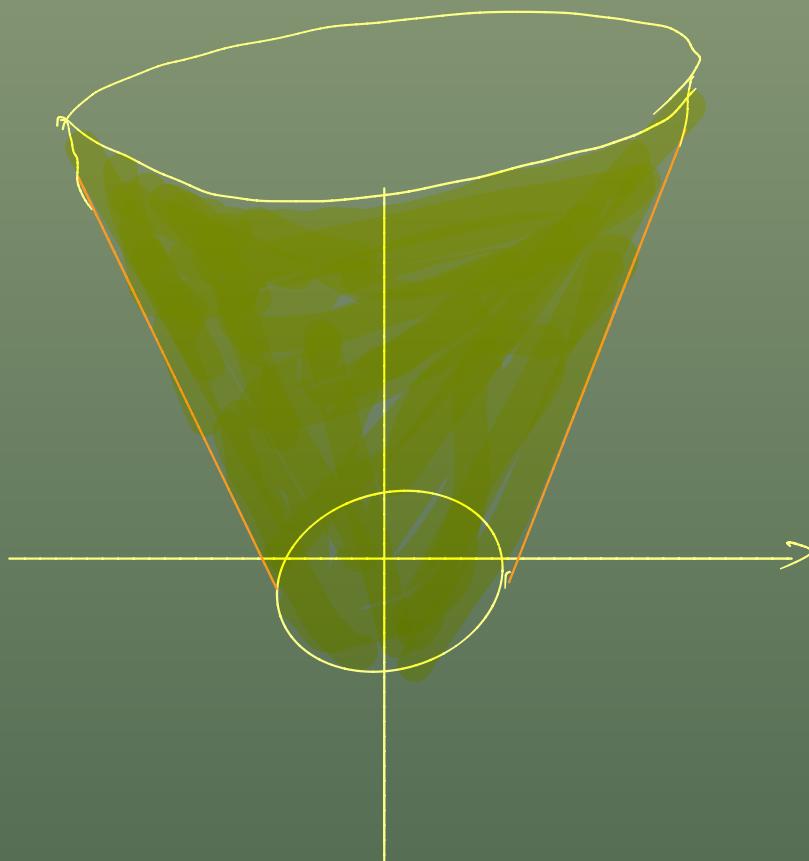


Green Functions.



Week 1.

Main Stream.

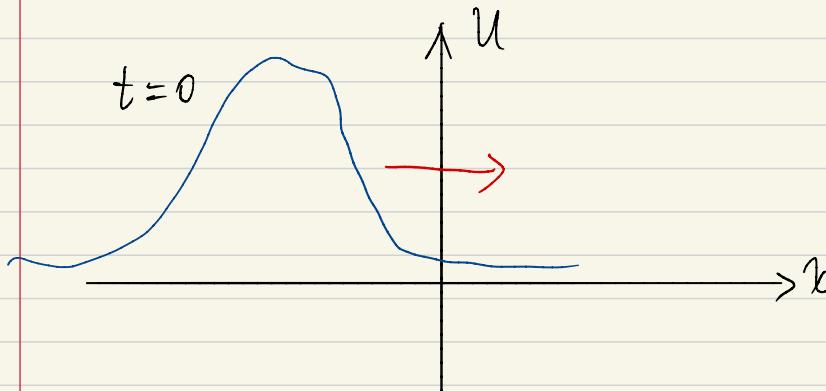
- Burger's Equation.

$$u_t + u u_x = u_{xx}, \quad x \in \mathbb{R}, t > 0$$

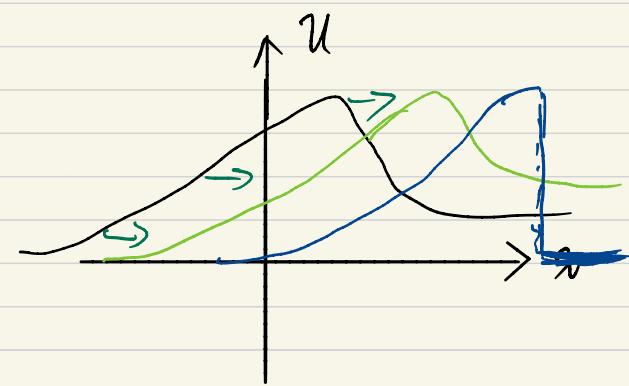
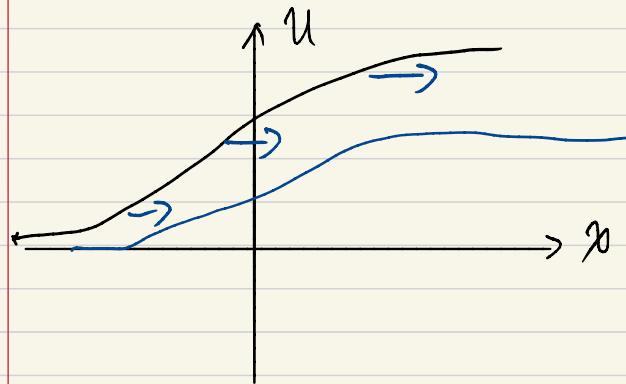
- First, $u_t + u u_x = 0$, hyperbolic equation

Transport eq.:

$$u_t + c u_x = 0$$



e.g.



$$\text{Then, } u_t + (\frac{u^2}{2})_x = u_{xx}$$

$$\text{Introduce: } B_x = u, \Rightarrow B_{xt} + (\frac{B_x^2}{2})_x = (B_{xx})_x \quad (*)$$

Integrate (*) in x , assume $B_t = 0$ in the infinity,

$$B_t + \frac{B_x^2}{2} = B_{xx}$$

Then,

$$\text{take } B = -2 \log \phi, \quad B_t = -2 \frac{\phi_t}{\phi}, \quad B_x = -2 \frac{\phi_x}{\phi}$$

$$B_x^2 = 4 \cdot \left(\frac{\phi_x}{\phi} \right)^2$$

$$\& \quad B_{xx} = -2 \frac{\phi_{xx}\phi - \phi_x^2}{\phi^2}$$

Note:

$$-2 \frac{\phi_t}{\phi} + 2 \frac{\phi_x^2}{\phi^2} = -2 \frac{\phi_{xx}}{\phi} + \frac{2\phi_x^2}{\phi^2}$$

$\Rightarrow \phi_t = \phi_{xx}$ heat equation!

$$\text{Thus, } \phi(x, t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} \phi(y, 0) dy = \underline{\underline{\int}} \underline{\underline{\int}} k * \phi(x, 0)$$

Since $\underline{\underline{\phi(x, t)}} = e^{-\frac{1}{2}B(x, t)}$

& $B_x = U$,
then

$$U(x, t) = \int_{-\infty}^{\infty} U(y, t) dy \quad \text{initial data.}$$

$$\phi(x, t) = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \int_{-\infty}^y U(z, 0) dz} \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} dy$$

Green function of
heat equation.

Hopf-Cole transform.

$$U(x, t) = \partial_x B = \partial_x (-2) \log \phi = -\frac{2\partial_x}{\phi}$$

This can be
solved by

Next, want to use a new method to do it!

Preliminaries:

• Fix Point Theorem:

Picard's iteration, $\begin{cases} y' = f(t, y) \\ y^{(0)} = y_0 \end{cases}$

$$y_1(t) = y_0 + \int_0^t f(z, y_0) dz$$

$$y_2(t) = y_0 + \int_0^t f(z, y_1(z)) dz$$

:

$$y_n(t) = y_0 + \int_0^t f(z, y_{n-1}(z)) dz$$

Want to have ratio test:

$$y_n = y_0 + \int_0^t f(z, y_{n-1}) dz$$

$$- y_{n-1} = y_0 + \int_0^t f(z, y_{n-2}) dz$$

$$y_n - y_{n-1} = \int_0^t f(z, y_{n-1}) - f(z, y_{n-2}) dz$$

MVT:

$$|f(z, y_{n-1}) - f(z, y_{n-2})| \leq |y_{n-1} - y_{n-2}| \cdot \max_{z \in [y_{n-1}, y_{n-2}]} |f_z(z, g)| \leq M$$

Then,

$$|y_n - y_{n-1}| \leq M \int_0^t |y_{n-1} - y_{n-2}| dz$$



Metric space:

$$\text{Fix } t_0, \|y_n - y_{n-1}\| = \sup_{z \in [0, t_0]} |y_n(z) - y_{n-1}(z)|$$

Let $t < t_0$.

$$\Rightarrow |y_n(t) - y_{n-1}(t)| \leq M \int_0^t \|y_{n-1} - y_{n-2}\| dz \leq M t_0 \|y_{n-1} - y_{n-2}\|$$

by

$$\Rightarrow \|y_n - y_{n-1}\| \leq M t_0 \|y_{n-1} - y_{n-2}\| \quad \text{How? By taking } t_0 \text{ small}$$

If choose $M t_0 < 1$, then $\lim_{n \rightarrow \infty} \|y_n - y_{n-1}\| = 0$

$$\text{Moreover, } \|y_n - y_{n-1}\| \leq C_0 (M t_0)^n \quad \text{--- (*)}$$

Now, just need to choose t_0 at beginning.

Then, consider:

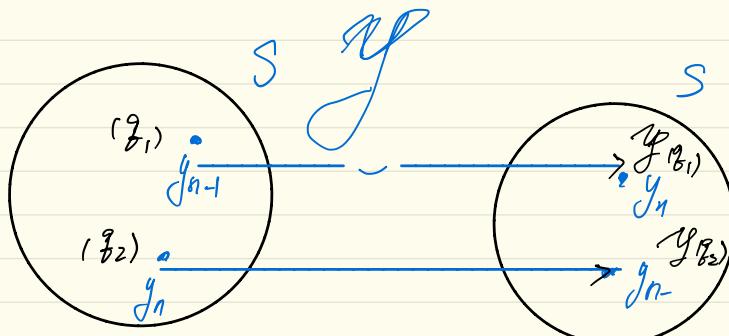
$y_1 + (y_2 - y_1) + \dots + (y_n - y_{n-1})$ is convergent by $(*)$

Implying:

$$\lim_{n \rightarrow \infty} y_n \text{ exists}$$

Consider fixed point thm:

$$y_n^{(t)} = y_0 + \int_0^t f(z, y_{n-1}(z)) dz$$



$$\|g_1 - g_2\|$$

$$\|\varphi(g_1) - \varphi(g_2)\|$$

If $\|\varphi(g_1) - \varphi(g_2)\| \leq \alpha \|g_1 - g_2\|$, with $\alpha < 1$,

then there exists a fixed point g^* .

Fix point Thm: $\begin{cases} f: S \rightarrow S, \text{ complete space } S. \\ f \text{ is a contract map.} \end{cases} \Rightarrow \exists! \text{ fixed point.}$

Want to show it satisfies the condition of fix point thm.

- $y_0 \in S$, & $y_n \in \varphi(y_{n-1}) \quad \forall n \geq 1$

$$\sum_{n=1}^{\infty} \|y_n - y_{n-1}\| \text{ converges.}$$

Proof:

$$\|y_n - y_{n+1}\| = \|\mathcal{F}(y_{n-1}) - \mathcal{F}(y_{n-2})\| < 2 \|y_{n-1} - y_{n-2}\|$$

$$\Rightarrow \|y_n - y_{n-1}\| < \alpha^{n-1} \|y_1 - y_0\|$$

$$\Rightarrow \sum_{n=1}^{\infty} \|y_n - y_{n+1}\| < \frac{\|y_1 - y_0\|}{\alpha}$$

What is the Green function or fundamental solutions?

$$\begin{cases} \vec{y}' = A\vec{y} \\ \vec{y}(0) = \vec{y}_0 \end{cases} \stackrel{\text{"ODE"}}{\Rightarrow} e^{At} (\vec{y}' - A\vec{y}) = \vec{0}$$

$$\Rightarrow (e^{-At} \vec{y})' = \vec{0}$$

$$\Rightarrow e^{-At} \vec{y} = \vec{c}, \quad \vec{c} = \vec{y}_0$$

$$\Rightarrow y(t) = e^{\lambda t} y_0$$

Now, A is a matrix

Q: What is e^{At} ? $A = S^{-1} \Lambda S$

$$\text{Ans: } e^A = g^{-1} e^A S \quad , \quad e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

"PDE",

$$\partial_t u = \partial_x u$$

$$\{a_0, a_1, \dots\} \in \mathbb{R}^\infty$$

$$u(x,t) = a_0(t) + a_1(t)x + \dots + a_n(t)x^n + \dots$$

By Taylor's expansion:

$$2\omega U = a_1(t) + 2a_2(t)x + \dots + n a_n(t)x^{n-1} + \dots$$

$$(a_1, 2a_2, \dots, n a_n, \dots)$$

So,

$$\frac{d}{dt} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

$$L^2(\mathbb{R}) = \left\{ f \mid \int_{-\infty}^{\infty} f_m^2 dx < \infty \right\}.$$

$$(f, g) = \int_{-\infty}^{\infty} f_m g_m dx - \text{Inner Product.}$$

$$\|f\|^2 = (f, f) - \|\cdot\|: \text{Norm for } L^2(\mathbb{R})$$

$$\|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x)|$$

$$\|\cdot\|_L = \int_{-\infty}^{\infty} |f| dx$$

Fourier transform:

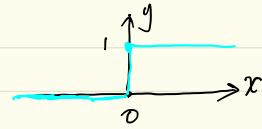
$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-iyx} f(x) dx$$

Inverse Fourier transform:

$$f(x) = \frac{1}{2\pi} \int \hat{f}(y) e^{iyx} dy$$

$$\delta(x) = \frac{d}{dx} H(x)$$

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(x) dx = \int_{-\infty}^{\infty} \frac{d}{dx} H(x) dx = H \Big|_{-\infty}^{\infty} = 1 - 0 = 1$$

$f \in C_0(\mathbb{R})$: then

$$\begin{aligned} & \int_{-\infty}^{\infty} \delta(x) f(x) dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dx} H(x) \cdot f(x) dx \stackrel{\text{I.B.P.}}{=} - \int_{-\infty}^{\infty} H(x) f'(x) dx \\ &= - \left(\int_{-\infty}^0 H(x) f'(x) dx + \int_0^{+\infty} H(x) f'(x) dx \right) \\ &= -f(x) \Big|_0^\infty = - (f(\infty) - f(0)) = f(0). \end{aligned}$$

Property:

For any continuous function $g(x)$,

$$\int_{\mathbb{R}} \delta(x-x_0) g(x) dx = g(x_0).$$

$$\text{Then, } \hat{\delta}(\eta) = \int e^{-i\eta x} \delta(x) dx = e^{-i\eta 0} = 1$$

Properties of Fourier Transform:

$$\begin{aligned} ① \widehat{\delta_x f} &= \int_{-\infty}^{\infty} e^{-i\eta x} \delta_x f dx \stackrel{\text{I.B.P.}}{=} - \int_{-\infty}^{+\infty} (e^{-i\eta x})'_x f dx = i\eta \int_{-\infty}^{\infty} f \cdot e^{-i\eta x} dx \\ &= i\eta \cdot \hat{f}(\eta) \end{aligned}$$

$$\Downarrow (\delta_x f)^{\wedge} = i\eta \cdot \hat{f}(\eta)$$

Application: $\begin{cases} c \frac{du}{dt} + c \frac{du}{dx} u = 0 \\ u(x, 0) = u_0(x) \end{cases}$

Take Fourier transform:

$$\begin{aligned} \partial_t \hat{u} + c \operatorname{ig} \hat{u} &= 0, \quad \operatorname{ig} \text{ fixed} \\ \Rightarrow \partial_t (e^{\operatorname{ig} y t} \hat{u}) &= 0 \Rightarrow \hat{u}(y, t) = e^{-\operatorname{ig} y t} \hat{u}(y, 0) \\ &= e^{-\operatorname{ig} y t} \cdot \int_{-\infty}^{\infty} e^{-\operatorname{ig} x y} u_0(x) dx \\ &= \int_{-\infty}^{\infty} e^{-\operatorname{ig} (x+ct)} u_0(x) dx \end{aligned}$$

$$\begin{aligned} \text{Change } x &= y+ct. \\ x &= y-ct \end{aligned}$$

$$\begin{aligned} \Rightarrow &= \int_{-\infty}^{\infty} e^{-\operatorname{ig} x} u_0(x-ct) dx \\ &= \int_{-\infty}^{\infty} e^{-\operatorname{ig} x} u_0(x-ct) dx \approx u_0(x-ct)^{\wedge} \end{aligned}$$

②
$$f(x-a)^{\wedge} = e^{-iax} \hat{f};$$

Laplace Transform / Fourier Transform

Convolution:

$$f * g(x) = \int_{\mathbb{R}} f(x-y) \cdot g(y) dy$$

③
$$(f * g)^{\wedge} = \hat{f} \cdot \hat{g};$$
 Notice, $\hat{u}(y, t) = e^{-\operatorname{ig} y t} \cdot \hat{u}(y, 0)$

$$\text{for } \delta(x-ct) = \hat{\delta}(y) \cdot e^{-icty} = e^{-icty}$$

$$\Rightarrow u(x, t) = \delta(x-ct) * u(x, 0)$$

$$\begin{aligned} &= \int \delta(x-ct-y) \cdot u(y, 0) dy = u(x-ct, 0) \\ &= u_0(x-ct). \# \end{aligned}$$

[R.K.]: thus, $(e^{-i\gamma ct}) = \widehat{\delta(x-ct)}$ is the Green Function of $u_t + c u_x = 0$. & $u(x, t) = \widehat{\delta}(x-ct) \neq u_0(x)$.

Recall :

$$\begin{aligned} \text{Fourier Transform: } \delta(x) &\longrightarrow 1 \\ \delta(x-ct) &\longrightarrow e^{-i\gamma ct} \\ f &\longrightarrow \hat{f}(y) \\ \hat{f}' &\longrightarrow i\gamma \hat{f}(y) \\ \hat{f} * g &\longrightarrow \hat{f}(y) \cdot \hat{g}(y) \end{aligned}$$

Inverse Fourier transform:

$$\hat{f} \longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(y) \cdot e^{iyx} dy$$

Now,

$$\begin{cases} g_t + c g_x = \delta(x, t) \\ g(x, 0) = g_0(x) \end{cases}$$

Take Fourier transform:

PDE \Rightarrow ODE



$$\begin{cases} \hat{g}_t + c i\gamma \hat{g} = \hat{\delta}(y, t) \\ \hat{g}(y, 0) = \hat{g}_0(y) \end{cases} \quad (2)$$

Next, $e^{i\gamma ct} \cdot (2)$

$$\Rightarrow \frac{d}{dt} (e^{i\gamma ct} \hat{g}) = e^{i\gamma ct} \widehat{s}(y, t)$$

Take integral from 0 to z ,

$$\int_0^z \frac{d}{dt} [e^{iyct} \hat{g}(y, t)] dt = \int_0^z e^{iyct} \hat{s}(y, t) dt$$

$$\Rightarrow e^{iycz} \hat{g}(y, z) - \hat{g}(y, 0) = \int_0^z e^{iyct} \hat{s}(y, t) dt$$

$$\Rightarrow \hat{g}(y, z) = e^{-iycz} \hat{g}(y, 0) = \int_0^z e^{iyc(t-z)} \hat{s}(y, t) dt$$

$$g(x, z) - \underbrace{\delta(x-cz) * g_0}_{\int \delta(x-c(z-y)) g(y) dy} = \int_0^z \delta(x-c(z-t)) * S(x, t) dt$$

$$\int \delta(x-c(z-y)) g(y) dy$$

$$g_0(x-cz)$$

final result.

$$\Rightarrow \boxed{g(x, z) = g_0(x-cz) + \int_0^z S(x-c(z-t), t) dt}$$

$$\text{i.e. } g(x, t) = g_0 * \delta(x-ct) + \int_0^t S(x, z) * \delta(x-c(z-t)) dz$$

Wave Equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x, 0) = u_0(x) \\ u_t(x, 0) = u_1(x) \end{cases}$$

Take Fourier Transform,

$$\hat{u}_{tt} - (i\gamma)^2 \hat{u} = 0 \rightarrow \hat{u}_{tt} + \gamma^2 \hat{u} = 0$$

$$\begin{cases} \hat{u}(x, 0) = \hat{u}_0(\gamma) & \text{---①} \\ \hat{u}_t(x, 0) = \hat{u}_1(\gamma) & \text{---②} \end{cases}$$

$$\lambda^2 + \gamma^2 = 0, \Rightarrow \lambda = \pm i\gamma$$

$$\hat{u}(\gamma, t) = A \cdot e^{i\gamma t} + B e^{-i\gamma t}$$

use condition ① & ② to solve A, B.

$$\hat{u}_0(\gamma) = A + B$$

$$\hat{u}_1(\gamma) = i\gamma A - i\gamma B \Leftrightarrow \frac{\hat{u}_1(\gamma)}{i\gamma} = A - B$$

$$\text{So, } \boxed{A = \frac{1}{2} (\hat{u}_0(\gamma) + \frac{\hat{u}_1(\gamma)}{i\gamma})}, \text{ & } \boxed{B = \frac{1}{2} (\hat{u}_0(\gamma) - \frac{\hat{u}_1(\gamma)}{i\gamma})}$$

$$\begin{aligned} \hat{u}(\gamma, t) &= \frac{1}{2} (\hat{u}_0(\gamma) + \frac{\hat{u}_1(\gamma)}{i\gamma}) \cdot e^{i\gamma t} + \\ &\quad \frac{1}{2} (\hat{u}_0(\gamma) - \frac{\hat{u}_1(\gamma)}{i\gamma}) \cdot e^{-i\gamma t} = \boxed{\frac{1}{2} \hat{u}_0(\gamma) (e^{i\gamma t} + e^{-i\gamma t}) +} \\ &\quad \boxed{\frac{1}{2} \hat{u}_1(\gamma) \frac{(e^{i\gamma t} - e^{-i\gamma t})}{i\gamma}} \\ &= \frac{1}{2} (\overbrace{u_0(x+t)}^{} + \overbrace{u_0(x-t)}^{}) \\ &\quad + \frac{1}{2} \left[\int u_1(x+t) - u_1(x-t) dx \right] \end{aligned}$$

$$\text{So, } u(x,t) = \frac{1}{2} (u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(\xi) d\xi.$$

For Inhomogeneous,

$$\begin{cases} u_{tt} - u_{xx} = S(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$\Rightarrow \hat{u}_{tt} + \eta^2 \hat{u} = \hat{S}(\eta, t)$$

$$\begin{cases} \hat{u}(\eta, 0) = 0 \\ \hat{u}_t(\eta, 0) = 0 \end{cases}$$

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u_t(0, x) = \delta(x) \end{cases}$$

Fourier
Transform. $u(0, x) = 0$

Want to solve $\hat{u}(\eta, t)$:

Introduce $G(z, \eta)$, $z \in (0, t)$

$$\text{s.t. } \begin{cases} G_z(t, \eta) = 1 \\ G_t(t, \eta) = 0 \end{cases}$$

$$(-\partial_z)^2 G + \eta^2 G = 0 \quad (*)$$

"Green function"

$$\int_0^t G(z, \eta) \cdot [\hat{u}_{zz}(\eta, z) + \eta^2 \hat{u}(\eta, z)] dz = \int_0^t G(z, \eta) \cdot \hat{S}(y, z) dz$$

$$G(z, \eta) \hat{u}_z \Big|_0^t - \int_0^t \hat{u}_z \cdot G_z(z, \eta) dz$$

$$\Downarrow$$

$$-(G_z \hat{u} \Big|_0^t - \int_0^t G_{zz} \hat{u} dz) +$$

$$= -G_z \hat{u} \Big|_t + \int_0^t G_{zz} \hat{u} dz +$$

$$= -\hat{u}(\eta, t) + \int_0^t G_{zz} \hat{u} dz + \int_0^t G(z, \eta) \eta^2 \hat{u}(y, z) dz$$

$$= -\hat{u}(\eta, t) + \int_0^t (G_{zz} + \eta^2 G_z) \hat{u} dz$$

\Downarrow by (*)

$$\Rightarrow \hat{u}(y, t) = - \int_0^t G(z, y) \cdot \hat{s}(z, y) dz.$$

W.T.S get $G(z, y)$:

$$\text{Let } \tilde{G}(z, y) = G(t+z, y).$$

Then,

$$\begin{cases} \tilde{G}_{yy} + y^2 \tilde{G} = 0 \\ \tilde{G}_y(0, y) = 1 \\ \tilde{G}(0, y) = 0 \end{cases}$$

$$\Rightarrow r^2 + y^2 = 0 \quad \text{i.e. } r = \pm iy$$

$$\Rightarrow \tilde{G}(z, y) = A \cdot e^{iyz} + B \cdot e^{-iyz}$$

$$\Rightarrow A = -B \quad \text{by initial value}$$

$$\Rightarrow \tilde{G}_y(0, y) = iyA + iyA = 1 \quad \text{implying} \quad A = \frac{1}{2iy}$$

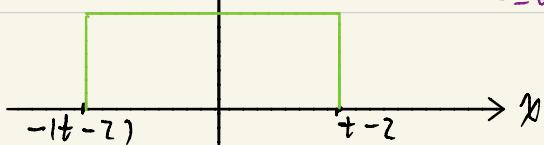
$$\Rightarrow \tilde{G}(z, y) = \frac{1}{2iy} (e^{iyz} - e^{-iyz}),$$

$\uparrow \quad \uparrow$
 $\delta(y+t) \quad \delta(y-t)$.
F.T.

$$\begin{aligned} \Rightarrow G(z, y) &= \frac{1}{2iy} (e^{iy(z-t)} - e^{-iy(z-t)}) \\ &= \frac{1}{2iy} (e^{-iy(t-z)} - e^{iy(t-z)}) \end{aligned}$$

$$\Rightarrow \hat{u}(t, y) = \int_0^t \frac{1}{2iy} (e^{iy(t-z)} - e^{-iy(t-z)}) \cdot \hat{s}(z, y) dz$$

$\uparrow \quad \uparrow$
 $\frac{1}{2} \int_{-\infty}^y (\delta(y+(t-z)) - \delta(y-(t-z))) dy$



$$\chi_{[-t+2, t-2]}(x) * s(x, x)$$

$$\Rightarrow \hat{u}(y, t) = \int_0^t G(y, z) \hat{S}(y, z) dz$$

$$G = \frac{e^{i(y-t)z} - e^{-i(y-t)z}}{2iz} \quad \text{solved by previous step.}$$

$$u(x, t) = \int_0^t \int_{x-(t-z)}^{x+(t-z)} \frac{S(\xi, z) dz}{z} d\xi$$

Heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x) \end{cases}$$

$$\begin{cases} \hat{u}_t + i^2 \hat{u} = 0 \\ \hat{u}(x, 0) = \hat{u}_0(y) \end{cases} \Rightarrow \frac{d}{dt} (e^{i^2 t} \hat{u}) = 0$$

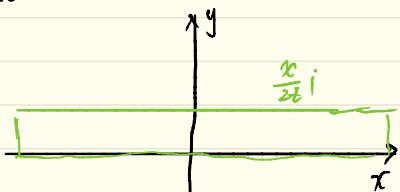
$$e^{i^2 t} \cdot \hat{u}(y, t) = \hat{u}(y, 0)$$

$$\Rightarrow \hat{u}(y, t) = \frac{e^{-i^2 t}}{\pi} \frac{\hat{u}(y, 0)}{\text{known}}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i^2 t + iyx} dy$$

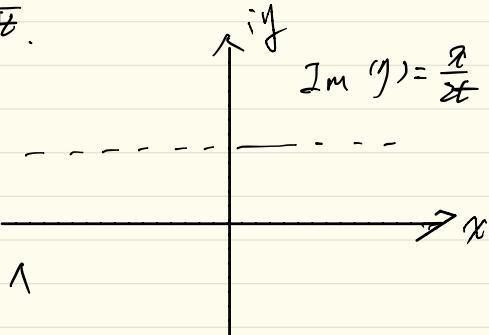
↙ look for its inverse.

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i^2 t + iyx + \frac{y^2}{4t} - \frac{x^2}{4t}} dy = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-t(y - \frac{ix}{2t})^2 - \frac{x^2}{4t}} dy$$



By Complex analysis

$$\begin{aligned}
 &= \frac{e^{-\frac{x^2}{4t}}}{2\pi} \cdot \int_{\text{Im }(\eta) = \frac{x}{2t}} e^{-t|\eta| - \frac{i\eta}{2t}} d\eta = \frac{e^{-\frac{x^2}{4t}}}{2\pi} \int_{-\infty}^{\infty} e^{-t\sqrt{\eta^2 + \frac{x^2}{4t}}} \frac{d\eta}{\sqrt{\eta^2 + \frac{x^2}{4t}}} \quad \nu = \eta - \frac{ix}{2t}, \\
 &= \frac{e^{-\frac{x^2}{4t}}}{2\pi\sqrt{t}} \cdot \int_{-\infty}^{\infty} e^{-w^2} dw, \quad w = \sqrt{\frac{x^2}{4t}}, \\
 &= \frac{\sqrt{\pi}}{2\pi\sqrt{t}} \cdot e^{-\frac{x^2}{4t}}
 \end{aligned}$$



$$\text{So, } \hat{u}(x, t) = \left[\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * u(x, 0) \right]^{\wedge}$$

$$u(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * u(x, 0) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u(y, 0) dy.$$

Heat kernel: $\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

$$\hat{k}(\eta, t) = e^{-\eta^2 t}.$$

Duhamel's Principle:

$$\begin{cases} u_t - u_{xx} = s(x, t) \\ u(x, 0) = 0 \end{cases}$$

By Fourier Transform

$$\begin{cases} \hat{u}_t + \eta^2 \hat{u} = \hat{s}(\eta, t) \\ \hat{u}(\eta, 0) = 0 \end{cases} \quad \text{--- } \textcircled{1}$$

Integrate $\textcircled{1}$ by $e^{\eta^2 t}$:

$$\Rightarrow \frac{d}{dt} (e^{\eta^2 t} \cdot \hat{u}) = \hat{s}(\eta, t) \cdot e^{\eta^2 t}$$

Change "t" to "z":

$$\int_0^t \left[\frac{d}{dz} (e^{\eta^2 z} \cdot \hat{u}) \right] dz = \hat{s}(\eta, z) e^{\eta^2 z} \Big|_0^t$$

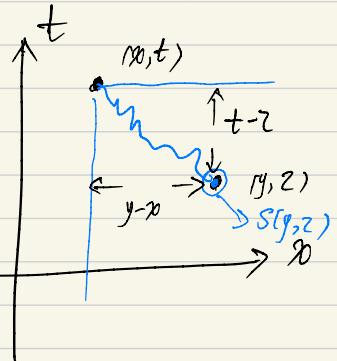
$$\Rightarrow e^{\eta^2 t} \hat{u}(\eta, t) = \int_0^t \hat{s}(\eta, z) e^{\eta^2 z} dz$$

$$\Rightarrow \hat{u}(\eta, t) = \int_0^t e^{\eta^2 (z-t)} \hat{s}(\eta, z) dz$$

$$\Rightarrow u(x, t) = \boxed{\int_0^t k(x, t-z) * s(x, z) dz}$$

i.e.

$$u(x, t) = \int_0^t \int_{-\infty}^{+\infty} k(x-y, t-z) \cdot s(y, z) dy dz$$



Now, For general,

$$(Forward Eq.) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = u_{xx} + s(x, t) \\ u(x, 0) = u_0(x) \end{cases} \quad \rightarrow \quad \text{(1)}$$

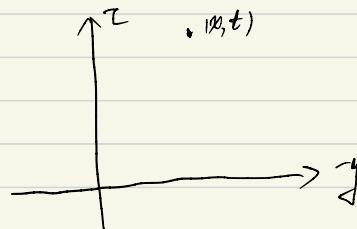
Then, $u(x, t) = R(x, t) * u_0(x) + \int_0^t k(x, t-z) * s(x, z) dz$

(ii) (-)

Introduce a function
 $G(y, z)$

s.t.

$$\text{Backward Eq. } \left\{ \begin{array}{l} \textcircled{1} \quad R(y, t) = \delta(x-y) \\ \textcircled{2} \quad [-\partial_z - (-\partial_y)^2] G(y, z) = 0 \end{array} \right.$$



For

$$\textcircled{1} \quad \int_{-\infty}^{+\infty} [u_z(y, z) - u_{yy}(y, z) - s(y, z)] G(y, z) dy dz = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} G(y, z) U(y, z) dy \Big|_{z=0}^{z=t} + \int_0^t \int_{-\infty}^{+\infty} (-\partial_z - (-\partial_y)^2) G(y, z) U(y, z) dy dz = 0$$

$$- G(y, z) S(y, z) dy dz = 0$$

$$\Rightarrow u(x, t) = \int_{-\infty}^{\infty} G(y, 0) U(y, 0) dy + \int_0^t \int_{-\infty}^{\infty} G(y, z) S(y, z) dy dz$$

$$\Rightarrow Q(y, z) = k(x-y, t-z).$$

"Hard"
A nonlinear problem:

$$\begin{cases} u_t + uu_x = u_{xx} \\ |u(x, 0)| \leq \varepsilon \cdot \frac{e^{-\frac{x^2}{4}}}{\sqrt{16\pi}}, \quad u(x, 0) = U(x) \end{cases}$$

How to construct a sol.?

Observe: uu_x is $\mathcal{O}(\varepsilon^2)$ & much smaller than other terms.

thus, write

$$u_t - u_{xx} = -\left(\frac{u^2}{2}\right)_x$$

By Duhamel's principle,

$$u(x, t) = k(x, t) * U(x) + \underbrace{\int_0^t k(x, t-z) * \left(-\left(\frac{u^2}{2}\right)_x\right) dz}_{-\int_0^t k(x, t-z) * \left[\frac{u^2 x_z}{2}\right]_x dz}$$

$$\text{So, } u(x, t) = k(x, t) * U(x) - \underbrace{\int_0^t k(x, t-z) * \left[\frac{u^2 x_z}{2}\right]_x dz}_{-\int_0^t \int_{-\infty}^{\infty} k(x-y, t-z) \left(\frac{u^2 y_z}{2}\right)_y dy dz}$$

$$-\int_0^t \int_{-\infty}^{\infty} k(x-y, t-z) \left(\frac{u^2 y_z}{2}\right)_y dy dz$$

$$+ \int_0^t \int_{-\infty}^{\infty} k_y(x-y, t-z) \cdot \frac{u^2 y_z}{2} dy dz$$

$$u(x, t) = \int_{-\infty}^{\infty} k(x-y, t) \cdot U(y) dy + \int_0^t \int_{-\infty}^{\infty} k_y(x-y, t-z) \cdot \frac{u^2 y_z}{2} dy dz$$

Construct iteration:

$$u_k(x, t) = \int_{-\infty}^{\infty} k(x-y, t) \cdot U(y) dy + \int_0^t \int_{-\infty}^{\infty} k_y(x-y, t-z) \cdot \frac{u^2 y_z}{2} dy dz, \quad k \geq 1$$

Initial u_0 : $u_0(x, t) = \int_{-\infty}^{\infty} k(x-y, t) U(y) dy = k \neq U(x)$

$$\Rightarrow |u_0(x, t)| \leq \varepsilon \int_{-\infty}^{\infty} k(x-y, t) |U(y)| dy = \varepsilon k(x, t+1)$$

Assume $|u_k(x, t)| \leq 2\varepsilon k(\frac{x}{2}, t+1)$.

By induction, this is true for $k \in \mathbb{N}$.

Define a norm,

$$\|f\| = \sup_{(x, t)} \frac{|f(x, t)|}{k(\frac{x}{2}, t+1)} \Rightarrow \|u_0\| \leq \sup_{(x, t)} \frac{|k(x, t+1)|\varepsilon}{k(\frac{x}{2}, t+1)}$$

$$\leq \varepsilon$$

By constructing,

$$|u_k - u_{k-1}| \leq \int_0^t \int_{-\infty}^{\infty} k(x-y, t-2) \left| \frac{u_{k-1} - u_{k-2}}{2} \right| \underbrace{[U_{k-1} + U_{k-2}]}_{4\varepsilon k(\frac{y}{2}, t+1)} dy dz$$

$$\leq 4\varepsilon \|u_{k-1} - u_{k-2}\| \cdot \int_0^t \int_{-\infty}^{+\infty} k_y(x-y, t-2) \underbrace{k(\frac{y}{2}, t+1)}_{K^2(\frac{y}{2}, t+1)} dy dz$$

$$\begin{aligned} |k_y(x-y, t-2)| &= \left| \frac{2(x-y)}{4(t-2)} k(x-y, t-2) \right| \\ &\leq \frac{C}{\sqrt{t-2}} \cdot k(\frac{x-y}{2}, t-2). \end{aligned}$$

$$\Rightarrow k_y(\frac{y}{2}, t+1) \leq C \cdot k(\frac{y}{2}, t+1) / \sqrt{2}$$

$$\begin{aligned} \text{Eventually, } \textcircled{E} &\leq 4\varepsilon \|u_{k-1} - u_{k-2}\| \cdot \int_0^t k(\frac{x}{2}, t+1) \cdot \frac{1}{\sqrt{t-2}} \frac{1}{\sqrt{t-2}} dz \cdot C \\ &\leq 4\varepsilon C \cdot \|u_{k-1} - u_{k-2}\| \cdot k(\frac{x}{2}, t+1) \end{aligned}$$

$$\Rightarrow \|u_k - u_{k-1}\| \leq 4\varepsilon C \cdot \|u_{k-1} - u_{k-2}\|$$

Observe:

$f: \{\|u\| \leq 2\varepsilon\} \rightarrow \{\|u\| \leq 2\varepsilon\}$: complete metric space.

$$f(u_k) = u_{k+1}$$

• f is a contraction:

$$\begin{aligned} \|f(u_{k-1}) - f(u_{k-2})\| &= \|u_k - u_{k-1}\| \\ \text{By fixed point thm,} \quad &\leq 4\varepsilon C_1 \cdot \|u_{k-1} - u_{k-2}\| \\ \Rightarrow \exists! u \text{ s.t. } f(u) &= u. \end{aligned}$$

Q.E.D.

Recall:

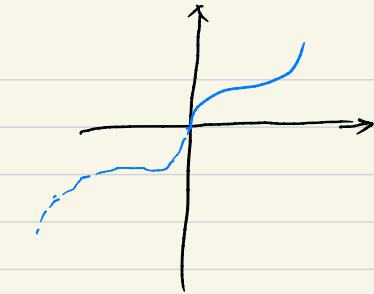
(Case 1):

$$\begin{cases} u_{tt} = u_{xx}, & x > 0 \\ u(0, t) = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1 \end{cases}$$

By odd extension,

$$f(x), \quad x > 0$$

$$f_{\text{odd}}(x) = \begin{cases} f(x), & \text{if } x > 0 \\ -f(-x), & \text{if } x < 0 \end{cases}$$

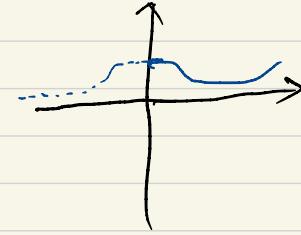


(Case 2):

$$\begin{cases} u_{tt} = u_{xx}, & x > 0 \\ u_x(0, t) = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1 \end{cases}$$

By even extension:

$$f_{\text{even}}(x) = \begin{cases} f(x), & \text{if } x > 0 \\ f(-x), & \text{if } x < 0 \end{cases}$$



What if for mixed bdd?

$$u_{tt} = u_{xx}, \quad x > 0$$

$$u_x(0, t) = a \cdot u(t), \quad u(0, 0) = u_0(x), \quad u_t(0, 0) = u_1(x)$$

Now, there is a new method, which can solve the problem via Laplace Transform.

$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \Rightarrow u(x, t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$

Laplace Transform:

$$\underline{u(x, s)} = \int_0^\infty e^{-st} \cdot u(x, t) dt, \quad \text{Re}(s) \geq 0.$$

Properties:

① $\underline{\underline{u}_t(x, s)} = \int_0^\infty e^{-st} \cdot u_t(x, t) dt = e^{-st} \cdot u(x, t) \Big|_0^\infty + s \int_0^\infty e^{-st} u(x, t) dt$

$\Rightarrow \underline{\underline{u}_t(x, s)} = -u(x, 0) + s \cdot \underline{u(x, s)}$

For ②, $\begin{cases} u_{tt} = \frac{\partial^2}{\partial x^2} u \\ u(x, 0) = \delta(x) \end{cases} \Rightarrow \underline{\underline{s}^2 u - \delta(x)} = \frac{\partial^2}{\partial x^2} \underline{u}.$

This is ODE.

So, $\underline{u} = \begin{cases} A_+ \cdot e^{-\sqrt{s}x} + B_+ \cdot e^{\sqrt{s}x}, & x > 0 \\ A_- \cdot e^{-\sqrt{s}x} + B_- \cdot e^{\sqrt{s}x}, & x < 0 \end{cases}$

By contin. at $x=0$, $A_+ = B_-$

$$\text{So, } \mathcal{L}u_0 = \begin{cases} -\sqrt{s}A + e^{-\sqrt{s}x}, & \text{if } x > 0 \\ \sqrt{s}A + e^{\sqrt{s}x}, & \text{if } x < 0 \end{cases} \Rightarrow \text{The size of the jump is: } -2\sqrt{s}A$$

$$\Rightarrow -2\sqrt{s}A = 1, \text{ i.e. } A_f = \frac{1}{2\sqrt{s}}$$

Finally, $\mathcal{L}u = \begin{cases} \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}x}, & x > 0 \\ \frac{1}{2\sqrt{s}} \cdot e^{\sqrt{s}x}, & x < 0 \end{cases}$

R.K.: Using this to solve lots of eq.s.

(*)

$\Rightarrow \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x|}, \Rightarrow u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$

Eg 1: $\begin{cases} u_t = u_{xx}, & x > 0 \\ u(0, t) = 0, & u(x, 0) = u_0(x). \end{cases}$

By odd extension, u^{odd} :

$$\begin{cases} u_0^{odd} = u_m^{odd}, & x \in \mathbb{R} \\ u_0(x, 0) = u_0^{odd}(x) \end{cases}$$

By Duhamel's principle:

$$u^{odd}(x, t) = \int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y) dy$$

$$\Rightarrow u^{odd}(x, t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u_0(y) dy = \int_0^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u_0(y) dy = \int_0^{\infty} \left[\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}} \right] u_0(y) dy$$

"Green function of"
↑ Eg 1:

$$\text{So, } \Rightarrow \begin{cases} g_t = g_{xx}, & x > 0 \\ g(0, t, y) = 0 \\ g(x, 0, y) = \delta(x-y) \end{cases}$$

Applying Lap. Transform, $\Rightarrow \mathcal{L}g - \delta(x-y) = \frac{\partial^2}{\partial x^2} \mathcal{L}g$

$$\begin{cases} \mathcal{L}g(0, s, y) = 0 \end{cases}$$

$$\Rightarrow \mathcal{L}g = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x-y|} \text{ defined } \alpha(x, s)$$

$$\Rightarrow S\alpha(x, s) - \alpha_{xx}(x, s) = 0 \quad \& \quad \alpha(x, s) = A_f e^{-\sqrt{s}x}, x > 0$$

$$\alpha(0, s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|} \Rightarrow A_f = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|}$$

$$\Rightarrow \alpha(x, s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x-y|} \Rightarrow \mathcal{L}g = \frac{1}{2\sqrt{s}} (e^{-\sqrt{s}|x-y|} - e^{-\sqrt{s}|x+y|})$$

$$\Rightarrow g(x, t, y) = \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{|x+y|^2}{4t}}}{\sqrt{4\pi t}} *$$

Verify:

$$L_u(x, s) = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s} \cdot |x|},$$

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$$

$$Lu = \int_0^\infty \frac{e^{-st} \cdot e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} dt,$$

Let $\tilde{t} = \sqrt{t}$

$$= \int_0^\infty \frac{1}{2\sqrt{\pi}\tilde{t}} \cdot e^{-st^2 - \frac{x^2}{4\tilde{t}^2}} \cdot 2\tilde{t} d\tilde{t}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-st^2 - \frac{x^2}{4\tilde{t}^2}} d\tilde{t}$$

$$= \frac{e^{-\sqrt{s}|x|}}{\sqrt{\pi}} \cdot \int_0^\infty e^{-(\sqrt{s}\tilde{t} + \frac{|x|}{2\tilde{t}})^2} d\tilde{t}, \text{ Let } y = \sqrt{s}\tilde{t} + \frac{|x|}{2\tilde{t}}$$

Note

$$\int_0^\infty e^{-(ax - \frac{b}{x})^2} dx = \frac{1}{a} \cdot \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2a}$$

$$\Rightarrow Lu = e^{-\sqrt{s} \cdot |x|} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2\sqrt{s}}$$

$$= \frac{e^{-\sqrt{s} \cdot |x|}}{2\sqrt{s}}$$

//

Solution for the following HW:

Problem:

$$\begin{cases} u_t = u_{xx}, & x > 0 \\ u(0,t) + u_x(0,t) = 0 \\ u(x,0) = \delta(x-y), & y > 0. \end{cases}$$

Homework: PDES.

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GENG Xizhi

Solution:

By Laplace transform: $\mathcal{L}u = \int_0^\infty u(x,t) \cdot e^{-st} dt$

$$\text{Thus, } \mathcal{L}u_t(x,s) = \int_0^\infty u_t \cdot e^{-st} dt \stackrel{\text{I.B.P}}{=} s \cdot \mathcal{L}u - u(x,0) = s \cdot \mathcal{L}u - \delta(x-y).$$

And we get

$$s \cdot \mathcal{L}u - \delta(x-y) = \partial_x^2 \mathcal{L}u$$

The boundary condition is

$$\mathcal{L}u(0,t) + \partial_x \mathcal{L}u(0,t) = 0 \quad \text{--- B.C.}$$

Remember, previously in class, we already solve:

$$\tilde{u}_t = \tilde{u}_{xx}, \quad x \in \mathbb{R}, t > 0$$

$$\tilde{u}(x,0) = \delta(x) \quad \tilde{u}(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

The solution is $\tilde{u}(x,t) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}$.

Consider $\tilde{u}(x-y,t) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}$,

It satisfies the Laplace transform:

$$\mathcal{L}\tilde{u}(x-y,s) = s \cdot \mathcal{L}\tilde{u}(x-y,s) - \delta(x-y)$$

Now, take $H(x,t) = u(x,t) - \tilde{u}(x-y,t)$

Observe:

$$s \mathcal{L}H = (\mathcal{L}u - \mathcal{L}\tilde{u}(x-y,s)) - s$$

$$\partial_x^2 H = \partial_x^2 \mathcal{L}u - \partial_x^2 \mathcal{L}\tilde{u}(x-y,s)$$

$$\text{Thus, } s \mathcal{L}H = \partial_x^2 H \quad \text{for } s \mathcal{L}u = \delta(x-y) + \partial_x^2 \mathcal{L}u$$

$$\text{and } s \mathcal{L}\tilde{u}(x-y) = \partial_x^2 \mathcal{L}\tilde{u}(x-y) + \delta(x-y)$$

So, $\mathcal{L}H = \alpha \cdot e^{-\sqrt{s}x}$, α - to be determined.

$$\text{Then, } \mathcal{L}u = \mathcal{L}H + \mathcal{L}\tilde{u}(x-y) = \alpha \cdot e^{-\sqrt{s}x} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}$$

$$\text{By B.C., } \alpha + \frac{e^{-\sqrt{s}y}}{2\sqrt{s}} + (-\sqrt{s}\alpha) + (\sqrt{s}) \cdot \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}} = 0$$

$$\Rightarrow \alpha = -\frac{(1+\sqrt{s})}{1-\sqrt{s}} \cdot \frac{e^{-\sqrt{s}y}}{2\sqrt{s}}$$

$$\text{So, } \mathcal{L}u = -\frac{(1+\sqrt{s})}{1-\sqrt{s}} \cdot \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}, \quad y > 0$$

$$\begin{aligned} \mathcal{L}U &= -\left(-1 + \frac{2}{1-s} + \frac{2}{1-s}\sqrt{s}\right) \cdot e^{-\sqrt{s}(y+x)} + \frac{e^{-\sqrt{s}(x-y)}}{2\sqrt{s}}, \quad y>0, x>0 \\ &= \left(1 - \frac{2}{1-s} + \frac{1}{1-s}\sqrt{s}\right) \cdot e^{-\sqrt{s}(x+y)} + \frac{e^{-\sqrt{s}(x-y)}}{2\sqrt{s}}. \end{aligned}$$

observe:

$$R(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}, \quad \mathcal{L}R(x, s) = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x|t}$$

$$\therefore 2\sqrt{s} \cdot \frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}} = -2 \partial_x \left(\frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}} \right)$$

$$= -2 \mathcal{L}(k_x|x+y|, t)$$

$$\begin{aligned} -\frac{2}{1-s} \cdot e^{-\sqrt{s}|x+y|} &= -2 \mathcal{L}(e^t), \quad (-2 \cdot \mathcal{L}k_x(x+y, t)) \\ &= 4 \mathcal{L}(e^t) \cdot \mathcal{L}k_x(x+y, t) \quad \text{for } \mathcal{L}f \cdot \mathcal{L}g \\ &= 4 \mathcal{L}(e^t * k_x(x+y, t)) \quad = \mathcal{L}(f * g) \\ , \frac{2}{1-s}\sqrt{s} \cdot e^{-\sqrt{s}|x+y|} &= \frac{4s}{1-s} \cdot \frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}} \\ &= \left(-4 \frac{4}{1-s}\right) \cdot \mathcal{L}k(x+y, t) \\ &= -4 \mathcal{L}R(x+y, t) + 4 \mathcal{L}(e^t) \cdot \mathcal{L}k(x+y, t) \\ &= -4 \mathcal{L}R(x+y, t) + 4 \mathcal{L}(e^t * k(x+y, t)). \end{aligned}$$

$$\begin{aligned} \text{So, } u(x, t) &= -2k_x(x+y, t) + 4e^t * k_x(x+y, t) \\ &\quad - 4R(x+y, t) + R(x-y, t), \quad \text{where } \\ &\quad R(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}. \end{aligned}$$

□ -

HW: Solve

$$\begin{cases} u_t - u_{xx} = 0, & x > 0 \\ u_x(0, t) + g(t) = 0 \\ u(x, 0; y) = \delta(x-y), & y > 0 \end{cases}$$

Class 6:

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > 0, t > 0 \\ u(x, 0) = 0, \quad u_t(x, 0) = \delta(x-y), \quad u(0, t) = 0, \quad y > 0 \end{cases}$$

Recall for a whole space problem:

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ u_t(x, 0) = u_1(x), \quad u(x, 0) = u_0(x) \end{cases}$$

$$u(x, t) = \frac{1}{2} (u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(s) ds$$

If $u_1(x) = \delta(x)$, & $u_0(x) = 0$, then

$$u(x, t) = \begin{cases} 1, & \text{if } |x| < t \\ 0, & \text{if } |x| > t \end{cases}$$



Consider

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ u_t(x, 0) = \delta(x), \quad u(x, 0) = 0 \end{cases}$$



Apply Laplace Transform to \square :

($s > 0$)

$$s^2 \mathcal{L}u - u_t(x, 0) + s u(x, 0) - \partial_x^2 \mathcal{L}u = 0$$

$$\Rightarrow s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u = \delta(x) \quad (*)$$

$$\Rightarrow \mathcal{L}u = \begin{cases} A_+ e^{sx} + B_+ e^{-sx}, & x > 0 \\ A_- e^{sx} + B_- e^{-sx}, & x < 0 \end{cases}$$

$$\Rightarrow \mathcal{L}u = \begin{cases} B_+ e^{-sx}, & x > 0 \\ B_- e^{sx}, & x < 0 \end{cases}$$

By contin. at $x=0$
 $\Rightarrow A_- = B_+$

$$\text{And } \partial_x \mathcal{L} u = \begin{cases} -sB_+ e^{sx}, & x>0 \\ s \cdot B_+ \cdot e^{sx}, & x<0 \end{cases}$$

$$\text{By (*) , } \partial_x \mathcal{L} u = -H(\pi) \Rightarrow 2sB_+ = 1$$

$$\Rightarrow \mathcal{L} u = \begin{cases} \frac{1}{2s} e^{-sx}, & x>0 \\ \frac{1}{2s} \cdot e^{sx}, & x<0 \end{cases} \quad \text{i.e. } \mathcal{L} u = \frac{e^{-|x|}}{2s}$$

Now,

$$\begin{cases} u_{tt} - u_{xx} = 0, & x>0, t>0 \\ u(x, 0) = 0, \quad u_t(x, 0) = \delta(x-y), \quad y>0 \\ u(0, t) = 0. \end{cases}$$

Laplace Transform:

$$\begin{cases} s^2 \mathcal{L} u - \partial_x^2 \mathcal{L} u = \delta(x-y), & x>0, y>0 \\ \mathcal{L} u(0, s) = 0 \end{cases}$$

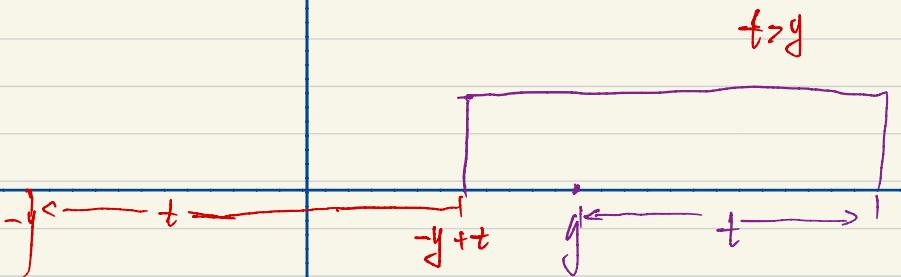
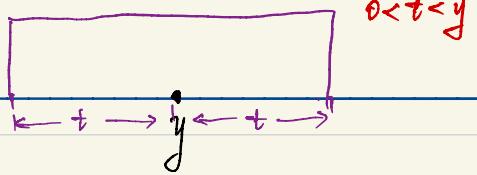
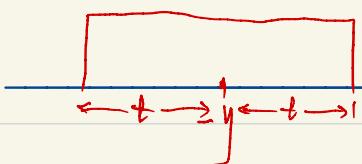
$$\Rightarrow s^2 (\mathcal{L} u - \frac{e^{-sy}}{2s}) - \partial_x^2 (\mathcal{L} u - \frac{e^{-sy}}{2s}) = 0, \quad x>0$$

$$\Rightarrow \textcircled{v} = A \cdot e^{-sx} \quad \text{for } x>0$$

$$\Rightarrow \textcircled{v} \Big|_{x=0} = -\frac{e^{-sy}}{2s} = A \quad \text{i.e. } A = -\frac{e^{-sy}}{2s} \quad \text{for } y>0$$

$$\Rightarrow \textcircled{v} = -\frac{e^{-s(y+x)}}{2s}$$

$$\Rightarrow \mathcal{L} u = \frac{t}{2s} e^{-sx-y} - \frac{e^{-sx-y}}{2s}, \quad x>0, y>0$$



Now,

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x > 0, \quad t > 0 \\ u(x, 0) = 0, \quad u_t(x, 0) = \delta(x-y), \quad y > 0 \\ u_x(0, t) = 0 \end{cases}$$

$$\Rightarrow s^2 \mathcal{F}u - \partial_x^2 \mathcal{F}u = \delta(x-y)$$

$$\underbrace{\partial_x \mathcal{F}u(0, s)}_{s > 0} = 0$$

$$\Rightarrow s^2 \left(\mathcal{F}u - \frac{e^{-sy}}{2s} \right) - \partial_x^2 \left(\mathcal{F}u - \frac{e^{-sy}}{2s} \right) = 0, \quad x > 0$$

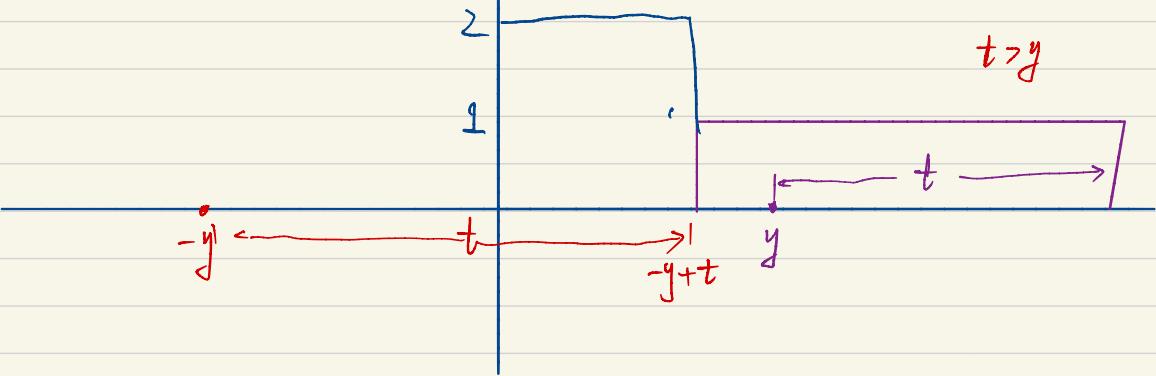
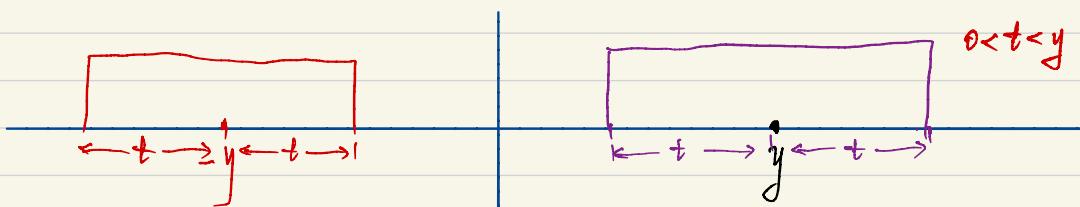
(*)

Then, $\mathcal{F}u = A \cdot e^{-sx}$

By B.C. $\partial_x \mathcal{F}u \Big|_{x=0} = \partial_x \left(\mathcal{F}u - \frac{e^{-sy}}{2s} \right) = \partial_x A \cdot e^{-sx}$

$$\Rightarrow -\frac{e^{-sy}}{2} = -sA \quad \text{i.e. } A = \frac{e^{-sy}}{2s}$$

$$\Rightarrow f_u = \frac{e^{-s(x-y)}}{2s} + \frac{e^{-s(x+y)}}{2s}, x > 0, y > 0$$



Functional Analysis:

$$H^1(\mathbb{R}) = \{f \mid \|f\|_2 + \|f_x\|_2 < \infty\}$$

$$\|f\|_{H^1} = \sqrt{\|f\|_2^2 + \|f_x\|_2^2} = (\int_{\mathbb{R}} |f|^2 + |f_x|^2 dx)^{1/2}$$

$$\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)|$$

$$\begin{aligned} \|f\|_\infty^2 &= \left| \int_{-\infty}^{\infty} \frac{d}{dx} (f^2) dx \right| = \left| \int_{-\infty}^{\infty} 2f f_x dx \right| \\ &\leq 2 \int_{-\infty}^{\infty} |f| \cdot |f_x| dx \leq \int_{-\infty}^{\infty} (|f|^2 + |f_x|^2) dx \\ &\leq (\|f\|_{H^1})^2 \end{aligned}$$

$\Rightarrow \|f\|_\infty \leq \|f\|_{H^1}$ ~~if~~.

Sobolev's space: $f(\vec{x}) : \vec{x} \in \mathbb{R}^n$

$$\hat{f}(\vec{y}) = \int_{\mathbb{R}^n} f(\vec{x}) \cdot e^{-i\vec{y}\vec{x}} d\vec{x} \quad \text{— Fourier transform}$$

$$\|f\|_{H^k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\vec{y}|^2)^k \cdot |\hat{f}(\vec{y})|^2 d\vec{y} \right)^{1/2}$$

$$f(\vec{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\vec{y}) \cdot e^{i\vec{y}\vec{x}} d\vec{y}$$

Suppose $f \in H^k(\mathbb{R}^n)$, with $k > \frac{n}{2}$. Then,

$$|f(\vec{x})| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\vec{y}\vec{x}} \cdot \hat{f}(\vec{y}) d\vec{y} \right|$$

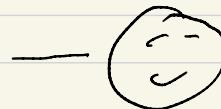
$$\begin{aligned} \text{Schwarz ineq.} \rightarrow &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\vec{y}\vec{x}} \cdot \hat{f}(\vec{y}) \cdot \frac{(1 + |\vec{y}|^2)^{k/2}}{(1 + |\vec{y}|^2)^{k/2}} d\vec{y} \right| \\ &\leq \frac{1}{(2\pi)^n} \cdot \left(\int_{\mathbb{R}^n} |\hat{f}(\vec{y})| \cdot (1 + |\vec{y}|^2)^{k/2} d\vec{y} \right)^{1/2} \cdot \left(\int_{\mathbb{R}^n} \frac{d\vec{y}}{(1 + |\vec{y}|^2)^{k/2}} \right)^{1/2} \\ &\leq \frac{1}{(2\pi)^n} \cdot \|f\|_{H^k} \cdot \left(\int_0^\infty \int_{S^{n-1}} \frac{r^{n-1}}{(1 + r^2)^k} dr d\Omega \right)^{1/2} \end{aligned}$$

Integrable!

Class 7: (Telegraph Equation)

Caution: In previous class, we can construct the sol. by Fourier transform, but in some cases, there exist things Telegraph cannot be transformed by inverse Fourier transform !!!

E.g. $\begin{cases} u_{tt} - u_{xx} + u_t = 0, x \in \mathbb{R}, t > 0, \\ u(x, 0) = 0 \\ u_t(x, 0) = g(x) \end{cases}$, Cauchy Problem



Take Fourier Transform:

$$\begin{cases} \hat{u}_{tt} + \eta^2 \hat{u} + \hat{u}_t = 0 \\ \hat{u}(y, 0) = 0 \\ \hat{u}_t(y, 0) = 1 \end{cases} \quad \text{"ODE"}$$

Fix η , solve "ODE": Characteristic eq. $\eta^2 + \eta + 1 = 0$

$$\Rightarrow \hat{u}(\eta, t) = (A e^{\frac{\eta_+ t}{2}} + B e^{\frac{\eta_- t}{2}})$$

$$\Rightarrow B = -A \quad \text{for } \hat{u}(y, 0) = 0$$

$$\& \hat{u}_t(y, t) = A \eta_+ e^{\frac{\eta_+ t}{2}} - A \eta_- e^{\frac{\eta_- t}{2}} \Rightarrow A(\eta_+ - \eta_-) = 1 \quad \text{for}$$

Get $A = \frac{1}{\sqrt{1-4\eta^2}}$ & $\hat{u}(y, t) = \frac{1}{\sqrt{1-4\eta^2}} (e^{\frac{\eta_+ t}{2}} - e^{\frac{\eta_- t}{2}}) \quad \tilde{u}_t(y, 0) = 1$

Q: How to understand the sol. $\hat{u}(y, t)$?

A: Need to find sol. in x -space. wave number

Intuition: $f = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\eta) \cdot e^{i\eta x} d\eta$ dispersion relationship.

low pitch: η small, $1-4\eta^2 > 0$ by Taylor's expansion of $\frac{1+\sqrt{1-4\eta^2}}{2} \approx \frac{1+(1-4\eta^2)/2}{2} + O(\eta^4)$

$$\frac{\eta_+ t}{2} = \frac{1+\sqrt{1-4\eta^2}}{2} t \Rightarrow e^{\eta_+ t} \sim e^{-2\eta^2 t} \Rightarrow \sim \frac{1}{\sqrt{8\pi t}} \cdot e^{-\frac{x^2}{8t}}$$

Task 1: Extract Singularity in the frequency domain:

① Compute $\hat{u}(y, t) = \frac{1}{\sqrt{1-4y^2}} (e^{\xi_+ t} - e^{\xi_- t})$;
 $\xi_{\pm} = \frac{-1 \pm \sqrt{1-4y^2}}{2}$

Try to understand the sol. as $y \rightarrow \infty$;

$$\sqrt{1-4y^2} = 2y \sqrt{1 - \frac{1}{4y^2}} \quad \begin{array}{l} \text{Taylor's} \\ \text{expansion} \end{array} \quad i2y \left(1 - \frac{1}{8y^2} + \frac{O(y)}{y^4} \right)$$

$$i2y \left(1 - \frac{1}{8y^2} + \frac{C}{y^4} + \frac{O(y)}{y^6} \right)$$

Now, find ξ_{\pm}^* to approximate ξ_{\pm} :

② Define

$$\xi_{\pm}^* = \frac{-1 \pm 2y; \left(1 - \frac{1}{8(y^2+1)} + \frac{C}{(y^2+1)^2} \right)}{2}$$

& $|S_{\pm} - \xi_{\pm}^*| = O(y) \cdot \frac{1}{y^5}$ as $y \rightarrow \infty$

Want to find approximating sol. of $\hat{u}(y, t)$:

Notice $(\sqrt{1-4y^2})^* = 2y \left(1 - \frac{1}{8(y^2+1)} + \frac{C}{(y^2+1)^2} \right)$

\Downarrow $\hat{u}^*(y, t) = \frac{1}{(\sqrt{1-4y^2})^*} \cdot (e^{\xi_+^* t} - e^{\xi_-^* t})$

③ Define $\hat{u}^*(y, t)$

The initial conditions are satisfied as follows:

- $\hat{u}_t^*(y, 0) = 1, \hat{u}^*(y, 0) = 0$;

- $\hat{u}_{tt}^* + y^2 \hat{u}^* \rightarrow \hat{u}_t^*$

$$= \left[\underbrace{(\xi_+^*)^2 \cdot e^{\xi_+^* t} - (\xi_-^*)^2 \cdot e^{\xi_-^* t}}_{= \frac{1}{(\sqrt{1-4y^2})^*} \left[\left((\xi_+^*)^2 + y^2 + \xi_-^* \right) \cdot e^{\xi_+^* t} - \left((\xi_-^*)^2 + y^2 + \xi_+^* \right) \cdot e^{\xi_-^* t} \right]} + \underbrace{y^2 e^{\xi_+^* t} - y^2 e^{\xi_-^* t}}_{= \frac{1}{(\sqrt{1-4y^2})^*} \left[\frac{1}{(\xi_+^*)^2 + y^2 + \xi_-^*} - \frac{1}{(\xi_-^*)^2 + y^2 + \xi_+^*} \right]} + \underbrace{\xi_+^* \cdot e^{\xi_+^* t}}_{= \frac{1}{(\sqrt{1-4y^2})^*} \cdot \xi_+^*} - \underbrace{\xi_-^* \cdot e^{\xi_-^* t}}_{= \frac{1}{(\sqrt{1-4y^2})^*} \cdot \xi_-^*} \right]$$

$$\text{Note: } [(\xi_+^*)^2 + \eta^2 + \zeta_+^*] \cdot e^{\xi_+^* t} \\ = (\xi_+^2 + \eta^2 + \zeta_+^* + (\underbrace{(\xi_+^*)^2 - \zeta_+^2}_{O(1) \frac{1}{\eta^5}} + (\xi_+^* - \zeta_+))) \cdot e^{\xi_+^* t}$$

$$\text{for } |\xi_+ - \xi_+^*| = O(1) \frac{1}{\eta^5}$$

$$\Rightarrow \hat{u}_{ttt}^* + \eta^2 \hat{u}^* + \hat{u}_t^* = \hat{S}(y, t) \leq O(1) \cdot \frac{e^{-t}}{(1\eta^1+1)^5}$$

Compare $|\hat{u}^*(y, t) - \hat{u}(y, t)|$:

$$\Rightarrow \begin{cases} (\hat{u} - \hat{u}^*)_t + \eta^2 (\hat{u} - \hat{u}^*) + (\hat{u} - \hat{u}^*)_t = O(1) \cdot \frac{e^{-t}}{c(y^1+1)^5} \\ (\hat{u} - \hat{u}^*)(y, 0) = (\hat{u}_t - \hat{u}_t^*)(y, 0) = 0 \end{cases}$$

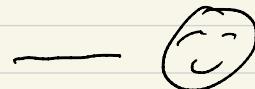
$$\text{Let } \hat{V} = \hat{u} - \hat{u}^*$$

$$(*) \quad \begin{cases} \hat{V}_{ttt} + \eta^2 \hat{V} + \hat{V}_t = O(1) \cdot \frac{e^{-t}}{(1\eta^1+1)^5} \leq e^{-t} \\ \hat{V}(y, 0) = \hat{V}_t(y, 0) = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |x|^3 S(x, t) dx \leq e^{-t}$$

Rewrite the ~~as~~ as:

$$\begin{cases} V_t = V_x + U_t = S(x, t) \\ V(x, 0) = U_t(x, 0) = 0; \end{cases}$$



Energy Estimate:

$$\int_{\mathbb{R}} V_t \cdot (U_t - V_x + U_t) dx = \int_{\mathbb{R}} S(x, t) U_t dx$$

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (V_t)^2 dx + \int_{\mathbb{R}} V_t x \frac{V_x}{2} dx + \int_{\mathbb{R}} U_t^2 dx = \int_{\mathbb{R}} S(x, t) U_t dx$$

$$\Rightarrow \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (V_t^2 + V_x^2) dx + \int_{\mathbb{R}} U_t^2 dx = \int_{\mathbb{R}} S(x, t) U_t dx$$

Integrate on t :

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}} \frac{1}{2} [(V_t)^2 + (V_x)^2] dx \Big|_0^T + \int_0^T \int_{\mathbb{R}} (V_t)^2 dx dt &= \int_0^T \int_{\mathbb{R}} S(x, t) V_t dx dt \\ &\leq \int_0^T \int_{\mathbb{R}} \frac{S^2(x, t) + (V_t)^2}{2} dx dt \end{aligned}$$

$$\int_0^T \int_{\mathbb{R}} \frac{S^2(x, t)}{2} dx dt$$

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}} \frac{1}{2} [V_t^2 + V_x^2] dx \Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}} V_t^2 dx dt &\\ \leq \int_0^T \int_{\mathbb{R}} \frac{S^2(x, t)}{2} dx dt & \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} \underbrace{\partial_x^k v_t}_{\text{Term 1}} \cdot \underbrace{\partial_x^k (v_{tt} - v_{xx} + v_t)}_{\text{Term 2}} dx = \int_{\mathbb{R}} \underbrace{\partial_x^k S(x,t)}_{\text{Term 3}} \cdot \underbrace{\partial_x^k v_t}_{\text{Term 4}} dx$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2} \left[(\partial_x^k v_t)^2 + (\partial_x^k v_x)^2 \right] dx \Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}} (\partial_x^k v_t)^2 dx dt \\ \leq \int_0^T \int_{\mathbb{R}} \frac{(\partial_x^k S(x,t))^2}{2} dx dt$$

$$k = 1, 2, 3.$$

Task 2: Find the structure $u^*(x, t)$.

Another Functional Thm:

If $f(\eta)$ satisfies $\begin{cases} \hat{f}(\eta) \text{ analytic in } |\operatorname{Im} \eta| < \eta_0 \\ |\hat{f}(\eta)| \leq \frac{O(1)}{\eta^2 + 1} \end{cases}$

Then, $|f(x)| \leq O(1) \cdot e^{-\eta_0 |x|}$ Cauchy Integrable Thm.

Proof: $f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\eta} \hat{f}(\eta) d\eta \stackrel{\eta = s + iv_0}{=} \frac{1}{2\pi} \int_{\mathbb{R} + i v_0} e^{i(x-s)v_0} \hat{f}(s + iv_0) ds$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(s + iv_0)x} \hat{f}(s + iv_0) ds = \frac{e^{-v_0 x}}{2\pi} \int_{-\infty}^{\infty} e^{isx} \hat{f}(s + iv_0) ds$$

$$\Rightarrow |f(x)| \leq \frac{e^{-v_0 x}}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(s + iv_0)| ds \leq \frac{e^{-v_0 x}}{2\pi} C \quad |v_0| < \eta_0$$

$$\leq \frac{e^{-v_0 x}}{2\pi} C \quad C \text{ const.}$$

$$\Rightarrow |f(x)| \leq \frac{e^{-|v_0 x|}}{2\pi} \cdot C \leq O(1) \cdot e^{-\eta_0 |x|} \quad \text{Q.E.D.}$$

Recall,

$$\zeta_{\pm}^* = \frac{-i \pm 2i\eta \left[1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right]}{2} \quad \text{analytic in } |\operatorname{Im} \eta| < \eta_0/2$$

- $(\sqrt{1-4\eta^2})^* = i2\eta \left(1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right)$
- $|\beta_{\pm} - \zeta_{\pm}^*| = O(1) \frac{1}{\eta^6}, \quad \text{as } \eta \rightarrow \infty$
- $\hat{u}^*(\eta, t) = \frac{1}{(\sqrt{1-4\eta^2})^*} \cdot (e^{\zeta_{+}^* t} - e^{\zeta_{-}^* t})$

$$I = e^{-\frac{t}{2}} \left[e^{2i\eta t} \left[1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right] t - e^{-2i\eta t} \left[1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right] \right]$$

$$II \rightarrow 2i\eta \cdot \left(1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right)$$

Consider

$$e^{2i\eta t} \left[1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2} \right] t \\ = e^{2i\eta t} \underbrace{\left(\frac{-2i\eta t}{8(\eta^2+1)} + \frac{C \cdot 2i\eta t}{(\eta^2+1)^2} \right)}_{\delta(x+2t)}$$

Reconsider:

$$e^{-\frac{2i\eta t}{8(\eta^2+1)} + \frac{C \cdot 2i\eta t}{(\eta^2+1)^2}} - 1 = \int_0^\infty e^{\alpha} d\alpha$$

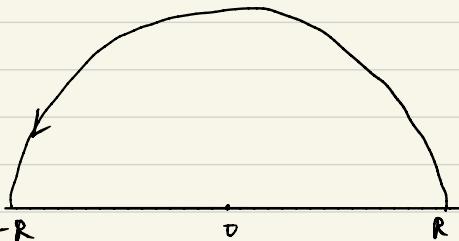
$$\delta(\infty)$$

$$= -\frac{2i\eta t}{8(\eta^2+1)} + \frac{C \cdot 2i\eta t}{(\eta^2+1)^2} + O(|t|) t^2$$

$$\underbrace{e^{-1\eta_0 x_0}}_{\uparrow}$$

By Complex Analysis:

$$\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} \cdot e^{i\eta x} dx$$



$$= \frac{1}{2\pi} \cdot \text{Res.} \left(\frac{e^{i\eta x}}{1+\eta^2} \right)_{x=0} \\ = \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{e^{-\eta x}}{2i} = K \cdot e^{-\eta x}, \text{ if } x > 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} e^{i\eta x} dx = C \cdot e^{-|\eta|}$$

Thus, $\frac{-2i\eta}{8(\eta^2+1)} \leftarrow \text{c.t. } (e^{-1\eta x})_x$.

Then, $I = e^{-\frac{t}{2}} \cdot \left[\delta(x+2t) * \left[(e^{-\eta x})_x + e^{-1\eta x} \right] C.t. - \delta(x-2t) * \left[(e^{-\eta x})_x + e^{-1\eta x} \right] C.t. \right]$

II is a polynomial $\Rightarrow u^*(x, t)$ is around $e^{-|\eta| - t/K}$.

Task 3: Verify $|u(x, t) - u^*(x, t)|$ is exponentially small in the region $|x| > \alpha t$ for any positive α .

By Energy Estimate

$$\text{on } V(x, t) = u(x, t) - u^*(x, t).$$

- $\int_{-\infty}^{\infty} (V_t^2(x, t) + V_x^2(x, t)) \cdot e^{-\varepsilon|x-\alpha t|} dx \leq O(1) \cdot e^{-\varepsilon^2 t}$ for $x > \alpha t$
- $\int_{-\infty}^{\infty} (V_t^2(x, t) + V_x^2(x, t)) \cdot e^{-\varepsilon|x+\alpha t|} dx \leq O(1) \cdot e^{-\varepsilon^2 t}$ for $x < -\alpha t$

Recall,

$$\begin{cases} V_{tt} - V_{xx} + U_t = S(x, t), \\ V(x, 0) = V_t(x, 0) = 0 \end{cases}$$

$$|\partial_y^3 S(x, t)|^2 \leq O(1) \cdot e^{-t - |x|/k} \quad \text{for } |\hat{S}(y, t)| \leq \frac{e^{-t}}{(1+y)^4} \cdot O(1)$$

Observe:

If $|x| > \frac{\alpha t}{2}$, then $\exists k, s.t.$

$$|V(x, t)| \leq O(1) \cdot e^{-(t+|x|)/k},$$

By method of energy:

$$\int_{-\infty}^{\infty} e^{\varepsilon|x - \frac{\alpha t}{2}|} \cdot V_t \cdot (V_{tt} - V_{xx} + U_t) dx \underset{\approx}{=} \int_{-\infty}^{\infty} S(x, t) \cdot e^{-\varepsilon|x - \frac{\alpha t}{2}|} dx$$

- W.T. know how $U = U^* - U$ looks like?

$$\begin{cases} U_{tt} - U_{xx} + U_t = 0 \\ U(x, 0) = 0 \\ U_t(x, 0) = \delta(x) \end{cases}$$

$$\text{ & } \begin{cases} U_{tt}^* - U_{xx}^* + U_t^* = S(x, t) \\ U^*(x, 0) = 0 \\ U_t^*(x, 0) = \delta(x). \end{cases}$$

Combine the two equations:

$$\Rightarrow \begin{cases} U_{tt} - U_{xx} + U_t = -S(x, t) \\ U(x, 0) = 0 \\ U_t(x, 0) = 0 \end{cases}$$

$$-\frac{t}{K}$$

$$|S(x, t)| \leq O(t)$$

If $|x| > t/2$, there exists k_1 s.t. $|U(x, t)| \leq k_1 \cdot e^{-\frac{(|x|+t)}{K}}$

Proof.: What we want to prove

Consider $e^{\xi(x - \frac{t}{8})}$,

$$\int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_t \cdot (U_{tt} - U_{xx} + U_t) dx = \int_{-\infty}^{\infty} -S(x, t) \cdot e^{\xi(x - \frac{t}{8})} \cdot U_t dx$$

$$U_t \cdot U_{tt} = (\frac{1}{2} U_t^2)_t$$

(I)

(III)

$$\therefore \int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot (\frac{1}{2} U_t^2)_t dx = \frac{d}{dt} \left[\frac{1}{2} \int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot (U_t)^2 dx + \frac{\xi}{16} \int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_t^2 dx \right]$$

$$\begin{aligned} - \int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_t \cdot U_{xx} dx &= \int_{-\infty}^{\infty} (e^{\xi(x - \frac{t}{8})} \cdot U_t)_x \cdot U_x dx \\ &= \underbrace{\int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_{tx} \cdot U_x dx}_{\int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_t \cdot U_x dx} + \underbrace{\int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_t \cdot U_{xx} dx}_{\int_{-\infty}^{\infty} e^{\xi(x - \frac{t}{8})} \cdot U_{tx} \cdot U_{xx} dx} \end{aligned}$$

$$\begin{aligned}
 &= \underbrace{\int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot \left(\frac{V_x^2}{2}\right)_t dx + \xi \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot u_t \cdot V_x dx}_{\text{blue line}} \\
 &= \frac{d}{dt} \frac{1}{2} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_x^2 dx + \frac{\xi}{16} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_x^2 dx + \text{blue circle}
 \end{aligned}$$

& (III) $\Rightarrow \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_t^2 dx$

In all,

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot (V_t)^2 dx + \frac{\xi}{16} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot (V_t)^2 dx \\
 &+ \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_x^2 dx + \frac{\xi}{16} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_x^2 dx + \xi \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_t V_x dx \\
 &+ \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot V_t^2 dx = \int_{-\infty}^{\infty} -S(x, t) \cdot e^{\xi(x - \frac{1}{8}t)} \cdot V_t dx
 \end{aligned}$$

If we choose ξ small enough, then \sim

$$\begin{aligned}
 &\Rightarrow \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{16} \xi V_x^2 + \xi V_t V_x + V_t^2 \right)}_{\geq \frac{1}{32} \xi \cdot (V_x^2 + V_t^2)} \cdot e^{\xi(x - \frac{1}{8}t)} dx
 \end{aligned}$$

Rewrite:

$$\begin{aligned}
 &\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot (V_t^2 + V_x^2) dx + \int_{-\infty}^{\infty} \frac{1}{32} \xi \cdot (V_x^2 + V_t^2) \cdot e^{\xi(x - \frac{1}{8}t)} \\
 &\leq 1 - \left| \int_{-\infty}^{\infty} S(x, t) \cdot e^{\xi(x - \frac{1}{8}t)} V_t dx \right| \\
 &\leq \int_{-\infty}^{\infty} \left[\frac{\xi^2(x, t)}{64} + \frac{1}{64} \xi \cdot (V_t)^2 \right] e^{\xi(x - \frac{1}{8}t)} dx \\
 &\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\xi(x - \frac{1}{8}t)} \cdot [(V_t)^2 + (V_x)^2] dx + \int_{-\infty}^{\infty} \frac{1}{64} \xi \cdot (V_x^2 + V_t^2) \cdot e^{\xi(x - \frac{1}{8}t)} \\
 &\leq \int_{-\infty}^{\infty} \frac{S^2(x, t)}{64 \cdot \xi} \cdot e^{\xi(x - \frac{1}{8}t)} dx
 \end{aligned}$$

$$\Rightarrow \int_{\mathbb{R}} e^{\zeta(x - \frac{1}{8}t)} [v_t^2 + v_x^2] dx \Big|_{t=T} \leq \int_0^T \int_{-\infty}^{\infty} O(1) \cdot S^2 \cdot e^{\zeta(x - \frac{1}{8}t)} dt$$

$$\text{Recall, } \hat{s}(y, t) = O(1) \cdot \frac{e^{-t}}{(1+y)^5}$$

$$\text{i.e. } s(x, t) = O(1) \cdot e^{-(|t| + |x|)/k}.$$

$$\Rightarrow \int_{\mathbb{R}} e^{\zeta(x - \frac{1}{8}t)} \cdot [v_t^2 + v_x^2] dx \Big|_{t=T} \leq O(1) \cdot e^{-\frac{\zeta T}{8}}$$

Class II:

Task 4: By Long wave - Shortwave Decomposition:

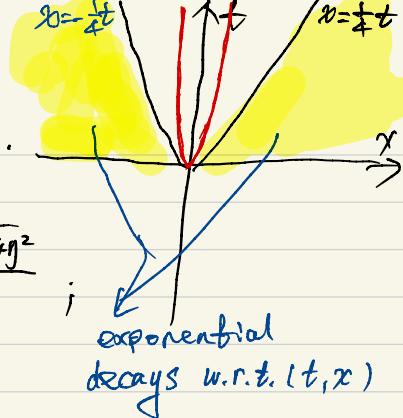
$$u_L(x, t) = \int_{|\gamma| < \varepsilon} e^{i\gamma x} \cdot \hat{u}(\gamma, t) d\gamma$$

$$u_S(x, t) = \int_{|\gamma| \geq \varepsilon} e^{i\gamma x} \cdot \hat{u}(\gamma, t) d\gamma;$$

$$\operatorname{Re}(\beta_{\pm}(\gamma)) < -\varepsilon_0 \Rightarrow \|u_S\|_{L^2} \leq O(1) \cdot e^{-\varepsilon_0 t}$$

$$\Rightarrow \|u_S\|_{\infty} \leq O(1) \cdot e^{-\varepsilon_0 t}.$$

Aim: Want to know what happens in the cone.



Recall,

$$\hat{u}(y, t) = \frac{1}{\sqrt{1-4\eta^2}} (e^{\xi_+ t} - e^{\xi_- t}), \quad \xi_{\pm} = \frac{-1 \pm \sqrt{1-4\eta^2}}{2}$$

$\eta \sim \text{wave #}$ (oscillation)²

Long wave - short wave decomposition:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \hat{u}(y, t) dy, \quad \cdot |y| > 1 \quad \begin{matrix} \text{short wave} \\ (\text{wave length} \\ \text{small}) \end{matrix}$$

$\cdot |y| < 1 \quad \begin{matrix} \text{long wave} \\ (\text{wave length big}) \end{matrix}$

$$u(x, t) = \frac{1}{2\pi} \left(\int_{|y| < \varepsilon} + \int_{|y| > \varepsilon} \right) \cdot e^{iyx} \cdot \hat{u}(y, t) dy$$

↑ "Long wave component" ↑ "short wave component"

$$= u_L(x, t) + u_S(x, t).$$

For $|y| > \varepsilon$, $\operatorname{Re}(\xi_-) < -\frac{1}{2}$ $\Rightarrow |e^{\xi_- y t}| < e^{-\frac{1}{2}t}$

$$\begin{aligned} \operatorname{Re}(-1 + \sqrt{1-4\eta^2}) &= \operatorname{Re}\left(\frac{1 - \sqrt{1-4\eta^2}}{1 - \sqrt{1-4\eta^2}}\right) \\ &= \operatorname{Re}\left(-\frac{4\eta^2}{1 + \sqrt{1-4\eta^2}}\right) \leq -\frac{4\varepsilon^2}{2} \\ \Rightarrow |e^{-\xi_-(y)t}| &\leq O(\varepsilon) e^{-2\varepsilon^2 t} \end{aligned}$$

$$|u_S(x, t)| = \left| \frac{1}{2\pi} \int_{|y| > \varepsilon} e^{iyx} \cdot \frac{1}{\sqrt{1-4\eta^2}} (e^{\xi_+ t} - e^{\xi_- t}) dy \right|$$

Recall

$$\|u_s(., t)\|_{L^2}^2 : f \in L^2(\Omega) \Rightarrow \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$
$$\int |\hat{f}|^2 = \int |f|^2$$

Thus, $\|u_s(., t)\|_{L^2}^2 = \int_{|y| > \varepsilon} \frac{1}{1-4y^2} \cdot (e^{S_+ t} - e^{S_- t})^2 dy$

$$\leq O(1) \cdot e^{-\frac{\varepsilon^2 t}{2}}$$

$$\|u_s^+ - u_s\|_\infty \leq O(1) \cdot e^{-\frac{\varepsilon^2 t}{2}} \text{ for all } \eta.$$

To find $u_L(x, t)$ with $x < \frac{1-t}{2}$

$$u_L(x, t) = \frac{1}{2\pi} \int_{|y| < \varepsilon} e^{iyx} \cdot \frac{1}{\sqrt{1-4y^2}} \cdot (e^{S_+ y t} - e^{S_- y t}) dy$$

Analytic & Cauchy formula

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{iyx}}{\sqrt{1-4y^2}} (e^{S_+ y t} - e^{S_- y t}) dy$$

$$S_{\pm}(y) = \frac{-1 \pm \sqrt{1-4y^2}}{2} = \underbrace{\frac{-1 \pm (1-2y^2) + \Theta(y^4)}{2}}_{\text{Q: Analytic around 0}};$$

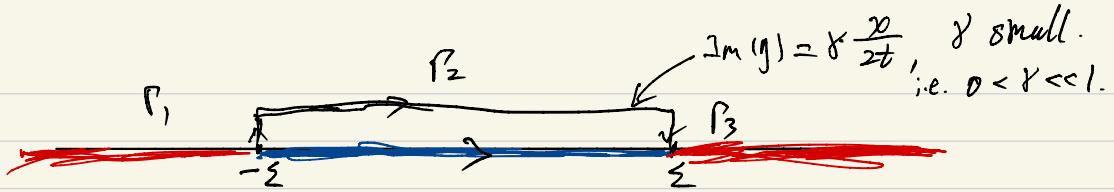
$$S_- = -1 + y^2 - \Theta(y^4) \Rightarrow \operatorname{Re}(S_-(y)) \leq -\frac{1}{2} \text{ for } |y| \leq \varepsilon$$

$S_+ = -y^2 + \Theta(y^4)$ — Problem for $|y|$ small enough

Focus on S_+ :

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{iyx + S_+(y)t}}{\sqrt{1-4y^2}} dy + \frac{x^2}{4t} - \frac{x^2}{4t} \\ &= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{e^{iyx + (-y^2 + \Theta(y^4))t}}{\sqrt{1-4y^2}} dy \\ &= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{1-4y^2}} \cdot e^{-\frac{x^2}{4t} - t(y - \frac{ix}{2t})^2 + \Theta(y^4, t)} dy \end{aligned}$$

How to balance the term?



$$\frac{1}{2\pi} \int_{P_2} e^{-\frac{x}{4t} - t(y - \frac{i\gamma x}{2t})^2 + O(\gamma^4)t} dy, \quad y = v + i\gamma \cdot \frac{x}{2t}, \quad \gamma \in (-\varepsilon, \varepsilon)$$

$$= \frac{1}{2\pi} \int_{P_2} e^{-\frac{x^2}{4t} - t(v + i\gamma \frac{x}{2t} - \frac{i\gamma x}{2t})^2 + O((1+v+i\gamma \frac{x}{2t})^4)t} dv$$

$$= \frac{1}{2\pi} \int_{P_2} e^{-\frac{x^2}{4t} - t(v + (1-\gamma)i\frac{x}{2t})^2 + O((v + i\gamma \frac{x}{2t})^4)t} dv$$

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{x^2}{4t} - t[v - i(1-\gamma)\frac{x}{24}]^2} dv = -(1-(1-\gamma)^2) \frac{x^2}{4t} - tv^2$$

$$= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{x^2}{4t} - (1-(1-\gamma)^2) \frac{x^2}{4t}} \cdot e^{-tv^2 - \gamma tv(1-\gamma) \frac{x}{2t} + O((v + i\gamma \frac{x}{2t})^2)} dv$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{x^2}{8t} - (1-(1-\gamma)^2) \frac{x^2}{8t}} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-\frac{tv^2}{2} - \frac{\gamma tv(1-\gamma)x}{2t} + O((v + i\gamma \frac{x}{2t})^2)} dv$$

$$= \frac{1}{2\pi} \cdot \int_{-\varepsilon}^{\varepsilon} \left(\text{circled term} \right) dv$$

$$\operatorname{Re} \left[-[1-(1-\gamma)^2] \frac{x^2}{8t} - \frac{tv^2}{2} + O((v + \frac{i\gamma x}{2t})^4)t \right] = 0 \quad \text{if } v \in (-\varepsilon, \varepsilon)$$

$$\leq \frac{e^{-[1-(1-\gamma)^2] \frac{x^2}{8t}}}{2\pi} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-\frac{tv^2}{2}} dv \quad \text{circled term} = \frac{dv_{NB}}{\sqrt{t}}$$

$$\leq O(1) \cdot \frac{e^{-[-(1-\gamma)^2] \frac{x^2}{8t}}}{\sqrt{t}}$$

$$1. \quad \begin{cases} V_{tt} - V_{xx} + V_t = 0 & , \quad x \in \mathbb{R} \\ V(x, 0) = 0 \\ V_t(x, 0) = \delta(x - x_0) , \quad x_0 > 0 \end{cases}$$

Take Laplace Transform:

$$s^2 \mathcal{L} u - \delta(x - x_0) - \partial_x^2 \mathcal{L} u + s \mathcal{L} u = 0. \quad (x \in \mathbb{R})$$

$$\Rightarrow (s^2 + s) \cdot \mathcal{L} u - \partial_x^2 \mathcal{L} u = \delta(x - x_0)$$

$$\text{Then, } \mathcal{L} u = \begin{cases} A \cdot e^{-\sqrt{s^2+s}(x-x_0)} & \text{if } x > x_0 \\ A \cdot e^{\sqrt{s^2+s}(x-x_0)} & \text{if } x < x_0 \end{cases}$$

$$\mathcal{L}_x u = \begin{cases} -\sqrt{s^2+s} \cdot A \cdot e^{-\sqrt{s^2+s}(x-x_0)} & \text{if } x > x_0 \\ \sqrt{s^2+s} \cdot A \cdot e^{\sqrt{s^2+s}(x-x_0)} & \text{if } x < x_0 \end{cases}$$

$$\Rightarrow A = \frac{1}{2\sqrt{s^2+s}}$$

$$\Rightarrow \mathcal{L} u = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}|x-x_0|}$$

$$2. \quad \begin{cases} V_{tt} - V_{xx} + V_t = 0, & x > 0 \\ V(x, 0) = 0, \quad V_t(x, 0) = \delta(x - x_0), \quad x_0 > 0 \\ V(0, t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} s^2 \mathcal{L} V - \delta(x - x_0) - \partial_x^2 \mathcal{L} V + s \mathcal{L} V = 0 & (x > 0) \\ \mathcal{L} V(0, s) = 0 \end{cases}$$

$$\Rightarrow \underbrace{s^2 \mathcal{L} u - \delta |x-x_0| - \partial_x^2 \mathcal{L} u + s \mathcal{L} u = 0}_{}$$

Then, $s^2 \mathcal{L}(V-u) - \partial_x^2 \mathcal{L}(V-u) + s \mathcal{L}(V-u) = 0$

$$\{ \mathcal{L}(V-u)(0,s) = -\mathcal{L}u(0,s)$$

$$\Rightarrow \mathcal{L}(V-u)(x,s) = -\mathcal{L}u(0,s) \cdot e^{-\sqrt{s^2+s}x}$$

$$f_u = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}|x-x_0|}$$

$$\Rightarrow \mathcal{L}V = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}|x-x_0|} - \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}(x_0+x)}$$

3. $\begin{cases} V_{tt} - V_{xx} + V_t = 0, & x > 0 \\ V(x,0) = 0, V_t(x,0) = \delta|x-x_0|, & x_0 > 0 \\ V(0,t) + V_x(0,t) = 0 \end{cases}$

$$\Rightarrow \mathcal{L}V + \partial_x \mathcal{L}V(0,s) = 0$$

$$\Rightarrow \begin{cases} s^2 \mathcal{L}V - \delta|x-x_0| - \partial_x^2 \mathcal{L}V + s \mathcal{L}V = 0 & (x > 0) \\ (\mathcal{L}V + \partial_x \mathcal{L}V)(0,s) = 0 \end{cases}$$

$$\Rightarrow \underbrace{s^2 \mathcal{L}u - \delta|x-x_0| - \partial_x^2 \mathcal{L}u + s \mathcal{L}u = 0}_{}$$

$$\mathcal{L}(V-u)(x,s) = Q(s) \cdot e^{-\sqrt{s^2+s}x}$$

$$(1 + \partial_x \mathcal{L})(V-u)(0,s) = -(1 + \partial_x) \mathcal{L}u(0,s)$$

$$\Rightarrow (1 - \sqrt{s^2+s})Q = (1 + \sqrt{s^2+s}) \cdot \frac{e^{-\sqrt{s^2+s} \cdot x_0}}{2\sqrt{s^2+s}}$$

$$\Rightarrow Q = \left(\frac{1 + \sqrt{s^2 + s}}{1 - \sqrt{s^2 + s}} \right) \cdot e^{-\sqrt{s^2 + s} \cdot x_0}$$

$$\Rightarrow V = u + \frac{1}{s^2 + s - 1} \cdot \underbrace{(1 + \sqrt{s^2 + s})^2}_{(1 - \sqrt{s^2 + s})^2} \cdot \frac{e^{-\sqrt{s^2 + s}(x + x_0)}}{2 \cdot \sqrt{s^2 + s^2}}$$

$$V = u + f^{-1}\left(\frac{1}{s^2 + s - 1}\right) * (1 - \sqrt{s^2 + s})^2 u(x - 2x_0, t).$$

High dimension problem:

$$u_{tt} - \Delta u = 0, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad (x, y, z) \in \mathbb{R}^3$$

$$\vec{\eta} = (\eta_1, \eta_2, \eta_3)$$

$$\hat{u}(\vec{\eta}, t) = \iiint_{\mathbb{R}^3} e^{-\vec{\eta} \cdot \vec{x}} \hat{u}(\vec{x}, t) d\vec{x} \quad \text{Fourier transform.}$$

then, $\hat{u}_{tt} - \iiint_{\mathbb{R}^3} e^{-i\vec{\eta} \cdot \vec{x}} \cdot (\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot \hat{u}(\vec{x}, t) d\vec{x} = 0$

I.B.P

$$\hat{u}_{tt} - \iiint_{\mathbb{R}^3} (\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot e^{-i\vec{\eta} \cdot \vec{x}} \cdot \hat{u}(\vec{x}, t) d\vec{x} = 0$$

$$\hat{u}_{tt} + |\vec{\eta}|^2 \hat{u} = 0 \quad \text{By computation.}$$

$$\Rightarrow \hat{u}(\vec{\eta}, t) = \frac{1}{2} (e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t}) \hat{u}(\vec{\eta}, 0) + \frac{e^{i|\vec{\eta}|t} - e^{-i|\vec{\eta}|t}}{2i|\vec{\eta}|} \cdot \hat{u}_t(\vec{\eta}, 0).$$

Inverse Fourier Transform,

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{(e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t})}{2} \cdot e^{i\vec{\eta} \cdot \vec{x}} d\vec{\eta}$$

$\stackrel{?}{=}$?
Can't be handled.

Fourier Transform Method fails !!!



Possion mean:

For 1-D:

$$w_{tt} - w_{xx} = 0, \quad x \in \mathbb{R}.$$

$$\Rightarrow w(x, t) = \frac{1}{2} [w(x+t, 0) + w(x-t, 0)] + \frac{1}{2} \int_{x-t}^{x+t} w_t(x, \sigma) dx.$$

by Taylor's expansion.

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u(\vec{x}, t) : \end{cases} \quad \Delta = \nabla \cdot \nabla$$

Define $\bar{u}(\vec{x}, t; r) = \frac{1}{4\pi r^2} \cdot \int_{B_r(\vec{x})} u(\vec{y}, t) dA_y$

Now, $\int_{B_r(\vec{x})} u_{tt}(\vec{y}, t) d\vec{y} = \int_{B_r(\vec{x})} \nabla \cdot \nabla u(\vec{y}, t) d\vec{y}$

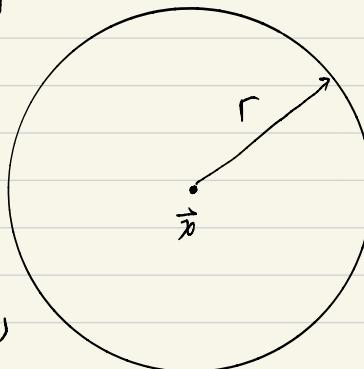
Divergence Thm

$$= \int_{\partial B_r(\vec{x})} \nabla u(\vec{y}, t) \cdot \vec{n} ds$$

$$= \int_{\partial B_r(\vec{x})} \frac{\partial u}{\partial r} ds, \quad dA = r^2 \cdot dS,$$

$$= \int_{B_r(\vec{x})} \frac{\partial u}{\partial r} \cdot r^2 dS,$$

$$= r^2 \cdot 4\pi \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r)$$



$$\iiint_{B_r(\vec{x})} u_{tt}(\vec{y}, t) d\vec{y} = \int_0^r \iint_{|w|=1} u_{tt}(r\vec{w} + \vec{x}, t) \cdot r^2 d\omega dp$$

$$= \int_0^r 4\pi r^2 \bar{u}_{tt}(\vec{x}, t; r) dr$$

Now, we have:

$$r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r) = \int_0^r 4\pi r^2 \bar{u}_{tt}(\vec{x}, t; r) dr$$

Take derivative w.r.t. r,

$$\frac{\partial}{\partial r} (r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r))$$

$$= 4\pi r^2 \bar{u}_{tt}(\vec{x}, t; r)$$

Simplified:

$$2r \cdot \frac{\partial \bar{u}}{\partial r} + r^2 \cdot \frac{\partial^2 \bar{u}}{\partial r^2} = r^2 \bar{u}_{tt}$$

$$\Rightarrow \bar{u}_{tt} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \bar{u}}{\partial r} \quad \leftarrow (\ast)$$

then, $(\ast) \cdot t$

$$\Rightarrow (r \cdot \bar{u})_{tt} = (r \bar{u})_{rr} \quad \text{with } r > 0.$$

Next, by odd extension for $r \cdot \bar{u} = 0$ at $r=0$

$$\Rightarrow (r \bar{u}) = (r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t) \bar{u}(\bar{x}, 0; r-t)$$

$$+ \frac{1}{2} \int_{r-t}^{r+t} p \bar{u}_t(x, 0; p) dp$$

Simplified:

$$\bar{u}(\bar{x}, t; r) = (r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t) \bar{u}(\bar{x}, 0; r-t)$$

$$+ \frac{1}{2r} \int_{r-t}^{r+t} p \bar{u}_t(x, 0; p) dp$$

Recall, for contin. solution,

$$\lim_{r \rightarrow 0} \bar{u}(\bar{x}, t; r) = u(\bar{x}, t) \text{ by Lebegues Thm.}$$

Then, $\lim_{r \rightarrow 0} \bar{u}(\bar{x}, t; r)$

$$= \lim_{r \rightarrow 0} \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t) \bar{u}(\bar{x}, 0; r-t)}{2r} \quad \leftarrow (\text{II})$$

$$+ \lim_{r \rightarrow 0} \frac{1}{2r} \int_{r-t}^{r+t} p \bar{u}_t(x, 0; p) dp$$

(I)

Consider (I):

$$\frac{1}{2r} \left(\int_0^{r+t} p \vec{u}_t(x, 0; p) dp + \underbrace{\int_{r-t}^0 p \vec{u}_t(x, 0; p) dp}_{//} \right)$$

$$- \int_0^{t-r} (-) p \vec{u}_t(x, 0; p) (-) dp \quad \text{for } \vec{u} \text{ is odd extension.}$$

$$= \frac{1}{2r} \int_{t-r}^{t+r} p \vec{u}_t(x, 0; p) dp$$

Take lim of r:

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{t-r}^{t+r} p \cdot \vec{u}_t(x, 0; p) dp = t \cdot \vec{u}_t(x, 0; t)$$

$$= t \cdot \int_{\partial B_t(x)} u_t ds / 4\pi t^2$$

$$= \frac{1}{4\pi t} \cdot \int_{\partial B_t(x)} u_t(x, 0; t) ds.$$

Consider (II) :

Take lim of r,

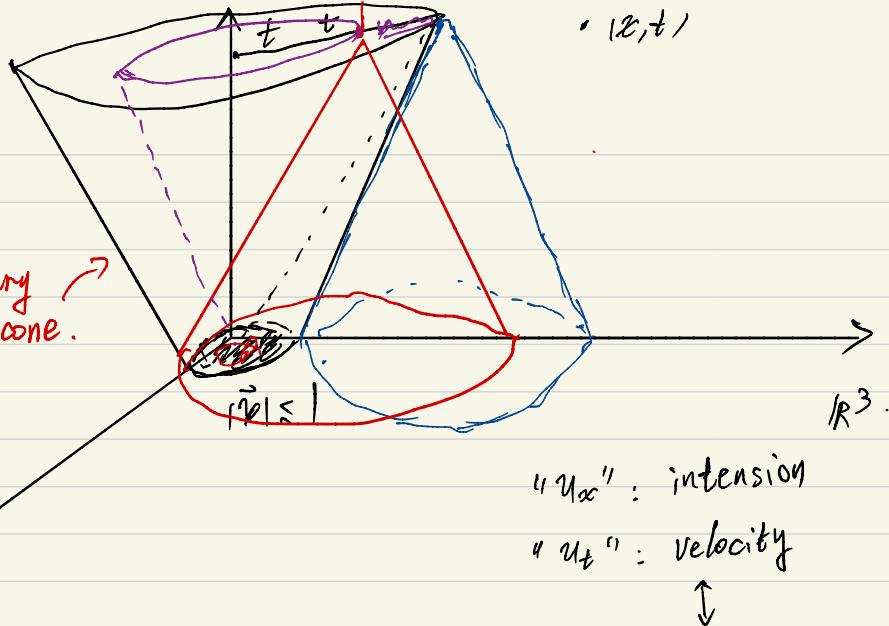
$$\lim_{r \rightarrow 0} \frac{(r+t) \cdot \vec{u}(x, 0; r+t) + (r-t) \vec{u}(x, 0; r-t)}{2r} - (-r+t) \cdot \vec{u}(x, 0; -t)$$

$$= \frac{\partial}{\partial t} - t \vec{u}(x, 0; t) = \vec{u}(x, 0; t) + t \cdot \vec{u}_t(x, 0; t)$$

$$u(x, t) = \frac{1}{4\pi t} \cdot \int_{\partial B_t(x)} u_t(x, 0; t) ds +$$

$$\frac{\partial}{\partial t} \cdot \frac{1}{4\pi t} \int_{\partial B_t(x)} u(x, 0; t) ds$$

Kirchhoff's formula:



Remark:

- The wave travels around the boundary of the cone.

" u_{xx} ": intension

" u_t ": velocity



kinetic energy

Q: What about wave eq. in 2-diml. space?

A: The method doesn't work.

By same Poisson mean,

$$\Rightarrow \frac{1}{r} \left(\frac{\partial \bar{u}}{\partial r} + r \cdot \frac{\partial^2 \bar{u}}{\partial r^2} \right) = \bar{u}_{tt}$$

this equation can't reduce to be a wave equation in 1-D.!

Another way to solve it: (See 2-diml. to be embedded in 3-diml.)

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x, y \in \mathbb{R}, t > 0 \\ u(x, y, 0) = u_0(x, y), & u_t(x, y, 0) = u_1(x, y) \end{cases}$$

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x, y \in \mathbb{R}, t > 0 \\ u(x, y, 0) = u_0(x, y), & u_t(x, y, 0) = u_1(x, y) \end{cases}$$

Define $\bar{u}(x, y, z, t) = u(x, y, t)$,

then $\bar{u}_{tt}(x, y, z, t) = u_{tt}(x, y, z)$

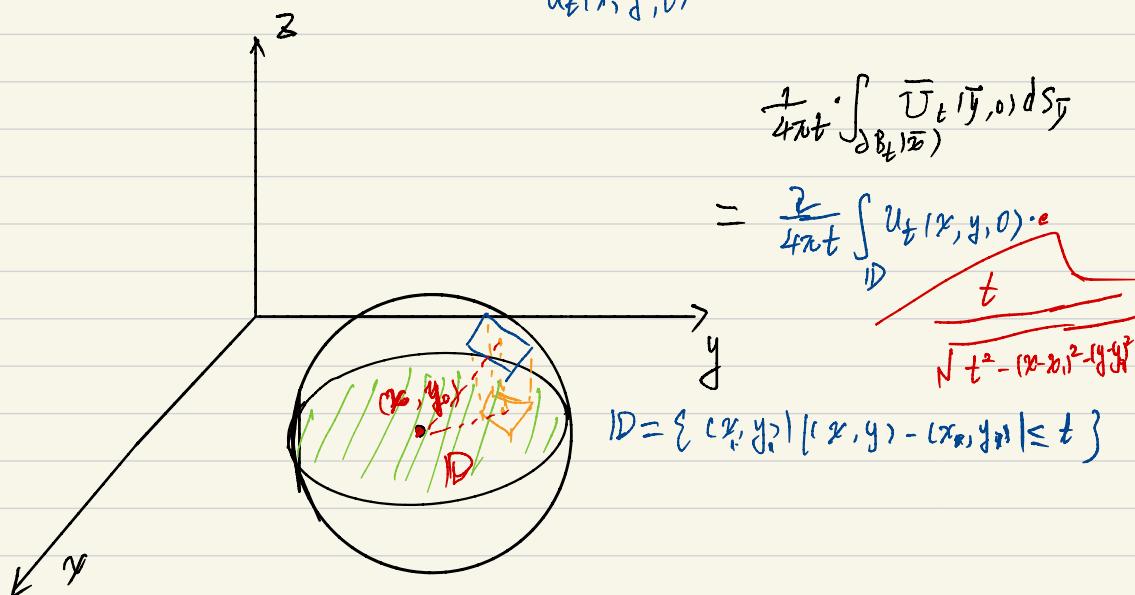
$$u(x, y, t) \quad \bar{U}_{xx} + \bar{U}_{yy} + \bar{U}_{zz} = u_{xx} + u_{yy} + 0$$

Then,

||

$$\bar{U}(\vec{x}, t) = \frac{1}{4\pi t} \int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y}, 0) dS_{\vec{y}} + \frac{\partial}{\partial t} \frac{1}{4\pi t} \int_{\partial B_t(\vec{x})} \bar{U}(\vec{y}, 0) dS_{\vec{y}}$$

$$\vec{x} = (x, y, z)$$



$$u_t(x, y, 0)$$

$$\frac{1}{4\pi t} \cdot \int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y}, 0) dS_{\vec{y}}$$

$$= \frac{1}{4\pi t} \int_D u_t(x, y, 0) \cdot e$$

$$t$$

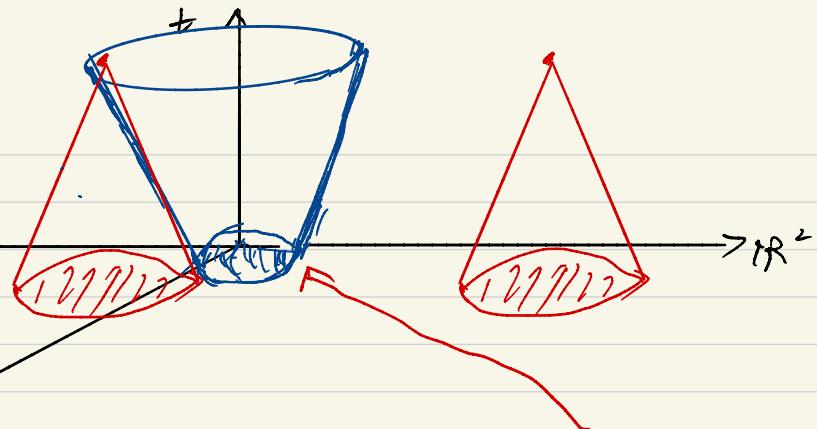
$$\sqrt{t^2 - (x - x_0)^2 - (y - y_0)^2}$$

$$D = \{(x, y) | |(x, y) - (x_0, y_0)| \leq t\}$$

$$\frac{1}{4\pi t} \cdot \int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y}, 0) dS_{\vec{y}} = \frac{1}{2\pi} \iint_{\substack{u_1(x_1, y_1) \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq t}} \frac{d\omega_1 dy_1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

Take $\vec{x} = (x, y)$

$$u(\vec{x}, t) = \frac{1}{2\pi} \iint_{\substack{u_1(\vec{y}) \\ |\vec{y} - \vec{x}| \leq \sqrt{t^2 - |\vec{x} - \vec{y}|^2}}} \frac{u_1(\vec{y})}{\sqrt{t^2 - |\vec{x} - \vec{y}|^2}} d\vec{y} + \frac{1}{2\pi} \iint_{\substack{u_0(\vec{y}) \\ |\vec{y} - \vec{x}| \leq t}} \frac{u_0(\vec{y})}{\sqrt{t^2 - |\vec{x} - \vec{y}|^2}} d\vec{y}$$



Remark: • The wave travels inside of the cone!
Big difference with 3-diml.

Now, define

$$f(u_1) = \frac{1}{2\pi} \iint_{|\vec{x}-\vec{y}| \leq t} \frac{u_1(\vec{y})}{\sqrt{t^2 - (\vec{x}-\vec{y})^2}} d\vec{y}$$

$$f(u_1) \equiv \text{solution of } \begin{cases} u_{tt} - u_{xx} - u_{yy} = 0 \\ u(x, y, 0) = 0 \\ u_t(x, y, 0) = u_1 \end{cases}$$

By Fourier Transform.

$$\Rightarrow \hat{u}(\vec{y}, t) = \frac{e^{i\vec{y}t} - e^{-i\vec{y}t}}{2i\vec{y}} \hat{u}_1(\vec{y}) = \frac{\sin |\vec{y}|t}{|\vec{y}|} \hat{u}_1(\vec{y})$$

\checkmark Must be equivalent

Fermat Last Thm:

$$x^n + y^n = z^n, n > 2$$

Half Space:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x > 0, y \in \mathbb{R}, t > 0 \\ u(0, y, t) = 0 \\ \cancel{u(x, y, 0) = u_0(x, y)} = 0 \\ u_t(x, y, 0) = u_1(x, y) = \delta(x-x_0) \delta(y) \end{cases}$$

Laplace - Fourier
Transform.

Define

$$\mathcal{L} u(x, y, s) = \int_0^\infty \int_{\mathbb{R}} e^{-st-iy} u(x, y, t) dy dt$$

$$\Rightarrow s^2 \mathcal{L} u - \hat{u}_1(x, y) - \partial_x^2 \mathcal{L} u + y^2 \mathcal{L} u = 0$$

$$\mathcal{L} u(0, y, s) = 0$$

$$\Rightarrow \begin{cases} (s^2 + y^2) \mathcal{L} u - \partial_x^2 \mathcal{L} u = \hat{u}_1 = \delta(x-x_0) \\ \mathcal{L} u(0, y, s) = 0 \end{cases} \quad (*)$$

We already have known:

$$\begin{cases} w_{tt} - w_{xx} - w_{yy} = 0, & x, y \in \mathbb{R} \\ w(x, y, 0) = 0 \\ w_t(x, y, 0) = \delta(x) \cdot \delta(y) \end{cases}$$

$$\Rightarrow (s^2 + y^2) \mathcal{L} w - \partial_x^2 \mathcal{L} w = \delta(x)$$

$$\mathcal{L} w = \frac{e^{-\sqrt{s^2+y^2}|x|}}{2\sqrt{s^2+y^2}}$$

Then, by (*), we get:

$$\mathcal{L} w = \frac{e^{-\sqrt{s^2+y^2}|x-x_0|}}{2\sqrt{s^2+y^2}} + A \cdot e^{-\sqrt{s^2+y^2}|x|}$$

$$\Rightarrow A = - \frac{e^{-\sqrt{s^2 + y^2} \cdot |x_k|}}{2 \cdot \sqrt{s^2 + y^2}}$$

$$\Rightarrow f_u = \frac{e^{-\sqrt{s^2 + y^2} \cdot |x - x_k|}}{2 \cdot \sqrt{s^2 + y^2}} - \frac{e^{-\sqrt{s^2 + y^2} \cdot (x + x_k)}}{2 \cdot \sqrt{s^2 + y^2}}, \quad x_k > 0$$

!

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x > 0, y \in \mathbb{R} \\ u(x, y, 0) = 0, \quad u_t(x, y, 0) = \delta(x - x_0) \delta(y) \\ u(0, y, t) - k u_x(0, y, t) = 0. \end{cases}$$

Transform in (t, y) .

Take Laplace - Fourier Transform,

$$\begin{cases} s^2 \mathcal{L}u - \frac{\partial^2}{\partial x^2} \mathcal{L}u + \eta^2 \mathcal{L}u = \delta(x - x_0) \\ \mathcal{L}u(0, y, s) - k \frac{\partial}{\partial x} \mathcal{L}u(0, y, s) = 0 \end{cases} \quad \text{B.C.}$$

$$\Rightarrow \mathcal{L}u(x, y, s) = \frac{e^{-\sqrt{s^2+\eta^2}|x-x_0|}}{2\sqrt{s^2+\eta^2}} + A \cdot e^{-\sqrt{s^2+\eta^2}x}$$

Use B.C.,

$$\Rightarrow \left. \frac{e^{-\sqrt{s^2+\eta^2}|x_0|}}{2\sqrt{s^2+\eta^2}} + A - k \left(\frac{e^{-\sqrt{s^2+\eta^2}(x-x_0)}}{2\sqrt{s^2+\eta^2}} + A \cdot e^{-\sqrt{s^2+\eta^2}x} \right) \right|_{x=0} = 0$$

$$\Downarrow \sqrt{s^2+\eta^2} \left(\frac{e^{-\sqrt{s^2+\eta^2}x_0}}{2\sqrt{s^2+\eta^2}} - A \right)$$

$$\Rightarrow (1 + k\sqrt{s^2+\eta^2})A + \left(1 - k\sqrt{s^2+\eta^2} \right) \cdot \frac{e^{-\sqrt{s^2+\eta^2}x_0}}{2\sqrt{s^2+\eta^2}} = 0$$

$$\Rightarrow A = \frac{-e^{-\sqrt{s^2+\eta^2}x_0} (1 - k\sqrt{s^2+\eta^2})}{2\sqrt{s^2+\eta^2} \cdot (1 + k\sqrt{s^2+\eta^2})}$$

$$\Rightarrow \mathcal{L}u(x, y, s) = \frac{e^{-\sqrt{s^2+\eta^2}|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1 - k\sqrt{s^2+\eta^2})}{(1 + k\sqrt{s^2+\eta^2})} \cdot \frac{e^{-\sqrt{s^2+\eta^2}(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

\Downarrow

$$= \frac{e^{-\sqrt{s^2+\eta^2} |x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1-k\sqrt{s^2+\eta^2})^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$= \frac{e^{-\sqrt{s^2+\eta^2} |x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1+\frac{x_0}{s})^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

Inverse Laplace Transform:

$$f(t) : F(s) = \int_0^\infty e^{-st} \cdot f(t) dt, \quad \operatorname{Re}(s) > 0, \quad s \in \mathbb{C}$$

$$f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=0} e^{st} \cdot F(s) ds$$

$$\mathcal{L}^{-1} \left[\frac{1}{(1-k^2(s^2+\eta^2))} \right] = \frac{1}{4\pi^2 i} \iint_{-\infty}^{\infty} e^{i\eta y + st} \frac{1}{[1-k^2(s^2+\eta^2)]} ds dy$$

Another boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x > 0, y \in \mathbb{R} \\ u(x, y, 0) = 0, \quad u_t(x, y, 0) = \delta(x - x_0) \delta(y) \\ u(0, y, t) - k u_x(0, y, t) = 0. \end{cases}$$

(B.C.)

B.C. $\Rightarrow S \int u = k \partial_x \int u.$

$$S \int u(x, y, s) = \frac{e^{-\sqrt{s^2+y^2}(x-x_0)}}{2\sqrt{s^2+y^2}} + A \cdot e^{-\sqrt{s^2+y^2}x}$$

By the process above, change "1" to "3":

$$\Rightarrow (S + K\sqrt{s^2+y^2}) A + \frac{e^{-\sqrt{s^2+y^2}x_0}}{2\sqrt{s^2+y^2}} (S - k\sqrt{s^2+y^2}) = 0$$

$$= \frac{e^{-\sqrt{s^2+y^2}(x-x_0)}}{2\sqrt{s^2+y^2}} - \frac{(S - k\sqrt{s^2+y^2})^2}{[s^2 - k^2(s^2+y^2)]} - \frac{e^{-\sqrt{s^2+y^2}(x+x_0)}}{2\sqrt{s^2+y^2}}$$



$$\frac{(S^2 - 2k\sqrt{s^2+y^2}s + k^2(s^2+y^2))}{[s^2 - k^2(s^2+y^2)]}$$

$$= \frac{e^{-\sqrt{s^2+y^2}(x-x_0)}}{2\sqrt{s^2+y^2}} - \frac{\cancel{[s^2 + 2ks \partial_x + k^2 \partial_x^2]}}{\cancel{(s^2 - k^2(s^2+y^2))}} \cdot \frac{e^{-\sqrt{s^2+y^2}(x+x_0)}}{2\sqrt{s^2+y^2}}$$

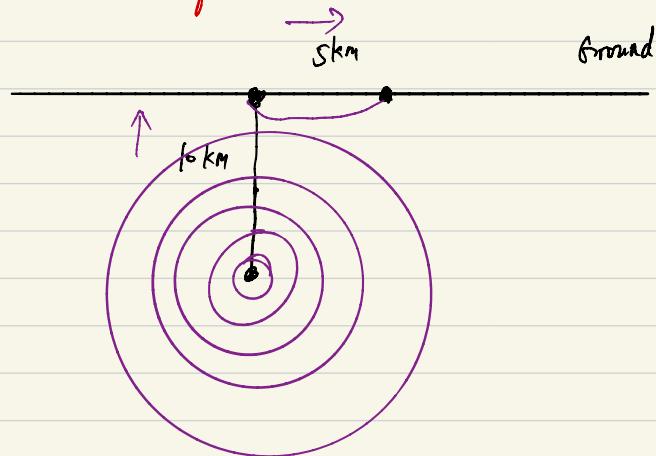
$$= \frac{e^{-\sqrt{s^2+y^2}(x-x_0)}}{2\sqrt{s^2+y^2}} - [s^2 + 2ks \partial_x + k^2 \partial_x^2] \cdot \frac{e^{-\sqrt{s^2+y^2}(x+x_0)}}{2\sqrt{s^2+y^2}} \cdot \frac{1}{(1-k^2)s^2 - k^2y^2}$$

$$\sim -\frac{1}{k^2} \int \left[\left(\frac{k^2-1}{k^2} \partial_t^2 - \partial_y^2 \right)^{-1} \right]$$

$$= \frac{-1/k^2}{\frac{k^2-1}{k^2} s^2 + g^2}$$

Physical meaning:

- ① Speed: $\frac{k^2}{k^2-1}$
- ② If k small, then no wave eq.



$$\begin{cases} u_t - u_x + v_y = u_{xx} + u_{yy}, & x>0 \\ v_t + v_x + u_y = v_{xx} + v_{yy}, & y \in \mathbb{R} \end{cases}$$

i.e. $\begin{cases} u_t - u_x + v_y = \Delta u \\ v_t + v_x + u_y = \Delta v \end{cases}$

$$\begin{pmatrix} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \Delta & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} ((\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \Delta) \cdot (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \Delta) - \frac{\partial^2}{\partial y^2}) u = 0 \\ ((\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \Delta) \cdot (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \Delta) - \frac{\partial^2}{\partial y^2}) v = 0 \end{cases}$$

$$\Rightarrow [(\frac{\partial}{\partial t} - \Delta)^2 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}] u = 0 \quad \&$$

$$[(\frac{\partial}{\partial t} - \Delta)^2 - \Delta] \cdot v = 0$$

Suppose $[(\frac{\partial}{\partial t} - \Delta)^2 - \Delta] u = 0$ is an equation on $\mathbb{R}^2 \times \mathbb{R}_+$

$\hat{u}(\vec{\eta}, t)$: Fourier transform of $u(x, y, t)$, $\vec{\eta} \in \mathbb{R}^2$

$$\Rightarrow [(\frac{\partial}{\partial t} + |\vec{\eta}|^2)^2 + |\vec{\eta}|^2] \hat{u} = 0$$

$$\Rightarrow (\frac{\partial}{\partial t} + |\vec{\eta}|^2 - i|\vec{\eta}|) (\frac{\partial}{\partial t} + |\vec{\eta}|^2 + i|\vec{\eta}|) \hat{u} = 0$$

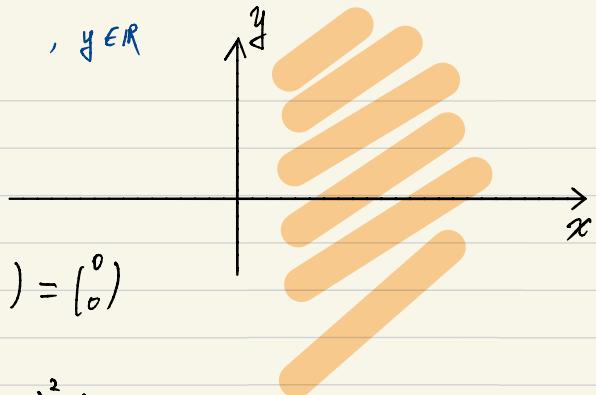
$$\text{Then, } \hat{u}(\vec{\eta}, t) = A \cdot e^{(-|\vec{\eta}|^2 + i|\vec{\eta}|)t} + B \cdot e^{-|\vec{\eta}|^2 t - i|\vec{\eta}| t}$$

$$= e^{-|\vec{\eta}|^2 t} \cdot (A \cdot e^{i|\vec{\eta}| t} + B \cdot e^{-i|\vec{\eta}| t})$$

$$? = e^{-|\vec{\eta}|^2 t} \cdot (A \cdot \frac{\sin |\vec{\eta}| t}{i|\vec{\eta}|} + B \cdot \cos |\vec{\eta}| t)$$

$$A = \hat{u}(\vec{\eta}, 0)$$

$$B = \hat{u}'(\vec{\eta}, 0)$$



Fourier - Laplace transform:

$$\mathcal{L} \begin{pmatrix} \partial_t - \partial_x - \Delta & \partial_y \\ \partial_y & \partial_t + \partial_x - \Delta \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{pmatrix} \mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{u}(x, \eta, 0) \\ \hat{v}(x, \eta, 0) \end{pmatrix}$$

Suppose $\begin{pmatrix} u(x, \eta, 0) \\ v(x, \eta, 0) \end{pmatrix} = \begin{pmatrix} \delta_{1x} & \delta_{1y} \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \delta_{1x}, \delta_{1y} \end{pmatrix}$.

If $\begin{pmatrix} u(x, \eta, 0) \\ v(x, \eta, 0) \end{pmatrix} = \begin{pmatrix} \delta_{1x} & \delta_{1y} \\ 0 & 0 \end{pmatrix}$,

$$\begin{pmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{pmatrix} \mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \delta_{1x} \\ 0 \end{pmatrix}$$

Consider

$$\begin{pmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{pmatrix} \Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{with}$$

Assume $\Psi = e^{\lambda x} \cdot \vec{V}_0$ is a solution

$$\Rightarrow \begin{pmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{pmatrix} e^{\lambda x} \cdot \vec{V}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} s - \cancel{\lambda} - \lambda^2 + \eta^2 & i\eta \\ i\eta & s + \cancel{\lambda} - \lambda^2 + \eta^2 \end{pmatrix} \vec{V}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (s + j^2 - \lambda^2)^2 - \lambda^2 + j^2 = 0$$

$$V_0 = \begin{bmatrix} -ij \\ s - \lambda - \lambda^2 + j^2 \end{bmatrix}$$

$$\alpha = \lambda^2 \Rightarrow (s + j^2 - \alpha)^2 - \alpha + j^2 = 0$$

$$\Rightarrow \alpha^2 - 2(s + j^2)\alpha - \alpha + (s + j^2)^2 + j^2 = 0$$

$$\Rightarrow \alpha = \frac{2(s + j^2) + 1 \pm \sqrt{(2(s + j^2) + 1)^2 - 4[(s + j^2)^2 + j^2]}}{2}$$

$$\text{Since } [2(s + j^2) + 1]^2 - 4[(s + j^2)^2 + j^2]$$

$$= 4s + 1$$

$$\text{then } \alpha = \frac{2(s + j^2) + 1 \pm \sqrt{4s + 1}}{2}$$

$$\lambda = \pm \sqrt{\frac{2(s + j^2) + 1 \pm \sqrt{4s + 1}}{2}}$$

$$\Rightarrow \lambda_{\pm}^R = -\sqrt{\frac{2(s + j^2) + 1 \pm \sqrt{4s + 1}}{2}}$$

$$\& \lambda_{\pm}^L = \sqrt{\frac{2(s + j^2) + 1 \pm \sqrt{4s + 1}}{2}}$$

$$\bullet V_0 = \begin{bmatrix} -ij \\ s - \lambda - \lambda^2 + j^2 \end{bmatrix},$$

$$e^{\lambda_{\pm}^L x} \left(\begin{bmatrix} -ij \\ s - \lambda_{\pm}^L - \lambda_{\pm}^L + j^2 \end{bmatrix} \right)$$

$$e^{\lambda_{\pm}^R x} \left(\begin{bmatrix} -ij \\ s - \lambda_{\pm}^R - \lambda_{\pm}^R + j^2 \end{bmatrix} \right)$$

$$Q(x) = \begin{cases} L_+ \cdot e^{\lambda_+^R x} \cdot \left(\frac{-iy}{s - \lambda_+^L - (\lambda_+^L)^2 + j^2} \right) + L_- \cdot e^{\lambda_-^R x} \cdot \left[\frac{-iy}{s - \lambda_-^L - (\lambda_-^L)^2 + j^2} \right], & x > 0 \\ R_+ \cdot e^{\lambda_+^R x} \cdot \left[\frac{-iy}{s - \lambda_+^R - (\lambda_+^R)^2 + j^2} \right] + R_- \cdot e^{\lambda_-^R x} \cdot \left[\frac{-iy}{s - \lambda_-^R - (\lambda_-^R)^2 + j^2} \right], & x < 0 \end{cases}$$

Continuity:

$$\begin{aligned} & L_+ \left[\frac{-iy}{s - \lambda_+^L - (\lambda_+^L)^2 + j^2} \right] + L_- \left[\frac{-iy}{s - \lambda_-^L - (\lambda_-^L)^2 + j^2} \right] \\ &= R_+ \left[\frac{-iy}{s - \lambda_+^R - (\lambda_+^R)^2 + j^2} \right] + R_- \left[\frac{-iy}{s - \lambda_-^R - (\lambda_-^R)^2 + j^2} \right] \end{aligned}$$

- Consider $\text{Arg}(s) \cdot e^{-\sqrt{\frac{2(s+j^2)+1 \pm \sqrt{4s+1}}{2}}x}, x > 0$
- How to compute inverse of $\text{Arg}(s) \cdot e^{-\sqrt{\frac{2(s+j^2)+1 \pm \sqrt{4s+1}}{2}}x}$?

$$\begin{aligned} \sqrt{\frac{2(s+j^2)+1 \pm \sqrt{4s+1}}{2}} &= \sqrt{2(s+j^2) - 2\sqrt{s+\frac{1}{4}} + 1} / 2 \\ &= \sqrt{s + \frac{1}{4} + \frac{1}{4} - \sqrt{s+\frac{1}{4}} + j^2} = \sqrt{(\sqrt{s+\frac{1}{4}} - \frac{1}{2})^2 + j^2} = \lambda. \end{aligned}$$

$$\Rightarrow \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\text{Re}(s)=0} \int_{\mathbb{R}} e^{-\lambda x + iy + st} \cdot dy ds$$

Fourier-Laplace Path:

$$\frac{1}{2\pi} \cdot \frac{1}{2\pi i} \cdot \int_{\mathbb{R}} \int_{|\operatorname{Re}(s)|=0}^{\infty} e^{-\lambda x + i\gamma y + st} ds dy$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \cdot \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\zeta x + i\gamma y + st} \underbrace{ds dy}_{\frac{ds(s)}{ds}}$$

Want: $\lambda(s(\zeta)) = i\zeta, \zeta \in \mathbb{R}$

$$\sqrt{(\sqrt{s+\frac{1}{4}} - \frac{1}{2})^2 + \gamma^2} = -i\zeta,$$

then, $(\sqrt{s+\frac{1}{4}} - \frac{1}{2})^2 + \gamma^2 = -\zeta^2$

$$\Rightarrow (\sqrt{s+\frac{1}{4}} - \frac{1}{2})^2 = -\zeta^2 - \gamma^2$$

$$\Rightarrow \sqrt{s+\frac{1}{4}} - \frac{1}{2} = \pm i\sqrt{\gamma^2 + \zeta^2}$$

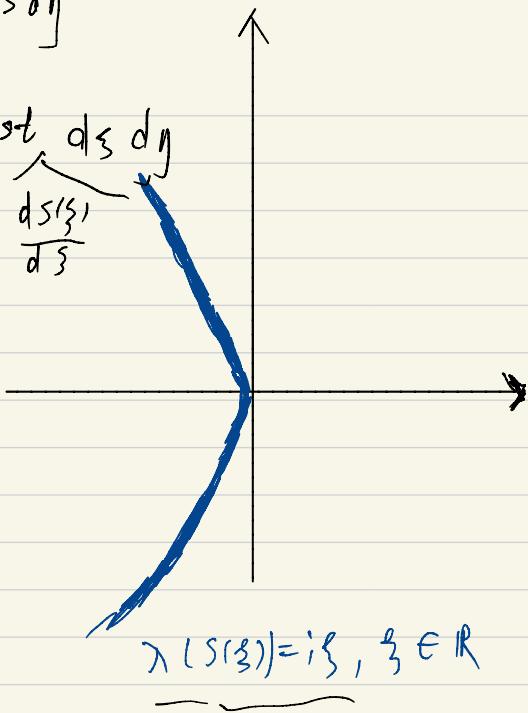
$$(\sqrt{s+\frac{1}{4}})^2 = (\pm i\sqrt{\gamma^2 + \zeta^2} + \frac{1}{2})^2$$

$$\therefore s = -(\gamma^2 + \zeta^2) \pm i\sqrt{\gamma^2 + \zeta^2}$$

thus, $\frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\zeta x + i\gamma y - (\gamma^2 + \zeta^2)t} \underbrace{\frac{ds(s)}{ds}}_{\text{heat eq.}} \underbrace{ds dy}_{\text{wave eq.}}$

$$\frac{ds}{d\zeta} = -2\zeta \pm i \frac{\zeta}{\sqrt{\gamma^2 + \zeta^2}}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\zeta x + i\gamma y - (\gamma^2 + \zeta^2)t} \pm i\sqrt{\gamma^2 + \zeta^2} t \cdot \left(-2\zeta \pm \frac{i\zeta}{\sqrt{\gamma^2 + \zeta^2}} \right) \frac{ds dy}{ds}$$



$$\begin{aligned}
&= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\text{R}} \int_{-\infty}^{\infty} e^{isx + iy - (j^2 + s^2)t \pm i\sqrt{j^2 + s^2}t} \Big|_{(-2j)} ds dy \\
&\quad + \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\text{R}} \int_{-\infty}^{\infty} e^{isx + iy - (j^2 + s^2)t \pm i\sqrt{j^2 + s^2}t} \Big|_{\frac{\pm is}{\sqrt{j^2 + s^2}}} ds dy \\
&= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\text{R}} \int_{-\infty}^{\infty} 2i \partial_x e^{isx + iy - (j^2 + s^2)t \pm i\sqrt{j^2 + s^2}t} ds dy \\
&\quad + \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\text{R}} \int_{-\infty}^{\infty} \partial_x e^{isx + iy - (j^2 + s^2)t \pm i\sqrt{j^2 + s^2}t} \cdot \frac{\pm 1}{\sqrt{j^2 + s^2}} ds dy \\
&\quad \underbrace{\partial_x \cdot e^{i(jx+iy)}}_{\text{II}} \cdot e^{-(-j^2+s^2)t} \cdot \frac{e^{\pm i\sqrt{j^2+s^2}t}}{\pm\sqrt{j^2+s^2}}
\end{aligned}$$

\downarrow
 $\gamma^{-1} \left(\frac{e^{-(-j^2+s^2)t}}{\pm\sqrt{j^2+s^2}} \right) * \gamma^{-1} \left(\frac{e^{\pm i\sqrt{j^2+s^2}t}}{\pm\sqrt{j^2+s^2}} \right)$
 \downarrow
 $\gamma^{-1} \left(\frac{\sin \sqrt{j^2+s^2} t}{\sqrt{j^2+s^2}} \right)$

Q.E.D.

Navier-Stokes Eq.:

$$\begin{cases} p_t + m_x = 0 \\ m_t + (um)_x + P(p)_x = u_{xx} \end{cases}, \quad \begin{array}{l} m: \text{momentum} \\ p: \text{density of fluid} \\ u: \text{fluid velocity} \end{array}$$

$P(p)$: Pressure

$$P(p) = p^\gamma, \gamma \in (1, \frac{5}{3})$$

A Linearized Eq.

$$\begin{cases} p_t + m_x = 0 \\ m_t + p_x = m_{xx} \end{cases}$$

$$\begin{array}{l} \textcircled{1} \quad \begin{cases} p_t + m_x = 0 \\ m_t + p_x = 0 \end{cases} \quad \text{wave eq.} \\ \Downarrow \end{array}$$

$$(p_t + m_{xx})_x = 0 = (m_t + p_x)_t$$

$$\Downarrow \\ m_{xx} - m_{tt} = 0.$$

$$\textcircled{2} \quad \left(\begin{array}{cc} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial t} \end{array} \right) \left(\begin{array}{c} p \\ m \end{array} \right) = 0.$$

$$\Rightarrow \partial_{tt} - \partial_{xx} = 0.$$

$$\textcircled{3} \quad \begin{cases} p_t + m_x = p_{xx} \\ m_t + p_x = m_{xx} \end{cases} \Rightarrow \left(\begin{array}{cc} \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \end{array} \right) \left(\begin{array}{c} p \\ m \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$$

$$[(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})^2 - \frac{\partial^2}{\partial x^2}] p = 0$$

$$[(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})^2 - \frac{\partial^2}{\partial x^2}] m = 0$$

By Fourier Transform

$$[(\alpha + \eta^2)^2 + \eta^2] \hat{P} = 0$$

$$\hat{P} = A_+ e^{-\eta^2 t + i \gamma t} + A_- e^{-\eta^2 t - i \gamma t}$$

heat eq. transport-

Weak solution: Nonlinear Problem:

$$u_t + f(u)_x = 0$$

$$\Rightarrow \iint \varphi(x, t) \cdot (u_t + f(u)_x) dx dt = 0, \quad \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$$



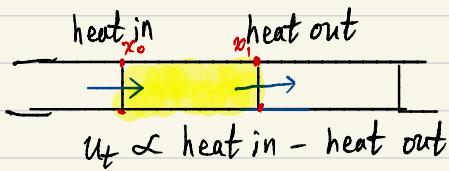
$$(1*) \quad \iint [-\varphi_t(x, t) \cdot u(x, t) - \varphi_x(x, t) f(u)] dx dt = 0,$$

A solution u satisfying $(1*)$ is called weak sol.

Heat Eq:

$$u_t = u_{xx}$$

Fourier's Law: Heat flux is proportion to temperature gradient.



• Flux: u at $x_0 \sim -k u_x(x_0, t)$

$$u_t = [-k_1 x_0 \cdot u_x(x_0, t) + k_1 x_1 \cdot u_x(x_1, t)]$$

$$\Rightarrow u_t = \lim_{x_1 \rightarrow x_0} \frac{[-k_1 x_0 \cdot u_x(x_0, t) + k_1 x_1 \cdot u_x(x_1, t)]}{x_1 - x_0}$$

$$\Rightarrow u_t = \partial_x(k \pi) u_x$$

Consider weak solution:

$$\iint p(u_t - \partial_x(k \pi) \cdot u_x) dx dt = 0,$$

$$\Rightarrow \iint -p_t u + p_x \cdot k(x) \cdot u_x \, dx \, dt = 0, \quad \forall p \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$$

u_x local integrable $\Rightarrow u$: continuous.

Condition for $k(x)$:

$k(x)$: B.V. function:

- Bounded Variation : • Piecewise continuous
- Jump is countable

Step 1: k is const. & $k > 0$;

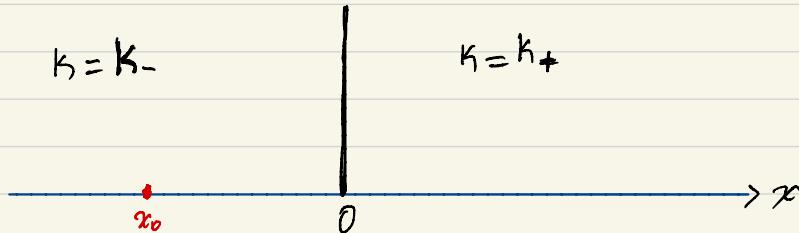
$$\begin{cases} u_t = k u_{xx} \\ u(x, 0) = \delta(x) \end{cases} \quad \text{Green Function.} \quad \Rightarrow \quad u(x, t) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi t}}$$

$$\mathcal{L} u(x, s) = \int_0^\infty e^{-st} \cdot u(x, t) \, dt \quad -\sqrt{s_k} |x| \\ \Rightarrow \mathcal{L} u(x, s) = \frac{e}{2 \sqrt{s_k}}$$

Step 2:

$$k = k_-$$

$$k = k_+$$



$$\begin{cases} u_t = \partial_x (k(x) u_x) \\ u(x, 0) = \delta(x - x_0), \quad x_0 < 0 \end{cases}$$

i) $u(x)$: continuous.

I.B.P.

$$\iint p y' \, dx = - \iint p' y \, dx$$

y : continuous.

(ii) $\lim_{x \rightarrow 0} u_x$: contin.

$$\begin{cases} S_L u = \partial_x (k(x) \partial_x L u) + f(x-x_0) \\ u(x, 0) = \delta(x-x_0), \quad x_0 < 0 \end{cases}$$

$$\begin{cases} S_L u = \partial_x (k_- \partial_x L u) + f(x-x_0) \quad \text{for } x < 0 \\ S_L u = \partial_x (k_+ \partial_x L u) \quad \text{for } x > 0 \end{cases}$$

$$\Rightarrow L u = \begin{cases} \frac{e^{-\sqrt{s/k_-} \cdot |x-x_0|}}{2\sqrt{s k_-}} + U_- \cdot e^{\sqrt{s/k_-} x}, & x < 0 \\ S_+ \cdot e^{-\sqrt{s/k_+} x} & x > 0 \end{cases}$$

By continuity of u at 0,

$$\frac{e^{-\sqrt{s/k_-} \cdot |x_0|}}{2\sqrt{s k_-}} + U_- = S_+ \quad \text{--- (I)}$$

By continuity of flux:

$$\begin{cases} \left[\frac{e^{-\sqrt{s/k_-} \cdot |x-x_0|}}{2\sqrt{s k_-}} + U_- \cdot e^{\sqrt{s/k_-} x} \right]_x, & x < 0 \\ (S_+ \cdot e^{-\sqrt{s/k_+} x})_x, & x > 0 \end{cases}$$

$$\Rightarrow k_- \left(-\sqrt{\frac{s}{k_-}} \cdot \frac{e^{-\sqrt{s/k_-} |x_0|}}{2\sqrt{s k_-}} + U_- \cdot \sqrt{\frac{s}{k_-}} \right)$$

$$= k_+ (-S_+ \cdot \sqrt{\frac{s}{k_+}})$$

$$\Rightarrow \frac{e^{-\sqrt{S/k_-}(x_0)}}{2\sqrt{S \cdot k_-}} = S_+ - U_-$$

transmission reflexion.

$$S_+ + \sqrt{\frac{k_-}{k_f}} U_- = (k_- \sqrt{\frac{1}{k_-}} \cdot \frac{e^{-\sqrt{S/k_-}(x_0)}}{2\sqrt{S \cdot k_-}}) / \sqrt{N k_f}$$

Physical meaning:

1. If $k_- = k_+$, $U_- = 0$ i.e. no reflexion

$$2. U_- = 0 \quad (1) \quad (\frac{\sqrt{F}}{k_f} - 1) \cdot \frac{e^{-\sqrt{S/k_-}(x_0)}}{\sqrt{N S k_-}}$$

$$S_+ = [1 + 0(1) \cdot (\frac{\sqrt{K_1}}{k_f} - 1)] \cdot \frac{e^{-\sqrt{S/k_-}(x_0)}}{\sqrt{N S k_-}}$$

$$\Rightarrow L_u = \frac{e^{-\sqrt{S/k_-}(x-x_0)}}{2\sqrt{S k_-}} + \underbrace{U_- \cdot e^{\sqrt{S/k_-}x}}_{\text{reflexion}}, \quad x < 0$$

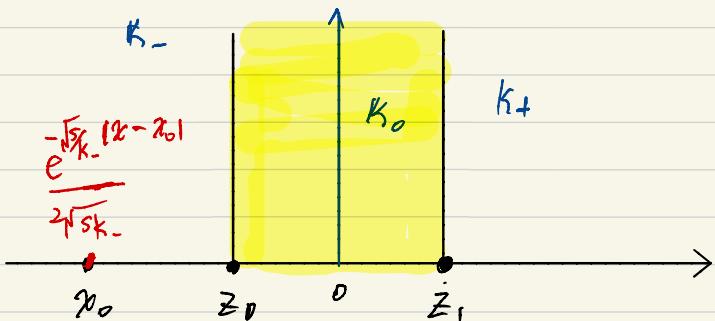
$$\underbrace{S_+ \cdot e^{-\sqrt{S/k_f}x}}_{\text{transmittion}}, \quad x > 0$$

$$\Rightarrow L_u = \frac{e^{-\sqrt{S/k_-}(x-x_0)}}{2\sqrt{S k_-}} + 0(1) \cdot (\frac{\sqrt{F}}{k_f} - 1) \cdot \frac{e^{\sqrt{S/k_-}(x-x_0)}}{\sqrt{N S k_-}} \quad x < 0$$

$$[1 + 0(1) \cdot (\frac{\sqrt{K_1}}{k_f} - 1)] \cdot \frac{e^{-\sqrt{S/k_-}(x_0)} - \sqrt{S/k_f}x}{\sqrt{N S k_-}}, \quad x > 0$$

$$\frac{e^{\sqrt{S/k_-}(x-x_0)}}{\sqrt{N S k_-}} = \frac{e^{-\sqrt{S} \cdot (\frac{|x_0|}{\sqrt{k_-}} + \frac{|x|}{\sqrt{k_-}})}}{\sqrt{N S k_-}}$$

$$\Rightarrow \frac{e^{-\frac{x^2}{4t}}}{\sqrt{\pi t} \cdot \sqrt{k_-}} = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t} \cdot \sqrt{k_-}} = \frac{e^{-\frac{(x-x_0)^2}{4t}}}{\sqrt{4\pi t k_-}}$$



Observe:

$$\sqrt{\frac{k_-}{k_+}} - 1 = \sqrt{\frac{k_- - k_+}{k_+}} + 1 - 1$$

↑ Small enough

By Taylor expansion,

$$\sqrt{\frac{k_- - k_+}{k_+}} + 1 = \frac{(k_- - k_+)}{k_+} + 1 + \dots$$

$\rightarrow = O(1) - (k_- - k_+).$

$$\begin{cases} u_t - \partial_x (K(x) u_x) = 0 \\ u(x_0) = \delta(x - x_0) \end{cases}$$

Take Laplace transform,

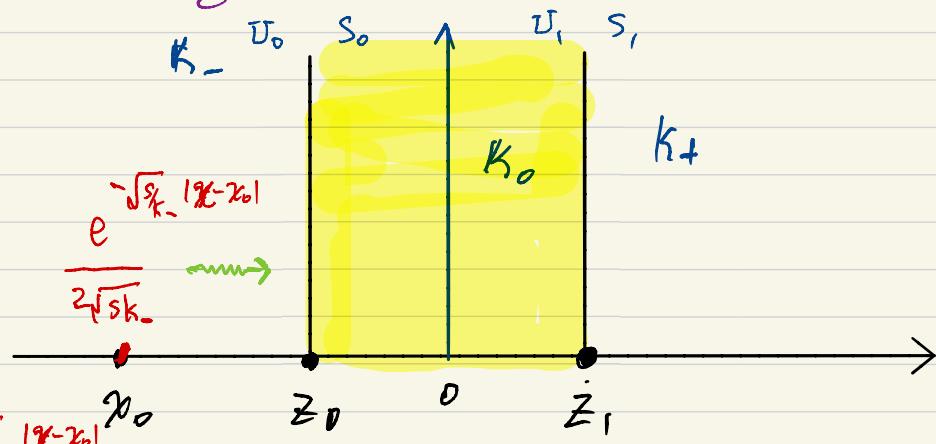
$$\Rightarrow s \mathcal{L} u - \partial_x (K(x) \partial_x \mathcal{L} u) = \delta(x - x_0)$$

$$(s - \partial_x (K(x) \partial_x)) \mathcal{L} u = \delta(x - x_0)$$

$$\Rightarrow " \mathcal{L} u = (s - \partial_x (K(x) \partial_x))^{-1} \delta(x - x_0)"$$

No understanding!)
See the operator as spectrum, then by F.A.

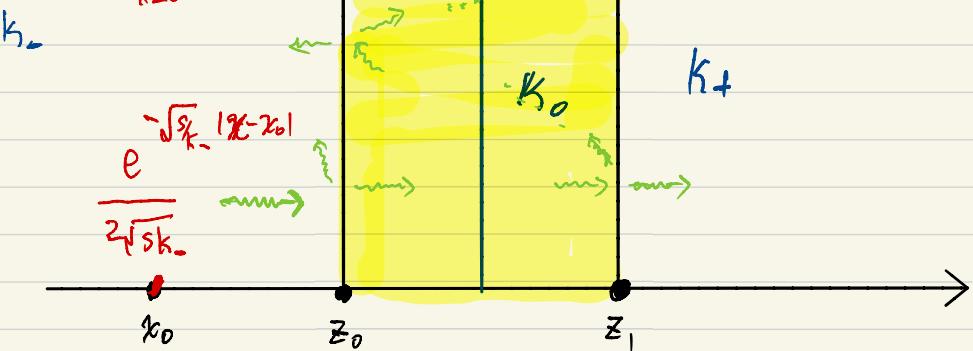
Do it with understanding:



$$u = \begin{cases} \frac{e}{2\sqrt{sk_-}} + U_0 \cdot e^{\sqrt{\frac{s}{k_-}}x}, & x < z_0 \\ S_0 \cdot e^{-\sqrt{\frac{s}{k_-}}(x-z_0)} + U_1 \cdot e^{\sqrt{\frac{s}{k_+}}(x-z_1)}, & z_0 \leq x \leq z_1 \\ S_1 \cdot e^{-\sqrt{\frac{s}{k_+}}(x-z_1)} & , \quad x > z_1 \end{cases}$$

$$\sum_{k=0}^{\infty} U_0^k = U_0 \quad \sum_{k=0}^{\infty} S_0^k = S_0$$

$$\sum_{k=0}^{\infty} U_1^k = U_1 \quad \sum_{k=0}^{\infty} S_1^k = S_1$$



$$R_{++}^0 \quad T_{+-}^0$$

$$R_{--}^0 \quad T_{-+}^0$$

$$U_0^0 = R_{--}^0 \cdot \frac{e^{-\sqrt{S_{k_-}} |z_0|}}{2\sqrt{S_{k_-}}},$$

$$R_{++}^1 \quad T_{+-}^1 \quad \text{all are const. about } \pi.$$

$$R_{--}^1 \quad T_{-+}^1$$

$$S_0^0 = T_{-+}^0 \cdot \frac{e^{-\sqrt{S_{k_-}} |z_0|}}{2\sqrt{S_{k_-}}}$$

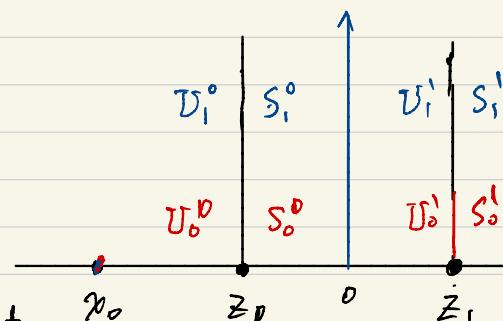
$$U_0^1 = 0, \quad S_0^1 = 0$$

$$S_1^0 = U_0^1 \cdot e^{\sqrt{S_{k_0}} (z_0 - z_1)} \cdot R_{++}^0$$

$$U_1^0 = U_0^1 \cdot e^{\sqrt{S_{k_0}} (z_0 - z_1)} \cdot T_{+-}^0$$

$$U_1^1 = S_0^0 \cdot e^{\sqrt{S_{k_0}} (z_0 - z_1)} \cdot R_{--}^1$$

$$S_1^1 = S_0^0 \cdot e^{\sqrt{S_{k_0}} (z_0 - z_1)} \cdot T_{-+}^1$$



⋮ ⋮ ⋮

Observe:

$$S_{2k}^o = e^{2\sqrt{s_{k_0}}(Z_0 - Z_1)k}$$
$$R_{--}^1 \cdot R_{++}^0 \cdot S_{2k-2}^o$$

$$S_k^o = U_{k+1}^1 \cdot e^{\sqrt{s_{k_0}}(Z_0 - Z_1)} \cdot R_{++}^0$$

$$U_k^o = U_{k-1}^1 \cdot e^{\sqrt{s_{k_0}}(Z_0 - Z_1)} \cdot T_{+-}^0$$

$$U_k^1 = S_{k+1}^o \cdot e^{\sqrt{s_{k_0}}(Z_0 - Z_1)} \cdot R_{--}^1$$

$$S_k^1 = S_{k+1}^o \cdot e^{\sqrt{s_{k_0}}(Z_0 - Z_1)} \cdot T_{-+}^1$$

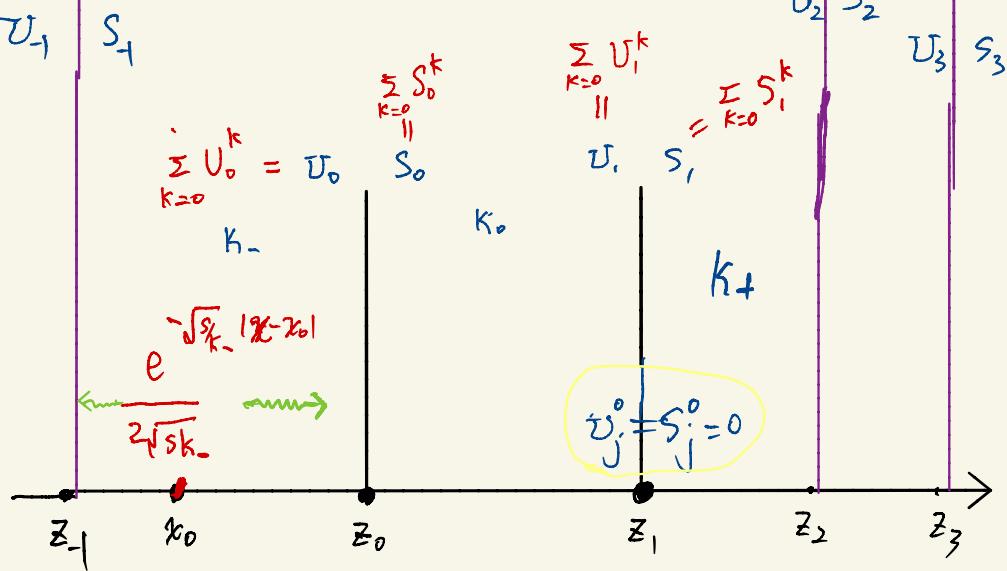
; ; ;

$$\Rightarrow S_{2k}^o = e^{2\sqrt{s_{k_0}}(Z_0 - Z_1)k} \cdot (R_{--}^1 \cdot R_{++}^0)^k \cdot \frac{e^{-\sqrt{s_{k_0}}|x_0|}}{2\sqrt{s_{k_0}}}$$

$$\sum_{k=0}^{\infty} S_{2k}^o = \sum \underbrace{(R_{--}^1 \cdot R_{++}^0)^k}_{\text{depending on } k} \cdot \frac{e^{2\sqrt{s_{k_0}}(Z_0 - Z_1)k - \sqrt{s_{k_0}}|x_0|}}{2\sqrt{s_{k_0}}}$$

$$\frac{e^{2\sqrt{s_{k_0}}(Z_0 - Z_1)k - \sqrt{s_{k_0}}|x_0|}}{2\sqrt{s_{k_0}}} = \ell$$

$$\ell \frac{- [2\sqrt{s_{k_0}}(Z_0 - Z_1)k - \sqrt{s_{k_0}}|x_0|]^2}{4t}$$
$$\leftrightarrow \ell \frac{\sqrt{s_{k_0}} \cdot \sqrt{4\pi t}}{4t}$$

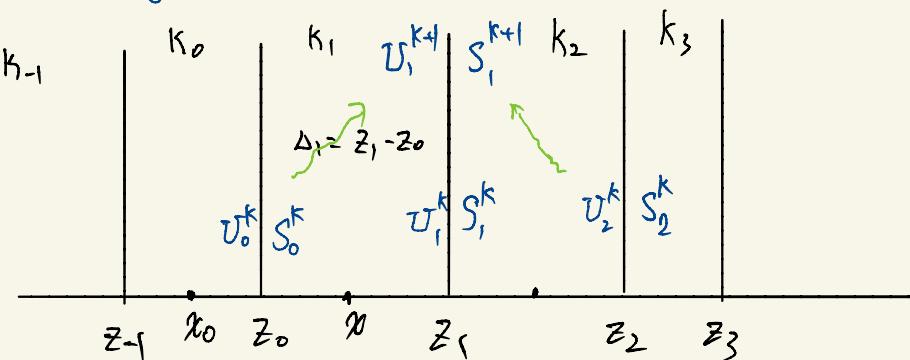


$$U_j = \sum_{k=0}^{\infty} U_j^k \quad \& \quad S_j = \sum_{k=0}^{\infty} S_j^k$$

- $U_j^0 = S_j^0 = 0$, if $j \geq 1$ or $j \leq -2$

$$S_0^0 = \frac{e^{-\sqrt{s_{k_-}}|z_0-x_0|}}{2\sqrt{s_{k_-}}} \cdot T_{-f}^0, \quad U_0^0 = \frac{e^{-\sqrt{s_{k_-}}|z_0-x_0|}}{2\sqrt{s_{k_-}}} \cdot R_{--}^0$$

- Suppose (S_j^k, U_j^k) , $\forall j$.



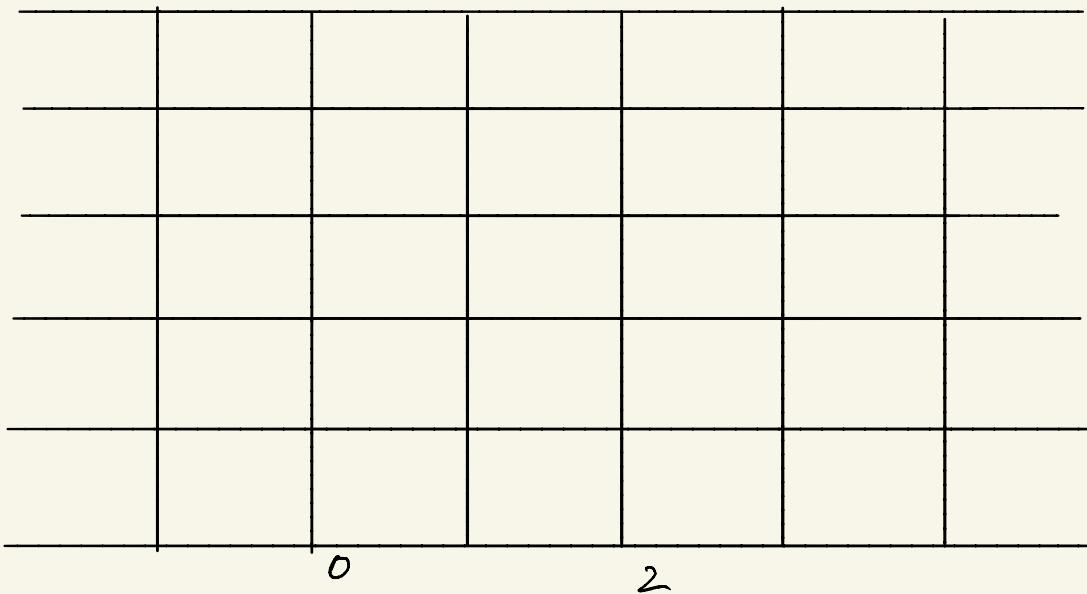
$$U_1^{k+1} = S_o^k \cdot e^{-\sqrt{S/k_1} \Delta_1} R_{--}^1 + U_2^k \cdot e^{-\sqrt{S/k_2} \Delta_2^1} \cdot T_{+-}^1$$

&

$$S_1^{k+1} = S_o^k \cdot e^{-\sqrt{S/k_1} \Delta_1} T_{-+}^1 + U_2^k \cdot e^{-\sqrt{S/k_2} \Delta_2^1} \cdot R_{++}^1$$

$$\Rightarrow U_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{S/k_j} \Delta_j} R_{--}^j + U_{j+1}^k \cdot e^{-\sqrt{S/k_{j+1}} \Delta_{j+1}} T_{+-}^j$$

$$S_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{S/k_j} \Delta_j} T_{-+}^j + U_{j+1}^k \cdot e^{-\sqrt{S/k_{j+1}} \Delta_{j+1}} R_{++}^j$$

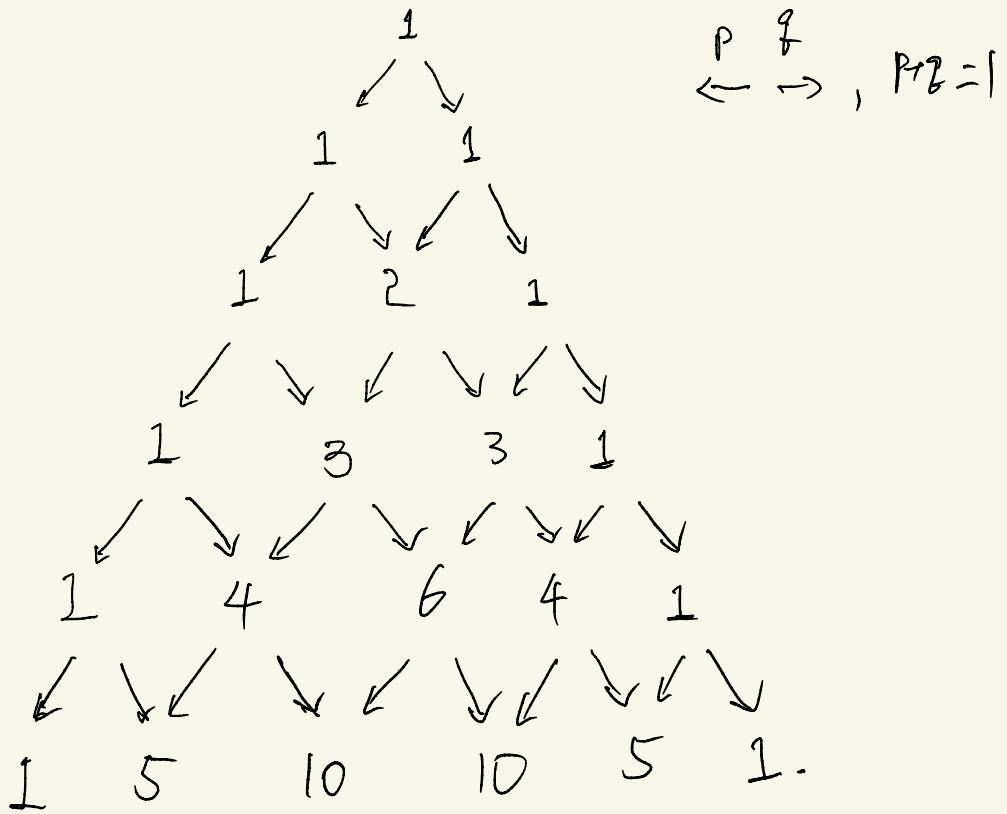
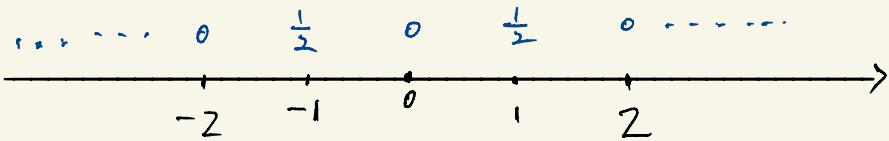


$\leftarrow \rightarrow$ $(n=6)$

$$\left(\frac{1}{2}x + \frac{1}{2}\frac{1}{x} \right)^6$$

Random Walk.

$$= \left(\frac{1}{2}x + \frac{1}{2}\frac{1}{x} \right) \cdots \cdots \left(\frac{1}{2}x + \frac{1}{2}\frac{1}{x} \right)$$



Numerical Solution

$$u_t + u_x = 0$$

Numerical Sol.

$$u(j\Delta x, n\Delta t) = u_j^n$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\Rightarrow u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$\begin{aligned} \Rightarrow u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0 \\ &= -\frac{\lambda}{2} u_{j+1}^n + u_j^n + \frac{\lambda}{2} u_{j-1}^n \end{aligned}$$

Need to be positive for using probability.

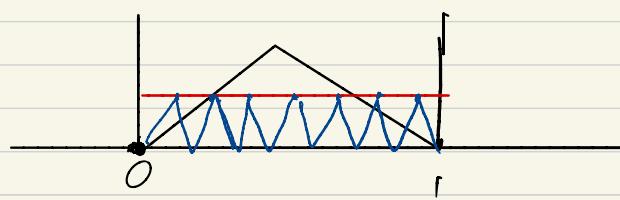
Another Scheme:

$$u_j^{n+1} - \frac{(u_{j+1}^n + u_j^n + u_{j-1}^n)}{3} + \frac{\Delta t}{\Delta x} \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$\frac{\lambda}{2} < \frac{1}{3} \Rightarrow u_j^{n+1} = \left(\frac{1}{3} - \frac{\lambda}{2}\right) u_{j+1}^n + \frac{1}{3} u_j^n + \left(\frac{1}{3} + \frac{\lambda}{2}\right) u_{j-1}^n$$

B.V. : Bounded Variation Function

$$\sup_{P \in \{P_1, \dots\}} \sum_i |k(x_i) - k(x_{i+1})| < \infty$$



Brown Motion

Now, assume $k(x)$ is B.V. function.

$$\|k\|_{B.V.} = \sup_{P \in \{P_1, \dots\}} \sum_i |k(x_i) - k(x_{i+1})| < \infty$$

If k is continuous, then $\|k\|_{B.V.} = \int_{\mathbb{R}} |k'_x| dx$

$$\begin{cases} U_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j} \Delta_j^- R_{--}^j} + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}} \Delta_{j+1}^- T_{+-}^j} \\ S_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j} \Delta_j^- T_{-+}^j} + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}} \Delta_{j+1}^- R_{++}^j} \end{cases}$$

$$\begin{bmatrix} U_j^{k+1} \\ S_j^{k+1} \end{bmatrix} = R_{j-1} \begin{bmatrix} U_{j-1}^k \\ S_{j-1}^k \end{bmatrix} + L_{j+1} \begin{bmatrix} U_{j+1}^k \\ S_{j+1}^k \end{bmatrix}$$

$$S^k := \{w^k(j) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$$

$$x_j(w^k) = \sum_{l=0}^k w^k(l)$$

$$\bigcup_{k=0}^{\infty} S^k$$

$$U_j^k = \sum_{w \in \mathcal{N}^k} "D_1(w) \cdot D_2(w) \cdots D_k(w)"$$

$x_k(w) = j$

$$D_l(w) = \begin{cases} R_{X_l} \\ L_{X_l} \end{cases}$$

$$\Rightarrow |D_1(w) \cdot D_2(w) \cdots D_k(w)| \leq C \cdot \prod_{i=1}^k |k_i - k_{i+1}|$$

$X = n$: change direction.

$$\sum_{k=0}^{\infty} U_j^k = \sum_{k=0}^{\infty} \sum_{w \in \Sigma^k} "D_1(w) \cdot D_2(w) \cdots D_k(w)"$$

$\Sigma_k(w) = j$

$\Sigma^k = \bigcup_{l=0}^{\infty} \Sigma_l^k$, Σ_l^k : l weak^k; w change direction l times exactly 3.

$$\Rightarrow \sum_{k=0}^{\infty} U_j^k = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\substack{w \in \Sigma_l^k \\ \Sigma_k(w) = j}} "D_1(w) \cdot D_2(w) \cdots D_k(w)"$$

$$\leq \sum_{l=0}^{\infty} \cdot \sum_{k=0}^{\infty} \cdot \sum_{\substack{w \in \Sigma_l^k \\ \Sigma_k(w) = j}} O(1) \pi |k_n - k_{n+1}|$$

m: where
 $\bar{x} = m$ change direction.

$$\sum_{i,j} a_i a_j = (\sum a_i)^2$$

$$\sum_{i,j,k} a_i a_j a_k = (\sum a_i)^3$$

; ;

$$= O(1) \cdot \sum_{l=0}^{\infty} (\sum_n |k_n - k_{n+1}|)^l \quad \text{Absolutely converge.}$$

$$\Rightarrow |\partial_x u(x,t)| \leq O(1) \frac{C}{t} e^{-\frac{|x|}{Ct}}$$

for some $C > 0$.

We prove when K is step function. What if $K(x)$ is B.V. funct.?

$$\begin{cases} u_t - \partial_x (K(x) u_x) = 0 \\ u(x, 0) = \delta(x - x_0) \end{cases}$$

$K(x)$: A B.V. function.

$\{k_n\}_{n \in \mathbb{N}}$: A step function. A Cauchy sequence in B.V.-norm.

$$\lim_{A \rightarrow \infty} \|k_n - k\|_\infty = 0$$

Consider $\begin{cases} u_t^n - \partial_x (k_n(x) u_x^n) = 0 \\ u(x, 0) = \delta(x - x_0) \end{cases}$ Sol. $\iff K(x, t; x_0, k_n) = u^n(x, t)$

Rewrite it as:

$$u_t^n - \partial_y (k_n(y) \cdot u_y^n) = 0$$

$$\Rightarrow \int_0^t \int_{\mathbb{R}} K(x, t-z; y, k_n) \cdot (u_z^n - \partial_y (k_n(y) \cdot u_y^n)) dy dz = 0$$

$$u^n(x, t) - \int_{\mathbb{R}} K(x, t; y, k_n) \cdot u(y, 0) dy$$

$$+ \int_0^t \int_{\mathbb{R}} -\partial_z K(x, t-z; y, k_n) u^n$$

$$\cdot + k_y(x, t-z; y, k_n) \underbrace{\cdot k_n(y)}_{(k_n)} \cdot u_y^n dy dz = 0$$

Question 1:

Consider the problem

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 \text{ for } x \in \mathbb{R}, t > 0 \\ u(x, 0) = 0 \\ u_t(x, 0) = \delta(x) \end{cases} \quad (1)$$

Use the singular-regular decomposition to decompose the solution

$u(x, t) = u_s(x, t) + u_r(x, t)$, where $u_s(x, t)$ is a singular part with an exponentially decaying structure in both x and t variable; and $u_r(x, t)$ is a regular part with a sufficient regularity in x variable.

- (1) Use energy estimate to show that $u_s(x, t)$ is exponentially decaying in x variable when $|x| > 2t$ with $t > 1$.
- (2) Use Long wave-short wave decomposition, Fourier transform, energy estimates, and complex analysis to show that there exists $C > 0$ s.t.

$$|u_s(x, t)| < C \cdot \left(\frac{e^{-\frac{x^2}{C(t+1)}}}{\sqrt{t+1}} \right) \quad (2).$$

Question 2:

Let $\mathcal{L}[u](x, s)$ be the Laplace transform of $u(x, t)$ w.r.t. t

$\mathcal{L}[u](x, s) = \int_0^\infty u(x, t) \cdot e^{-st} dt$ with $\operatorname{Re}(s) \geq 0$. Let $u(x, t)$ be the solution of (1) and compute $\mathcal{L}[u](x, s)$ in terms of Laplace wave

trains:

$$\mathcal{L}[u](x, s) = \begin{cases} A_+(s) \cdot e^{\lambda_+(s)x} & \text{for } x > 0 \\ A_-(s) \cdot e^{\lambda_-(s)x} & \text{for } x < 0 \end{cases}$$

- (1) Find $\lambda_{\pm}(s)$ and $A_{\pm}(s)$. Let $w(x, t)$ be the solution of the initial boundary value problem:

$$\begin{cases} w_{tt} - w_{xx} + w_t = 0 \text{ for } x, t > 0, \\ w(x, 0) = w(0, t) = 0, \\ w_t(x, 0) = \delta(x - x_0) \text{ with } x_0 > 0 \end{cases}$$

- (2) Find the solution $\mathcal{L}[w](x, s)$ in terms of the Laplace wave trains $e^{\lambda_+(s)(x-x_0)}$, $e^{\lambda_-(s)(x-x_0)}$ and $e^{\lambda_+(s)x}$.

Question 3:

Let $u(x, t)$ be the weak solution of the heat equation

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 \\ u(x, 0) = \delta(x) \end{cases}$$

where $\mu(x) \equiv 1 + H(x+1) - H(x-1)$, and $H(x)$ is the Heaviside funct.

i.e. $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

* Show that $\exists C > 0$ s.t.

$$|u(x, t)| \leq C \cdot \frac{e^{-\frac{x^2}{Ct}}}{t}.$$