NON-TRANSITIVE SET OF DICE WITH RDICE

GENNARO TEDESCO

Abstract

We present a simple construction of non-transitive set of dice in some particular cases, deriving a general formula for the corresponding non-transitive probabilities. In particular we introduce the concept of fully perfect non-transitive set and show that non-transitive constructions always exist.

Moreover, we make use of the `R` package `Rdice` (developed by the author) to show how constructions can be iteratively found, provide several example for different cases.

CONTENTS

1 PRELIMINARIES

Let $D_1, \ldots, D_Z$ be Z finite sets, where $D_k = \{ d_k^1 \leq \ldots \leq d_k^N \}$, $k = 1, \ldots, Z$. Should the elements of such sets be natural numbers, we would then refer to the collection of $D_1, \ldots, D_Z$ as to a set of dice, whose cardinalities $|D_k|$ denote the number of faces thereof. As to do so, picking a random element d_k^P will be the equivalent of rolling the k^{th} die on its P^{th} face with outcome d_k^P. A game of two dice (I, J) is the cartesian product $D_I \times D_J$. Likewise, a game of Z dice is the pairwise union of all the possible $\binom{Z}{2}$ games of two dice each, with cardinality $\binom{Z}{2} \cdot N^Z$.

Let us henceforth consider a set of Z dice. Without loss of generality we may assume N to be even (though the odd case works along the same lines). For each pair of dice (D_I, D_J) let $M_{IJ} := |\{ (i,j) \mid d_I^i > d_J^j \}|$. The die D_I is said to win against the die D_J (denoted as $D_J \rhd D_J$) whenever $M_{IJ} > N/2$, otherwise the two dice are said to tie. Replacing $I \to J$ exhausts the one more remaining case. In a nutshell, the former die wins if it rolls higher than the latter more than a half times; in particular, counting all the possible outcomes, the probability of such to be achieved is given by the overall possible combinations

$$p(D_I \rhd D_J) = \frac{M_{IJ}}{N^2}$$

Example 1 *Given two three-sided dice as*

$$D_1 = \{1, 5, 6\}$$
$$D_2 = \{2, 3, 4\}$$

the former dice wins against the latter $M_{12} = 6$ times, hence with $p(D_1 \rhd D_2) = 2/3$.

Definition 1 *A set of Z N-sided dice is said to be fully Efron's of order (Z, N) if the cyclicity condition*

$$D_1 \rhd D_2 \rhd \ldots \rhd D_Z \rhd D_1$$

holds with $p(D_{I-1} \rhd D_I) = p(D_I \rhd D_{I+1}), \quad I = 2, \ldots, Z-1$. Should the mutual winning probabilities be different from one other, such set is then said to be partially Efron's. Also, a fully Efron's set of dice of order (Z, N) is said to be perfect if the dice entries exhaust all the natural numbers $\{1, \ldots, ZN\}$.

2 Construction of fully perfect Efron's dice

The simplest case $Z = N$ provides a standard construction of fully perfect Efron's dice, as we shall show by iteration. For $N = 2$ the dice always tie, therefore we start with $N = 3$. As it will be later on clarified, a standard constuction may proceed by cyclicity as follows: we pick a different starting die for each face and fill the values up incrementing each face by 1 upwards.

Example 2 (Three three-sided perfect Efron's dice) *The following choice*

$$D_1 = \{\odot, (\,\cdot\,), (\,\cdot\,)\}$$
$$D_2 = \{(\,\cdot\,), (\,\cdot\,), \odot\}$$
$$D_3 = \{(\,\cdot\,), \odot, (\,\cdot\,)\}$$

allows

$$D_1 = \{1, 6, 8\}$$
$$D_2 = \{3, 5, 7\}$$
$$D_3 = \{2, 4, 9\}$$

with winning probability $p(D_I \rhd D_{I+1}) = 5/9$. It is easy to see that any other choice would lead to the same result.

Example 3 (Four four-sided perfect Efron's dice) *The following choice*

$$D_1 = \{\odot, (\,\cdot\,), (\,\cdot\,), (\,\cdot\,)\}$$
$$D_2 = \{(\,\cdot\,), (\,\cdot\,), \odot, (\,\cdot\,)\}$$
$$D_3 = \{(\,\cdot\,), \odot, (\,\cdot\,), (\,\cdot\,)\}$$
$$D_4 = \{(\,\cdot\,), (\,\cdot\,), (\,\cdot\,), \odot\}$$

allows

$$D_1 = \{1, 7, 10, 16\}$$
$$D_2 = \{4, 6, 9, 15\}$$
$$D_3 = \{3, 5, 12, 14\}$$
$$D_4 = \{2, 8, 11, 13\}$$

with winning probability $p(D_I \rhd D_{I+1}) = 9/16$.

Example 4 (Five five-sided perfect Efron's dice) *Along the same lines*

$$\begin{aligned} D_1 &= \{1, 7, 13, 18, 25\} \\ D_2 &= \{5, 6, 12, 19, 24\} \\ D_3 &= \{4, 10, 11, 20, 23\} \\ D_4 &= \{3, 9, 15, 16, 22\} \\ D_5 &= \{2, 8, 14, 17, 21\} \end{aligned}$$

with winning probability $p(D_I \rhd D_{I+1}) = 14/25, I = 3, 4$ *and* $p(D_I \rhd D_{I+1}) = 13/25$ *elsewise.*

Example 5 (Six six-sided perfect Efron's dice) *Also*

$$\begin{aligned} D_1 &= \{1, 9, 14, 24, 28, 35\} \\ D_2 &= \{6, 8, 13, 23, 27, 34\} \\ D_3 &= \{5, 7, 18, 22, 26, 33\} \\ D_4 &= \{4, 12, 17, 21, 25, 32\} \\ D_5 &= \{3, 11, 16, 20, 30, 31\} \\ D_6 &= \{2, 10, 15, 19, 29, 36\} \end{aligned}$$

with winning probability $p(D_I \rhd D_{I+1}) = 20/36 = 5/9$.

With no need to go any further, the probability path is easy to recognise. In fact, for each N we have at most $N(N+1)/2 - 1$ mutual winning combinations, giving rise to

$$p(D_I \rhd D_{I+1})|_{\max} = \frac{N(N+1)/2 - 1}{N^2} = \frac{1}{2} + \frac{1}{2N} - \frac{1}{N^2}$$

which decreases monotonously to $1/2$ as $N \to \infty$. The maximal winning probability is obtained for $N = 4$.

The package `Rdice`[1] provides the function `is.nonTransitive` to check whether a given set of dice is non-transitive; as an example we can prove if on the above $N = 5$ fully Efron's set as follows:

```
fullyPerfectFive <- data.frame(
          die1 = c(1,7,13,18,25),
          die2 = c(5,6,12,19,24),
          die3 = c(4,10,11,20,23),
          die4 = c(3,9,15,16,22),
          die5 = c(2,8,14,17,21)
        )

> is.nonTransitive(fullyPerfectFive)
[1] TRUE
```

2.1 *The case $Z < N$: minimal winning probabilities*

We will now look at the case $Z < N$; in particular we shall try to generate non-transitive set of dice with *minimal* winning probability.

It is straightforward to see that once we fix the number of dice Z, the minimal winning probabilities one can achieve are given by

$$\tilde{p}(N) \equiv p(N)|_{\min} = \begin{cases} \dfrac{\dfrac{N^2}{2} + 1}{N^2}, & N \text{ even} \\[2ex] \dfrac{\dfrac{N^2 - 1}{2} + 1}{N^2}, & N \text{ odd.} \end{cases} \tag{1}$$

[1] `https://cran.r-project.org/web/packages/Rdice/index.html`

In particular, the minimal probability of any given odd integer N is always smaller than the one achieved by its next even integer, namely one has $\tilde{p}(2K-1) < \tilde{p}(2K)$, $K \in \mathbb{N}$. As such, for the minimal probability examples we will only be looking at the case of N odd.

The package Rdice provides the function `nonTransitive.generator`, that generates a random set of non-transitive dice with given probability, once one specifies conditions on the total number of faces and maximum integer value allowed for any thereof. For the sake of simplicity we can start looking at the case of $Z = 3$ and vary N in order to minimise the winning probability according to (1). For $N = 5$ (we look at odd integers only, as specified before) one has $\tilde{p} = 13/25$: with `nonTransitive.generator(dice = 3, faces = 5, max_value = 6, prob = 13/25)`

```
# set prob = 13/25
is.nonTransitive(
        data.frame(
        die1 = c(0,2,6,1,6),
        die2 = c(5,4,0,1,4),
        die3 = c(2,4,2,3,3)
        ),
prob = 13/25
)
[1] TRUE
```

Choosing different values for N generates non-transitive sets with smaller winning probabilities, as N increases, due to the fact that (1) decreseas monotonously for $N \to \infty$; with $N = 13$ faces and using the function `nonTransitive.generator(dice = 3, faces = 13, max_value = 20)` we find the smallest possible winning probability for a non-transitive set of $Z = 3$ dice to be given by:

```
is.nonTransitive(
        data.frame(
        die1 = c(5,8,11,6,6,8,7,18,5,4,16,18,16),
        die2 = c(3,11,0,4,17,15,0,13,2,14,17,11,19),
        die3 = c(16,0,8,9,4,9,11,17,12,12,9,1,18)
        ),
)
[1] TRUE
```

where the second die beats the third with $\tilde{p}^* = 84/169 = 0.5029586$. The other combinations, however, beat each other with $p > \tilde{p}^*$. For a set of $Z = 4$ we have produced the following results: $N = 7$-faced dice

```
is.nonTransitive(
        data.frame(
        die1 = c(3,6,8,3,9,0,2),
        die2 = c(8,9,2,1,1,8,1),
        die3 = c(6,5,6,0,6,4,7),
        die4 = c(4,0,0,4,10,9,4)
        ),
)
[1] TRUE
```

with the first dice realising a non-transivitive winning probability of 25/49. For $N = 8$ we reach the least possible non-transitive winning probability, namely, due to ties,

```
is.nonTransitive(
        data.frame(
```

```
        die1 = c(5,5,7,10,4,4,5,1),
        die2 = c(10,2,2,5,6,2,6,3),
        die3 = c(0,1,9,1,0,10,10,9),
        die4 = c(6,7,0,7,0,7,5,6)
        ),
)
[1] TRUE
```

presents the first and the third dice winning the corresponding next one with $p = 1/2$.

3 Maximal probabilities for general Efron's dice

The case of maximal probabilities is much harder to deal with and we refer the reader to the corresponding literature. In particular the author in [1] shows that the maximal probability for a set of Z dice cannot exceed the value

$$p_{\max} = 1 - \frac{1}{4\cos^2(\pi/(Z+2))}$$

providing examples where the above is just an upper bound as its irrational value cannot be realised with regular set of dice as the total number of success and possible cases must still be integers.

References

[1] Igor Bogdanov. Intransitive roulettes. 14(III):240–255, 2010.

[2] S. Trybuła. On the paradox of n random variables. *Applicationes Mathematicae*, 8(2):143–156, 1965.

[3] Zalman Usiskin. Max-min probabilities in the voting paradox. *Ann. Math. Statist.*, 35(2):857–862, 06 1964.