MEK4100 STOKES WAVES.

A real, and hence somewhat messy, application of the Poincaree-Lindstedt's method

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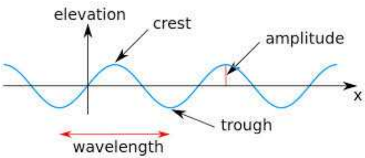
Motivation

A first step from this...



Surface gravity waves

- From ME3230, ME4320 etc: Linear periodic solution
- Smooth, rounded, regular.
- Real waves seldom look like this.

to this...

and this...

Nonlinear periodic surface gravity waves

Stokes waves

- Nonlinear periodic surface gravity waves of permanent shape
- First presented in 1847
- Stability and modulation still an active field of research
- Here: basic solution in infinite depth.

Described by two parameters:

- Amplitude $a^{\star}$
- Wavelength $\lambda^{*} = 2\pi/k^{*}$

Wave celerity, c^{*}, surface elevation, $\zeta^{\star}$ and velocity potential, ϕ^{*}, must be found.
$*$ marks dimensional coordinates.

Basic equations, scaling, introduction of expansion parameter (ϵ) and eigenvalue (c)

Basic equations

Two surface conditions, the Laplace's equation in the fluid, vanishing velocity in inifinte depth.

$$\zeta^*{}_{t^*} + \phi^*{}_{x^*}\zeta^*{}_{x^*} = \zeta^*{}_{z^*}, \quad z^* = \zeta^*; \tag{1}$$

$$\phi^*{}_{t^*} + \frac{1}{2}(\nabla^*\phi^*)^2 + g\zeta^* = 0, \quad z^* = \zeta^*; \tag{2}$$

$$(\nabla^*)^2\phi^* = 0, \quad z^* < \zeta^*; \tag{3}$$

$$\nabla^*\phi^* \to 0, \quad z^* \to -\infty; \tag{4}$$

Indices after comma: partial derivation.

Dimensional analysis

4 parametere, apart from field variables,:

$$k^*, \quad a^*, \quad g, \quad c^* \tag{5}$$

Then

$$c^* = U^* f(\epsilon) \tag{6}$$

where $U^* \equiv \sqrt{g/k^*}$ and $\epsilon \equiv a^* k^*$ is **wave steepness**.

Linear celerity:
$\epsilon \to 0 \Rightarrow c^* = const \times U^*$

Scaling, simplification

Non-dimensional variables

$$
\begin{aligned}
z &= k^* z^* \qquad & x &= k^* x^* \\
t &= k^* U^* t^* \qquad & \zeta^* &= a^* \zeta \\
\phi^* &= U^* a^* \phi
\end{aligned}
\tag{7}
$$

Waves of permanent form

Phase variable: $\theta = x - ct$

$\zeta = \zeta(\theta)$, $\phi = \phi(\theta, z)$

Only two free variables.

Dimensionless equations

Boundary conditions at $z = \epsilon\zeta$:

$$-c\phi_{,\theta} + \frac{1}{2}\epsilon(\phi_{,\theta}^2 + \phi_{,z}^2) + \zeta = 0 \tag{8}$$

$$-c\zeta_{,\theta} + \epsilon\zeta_{,\theta}\phi_{,\theta} = \phi_{,z} \tag{9}$$

In the bulk of the fluid $z < \epsilon\zeta$:

$$\phi_{,\theta\theta} + \phi_{,zz} = 0 \tag{10}$$

At infinite depth $z \to -\infty$:

$$\phi_{,\theta},\ \phi_{,z} \to 0 \tag{11}$$

c is an explicit unknown (eigenvalue)

The expansion

Power series

$$\zeta = \zeta_0(\theta) + \epsilon\zeta_1(\theta) + \epsilon^2\zeta_2(\theta) + \cdots \quad (12)$$

$$\phi = \phi_0(\theta, z) + \epsilon\phi_1(\theta, z) + \epsilon^2\phi_2(\theta, z) + \cdots \quad (13)$$

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots \quad (14)$$

Requirement: ϕ_i og ζ_i must inherit period 2π in θ

Geometrical nonlinearity

Challenges

1. Domain in which Laplace's equation is solved is unknown and must be established as part of the solution.
2. Surface conditions apply at unknown position $z = \epsilon\zeta$.

Point 1 is no problem in this particular case, but terms in surface conditions must be expanded. Example

$$
\begin{aligned}
\phi_{,\theta}(\theta, \epsilon\zeta) = &\quad \phi_{,\theta}(\theta, 0) + \epsilon\phi_{,\theta z}(\theta, 0)\zeta + \tfrac{1}{2}\epsilon^2\phi_{,\theta zz}(\theta, 0)\zeta^2 \\
= &\quad \phi_{0,\theta} + \epsilon\left(\phi_{0,\theta z}\zeta_0 + \phi_{1,\theta}\right) \\
&\quad +\epsilon^2\left(\tfrac{1}{2}\phi_{,\theta zz}\zeta_0^2 + \phi_{1,\theta z}\zeta_0 + \phi_{0,\theta z}\zeta_1 + \phi_{2,\theta}\right)
\end{aligned}
$$

where the argument $(\theta, 0)$ is implicit in the lower two lines.
With expansion of c and nonlinearities: quite some book keeping.

Hierarchy of equations and solutions

$O(\epsilon^0)$; linear approximation

$$\begin{aligned} -c_0\phi_{0,\theta} + \zeta_0 &= 0 && \text{for} \quad z = 0 \\ c_0\zeta_{0,\theta} + \phi_{0,z} &= 0 && \text{for} \quad z = 0 \\ \phi_{0,\theta\theta} + \phi_{0,zz} &= 0 && \text{for} \quad z < 0 \\ \phi_{0,\theta}, \phi_{0,z} &\to 0 && \text{for} \quad z \to -\infty \end{aligned} \tag{15}$$

The leading order periodic solution

$$\zeta_0 = \cos\theta \quad \phi_0 = e^z \sin\theta \quad c_0 = 1 \tag{16}$$

$O(\epsilon^1)$

For $z = 0$:

$$
\begin{aligned}
-c_0\phi_{1,\theta} + \zeta_1 &= c_1\phi_{0,\theta} - \tfrac{1}{2}(\phi_{0,\theta}^2 + \phi_{0,z}^2) + c_0\phi_{0,\theta z}\zeta_0 \\
c_0\zeta_{1,\theta} + \phi_{1,z} &= -c_1\zeta_{0,\theta} + \zeta_{0,\theta}\phi_{0,\theta} - \phi_{0,\theta zz}\zeta_0
\end{aligned}
$$

For $z < 0$ (this part is equal for all $O(\epsilon^n)$):

$$
\begin{aligned}
&\phi_{1,\theta\theta} + \phi_{1,zz} = 0 && \text{for} \quad z < 0 \\
&\phi_{1,\theta}, \phi_{1,z} \to 0 && \text{for} \quad z \to -\infty
\end{aligned}
$$

Elimination of ζ_1 from surface condtions$\Rightarrow$

$$
\phi_{1,\theta\theta} + \phi_{1,z} = 2c_1 \sin\theta \qquad \text{for} \quad z = 0 \tag{17}
$$

may be interpreted as resonant forcing by surface pressure
Periodic solution $\Rightarrow$ $c_1 = 0$ and hence $\phi_1 = 0$

$O(\epsilon^1)$

Surface elevation from surface condition

$\zeta_1 = \frac{1}{2}\cos 2\theta, \qquad \phi_1 = 0, \qquad c_1 = 0$

$O(\epsilon^2)$

Same structure as $O(\epsilon)$, but $\phi_1 = c_1 = 0$ helps

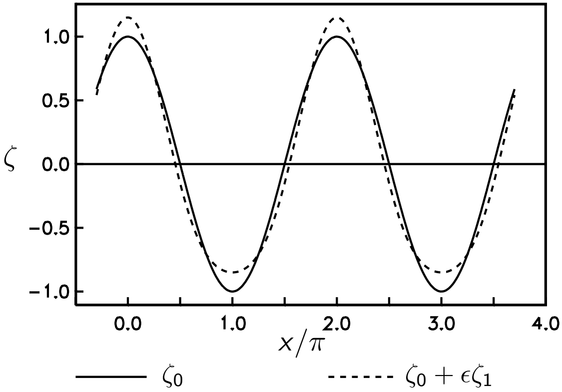
Surface condition $\Rightarrow \phi_{2,\theta\theta} + \phi_{2,z} = (2c_2 - 1)\sin\theta \text{ for } z = 0$

$\Rightarrow c_2 = \frac{1}{2}$ and $\phi_2 = 0$.

The surface contribution $\zeta_2 = \frac{3}{8}\cos(3\theta)$

$$c^* = \sqrt{\frac{g}{k^*}}\left(1 + \frac{1}{2}(a^*k^*)^2 + \cdots\right)$$

Wave form



Higher orders

The expansions take the form of Fourier series

$$\zeta = \sum_{k=0}^{\infty} b_k \epsilon^k \cos(k+1)\theta \quad (18)$$

$$\phi = \sum_{k=0}^{\infty} B_k \epsilon^k e^{(k+1)z} \sin(k+1)\theta \quad (19)$$

The highest possible wave-steepness is $\epsilon \approx 0.44$, when the crests approach apexes of 120°.
The solution is always unstable in infinite depth.