Geomorphic (SPIM) fundamental function in canonical, generalized Kropina metric form

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Summary

The geomorphic fundamental "metric" function $\mathcal{F}^*$ prescribes the distance traveled by an erosion front in unit time. It takes the form (for the bedrock channel SPIM model) of power functions of the surface-normal erosion slowness covector components $\mathbf{p} = [p_x, p_z]$ multiplied by a position dependent rate variable $\varphi(\mathbf{r})$. This kind of fundamental function is called a generalized Kropina metric on $\mathsf{T}^*\mathsf{M}$. This canonical metric is one of the family of so-called (α^*, β^*) Finsler (Cartan) metrics that, when defined on the cotangent manifold indicated by the $*$ annotation, combine a contravariant Riemannian metric tensor a^{ij} with a vector field b^i (the equivalent on the tangent manifold employs a 1-form b_i along with a covariant metric tensor a_{ij}). In this geomorphic case, the Riemannian tensor is just the Euclidean metric tensor scaled by $\sqrt{\varphi(\mathbf{r})}$, while the vector field is simply $\varphi(\mathbf{r})$ applied to the horizontal unit vector.

Fundamental function

The co-Finsler/Cartan fundamental function for the geomorphic (SPIM) Hamiltonian, derived from first principles, is:

$$\mathcal{F}_* = p_x^{\eta} \left(p_x^2 + p_z^2\right)^{\frac{1}{2} - \frac{\eta}{2}} \varphi(\mathbf{r})$$

For a gradient-scaling exponent of $\eta = \frac{1}{2}$ in the SPIM, we have

$$\mathcal{F}_* = \sqrt{p_x} \sqrt[4]{p_x^2 + p_z^2} \varphi(\mathbf{r})$$

while for a gradient-scaling exponent of $\eta = \frac{3}{2}$ in the SPIM, we have

$$\mathcal{F}_* = \frac{p_x^{\frac{3}{2}} \varphi(\mathbf{r})}{\sqrt[4]{p_x^2 + p_z^2}}$$

Conversion to generalized Kropina form

$$\left(\alpha^*_{\mathrm{Kr}} = \sqrt{p_x^2 + p_z^2} \varphi(\mathbf{r}),\ \beta^*_{\mathrm{Kr}} = p_x \varphi(\mathbf{r})\right)$$

So we have the components of a Kropina-type metric on our Cartan/co-Finsler space, where

$$\alpha^*_{\mathrm{Kr}} = \left(a^{ij}(\mathbf{r})\, p_i\, p_j\right)^{\frac{1}{2}}, \quad \beta^*_{\mathrm{Kr}} = p_i b^i(\mathbf{r}) \tag{1}$$

The Riemannian metric component a^{ij} is just a Euclidean metric

$$a^{ij}(\mathbf{r}) = \sqrt{\varphi(\mathbf{r})} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

and the vector field component b^i (the dual of what would be a 1-form for a Kropina-type metric on a Finsler space) is

$$b^i(\mathbf{r}) = \varphi(\mathbf{r}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

$(\alpha^*_{\mathrm{Kr}})^{1-\eta}\, (\beta^*_{\mathrm{Kr}})^{\eta} = p_x^{\eta} \left(p_x^2 + p_z^2\right)^{\frac{1}{2}-\frac{\eta}{2}} \varphi(\mathbf{r})$

$\left(\mathcal{F}_* = p_x^{\eta} \left(p_x^2 + p_z^2\right)^{\frac{1}{2}-\frac{\eta}{2}} \varphi(\mathbf{r}),\ \text{True}\right)$

Therefore F^* is a generalized m-Kropina metric, which has the standard form:

$$F^* = \alpha^{m+1}\beta^{-m} \tag{4}$$

where $m := -\eta$, with the constraint that $m \neq 0, -1$, and where $\alpha := \alpha^*_{\mathrm{Kr}}$ and $\beta := \beta^*_{\mathrm{Kr}}$ (don't confuse these α, β with the ray and surface tilt angles).

For $\eta = \frac{1}{2}$, we have a generalized $\left(-\frac{1}{2}\right)$-Kropina metric function $F^* = \alpha^{1/2}\beta^{1/2}$.

For $\eta = \frac{3}{2}$, we have a generalized $\left(-\frac{3}{2}\right)$-Kropina metric function $F^* = \alpha^{-1/2}\beta^{3/2}$.

For $\eta = 1$, the metric function is ill-defined.

Note: the canonical Kropina metric is usually used to describe F not F^*, but not exclusively.

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