INTRODUCTION TO THE THEORY OF CONNECTIONS ON PRINCIPAL BUNDLES

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The following notes are meant to be a brief introduction to the theory of connections on principal bundles. The main focus is on the transformation properties of gauge potentials in terms of the geometrical structure of fibre bundles. Electromagnetism as a U(1)-bundle gauge theory is provided as an example in the abelian case.

The reader is assumed to be familiar with the ordinary differential calculus and tensor analysis on manifolds.

CONTENTS

1 CONNECTIONS ON PRINCIPAL BUNDLES

Let $P(M, G)$ be a principal bundle whose fibre F is identical to the structure group G. Take $u \in P$ and let $T_u P$ be the tangent space at u. The set of vertical vectors $V_u P$ at u is defined as the subset of $T_u P$ which is also tangent to the fibre F_m in $m = \pi(u)$. According to this definition the subspace $V_u P$ easily emerges in the following way: let $A \in \mathfrak{g}$ be an element of the lie algebra $\mathfrak{g}$ of G; e^{tA} is an element of a one-parameter subgroup of G genereted by A. The right action of G on itself, given by

$$\begin{gathered} G \times G \to G \\ (u = \pi^{-1}(m), \mathrm{e}^{tA}) \to u\, \mathrm{e}^{tA} \end{gathered}$$

defines a curve through u belonging to the fibre, because $\pi(u) = \pi(u\, \mathrm{e}^{tA})$. As a consequence any tangent vector defined by means of such a curve is tangent to G_m. We define the vector $A^{\#} \in V_u P$ by setting:

$$A^{\#} f(u) = \frac{d}{dt} f(u\, \mathrm{e}^{tA}) \bigg|_{t=0}$$

f being an arbitrary smooth function $\in \mathscr{F}(P)$. Since in this way a vector $A^{\#}$ is defined at each $u \in P$, a vector field $A^{\#} \in \mathfrak{X}(P)$ can be constructed. It's easy to show the vector space isomorphism between $\mathfrak{g}$ and $V_u P$ defined by #

$$\begin{gathered} \#\colon \mathfrak{g} \to V_u P \\ A \to A^{\#}. \end{gathered}$$

Once the subspace $V_u P$ is constructed, the horizontal subspace $H_u P$ is defined as a complement of $V_u P$ in $T_u P$.

THEOREM 1: *The set of vertical vectors $V_u P$ could equivalently be defined as the kernel of the map π_* at $u \in P$, that is:*

$$X \in V_u P \iff \pi_* X = 0.$$

Proof. First note that the statement $\pi_* X = 0 \Rightarrow X \in V_u P$ is equivalent to $X \notin V_u P \implies \pi_* X \neq 0$. Then take $g \in \mathscr{F}(M)$:

$$\pi_* X[g] = X[g \circ \pi] = \frac{d}{dt} (g \circ \pi)(u(t)) \bigg|_{t=0}$$

The derivative $\frac{d}{dt} (g \circ \pi)(u(t)) \Big|_{t=0}$ is equal to

$$\frac{dg}{d\pi} \frac{d}{dt} \pi(u(t)) = \frac{dg}{d\pi} \frac{d}{dt} m(t) \bigg|_{t=0} \iff m(t) = m_0.$$

But the condition $\pi(u(t)) = m_0$ implies the curve $u(t)$ to belong to the fibre in m_0 for each value of t, hence the tangent vector will act as a vector tangent to a curve in the fibre, i.e., by definition, as a vector of $V_u P$. Now comes the second part of the theorem: $X^{\#} \in V_u P \implies \pi_* X^{\#} = 0$. We have:

$$\begin{aligned} \pi_* X^{\#}[g] &= X^{\#}[g \circ \pi] = 0 && \forall g \in \mathscr{F}(M) \text{ if } X \in V_u P \\ X^{\#}[g \circ \pi] &= \frac{d}{dt} (g \circ \pi)(u\, \mathrm{e}^{tX}) \bigg|_{t=0} && \text{because } X \in V_u P \end{aligned}$$

But

$$\frac{d}{dt}(g \circ \pi)(u\, \mathrm{e}^{tX})\Big|_{t=0} = \frac{dg}{d\pi}\frac{d}{dt}\,\pi(u\, \mathrm{e}^{tX}) = \frac{dg}{d\pi}\frac{d}{dt}\,\pi(u) = 0$$

because $X^\# \in V_u P$, that is the action of a flow generated by X leaves the fibre invariant. □

DEFINITION 1: *Let $P(M, G)$ be a principal bundle. A connection on P is a unique splitting of the tangent space $T_u P$ such that the following conditions hold:*

- $T_u P = V_u P \oplus H_u P \quad (u \to H_u P$ being smooth on $u)$;
- $\forall\, X \in T_u P \Rightarrow X = X^v + X^h; \quad X^v \in V_u P, X^h \in H_u P$
- $H_{ug} P = (R_{g_*}) H_u P \quad \forall\, u \in P, g \in G$

the last equality means that horizontal subspaces on the same fibre are related one other by means of the push forward map induced by the right action of the group on itself. So, given a horizontal subspace on a fibre, any horizontal subspace on the same fibre is constructed by right action. In order to obtain such a splitting, a Lie–algebra valued one-form $\omega \in \mathfrak{g} \otimes T^* P$ can be introduced. We require:

- $\omega(A^\#) = A \qquad (A \in \mathfrak{g}$ isomorphic to $V_u P)$
- $H_u P \overset{\text{def}}{=} \mathrm{Ker}(\omega)$, that is $\{\, X \in T_u P \mid \omega(X) = 0 \,\}$

Another property of ω has to be carried out, in order $H_{ug} P = (R_{g_*}) H_u P$ to be satisfied $\forall\, u \in P, g \in G$; namely the requirement is $(R_g{}^*)\omega = \mathrm{Ad}_{g^{-1}}\, \omega$.

PROPOSITION 1: *If $(R_g{}^*)\omega = \mathrm{Ad}_{g^{-1}}\, \omega$ holds then*

$$(R_{g_*}) H_u P = H_{ug} P \qquad \forall\, u \in P, g \in G$$

Proof. We first prove that $(R_{g_*}) H_u P \in H_{ug} P$. Take $X \in H_u P$, then $(R_{g_*}) X$ belongs to $H_{ug} P$, in fact

$$\omega(R_{g_*} X) = R_g{}^* \omega(X) \qquad \text{by definition of pullback map}$$
$$\omega(R_{g_*} X) = R_g{}^* \omega(X) = \mathrm{Ad}_{g^{-1}}\, \omega = g^{-1} \omega(X) g = 0$$

thus $(R_{g_*}) H_u P \in H_{ug} P$. Now it's left to prove that if $Y \in H_{ug} P$, then there exists a vector $X \in H_u P$ such that $Y = (R_{g_*}) X$; this assertion widely holds because R_{g_*} is an invertible linear map. □

2 GAUGE POTENTIALS

Let $P(M, G)$ be a principal bundle. Let $\{U_i\}$ be an open covering of M with local section σ_i defined on each U_i [1].

1 In the following we also refer to the couple (U_i, σ_i) as to a covering atlas of M.

Given a connection one-form ω, a Lie-algebra valued one form $\mathcal{A}_i$ on each U_i is automatically defined as

$$\mathcal{A}_i = {\sigma_i}^* \omega \in \mathfrak{g} \otimes \Omega^1(M)$$

depending on the section σ_i. We refer to the $\mathcal{A}_i$'s as to the gauge potentials defined on M by ω, once a covering atlas (U_i, σ_i) is assigned on M.

Now a question automatically arises. It is important to know whether, conversely, the knowledge of the gauge potentials on M uniquely identifies the connection one-form ω on P. In order to answer this question, we explicitly construct such a form, and then we show that it will be the connection form required.

To construct ω one needs to introduce the canonical local trivialization on a fibre bundle. Let $P(M, G)$ be a principal bundle with a set of open covering $\{U_i\}$ of M, then consider the following:

Definition 2: *Consider the map $\phi_i \colon U_i \times F \to \pi^{-1}(U_i)$ defined in such a way that $\pi \circ \phi_i\ (m, f) = m$; this means that*

$$U_i \times F \xrightarrow{\phi_i} \pi^{-1}(U_i) \xrightarrow{\pi} U_i$$

sends the pair $(m, f) \to m$. The map ${\phi_i}^{-1}$ is called the canonical local trivialization of $\pi^{-1}(U_i)$, because it maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$

$${\phi_i}^{-1} \colon \pi^{-1}\ (U_i) \to\ U_i \times F$$

so it splits locally the principal bundle $P(M, G)$.

On the intersection $U_i \cap U_j \neq 0$ the map

$$t_{ij}(m) \stackrel{\text{def}}{=} {\phi_i}^{-1} \circ \phi_j \colon U_j \times F \to U_i \times F$$

is required to be smooth and, once $m \in M$ is fixed, to belong to G. The t_{ij} are referred to as the transition functions from $U_j \times F$ to $U_i \times F$. A fibre bundle is called trivial if it exists a trivialization of the form

$$\pi^{-1}\ (U_i) \quad \longrightarrow \quad U_i\ \times\ F$$

which acts globally on M, i.e. $\exists$ a map $P \to M \times F$. In that case it is a common use to identify P itself with the cartesian product $M \times F$. It can be shown that such a map always exists if we restric all the transition functions t_{ij} to be the identity maps; this can be made only if the base space M is contractible to a point.

Theorem 2: *Given an open covering atlas (U_i, σ_i) on M and a set of gauge potentials $\mathcal{A}_i$, then*

$$\omega_i = {g_i}^{-1} \pi^* \mathcal{A}_i\, g_i + {g_i}^{-1} d g_i$$

is a connection one-form on P if $g_i \in G$ is the canonical local trivialization on $\pi^{-1}(U_i)$ defined by ${\phi_i}^{-1}(u = \sigma_i(m) g_i) = (m, g_i)$

Proof. We first prove that ${\sigma_i}^*\omega_i = \mathcal{A}_i$. Take $X \in T_m M$ and evaluate

$$({\sigma_i}^*\omega_i)(X) = \omega(\sigma_{i*}X) = {g_i}^{-1}\pi^*\mathcal{A}_i(\sigma_{i*}X)g_i + {g_i}^{-1}\mathrm{d}g_i(\sigma_{i*}X)$$

If $\mathcal{A}_i$ is defined on $m \in M \implies {\sigma_i}^*\omega_i \in T_m^*M$, then $u \in P$ must be $\sigma_i(m)$. But in the definition of local trivialization we have set $u = \sigma_i(m)g_i$, so it follows that the g_i's must be the identity elements if we act on ω with the pullback of σ_i. It follows that

$$\begin{aligned}({\sigma_i}^*\omega_i)(X) &= \pi^*\mathcal{A}_i(X) + \mathrm{d}g_i(\sigma_{i*}X)\\ ({\sigma_i}^*\omega_i)(X) &= \mathcal{A}_i(\pi_*\sigma_{i*}X) + \mathrm{d}g_i(\sigma_{i*}X)\end{aligned}$$

Note that $\pi_*\ \sigma_{i*} = \mathrm{id}_{T_m M}$ by definition of local section. Moreover the element g_i is forced to be the identity on all the flow carried by σ_{i*}, so it cames out that

$$({\sigma_i}^*\omega_i)(X) = \mathcal{A}_i(X) \qquad \forall\, X \in T_m M$$

that is ${\sigma_i}^*\omega_i = \mathcal{A}_i$. Now the proof ought to go on by showing that ω really satisfies the property required to be a correct connection one-form on a fibre bundle, but this will be omitted here. □

We have constructed ω with a pullback on each U_i on M, but in order ω to be uniquely defined on P, i.e. in order the splitting $T_u P = V_u P \oplus H_u P$ to be unique, as required in the definition of the connection, it must be

$$\omega_i = \omega_j \quad \text{on } U_i \cap U_j \neq 0 \tag{2.1}$$

By means of the transformation properties of the push forward map σ_{i*} of local sections and of the definition of the transition functions t_{ij}, it can be shown that condition (2.1) is satisfied if and only if the gauge potentials, as seen in two different charts, transform as

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij} \tag{2.2}$$

So the gauge potentials as seen in different charts must be related by (2.2) in order to give raise to a well defined connection one-form on P. It means that a singular $\mathcal{A}_i$ cannot contain all the information on the separation of the bundle; this is achieved by the set of all $\mathcal{A}_i$'s related by (2.2).

Example: the $U(1)$ bundle. Let $P(M, \mathrm{U}(1))$ be a principal bundle. The structure group $\mathrm{U}(1)$ is the circle S^1 and its Lie algebra is a line spanned by the element $i \in \mathbb{C}$. Let a covering atlas (U_i, σ_i) be given on M, such that $U_i \cap U_j \neq 0$; $\mathcal{A}_i$ are gauge potentials as seen in each chart on M. According to (2.2) they

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}$$

where $t_{ij,m} \in \mathrm{U}(1)$ is forced to be of the form $\mathrm{e}^{if(m)}$. Thus

$$\begin{aligned}\mathcal{A}_j &= \mathrm{e}^{-if(m)}\,\mathcal{A}_i\,\mathrm{e}^{if(m)} + \mathrm{e}^{-if(m)}\,\mathrm{e}^{if(m)}\,i\mathrm{d}f(m)\\ \mathcal{A}_j &= \mathcal{A}_i + i\mathrm{d}f\end{aligned} \tag{2.3}$$

$\mathcal{A}_j$ by definition $\in \mathfrak{u}(1) \times \Omega^1(M)$, so it can be decomposed using a basis on $\mathfrak{u}(1)$; then $\mathcal{A}_j = i\, A_j$, where A_j is a real valued one-form $\in \Omega^1(M)$. It follows that

$$A_j = A_i + \mathrm{d}f \tag{2.4}$$

is the transformation law admitted for the component $\Omega^1(M)$ of the gauge one-form.

3 COVARIANT EXTERIOR DERIVATIVE AND CURVATURE IN PRINCIPAL BUNDLES

Given a vector space V and a manifold M, a completely antisimmetric map

$$\omega\colon \underbrace{TM \times \cdots \times TM}_{r \text{ times}} \to V$$

is said to be a vector valued r-form. It's easy to see that ω can always be decomposed in the form

$$\omega = \phi^\alpha \otimes \underline{e}_\alpha$$

$\{\underline{e}_\alpha\}$ being a basis of V, $\phi^\alpha \in \Omega^r(M)$.

DEFINITION 3: *Let $P(M,G)$ be a principal bundle whit a connection one-form separating T_uP into $V_uP \oplus H_uP$. The covariant exterior derivative of a vector valued r-form is defined as:*

$$\mathrm{D}\colon \Omega^r(M) \times V \to \Omega^{r+1}(M) \times V$$

such that

$$\mathrm{D}\,\omega\;(X_1, \ldots, X_{r+1}) = \mathrm{d}\,\omega\;({X_1}^h, \ldots, {X_{r+1}}^h)$$

where $X_i \in T_uP$ can be decomposed into $X = X^v + X^h$.

DEFINITION 4: *The covariant extrerior derivative of the connection one-form ω*

$$\Omega = \mathrm{D}\,\omega$$

is said to be the curvature two-form of the bundle generated by ω.

It can be shown that, given $X, Y \in T_uP$, the curvature two form satisfies

$$\Omega(X,Y) = \mathrm{d}\,\omega(X,Y) + [\omega(X), \omega(Y)] \tag{3.1}$$

known as the Cartan's structure equation. (3.1) can be also written as

$$\Omega = \mathrm{d}\,\omega + \omega \wedge \omega. \tag{3.2}$$

Let the manifold M be equipped with a covering atlas (U_i, σ_i); by conside-ring the definition of gauge potential as the local form of the connection ω, it is obvious that an expression such

$$\mathcal{F}_i = {\sigma_i}^*\,\Omega \tag{3.3}$$

must give the local form (or field strenght) on a chart U_i of the curvature two-form. Let's explicitly write (3.3), we get

$$\begin{aligned}\mathcal{F}_i &= {\sigma_i}^*(\mathrm{d}\,\omega + \omega \wedge \omega) = {\sigma_i}^*(\mathrm{d}\,\omega) + {\sigma_i}^*\omega \wedge {\sigma_i}^*\omega \\ \mathcal{F}_i &= \mathrm{d}({\sigma_i}^*\omega) + {\sigma_i}^*\omega \wedge {\sigma_i}^*\omega \\ \mathcal{F}_i &= \mathrm{d}\,\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \end{aligned} \tag{3.4}$$

which is the local expression of $\mathcal{F}_i$ in terms of gauge potentials $\mathcal{A}_i$ on a chart U_i. It's useful to write (3.4) by using a set of coordinate $x^\mu = (\varphi(m))^\mu$ on a chart U_i

$$(\mathcal{F}_i)_{\mu\nu} = \partial_\mu(\mathcal{A}_i)_\nu - \partial_\nu(\mathcal{A}_i)_\mu + [(\mathcal{A}_i)_\mu, (\mathcal{A}_i)_\nu]$$

Since $(\mathcal{A}_i)_\mu$ and $(\mathcal{F}_i)_{\mu\nu}$ are both $\mathfrak{g}$-valued functions, they can be expanded using a basis of $\mathfrak{g}$, so:

$$(\mathcal{F}_i)_{\mu\nu} = (F_i)^\alpha_{\mu\nu}\underline{e}_\alpha \qquad (\mathcal{A}_i)_\mu = (A_i)^\alpha_\mu\underline{e}_\alpha$$

to obtain

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + {c_{\beta\gamma}}^\alpha A^\beta_\mu A^\gamma_\nu \tag{3.5}$$

where $[\underline{e}_\beta, \underline{e}_\gamma] = {c_{\beta\gamma}}^\delta \underline{e}_\delta$ is the Lie bracket of the basis elements of $\mathfrak{g}$.

In section 2 it has been shown that, in order ω to be uniquely defined by means of the $\mathcal{A}_i$'s on (U_i, σ_i), the $\mathcal{A}_i$'s are required to transform as

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}$$

on $U_i \cap U_j \neq 0$. By plugging this condition into (3.4) the relation between the local curvature two-form as seen in different charts reads:

$$\mathcal{F}_j = t_{ij}^{-1}\mathcal{F}_i t_{ij}. \tag{3.6}$$

Theorem 3 (Bianchi): *Given a curvature two form Ω, then* $\mathrm{D}\,\Omega = 0$.

Proof. Take $X, Y, Z \in T_uP$, then

$$\begin{aligned}\mathrm{D}\,\Omega(X, Y, Z) &\overset{\text{def}}{=} \mathrm{d}\,\Omega(X^h, Y^h, Z^h) \\ &= \mathrm{d}(\mathrm{d}\,\omega + \omega \wedge \omega)(X^h, Y^h, Z^h) \\ &= (\mathrm{d}\,\omega \wedge \omega + (-1)^{\deg \omega}\omega \wedge \mathrm{d}\,\omega)(X^h, Y^h, Z^h) = 0\end{aligned}$$

because $\omega(X^h) = 0$ by definition of X^h. $\square$

By acting with ${\sigma_i}^*$ on $\mathrm{D}\,\Omega$ we obtain the local form of the Bianchi identity as written on a chart (U_i, σ_i):

$$\mathcal{D}\mathcal{F}_i = {\sigma_i}^*(\mathrm{D}\,\Omega) = \mathrm{d}\,\mathcal{F}_i + [\mathcal{A}_i, \mathcal{F}_i] = 0.$$

To conclude this section we note that if the Lie algebra $\mathfrak{g}$ is one dimensional, then relation (3.4) can be written

$$(\mathcal{F}_i)_{\mu\nu} = (F_i)_{\mu\nu}\underline{u} \qquad (\mathcal{A}_i)_\mu = (A_i)_\mu\underline{u}$$

so we can omitt the supercript α in writing the component expression and get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.7}$$

provided ${c_{\beta\gamma}}^{\alpha}$ to be identically zero. Furthermore in the particular example of $\mathrm{U}(1)$ bundles, we find that the transformation law for the field strenght is

$$\mathcal{F}_j = \mathcal{F}_i$$

therefore, in different charts of a $\mathrm{U}(1)$ bundle, the curvature two-form has the same expression; its components are related to the gauge potential by means of (3.7).

4 COVARIANT DERIVATIVE OF VECTOR FIELDS

In this section we introduce the notion of horizontal lift of a path and define a covariant derivative for vector fields depending on the connection one-form defined on $P(M, G)$.

DEFINITION 5: *Let $\gamma\colon]0,1[\to M$ be a curve on M. A curve $\widetilde{\gamma}\colon]0,1[] \to P$ is said to be the horizontal lift of γ if*

$$\pi \circ \widetilde{\gamma} = \gamma$$

and

$$X|_{\widetilde{\gamma}(t)} \in H_{\widetilde{\gamma}(t)}P.$$

The second condition is necessary in order the curve to be uniquely defined (Cauchy's theorem on ordinary differential equation needs a condition on the tangent vectors). The following result holds:

THEOREM 4: *Let $\gamma\colon]0,1[\to M$ be a curve on M, with $u_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\widetilde{\gamma}$ such that $\widetilde{\gamma}(0) = u_0$.*

It is easy to realize that this theorem can be proven by using the fundamental theorem for ordinary differential equations that ensures the local existence and uniqueness of the solution, once "initial conditions" are provided. Note that the horizontal lift depends on the connection one-form through the condition

$$X|_{\widetilde{\gamma}(t)} \in H_{\widetilde{\gamma}(t)}P$$

which means $\omega(X) = 0$ for any tangent vector to the curve $\widetilde{\gamma}$.

Now consider a curve $\gamma\colon]0,1[\to M$ and take a point $u_0 \in \pi^{-1}(\gamma(0))$. The horizontal lift of γ provides a curve $\widetilde{\gamma}$ such that $\widetilde{\gamma}(0) = u_0$. Since this curve is unique, then there exists a unique $u_1 = \widetilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$. Thus $\widetilde{\gamma}$ provides a corrispondence between $u_0 \to u_1$, u_1 referred to as the parallel transport of u_0 along $\widetilde{\gamma}$. In general a map can be introduced

$$\begin{aligned} \Gamma_{\widetilde{\gamma}}\colon]0,1[\times P &\to P \\ (t, u_0) &\to u_t \stackrel{\text{def}}{=} \widetilde{\gamma}(t) \end{aligned} \tag{4.1}$$

relating each point u_0 to a unique point u_t through the curve $\widetilde{\gamma}$. It's important to stress that this assignement depends on the choice of horizontal lift, i.e. on the choice of ω. The map $\Gamma_{\widetilde{\gamma}}$ provides a transport rule of points $\in P$ once a curve on M is taken. This rule can be used to associate a unique point u_t to each point u_0 in a way depending on the connection. If we choose a loop on M, i.e. a curve such $\gamma(0) = \gamma(1)$, the points u_0 and its parallel transport $u_1 = \widetilde{\gamma}(1)$ must lie on the same fibre $\pi^{-1}(\gamma(0))$, but need not to be the same point. So any loop $\mathcal{C}$ in M provides couples $(u_0, u_1)_{\mathcal{C}}$ related by a transformation $\tau_{\mathcal{C}}\colon u_1 = \tau_{\mathcal{C}} u_0$. The set of all $\tau_{\mathcal{C}}$, $\mathcal{C}$ being a loop through $m_0 = \pi(u_0)$, is a subgroup of G and is identified with the holonomy group at u_0 generated by ω.

We are now in the position to define transport rules for vectors, depending on the connection form ω, that enables the definition of a covariant derivative. Let $P(M, G)$ be a tangent bundle on M. The elements of the total space TM are tangent vectors to the manifold M. Transormation laws for point in M are defined onoce a curve on M is provided; but by means of horizontal lift transport rules are provided also for elements in TM:

$$\begin{aligned} \gamma &\colon m \to m_t \\ \widetilde{\gamma} &\colon Y \to Y_t \end{aligned}$$

In this way we can give sense to the expression

$$\lim_{t\to 0} \frac{1}{t}\left(\overline{Y}\big|_{m_t} - Y\big|_{m_0}\right)$$

where $\overline{Y}\big|_{m_t}$ must belong to the same fibre of $Y|_{m_0}$. The above expression makes sense if we set

$$\overline{Y}\big|_{m_t} \stackrel{\text{def}}{=} (\Gamma_{\widetilde{\gamma},t})^{-1}\, Y|_{m_t}$$

so we obtain

$$Y'\big|_{m_0} = \lim_{t\to 0} \frac{1}{t}\left[(\Gamma_{\widetilde{\gamma},t})^{-1}\, Y|_{m_t} - Y|_{m_0}\right] \tag{4.2}$$

where $\Gamma_{\widetilde{\gamma}}$ is the transport rule associated to the curve γ by means of a connection form ω on TM. If γ is a flow generated by the field X, we refer to the previous as to the covariant derivative of the vector field Y through X, evaluated at m_0. By changing the point of evaluation m_0 we get a vector field corrispondence:

$$\begin{aligned} \nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\to \nabla_X Y \end{aligned}$$

Such a map defines an operator, called the covariant derivative ∇_X, which acting on Y gives raise to $\nabla_X Y$. This operator satisfies

- linearity in X and Y
- $\nabla_X (fY) = f\nabla_X Y + (\mathrm{L}_X f)\, Y$ Leibniz rule on $f \in \mathscr{F}(M)$
- $\nabla_{(fX)} Y = f\nabla_X Y$ invariance under reparametrization of the field

the last relation being very useful in employing the covariant derivative as a correct derivation rule for physical field theories.

5 ALTERNATIVE DEFINITION OF CONNECTIONS

In this section we introduce a more general definition of connection on a bundle which does not involve the structure group G and its action on a fibre of P. This way of introducing connections is based on a particular choice of "projectable fields" in a sense that we are going to define as follows:

DEFINITION 6: *Let $X \in \mathfrak{X}(P)$ be a vector field on P; X is said to be vertical if, for any $f \in \mathscr{F}(M)$, it happens that*

$$\mathrm{L}_X(\pi^* f) = 0.$$

This equation means

$$\mathrm{L}_X(\pi^* f) = \mathrm{L}_X(f \circ \pi) = X[f \circ \pi] = 0$$

for any point $u \in P$; as a consequence, if we evaluate the previous relation at $u \in P$ we get

$$(X\,[f \circ \pi])|_u = X|_u[f \circ \pi] = (\pi_* X|_u)[f] = 0$$

which means that $X|_u$ belongs to the kernel of π_* at u. This definition of vertical fields is then in agreement with the definition given in Section 1, which introduces the vertical spaces $V_u P$ as the kernel of the map π_* at each point $u \in P$.

DEFINITION 7: *A field $Y \in \mathfrak{X}(P)$ is said to be projectable if $\exists\, X \in \mathfrak{X}(M)$ such that, for any $f \in \mathscr{F}(M)$*

$$\mathrm{L}_Y(\pi^* f) = \pi^*(\mathrm{L}_X f).$$

In that case we refer to the field Y as to $X^{\uparrow}$, and the previous relation may be written as

$$\mathrm{L}_{X^{\uparrow}}(\pi^* f) = \pi^*(\mathrm{L}_X\, f)$$

and it seems that the operator $\uparrow$ "brings" X to act on f before π^*. With this definition we can regard a vertical field as a field which projects on the null field in $\mathfrak{X}(M)$. The following proposition shows that the set of vertical fields $\mathfrak{X}^v(P)$ is an ideal in $\mathfrak{X}^\pi(P)$.

PROPOSITION 2: *The set of projectable fields could be equivalently defined as the set of fields which leaves $\mathfrak{X}^v(P)$ invariant under the action of the Lie bracket, that is, given $Z \in \mathfrak{X}^v(P)$*

$$Y \in \mathfrak{X}^\pi(P) \iff [Z, Y] \in \mathfrak{X}^v(P)$$

Proof. Let's first prove: $Y \in \mathfrak{X}^\pi(P) \implies [Z, Y] \in \mathfrak{X}^v(P)$. Since Y is projectable then there exists a field $X \in \mathfrak{X}(M)$ such that

$$\begin{aligned} \mathrm{L}_Y(\pi^* f) &= \pi^*(\mathrm{L}_X\, f) \qquad\qquad \forall f \in \mathscr{F}(M) \\ \mathrm{L}_Z\, \mathrm{L}_Y(\pi^* f) &= \mathrm{L}_Z \pi^*(\mathrm{L}_X f) = 0 \end{aligned}$$

because Z is vertical. We can add a null term to this equation without violating the identity, and we choose it to be of the form $-\mathrm{L}_Y\mathrm{L}_Z(\pi^* f)$. Therefore we get

$$\mathrm{L}_Z\mathrm{L}_Y(\pi^* f) - \mathrm{L}_Y\mathrm{L}_Z(\pi^* f) = \mathrm{L}_{[Z,Y]}(\pi^* f) = 0$$

which means that $[Z, Y]$ is a vertical field. Conversely, if $[Z, Y]$ is a vertical field, then

$$\begin{aligned} \mathrm{L}_{[Z,Y]}(\pi^* f) &= 0 \qquad \forall f \in \mathscr{F}(M) \\ \mathrm{L}_Z\mathrm{L}_Y(\pi^* f) - \mathrm{L}_Y\mathrm{L}_Z(\pi^* f) &= \mathrm{L}_Z\mathrm{L}_Y(\pi^* f) = 0. \end{aligned}$$

Z is a vertical field for hypothesis, so this condition implies $L_Y\,(\pi^* f)$ to be of the form $\pi^*(\,\cdot\,)$, i.e. Y must be projectabile on a particular field in $\mathfrak{X}(M)$. □

It is obvious that if we fix $X \in \mathfrak{X}(M)$ there exist different $Y \in \mathfrak{X}(P)$ such that

$$Y = X^{\uparrow}$$

in fact we can choose a particular $X^{\uparrow}$ and add a generical vertical field to obtain another field which projects on the same $X \in \mathfrak{X}(M)$. We say the lift of a vector field $X \in \mathfrak{X}(M)$ is determined up to vertical fields. The particular choise of the lift $X^{\uparrow}$, once X is assigned, defines the choise of connection. Let's define a map

$$\begin{aligned} \sigma\colon \mathfrak{X}(M) &\to \mathfrak{X}^{\pi}(P) \\ X &\to X^{\uparrow} \end{aligned} \tag{5.1}$$

which is required to be an $\mathscr{F}(M)$-module, that is

$$(fX)^{\uparrow} = (\pi^* f)X^{\uparrow}$$

with the further requirement $\pi_* \circ \sigma = \mathrm{id}_{\mathfrak{X}(M)}$. Under this assumption we call the map (5.1) a connection on the bundle $P(M, G)$. Notice that this definition of connection does not involve the structure group G and its action on the fibre, conversely it just makes use of the spaces P and M.

Definition 8: *The set of vector fields spanned as a module on P by the lift $X^{\uparrow}$ are referred to as the horizontal fields $\mathfrak{X}^h(P)$.*

The map (5.1) need not to be a Lie-algebra homomorphism. The failure of this condition is taken into account by the quantity

$$\Omega(X_1, X_2) \stackrel{\text{def}}{=} [X_1, X_2]^{\uparrow} - [X_1{}^{\uparrow}, X_2{}^{\uparrow}]$$

called the *curvature* of the connection.

Proposition 3: *The following holds:* $\Omega(X_1, X_2) \in \mathfrak{X}^v(P)$

Proof. Take $f \in \mathscr{F}(M)$ and evaluate

$$\begin{aligned} \mathrm{L}_{\Omega}(\pi^* f) &= \mathrm{L}_{[X_1,X_2]^{\uparrow} - [X_1{}^{\uparrow},X_2{}^{\uparrow}]}(\pi^* f) \\ &= \mathrm{L}_{[X_1,X_2]^{\uparrow}}(\pi^* f) - \mathrm{L}_{[X_1{}^{\uparrow},X_2{}^{\uparrow}]}(\pi^* f) \\ &= \pi^*(\mathrm{L}_{[X_1,X_2]} f) - (\mathrm{L}_{X_1{}^{\uparrow}}\mathrm{L}_{X_2{}^{\uparrow}} - \mathrm{L}_{X_2{}^{\uparrow}}\mathrm{L}_{X_1{}^{\uparrow}})(\pi^* f) \\ &= \pi^*(\mathrm{L}_{[X_1,X_2]} f) - \pi^*(\mathrm{L}_{[X_1,X_2]} f) = 0 \end{aligned}$$

how can be easily checked by means of the definition of $X^{\uparrow}$. □

The map π between P and M induces a pullback of forms at each point of M. Let's now consider the pullback $\pi^*\,\Omega$ evaluated in a point $m = \pi(u) \in M$ and let's calculate its action on vertical vectors $Y_1, Y_2 \in V_u P$

$$(\pi^*\,\Omega)(Y_1, Y_2) = \Omega(\pi_* Y_1, \pi_* Y_2).$$

This quantity actually vanishes, because vertical vectors $\in V_u P$ are in the kernel of π_* at each point $u \in P$; in this sense we often refer to this condition by saying the curvature is "horizontal" (note the abuse: indeed the curvature is a vertical field because its action on fields in $\mathfrak{X}(M)$ gives raise to a field in $\mathfrak{X}^v(P)$; we say it is "horizontal" in the sense that its pullback at each point $\pi^*\Omega$ vanishes when evaluated on vertical vectors). This behavior can be compared with the same behavior of the curvature form defined in Section 3 as the covariant exterior derivative of the connection one-form; the map

$$\underline{\Omega} \in \Omega^2(P) \otimes \mathfrak{g}$$

also vanishes on vertical vectors $\in V_u P$ at each point $u \in P$, because it is defined as a horizontal exterior derivative. So if we could identify the sets $\mathfrak{g}$ and $\mathfrak{X}^v(P)$ the two different maps we introduced in the description of a connection theory, namely the two different definition of "curvature", could be regarded as the same object as described by using different formalism. In order to achieve this step let's take a basis $(V_1, \ldots, V_n)$ in $\mathfrak{g}$. Each basis vector V_i generates a one-parameter subgroup $g_i(t) = \mathrm{e}^{tV_i}\, g(0)$ on G. By means of the right action on the manifold P, this induces a flow $\sigma(t) \in P$; the map relating each point on P to the tangent vector to the flow induced by G is called the induced vector field $V_i^\#$ generated by $V_i \in \mathfrak{g}$. The corrispondence

$$\#\colon \mathfrak{g} \to \mathfrak{X}(P)$$

is an isomorphism. The set of induced vector fields $V_i^\#$ span as a module the entire set $\mathfrak{X}^v(P)$; hence, up to an isomorphism #, we can identify $\mathfrak{g}$ and $\mathfrak{X}^v(P)$. By using this formulation one can easily obtain the field strenght as the curvature of the connection. Let's perform the calculation in detail for an example involving a spacetime bundle.

Example. Let $P(M, G)$ be a bundle with $P = \mathbb{R}^5$ and $M = \mathbb{R}^4$. Charts on P and M have coordinate representations as $(x_1, \ldots, x_4)$ in $\mathbb{R}^4$ and $(y_1, \ldots, y_4, z)$ in $\mathbb{R}^5$. The assigned connection is

$$\begin{aligned} \sigma\colon \mathfrak{X}(\mathbb{R}^4) &\to \mathfrak{X}(\mathbb{R}^5) \\ \frac{\partial}{\partial x^\mu} &\to \frac{\partial}{\partial y^\mu} - A_\mu(y^\nu) z \frac{\partial}{\partial z} \end{aligned}$$

A_μ being a real function on $\mathbb{R}^4$. The curvature of the connection is defined as the failure of σ in being a Lie-algebra homomorphism

$$\Omega\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \stackrel{\text{def}}{=} \left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right]^\uparrow - \left[\frac{\partial}{\partial x^\mu}^\uparrow, \frac{\partial}{\partial x^\nu}^\uparrow\right]$$

The Lie brackets are evaluated by direct application on a function $f \in \mathscr{F}(\mathbb{R}^5)$. The calculation yields

$$\Omega\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \left(\frac{\partial A_\nu}{\partial y^\mu} - \frac{\partial A_\mu}{\partial y^\nu}\right) z\frac{\partial}{\partial z}.$$

The component $\Omega_{\mu\nu}$ is called the field strenght of the connection.

6 ELECTROMAGNETISM AS AN U(1) GAUGE THEORY

In this last section we introduce a formal theory of electromagnetism coherent with Maxwell equations by employing the minimum number of initial hyphotheses and using the formalism of connections on bundles. We use the description developed in Sections 1, 2, 3 because we want to recover the formal structure of electromagnetic theory by taking into account only the transformation properties required for the potentials.

Hypothesis 1. Assume that a one-form A exists such that the interaction of a charged particle with the field appears in the action with the term

$$S_{\text{int}} = \int qA \tag{6.1}$$

q being the charge of the particle, $A = A_\mu(x^\nu)\mathrm{d}x^\mu$ the potential of the field. Equations of motion for a charged particle are obtained by making variations on

$$S = S_{\text{free}} + S_{\text{int}} = -mc\int_{\mathrm{P}_1}^{\mathrm{P}_2} \mathrm{d}s + \int_{\mathrm{P}_1}^{\mathrm{P}_2} qA$$

with respect to the paths γ whose boundaries P_1 and P_2 are fixed. In making variations on the integral (6.1) it is straightforward to realize that derivatives of the components A_μ of A with respect to the coordinate x^ν will appear. Hence the equations of motion will be written in a form containing derivatives of A; in particular we have

$$\frac{d}{ds}\, p_\mu = q(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu.$$

We want the electromagnetic field to be defined as the object appearing in the RHS of the equations of motion, so it is a natural choice to define

$$F = \mathrm{d}\, A.$$

With this definition we get $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A trivial consequence of the above definition is that

$$\mathrm{d}^2 A = \mathrm{d}\, F = 0$$

identically. Evaluation of this expression in components leads to

$$\partial_\lambda F_{\mu\nu} + \text{cyc. perm.}(\lambda, \mu, \nu) = 0 \tag{6.2}$$

which we recognize to be a set of four Maxwell equations. This set of equations has been obtained just employing *hypothesis 1* and they contain no physical information about how the fields are related to the sources of propagation. They just recover the nature of the potential as a one-form. Indeed there may exist different one-forms giving raise to the same equations of motion. These different potentials must be related in a way such that making variations on (6.1) as expressed with "another" potential A_2 gives raise to no other conditions apart from the ones obtained in making variations with the initial potential A_1. In formulae, let

$$S = S_{\text{free}} + S_{\text{int}}^{(1)}$$

be the action as expressed in terms of the potential A_1, which means that the equations of motion are $\delta S = \delta S_{\text{free}} + S_{\text{int}}^{(1)} = 0$. We want to know how a different potential A_2 can be related to A_1 in order to get the same equations. If $A_2 = f(A_1)$, with f any function (we exclude a trivial change of scale) we get a new action

$$S' = S_{\text{free}} + S_{\text{int}}^{(2)}$$

and equations of motion of the form $\delta S = \delta S_{\text{free}} + S_{\text{int}}^{(2)} = 0$. In order this equation to be the same we obtained by taking the potential A_1, we have to require that

$$\delta S_{\text{int}}(A_2) = \delta S_{\text{int}}(f(A_1)) = \delta S_{\text{int}}(A_1).$$

It is easy realized that the previous equation imposes a new condition on A_1 to hold, different from the ones obtained in the equations of motion. Hence a transformation rule of the potential with a generical function is not allowed because it woulde change the action and then the condition required by the dynamics. We can allow transformation laws as

$$A_2 = A_1 + B$$

where B is a generical one-form. Let's insert this condition in the action and evaluate

$$\delta S_{\text{int}}(A_2) = \delta S_{\text{int}}(A_1) + \delta S_{\text{int}}(B) = \delta S_{\text{int}}(A_1)$$

i.e. the one-form B must be such that $\delta S_{\text{int}}(B) = 0$.

$$\begin{aligned}
\delta \int B \overset{\text{def}}{=} \int \delta B &= \int \delta B_\mu \mathrm{d}x^\mu + B_\mu \delta \mathrm{d}x^\mu \\
&= \int \partial_\nu B_\mu \delta x^\nu \mathrm{d}x^\mu + B_\mu \mathrm{d}\delta x^\mu \\
&= \int \partial_\nu B_\mu \delta x^\nu \mathrm{d}x^\mu - \delta x^\mu \partial_\nu B_\mu \mathrm{d}x^\nu \\
&= \int \delta x^\nu \mathrm{d}x^\mu \left(\partial_\nu B_\mu - \partial_\mu B_\nu\right)
\end{aligned}$$

which implies the component of B to satisfy $\partial_\nu B_\mu - \partial_\mu B_\nu = 0$, i.e. the form B has to be exact: $B = \mathrm{d}f$. Hence the only admitted transformation law for the potentials is

$$A_2 = A_1 + \mathrm{d}f.$$

Notice that this transformation rule has been obtained only by fixing the interaction term in the action as

$$\int qA$$

so this action hypothesis uniquely fixes the properties of transformation for the potentials. The equation relating two potentials giving raise to the same dynamics of the system is nothing but the equation (2.4) that relates the components $\in \Omega^1(M)$ of two gauge potentials as seen in different charts on a $\mathrm{U}(1)$ bundle on a manifold $M = \mathbb{R}^4$. It is clear then that the set of all potentials able to describe the dynamic of a charged particle in an electromagnetic field is nothing but the set of all gauge potentials on a $\mathrm{U}(1)$-bundle as seen in different charts U_i; thus electromagnetism can be regarded as a theory on a principal bundle $P(\mathbb{R}^4, \mathrm{U}(1))$. Any potential represents the gauge potential giving raise to the connection one-form $\omega \in \mathfrak{u}(1) \otimes T^*P$ as seen in a chart $U_i \in \mathbb{R}^4$.

Let's now evaluate the local form of the curvature of the connection generated by the gauge potentials. By definition (3.3)

$$\mathcal{F}_i = \sigma_i{}^* \, \Omega = \mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i.$$

The structure group $\mathrm{U}(1)$ is one dimensional, so the field strenght has the simplified expression

$$\mathcal{F}_i = \mathrm{d}\mathcal{A}_i$$

which expanded on the base $i \in \mathfrak{u}(1)$ gives raise to the expression (3.7) for the components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which is identical to the "physical" definition of the electromagnetic tensor we introduced before. Then, by using the theory of connections on principal bundles, we can regard the electromagnetic field as the curvature of the connection form on a $\mathrm{U}(1)$ bundle. The transformation law for field strenghts on $\mathrm{U}(1)$ is

$$\mathcal{F}_i = \mathcal{F}_j$$

which is the mathematical proof that the electromagnetic tensor must not transform under a gauge potential transformation. It is also easy to show that Bianchi identity can be written in the form

$$\mathrm{d}\,\mathcal{F} = \mathrm{d}^2\mathcal{A} = 0$$

which is nothing but equation (6.2).

Obviously no relation between the field F and the sources may emerge from the geometrical structure of the $\mathrm{U}(1)$ bundle on $\mathbb{R}^4$, so we are led to introduce another hypothesis in order to obtain the relation between sources and field. In order to achieve this purpose some considerations about classical electrodynamics are needed.
The entire classical electromagnetic theory is described once the fields (in this formalism regarded as forms) E, D, B, H are assigned in each point of a spacetime manifold $M = \mathbb{R}^4$. These fields can be expressed by means of

the 2-form F we have introduced as the derivative of the potential form. In particular if we define

$$B = (\partial_i A_j - \partial_j A_i)\mathrm{d}x^i \wedge \mathrm{d}x^j$$
$$E = (\partial_i A_0 - \partial_0 A_i)\mathrm{d}x^i$$

the 2-form $F = (\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = B + E \wedge \mathrm{d}t$. These definitions are nothing but the expressions of the fields B and E in terms of the potential A, as derived from standard equations written in the form

$$\operatorname{div} \mathbf{B} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0$$

Notice that in classical electromagnetic theory, equations containing sources only involve the fields D and H, instead of the equations without sources, which only involve the fields B and E. Maxwell equations for fields cannot provide the complete set of solutions for propagation, because the number of unknows is more than the number of equations. So we need to specify a relation between fields E, D, B, H, in order the equations to be uniquely solved, up to initial and boundary conditions; generally such kind of relations are assumed to be true by experience. We write the constitutive equations for fields as

$$\begin{aligned} B &= *H \\ D &= *E \end{aligned} \tag{6.3}$$

Notice that in writing these equations a metric (expressed by means of the Hodge $*$ operator) must be introduced. So the constitutive equations (depending on the continuum media in which fields propagation takes part) uniquely fix the metric to use. Once we have (6.3) we can easily express a field as a function of the other. Now the task is to build an action principle from which the remaining Maxwell equations can be derived by making variations. Usually if the dynamics of a system is assigned, it is not obvious that an action S can be written in order to derive the dynamics by making variations on S; in other words, we are not sure that such a system admits a lagrangian description in terms of variational principles. The inverse problem of motion, i.e. providing a lagrangian description once the dynamic is assigned, could not admit general solutions, or, anyway, such a solution could not emerge in an easy way.
Anyway we know by experience that electromagnetism does admit such a lagrangian description, and we introduce the action principle by means of direct construction from Maxwell equations.

Hyphothesis 2. We postulate the action for field propagation to be

$$S_{\text{field}} = \int \mathrm{d}A \wedge *\mathrm{d}A + A \wedge \mathcal{J} \tag{6.4}$$

with the current 3-form $\mathcal{J}$ defined as $\mathcal{J} = 1/3!\, \varepsilon_{\mu\nu\lambda\rho} j^\mu \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\rho$. Remark by the way that a metric on $\mathbb{R}^4$ (expressed by means of the Hodge $*$ operator) has been introduced to derive the equations for the field. Now in making variations on (6.4) we assume the current to be assigned, so variations must be carried out by varying only the field: $A \to A + \delta A$.

With the help of some algebra and of the definition of Frechet derivative for functionals:

$$\delta S(\phi) \stackrel{\text{def}}{=} \lim_{s\to 0} \frac{S(\phi + s\delta\phi) - S(\phi)}{s}$$

we compute the result assuming the variation $\delta S\,(\phi)$ to be zero if ϕ is a solution of the equations of motion.

$$\delta S(A) = \lim_{s\to 0} \frac{1}{s} \left[\int \mathrm{d}(A + s\delta A) \wedge *\mathrm{d}(A + s\delta A) + \right.$$
$$\left. + \; (A + s\delta A) \wedge \mathcal{J} - \mathrm{d}A \wedge *\mathrm{d}A - A \wedge \mathcal{J}\right] = 0$$

$$\begin{aligned} \delta S(A) &= \int \mathrm{d}\delta A \wedge *\mathrm{d}A + \mathrm{d}A \wedge *\mathrm{d}\delta A + \delta A \wedge \mathcal{J} &&= 0 \\ \delta S(A) &= \int 2\mathrm{d}\delta A \wedge *\mathrm{d}A + \delta A \wedge \mathcal{J} &&= 0 \end{aligned}$$

The action may be defined so that the factor 2 does not appear in the last line. Calculations lead to

$$\begin{aligned} \delta S(A) &= \int \mathrm{d}(\delta A \wedge *\mathrm{d}A) - \delta A \wedge \mathrm{d} * \mathrm{d}A + \delta A \wedge \mathcal{J} &&= 0 \\ \delta S(A) &= \int -\delta A \wedge \mathrm{d} * \mathrm{d}A + \delta A \wedge \mathcal{J} &&= 0 \end{aligned}$$

where boundary conditions assumed in the action principle have been used. If we choose the metric to be Lorentian, then $-1 = **$, so the equation becomes

$$\begin{aligned} \delta S(A) &= \int \delta A \wedge \; * * \mathrm{d} * \mathrm{d}A - \delta A \wedge * * \mathcal{J} &&= 0 \\ \delta S(A) &= \int \delta A \wedge *(*\mathrm{d} * \mathrm{d}A - *\mathcal{J}) &&= 0 \end{aligned}$$

since this condition must hold for any δA we choose, the non-vanishing requirement on the scalar product leads to

$$\mathrm{d}^\dagger \mathrm{d}A = *\mathcal{J}. \tag{6.5}$$

By means of equations (6.3) we can easily write the previous relation in a form containing D and H. We introduce a 2-form $G = D - H \wedge \mathrm{d}t$, where D and H are given by (6.3); with the help of this definition (6.5) can be written as

$$\mathrm{d}G = \mathcal{J} \;.$$

If we choose the one-form A such that $\mathrm{d}^\dagger A = 0$, equation (6.5) may be written as

$$(\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)A = *\mathcal{J}$$

which is the common wave equation for the potential A:

$$\Box A = *\mathcal{J}.$$

Condition $\mathrm{d}^\dagger A = 0$ may be employed if we consider the gauge transformation $A = A' - \mathrm{d}f$, so

$$\begin{aligned} \mathrm{d}^\dagger A &= \mathrm{d}^\dagger A' - \mathrm{d}^\dagger \mathrm{d}f &&= 0 \\ \mathrm{d}^\dagger A &= \mathrm{d}^\dagger A' - \Box f &&= 0 \end{aligned}$$

which holds whenever a function f can be found such that

$$\Box f = \mathrm{d}^\dagger A'.$$

We assume the last equation to be always satisfied for any potential A'.

In this way one ends up with the electromagnetic equations in the form

$$\begin{aligned} \mathrm{d}^2 A &= 0 \\ \mathrm{d}G &= \mathcal{J} \end{aligned}$$

and

$$\begin{aligned} B &= *H \\ D &= *E \end{aligned}$$

Usually a physical theory is described by means of "equations of motion" and "constraint equations", the former being equations containing the derivatives with respect to a parameter describing the flow on a manifold M (in point mechanics d/dt). In this theory the dynamical fields are maps on a manifold $M = \mathbb{R}^4$, so the derivatives with respect to each component $\partial/\partial x^\mu$ all play the same role. The "true equations of motion" are then the only equations which are not "geometrical", i.e. not derived by means of geometrical properties of $\mathrm{U}(1)$ bundles.

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