Problems for MEK4320

Autumn 2012

Problem 1 ***Numerical dispersion relation .***
We assume constant depth which equals h_0 in dimensionless coordinates. Then there are solutions of the linear shallow water equations on the form

$$\eta = \mathrm{Re}\, \hat{\eta} e^{\imath(kx-\omega t)}, \quad u = \mathrm{Re}\, \hat{u} e^{\imath(kx-\omega t)}. \tag{1}$$

a) Find the relations between ω and k and between $\hat{u}$ and $\hat{\eta}$.

b) In the discrete equations (i) and (ii) in the description of the model we may correspondingly insert

$$\eta^{(n)}_{j-\frac{1}{2}} = \mathrm{Re}\, \hat{\eta} e^{\imath(k(j-\frac{1}{2})\Delta x - \hat{\omega} n \Delta t)}, \quad u_j^{(n+\frac{1}{2})} = \mathrm{Re}\, \hat{u} e^{\imath(kj\Delta x - \omega(n+\frac{1}{2})\Delta t)}. \tag{2}$$

Show that the dispersion relation becomes

$$\frac{2}{\Delta t} \sin\left(\frac{\hat{\omega}\Delta t}{2}\right) = \pm\sqrt{h_0}\frac{2}{\Delta x}\sin\left(\frac{k\Delta x}{2}\right). \tag{3}$$

What is the shortest wave for which (3) is meaningful ? Are the waves in the numerical description dispersive ?

c) In this subproblem we will investigate the stability of the numerical method. Stability requires that no mode will grow in time regardless of the value of k. Explain briefly why this is necessary and sufficient for stability. Moreover, instability will occur when (3) yields an ω with a positive imaginary part. When ω fulfills (3), so does the complex conjugate of ω (show this), implying that the existence any complex ω implies instability.

Then, show that the numerical solution is unstable (grows exponentially in time) when

$$\mathrm{Co} \equiv \frac{\sqrt{h_0}\Delta t}{\Delta x} > 1. \tag{4}$$

The quantity Co is called the Courant number. (Tip: you may assume that $|\sin(\beta)| \leq 1$ is equvalent to β being real.)

d) Show that the numerical dispersion relation is exact for

$$\mathrm{Co} = 1. \tag{5}$$

e) For the Boussinesq equations we have

$$\omega = \pm h_0^{\frac{1}{2}} k(1 - \frac{1}{6}(kh_0)^2 + O((kh_0)^4)). \tag{6}$$

Show this.
From (3), which is the numerical dispersion relation for the shallow water equations, we may find a similar expression

$$\hat{\omega} = \pm h_0^{\frac{1}{2}} k(1 - \kappa(kh_0)^2 + O((kh_0)^4)). \tag{7}$$

Find κ and a relation between Δx and Δt for which (7) becomes equal to (6), meaning that a numerical solution of the shallow water equations possesses approximately the same dispersion properties as an exact solution of the linear Boussinesq equation.
Tip: expand the left and right hand sides of (3) in Taylor series to obtain $\hat{\omega} + ()\hat{\omega}^3 + ... = h_0^{\frac{1}{2}}(k + ()k^3 + ...)$. Leading order then implies $\hat{\omega} = h_0^{\frac{1}{2}} k + O(k)^3$. Insert this into the ω^3 term and explain why we then obtain the correct relation (7).

Problem 2 ***Solitary wave propagation and grid effects.***
Use the sbouss program to simulate the propagation of a solitary wave, with amplitude $\alpha = 0.1$, starting at $x = 20$ until $t = 70$ on constant depth. Make certain that the numerical wave tank is long enough. Use Matlab, or some other tool, to find the maximum η at $t = 70$ when you choose $\Delta x = 2$, $\Delta x = 1$ and $\Delta x = 0.5$, while the time step reduction factor is set to 0.5 and the discrete correction term is turned off. Do also depict all three solutions in the same diagram. Comment on what resolution that seems relevant.

Problem 3 ***Numerical instability.***
Make a LSW simulation with $\Delta x = 0.5$ and $\Delta t = 1$ and a solitary wave, $A = 0.1$, as initial condition. Run the simulation until $t = 40$. What happens and why ?

Problem 4 ***Wave dispersion.***
Use Matlab, for instance, to produce an initial condition

$$\eta(x,0) = \begin{cases} 2A\cos^2\left(\frac{\pi(x-x_0)}{\lambda}\right) & \text{if} \quad -\frac{1}{2}\lambda < x - x_0 < \frac{1}{2}\lambda \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Explain why this initial condition has continuous first derivative and why it to leading order yields a wave in each direction with amplitude A and length λ.
Choose $\lambda = 20$, $x_0 = 0$, a tank length $L = 100$ and simulation time $t = 80$.

a) Solve the LSW equations. What should the result be ? Adjust the resolution to obtain this result well enough (explain what you mean).

b) Solve the linearized Boussinesq equations. Plot the solution at $t = 80$. Relate the solution to results given in the course leaflet.

c) Solve the full Boussinesq equations. What is the difference from the previous case ? What may the leading wave be ? Depict the solution together with the previous one.

d) Solve the NLSW equations. You will see some short features. Use grid refinement to see if they change dramatically. If they do they are artifacts and the solution is invalid.

e) Solve the NLSW equations again, but focus on the times before the artifacts appear. What happens to the shape ? Illustrate with figures. What may be the physical interpretation of this shape change ?

Problem 5 ***Bores.***
Again we use (8) as initial condition with $x_0 = 0$. However, this time we employ a long initial condition and a long wave tank according to $L = 1000$ and $\lambda = 100$.

a) We focus on $t < 200$. Do simulations with NLSW and Boussinesq equations. How long are the results similar ? Illustrate with graphs.

b) Do the Boussinesq simulation until $t = 800$. What happens? Use Matlab to compare the the second crest to the solitary wave profile in the leaflet.

c) A sequence of crests like the ones in the previous subproblem is called an undular bore. It may also be generated due to shoaling. make a depth file to sbouss which corresponds to (i) $h = 1$ for $x < 40$, $h = 0.2$ for $150 > x > 50$ and a linear slope in between. Run a solitary wave with amplitude $A = 0.05$ from the deep to the shallow region (you need a fine grid due to the shallow shelf). Show that an undular bore is generated. Explain the relation to the previous sub problem. Undular bores from tsunami simulations and bores generated by tides in rivers may be seen on the slides from the presentation on November 4th.

Problem 6 ***Numerical dispersion.***
In (8) we choose $\lambda = 16$ and $x_0 = 0$, whereas the choice of A is arbitrary. Moreover, we select a flat bottom with $L = 130$. Do two simulations with

(i) Boussinesq equations and $\Delta x = \Delta t = \frac{1}{2}$. Set the time step explicitly, as explained in the documentation of sbouss.

(ii) LSW equations, $\Delta x = \frac{1}{2}\sqrt{17}$ and $\Delta t = \frac{1}{2}$.

Depict both solutions at a time just before they reach the farther end of the wave tank. How can they be so similar ? (tip: look at the first problem)