is obtained by setting ϵ equal to zero

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

If $\lambda > 0$ the solution of the differential equations is

$$y = A \cos \lambda^{\frac{1}{2}} x + B \sin \lambda^{\frac{1}{2}} x, \qquad \int_0^1 y^2 dx = \frac{1}{2}.$$

From $y(0) = 0$ it follows that $A = 0$. For $y(1) = 0$ to be fulfilled we must require either $B = 0$, which yields the trivial solution, or that $\lambda^{\frac{1}{2}}$ is a natural number times π.

For $\lambda = 0$ we have a linear polynomial as solution of the differential equation, while $\lambda < 0$ yields $y = A \cosh |\lambda|^{\frac{1}{2}} x + B \sinh |\lambda|^{\frac{1}{2}} x$. In neither case can we find a non-trivial solution that agrees with the boundary conditions.

Then we have all the eigensolutions on the form

$$y_n(x) = B \sin(n\pi x), \quad \lambda_n = n^2 \pi^2.$$

Emploing the normalization conditon

$$\int_0^1 y^2 dx = \frac{1}{2}$$

we obtain $B = 1$ for all n. In fact, concerning this point we have done more than we were asked. It would suffice to verify the proposed solutions by substituting them into the differential equation, the boundary conditions and the normalization condition.

It mus be expected that λ depends on ϵ. Employing the Poincare-Lindstedt's method we then assume the series

$$y = u_0 + \epsilon u_1 + ..., \quad \lambda = \lambda_0 + \epsilon\lambda_1 + ...,$$

where λ_i is redefined as the coefficients in the series expansion for the perturbed smallest eigenvalue instead of being eigenvalue number i. Substitution into the differential equation yields

$$\begin{aligned} 0 &= (u_0 + \epsilon u_1 + ...)'' + (\lambda_0 + \epsilon\lambda_1 + ...)(u_0 + \epsilon u_1 + ...) + \epsilon(u_0 + \epsilon u_1 + ...)^3 \\ &= u_0'' + \lambda_0 u_0 + \epsilon(u_1'' + \lambda_0 u_1 + \lambda_1 u_0 + u_0^3) + O(\epsilon^2) \end{aligned}$$

For the normalisation condition we obtain

$$\frac{1}{2} = \int_0^1 (u_0 + \epsilon u_1 + ...)^2 dx = \int_0^1 u_0^2 dx \, + \, 2\epsilon \int_0^1 u_0 u_1 dx + O(\epsilon^2)$$

The boundary conditions become $u_i(0) = u_i(1) = 0$ for all i.

Order ϵ^0

We must solve

$$\begin{aligned} &u_0'' + \lambda_0 u_0 = 0, \\ &u_0(0) = 0, \quad u_0(1) = 0, \\ &\textstyle\int_0^1 u_0^2 dx = \frac{1}{2}. \end{aligned}$$

We now simply observe that $\lambda_0 = \pi$ and $u_0 = \sin \pi x$ fulfil all the equations.

Order ϵ

We must solve

$$\begin{aligned} &u_1'' + \lambda_0 u_1 = -\lambda_1 u_0 - u_0^3, \\ &u_1(0) = 0, \quad u_1(1) = 0, \\ &\textstyle\int_0^1 u_0 u_1 dx = 0. \end{aligned}$$

We substitute λ_0 and u_0 into the differential equation and use the formulae on the sheet

$$\begin{aligned} u_1'' + \pi^2 u_1 &= -\lambda_1 u_0 - u_0^3 \\ &= -\lambda_1 \sin \pi x - \sin^3 \pi x \\ &= -(\lambda_1 + \tfrac{3}{4}) \sin \pi x + \tfrac{1}{4} \sin 3\pi x. \end{aligned}$$

We cannot discard the $\sin \pi x$ terms at this stage because y_1 does not need to be periodic and x may not be larger than 1. Hence, we solve the equation (by using the particular solutions on the formula sheet) and obtain

$$u_1 = \frac{(\lambda_1 + \frac{3}{4})}{2\pi} x \cos \pi x - \frac{1}{32\pi^2} \sin 3\pi x + A_1 \cos \pi x + B_1 \sin \pi x,$$

where the constant in the homogeneous part of the solution must be determined from the boundary and normalisation conditions. The boundary conditions yield

$$0 = u_1(0) = A_1, \quad 0 = u_1(1) = -\frac{(\lambda_1 + \frac{3}{4})}{2\pi} - A_1.$$

This implies $A_1 = 0$ and determine the correction to eigenvalue

$$\lambda_1 = -\frac{3}{4}.$$

The expression for u_1 is then reduced to

$$u_1 = -\frac{1}{32\pi^2} \sin 3\pi x + B_1 \sin \pi x.$$

Normalisation

$$0 = \int_0^1 u_0 u_1 \, dx = \frac{1}{2} B_1 - \frac{1}{32\pi^2} \int_0^1 \sin 3\pi x \sin \pi x \, dx.$$

We can show that the last integral is zero either by rewriting according to $\sin 3t \sin t = \frac{1}{2}(\cos 2t + \cos 4t)$, integration by parts (twice) or use of complex exponentials. Anyhow we find $B_1 = 0$ and

$$u_1 = -\frac{1}{32\pi^2} \sin 3\pi x, \quad \lambda_1 = -\frac{3}{4}.$$

A good alternative in point b is to apply Helmholtz rule (for those who remember).