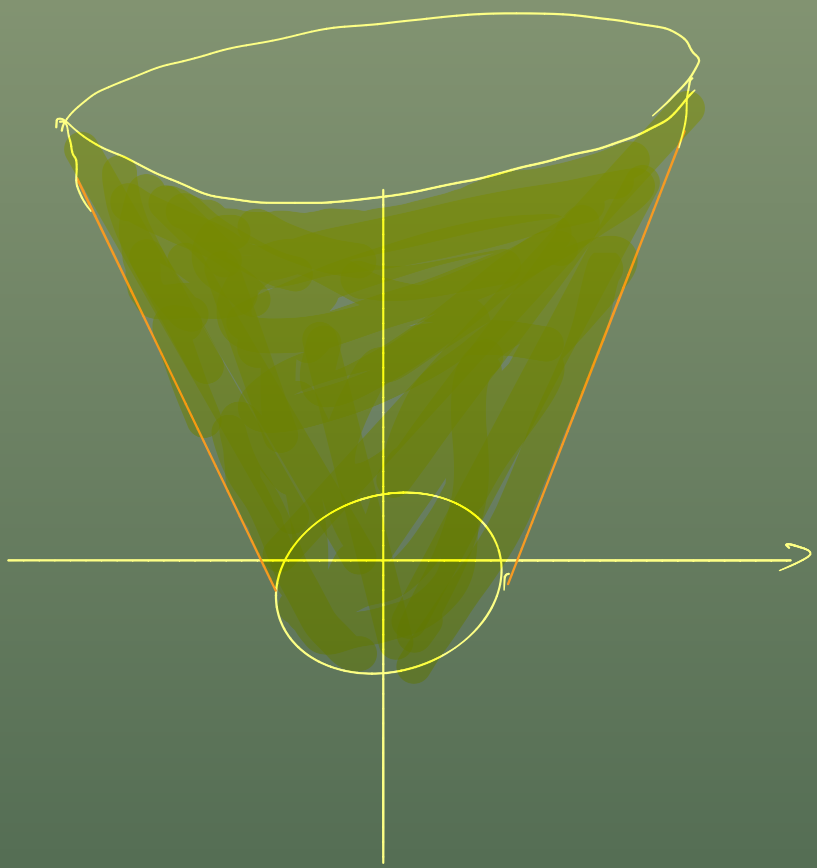
Green Functions.



Week 1

Main Stream.

一、 Burger's Equation.

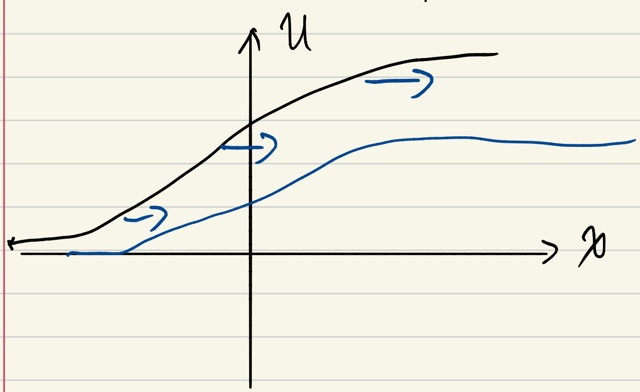
$$u_t + u u_x = u_{xx}, \quad x \in \mathbb{R}, \ t > 0$$

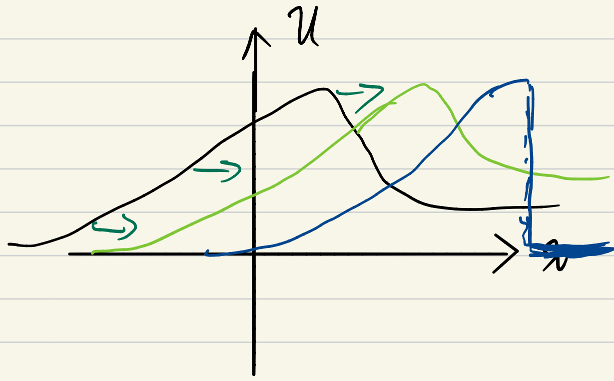
- First,

$$u_t + u u_x = 0, \quad \text{hyperbolic equation}$$

Transport eq.: $u_t + \underline{c}\, u_x = 0$

e.g.





Then, $u_t + (\frac{u^2}{2})_x = u_{xx}$

Introduce: $B_x = u$, $\Rightarrow B_{xt} + (\frac{B_x^2}{2})_x = (B_{xx})_x \quad (*)$

Integrate $(*)$ in x, assume $B_t = 0$ in the infinity,

$$B_t + \frac{B_x^2}{2} = B_{xx}$$

Then, take $B = -2\log\phi$, $B_t = -2\frac{\phi_t}{\phi}$, $B_x = -2\frac{\phi_x}{\phi}$

$$B_x^2 = 4 \cdot (\frac{\phi_x}{\phi})^2$$

$$\& \quad B_{xx} = -2\,\frac{\phi_{xx}\phi - \phi_x^2}{\phi^2}$$

Note:

$$-2\frac{\phi_t}{\phi} + 2\frac{\phi_x^2}{\phi^2} = -2\frac{\phi_{xx}}{\phi} + \frac{2\phi_x^2}{\phi^2}$$

$\Rightarrow \phi_t = \phi_{xx}$ heat equation !

Thus, $$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}}\, \phi(y,0)\, dy = \underline{\underline{K}} * \phi(x,0)$$

Green function of heat equation.

Since $\underline{\phi(x,t) = e^{-\frac{1}{2}B(x,t)}}$

& $B_x = u$,

then

Hopf-Cole transform.

$$u(x,t) = \int_{-\infty}^{x} u(y,t)\, dy$$

initial data.

$$\phi(x,t) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\int_{-\infty}^{x} u(z,0)\, dz} \cdot \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}}\, dy$$

$$u(x,t) = \partial_x B = \partial_x (-2) \log \phi = -\frac{2\phi_x}{\phi}$$

This can be solved by

Next, want to use a new method to do it!

Preliminaries :

- Fix Point Theorem:

Picard's iteration, $\begin{cases} y' = f(t,y) \\ y(0) = y_0 \end{cases}$

$$y_1(t) = y_0 + \int_0^t f(z, y_0)\, dz$$

$$y_2(t) = y_0 + \int_0^t f(z, y_1(z))\, dz$$

$$\vdots$$

$$y_n(t) = y_0 + \int_0^t f(z, y_{n-1}(z))\, dz$$

Want to have ratio test:

$$y_n = y_0 + \int_0^t f(z, y_{n-1})\, dz$$

$$- \quad y_{n-1} = y_0 + \int_0^t f(z, y_{n-2})\, dz$$

$$y_n - y_{n-1} = \int_0^t f(z, y_{n-1}) - f(z, y_{n-2})\, dz$$

MVT: $|f(z, y_{n-1}) - f(z, y_{n-2})| \le |y_{n-1} - y_{n-2}| \cdot \max\limits_{\xi \in (y_{n-1}, y_{n-2})} |f_y(z, \xi)| \le M$ assume

Then,

$$|y_n(t) - y_{n-1}(t)| \le M \int_0^t |y_{n-1}(z) - y_{n-2}(z)|\, dz$$

Metric space:

Fix t_0, $\quad \|y_n - y_{n-1}\| = \sup\limits_{z \in [0, t_0]} |y_n(z) - y_{n-1}(z)|$

Let $t < t_0$

$$\Rightarrow \underbrace{|y_n(t) - y_{n-1}(t)|}_{\sup_{t \in [0, t_0)}} \le \underbrace{M \int_0^t \|y_{n-1} - y_{n-2}\|\, dz}_{\sup_{t \in (0, t_0)}} \le M t_0 \|y_{n-1} - y_{n-2}\|$$

by

$$\Rightarrow \|y_n - y_{n-1}\| \le M t_0 \|y_{n-1} - y_{n-2}\|$$

How ? : By taking t_0 small

If choose $M t_0 < 1$, then $\lim\limits_{n \to \infty} \|y_n - y_{n-1}\| = 0$

Moreover, $\|y_n - y_{n-1}\| \le C_0 (M t_0)^n$ (*)

Now, just need to choose t_0 at beginning.

Then, consider:

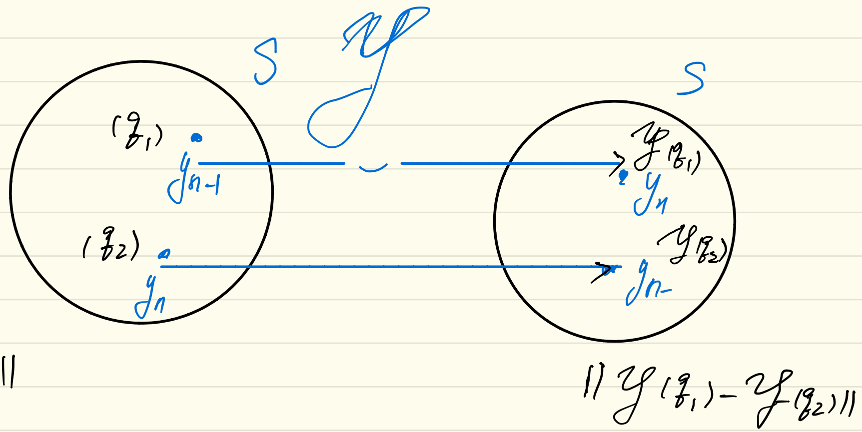
$$y_1 + (y_2 - y_1) + \dots (y_n - y_{n-1}) \text{ is convergent by } (*)$$

Implying:

$$\lim_{n\to\infty} y_n \text{ exists}$$

Consider fixed point thm:

$$y_n(t) = y_0 + \int_0^t f(z, y_{n-1}(z))\,dz$$



If $\|\mathcal{Y}(q_1) - \mathcal{Y}(q_2)\| \le \alpha \|q_1 - q_2\|$, with $\alpha < 1$,

then there exists a fixed point q^*.

Fix point Thm: $\begin{cases} f: S \to S, \text{ complete space } S. \\ f \text{ is } \alpha \text{ contract map.} \end{cases} \Rightarrow \exists!$ fixed point.

Want to show it satisfies the condition of fix point thm:

- $y_0 \in S$, & $y_n \in \mathcal{Y}(y_{n-1})$ $\forall n \ge 1$

$$\sum_{n=1}^{\infty} \|y_n - y_{n-1}\| \text{ converges.}$$

Proof:

$$\|y_n - y_{n-1}\| = \|\mathcal{F}(y_{n-1}) - \mathcal{F}(y_{n-2})\| < \alpha \|y_{n-1} - y_{n-2}\|$$

$$\Rightarrow \quad \|y_n - y_{n-1}\| < \alpha^{n-1} \|y_1 - y_0\|$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \|y_n - y_{n-1}\| < \frac{\|y_1 - y_0\|}{\alpha}$$

What is the Green function or fundamental solutions?

"ODE"

$$\begin{cases} \vec{y}' = A\vec{y} \\ \vec{y}(0) = \vec{y}_0 \end{cases} \quad \Rightarrow \quad e^{-At}(y' - Ay) = 0$$

$$\Rightarrow \quad (e^{-At} y)' = 0$$

$$\Rightarrow \quad e^{-At} y = c, \qquad c = y_0$$

$$\Rightarrow \quad y(t) = e^{At} y_0$$

Now, A is a matric

Q: What is e^{At}? $\qquad A = S^{-1} \Lambda S$

A: $\quad e^{A} = S^{-1} e^{\Lambda} S, \qquad e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots$

"PDE":

$$\partial_t u = \partial_x u \qquad (a_0, a_1, \cdots) \in \mathbb{R}^{\infty}$$

$$u(x,t) = a_0(t) + a_1(t) x + \cdots a_n(t) x^n + \cdots$$

By Talyer's expansion:

$$\partial_x u = a_1(t) + 2a_2(t)x + \cdots n a_n(t) x^{n-1} + \cdots$$

$$(a_1, 2a_2, \cdots, n a_n, \cdots)$$

So,

$$\frac{\partial}{\partial t}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 2 & \cdots & \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & & & & \\ 0 & \cdots & & & \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

$$L^2(\mathbb{R}) = \left\{ f \mid \int_{-\infty}^{\infty} f^2_{(x)}\, dx < \infty \right\}.$$

$$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$ — Inner Product.

$$\|f\|^2 = (f, f)$$ — $\|\cdot\|$: Norm for $L^2(\mathbb{R})$

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

$$\|\cdot\|_{L^1} = \int_{-\infty}^{\infty} |f|\, dx$$

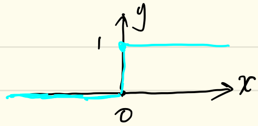
Fourier transform:

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-iyx} f(x)\, dx$$

Inverse Fourier transform:

$$f(x) = \frac{1}{2\pi}\int \hat{f}(y)\, e^{iyx}\, dy$$

$\delta(x) = \partial_x H(x)$: $H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0 \end{cases}$



$$\int_{-\infty}^{\infty} \delta(x)\, dx = \int_{-\infty}^{\infty} \partial_x H(x)\, dx = H\Big|_{-\infty}^{\infty} = 1 - 0 = 1$$

$f \in C_0(\mathbb{R})$: then $\displaystyle\int_{-\infty}^{\infty} \delta(x) f(x)\, dx$

$$= \int_{-\infty}^{\infty} \partial_x H(x) \cdot f(x)\, dx \overset{\text{I.B.P}}{=} -\int_{-\infty}^{\infty} H(x) f'(x)\, dx$$

$$= -\left(\int_{-\infty}^{0} H(x) f'(x)\, dx + \int_{0}^{+\infty} H(x) f'(x)\, dx\right)$$

$$= -f(x)\Big|_0^{\infty} = -(f(\infty) - f(0)) = f(0).$$

Property:

For any continuous function $g(x)$,

$$\int_{\mathbb{R}} \delta(x - x_0) g(x)\, dx = g(x_0).$$

Then, $\hat{\delta}(\eta) = \displaystyle\int e^{-i\eta x} \delta(x)\, dx = e^{-i\eta 0} = 1$

Properties of Fourier Transform:

① $\widehat{\partial_x f} = \displaystyle\int_{-\infty}^{\infty} e^{-i\eta x} \partial_x f\, dx \overset{\text{I.B.P}}{=} -\int_{-\infty}^{+\infty} (e^{-i\eta x})_x f\, dx = i\eta \int_{-\infty}^{\infty} f \cdot e^{-i\eta x} dx$

$= i\eta \cdot \hat{f}(\eta)$

$\Downarrow$ $\boxed{(\partial_x f)^\wedge = i\eta \cdot \hat{f}(\eta)}$

Application; $\begin{cases} (\partial_t + c\,\partial_x) u = 0 \\ u(x, 0) = u_0(x) \end{cases}$

Take Fourier transform:

$$\partial_t \hat{u} + ci\eta \hat{u} = 0 \quad , \eta \text{ fixed}$$

$$\Rightarrow \partial_t (e^{ci\eta t} \hat{u}) = 0 \Rightarrow \hat{u}(\eta, t) = e^{-ic\eta t} \hat{u}(\eta, 0)$$

$$= e^{-ic\eta t} \cdot \int_{-\infty}^{\infty} e^{-i\eta x} u_0(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-i\eta(x+ct)} u_0(x)\, dx$$

Change $z = x + ct$.
$x = z - ct$

$$\Rightarrow = \int_{-\infty}^{\infty} e^{-i\eta z} u_0(z - ct)\, dz$$

$$= \int_{-\infty}^{\infty} e^{-i\eta x} u_0(x - ct)\, dx = u_0(x-ct)^{\wedge}$$

② $\boxed{f(x-a)^{\wedge} = e^{-ia\eta} \hat{f}}$;

Laplace Transform / Fourier Transform

- Convolution:

$$f * g(x) = \int_{\mathbb{R}} f(x-y) \cdot g(y)\, dy$$

③ $\boxed{(f*g)^{\wedge} = \hat{f} \cdot \hat{g}}$; Notice, $\hat{u}(\eta, t) = e^{-ic\eta t} \cdot \hat{u}(\eta, 0)$

$$= \delta(x-ct)^{\wedge} \cdot \hat{u}(\eta, 0)$$

for $\delta(x-ct) = \hat{\delta}(\eta) \cdot e^{-ict\eta} = e^{-ict\eta}$

$$\Rightarrow u(x,t) = \delta(x-ct) * u(x,0)$$

$$= \int \delta(x-ct-y) \cdot u(y,0)\, dy = u(x-ct, 0)$$

$$= u_0(x-ct) . \#$$

[R.K]: Thus, $\widehat{(e^{-i\eta ct})} = \delta(x-ct)$ is the Green Function of $u_t + c u_x = 0$. & $u(x,t) = \delta(x-ct) * u_0(x)$.

Recall:

Fourier Transform:

$$\delta(x) \longrightarrow 1$$
$$\delta(x-ct) \longrightarrow e^{-i\eta ct}$$
$$f \longrightarrow \hat{f}(\eta)$$
$$\hat{f'} \longrightarrow i\eta \hat{f}(\eta)$$
$$\widehat{f*g} \longrightarrow \hat{f}(\eta)\cdot \hat{g}(\eta)$$

Inverse Fourier transform:

$$\hat{f} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)\cdot e^{i\eta x}\, d\eta$$

Now, $\begin{cases} g_t + c g_x = S(x,t) \\ g(x,0) = g_0(x) \end{cases}$ Take Fourier transform: "PDE ⇒ ODE"

$$\Longrightarrow \begin{cases} \hat{g}_t + ci\eta \hat{g} = \hat{S}(\eta,t) \\ \hat{g}(\eta,0) = \hat{g}_0(\eta) \end{cases}$$

Next, $e^{i\eta ct}$.

$$\Rightarrow \frac{d}{dt}(e^{i\eta ct}\hat{g}) = e^{i\eta ct}\hat{S}(\eta,t)$$

Take integral from 0 to τ,

$$\int_0^\tau \frac{d}{dt}\left(e^{i\eta ct}\cdot \hat{g}\right)dt = \int_0^\tau e^{i\eta ct}\,\hat{S}(\eta,t)\,dt$$

$$\Rightarrow \quad e^{i\eta c\tau}\cdot\hat{g}(\eta,\tau) - \hat{g}_0(\eta) = \int_0^\tau e^{i\eta ct}\,\hat{S}(\eta,t)\,dt$$

$$\Rightarrow \quad \hat{g}(\eta,\tau) - e^{-i\eta\tau c}\hat{g}_0(\eta) = \int_0^\tau e^{i\eta c(t-\tau)}\,\hat{S}(\eta,t)\,dt$$

$\delta(x-c\tau)$ $\qquad$ $\delta(x-c(\tau-t))$

$$g(x,\tau) - \underbrace{\delta(x-c\tau)*g_0}_{\int \delta(x-c\tau-y)\,g_0(y)\,dy \;=\; g_0(x-c\tau)} = \int_0^\tau \delta(x-c(\tau-t))*S(x,t)\,dt$$

$$\Rightarrow \quad \boxed{g(x,\tau) = g_0(x-c\tau) + \int_0^\tau S(x-c(\tau-t),t)\,dt}$$

final result.

i.e. $g(x,t) = g_0*\delta(x-ct) + \int_0^t S(x,\tau)*\delta(x-c(t-\tau))\,d\tau$

Wave Equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$$

Take Fourier Transform,

$$\begin{cases} \hat{u}_{tt} - (i\eta)^2 \hat{u} = 0 \longrightarrow \hat{u}_{tt} + \eta^2 \hat{u} = 0 \\ \hat{u}(x,0) = \hat{u}_0(\eta) \quad \text{—①} \\ \hat{u}_t(\eta,0) = \hat{u}_1(\eta) \quad \text{—②} \end{cases}$$

$$\lambda^2 + \eta^2 = 0, \Rightarrow \lambda = \pm i\eta$$

$$\Downarrow$$

$$\hat{u}(\eta,t) = A \cdot e^{i\eta t} + B e^{-i\eta t}$$

Use condition ① & ② to solve A, B.

$$\begin{cases} \hat{u}_0(\eta) = A + B \\ \hat{u}_1(\eta) = i\eta A - i\eta B \iff \frac{\hat{u}_1(\eta)}{i\eta} = A - B \end{cases}$$

So, $A = \frac{1}{2}\left(\hat{u}_0(\eta) + \frac{\hat{u}_1(\eta)}{i\eta}\right)$, & $B = \frac{1}{2}\left(\hat{u}_0(\eta) - \frac{\hat{u}_1(\eta)}{i\eta}\right)$

$$\Rightarrow \hat{u}(\eta,t) = \frac{1}{2}\left(\hat{u}_0(\eta) + \frac{\hat{u}_1(\eta)}{i\eta}\right) \cdot e^{i\eta t} + \frac{1}{2}\left(\hat{u}_0(\eta) - \frac{\hat{u}_1(\eta)}{i\eta}\right) \cdot e^{-i\eta t} = \boxed{\frac{1}{2}\hat{u}_0 (e^{i\eta t} + e^{-i\eta t}) + \frac{1}{2}\hat{u}_1 \frac{(e^{i\eta t} - e^{-i\eta t})}{i\eta}}$$

$$= \frac{1}{2}\left(\widehat{u_0(x+t)} + \widehat{u_0(x-t)}\right) + \frac{1}{2}\left[\int u_1(x+t) - u_1(x-t)\, dx\right]^{\wedge}$$

So, $u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\,d\xi.$

For Inhomogeneous,

$$\boxed{\begin{cases} u_{tt} - u_{xx} = S(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}} \Rightarrow \hat{u}_{tt} + \eta^2 \hat{u} = \hat{S}(\eta,t), \quad \begin{cases} \hat{u}(\eta,0) = 0 \\ \hat{u}_t(\eta,0) = 0. \end{cases}$$

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u_t(0,x) = \delta(x) \\ u(0,x) = 0 \end{cases}$$

Fourier Transform.

Want to solve $\hat{u}(\eta,t)$:

Introduce $G(z,\eta)$, $z \in (0,t)$ s.t. $\begin{cases} G_z(t,\eta) = 1 \\ G(t,\eta) = 0 \\ (-\partial_z)^2 G + \eta^2 G = 0 \quad —(*) \end{cases}$

"Backward equation"

"Green function"

$$\int_0^t G(z,\eta)\cdot\left[\hat{u}_{zz}(\eta,z) + \eta^2\hat{u}(\eta,z)\right]dz = \int_0^t G(z,\eta)\,\hat{S}(\eta,z)\,dz$$

$$G(z,\eta)\,\hat{u}_z\Big|_0^t - \int_0^t \hat{u}_z\cdot G_z(z,\eta)\,dz$$

$\Downarrow$ 0

$$\Downarrow$$

$$\text{"} -\left(G_z\,\hat{u}\Big|_0^t - \int G_{zz}\,\hat{u}\,dz\right) \text{"} +$$

$$= -G_z\,\hat{u}\Big|_t + \int_0^t G_{zz}\,\hat{u}\,dz +$$

$$= -\hat{u}(\eta,t) + \int_0^t G_{zz}\,\hat{u}\,dz + \int_0^t G(z,\eta)\,\eta^2\,\hat{u}(\eta,z)\,dz$$

$$= -\hat{u}(\eta,t) + \int_0^t \underbrace{(G_{zz} + \eta^2 G)}_{0 \;\Leftarrow \text{ by } (*)}\,\hat{u}\,dz$$

$$\Rightarrow \hat{u}(\eta,t) = -\int_0^t G(z,\eta)\cdot \hat{S}(z,\eta)\,dz.$$

W.T.S get $G(z,\eta)$:

Let $\tilde{G}(z,\eta) = G(t+z,\eta)$.

Then,

$$\begin{cases} \tilde{G}_{zz} + \eta^2 \tilde{G} = 0 \\ \tilde{G}_z(0,\eta) = 1 \\ \tilde{G}(0,\eta) = 0 \end{cases}$$

$\Rightarrow r^2 + \eta^2 = 0$ i.e $r = \pm i\eta$

$\Rightarrow \tilde{G}(z,\eta) = A\cdot e^{i\eta z} + B\cdot e^{-i\eta z}$

$\Rightarrow A = -B$ by initial value

$\Rightarrow \tilde{G}_z(0,\eta) = i\eta A + i\eta A = 1$ implying $A = \dfrac{1}{2i\eta}$

$\Rightarrow \tilde{G}(z,\eta) = \dfrac{1}{2i\eta}\left(e^{i\eta z} - e^{-i\eta z}\right)$

F.T.

$\delta(x+t)$ $\delta(x-t)$.

$$\Rightarrow G(z,\eta) = \frac{1}{2i\eta}\left(e^{i\eta(z-t)} - e^{-i\eta(z-t)}\right)$$

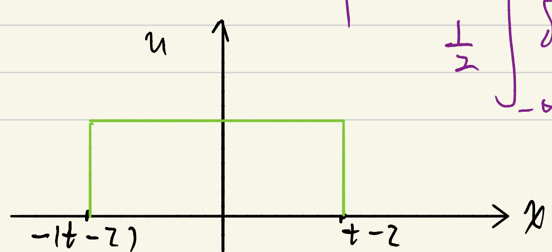
$$= \frac{1}{2i\eta}\left(e^{-i\eta(t-z)} - e^{i\eta(t-z)}\right)$$

$$\Rightarrow \hat{u}(t,\eta) = \int_0^t \frac{1}{2i\eta}\left(e^{i\eta(t-z)} - e^{-i\eta(t-z)}\right)\cdot \hat{S}(z,\eta)\,dz$$

$$\frac{1}{2}\int_{-\infty}^{x} \delta(y+(t-z)) - \delta(y-(t-z))\,dy$$

$\Downarrow$

$\chi_{[-(t-z),\, t-z]}(x) * S(z,x)$



$$\Rightarrow \hat{u}(\eta,t) = \int_0^t G(\eta,\tau)\,\hat{S}(\eta,\tau)\,d\tau$$

$$G = \frac{e^{i(t-\tau)\eta} - e^{-i\eta(t-\tau)}}{2i\eta}$$ solved by privious step.

$$\boxed{u(x,t) = \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \frac{S(\xi,\tau)\,d\xi}{2}\,d\tau}$$

Heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x) \end{cases}$$

$$\begin{cases} \hat{u}_t + \eta^2 \hat{u} = 0 \\ \hat{u}(x,0) = \hat{u}_0(\eta) \end{cases} \Longrightarrow \frac{d}{dt}\left(e^{\eta^2 t}\cdot \hat{u}\right) = 0$$

$$\Downarrow$$

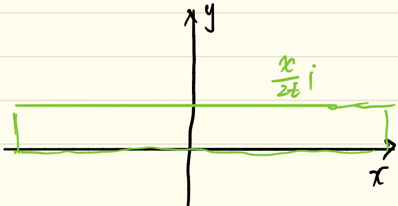
$$e^{\eta^2 t}\cdot \hat{u}(\eta,t) = \hat{u}(\eta,0)$$

$$\Rightarrow \hat{u}(\eta,t) = e^{-\eta^2 t}\ \underbrace{\hat{u}(\eta,0)}_{\text{known}}$$

Look for its inverse.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\eta^2 t + i\eta x}\,d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{-\eta^2 t + i\eta x + \frac{x^2}{4t} - \frac{x^2}{4t}}}_{e^{-t(\eta - \frac{ix}{2t})^2}\cdot\, e^{-\frac{x^2}{4t}}}\,d\eta = \frac{e^{-x^2/4t}}{2\pi}\cdot\int_{-\infty}^{\infty} e^{-t(\eta - \frac{ix}{2t})^2}\,d\eta$$

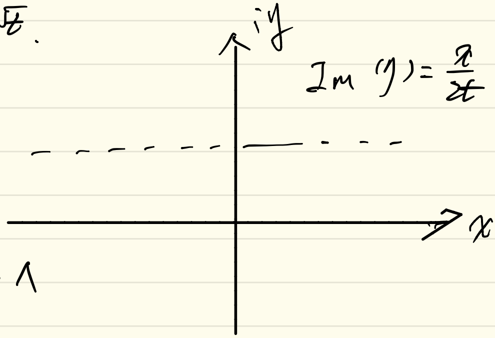


By complex analysis

$$= \frac{e^{-\frac{x^2}{4t}}}{2\pi}\cdot\int_{\mathrm{Im}(y)=\frac{x}{2t}} e^{-t(y-\frac{ix}{2t})^2}\,dy = \frac{e^{-x^2/4t}}{2\pi}\int_{-\infty}^{\infty} e^{-tv^2}\,\frac{d(v\sqrt{t})}{\sqrt{t}}, \qquad v = y-\frac{ix}{2t}$$

$$= \frac{e^{-x^2/4t}}{2\pi\sqrt{t}}\cdot\int_{-\infty}^{\infty} e^{-w^2}\,dw, \qquad w = v\sqrt{t}.$$

$$= \frac{\sqrt{\pi}}{2\pi\sqrt{t}}\cdot e^{-x^2/4t}$$



So, $\hat{u}(x,t) = \left[\frac{e^{-x^2/4t}}{\sqrt{4\pi t}} * u(x,0)\right]^{\wedge}$

$$u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} * u(x,0) = \int_{-\infty}^{\infty}\frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}}\cdot u(y,0)\,dy.$$

$$\boxed{\text{Heat kernel: } k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}}$$

$$\underline{\hat{k}(y,t) = e^{-y^2 t}.}$$

Duhamel's Principle:

$$(*)\ \begin{cases} u_t - u_{xx} = S(x,t) \\ u(x,0) = 0 \end{cases} \quad \xRightarrow{\text{By Fourier Transform}} \quad \begin{cases} \hat{u}_t + y^2\hat{u} = \hat{S}(y,t) \\ \hat{u}(y,0) = 0 \end{cases}$$

Integrate by $e^{y^2 t}$:

$$\Rightarrow \frac{d}{dt}(e^{y^2 t}\cdot\hat{u}) = \hat{S}(y,t)\cdot e^{y^2 t}$$

Change "t" to "τ",

$$\int_0^t \left[\frac{d}{d\tau}(e^{y^2\tau}\cdot\hat{u}) = \hat{S}(y,\tau)\, e^{y^2\tau}\right] d\tau$$

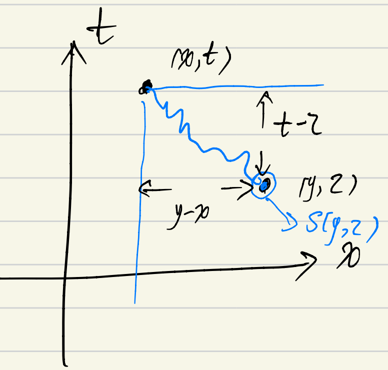
$$\Rightarrow \quad e^{\eta^2 t}\hat{u}(\eta,t) = \int_0^t \hat{S}(\eta,\tau)e^{\eta^2\tau}d\tau$$

$$\Rightarrow \quad \hat{u}(\eta,t) = \int_0^t \underbrace{e^{\eta^2(\tau-t)}}_{k(x,\,t-\tau)} \cdot \hat{S}(\eta,\tau)d\tau$$

$$\Rightarrow \quad \boxed{u(x,t) \equiv \int_0^t k(x,t-\tau) * S(x,\tau)d\tau}$$

i.e.

$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{+\infty} k(x-y,t-\tau)\cdot S(y,\tau)dy\,d\tau}$$



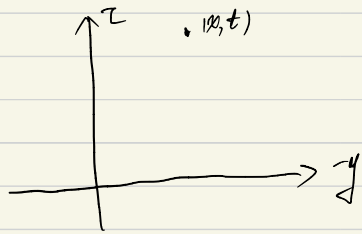
Now, For general,

(Forward Eq.) $\left\{\begin{array}{l} u_t(x,t) = u_{xx} + S(x,t) \\ u(x,0) = u_0(x) \end{array}\right\} \longrightarrow$ ☺

Then, $u(x,t) = k(x,t) * u_0(x) + \int_0^t k(x,t-\tau) * S(x,\tau)d\tau$

(ii) ☺

Introduce a function $G(y,\tau)$



s.t.

Backward Eq. $\left\{\begin{array}{l} ① \ G(y,t) = \delta(x-y) \\ ② \ [-\partial_\tau - (-\partial_y)^2]\,G(y,\tau) = 0 \end{array}\right.$

For ☺ $\int_0^t \int_{-\infty}^{+\infty} [u_\tau(y,\tau) - u_{yy}(y,\tau) - S(y,\tau) = 0]\, G(y,\tau)\,dy\,d\tau$

$$\Rightarrow \int_{-\infty}^{+\infty} G(y,\tau)\,U(x,\tau)\,dy \Big|_{\tau=0}^{\tau=t} + \int_0^t \int_{-\infty}^{+\infty} ((-\partial_\tau - (-\partial_y)^2)\,G\,U(y,\tau)$$

$$- G(y,\tau)\,S(y,\tau)\,dy\,d\tau = 0$$

$$\Rightarrow \quad u(x,t) - \int_{-\infty}^{\infty} G(y,0)\,U(y,0)\,dy + \int_0^t\int_{-\infty}^{\infty} G(y,\tau)\,S(y,\tau)\,dy\,d\tau$$

$$\Rightarrow \quad G(y,\tau) = k(x-y,\, t-\tau).$$

"Hard" $\boxed{\text{A nonlinear problem:}}$

$$\begin{cases} u_t + u u_x = u_{xx}. \\ |u(x,0)| \le \varepsilon \cdot \dfrac{e^{-\frac{x^2}{4}}}{\sqrt{16\pi}}, \quad u(x,0) = U(x). \end{cases}$$

How to construct a sol. ?

Observe: $u u_x$ is $O(\varepsilon^2)$ & much smaller then other terms.

Thus, write

$$u_t - u_{xx} = -\left(\frac{u^2}{2}\right)_x$$

By Duhamel's principle,

$$u(x,t) = k(x,t) * U(x) + \underbrace{\int_0^t k(x,t-\tau) * \left(-\left(\frac{u^2}{2}\right)_x\right) d\tau}_{-\int_0^t k(x,t-\tau) * \left[\frac{u^2(x,\tau)}{2}\right]_x d\tau}$$

So, $u(x,t) = k(x,t) * U(x) \underbrace{- \int_0^t k(x,t-\tau) * \left[\frac{u^2(x,\tau)}{2}\right]_x d\tau}$

$$\Downarrow$$

$$\underbrace{-\int_0^t \int_{-\infty}^{\infty} k(x-y,t-\tau)\left(\frac{u^2(y,\tau)}{2}\right)_y dy\,d\tau}$$

$$\Downarrow$$

$$+\int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-\tau)\cdot\frac{u^2(y,\tau)}{2}\,dy\,d\tau$$

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\cdot U(y)\,dy + \int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-\tau)\cdot\frac{u^2(y,\tau)}{2}\,dy\,d\tau$$

Construct iteration:

$$u_{k}(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\cdot U(y)\,dy + \int_0^t\int_{-\infty}^{\infty} k_y(x-y,t-\tau)\,\frac{u_{k-1}^2(y,\tau)}{2}\,dy\,d\tau,$$

$k \ge 1$

Initial u_0: $\boxed{u_0(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\, U(y)\, dy = k * U(x)}$

$\Rightarrow$ $|u_0(x,t)| \le \varepsilon \int_{-\infty}^{\infty} k(x-y,t)\, k(y,1)\, dy = \varepsilon\, k(x, t+1)$

Assume $|u_k(x,t)| \le 2\varepsilon\, k(x/2, t+1)$.

By introduction, this is true for $k \in \mathbb{N}$.

Define a norm,

$$|||f||| = \sup_{(x,t)} \frac{f(x,t)}{k(\frac{x}{2}, t+1)} \Rightarrow |||u_0||| \le \sup_{(x,t)} \frac{|k(x,t+1)|\varepsilon}{k(x/2, t+1)}$$

$$\le \varepsilon$$

By constructing,

$$|u_k - u_{k-1}| \le \int_0^t \int_{-\infty}^{\infty} |k_y(x-y, t-z)|\, \left|\left[\frac{(u_{k-1} - u_{k-2})}{\cancel{2}}\right]\right| \cdot \left|[u_{k-1} + U_{k-2}]\right| dy\, dz$$

$4\varepsilon k(\frac{y}{2}, z+1)$ $\qquad$ $|||u_{k-1} + u_{k-2}|||$.

$$\le 4\varepsilon\, |||u_{k-1} - u_{k-2}||| \cdot \int_0^t \int_{-\infty}^{+\infty} k_y(x-y, t-z) \cdot k(y/2, z+1). \; k^2(y/2, z+1)\, dy\, dz$$

$$|k_y(x-y, t-z)| = \left|\frac{2(x-y)}{2(t-z)}\, k(x-y, t-z)\right|$$

$$\le \frac{C}{\sqrt{t-z}} \cdot k\left(\frac{x-y}{2}, t-z\right).$$

$\Rightarrow$ $k_y^2(y/2, z+1) \le C \cdot k(\frac{y}{2}, z+1)/\sqrt{z}$

Eventually, $\le 4\varepsilon\, |||u_{k-1} - u_{k-2}||| \cdot \int_0^t k(\frac{x}{2}, t+1) \cdot \frac{1}{\sqrt{z+1}} \frac{1}{\sqrt{t-z}}\, dz \cdot C$

$\le 4\varepsilon C_1 \cdot |||u_{k-1} - u_{k-2}||| \cdot k(\frac{x}{2}, 1+t)$

$\Rightarrow$ $|||u_k - u_{k-1}||| \le 4\varepsilon C_1 \cdot |||u_{k-1} - u_{k-2}|||$

Observe:

$\mathcal{F}: \{|||u||| \le 2\varepsilon\} \longrightarrow \{|||u||| \le 2\varepsilon\}$: complete metric space.

- $\mathcal{F}(u_k) = u_{k+1}$

- $\mathcal{F}$ is a contraction:

$$|||\mathcal{F}(u_{k-1}) - \mathcal{F}(u_{k-2})||| = ||| u_k - u_{k-1} |||$$

By fixed point Thm,

$$\leq 4\varepsilon C_1 \cdot ||| u_{k-1} - u_{k-2} |||$$

$\Longrightarrow$ $\exists !\ u$ s.t.

$$\mathcal{F}(u) = u.$$

Q.E.D.

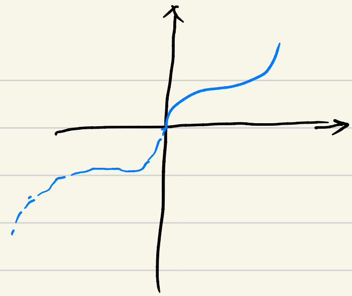
Recall:

$\boxed{\text{Case 1}}$: $\begin{cases} u_{tt} = u_{xx}, & x>0 \\ u(0,t)=0, & u(x,0)=u_0(x), \quad u_t(x,0)=u_1 \end{cases}$

By odd extension,

$f(x), \quad x>0$

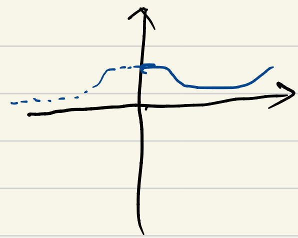
$$f_{odd}(x) = \begin{cases} f & \text{if } x>0 \\ -f(-x), & \text{if } x<0 \end{cases}$$



$\boxed{\text{Case 2}}$: $\begin{cases} u_{tt} = u_{xx}, & x>0 \\ u_x(0,t)=0, & u(x,0)=u_0(x), \quad u_t(x,0)=u_1 \end{cases}$

By even extension:

$$f^{even}(x) = \begin{cases} f(x), & \text{if } x>0 \\ f(-x), & \text{if } x<0 \end{cases}$$



- What if for mixed bdd?

$$\begin{cases} u_{tt} = u_{xx}, \quad x>0 \\ u_x(0,t) = a\cdot u(0,t), \quad u(x,0)=u_0(x), \quad u_t(x,0) = v_1(x) \end{cases}$$

Now, there is a new method, which can solve the problem via $\boxed{\text{Laplace Transform}}$.

☺ $\begin{cases} u_t = u_{xx}, & x\in\mathbb{R},\ t>0 \\ u(x,0)=\delta(x) \end{cases} \Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi t}}\cdot e^{-\frac{x^2}{4t}}$

$\boxed{\text{Laplace Transform: } \quad \mathcal{L}u(x,s) = \int_0^\infty e^{-st}\cdot u(x,t)\,dt, \quad \mathrm{Re}(s)\geq 0.}$

Properties:

① $$\mathcal{L}u_t(x,s) = \int_0^\infty e^{-st}\cdot u_t(x,t)\,dt = e^{-st}\cdot u(x,t)\Big|_0^\infty + s\cdot\int_0^\infty e^{-st}u(x,t)dt$$

$$\Rightarrow \boxed{\mathcal{L}u_t(x,s) = -u(x,0) + s\cdot\mathcal{L}u(x,s)}$$

For ☺, $\begin{cases} \mathcal{L}u_t = \partial^2_{xx}\mathcal{L}u \\ u(x,0) = \delta(x) \end{cases} \Rightarrow \underline{s\mathcal{L}u - \delta(x) = \partial^2_{xx}\mathcal{L}u}.$ This is ODE.

So, $\mathcal{L}u = \begin{cases} A_+\cdot e^{-\sqrt{s}x} + B_+\cdot e^{\sqrt{s}x}, & x>0 \\ A_-\cdot e^{-\sqrt{s}x} + B_-\,e^{\sqrt{s}x}, & x<0 \end{cases} \Rightarrow$ By contin. at $x=0$, $A_+ = B_-$

So, $\mathcal{L} u_x = \begin{cases} -\sqrt{s} A_+ e^{-\sqrt{s}x}, & \text{if } x>0 \\ \sqrt{s} A_+ e^{\sqrt{s}x}, & \text{if } x<0 \end{cases}$ $\Rightarrow$ The size of the jump is: $-2\sqrt{s} \cdot A_+$

$\Rightarrow -2\sqrt{s} \cdot A_+ = -1$, i.e. $A_+ = \frac{1}{2\sqrt{s}}$

Finally, $\mathcal{L} u = \begin{cases} \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}x}, & x>0 \\ \frac{1}{2\sqrt{s}} \cdot e^{\sqrt{s}x}, & x<0 \end{cases}$ — (*)

R.k: Using this to solve lots of eq-s.

$\Leftrightarrow \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x|}$, $\Rightarrow u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$.

Eg 1: $\begin{cases} u_t = u_{xx}, & x>0 \\ u(0,t)=0, & u(x,0) = u_0(x). \end{cases}$

By odd extension, u^{odd}; $\begin{cases} u_t^{odd} = u_{xx}^{odd}, & x \in \mathbb{R}^1 \\ u(x,0) = u_0^{odd}(x) \end{cases}$

By Duhamel's principle:

$$u^{odd}(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * u_0^{odd}(x)$$

$$\Rightarrow u^{odd}(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u_0(y)\,dy = \int_{-\infty}^{0} + \int_{0}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \cdot u_0(y)\,dy = \int_0^\infty \boxed{\left(\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}}\right)} u_0(y)\,dy$$

$g(x,t)$. "Green function of Eg 1."

So, $\Rightarrow \begin{cases} g_t = g_{xx}, & x>0 \\ g(0,t,y) = 0 \\ g(x,0,y) = \delta(x-y) \end{cases}$ Applying Lap. Transform, $\Rightarrow \begin{cases} s\mathcal{L}g - \delta(x-y) = \partial_x^2 \mathcal{L}g \\ \mathcal{L}g(0,s,y) = 0 \end{cases}$

$\Rightarrow \mathcal{L}g - \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}\cdot|x-y|} \underset{\text{defined}}{=} \alpha(x,s)$

$\Rightarrow \begin{cases} s\alpha(x,s) - \alpha_{xx}(x,s) = 0 \\ \alpha(0,s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|} \end{cases}$ & $\alpha(x,s) = A_+ e^{-\sqrt{s}x}$, $x>0$

$\longrightarrow A_+ = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|y|}$

$\Rightarrow \alpha(x,s) = -\frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}(x+y)}$, $\Rightarrow \mathcal{L}g = \frac{1}{2\sqrt{s}} \left(e^{-\sqrt{s}|x-y|} - e^{-\sqrt{s}|x+y|}\right)$

$\Rightarrow g(x,t;y) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} - \frac{e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}}$. ※

verify: $\quad Lu(x,s) = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}\cdot|x|}$,

$\qquad\quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$

$$Lu = \int_0^{+\infty} \frac{e^{-st} \cdot e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\, dt,$$

Let $\tilde{t} = \sqrt{t}$

$$= \int_0^{\infty} \frac{1}{\cancel{2}\sqrt{\pi}\,\cancel{\tilde{t}}} \cdot e^{-s\tilde{t}^2 - \frac{x^2}{4\tilde{t}^2}} \cdot \cancel{2}\cancel{\tilde{t}}\, d\tilde{t}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s\tilde{t}^2 - \frac{x^2}{4\tilde{t}^2}}\, d\tilde{t}$$

$$= \frac{e^{-\sqrt{s}|x|}}{\sqrt{\pi}} \cdot \int_0^{\infty} e^{-(\sqrt{s}\,\tilde{t} + \frac{|x|}{2\tilde{t}})^2}\, d\tilde{t}, \quad \text{Let } y = \sqrt{s}\,\tilde{t} + \frac{|x|}{2\tilde{t}}$$

Note $\boxed{\int_0^{\infty} e^{-(ax - \frac{b}{x})^2}\, dx = \frac{1}{a} \cdot \int_0^{\infty} e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2a}}$

$$\Rightarrow Lu = e^{-\sqrt{s}\cdot|x|} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2\sqrt{s}}$$

$$= \frac{e^{-\sqrt{s}\cdot|x|}}{2\sqrt{s}}$$

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Solution for the following HW:

Homework: PDEs. A0187036X

GENG Xingri

Problem:
$$\begin{cases} u_t = u_{xx}, \ x>0 \\ u(0,t) + u_x(0,t) = 0 \\ u(x,0) = \delta(x-y), \ y>0. \end{cases}$$

Solution: By Laplace transform: $\mathcal{L}u = \int_0^\infty u(x,t)\cdot e^{-st}\,dt$

Thus, $\mathcal{L}u_t(x,s) = \int_0^\infty u_t \cdot e^{-st}\,dt \overset{\text{I.B.P}}{=} s\cdot\mathcal{L}u - u(x,0). = s\cdot\mathcal{L}u - \delta(x-y).$

And we get
$$\underline{s\cdot\mathcal{L}u - \delta(x-y) = \partial_x^2\mathcal{L}u}$$

The boundary condition is
$$\underline{\mathcal{L}u(0,t) + \partial_x\mathcal{L}u(0,t) = 0} \quad \text{— B.C.}$$

Remember, previously in class, we already solve:
$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx}, \ x\in\mathbb{R}, t>0 \\ \tilde{u}(x,0) = \delta(x) \end{cases}$$

The solution is $\tilde{u}(x,t) = \dfrac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$

Consider $\tilde{u}(x-y,t) = \dfrac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}$,

It satisfies the Laplace transform:
$$\mathcal{L}\tilde{u}_t(x-y,s) = s\cdot\mathcal{L}\tilde{u}(x-y,s) - \delta(x-y) = \partial_x^2\hat{\tilde{u}}(x-y,s)$$

Now, take $H(x,t) = u(x,t) - \tilde{u}(x-y,t)$

Observe:
$$s\,\mathcal{L}H = (\mathcal{L}u - \mathcal{L}\tilde{u}(x-y,s))\cdot s \ \&$$
$$\partial_x^2\mathcal{L}H = \partial_x^2\mathcal{L}u - \partial_x^2\mathcal{L}\tilde{u}(x-y,s)$$

Thus, $\underline{s\mathcal{L}H = \partial_x^2\mathcal{L}H}$ for $s\mathcal{L}u = \delta(x-y) + \partial_x^2\mathcal{L}u$

$\& \ s\mathcal{L}\tilde{u}(x-y) = \partial_x^2\mathcal{L}\tilde{u}(x-y) + \delta(x-y)$

So, $\mathcal{L}H = \alpha\cdot e^{-\sqrt{s}x}$, α — to be determined.

Then, $\mathcal{L}u = \mathcal{L}H + \mathcal{L}\tilde{u}(x-y) = \alpha\cdot e^{-\sqrt{s}x} + \dfrac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}$

By B.C.,
$$\alpha + \frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}} + (-\sqrt{s}\alpha) + (\sqrt{s})\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}} = 0$$

$$\Rightarrow \alpha = -\frac{(1+\sqrt{s})}{1-\sqrt{s}}\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}}$$

So,
$$\mathcal{L}u = -\frac{(1+\sqrt{s})}{1-\sqrt{s}}\cdot\frac{e^{-\sqrt{s}|y|}}{2\sqrt{s}}\cdot e^{-\sqrt{s}x} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}, \quad y>0.$$

$$\mathcal{L}u = -(-1 + \frac{2}{1-s} + \frac{2}{1-s}\sqrt{s}) \cdot e^{-\sqrt{s}(y+x)} + \frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}, \quad y>0,\ x>0$$

$$= (1 - \frac{2}{1-s} + \frac{1}{1-s}\sqrt{s}) \cdot e^{-\sqrt{s}|x+y|} + \underbrace{\frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}}}$$

observe: $k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$, • $\mathcal{L}k(x,s) = \frac{1}{2\sqrt{s}} \cdot e^{-\sqrt{s}|x|}$

$$\bullet\ 2\sqrt{s} \cdot \frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}} = -2\partial_x \left(\frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}}\right)$$

$$= -2\mathcal{L}(k_x(x+y,t))$$

$$\bullet\ -\frac{2}{1-s} \cdot e^{-\sqrt{s}|x+y|} = -2\mathcal{L}(e^t) \cdot (-2 \cdot \mathcal{L}k_x(x+y,t))$$

$$= 4\mathcal{L}(e^t) \cdot \mathcal{L}k_x(x+y,t) \quad \text{for } \mathcal{L}f \cdot \mathcal{L}g = \mathcal{L}(f*g)$$

$$= 4\mathcal{L}(e^t * k_x(x+y,t))$$

$$\bullet\ \frac{2}{1-s}\sqrt{s} \cdot e^{-\sqrt{s}|x+y|} = \frac{4s}{1-s} \cdot \frac{e^{-\sqrt{s}|x+y|}}{2\sqrt{s}}$$

$$= (-4 + \frac{4}{1-s}) \cdot \mathcal{L}k(x+y,t)$$

$$= -4\mathcal{L}k(x+y,t) + 4\mathcal{L}(e^t) \cdot \mathcal{L}k(x+y,t)$$

$$= -4\mathcal{L}k(x+y,t) + 4\mathcal{L}(e^t * k(x+y,t)).$$

$$= -4\mathcal{L}k(x+y,t)$$

So, $u(x,t) = -2k_x(x+y,t) + 4e^t * k_x(x+y,t)$

$-4k(x+y,t) + k(x-y,t)$, where $k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$.

$+4e^t * k(x+y,t)$.

$\square$.

HW: Solve $\begin{cases} g_t - g_{xx} = 0, & x>0 \\ g_x(0,t) + g(0,t) = 0 \\ g(x,0;y) = \delta(x-y), & y>0 \end{cases}$

Class 6.

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x>0,\ t>0 \\ u(x,0)=0, \quad u_t(x,0) = \delta(x-y), \quad u(0,t)=0, \quad y>0 \end{cases}$$

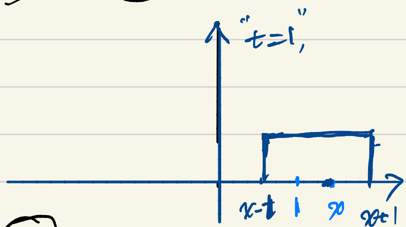
Recall for a whole space problem:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x\in\mathbb{R},\ t>0 \\ u_t(x,0) = u_1(x), \quad u(x,0) = u_0(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\,d\xi$$

If $u_1(x) = \delta(x)$, & $u_0(x)=0$, then

$$u(x,t) = \begin{cases} 1, & \text{if } |x|<t \\ 0, & \text{if } |x|>t \end{cases}$$



Consider $\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R},\ t>0 \\ u_t(x,0) = \delta(x), & u(x,0)=0 \end{cases}$

Apply Laplace Transform to :
(s>0)

$$\underbrace{s^2\mathcal{L}u - u_t(x,0) - s\,u(x,0)}_{\mathcal{L}u_{tt}} - \partial_x^2\mathcal{L}u = 0$$

$$\Rightarrow s^2\mathcal{L}u - \partial_x^2\mathcal{L}u = \delta(x) \quad -(*)$$

$$\Rightarrow \mathcal{L}u = \begin{cases} \cancel{A_+ e^{sx}} + B_+ e^{-sx}, & x>0 \\ A_- e^{sx} + \cancel{B_- e^{-sx}}, & x<0 \end{cases}$$

By contin. at $x=0$
$\Rightarrow A_- = B_+$

$$\Rightarrow \mathcal{L}u = \begin{cases} B_+ e^{-sx}, & x>0 \\ B_+ e^{sx}, & x<0 \end{cases}$$

And $\partial_x \mathcal{L} u = \begin{cases} -s B_+ e^{-sx}, & x>0 \\ s \cdot B_+ \cdot e^{sx}, & x<0 \end{cases}$,

By (*), $\partial_x \mathcal{L} u = -H(x) \Rightarrow 2 s B_+ = 1$

$$\Rightarrow \mathcal{L} u = \begin{cases} \frac{1}{2s} e^{-sx}, & x>0 \\ \frac{1}{2s} \cdot e^{sx}, & x<0 \end{cases} \qquad \text{i.e. } \mathcal{L} u = \frac{e^{-s|x|}}{2s}$$

Now,

$$\begin{cases} u_{tt} - u_{xx} = 0, & x>0, \ t>0 \\ u(x,0) = 0, & u_t(x,0) = \delta(x-y), \ y>0 \\ u(0,t) = 0. \end{cases}$$

Laplace Transform:

$$\begin{cases} s^2 \mathcal{L} u - \partial_x^2 \mathcal{L} u = \delta(x-y), & x>0, \ y>0 \\ \mathcal{L} u(0,s) = 0 \end{cases}$$

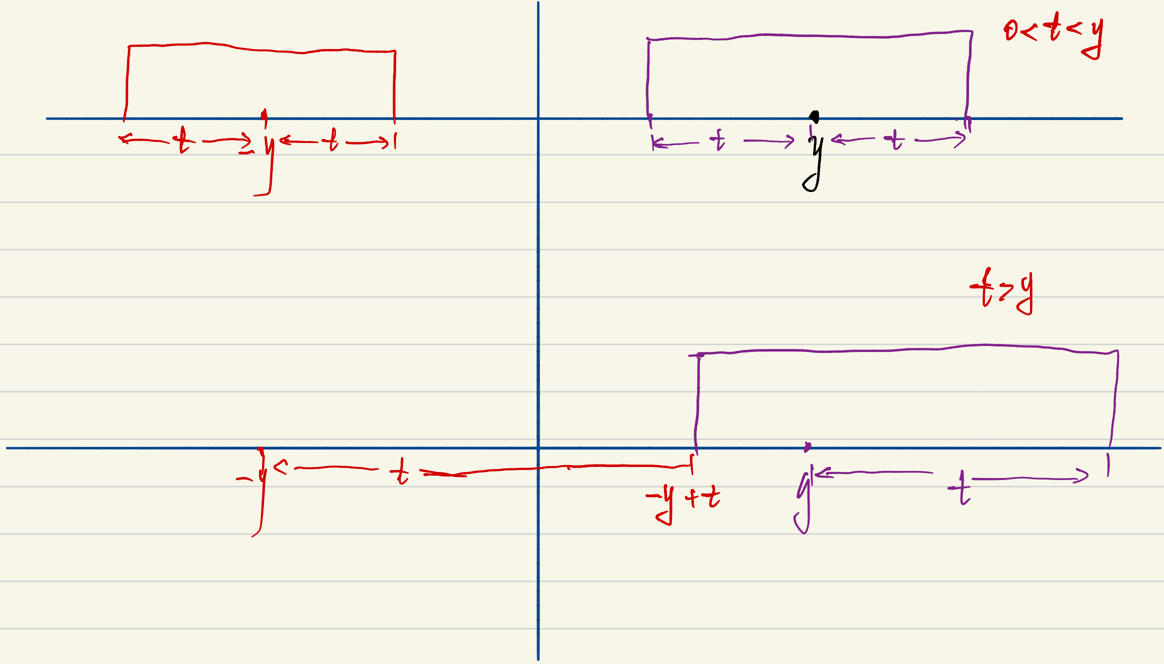
$$\Rightarrow s^2 \underbrace{\left(\mathcal{L} u - \frac{e^{-s|x-y|}}{2s}\right)}_{\text{☺}} - \partial_x^2 \underbrace{\left(\mathcal{L} u - \frac{e^{-s|x-y|}}{2s}\right)} = 0, \quad x>0$$

$$\Rightarrow \text{☺} = A \cdot e^{-sx} \quad \text{for } x>0$$

$$\Rightarrow \text{☺}\Big|_{x=0} = -\frac{e^{-s|y|}}{2s} = A \quad \text{i.e. } A = -\frac{e^{-sy}}{2s} \quad \text{for } y>0$$

$$\Rightarrow \text{☺} = -\frac{e^{-s(y+x)}}{2s}$$

$$\Rightarrow \mathcal{L} u = \frac{e^{-s|x-y|}}{2s} - \frac{e^{-s(x+y)}}{2s}, \quad x>0, \ y>0$$



Now, $\begin{cases} u_{tt} - u_{xx} = 0, & x>0,\ t>0 \\ u(x,0) = 0, & u_t(x,0) = \delta(x-y),\ y>0 \\ u_x(0,t) = 0 \end{cases}$

$$\Rightarrow \begin{cases} s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u = \delta(x-y) \\ \partial_x \mathcal{L}u(0,s) = 0 \end{cases}$$

$$\Rightarrow s^2\underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)} - \partial_x^2\underbrace{\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right)} = 0, \quad x>0$$

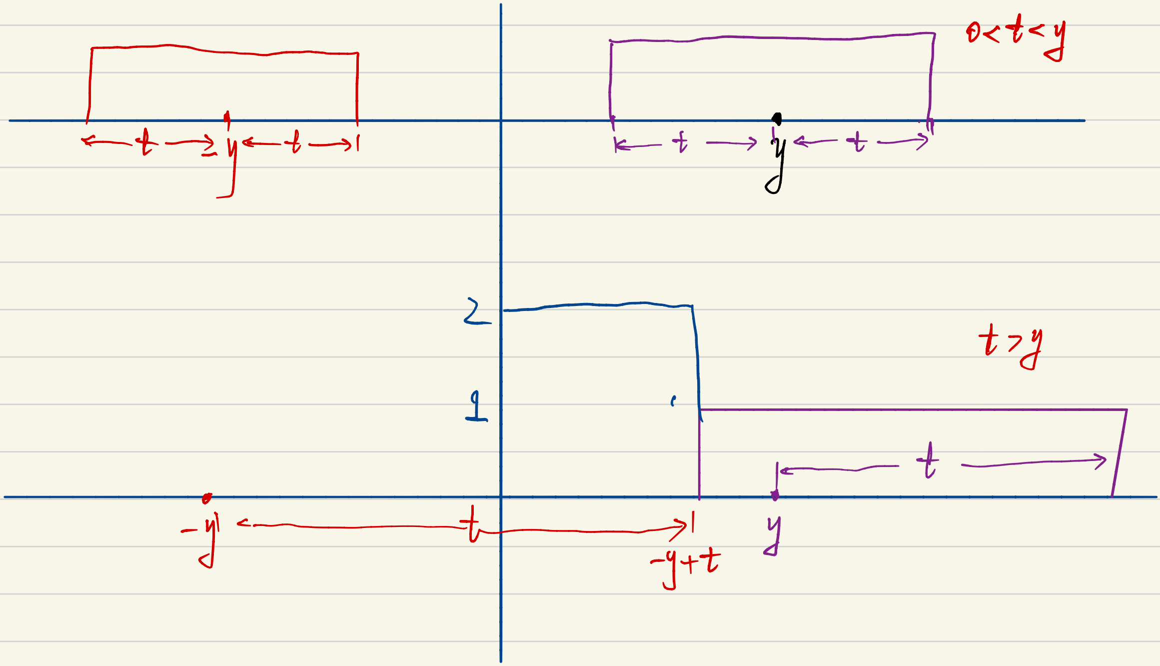
☺

Then, ☺ $= A\cdot e^{-sx}$

By B.C. ∂_x ☺ $\Big|_{x=0} = \partial_x\left(\mathcal{L}u - \frac{e^{-s|x-y|}}{2s}\right) = \partial_x A\cdot e^{-sx}$

$$\Rightarrow -\frac{e^{-sy}}{2} = -sA \quad \text{i.e. } A = \frac{e^{-sy}}{2s}$$

$$\Rightarrow \mathcal{L}u = \frac{e^{-s|x-y|}}{2s} + \frac{e^{-s(x+y)}}{2s}, \quad x>0,\ y>0$$



Functional Analysis:

$$H^1(\mathbb{R}) = \{ f \mid \|f\|_2 + \|f_x\|_2 < \infty \}$$

- $\|f\|_{H^1} = \sqrt{\|f\|_2^2 + \|f_x\|_2^2} = \left(\int_{\mathbb{R}} |f|^2 + |f_x|^2 \, dx \right)^{1/2}$
- $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)|$

$\forall f$,

$$f^2(x) = \left| \int_{-\infty}^{x} \frac{d}{dx} (f^2(x)) \, dx \right| = \left| \int_{-\infty}^{x} 2 f f_x \, dx \right|$$

$$\le 2 \int_{-\infty}^{x} |f| \cdot |f_x| \, dx \le \int_{-\infty}^{x} |f|^2 + |f_x|^2 \, dx$$

$$\le (\|f\|_{H^1})^2$$

$\Rightarrow$ $\|f\|_\infty \le \|f\|_{H^1}$ #.

Sobolev's space: $f(\vec{x})$: $\vec{x} \in \mathbb{R}^n$

$$\hat{f}(\vec{\eta}) = \int_{\mathbb{R}^n} f(\vec{x}) \cdot e^{-i \vec{\eta} \vec{x}} \, d\vec{x} \quad \text{— Fourier transform}$$

$$\|f\|_{H^k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\vec{\eta}|^2)^k \cdot |\hat{f}(\vec{\eta})|^2 \, d\eta \right)^{1/2}$$

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\vec{\eta}) \cdot e^{i \vec{\eta} \vec{x}} \, d\vec{\eta}$$

Suppose $f \in H^k(\mathbb{R}^n)$, with $k > n/2$. Then,

$$|f(x)| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \vec{\eta} \vec{x}} \cdot \hat{f}(\vec{\eta}) \, d\vec{\eta} \right|$$

$$= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \vec{\eta} \cdot \vec{x}} \cdot \hat{f}(\vec{\eta}) \cdot \frac{(1 + \vec{\eta}^2)^{k/2}}{(1 + \vec{\eta}^2)^{k/2}} \, d\vec{\eta} \right|$$

Schwarz ineq.

$$\le \frac{1}{(2\pi)^n} \cdot \left(\int_{\mathbb{R}^n} |\hat{f}(\vec{\eta}) \cdot (1 + \vec{\eta}^2)^{k/2}|^2 \, d\vec{\eta} \right)^{1/2} \cdot \left(\int_{\mathbb{R}^n} \frac{d\vec{\eta}}{(1 + \vec{\eta}^2)^k} \right)^{1/2}$$

$$\le \frac{1}{(2\pi)^n} \cdot \|f\|_{H^k} \cdot \left(\int_0^\infty \int_{S^{n-1}} \frac{r^{n-1}}{(1 + r^2)^k} \, d\Omega \right)^{1/2}$$

Integrable!

Class 7: (Telegraph Equation)

Caution: In previous class, we can construct the sol. by Fourier Transform, but in some cases, there exist things can not be transformed by inverse Fourier transform!!!

Telegraph Eq.

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0, & x \in \mathbb{R},\ t > 0, \quad \text{Cauchy Problem} \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases}$$

— 

Take Fourier Transform:

$$\begin{cases} \hat{u}_{tt} + \eta^2 \hat{u} + \hat{u}_t = 0 \\ \hat{u}(\eta,0) = 0 \\ \hat{u}_t(\eta,0) = 1 \end{cases} \qquad \text{"ODE"}$$

Fix η, solve "ODE": Characteristic eq. $\xi^2 + \xi + \eta^2 = 0$

$\Rightarrow \xi_\pm = \dfrac{-1 \pm \sqrt{1-4\eta^2}}{2}$

$\Rightarrow \hat{u}(\eta,t) = (A e^{\xi_+ t} + B e^{\xi_- t})$

$\Rightarrow B = -A$ for $\hat{u}(\eta,0) = 0$

& $\hat{u}_t(\eta,t) = A\xi_+ e^{\xi_+ t} - A\xi_- e^{\xi_- t} \Rightarrow A(\xi_+ - \xi_-) = 1$ for $\hat{u}_t(\eta,0) = 1$

Get $A = \dfrac{1}{\sqrt{1-4\eta^2}}$ & $\boxed{\hat{u}(\eta,t) = \dfrac{1}{\sqrt{1-4\eta^2}} (e^{\xi_+ t} - e^{\xi_- t})}$

Q: How to understand the sol. $\hat{u}(\eta,t)$?

A: Need to find sol. in x-space.

Intuition: $f = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\eta) \cdot e^{i\eta x} d\eta$ wave number

dispersion relationship.

low pitch: η small, $1 - 4\eta^2 > 0$ (smooth).

$e^{\xi_+ t} = e^{\frac{-1+\sqrt{1-4\eta^2}}{2} t}$ by Talyor's expansion of $\frac{-1+\sqrt{1-4\eta^2}}{2} \approx -1 + (1-4\eta^2)/2 + O(1)\eta^4$

$\Rightarrow e^{\xi_+ t} \sim e^{-2\eta^2 t} \Rightarrow \sim \dfrac{1}{\sqrt{8\pi t}} \cdot e^{-\frac{x^2}{8t}}$

Task 1: Extract Singularity in the frequence domain:

① Compute
$$\begin{cases} \hat{u}(\eta,t) = \frac{1}{\sqrt{1-4\eta^2}}\,(e^{\xi_+ t} - e^{\xi_- t}) \\ \xi_\pm = \frac{-1 \pm \sqrt{1-4\eta^2}}{2} \end{cases};$$

Try to understand the sol. as $\eta \to \infty$;

$$\sqrt{1-4\eta^2} = 2\eta i \sqrt{1-\frac{1}{4\eta^2}} \xrightarrow[\text{expansion}]{\text{Talyor's}} i2\eta\left(1-\frac{1}{8\eta^2}+\frac{O(1)}{\eta^4}\right)$$
$$= i2\eta\left(1-\frac{1}{8\eta^2}+\frac{C}{\eta^4}+\frac{O(1)}{\eta^6}\right)$$

Now, find $\xi_\pm^*$ to approximate $\xi_\pm$:

② Define
$$\boxed{\xi_\pm^* = \frac{-1 \pm 2\eta i\left(1-\frac{1}{8(\eta^2+1)}+\frac{C}{(\eta^2+1)^2}\right)}{2}}$$

& $|\xi_\pm - \xi_\pm^*| = O(1)\cdot\frac{1}{\eta^5}$ as $\eta \to \infty$

Want to find approximating sol. of $\hat{u}(\eta,t)$:

Notice $(\sqrt{1-4\eta^2})^* = 2\eta i\left(1-\frac{1}{8(\eta^2+1)}+\frac{C}{(\eta^2+1)^2}\right)$

$\Downarrow$

$$\boxed{\hat{u}^*(\eta,t) = \frac{1}{(\sqrt{1-4\eta^2})^*}\cdot(e^{\xi_+^* t} - e^{\xi_-^* t})}$$

③ Define $\hat{u}^*(\eta,t)$

The initial conditions are satisfied as follows:

- $\hat{u}_t^*(\eta,0) = 1,\ \hat{u}^*(\eta,0) = 0$;

- $\hat{u}_{tt}^* + \eta^2\hat{u}^* + \hat{u}_t^*$

$$= \left[(\xi_+^*)^2\cdot e^{\xi_+^* t} - (\xi_-^*)^2\cdot e^{\xi_-^* t} + \eta^2 e^{\xi_+^* t} - \eta^2 e^{\xi_-^* t} + \xi_+^*\cdot e^{\xi_+^* t} - \xi_-^* e^{\xi_-^* t}\right]\frac{1}{(\sqrt{1-4\eta^2})^*}$$

$$= \frac{1}{(\sqrt{1-4\eta^2})^*}\left[\left((\xi_+^*)^2+\eta^2+\xi_+^*\right)\cdot e^{\xi_+^* t} - \left((\xi_-^*)^2+\eta^2+\xi_-^*\right)\cdot e^{\xi_-^* t}\right]$$

Note: $[(\xi_+^*)^2 + \eta^2 + \xi_+^*] \cdot e^{\xi_+^* t}$

$$= (\cancel{\xi_+^2 + \eta^2 + \xi_+} + \underbrace{((\xi_+^*)^2 - \xi_+^2) + (\xi_+^* - \xi_+)}_{O(1)\frac{1}{\eta^5}}) \cdot e^{\xi_+^* \cdot t}$$

0

for $|\xi_\pm - \xi_\pm^*| = O(1)\frac{1}{\eta^5}$

$$\Rightarrow \hat{u}_{tt}^* + \eta^2 \hat{u}^* + \hat{u}_t^* = \hat{S}(\eta, t) \leq O(1) \cdot \frac{e^{-t}}{(|\eta|+1)^5}$$

Compare $|\hat{u}^*(\eta,t) - \hat{u}(\eta,t)|$:

$$\Rightarrow \begin{cases} (\hat{u} - \hat{u}^*)_{tt} + \eta^2(\hat{u} - \hat{u}^*) + (\hat{u} - \hat{u}^*)_t = O(1) \cdot \frac{e^{-t}}{c(|\eta|+1)^5} \\ (\hat{u} - \hat{u}^*)(\eta, 0) = (\hat{u}_t - \hat{u}_t^*)(\eta, 0) = 0 \end{cases}$$

Let $\hat{V} = \hat{u} - \hat{u}^*$

$$(*) \begin{cases} \hat{V}_{tt} + \eta^2 \hat{V} + \hat{V}_t = O(1) \cdot \frac{e^{-t}}{(|\eta|+1)^5} \leq e^{-t} \\ \hat{V}(\eta, 0) = \hat{V}_t(\eta, 0) = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |\partial_x^3 S(x,t)|^2 dx \le e^{-t}$$

Rewrite the ✱ as:

$$\begin{cases} v_{tt} - v_{xx} + v_t = S(x,t) \\ v(x,0) = v_t(x,0) = 0; \end{cases}$$

Energy Estimate:

$$\int_{\mathbb{R}} v_t \cdot (v_{tt} - v_{xx} + v_t)\, dx = \int_{\mathbb{R}} S(x,t)\, v_t\, dx$$

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(v_t)^2 dx + \int_{\mathbb{R}} \underbrace{v_{tx} v_x}_{\left(\frac{v_x^2}{2}\right)_t} dx + \int_{\mathbb{R}} v_t^2 dx = \int_{\mathbb{R}} S(x,t) v_t\, dx$$

$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(v_t^2 + v_x^2)\, dx + \int_{\mathbb{R}} v_t^2 dx = \int_{\mathbb{R}} S(x,t)\, v_t\, dx$$

Integrate on t:

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2}[(v_t)^2 + (v_x)^2]\, dx \Big|_0^T + \int_0^T \int_{\mathbb{R}} (v_t)^2 dx\, dt = \int_0^T \int_{\mathbb{R}} S(x,t)\, v_t(x,t)\, dx\, dt$$

$$\le \int_0^T \int_{\mathbb{R}} \frac{S^2(x,t) + (v_t)^2}{2}\, dx\, dt$$

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} (v_t)^2 dx\, dt$$

$$\int_0^T \int_{\mathbb{R}} \frac{S^2(x,t)}{2}\, dx\, dt$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2}[v_t^2 + v_x^2]\, dx \Big|_0^T + \frac{1}{2}\int_0^T \int_{\mathbb{R}} v_t^2 dx\, dt \le \int_0^T \int_{\mathbb{R}} \frac{S^2(x,t)}{2}\, dx\, dt$$

Similarly,

$$\int_{\mathbb{R}} \partial_x^k v_t \cdot \partial_x^k (v_{tt} - v_{xx} + v_t) \, dx = \int_{\mathbb{R}} \partial_x^k S(x,t) \cdot \partial_x^k v_t \, dx$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{2} [(\partial_x^k v_t)^2 + (\partial_x^k v_x)^2] dx \Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}} (\partial_x^k v_t)^2 \, dx dt$$

$$\leq \int_0^T \int_{\mathbb{R}} \frac{(\partial_x^k S(x,t))^2}{2} \, dx dt$$

$k = 1, 2, 3.$

Task 2: Find the structure $u^*(x,t)$.

Another Functional Thm:

If $f(x)$ satisfies $\begin{cases} \hat{f}(\eta): \text{Analytic in } |\operatorname{Im}\eta| < \eta_0 \\ |\hat{f}(\eta)| \lesssim \dfrac{O(1)}{\eta^2+1} \end{cases}$

Then, $|f(x)| \lesssim O(1)\cdot e^{-\eta_0 |x|}$

Proof:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x}\cdot \hat{f}(\eta)\,d\eta \overset{\text{Cauchy Integrable Thm.}}{\underset{\eta=\xi+iv_0}{=\!=}} \frac{1}{2\pi}\int_{\mathbb{R}+iv_0} e^{i\eta x}\hat{f}(\eta)\,d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi+iv_0)x}\hat{f}(\xi+iv_0)\,d\xi = \frac{e^{-v_0 x}}{2\pi}\int_{-\infty}^{\infty} e^{i\xi}\cdot\hat{f}(\xi+iv_0)\,d\xi$$

$|v_0| < \eta_0$

$$\Rightarrow |f(x)| \le \frac{e^{-v_0 x}}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi+iv_0)|\,d\xi$$

$$\lesssim \frac{e^{-v_0 x}}{2\pi}\, C \quad \leftarrow \text{const.}$$

$$\Rightarrow |f(x)| \le \frac{e^{-|v_0 x|}}{2\pi}\cdot C \le O(1)\cdot e^{-\eta_0|x|}$$

Q.E.D.

Recall,

$$\begin{cases} \cdot\ \xi_\pm^* = \dfrac{-1 \pm 2i\eta\left[1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2}\right]}{2} \\ \cdot\ (\sqrt{1-4\eta^2})^* = i2\eta\left(1 - \frac{1}{8(\eta^2+1)} + \frac{C}{(\eta^2+1)^2}\right) \\ \cdot\ |\xi_\pm - \xi_\pm^*| = O(1)\,\dfrac{1}{\eta^6}, \quad \text{as } \eta\to\infty \\ \cdot\ \hat{u}^*(\eta,t) = \dfrac{1}{(\sqrt{1-4\eta^2})^*}\cdot\left(e^{\xi_+^* t} - e^{\xi_-^* t}\right) \end{cases}$$

analytic in $|\operatorname{Im}(\eta)| < \frac{\alpha}{2}$

$$= \frac{\overbrace{e^{-\frac{t}{2}}\left[e^{2i\eta\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]t} - e^{-2i\eta\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]} \right]}^{\mathrm{I}}}{\underbrace{2i\eta\cdot\left(1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right)}_{\mathrm{II}}}$$

Consider

$$e^{2i\eta\cdot\left[1-\frac{1}{8(\eta^2+1)}+\frac{c}{(\eta^2+1)^2}\right]t}$$

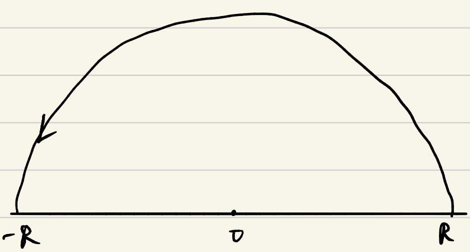
$$= \underbrace{e^{2i\eta t}}_{\delta(x+2t)} \cdot \underbrace{e^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}}}$$

Reconsider:

$$e^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}} \underbrace{-1}_{\delta(x)} = \int_0^{-\frac{2i\eta t}{8(\eta^2+1)}+\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}} e^{\alpha}\, d\alpha$$

$$= \underbrace{-\frac{2i\eta t}{8(\eta^2+1)}} + \underbrace{\frac{c\cdot 2i\eta t}{(\eta^2+1)^2}}_{e^{-|\eta_0 x|}} + O(1)\,t^2$$

By Complex Analysis:



$$\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\frac{1}{1+\eta^2}\cdot e^{i\eta x}\,d\eta$$

$$= \frac{1}{2\pi}\cdot \mathrm{Res.}\left(\frac{e^{i\eta x}}{1+\eta^2}\right)$$

$$= \frac{1}{2\pi}\cdot 2\pi i\cdot\frac{e^{-x}}{2i} = k\cdot e^{-x}, \text{ if } x>0$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{1+\eta^2}e^{i\eta x}\,dx = c\cdot e^{-|x|}.$$

Thus, $\quad \dfrac{-2i\eta}{8(\eta^2+1)} \longleftarrow c\cdot t\cdot\left(e^{-|x|}\right)_x.$

Then, $$\mathrm{I} = e^{-t/2}\cdot\Big[\delta(x+2t) * \left[\left(e^{-|x|}\right)_x + e^{-|\eta_0 x|}\right]c\cdot t$$
$$- \delta(x-2t) * \left[\left(e^{-|x|}\right)_x + e^{-|\eta_0 x|}\right]c\cdot t\Big]$$

II is a polynomial $\Rightarrow u^*(x,t)$ is around $e^{-|x|-t/k}$.

Task 3: Verify $|u(x,t) - u^*(x,t)|$ is exponentially small in the region $|x| > \alpha t$ for any positive α.

By Energy Estimate on $v(x,t) = u(x,t) - u^*(x,t)$.

- $\int_{-\infty}^{\infty} (v_t^2(x,t) + v_x^2(x,t)) \cdot e^{\varepsilon(x-\alpha t)}\, dx \le O(1) \cdot e^{-\varepsilon^2 t}$ for $x > \alpha t$
- $\int_{-\infty}^{\infty} (v_t^2(x,t) + v_x^2(x,t)) \cdot e^{-\varepsilon(x+\alpha t)}\, dx \le O(1) \cdot e^{-\varepsilon^2 t}$ for $x < -\alpha t$

Recall, $\begin{cases} v_{tt} - v_{xx} + v_t = s(x,t), \\ v(x,0) = v_t(x,0) = 0 \end{cases}$

$|\partial_x^3 S(x,t)|^2 < O(1) \cdot e^{-t-|x|/k}$ for $|\hat{S}(\eta,t)| \le \frac{e^{-t}}{(|\eta|+1)^4} \cdot O(1)$

Observe:

If $|x| > \frac{\alpha t}{2}$, then $\exists\ k_1$ s.t.

$$|v(x,t)| \le O(1) \cdot e^{-(t+|x|)/k_1}$$

By method of energy:

$\forall \varepsilon > 0$,

$$\int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{\alpha}{2}t)} \cdot v_t \cdot (v_{tt} - v_{xx} + v_t)\, dx = \int_{-\infty}^{\infty} S(x,t) \cdot e^{\varepsilon(x-\frac{\alpha}{2}t)} \cdot v_t\, dx$$

- W.T. know how $v = u^* - u$ looks like?

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases} \quad \& \quad \begin{cases} u^*_{tt} - u^*_{xx} + u^*_t = S(x,t). \\ u^*(x,0) = 0 \\ u^*_t(x,0) = \delta(x). \end{cases}$$

Combine the two equations:

$$\Rightarrow \begin{cases} v_{tt} - v_{xx} + v_t = -S(x,t) \\ v(x,0) = 0 \\ v_t(x,0) = 0, \end{cases} \qquad |\partial_x^3 S(x,t)| \le O(1) \cdot e^{-t - \frac{|x|}{k}}$$

If $|x| > t/2$, there exists k_1 s.t. $|v(x,t)| \le k_1 \cdot e^{-(|x|+t)/k_1}$

What we want to prove

Proof: Consider $e^{\varepsilon(x - \frac{1}{8}t)}$,

$$\int_{-\infty}^{\infty} e^{\varepsilon(x - \frac{1}{8}t)} \cdot v_t \cdot (\underbrace{v_{tt}}_{(\mathrm{I})} - \underbrace{v_{xx}}_{(\mathrm{II})} + \underbrace{v_t}_{(\mathrm{III})})\, dx = \int_{-\infty}^{\infty} -S(x,t) \cdot e^{\varepsilon(x - \frac{t}{8})} \cdot v_t\, dx$$

(I)

(III)

$v_t \cdot v_{tt} = (\frac{1}{2} v_t^2)_t$

(II)

$\Downarrow$

$$\int_{-\infty}^{\infty} e^{\varepsilon(x - \frac{1}{8}t)} \cdot (\frac{v_t^2}{2})_t\, dx = \frac{d}{dt}\left[\frac{1}{2}\int e^{\varepsilon(x - \frac{1}{8}t)} \cdot (v_t)^2 dx\right] + \frac{\varepsilon}{16}\int e^{\varepsilon(x - \frac{t}{8})} \cdot v_t^2\, dx$$

- $$-\int e^{\varepsilon(x - \frac{t}{8})} \cdot v_t \cdot v_{xx}\, dx \overset{\text{I.B.P}}{=} \int_{-\infty}^{\infty} (e^{\varepsilon(x - \frac{t}{8})} \cdot v_t)_x \cdot v_x\, dx$$

$$= \underline{\int_{-\infty}^{\infty} e^{\varepsilon(x - \frac{t}{8})} \cdot v_{tx} v_x\, dx} + \varepsilon \int_{-\infty}^{\infty} e^{\varepsilon(x - \frac{t}{8})} \cdot v_t \cdot v_x\, dx$$

$$= \underline{\int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot \left(\frac{V_x^2}{2}\right)_t \, dx} + \underline{\varepsilon \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t \cdot V_x \, dx}$$

$$= \frac{d}{dt} \frac{1}{2} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 \, dx + \frac{\varepsilon}{16} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 \, dx +$$

& (III) $\Rightarrow \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t^2 \, dx$.

In all,

$$\boxed{\begin{aligned} &\underline{\frac{1}{2} \frac{d}{dt} \cdot \int \cdot e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t)^2 \, dx} + \frac{\varepsilon}{16} \cdot \int e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t)^2 \, dx \\ &+ \underline{\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2} + \underline{\frac{\varepsilon}{16} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_x^2 \, dx + \varepsilon \int_{-\infty}^{\infty} e \cdot V_t V_x dx} \\ &+ \underline{\int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t^2 \, dx} = \int_{-\infty}^{\infty} -S(x,t) \, e^{\varepsilon(x-\frac{1}{8}t)} \cdot V_t \, dx \end{aligned}}$$

If we choose ε small enough, then

$$\Rightarrow \int_{-\infty}^{\infty} \underline{\left(\frac{1}{16}\varepsilon V_x^2 + \varepsilon V_t V_x + V_t^2\right)} \cdot e^{\varepsilon(x-\frac{1}{8}t)} \, dx$$

$$\geq \frac{1}{32} \varepsilon \cdot (V_x^2 + V_t^2)$$

Rewrite:

$$\Rightarrow \underline{\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} e^{\varepsilon(x-\frac{1}{8}t)} \cdot (V_t^2 + V_x^2) \, dx + \int_{-\infty}^{\infty} \frac{1}{32} \varepsilon (V_x^2 + V_t^2) \cdot e^{\varepsilon(x-\frac{1}{8}t)}}$$

$$\leq \left| -\int_{-\infty}^{\infty} S(x,t) \cdot e^{\varepsilon(x-\frac{1}{8}t)} V_t \, dx \right|$$

$$\leq \int_{-\infty}^{\infty} \left[\frac{S^2(x,t)}{1/64 \, \varepsilon} + \frac{1}{64} \varepsilon (V_t)^2 \right] e^{\varepsilon(x-\frac{1}{8}t)} \, dx$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \cdot e^{\varepsilon(x-\frac{1}{8}t)} \cdot [(V_t)^2 + (V_x)^2] \, dx + \int_{-\infty}^{\infty} \frac{1}{64} \varepsilon \cdot (V_x^2 + V_t^2) \cdot e^{\varepsilon(x-\frac{1}{8}t)} \, dx$$

$$\leq \int_{-\infty}^{\infty} \frac{S^2(x,t)}{1/64 \cdot \varepsilon} \cdot e^{\varepsilon(x-\frac{1}{8}t)} \, dx.$$

$$\Rightarrow \int_{\mathbb{R}} e^{\varepsilon(x-\frac{1}{8}t)} [\cdot V_t^2 + V_x^2] dx \Big|_{t=T} \le \int_0^T \int_{-\infty}^{\infty} O(1) \cdot S^2 \cdot e^{\varepsilon(x-\frac{1}{8}t)} dx \cdot dt$$

Recall, $\hat{S}(\eta, t) = O(1) \cdot \frac{e^{-t}}{(|\eta|+1)^5}$

i.e. $S^2(x,t) = O(1) \cdot e^{-(t+|x|)/K}$.

$$\Rightarrow \int_{\mathbb{R}} e^{\varepsilon(x-\frac{1}{8}t)} \cdot [V_t^2 + V_x^2] dx \Big|_{t=T} \le O(1) \cdot e^{-\frac{\varepsilon T}{8}}$$

Class II:

Task 4: By Longwave – Shortwave Decomposition:

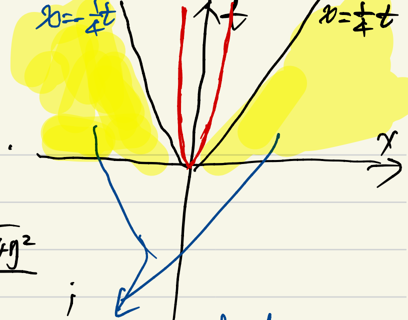
$$u_L(x,t) = \int_{|\eta|<\varepsilon} e^{i\eta x} \cdot \hat{u}(\eta,t)\, d\eta$$

$$u_S(x,t) = \int_{|\eta|\ge\varepsilon} \cdot e^{i\eta x} \cdot \hat{u}(\eta,t)\, d\eta;$$

$\mathrm{Re}(\beta_1(\eta)) < -\varepsilon_0 \quad \Rightarrow \quad \|u_S\|_{L^2} \le O(1) \cdot e^{-\varepsilon_0 t}$

$\Rightarrow \|u_S\|_\infty \le O(1) \cdot e^{-\varepsilon_0 t}.$

Aim: Want to know what happens in the cone.



Recall,

$$\hat{u}(\eta,t)=\frac{1}{\sqrt{1-4\eta^2}}\left(e^{\xi_+ t}-e^{\xi_- t}\right),\quad \xi_\pm=\frac{-1\pm\sqrt{1-4\eta^2}}{2};$$

η :~ wave # (ossilation)

exponential decays w.r.t. (t,x)

Long wave - short wave decomposition:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\eta x}\hat{u}(\eta,t)\,d\eta,$$

- $|\eta|>>1$ — short wave (wave length small)
- $|\eta|<<1$ — Long wave (wave length big)

$$u(x,t)=\frac{1}{2\pi}\Big(\underbrace{\int_{|\eta|<\varepsilon}}_{\text{"Long wave component"}}+\underbrace{\int_{|\eta|>\varepsilon}}_{\text{"short wave component"}}\Big)\cdot e^{i\eta x}\cdot\hat{u}(\eta,t)\,d\eta$$

$$=u_L(x,t)+u_S(x,t).$$

For $|\eta|>\varepsilon$, $\mathrm{Re}(\xi_-)<-\frac{1}{2}$ $\Rightarrow$ $|e^{\xi_-(\eta)t}|<e^{-\frac{1}{2}t}$

$$\mathrm{Re}(-1+\sqrt{1-4\eta^2})=\mathrm{Re}\left(\frac{1-(1-4\eta^2)}{-1-\sqrt{1-4\eta^2}}\right)$$

$$=\mathrm{Re}\left(-\frac{4\eta^2}{1+\sqrt{1-4\eta^2}}\right)\le-\frac{4\varepsilon^2}{2}$$

$$\Rightarrow\quad |e^{-\xi_+(\eta)t}|\le O(1)\,e^{-2\varepsilon^2 t}$$

$$|u_S(x,t)|=\left|\frac{1}{2\pi}\int_{|\eta|>\varepsilon}e^{i\eta x}\cdot\frac{1}{\sqrt{1-4\eta^2}}\left(e^{\xi_+ t}-e^{\xi t}\right)d\eta\right|$$

Recall

$\|u_s(\cdot,t)\|_{L^2}^2$: $f\in L^2(\Omega) \Rightarrow \|\hat{f}\|_{L^2} = \|f\|_{L^2}$

$$\int |\hat{f}|^2 = \int |f|^2$$

Thus, $\|u_s(\cdot,t)\|_{L^2}^2 = \int_{|\eta|>\varepsilon} \frac{1}{|1-4\eta^2|} \cdot |e^{\xi_+ t} - e^{\xi t}|^2 \, d\eta$

$$\le O(1)\cdot e^{-\varepsilon^2 t/2}$$

$\|u_s^* - u_s\|_\infty < O(1)\cdot e^{-\varepsilon^2 t/2}$ for all x.

To find $u_L(x,t)$ with $x < \frac{1}{2}t$

$$u_L(x,t) = \frac{1}{2\pi}\int_{|\eta|<\varepsilon} e^{i\eta x} \cdot \frac{1}{\sqrt{1-4\eta^2}} \cdot (e^{\xi_+(\eta)t} - e^{\xi(\eta)t})\, d\eta$$

Analytic & Cauchy formula

$$= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x}}{\sqrt{1-4\eta^2}} (e^{\xi_+ t} - e^{\xi_- t})\, d\eta$$

$\xi_\pm(\eta) = \frac{-1\pm\sqrt{1-4\eta^2}}{2} = \frac{-1\pm(1-2\eta^2)+Q(\eta^4)}{2}$; Q : Analytic around 0

$$\begin{cases} \xi_- = -1+\eta^2 - Q(\eta^4) \Rightarrow \mathrm{Re}(\xi_-(\eta)) \le -\frac{1}{2} \text{ for } |\eta|<\varepsilon \\ \xi_+ = -\eta^2 + O(\eta^4) \text{ — Problem for } |\eta| \text{ small enough} \end{cases}$$

Focus on ξ_+:

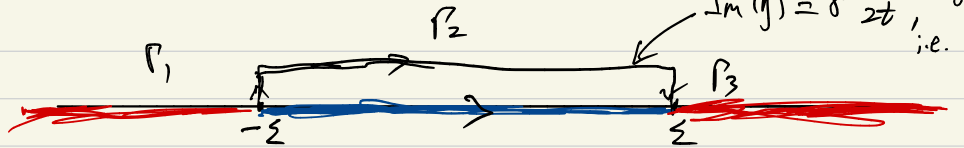
$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x + \xi_+(\eta)t}}{\sqrt{1-4\eta^2}}\, d\eta$$

$$= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{e^{i\eta x + (-\eta^2 + O(\eta^4))t + \frac{x^2}{4t} - \frac{x^2}{4t}}}{\sqrt{1-4\eta^2}}\, d\eta$$

$$= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{1-4\eta^2}} \cdot e^{-\frac{x^2}{4t} - t\left(\eta - \frac{ix}{2t}\right)^2 + O(\eta^4)t}\, d\eta$$

How to balance the term?

$\mathrm{Im}(\eta) = \gamma\frac{x}{2t}$, γ small. i.e. $0<\gamma<<1$.



$$\frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\eta-\frac{ix}{2t})^2 + O(\eta^4)t}\, d\eta, \quad \eta = \nu + i\cdot\gamma\cdot\frac{x}{2t},$$

$\gamma\in(-\varepsilon,\varepsilon)$

$$= \frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\nu + i\frac{\gamma x}{2t} - \frac{ix}{2t})^2 + O((\nu+i\gamma\frac{x}{2t})^4)t}\, d\nu$$

$$= \frac{1}{2\pi}\int_{\Gamma_2} e^{-\frac{x^2}{4t} - t(\nu + (1-\gamma)i\frac{x}{2t})^2 + O((\nu+i\gamma\frac{x}{2t})^4)t}\, d\nu$$

$$-\frac{x^2}{4t} - t[\nu - i(1-\gamma)\frac{x}{2t}]^2 = -(1-(1-\gamma)^2)\frac{x^2}{4t} - t\nu^2 - \{t\nu(1-\gamma)\frac{x}{2t}\}.$$

$$= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-(1-(1-\gamma)^2)\frac{x^2}{4t}} \cdot e^{-t\nu^2 - \gamma t\nu(1-\gamma)\frac{x}{2t} + tO((\nu+\frac{ix}{2t}\gamma)^4)}\, d\nu$$

$$-\frac{t\nu^2}{2} - \frac{t\nu^2}{2}$$

$$= \frac{1}{2\pi}\cdot e^{-(1-(1-\gamma)^2)\frac{x^2}{4t}} \cdot \int_{-\varepsilon}^{\varepsilon} \underbrace{e^{-t\nu^2 - \gamma t\nu(1-\gamma)\frac{x}{2t} + tO((\nu+\frac{ix}{2t}\gamma)^4)}}\, d\nu$$

$$= \frac{e^{-[1-(1-\gamma)^2]\frac{x^2}{8t} - [1-(1-\gamma)^2]\frac{x^2}{8t}}}{2\pi}\cdot\int_{-\varepsilon}^{\varepsilon} (\qquad)\, d\nu$$

$$\mathrm{Re}\left(-[1-(1-\gamma)^2]\frac{x^2}{8t} - \frac{t\nu^2}{2} + O((\nu+\frac{ix}{2t}\gamma)^4)t\right) < 0$$

if $\nu\in(-\varepsilon,\varepsilon]$, $\gamma << 1$

$$\le \frac{e^{-[1-(1-\gamma)^2]\frac{x^2}{8t}}}{2\pi}\cdot\int_{-\varepsilon}^{\varepsilon} e^{-\frac{t\nu^2}{2}}\, d\nu, \qquad d\nu = \frac{d\,\nu\sqrt{t}}{\sqrt{t}}.$$

$$\le O(1)\cdot\frac{e^{-[-(1-\gamma)^2]\frac{x^2}{8t}}}{\sqrt{t}}$$

1.
$$\begin{cases} U_{tt} - U_{xx} + U_t = 0 \quad , \quad x \in R \\ U(x,0) = 0 \\ U_t(x,0) = \delta(x - x_0) \quad , \quad x_0 > 0 \end{cases}$$

Take Laplace Transform:

$$s^2 \mathcal{L} u - \delta(x - x_0) - \partial_x^2 \mathcal{L} u + s \mathcal{L} u = 0. \qquad (x \in R)$$

$$\Rightarrow \quad (s^2 + s) \cdot \mathcal{L} u - \partial_x^2 \mathcal{L} u = \delta(x - x_0)$$

Then, $\mathcal{L} u = \begin{cases} A \cdot e^{-\sqrt{s^2+s}\,(x - x_0)} & \text{if } x > x_0 \\ A \cdot e^{\sqrt{s^2+s}\,(x - x_0)} & \text{if } x < x_0 \end{cases}$

$$\mathcal{L}_x u = \begin{cases} -\sqrt{s^2+s} \cdot A \cdot e^{-\sqrt{s^2+s}\,(x - x_0)}, & \text{if } x > x_0 \\ \sqrt{s^2+s} \cdot A \cdot e^{\sqrt{s^2+s}\,(x - x_0)}, & \text{if } x < x_0 \end{cases}$$

$$\Rightarrow \quad A = \frac{1}{2\sqrt{s^2+s}}$$

$$\Rightarrow \quad \mathcal{L} u = \frac{1}{2\sqrt{s^2+s}} \cdot e^{-\sqrt{s^2+s}\,|x - x_0|}$$

2.
$$\begin{cases} U_{tt} - U_{xx} + U_t = 0, \quad x > 0 \\ U(x,0) = 0, \quad U_t(x,0) = \delta(x - x_0), \quad x_0 > 0 \\ U(0,t) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} s^2 \mathcal{L} V - \delta(x - x_0) - \partial_x^2 \mathcal{L} V + s \mathcal{L} V = 0 & (x > 0) \\ \mathcal{L} V(0, s) = 0 \end{cases}$$

$$\Rightarrow \quad s^2\mathcal{L}u - \delta(x-x_0) - \partial_x^2\mathcal{L}u + s\mathcal{L}u = 0$$

Then, $s^2\mathcal{L}(v-u) - \partial_x^2\mathcal{L}(v-u) + s\mathcal{L}(v-u) = 0$

$\{ \mathcal{L}(v-u)(0,s) = -\mathcal{L}u(0,s)$

$$\Rightarrow \mathcal{L}(v-u)\ (x,s) = -\mathcal{L}u(0,s)\cdot e^{-\sqrt{s^2+s}\,x}$$

$$\mathcal{L}u = \frac{1}{2\sqrt{s^2+s}}\cdot e^{-\sqrt{s^2+s}\,|x-x_0|}$$

$$\Rightarrow \mathcal{L}v = \frac{1}{2\sqrt{s^2+s}}\cdot e^{-\sqrt{s^2+s}\,|x-x_0|} - \frac{1}{2\sqrt{s^2+s}}\cdot e^{-\sqrt{s^2+s}\,(x_0+x)}$$

3. $$\begin{cases} v_{tt} - v_{xx} + v_t = 0, & x>0 \\ v(x,0)=0,\ v_t(x,0) = \delta(x-x_0), & x_0>0 \\ v(0,t) + v_x(0,t) = 0 \end{cases} \Rightarrow (\mathcal{L}v + \partial_x\mathcal{L}v)(0,s) = 0$$

$$\Rightarrow \begin{cases} s^2\mathcal{L}v - \delta(x-x_0) - \partial_x^2\mathcal{L}v + s\mathcal{L}v = 0 & (x>0) \\ (\mathcal{L}v + \partial_x\mathcal{L}v)(0,s) = 0 \end{cases}$$

$$\Rightarrow \quad s^2\mathcal{L}u - \delta(x-x_0) - \partial_x^2\mathcal{L}u + s\mathcal{L}u = 0$$

$$\mathcal{L}(v-u)\ (x,s) = Q(s)\cdot e^{-\sqrt{s^2+s}\,x}$$

$$(\mathcal{L} + \partial_x\mathcal{L})(v-u)\ (0,s) = -(1+\partial_x)\mathcal{L}u(0,s)$$

$$\Rightarrow (1-\sqrt{s^2+s})\,Q = (1+\sqrt{s^2+s})\cdot\frac{e^{-\sqrt{s^2+s}\cdot x_0}}{2\sqrt{s+s^2}}$$

$$\Rightarrow \quad Q = -\left(\frac{1+\sqrt{s^2+s}}{1-\sqrt{s^2+s}}\right)\cdot e^{-\sqrt{s^2+s}\cdot x_0}$$

$$\Rightarrow \mathcal{L}V = \mathcal{L}u + \frac{1}{s^2+s-1}\cdot \underbrace{(1+\sqrt{s^2+s})}_{(1-\partial x)}{}^2 \cdot \frac{e^{-\sqrt{s^2+s}\,(x+x_0)}}{2\cdot\sqrt{s+s^2}}$$

$$V = u + \mathcal{L}^{-1}\left(\frac{1}{s^2+s-1}\right) * (1-\partial_x)^2\, u(x-2x_0, t).$$

High dimension problem:

$$u_{tt} - \Delta u = 0 \quad , \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad , \quad (x, y, z) \in \mathbb{R}^3$$

$$\vec{\eta} = (\eta_1, \eta_2, \eta_3)$$

$$\hat{u}(\vec{\eta}, t) = \iiint_{\mathbb{R}^3} e^{-\vec{\eta}\cdot\vec{x}\, i}\, u(\vec{x}, t)\, d\vec{x} \longleftarrow \text{Fourier transform.}$$

then, $\hat{u}_{tt} - \iiint_{\mathbb{R}^3} e^{-i\vec{\eta}\cdot\vec{x}} \cdot (\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot u(\vec{x}, t)\, d\vec{x} = 0$

$$\xRightarrow{\text{I.B.P}}$$

$$\hat{u}_{tt} - \iiint_{\mathbb{R}^3} [(\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot e^{-i\vec{\eta}\cdot\vec{x}}] \cdot u(\vec{x}, t)\, d\vec{x} = 0$$

$$\hat{u}_{tt} + |\vec{\eta}|^2 \hat{u} = 0 \quad \longleftarrow \text{By computation.}$$

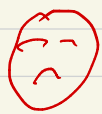
$$\Rightarrow \hat{u}(\vec{\eta}, t) = \frac{1}{2}(e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t})\, \hat{u}(\eta, 0) + \frac{e^{i|\vec{\eta}|t} - e^{-i|\vec{\eta}|t}}{2i|\vec{\eta}|} \cdot \hat{u}_t(\vec{\eta}, 0).$$

Inverse Fourier Transform,

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{(e^{i|\vec{\eta}|t} + e^{-i|\vec{\eta}|t})}{2} \cdot e^{i\vec{\eta}\cdot\vec{x}}\, d\vec{\eta} \longleftarrow$$

Q: ? Can't be handled.

Fourier Transform Method fails !!!



Possion mean:

For 1-D:

$$W_{tt} - W_{xx} = 0, \quad x \in \mathbb{R}.$$

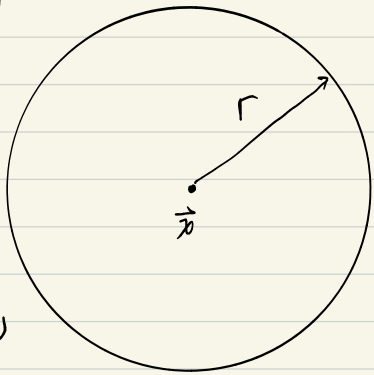
$$\Rightarrow W(x, t) = \frac{1}{2}[W(x+t, 0) + W(x-t, 0)] + \frac{1}{2}\int_{x-t}^{x+t} W_t(x, 0)\, dx.$$

by Taylor' expansion.

$$\begin{cases} u_{tt} - \Delta u = 0, & \Delta = \nabla \cdot \nabla \\ u(\vec{x}, t): \end{cases}$$

Define $\bar{u}(\vec{x}, t; r) = \frac{1}{4\pi r^2} \cdot \int_{\partial B_r(\vec{x})} u(\vec{y}, t)\, dA_y$

Now, $\int_{B_r(\vec{x})} u_{tt}(\vec{y}, t)\, d\vec{y} = \int_{B_r(\vec{x})} \nabla \cdot \nabla u(\vec{y}, t)\, d\vec{y}$



$$\overset{\text{Divergence thm}}{=} \int_{\partial B_r(\vec{x})} \nabla u(\vec{y}, t) \cdot \vec{n}\, ds$$

$$= \int_{\partial B_r(\vec{x})} \frac{\partial u}{\partial r}\, ds, \qquad dA = r^2 \cdot d\underset{\partial B_1(\vec{x})}{\underset{\shortparallel}{S_1}}$$

$$= \int_{B_1(\vec{x})} \frac{\partial u}{\partial r} \cdot r^2\, dS_1$$

$$= \underline{r^2 \cdot 4\pi \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r)}$$

$$\iiint_{B_r(\vec{x})} u_{tt}(\vec{y}, t)\, d\vec{y} = \int_0^r \iint_{|w|=1} u_{tt}(\rho\vec{\omega} + \vec{x}, t) \cdot \rho^2\, d\Omega\, d\rho$$

$$= \int_0^r 4\pi \rho^2\, \bar{u}_{tt}(\vec{x}, t; \rho)\, d\rho$$

Now, we have:

$$\underline{r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r) = \int_0^r 4\pi \rho^2\, \bar{u}_{tt}(\vec{x}, t; \rho)\, d\rho}$$

Take derivative w.r.t. r,

$$\frac{\partial}{\partial r}\left(r^2 \cdot 4\pi \cdot \frac{\partial}{\partial r} \bar{u}(\vec{x}, t; r)\right)$$

$$= 4\pi r^2\, \bar{u}_{tt}(\vec{x}, t; r)$$

Simplified:

$$2r \cdot \frac{\partial \bar{u}}{\partial r} + r^2 \cdot \frac{\partial^2 \bar{u}}{\partial r^2} = r^2\, \bar{u}_{tt}$$

$$\Rightarrow \bar{u}_{tt} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \bar{u}}{\partial r} \qquad \leftarrow (*)$$

Then, $(*) \cdot r$

$$\Rightarrow \boxed{(r \cdot \bar{u})_{tt} = (r\bar{u})_{rr}} \qquad \text{with } r > 0.$$

Next, by odd extension for $r \cdot \bar{u} = 0$ at $r = 0$

$$\Rightarrow (r\bar{u}) = \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\, \bar{u}(\bar{x}, 0; r-t)}{2} + \frac{1}{2}\int_{r-t}^{r+t} \rho\, \bar{u}_t(x, 0; \rho)\, d\rho$$

Simplified:

$$\bar{u}(\bar{x}, t; r) = \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\, \bar{u}(\bar{x}, 0; r-t)}{2r} + \frac{1}{2r}\int_{r-t}^{r+t} \rho\, \bar{u}_t(x, 0; \rho)\, d\rho$$

Recall, for contin. solution, $\lim_{r \to 0} \bar{u}(\bar{x}, t; r) = u(\bar{x}, t)$ by Lebegues Thm.

Then, $\lim_{r \to 0} \bar{u}(\vec{x}, t; r)$

$$= \lim_{r \to 0} \frac{(r+t) \cdot \bar{u}(\bar{x}, 0; r+t) + (r-t)\, \bar{u}(\bar{x}, 0; r-t)}{2 \cdot r} \leftarrow (\text{II})$$

$$+ \underbrace{\lim_{r \to 0} \frac{1}{2r}\int_{r-t}^{r+t} \rho\, \bar{u}_t(x, 0; \rho)\, d\rho}_{(\text{I})}$$

Consider (I):

$$\frac{1}{2r}\left(\int_0^{r+t}\rho\,\vec{u}_t(x,0;\rho)\,d\rho+\underline{\int_{r-t}^{0}\rho\,\vec{u}_t(x,0;\rho)\,d\rho}\right)$$

$$\Big\|$$

$$-\int_0^{t-r}(-)\rho\,\vec{u}_t(x,0;\rho)(-1)\,d\rho \quad \text{for } \vec{u} \text{ is odd extention.}$$

$$=\frac{1}{2r}\int_{t-r}^{t+r}\rho\,\vec{u}_t(x,0;\rho)\,d\rho$$

Take lim of r:

$$\lim_{r\to 0}\frac{1}{2r}\int_{t-r}^{t+r}\rho\cdot\vec{u}_t(x,0;\rho)\,d\rho=t\cdot\vec{u}_t(x,0;t)$$

$$=t\cdot\int_{\partial B_t(x)}u_t\,ds\Big/4\pi t^2$$

$$=\frac{1}{4\pi t}\cdot\int_{\partial B_t(x)}u_t(\ ,0;t)\,ds.$$

Conside (II) :

Take lim of r,

$$-(-r+t)\cdot\vec{u}(\vec{x},0;-r+t)$$

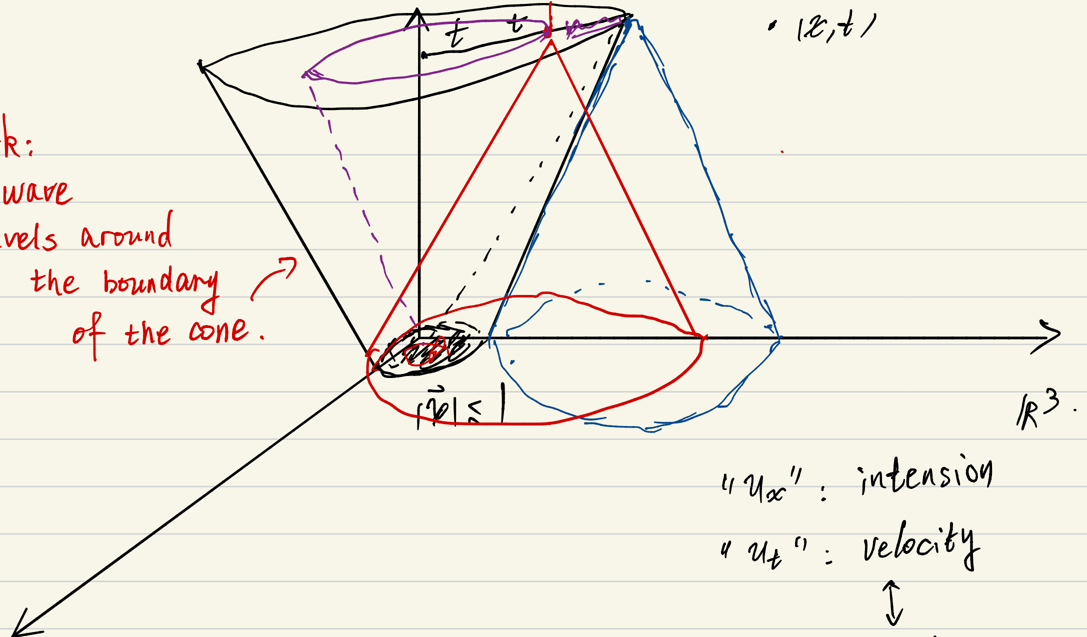
$$\lim_{r\to 0}\frac{(r+t)\cdot\vec{u}(\vec{x},0;r+t)+(r-t)\vec{u}(\vec{x},0;r-t)}{2r}$$

$$=\frac{\partial}{\partial t}t\,\vec{u}(x,0;t)=\vec{u}(x,0;t)+t\cdot\vec{u}_t(x,0;t)$$

$$\underline{u(\vec{x},t)=\frac{1}{4\pi t}\cdot\int_{\partial B_t(x)}u_t(\ ,0;t)\,ds+}$$

$$\underline{\frac{\partial}{\partial t}\cdot\frac{1}{4\pi t}\int_{\partial B_t(x)}u(\ ,0;t)\,ds}$$

<u>Kirchhoff's formular.</u>



Remark:
- The wave travels around the boundary of the cone.

"u_x": intension

"u_t": velocity $\updownarrow$ kinetic energy

Q: What about wave eq. in 2-dimal. space?

A: The method doesn't work.

By same Poisson mean,

$$\Rightarrow \underline{\frac{1}{r}\left(\frac{d\bar{u}}{dr} + r\cdot\frac{d^2\bar{u}}{dr^2}\right) = \bar{u}_{tt}}$$

This equation can't reduce to be a wave equation in 1-D.!

Another way to solve it: (See 2-diml. to be embeded in 3-diml.)

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x, y \in \mathbb{R},\ t > 0 \\ u(x,y,0) = u_0(x,y), & u_t(x,y,0) = u_1(x,y) \end{cases}$$

Define $\bar{U}(x,y,z,t) = u(x,y,t)$,

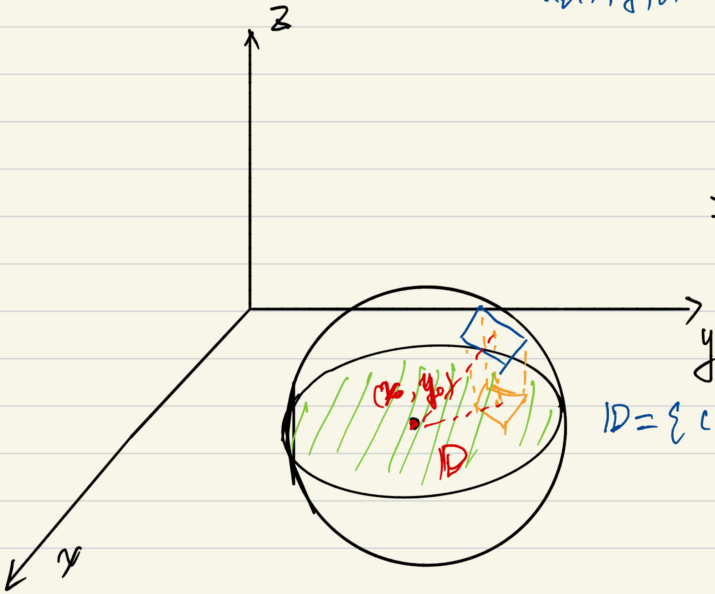
then $\bar{U}_{tt}(x,y,z,t) = u_{tt}(x,y,z)$

$$\bar{U}_{xx} + \bar{U}_{yy} + \bar{U}_{zz} = u_{xx} + u_{yy} + 0$$

Then,

$$\underbrace{\bar{U}(\vec{x}, t)}_{u(x,y,t)} = \frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \underbrace{\bar{U}_t(\vec{y}, 0)}_{u_t(x,y,0)}\, dS_{\vec{y}} + \frac{\partial}{\partial t}\frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \bar{U}(\vec{y}, 0)\, dS_{\vec{y}}$$

$\vec{x} = (x, y, z)$



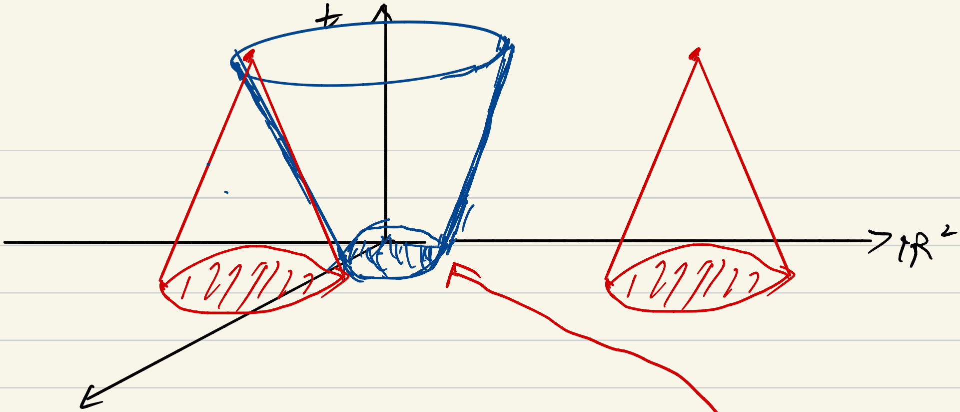
$$\frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y}, 0)\, dS_{\vec{y}} = \frac{2}{4\pi t}\int_{D} u_t(x, y, 0)\cdot \frac{t}{\sqrt{t^2 - (x-x_1)^2 - (y-y_1)^2}}$$

$D = \{ (x_1, y_1) \mid |(x, y) - (x_1, y_1)| \le t \}$

$$\frac{1}{4\pi t}\int_{\partial B_t(\vec{x})} \bar{U}_t(\vec{y}, 0)\, dS_{\vec{y}} = \frac{1}{2\pi}\iint_{\sqrt{(x_1-x)^2 + (y_1-y)^2} \le t} \frac{u_1(x_1, y_1)}{\sqrt{t^2 - (x-x_1)^2 - (y-y_1)^2}}\, dx_1 dy_1$$

Take $\vec{x} = (x, y)$

$$u(\vec{x}, t) = \frac{1}{2\pi}\iint_{|\vec{y}-\vec{x}| \le t} \frac{u_1(\vec{y})}{\sqrt{t^2 - |\vec{x}-\vec{y}|^2}}\, d\vec{y} + \frac{\partial}{\partial t}\frac{1}{2\pi}\iint_{|\vec{y}-\vec{x}| \le t} \frac{u_0(\vec{y})}{\sqrt{t^2 - |\vec{x}-\vec{y}|^2}}\, d\vec{y}$$



Remark: • The wave travels inside of the cone!

Big difference with 3-dim.

Now, define

$$\mathcal{Y}(u_1) = \frac{1}{2\pi}\iint_{|\vec{x}-\vec{y}|\le t} \frac{u_1(\vec{y})}{\sqrt{t^2-(\vec{x}-\vec{y})^2}}\,d\vec{y}$$

$$\mathcal{Y}(u_1) \equiv \text{solution of } \begin{cases} u_{tt} - u_{xx} - u_{yy} = 0 \\ u(x,y,0) = 0 \\ u_t(x,y,0) = u_1 \end{cases}$$

By Fourier Transform.

$$\Rightarrow \hat{u}(\vec{\eta},t) = \frac{e^{i|\vec{\eta}|t} - e^{-i|\vec{\eta}|t}}{2i|\vec{\eta}|}\hat{u}_1(\vec{\eta}) = \frac{\sin|\vec{\eta}|t}{|\vec{\eta}|}\hat{u}_1(\vec{\eta})$$

$$\Updownarrow \leftarrow \text{Must be equivalent}$$

$$\mathcal{Y}(u_1)$$

Fermat Last Thm:

$x^n + y^n = z^n$, $n > 2$

Half Space:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x>0,\ y\in\mathbb{R},\ t>0 \\ u(0,y,t) = 0 \\ \cancel{u(x,y,0) = u_0(x,y) = 0} \\ u_t(x,y,0) = u_1(x,y) = \delta(x-x_*)\,\delta(y) \end{cases}$$

Laplace – Fourier Transform.

Define

$$\mathcal{L}u(x,\eta,s) = \int_0^\infty \int_{\mathbb{R}} e^{-st-i\eta y}\, u(x,y,t)\, dy\, dt$$

$$\Rightarrow \quad s^2 \mathcal{L}u - \hat{u}_1(x,\eta) - \partial_x^2 \mathcal{L}u + \eta^2 \mathcal{L}u = 0$$

$$\mathcal{L}u(0,\eta,s) = 0$$

$$\Rightarrow \begin{cases} (s^2+\eta^2)\mathcal{L}u - \partial_x^2 \mathcal{L}u = \hat{u}_1 = \delta(x-x_*) & (*) \\ \mathcal{L}u(0,\eta,s) = 0 \end{cases}$$

We already have known:

$$\begin{cases} w_{tt} - w_{xx} - w_{yy} = 0, & x,y\in\mathbb{R} \\ w(x,y,0) = 0 \\ w_t(x,y,0) = \delta(x)\cdot\delta(y) \end{cases}$$

$$\Rightarrow \quad (s^2+\eta^2)\mathcal{L}w - \partial_x^2 \mathcal{L}w = \delta(x)$$

$$\mathcal{L}w = \frac{e^{-\sqrt{s^2+\eta^2}\,|x|}}{2\sqrt{s^2+\eta^2}}$$

Then, by $(*)$, we get:

$$\mathcal{L}w = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_*|}}{2\cdot\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

$$\Rightarrow A = -\frac{e^{-\sqrt{s^2+\eta^2}\cdot|x_*|}}{2\cdot\sqrt{s^2+\eta^2}}$$

$$\Rightarrow \mathcal{L}u = \frac{e^{-\sqrt{s^2+\eta^2}\cdot|x-x_*|}}{2\sqrt{s^2+\eta^2}} - \frac{e^{-\sqrt{s^2+\eta^2}\cdot(x+x_*)}}{2\cdot\sqrt{s^2+\eta^2}}, \quad x_* > 0$$

!

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x>0,\ y\in\mathbb{R} \\ u(x,y,0) = 0, & u_t(x,y,0) = \delta(x-x_0)\,\delta(y) \\ u(0,y,t) - k\, u_x(0,y,t) = 0. \end{cases}$$

Transform in (t, y).

Take Laplace – Fourier Transform,

$$\begin{cases} s^2 \mathcal{L}u - \partial_x^2 \mathcal{L}u + \eta^2 \mathcal{L}u = \delta(x-x_0) \\ \mathcal{L}u(0,\eta,s) - k\,\partial_x \mathcal{L}u(0,\eta,s) = 0 \quad \text{— B.C.} \end{cases}$$

$$\Rightarrow \mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

Use B.C.

$$\Rightarrow \frac{e^{-\sqrt{s^2+\eta^2}\,|x_0|}}{2\sqrt{s^2+\eta^2}} + A - k\left(\frac{e^{\sqrt{s^2+\eta^2}\,(x-x_0)}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+t^2}\,x}\right)_x\Bigg|_{x=0} = 0$$

$$\Downarrow$$

$$\sqrt{s^2+\eta^2}\left(\frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}} - A\right)$$

$$\Rightarrow (1+k\sqrt{s^2+\eta^2})\,A + (1-k\sqrt{s^2+\eta^2})\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}} = 0$$

$$\Rightarrow A = \frac{-e^{-\sqrt{s^2+\eta^2}\,x_0}\,(1-k\sqrt{s^2+\eta^2})}{2\sqrt{s^2+\eta^2}\cdot(1+k\sqrt{s^2+\eta^2})}$$

$$\Rightarrow \mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1-k\sqrt{s^2+\eta^2})}{(1+k\sqrt{s^2+\eta^2})}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,|x+x_0|}}{2\sqrt{s^2+\eta^2}}$$

$\times\ (1-k\sqrt{s^2+\eta^2})$

$$\Downarrow$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1-k\sqrt{s^2+\eta^2})^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(1+\cancel{dx})^2}{[1-k^2(s^2+\eta^2)]} \cdot \frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

Inverse Laplace Transform:

$f(t)$: $\quad F(s) = \int_0^{\infty} e^{-st} \cdot f(t)\, dt, \quad Re(s) \geq 0, \quad s \in \mathbb{C}$

$$f(t) = \frac{1}{2\pi i} \int_{Re(s)=0} e^{st} \cdot F(s)\, ds$$

$$\mathcal{L}^{-1}\left[\frac{1}{(1-k^2(s^2+\eta^2))}\right] = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty}\int_{Re(s)=0} e^{i\eta y+st} \cdot \frac{1}{[1-k^2(s^2+\eta^2)]}\, ds\, d\eta$$

Another boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, \quad x>0,\ y\in\mathbb{R} \\ u(x,y,0) = 0, \quad u_t(x,y,0) = \delta(x-x_0)\,\delta(y) \\ u_t(0,y,t) - k\, u_x(0,y,t) = 0. \end{cases} \qquad \text{(B.C.)}$$

B.C. $\Rightarrow \quad s\,\mathcal{L}u = k\,\partial_x \mathcal{L}u.$

$$s\,\mathcal{L}u(x,\eta,s) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} + A\cdot e^{-\sqrt{s^2+\eta^2}\,x}$$

By the process above, change "1" to "s":

$$\Rightarrow \quad (s + k\sqrt{s^2+\eta^2})\,A + \frac{e^{-\sqrt{s^2+\eta^2}\,x_0}}{2\sqrt{s^2+\eta^2}}\,(s - k\sqrt{s^2+\eta^2}) = 0$$

$$\mathcal{L}u(x,y,t) = \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{(s-k\sqrt{s^2+\eta^2})^2}{[s^2-k^2(s^2+\eta^2)]}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$\Downarrow$$

$$\frac{(s^2 - 2k\sqrt{s^2+\eta^2}\,s + k^2(s^2+\eta^2))}{[s^2-k^2(s^2+\eta^2)]}$$

$$\Downarrow$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\sqrt{s^2+\eta^2}} - \frac{[s^2+2ks\,\partial_x + k^2\partial_x^2]}{(s^2-k^2(s^2+\eta^2))}\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\sqrt{s^2+\eta^2}}$$

$$= \frac{e^{-\sqrt{s^2+\eta^2}\,|x-x_0|}}{2\cdot\sqrt{s^2+\eta^2}} - [s^2+2ks\,\partial_x + k^2\partial_x^2]\cdot\frac{e^{-\sqrt{s^2+\eta^2}\,(x+x_0)}}{2\cdot\sqrt{s^2+\eta^2}}\cdot\frac{1}{(1-k^2)s^2-k^2\eta^2}$$

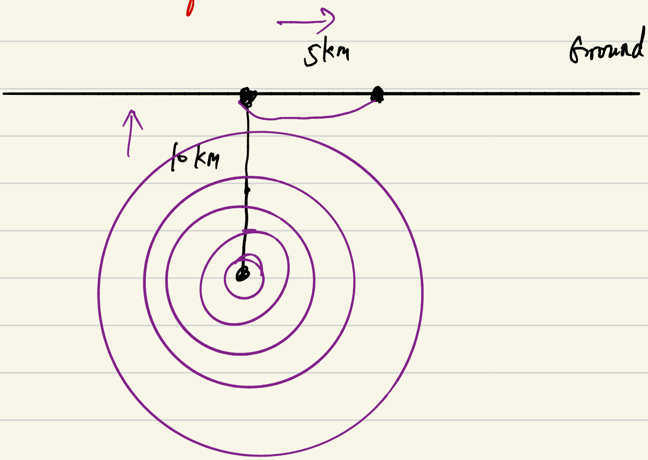
$$\rightsquigarrow -\frac{1}{k^2}\mathcal{L}\left[\left(\frac{k^2-1}{k^2}\partial_t^2-\partial_y^2\right)^{-1}\right]$$

$$\frac{-1/k^2}{-[(1-k^2)s^2-k^2\eta^2]/k^2} = \frac{-1/k^2}{\frac{k^2-1}{k^2}s^2+\eta^2}$$

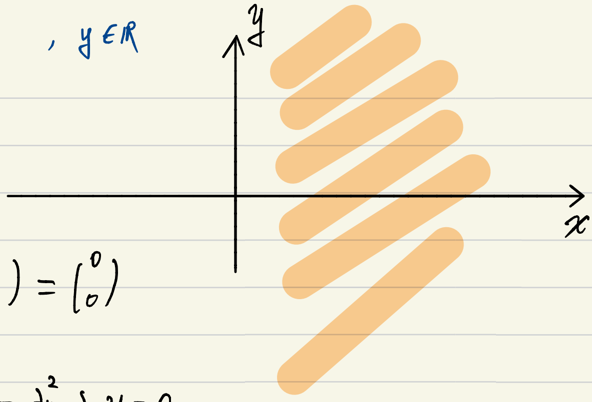
Remark: ① speed: $\frac{k^2}{k^2-1}$

② If k small, then no wave eq.

Physical meaning:



$$\begin{cases} u_t - u_x + v_y = u_{xx} + u_{yy}, & x>0 \\ v_t + v_x + u_y = v_{xx} + v_{yy}, & y \in \mathbb{R} \end{cases}$$



i.e. $\begin{cases} u_t - u_x + v_y = \Delta u \\ v_t + v_x + u_y = \Delta v \end{cases}$

$$\begin{pmatrix} \partial_t - \partial_x - \Delta & \partial_y \\ \partial_y & \partial_t + \partial_x - \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} ((\partial_t - \partial_x - \Delta)\cdot(\partial_t + \partial_x - \Delta) - \partial_y^2)\, u = 0 \\ ((\partial_t - \partial_x - \Delta)\cdot(\partial_t + \partial_x - \Delta) - \partial_y^2)\, v = 0 \end{cases}$$

$$\Rightarrow \left[(\partial_t - \Delta)^2 - \partial_x^2 - \partial_y^2\right] u = 0 \quad \&$$

$$[(\partial_t - \Delta)^2 - \Delta]\cdot u = 0$$

Suppose $[(\partial_t - \Delta)^2 - \Delta]\, u = 0$ is an Equation on $\mathbb{R}^2 \times \mathbb{R}_+$

$\hat{u}(\vec{\eta}, t)$: Fourier transform of $u(x, y, t)$, $\vec{\eta} \in \mathbb{R}^2$

$$\Rightarrow [(\partial_t + |\vec{\eta}|^2)^2 + |\vec{\eta}|^2]\, \hat{u} = 0$$

$$\Rightarrow (\partial_t + |\vec{\eta}|^2 - i|\vec{\eta}|)(\partial_t + |\vec{\eta}|^2 + i\cdot|\vec{\eta}|)\, \hat{u} = 0$$

Then, $\hat{u}(\vec{\eta}, t) = A\cdot e^{(-|\vec{\eta}|^2 + i|\vec{\eta}|)t} + B\cdot e^{-|\vec{\eta}|^2 t - i|\vec{\eta}|t}$

$$= e^{-|\vec{\eta}|^2 t}\cdot(A\cdot e^{i|\vec{\eta}|t} + B\cdot e^{-i|\vec{\eta}|t})$$

$$\overset{?}{=} e^{-|\vec{\eta}|^2 t}\cdot\left(A\cdot\frac{\sin|\vec{\eta}|t}{|\vec{\eta}|} + B\cdot\cos|\vec{\eta}|t\right)$$

$A = \hat{u}_t(\vec{\eta}, 0)$

$B = \hat{u}(\vec{\eta}, 0)$

Fourier - Laplace transform:

$$\mathcal{L}\begin{bmatrix} \partial_t - \partial_x - \Delta & \partial_y \\ \partial_y & \partial_t + \partial_x - \Delta \end{bmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \hat{u}(x,\eta,0) \\ \hat{v}(x,\eta,0) \end{bmatrix}$$

Suppose $\begin{bmatrix} u(x,y,0) \\ v(x,y,0) \end{bmatrix} = \begin{bmatrix} \delta(x)\,\delta(y) \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \delta(x)\cdot\delta(y) \end{bmatrix}$.

If $\begin{bmatrix} u(x,y,0) \\ v(x,y,0) \end{bmatrix} = \begin{bmatrix} \delta(x)\,\delta(y) \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \mathcal{L}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \delta(x) \\ 0 \end{bmatrix}$$

Consider

$$\begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} \Psi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x > 0$$

Assume $\Psi = e^{\lambda x} \cdot \vec{V}_0$ is a solution

$$\Rightarrow \begin{bmatrix} s - \partial_x - \partial_x^2 + \eta^2 & i\eta \\ i\eta & s + \partial_x - \partial_x^2 + \eta^2 \end{bmatrix} e^{\lambda x} \cdot \vec{V}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s - \lambda - \lambda^2 + \eta^2 & i\eta \\ i\eta & s + \lambda - \lambda^2 + \eta^2 \end{bmatrix} \vec{V}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (s+\eta^2-\lambda^2)^2-\lambda^2+\eta^2=0$$

$$V_0=\begin{bmatrix}-i\eta\\ s-\lambda-\lambda^2+\eta^2\end{bmatrix}$$

$$\alpha=\lambda^2 \qquad \Rightarrow \quad (s+\eta^2-\alpha)^2-\alpha+\eta^2=0$$

$$\Rightarrow \quad \alpha^2-2(s+\eta^2)\alpha-\alpha+(s+\eta^2)^2+\eta^2=0$$

$$\Rightarrow \quad \alpha=\frac{2(s+\eta^2)+1\pm\sqrt{[2(s+\eta^2)+1]^2-4[(s+\eta^2)^2+\eta^2]}}{2}$$

Since $[2(s+\eta^2)+1]^2-4[(s+\eta^2)^2+\eta^2]$

$$=4s+1,$$

then
$$\alpha=\frac{2(s+\eta^2)+1\pm\sqrt{4s+1}}{2}$$

$$\lambda=\pm\sqrt{\frac{2(s+\eta^2)+1\pm\sqrt{4s+1}}{2}}$$

$$\Rightarrow \quad \lambda^R_\pm=-\sqrt{\frac{2(s+\eta^2)+1\pm\sqrt{4s+1}}{2}}$$

&

$$\lambda^L_\pm=\sqrt{\frac{2(s+\eta^2)+1\pm\sqrt{4s+1}}{2}}$$

• $V_0=\begin{bmatrix}-i\eta\\ s-\lambda-\lambda^2+\eta^2\end{bmatrix}$,

$$e^{\lambda^L_\pm x}\begin{pmatrix}-i\eta\\ s-\lambda^L_\pm-(\lambda^L_\pm)^2+\eta^2\end{pmatrix}$$

$$e^{\lambda^R_\pm x}\cdot\begin{pmatrix}-i\eta\\ s-\lambda^R_\pm-(\lambda^R_\pm)^2+\eta^2\end{pmatrix}$$

$$Q(x) = \begin{cases} L_+ \cdot e^{\lambda_+^L x} \cdot \left(\frac{-i\eta}{s - \lambda_+^L - (\lambda_+^L)^2 + \eta^2} \right) + L_- e^{\lambda_-^L \cdot x} \left[\frac{-i\eta}{s - \lambda_-^L - (\lambda_-^L)^2 + \eta^2} \right], & x > 0 \\ R_+ \cdot e^{\lambda_+^R x} \cdot \left[\frac{-i\eta}{s - \lambda_+^R - (\lambda_+^R)^2 + \eta^2} \right] + R_- \cdot e^{\lambda_-^R \cdot x} \cdot \left[\frac{-i\eta}{s - \lambda_-^R - (\lambda_-^R)^2 + \eta^2} \right], & x < 0, \end{cases}$$

Continuity:

$$L_+ \left(\frac{-i\eta}{s - \lambda_+^L - (\lambda_+^L)^2 + \eta^2} \right) + L_- \cdot \left(\frac{-i\eta}{s - \lambda_-^L - (\lambda_-^L)^2 + \eta^2} \right)$$

$$= R_+ \left[\frac{-i\eta}{s - \lambda_+^R - (\lambda_+^R)^2 + \eta^2} \right] + R_- \left[\frac{-i\eta}{s - \lambda_-^R - (\lambda_-^R)^2 + \eta^2} \right]$$

- Consider $A(\eta, s) \cdot e^{-\sqrt{\frac{2(s+\eta^2)+1 \pm \sqrt{4s+1}}{2}}\, x}, \quad x > 0$

- How to compute inverse of $A(\eta, s) \cdot e^{-\sqrt{2(s+\eta^2)+1 \pm \sqrt{4s+1}}\, x}$?

$$\sqrt{\frac{2(s+\eta^2)+1-2\sqrt{4s+1}}{2}} = \sqrt{2(s+\eta^2) - 2\sqrt{s+1/4} + 1 \,/\, 2}$$

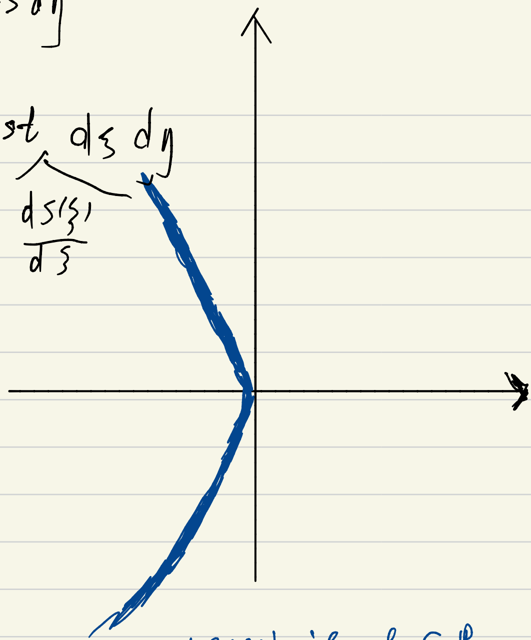
$$= \sqrt{s + \frac{1}{4} + \frac{1}{4} - \sqrt{s+1/4} + \eta^2} = \sqrt{(\sqrt{s+1/4} - \tfrac{1}{2})^2 + \eta^2} = \lambda.$$

$$\Rightarrow \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=0} \int_{\mathbb{R}} e^{-\lambda x + i\eta y + st} \cdot d\eta\, ds$$

Fourier - Laplace Path:

$$\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\cdot\int_{\mathbb{R}}\int_{\mathrm{Re}(s)=0} e^{-\lambda x + i\eta y + st}\, ds\, d\eta$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2\pi i}\cdot\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x + i\eta y + st}\, \underbrace{}_{\frac{ds(\xi)}{d\xi}} d\xi\, d\eta$$



$\lambda(s(\xi)) = i\xi,\ \xi \in \mathbb{R}$

Want : $\lambda(s(\xi)) = i\xi,\ \xi \in \mathbb{R}$

$$\sqrt{\left(\sqrt{s+\tfrac{1}{4}}-\tfrac{1}{2}\right)^2+\eta^2} = -i\xi,$$

then,

$$\left(\sqrt{s+1/4}-\tfrac{1}{2}\right)^2+\eta^2 = -\xi^2$$

$$\Rightarrow \left(\sqrt{s+1/4}-1/2\right)^2 = -\xi^2-\eta^2$$

$$\Rightarrow \sqrt{s+1/4}-1/2 = \pm i\sqrt{\eta^2+\xi^2}$$

$$\left(\sqrt{s+1/4}\right)^2 = \left(\pm i\sqrt{\eta^2+\xi^2}+\tfrac{1}{2}\right)^2$$

$$\Rightarrow s = -(\eta^2+\xi^2) \pm i\sqrt{\eta^2+\xi^2}$$

Thus,

$$\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x + i\eta y \underbrace{-(\eta^2+\xi^2)t}_{\text{heat eq.}} \pm \underbrace{i\sqrt{\eta^2+\xi^2}\,t}_{\text{wave eq.}}} \cdot \frac{ds(\xi)}{d\xi}\, d\xi\, d\eta$$

$$\frac{ds}{d\xi} = -2\xi \pm i\frac{\xi}{\sqrt{\eta^2+\xi^2}}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{-\infty}^{\infty} e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\,t} \cdot \left(-2\xi \pm \frac{i\xi}{\sqrt{\eta^2+\xi^2}}\right) d\xi\, d\eta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot (-2\xi) \, d\xi \, d\eta$$

$$+ \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot \left(\frac{\pm i \xi}{\sqrt{\eta^2+\xi^2}} \right) d\xi \, d\eta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} 2i \, \partial_x e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \, d\xi \, d\eta$$

$$+ \frac{1}{2\pi} \cdot \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{\infty} \underbrace{\partial_x e^{i\xi x + i\eta y - (\eta^2+\xi^2)t \pm i\sqrt{\eta^2+\xi^2}\, t} \cdot \frac{\pm 1}{\sqrt{\eta^2+\xi^2}}}_{} \, d\xi \, d\eta$$

$$\parallel$$

$$\underbrace{\partial_x \cdot e^{i\xi x + i\eta y} \cdot e^{-(\eta^2+\xi^2)t} \cdot \frac{e^{\pm i\sqrt{\eta^2+\xi^2}\, t}}{\pm\sqrt{\eta^2+\xi^2}}}_{}$$

$$\Downarrow$$

$$\partial_x \mathcal{F}^{-1}\left(e^{-(\eta^2+\xi^2)t}\right) * \mathcal{F}^{-1}\left(\frac{e^{\pm i\sqrt{\eta^2+\xi^2}\, t}}{\pm\sqrt{\eta^2+\xi^2}} \right)$$

$$\downarrow$$

$$\mathcal{F}^{-1}\left(\frac{\sin \sqrt{\eta^2+\xi^2}\, t}{\sqrt{\eta^2+\xi^2}} \right)$$

Q.E.D.

Navier-Stokes Eq.:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (u m)_x + P(\rho)_x = u_{xx}, \end{cases}$$

m: momentum

ρ: density of fluid

u: fluid velocity

$P(\rho)$: Pressure

$P(\rho) = \rho^\gamma,\ \gamma \in (1, 5/3)$

A Linearized Eq.

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \rho_x = m_{xx} \end{cases}$$

① $$\begin{cases} \rho_t + m_x = 0 \\ m_t + \rho_x = 0 \end{cases}$$ wave eq.

$$\Downarrow$$

$$(\rho_t + m_x)_x = 0 = (m_t + \rho_x)_t$$

$$\Downarrow$$

$$m_{xx} - m_{tt} = 0.$$

② $$\begin{pmatrix} \partial_t & \partial_x \\ \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \rho \\ m \end{pmatrix} = 0.$$

$$\Rightarrow \quad \partial_{tt} - \partial_{xx} = 0.$$

③ $$\begin{cases} \rho_t + m_x = \rho_{xx} \\ m_t + \rho_x = m_{xx} \end{cases} \Rightarrow \begin{pmatrix} \partial_t - \partial_x^2 & \partial_x \\ \partial_x & \partial_t - \partial_x^2 \end{pmatrix} \begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$[(\partial_t - \partial_x^2)^2 - \partial_x^2]\, \rho = 0$$

$$[(\partial_t - \partial_x^2)^2 - \partial_x^2]\, m = 0$$

By Fourier Transform

$$\Downarrow$$

$$\left[(\partial_t + \eta^2)^2 + \eta^2 \right] \hat{\rho} = 0$$

$$\Downarrow$$

$$\hat{\rho} = A_+ e^{\underbrace{-\eta^2 t}_{\text{heat eq.}} + \underbrace{i\eta t}_{\text{transport.}}} + A_- e^{-\eta^2 t - i\eta t}$$

Weak solution: Nonlinear Problem:

$$u_t + f(u)_x = 0$$

$$\Rightarrow \iint \varphi(x,t)\cdot (u_t + f(u)_x)\, dx\, dt = 0 \quad , \quad \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$$

$$\Downarrow$$

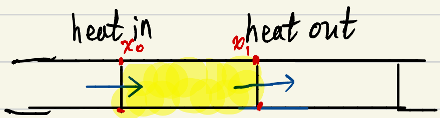
$$(*) \qquad \iint [-\varphi_t(x,t)\cdot u(x,t) - \varphi_x(x,t) f(u)]\, dx\, dt = 0,$$

A solution u satisfying $(*)$ is called weak sol.

Heat Eq:

$$u_t = u_{xx}$$

Fourier's Law: Heat flux is proportion to temperature gradient.



$$u_t \propto \text{heat in} - \text{heat out}$$

- Flux: u at x_0 $\sim -k\, u_x(x_0, t)$

$$u_t = [-k(x_0)\cdot u_x(x_0,t) + k(x_1)\cdot u_x(x_1,t)]$$

$$\Rightarrow u_t = \lim_{x_1\to x_0} \frac{[-k(x_0)\cdot u_x(x_0,t) + k(x_1)\cdot u_x(x_1,t)]}{x_1 - x_0}$$

$$\Rightarrow u_t = \partial_x(k(x)\, u_x)$$

Consider weak solution: $\iint \varphi(u_t - \partial_x(k(x)\cdot u_x))\, dx\, dt = 0,$

$$\Rightarrow \iint -\varphi_t u + \varphi_x \cdot k(x) \cdot u_x \, dx\, dt = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$$

u_x local integrable $\Rightarrow$ u: continuous.

Condition for $k(x)$:

$k(x)$: B. V. function :

Bounded Variation : • Piecewise continuous

• Jump is countable

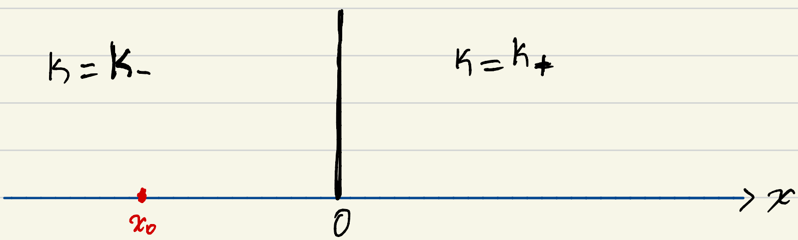
Step 1: k is const. & $k > 0$;

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \delta(x) \end{cases} \xRightarrow{\text{Green Function.}} u(x,t) = \frac{e^{-\frac{x^2}{4k \cdot t}}}{\sqrt{4\pi t}}$$

$$\mathcal{L}\, u(x,s) = \int_0^\infty e^{-st} \cdot u(x,t)\, dt$$

$$\Rightarrow \mathcal{L}\, u(x,s) = \frac{e^{-\sqrt{s/k}\,|x|}}{2\sqrt{sk}}$$

Step 2:



$$\begin{cases} u_t = \partial_x (k(x)\, u_x) \\ u(x,0) = \delta(x - x_0), \quad x_0 < 0 \end{cases}$$

I.B.P.: $\iint \varphi\, y'\, dx = -\iint \varphi' y\, dx$

y: continuous.

(i) $u(x)$: continuous.

(ii) $K(x)\, u_x$: contin.

$$\begin{cases} s\mathcal{L}u = \partial_x\left(K(x)\,\partial_x \mathcal{L}u\right) + \delta(x-x_0) \\ u(x,0) = \delta(x-x_0), \quad x_0 < 0 \end{cases}$$

$$\begin{cases} s\mathcal{L}u = \partial_x\left(K_-\ \partial_x \mathcal{L}u\right) + \delta(x-x_0) & \text{for } x<0 \\ s\mathcal{L}u = \partial_x\left(K_+\ \partial_x \mathcal{L}u\right) & \text{for } x>0 \end{cases}$$

$$\Rightarrow \mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/K_-}\cdot|x-x_0|}}{2\sqrt{sK_-}} + U_-\cdot e^{\sqrt{s/K_-}\,x}, & x<0 \\ S_+\cdot e^{-\sqrt{s/K_+}\,x} & x>0 \end{cases}$$

By continuity of u at 0,

$$\frac{e^{-\sqrt{s/K_-}\cdot|x_0|}}{2\sqrt{sK_-}} + U_- = S_+ \qquad \text{— (I)}$$

By continuity of flux:

$$\begin{cases} \left[\dfrac{e^{-\sqrt{s/K_-}\cdot|x-x_0|}}{2\sqrt{sK_-}} + U_-\cdot e^{\sqrt{s/K_-}\,x}\right]_x, & x<0 \\ \left(S_+\cdot e^{-\sqrt{s/K_+}\,x}\right)_x, & x>0 \end{cases}$$

$$\Rightarrow \quad K_-\left(-\sqrt{\frac{s}{K_-}}\cdot\frac{e^{-\sqrt{s/K_-}\,|x_0|}}{2\sqrt{sK_-}} + U_-\sqrt{s/K_-}\right)$$

$$= K_+\left(-S_+\cdot\sqrt{\frac{s}{K_+}}\right) \qquad \text{— (II)}$$

$$\Rightarrow \quad \frac{e^{\sqrt{s/k_-}\,|x_0|}}{2\sqrt{s\cdot k_-}} = \overset{\text{transmittion}}{S_+} - \overset{\text{reflexsion.}}{U_-}$$

$$\begin{cases} S_+ + \sqrt{\dfrac{k_-}{k_+}}\, U_- = \left(k_- \sqrt{\dfrac{1}{k_-}} \cdot \dfrac{e^{-\sqrt{s/k_-}\,|x_0|}}{2\sqrt{sk_-}} \right) \Big/ \sqrt{k_+} \end{cases}$$

Physical meaning:

1. If $k_- = k_+$, $U_- = 0$ i.e. no reflexsion

2. $U_- = O(1)\left(\sqrt{\frac{k_-}{k_+}} - 1\right) \cdot \dfrac{e^{-\sqrt{s/k_-}\,|x_0|}}{\sqrt{sk_-}}$

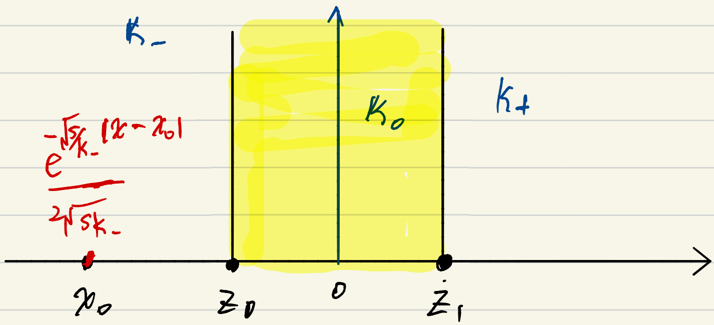
$S_+ = \left[1 + O(1)\cdot\left(\sqrt{\frac{k_1}{k_+}} - 1\right)\right] \dfrac{e^{-\sqrt{s/k_-}\,|x_0|}}{\sqrt{sk_-}}$

$$\Rightarrow \quad \mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\cdot|x-x_0|}}{2\sqrt{sk_-}} + \underbrace{U_- \cdot e^{\sqrt{s/k_-}\,x}}_{\text{reflexsion}}, & x<0 \\ \underbrace{S_+ \cdot e^{-\sqrt{s/k_+}\,x}}_{\text{transmittion.}}, & x>0 \end{cases}$$

$$\Rightarrow \mathcal{L}_e u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\cdot|x-x_0|}}{2\sqrt{sk_-}} + O(1)\left(\sqrt{\frac{k_-}{k_+}} - 1\right)\cdot \dfrac{e^{\sqrt{s/k_-}(x-|x_0|)}}{\sqrt{sk_-}}, & x<0 \\ \left[1 + O(1)\cdot\left(\sqrt{\frac{k_1}{k_+}} - 1\right)\right] \dfrac{e^{-\sqrt{s/k_-}\,|x_0| - \sqrt{s/k_+}\,x}}{\sqrt{sk_-}}, & x>0 \end{cases}$$

- $$\frac{e^{\sqrt{s/k_-}(x-|x_0|)}}{\sqrt{sk_-}} = \frac{e^{-\sqrt{s}\cdot\left(\underbrace{\frac{|x_0|}{\sqrt{k_-}} + \frac{|x|}{\sqrt{k_-}}}_{\delta.}\right)}}{\sqrt{sk_-}}$$

$$\Rightarrow \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}\cdot\sqrt{k_-}} = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}\cdot\sqrt{k_-}} = \frac{e^{-\left(\frac{x_0}{\sqrt{k_-}}+\frac{x}{\sqrt{k_-}}\right)^2}}{\sqrt{4\pi t k_-}}$$



• Observe:

$$\sqrt{\frac{k_-}{k_+}}-1 = \sqrt{\underbrace{\frac{k_--k_+}{k_+}}_{\text{small enough}}+1}\ -1$$

By Taylor' expansion,

$$\sqrt{\frac{k_--k_+}{k_+}+1} = \frac{(k_--k_+)}{k_+}+1+\cdots$$

$$\Rightarrow = O(1)\cdot(k_--k_+).$$

$$\begin{cases} u_t - \partial_x(k(x)\,u_x) = 0 \\ u(x_0) = \delta(x-x_0) \end{cases}$$

Take Laplace transform,

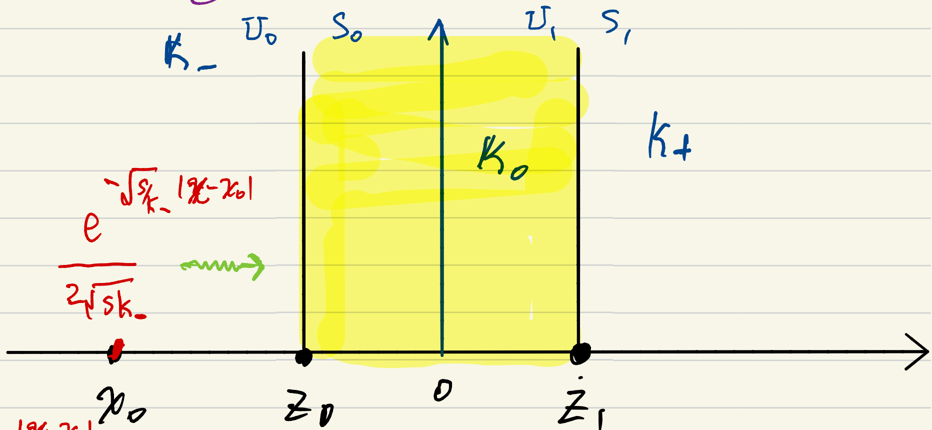
$$\Rightarrow s\mathcal{L}u - \partial_x(k(x)\,\partial_x\mathcal{L}u) = \delta(x-x_0)$$

$$(s-\partial_x k(x)\partial_x)\,\mathcal{L}u = \delta(x-x_0)$$

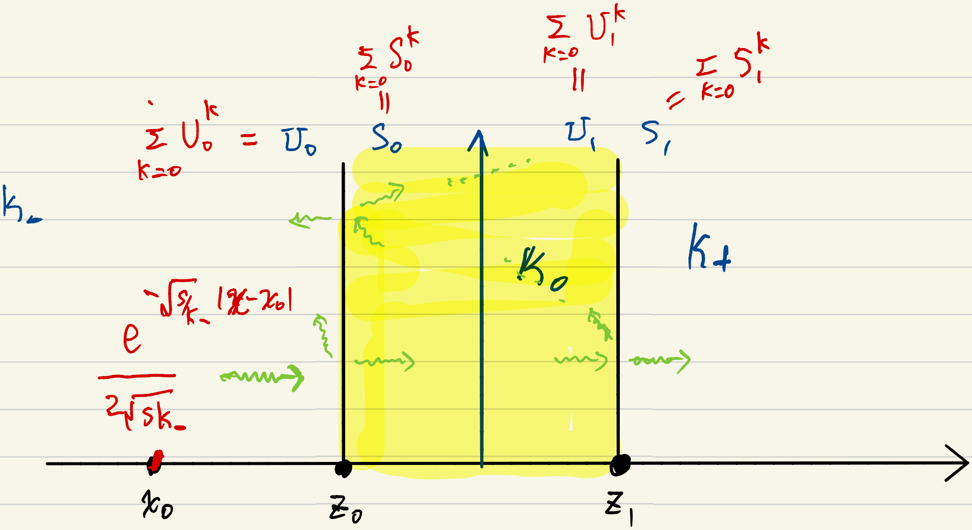
$$\Rightarrow \text{“}\mathcal{L}u = (s-\partial_x k(x)\partial_x)^{-1}\delta(x-x_0)\text{”}.$$

No understanding ! See the operator as spectrum, then by F.A.

Do it with understanding:



$$\mathcal{L}u = \begin{cases} \dfrac{e^{-\sqrt{s/k_-}\,|x-x_0|}}{2\sqrt{sk_-}} + U_0\cdot e^{\sqrt{s/k_-}\,x}, & x<z_0 \\ S_0\cdot e^{-\sqrt{\frac{s}{k_-}}(x-z_0)} + U_1\cdot e^{\sqrt{\frac{s}{k_0}}(x-z_1)}, & z_0\le x\le z_1 \\ S_1\cdot e^{-\sqrt{\frac{s}{k_0}}(x-z_1)}, & x\ge z_1 \end{cases}$$



R^0_{++} T^0_{+-} R^1_{++} T^1_{+-} all are const. about x.

R^0_{--} T^0_{-+} R^1_{--} T^0_{-+}

$U_0^0 = R^0_{--} \cdot \frac{e^{-\sqrt{s/k_-}|x_0|}}{2\sqrt{sk_-}}$, $S_0^0 = T^0_{-+} \frac{e^{-\sqrt{s/k_-}|x_0|}}{2\sqrt{sk_-}}$

$U_0^1 = 0$, $S_0^1 = 0$

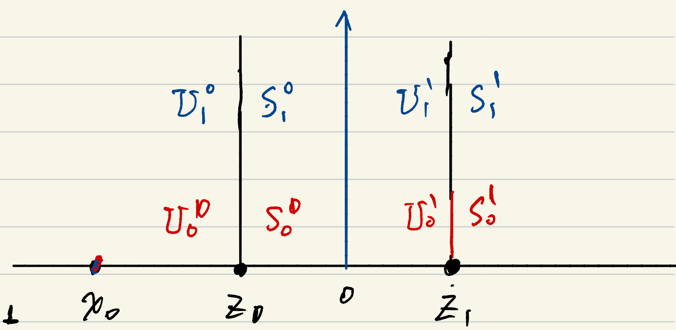
- $S_1^0 = U_0^1 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot R^0_{++}$

$U_1^0 = U_0^1 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T^0_{+-}$

$U_1^1 = S_0^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} R^1_{--}$

$S_1^1 = S_0^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T^1_{-+}$

$\vdots \quad \vdots \quad \vdots$



$$S_k^0 = U_{k-1}^1 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot R_{++}^0$$
$$U_k^0 = U_{k-1}^1 e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T_{+-}^0$$
$$U_k^1 = S_{k-1}^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} R_{--}^1$$
$$S_k^1 = S_{k-1}^0 \cdot e^{\sqrt{s/k_0}(z_0 - z_1)} \cdot T_{-+}^1$$

$$\vdots \qquad \vdots \qquad \vdots$$

Observe:

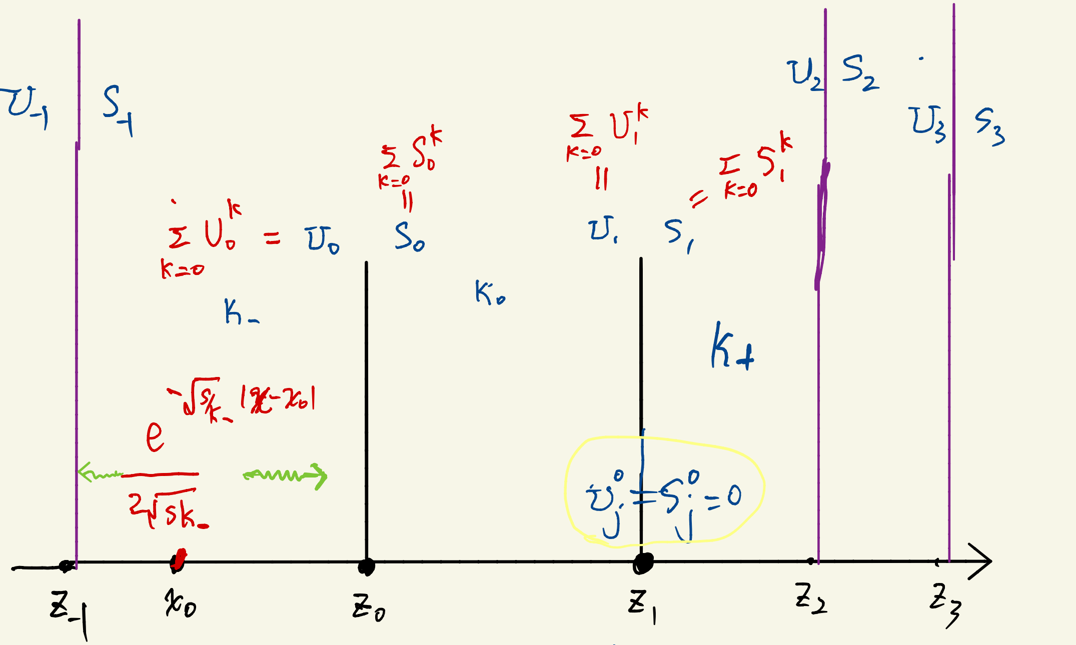
$$S_{2k}^0 = e^{2\sqrt{s/k_0}(z_0 - z_1)k} \cdot R_{--}^1 \cdot R_{++}^0 \cdot S_{2k-2}^0$$

$$\Rightarrow \quad S_{2k}^0 = e^{2\sqrt{s/k_0}(z_0 - z_1)k} \cdot (R_{--}^1 R_{++}^0)^k \cdot \frac{e^{-\sqrt{s/k_-}|x_0|}}{2\sqrt{s k_-}}$$

$$\sum_{k=0} S_{2k}^0 = \sum \underbrace{(R_{--}^1 R_{++}^0)^k}_{\text{depending on } k.} \cdot \frac{e^{2\sqrt{s/k_0}(z_0 - z_1)k - \sqrt{s/k_0}|x_0|}}{2\sqrt{s k_-}}$$

$$\frac{e^{2\sqrt{s/k_0}(z_0 - z)k - \sqrt{s/k_0}|x_0|}}{2\sqrt{s k_-}} = \frac{e^{\sqrt{s}\left(2\sqrt{1/k_0}(z_0 - z)k - \sqrt{1/k_0}|x_0|\right)}}{2\sqrt{s k_-}}$$

$$\longleftrightarrow \quad \frac{e^{-\frac{\left[2\sqrt{\frac{t}{k_0}}(z - z_0) - \sqrt{\frac{t}{k_0}}|x_0|\right]^2}{4t}}}{\sqrt{k_-} \cdot \sqrt{4\pi t}}.$$

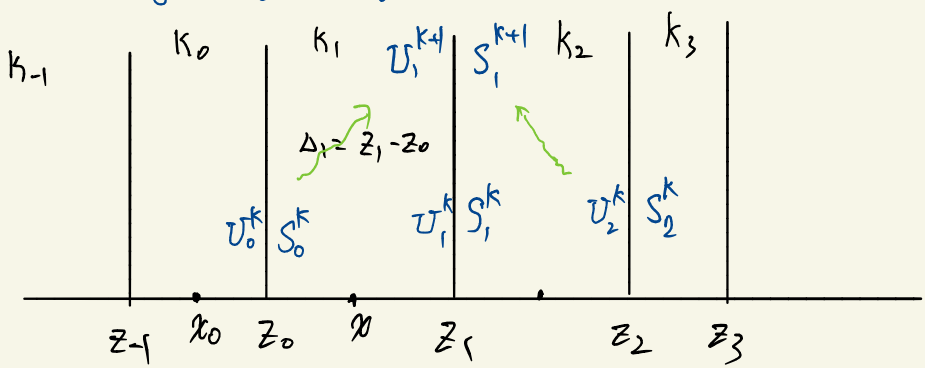


$$U_j = \sum_{k=0}^{\infty} U_j^k \quad \& \quad S_j = \sum_{k=0}^{\infty} S_j^k$$

• $U_j^0 = S_j^0 = 0$, if $j \geq 1$ or $j \leq -2$

$$S_0^0 = \frac{e^{-\sqrt{s/k_-}\,|z_0 - x_0|}}{2\sqrt{sk_-}} \cdot T_{1-f}^0, \quad U_0^0 = \frac{e^{-\sqrt{s/k_-}\,|z_0 - x_0|}}{2\sqrt{sk_-}} \cdot R_{--}^0$$

• Suppose (S_j^k, U_j^k), $\forall j$.



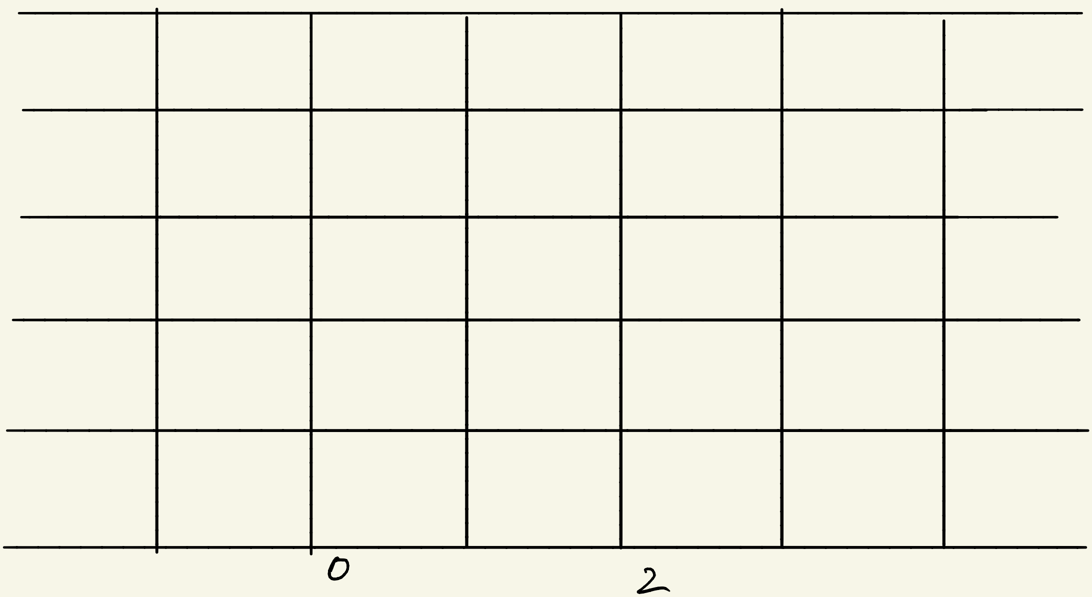
$$U_1^{k+1} = S_0^k \cdot e^{-\sqrt{s/k_1}\,\Delta_1} R_{--}^1 + U_2^k \cdot e^{-\sqrt{s/k_2}\,\Delta_2} \cdot T_{+-}^1$$

&

$$S_1^{k+1} = S_0^k \cdot e^{-\sqrt{s/k_1}\,\Delta_1} T_{-+}^1 + U_2^k \cdot e^{-\sqrt{s/k_2}\,\Delta_2} \cdot R_{++}^1$$

$$\Rightarrow U_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j}\,\Delta_j} R_{--}^j + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}}\,\Delta_{j+1}} \cdot T_{+-}^j$$

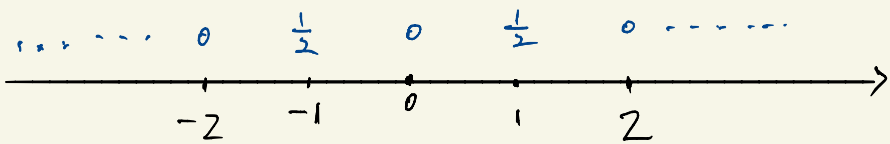
$$S_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j}\,\Delta_j} T_{-+}^j + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}}\,\Delta_{j+1}} \cdot R_{++}^j$$

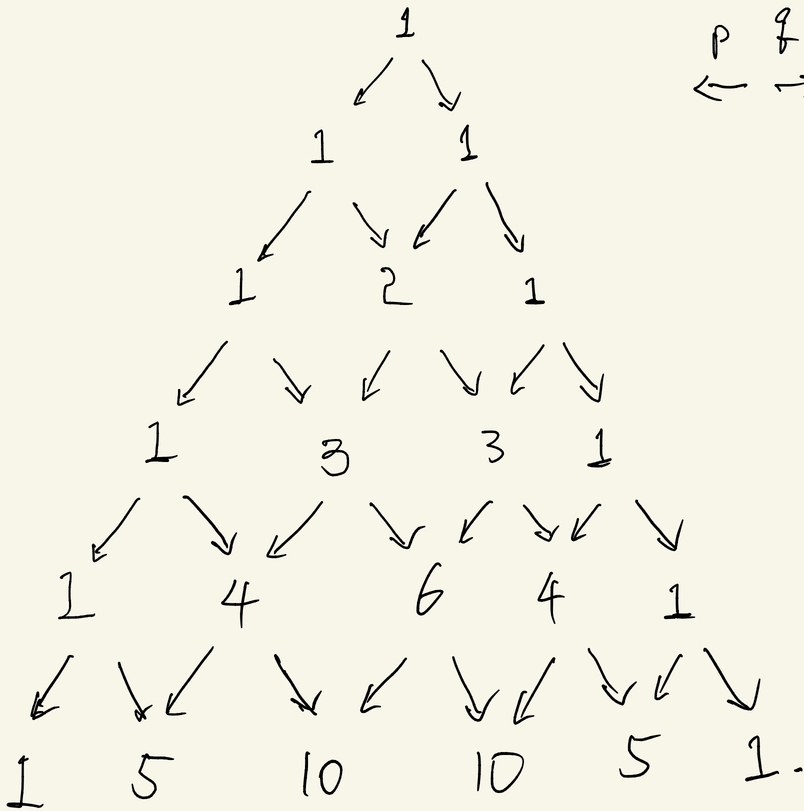


$\longleftarrow \; \longrightarrow$ $(n=6)$

$$\left(\frac{1}{2}x + \frac{1}{2}\frac{1}{x}\right)^6$$ Random Walk.

$$= \left(\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x}\right) \cdots\cdots \left(\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x}\right)$$





$\overset{p}{\longleftarrow} \quad \overset{q}{\longrightarrow}$, $p+q=1$

Numerical Solution

$$u_t + u_x = 0$$

Numerical Sol.

$$u(j\Delta x, n\Delta t) = u_j^n$$

$$\Rightarrow \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\Rightarrow \quad u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$\Rightarrow \quad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

$$= \underbrace{-\frac{\lambda}{2}} u_{j+1}^n + u_j^n + \frac{\lambda}{2} u_{j-1}^n$$

Need to be positive for using probility.

Another Scheme:

$$U_j^{n+1} - \frac{(u_{j+1}^n + u_j^n + u_{j-1}^n)}{3} + \frac{\Delta t}{\Delta x} \frac{u_{j+1}^n - u_{j-1}^n}{2} = 0$$

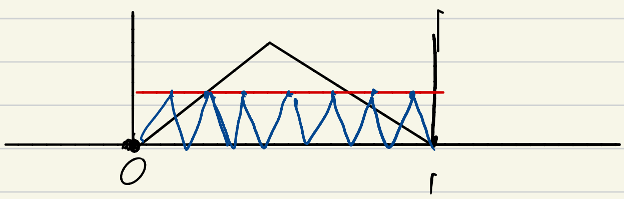
$\frac{\lambda}{2} < 1/3$

$$\Rightarrow \quad u_j^{n+1} = \left(\frac{1}{3} - \frac{\lambda}{2}\right) U_{j+1}^n + \frac{1}{3} U_j^n + \left(\frac{1}{3} + \frac{\lambda}{2}\right) U_{j-1}^n$$

B.V. : Bounded Variation Function

$$\sup_{\substack{P \\ \{x_1, \dots\}}} \sum_i |k(x_i) - k(x_{i+1})| < \infty$$

Brown Motion



Now, assume $k(x)$ is B.V. function.

$$\|k\|_{B.V.} = \sup_P \sum_i |k(x_i) - k(x_{i+1})| < \infty$$

$P = \{x_1, \dots\}$.

- If k is continuous, then $\|k\|_{B.V} = \int_{\mathbb{R}} |k_x| \, dx$

$$\begin{cases} U_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j}\,\Delta_j} R_{--}^j + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}}\,\Delta_{j+1}} \cdot T_{+-}^j \\ S_j^{k+1} = S_{j-1}^k \cdot e^{-\sqrt{s/k_j}\,\Delta_j} T_{-+}^j + U_{j+1}^k \cdot e^{-\sqrt{s/k_{j+1}}\,\Delta_{j+1}} \cdot R_{++}^j \end{cases}$$

$$\begin{bmatrix} U_j^{k+1} \\ S_j^{k+1} \end{bmatrix} = R_{j-1} \begin{bmatrix} U_{j-1}^k \\ S_j^k \end{bmatrix} + \mathcal{L}_{j+1} \cdot \begin{bmatrix} U_{j+1}^k \\ S_{j+1}^k \end{bmatrix}$$

$$\Omega^k := \{w^k(j) = \{ {-1 \atop 1} \quad \}$$

$$\Sigma_j(w^k) = \sum_{l=0}^{j} w^k(l)$$

$$\bigcup_{k=0}^{\infty} \Omega^k$$

$$U_j^k = \sum_{\substack{w \in \Omega^k \\ \Sigma_k(w) = j}} \text{“} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{”}$$

$$D_l(w) = \begin{cases} R_{xl} \\ \mathcal{L}_{xl} \end{cases}$$

$$\Rightarrow \quad |D_1(w) \cdot D_2(w) \cdots D_k(w)| \le O(1) \cdot \prod |k_n - k_{n+1}|$$

$\Sigma = n$: change direction.

$$\sum_{k=0}^{\infty} U_j^k = \sum_{k=0}^{\infty} \sum_{\substack{w \in \Omega^k \\ \Sigma_k(w) = j}} \text{“} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{”}$$

$$\Omega^k = \bigcup_{l=0}^{\infty} \Omega_l^k , \qquad \Omega_l^k : \{ w \in \Omega^k ; \ w \text{ change direction } l \text{ times exactly} \}.$$

$$\Rightarrow \sum_{k=0}^{\infty} U_j^k = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\substack{w \in \Omega_l^k \\ \Sigma_k(w) = j}} \text{“} D_1(w) \cdot D_2(w) \cdots D_k(w) \text{”}$$

$$\leq \sum_{l=0}^{\infty} \cdot \sum_{k=0}^{\infty} \cdot \sum_{\substack{w \in \Omega_l^k \\ \Sigma_k(w) = j}} O(1) \prod |k_n - k_{n+1}|$$

m: where $\Sigma = m$ change direction.

$$\sum_{i,j} a_i a_j = (\Sigma a_i)^2$$

$$\sum_{i,j,k} a_i a_j a_k = (\Sigma a_i)^3$$

$$\vdots \quad \vdots$$

$$= O(1) \cdot \sum_{l=0}^{\infty} \left(\sum_n |k_n - k_{n+1}| \right)^l \quad \text{Absolutely converge.}$$

$$\Rightarrow |\partial_x u(x,t)| \leq O(1) \frac{e^{-\frac{x^2}{C_* t}}}{t} \quad \text{for some } C_* > 0.$$

We prove when k is step function. What if $k(x)$ is B.V. funct.?

$$\begin{cases} u_t - \partial_x (\overset{k_n(x)}{k(x)} u_x) = 0 \\ u(x,0) = \delta(x - x_0) \end{cases}$$

$k(x)$: A B.V. function.

$\{k_n(x)\}_{n\in\mathbb{N}}$: A step function. A Cauchy sequence in B.V.-norm.

$$\lim_{n\to\infty} \| k_n - k \|_\infty = 0$$

Consider
$$\begin{cases} u^n_t - \partial_x (k_n(x) u^n_x) = 0 \\ u^n(x,0) = \delta(x - x_0) \end{cases} \Longleftarrow \overset{\text{Sol.}}{K(x,t;x_0,k_n)} \equiv u^n(x,t)$$

Rewrite it as:
$$u^n_\tau - \partial_y (k_n(y) \cdot u^n_y) = 0$$

$$\Rightarrow \int_0^t \int_{\mathbb{R}} K(x,t-\tau;y;k_n) \cdot (u^n_\tau - \partial_y(k_n(y) \cdot u^n_y)) \, dy \, d\tau = 0$$

$$u^n(x,t) - \int_{\mathbb{R}} K(x,t;y,k_n) \cdot u(y,0) \, dy$$

$$+ \int_0^t \int_{\mathbb{R}} -\partial_\tau K(x,t-\tau;y,k_n) u^n$$

$$+ K_y(x,t-\tau;y,k_n) \cdot \underline{k_n(y)} \cdot u^n_y \, dy \, d\tau = 0$$

$(k_m$

Question 1:

Consider the problem

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases} \qquad (1)$$

Use the singular-regular decomposition to decompose the solution $u(x,t) = u_*(x,t) + u_\#(x,t)$, where $u_*(x,t)$ is a singular part with an exponentially decaying structure in both x and t variable; and $u_\#(x,t)$ is a regular part with a sufficient regularity in x variable.

(1) Use energy estimate to show that $u_\#(x,t)$ is exponentially decaying in x variable when $|x| > 2t$ with $t > 1$.

(2) Use Long wave-short wave decomposition, Fourier transform, energy estimates, and complex analysis to show that there exists $C > 0$ s.t.

$$|u_\#(x,t)| < C \cdot \left(\frac{e^{-\frac{x^2}{C(t+1)}}}{\sqrt{t+1}} \right) \qquad (2).$$

Question 2:

Let $\mathcal{L}_o[u](x,s)$ be the Laplace transform of $u(x,t)$ w.r.t. t

$\mathcal{L}_o[u](x,s) = \int_0^\infty u(x,t) \cdot e^{-st} dt$ with $\Re(s) \geq 0$. Let $u(x,t)$ be the

Solution of (1) and compute $\mathcal{L}_o[u](x,s)$ in terms of Laplace wave

trains:

$$\mathcal{L}_o[u](x,s) = \begin{cases} A_+(s) \cdot e^{\lambda_+(s) x} & \text{for } x > 0 \\ A_-(s) \cdot e^{\lambda_-(s) x} & \text{for } x < 0 \end{cases}$$

(1) Find $\lambda_\pm(s)$ and $A_\pm(s)$. Let $w(x,t)$ be the solution of the initial boundary value problem:

$$\begin{cases} w_{tt} - w_{xx} + w_t = 0 & \text{for } x, t > 0, \\ w(x,0) = w(0,t) = 0, \\ w_t(x,0) = \delta(x - x_0) & \text{with } x_0 > 0 \end{cases}$$

(2) Find the solution $\mathcal{L}[w](x,s)$ in terms of the Laplace wave trains $e^{\lambda_+(s)(x-x_0)}$, $e^{\lambda_-(s)(x-x_0)}$ and $e^{\lambda_+(s)x}$.

Question 3:

Let $u(x,t)$ be the weak solution of the heat equation

$$\begin{cases} \partial_t u - \partial_x \mu(x) \partial_x u = 0 \\ u(x,0) = \delta(x) \end{cases}$$

where $\mu(x) \equiv 1 + H(x+1) - H(x-1)$, and $H(x)$ is the Heaveside funct.

i.e. $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

- Show that $\exists\, C > 0$ s.t.

$$|u_x(x,t)| \leq C \cdot \frac{e^{-\frac{x^2}{Ct}}}{t}.$$