Effective Boundary Conditions for the Heat Equation with Three-dimensional Anisotropic and Optimally Aligned Coatings

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Overview

Motivations : cell

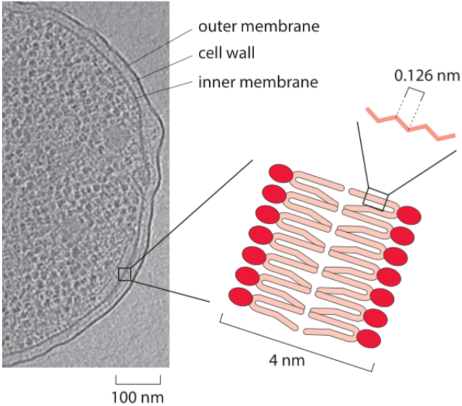
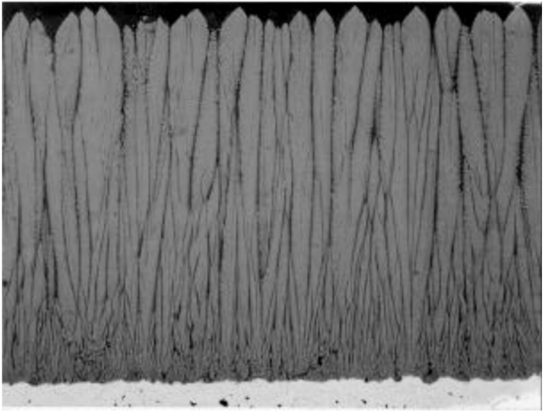


Figure: E. coli cell; membrane thick and diameter ratio 1:500; red-headed molecules are phospholipids

Coatings may be Anisotropic : anisotropy for TBC is caused by the fashions in which the ceramic “YSZ”(yttria-stabilized zirconia) is deposited on the blade :

If YSZ is sprayed on by electron beam physical vapor deposition (EB-PVD) method, then parallel crystal columns that are perpendicular to the boundary form ; and between these columns a small volume fraction of elongated pores also form. (Picture taken from J.R. Nicholls K.J. Lawson, A. Johnstone, D.S. Rickerby)



Motivations :

Common features :

- Domain contains a thin component ;
- Diffusion tensors on different components are drastically different.

Issues :

- the multi-scale in size and different diffusion tensors lead to computational difficulty ;
- It is hard to see the effect of the thin component ;

Resolution :

- think of the thin component as widthless surface and impose "effective boundary conditions"(EBCs).

History

- As early as 1959, Carlaw and Jaeger, in their classic book *Conduction of Heat in Solids*, first derived EBCs formally ;
- Rigorous derivation was initiated by Sanchez-Palencia in 1974, to study Laplace equation and the heat equation with thin diamond-shaped inclusions ;
- In 1980, Brezis, Caffarelli and Friedman studied the case of Poisson equation ;
- In 1987, Buttazo and Kohn studied the case of thin layer of oscillating thickness ;
- Lots of follow-up work on elastic equations, electromagnetic equations, etc ;

Mathematical model

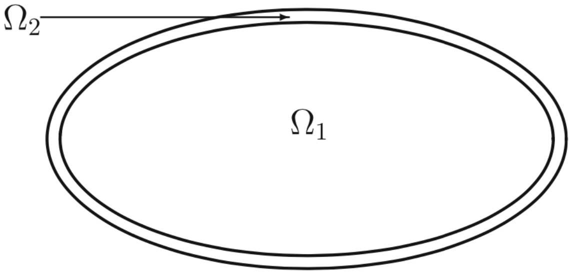


Fig. 1. The domain $\Omega = \overline{\Omega}_1 \cup \Omega_2 \subset \mathbb{R}^N$ consists of an isotropic body Ω_1 surrounded by a layer Ω_2 of uniform thickness δ

Mathematical model

For any fixed $T > 0$,

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(x,t), & (x,t) \in Q_T = \Omega \times (0,T), \\ u = 0, & (x,t) \in S_T = \partial\Omega \times (0,T), \\ u = u_0, & (x,t) \in \Omega \times \{0\} \end{cases} \tag{1}$$

where Ω_1 is fixed and $u_0 \in L^2(\Omega)$, $f \in L^2(Q_T)$. $A(x)$ is symmetric and positive definite :

$$A(x) = \begin{cases} kI_{N\times N}, & x \in \Omega_1 \\ (a_{ij}(x))_{N\times N}, & x \in \Omega_2 \end{cases}$$

with transmission conditions in $\partial\Omega_1$:

$$u_1 = u_2; \quad k\frac{\partial u_1}{\partial \mathbf{n}} = A(x)\nabla u_2 \cdot \mathbf{n}$$

and $\mathbf{n} = (n_1, ..., n_N)$ is the unit outer norm vector on $\partial\Omega_1$.

Results by Li, Rosencrans, Zhang and Wang [1]

- Suppose $a_{ij}(x) = \sigma(\overline{a}_{ij}(x))$ with $\overline{a}_{ij}(x) \in C^1(\overline{\Omega}_2)$ and $\sigma(\delta)$ is a positive parameter.
- If σ is bounded and $\lim_{\delta\to 0} \frac{\sigma}{\delta} = \alpha$, then $u \to v$ in $L^2(\Omega_1 \times [0,T])$, where v is the weak solution of

$$\begin{cases} v_t - k\Delta v = f(x,t), & (x,t) \in Q_T, \\ k\frac{\partial v}{\partial \mathbf{n}} + \alpha(\sum_{i,j} \overline{a}_{ij}(x) n_i n_j) v = 0, & (x,t) \in S_T, \\ v = u_0, & x \in \Omega, t = 0 \end{cases} \tag{2}$$

- A nature question : what is the effective boundary condition if $\sigma \to \infty$ as $\delta \to 0$?

1. J. Li, S. Rosencrans, K. Zhang, and X. Wang, 2009, Proc. AMS

Interior Inclusion in 3-D

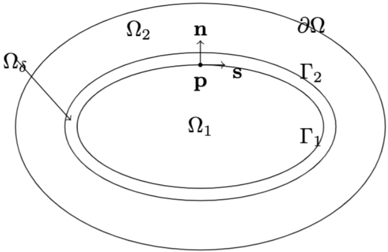
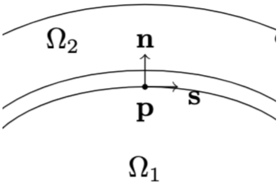


Figure.1: $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_\delta \cup \Omega_2$.

$$A(x) = \begin{cases} k_1, & x \in \Omega_1 \\ (a_{ij})_{3\times 3}, & x \in \Omega_\delta \\ k_2, & x \in \Omega_2 \end{cases}$$

where k_1 and k_2 are two positive constants independent of $\delta > 0$; σ is a positive function of δ ; Ω and Ω_1 are fixed with $\Gamma_1 \in C^3$.



Optimally aligned coating[2] :

- For any $x \in \Omega_\delta$, $\mathbf{n}(p)$ is always an eigenvector ;

Use curvilinear coordinates (s, r), $x = p + r\mathbf{n}(p) \in \Omega_\delta$, suppose

$$A(x)\mathbf{n}(p) = \sigma \mathbf{n}(p); \quad A(x)\mathbf{s}(p) = \mu \mathbf{s}(p) \tag{3}$$

$\mathbf{p}-$ the projection of x on $\Gamma_1 = \partial\Omega_1$; $r-$ distance from x to Γ_1 ; $\mathbf{s}(p)$ is an arbitrary tangent vector at p on $\partial\Omega_1$.

2. S. Rosencrans, and X. Wang, SIAM J. Appl. Math, 2006

Sobolev spaces

$$
\begin{aligned}
W_2^{1,0}(Q_T) &= \{u \in L^2(Q_T) : \nabla u \in L^2(Q_T)\}; \\
W_{2,0}^{1,0}(Q_T) &= \{u \in W_2^{1,0}(Q_T) : \text{ with trace 0 on} S_T\}; \\
W_2^{1,1}(Q_T) &= \{u \in L^2(Q_T) : u_t, \nabla u \in L^2(Q_T)\}; \\
W_{2,0}^{1,1}(Q_T) &= \{u \in W_2^{1,1}(Q_T) : \text{ with trace 0 on} S_T\}; \\
V_2^{1,0}(Q_T) &= \{u \in W_2^{1,0}(Q_T) : u \in C([0,T], L^2(\Omega)\}; \\
V_{2,0}^{1,0}(Q_T) &= \{u \in V_2^{1,0}(Q_T) : \text{ with trace 0 on} S_T\};
\end{aligned}
$$

Weak solution :

Definition

u is a weak solution of (1), if $u(x,t) \in V_{2,0}^{1,0}(Q_T)$ and it holds that

$$\begin{aligned} &\mathcal{A}[u,\xi] \\ =& -\int_\Omega u_0(x)\xi(x,0)dx + \int_{Q_T} (A(x)\nabla u)\cdot\nabla\xi - u\xi_t - f\xi dtdx \\ =&0 \end{aligned} \tag{4}$$

for any $\xi \in W_{2,0}^{1,1}(Q_T)$ satisfying $\xi = 0$ at $t = T$,

Energy estimates :

Lemma (1)

$$
\begin{aligned}
&(i) \max_{t\in[0,T]} \int_\Omega u^2(x,t)dx + \int_{Q_T} \nabla u \cdot A(x)\nabla u dxdt \\
\leq & C(T)(\int_\Omega u_0^2 dx + \int_{Q_T} f^2 dxdt); \\
&(ii) \max_{t\in[0,T]} t \int_\Omega \nabla u \cdot A(x)\nabla u(x,t)dx + \int_{Q_T} tu_t^2 dxdt \\
\leq & C(T)(\int_\Omega u_0^2 dx + \int_{Q_T} f^2 dxdt);
\end{aligned}
\tag{5}
$$

Second order estimates :

Lemma (2)

Suppose $\Gamma_1 \in C^3$ *and* $f \in L^2(Q_T)$ *with* $u_0 \in L^2(\Omega)$*. Then, for any fixed* $t_0 > 0$*, the weak solution* u *of (1) satisfies the following inequalities :*

$$\int_{t_0}^{T} \int_{\Omega_\delta} \mu(\Delta_\Gamma u)^2 + \sigma(\nabla_\Gamma u_r)^2 \leq O(1) + O(\frac{\sigma}{\mu}) + O(\frac{1}{\mu}) \tag{6}$$

and

$$\int_{t_0}^{T} \int_{\Omega_\delta} \sigma u_{rr}^2 \leq O(1) + O(\frac{1}{\sigma}) + O(\frac{\mu}{\sigma}) \tag{7}$$

General results [3]

Theorem (X.Chen, C.Pond and X.Wang)

Let $m \geq 2$ be an integer and $\alpha \in (0,1)$. Suppose that $\partial\Omega_1 \in C^{m+\alpha}$ and $f \in C^{m-2+\alpha,(m-2+\alpha)/2}(\overline{\Omega}_h \times [0,T])(h = 1, \delta, 2)$, and $a_{ij} \in C^{m-1+\alpha,(m-1+\alpha)/2}(\overline{\Omega}_h \times [0,T])$, then for any $t_0 > 0$, the weak solution u of (1) satisfies

$$u \in C^{m+\alpha,(m+\alpha)/2)}(\overline{\mathcal{N}}_h \times [t_0, T])$$

where $\mathcal{N}$ is a narrow neighborhood of $\partial\Omega_1$ and $\mathcal{N}_h = \mathcal{N} \cap \Omega_h$.

3. X.Chen, C.Pond and X.Wang, Arch.Ration.Mech.Anal.(2012)

Main results

Theorem (Geng)

[a] *Suppose that* $\Gamma_1 \in C^3$ *with*

$$\lim_{\delta\to 0}\frac{\sigma}{\delta} = b \in [0,\infty], \quad \lim_{\delta\to 0}\sigma\delta = a \in [0,\infty],$$
$$\lim_{\delta\to 0}\sigma\mu = \gamma \in [0,\infty], \quad \lim_{\delta\to 0}\mu\delta = \beta \in [0,\infty]. \tag{8}$$

As $\delta \to 0$, $u \to v$ *weakly in* $W_2^{1,0}(\Omega)$, *strongly in* $C([0,T];L^2(\Omega))$, *where* v *is the weak solution of the effective equation :*

$$\begin{cases} v_t - \nabla\cdot(A_0(x)\nabla v) = f(x,t), & (x,t)\in Q_T,\\ v = 0, & (x,t)\in S_T,\\ v = u_0, & x\in\Omega, t=0, \end{cases} \tag{9}$$

where $A_0(x) = k_1, x \in \Omega_1$ *and* $A_0 = k_2, x \in \Omega\backslash\overline{\Omega}_1$

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a. Xingri Geng, submitted

Main results

Theorem (Geng)

subject to the effective boundary conditions on $\Gamma_1 \times (0,T)$ *:*

Case 1: $b \in [0,\infty)$, as $\delta \to 0$.

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $b(v_2 - v_1) = k_1 \frac{\partial v_1}{\partial \mathbf{n}}$	$------$	$------$
$\beta \in (0,\infty)$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ $= \gamma \mathcal{J}^{\beta/\gamma}[v_1 + v_2]$	$------$
$\beta = \infty$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}}$ $= \gamma \mathcal{J}^{\infty}[v_1 + v_2]$	$\nabla_\Gamma v_1 = \nabla_\Gamma v_2 = 0$ $\int_{\Gamma_1} k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = 0$

Case 2: $\sigma \to 0$ and $b = \infty$, as $\delta \to 0$.

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$
$\beta \in (0,\infty)$	$------$	$------$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \beta \Delta_\Gamma v$
$\beta = \infty$	$------$	$------$	$v_1 = v_2$, $\nabla_\Gamma v = 0$, $\int_{\Gamma_1} k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = 0$

Main results

Case 3: $\sigma \geq O(1) > 0$ and $a \in [0, \infty)$ as $\delta \to 0$.

	$\gamma = 0$	$\gamma \in (0, \infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$
$\beta \in (0, \infty)$	$-------$	$-------$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = a \Delta_\Gamma v$
$\beta = \infty$	$-------$	$-------$	$v_1 = v_2$, $\nabla_\Gamma v = 0$, $\int_{\Gamma_1} k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = 0$

Case 4: $\sigma\delta \to \infty$ but $\sigma\delta^3 \to 0$ as $\delta \to 0$.

	$\gamma = 0$	$\gamma \in (0, \infty)$	$\gamma = \infty$
$\beta = 0$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$	$k_1 \frac{\partial v_1}{\partial \mathbf{n}} = k_2 \frac{\partial v_2}{\partial \mathbf{n}}$, $v_1 = v_2$
$\beta \in (0, \infty)$	$-------$	$-------$	$v_1 = v_2$, $k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = \beta \Delta_\Gamma v$
$\beta = \infty$	$-------$	$-------$	$v_1 = v_2$, $\nabla_\Gamma v = 0$, $\int_{\Gamma_1} k_1 \frac{\partial v_1}{\partial \mathbf{n}} - k_2 \frac{\partial v_2}{\partial \mathbf{n}} = 0$

Exotic BCs

Remarks :

- $\sigma\delta^3 \to 0$ can be removed if $\frac{\mu}{\sigma}$ does not vanish as $\delta \to 0$;
- Lots of new and exotic BCs will emerge, including Laplacian-Beltrami operator Δ_Γ and nonlocal operator $\mathcal{J}^H$;
- $\nabla_\Gamma v = 0$ but v can be a function of t.
- $\mathcal{J}^H$ is called Dirichlet-to-Neumann map.

Dirichlet-to-Neumann map :

Let $g(s)$ be a function on $\partial\Omega_1$ and $H = \lim_{\delta\to 0} h = \lim_{\delta\to 0} \sqrt{\mu/\sigma}\delta$

$$\begin{cases} \Psi_{RR} + \Delta_\Gamma \Psi = 0, & \partial\Omega_1 \times (0, H), \\ \Psi(s, 0) = g(s), & \Psi(s, H) = g(s) \end{cases} \tag{10}$$

Define

$$\mathcal{J}^H[g] = \Psi_R(s, 0) \tag{11}$$

$\mathcal{J}^H$ is symmetric for $H \in (0, \infty]$ and

$$\mathcal{J}^\infty = \lim_{H\to\infty} \mathcal{J}^H = -(-\Delta_\Gamma)^{1/2} \tag{12}$$

The formula of $\mathcal{J}^H$ can be obtained by using eigenfunctions of the Laplace-Beltrami operator : $-\Delta_\Gamma$.

Specical case : $\sigma = \mu$

Theorem (Isotropic case)

(i) If $\lim_{\delta\to 0}\frac{\sigma}{\delta} = b \in [0,\infty]$ *and* $\sigma \to 0$*, then*

$$k_1\frac{\partial v_1}{\partial \boldsymbol{n}} = b(v_1 - v_2), \quad k_1\frac{\partial v_1}{\partial \boldsymbol{n}} = k_2\frac{\partial v_2}{\partial \boldsymbol{n}}$$

(ii)If $\lim_{\delta\to 0}\sigma\delta = a \in [0,\infty)$ *and* $\sigma \geq O(1) > 0$*, then*

$$v_1 = v_2, \quad k_1\frac{\partial v_1}{\partial \boldsymbol{n}} - k_2\frac{\partial v_2}{\partial \boldsymbol{n}} = a\Delta_{\Gamma_1} v$$

(iii) If $\lim_{\delta\to 0}\sigma\delta = \infty$*, then*

$$v_1 = v_2, \quad \nabla_{\Gamma_1} v = 0, \quad \int_{\Gamma_1}(k_1\frac{\partial v_1}{\partial \boldsymbol{n}} - k_2\frac{\partial v_2}{\partial \boldsymbol{n}})ds = 0$$

where v_1 *and* v_2 *are the restrictions of* v *on* $\Omega_1 \times (0,T)$ *and* $(\Omega\backslash\Omega_1) \times (0,T)$ *respectively;*

Proof of the theorem

Main Steps :

- Step 1 : existence and uniqueness of weak solution of (1) ;
- Step 2 : energy estimates of the weak solution of (1) and prove $u \to v$ strongly in $C([0,T]; L^2(\Omega))$ as $\delta \to 0$;
- Step 3 : show that v is the exact weak solution of (9) with related EBCs ;
    - Construct a test function such that $\bar{\xi}(s,r,t) = \psi(s,r,t)$ in Ω_δ,

$$\begin{cases} \sigma\psi_{rr} + \mu\Delta_\Gamma\psi = 0, & \Gamma_1 \times (0,\delta), \\ \psi(s,0,t) = g_1(s) & \psi(s,\delta,t) = g_2(s) \end{cases} \tag{13}$$

    - By rescaling, $\Psi(s,R) = \psi(s, \sqrt{\mu/\sigma}r, t)$
- Step 4 : existence and uniqueness of the weak solution of the effective equation (9) with related EBCs ;

Boundary coatings

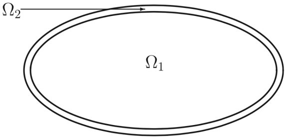


Fig. 1. The domain $\Omega = \overline{\Omega}_1 \cup \Omega_2 \subset \mathbb{R}^N$ consists of an isotropic body Ω_1 surrounded by a layer Ω_2 of uniform thickness δ

Theorem (Boundary case)

[a] *Let u be the weak solution of (1), then as $\delta \to 0$, $u \to v$ strongly in $C([0,T];L^2(\Omega))$, where v is the weak solution of the effective equation :*

$$\begin{cases} v_t - k\Delta v = f(x,t), & (x,t) \in \Omega_1 \times (0,T), \\ v(x,0) = u_0, & x \in \partial\Omega_1, \end{cases} \tag{14}$$

a. Xingri Geng, ready to submit

Theorem (Boundary case)

subject to the following effective boundary conditions :

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu} \to \gamma \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma \mathcal{J}_D^{\gamma/\alpha}[v]$	$v = 0$
$\sigma\mu \to \infty$	$\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_{\partial\Omega_1} (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v)dx = 0$	$v = 0$

Assume $\partial\Omega_1$ is a topological torus, namely, $\partial\Omega_1 = \Gamma_1 \times \Gamma_2$ and $A(x)$ satisfies

$$\begin{aligned} A(x)\mathbf{n}(p) &= \sigma\mathbf{n}(p); \\ A(x)\mathbf{t}_1(p) &= \mu_1\mathbf{t}_1(p); \\ A(x)\mathbf{t}_2(p) &= \mu_2\mathbf{t}_2(p) \end{aligned} \tag{15}$$

- $\mathbf{t}_1, \mathbf{t}_2-$ two orthonormal eigenvectors of $A(x)$ in the tangent plane ;

WOLG, suppose $\mu_1 > \mu_2$ and $\frac{\mu_2}{\mu_1} \to c \in [0,1]$.

- If $c \in (0,1]$, EBCs are similar as above ;
- If $c = 0$ with $\frac{\mu_2/\mu_1}{\delta^2} \to 0$, new results arise.

EBCs

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu_1 \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \mathcal{K}_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma (k\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$

Figure – EBCs on $\partial\Omega_1$ for $c \neq 0$

As $\delta \to 0$	$\frac{\sigma}{\delta} \to 0$	$\frac{\sigma}{\delta} \to \alpha \in (0,\infty)$	$\frac{\sigma}{\delta} \to \infty$
$\sigma\mu_1 \to 0$	$\frac{\partial v}{\partial \mathbf{n}} = 0$	$k\frac{\partial v}{\partial \mathbf{n}} = -\alpha v$	$v = 0$
$\sqrt{\sigma\mu_1} \to \gamma_1 \in (0,\infty)$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\infty}[v]$	$k\frac{\partial v}{\partial \mathbf{n}} = \gamma_1 \Lambda_D^{\gamma_1/\alpha}[v]$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \frac{\partial v}{\partial \mathbf{n}} = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} (\frac{\partial v}{\partial \mathbf{n}} + \alpha v) = 0$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sqrt{\sigma\mu_2} \to \gamma_2 \in (0,\infty)$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\infty}[v]\right) = 0$	$\frac{\partial v}{\partial \boldsymbol{\tau}_1} = 0$, $\int_{\Gamma_1} \left(k\frac{\partial v}{\partial \mathbf{n}} - \gamma_2 \mathcal{D}_D^{\gamma_2/\alpha}[v]\right) = 0$	$v = 0$
$\sigma\mu_1 \to \infty$, $\sigma\mu_2 \to \infty$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$\nabla_\Gamma v = 0$, $\int_\Gamma \frac{\partial v}{\partial \mathbf{n}} = 0$	$v = 0$

Figure – EBCs on $\partial\Omega_1$ for $c = 0$

Dirichlet-to-Neumann map

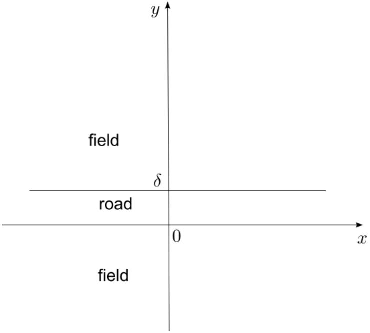
For $H \in (0, \infty]$, consider the degenerate equation :

$$\begin{cases} \Phi_{RR} + \Phi_{s_1 s_1} = 0, & \partial\Omega_1 \times (0, H) \\ \Phi(s, 0) = g(s), & \Phi(s, H) = 0 \end{cases} \tag{16}$$

Define

$$\Lambda_D^H[g] = \Phi_R(s, 0)$$

Ongoing Work



- Use EBC method to derive new model for other nonlinear equations, especially, Fisher-KPP.

THANK YOU !