

# Generic geomorphic Hamiltonian

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## Summary

The geomorphic Hamiltonian in Stark & Stark (2022) was derived from and equivalent to the Stream-Power Incision Model (SPIM). However, a more generic form for a geomorphic Hamiltonian can be derived using the same techniques. This notebook demonstrates this generality.

If we want to map an erosion rate equation into a geomorphic Hamiltonian, we need to make three conceptual leaps:

1. accept that *erosion takes place normal to the erosion surface*
  - so erosion rate is naturally modeled as an *erosion speed in the surface-normal direction*;
2. accept that the rate of motion of a surface is correctly expressed as a covector, aka a cotangent vector, aka a covariant rank-1 tensor, rather than as a tangent vector, aka a contravariant rank-1 tensor
  - so the erosion equation should in fact be written in terms of an *erosion slowness covector oriented in the surface-normal direction*;
3. realize that by writing normal slowness as a function of local properties we actually have an eikonal, or static Hamilton-Jacobi equation
  - which means we have a geomorphic Hamiltonian.

## Derivation

Here I demonstrate how to derive a *static* geomorphic Hamiltonian for a general erosion equation that satisfies some simple constraints. If we relax some of those constraints, and allow time-dependence and non-locality, a Hamiltonian could still be derived, but it would be more complex and likely non-classical (e.g., non-local Hamiltonians are actually a thing, but a rare thing).

### In 3-D or 2-D

Let's define a general equation for the speed of surface erosion  $\zeta^\perp$  in the normal direction that is a function of the local surface angle(s)  $\{\beta_k\}$  and position  $\mathbf{r}$ :

$$\zeta^\perp(\{\beta_k\}, \mathbf{r}) = f(\{\tan \beta_k\}, \{r^i\}) \quad (1)$$

Here the curly brackets  $\{\cdot\}$  mean a set of elements (indexed as appropriate), such as the set of local angles  $\{\beta_k\}$ , indexed by  $k \in \{1, 2\}$  in 3D and  $k \in \{1\}$  in 2D.

Such an erosion model encompasses the SPIM (in a surface-normal erosion rate form), “granular flow hillslopes” (in the sense of Pauli & Gioia, 2007), and others. It assumes that nonlocal

properties such as water volume flux are spatially constant properties that can be parameterized using  $\mathbf{r}$  alone. It disallows, for example, changes in drainage network pattern or extent – unless the erosion model erosion rate is independent of drainage pattern, which is the case for “granular flow hillslopes”. It is essentially a static model, which is why we can convert it into a static Hamiltonian, and can rewrite it as an eikonal equation (a static Hamilton-Jacobi equation).

We recognize that surface erosion rate direction and magnitude are best expressed using the surface-normal erosion *slowness* covector  $\mathbf{p}$ . Surface erosion slowness is the reciprocal of surface erosion speed, and since erosion speed scalar is obtained from the erosion velocity vector (in Euclidean space) using the  $L_2$  norm, we also use the  $L_2$  norm (in Euclidean space) to measure the length of the slowness covector:

$$\|\mathbf{p}\|_{L_2} = \frac{1}{\xi_{\perp}} \quad (2)$$

We realize that trigonometric functions of local surface angles can all be written using  $\{\tan \beta_i\}$  and that these tangent values can be written as ratios of slowness covector components:

$$\tan \beta_k = p_i / p_j \quad \text{for } i \neq j, i < j \quad (3)$$

for appropriate choices of  $i, j, k$ .

Using Okubo’s trick where all the surface erosion slowness covector components are scaled by a positive factor  $\mathcal{F}_*$ ,

$$\{p_i\} \rightarrow \left\{ \frac{p_i}{\mathcal{F}_*} \right\} \quad (4)$$

substituted into the rate equation, the equation rearranged, and  $\mathcal{F}_*$  equated with the fundamental function, we get

$$\mathcal{F}_*(\mathbf{p}, \mathbf{r}) = \|\mathbf{p}\|_{L_2} f\left(\left\{ \frac{p_i}{p_j} \right\}, \{r^k\}\right) \quad (5)$$

where  $\mathcal{F}_*(\mathbf{p}, \mathbf{r}) = 1$ , obviously. So we have the eikonal equation

$$\|\mathbf{p}\|_{L_2} = \frac{1}{f(\cdot, \cdot)} \quad (6)$$

which we could have obtained simply from the definition of  $\|\mathbf{p}\|_{L_2}$  (eqn. 2), and which is equivalent to saying that

$$\text{observed surface-normal erosion slowness} = \frac{1}{\text{required surface-normal erosion speed}} \quad (7)$$

From the fundamental function we get the generic geomorphic Hamiltonian

$$\mathcal{H}(\mathbf{p}, \mathbf{r}) := \frac{1}{2} \|\mathbf{p}\|_{L_2}^2 f^2\left(\left\{ \frac{p_i}{p_j} \right\}, \{r^k\}\right) = \frac{1}{2} \quad (8)$$

Given how the Hamiltonian  $\mathcal{H}$  is constructed from  $\mathcal{F}_*$ , which was obtained by the Okubo substitution, it is inevitable that  $\mathcal{H}$  is Euler order-2 homogeneous in  $\mathbf{p}$ :

$$\mathcal{H}(\lambda \mathbf{p}, \mathbf{r}) = \frac{1}{2} \|\lambda \mathbf{p}\|_{L_2}^2 f^2 \left( \left\{ \frac{\lambda p_i}{\lambda p_j} \right\}, \{r^k\} \right) \quad (9)$$

$$= \frac{1}{2} \lambda^2 \|\mathbf{p}\|_{L_2}^2 f^2 \left( \left\{ \frac{p_i}{p_j} \right\}, \{r^k\} \right) \quad (10)$$

$$= \lambda^2 \mathcal{H}(\mathbf{p}, \mathbf{r}) \quad (11)$$

This order-2 homogeneity is *EXTREMELY IMPORTANT* because it's essentially the origin of all the interesting behaviour of the geomorphic Hamiltonian, such as the close correspondence with geometric optics, the conjugacy of slowness covector and velocity vector  $\mathbf{p}(\mathbf{v}) = 1$ , the amazing utility of the (semi-)metric tensor  $g$ , the parametric form of the geomorphic Lagrangian, the geodesic equation form of the Euler-Lagrange equations, and the consequent geodesic spray.

Hamilton's equations are given in the usual way:

$$v^i = \dot{r}^i = \frac{dr^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad (12)$$

$$\dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial r^i} \quad (13)$$

### Simpler treatment in 2D (vertical section)

The above explanation is a bit abstract, so to make it more concrete here it is for 2D. Using the same coordinates  $x$ - $z$  as Stark & Stark (2022), and having only one local angle to worry about ( $\beta$ ), we write the erosion equation as:

$$\xi^\perp(\beta, \mathbf{r}) = f(\tan \beta, r^x, r^z) \quad (14)$$

We could just write the eikonal equation directly, given the definition of the surface-normal erosion slowness covector  $\mathbf{p}$ :

$$\|\mathbf{p}\|_{L_2} = \frac{1}{f(\tan \beta, \mathbf{r})} \quad (15)$$

or we can use the Okubo trick to obtain the fundamental function

$$\mathcal{F}_*(\mathbf{p}, \mathbf{r}) = \sqrt{p_x^2 + p_z^2} f\left(\frac{p_x}{p_z}, r^x, r^z\right) \quad (16)$$

where we know that  $\mathcal{F}_*(\mathbf{p}, \mathbf{r}) = 1$ . The generic 2D geomorphic Hamiltonian is then defined as (half) the square of  $\mathcal{F}_*$ :

$$\mathcal{H}(\mathbf{p}, \mathbf{r}) := \frac{1}{2} (p_x^2 + p_z^2) f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) = \frac{1}{2} \quad (17)$$

Euler order-2 homogeneity for  $\mathcal{H}$  is inevitable given the manner in which  $\mathcal{F}_*$  was obtained

$$\mathcal{H}(\lambda \mathbf{p}, \mathbf{r}) = \frac{1}{2}(\lambda^2 p_x^2 + \lambda^2 p_z^2) f^2\left(\frac{\lambda p_x}{\lambda p_z}, r^x, r^z\right) \quad (18)$$

$$= \frac{1}{2}\lambda^2(p_x^2 + p_z^2) f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) \quad (19)$$

$$= \lambda^2 \mathcal{H}(\mathbf{p}, \mathbf{r}) \quad (20)$$

Hamilton's equations are then:

$$v^x = \frac{dr^x}{dt} = \frac{\partial \mathcal{H}}{\partial p_x} = 2p_x f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) + (p_x^2 + p_z^2) f\left(\frac{p_x}{p_z}, r^x, r^z\right) \frac{\partial f}{\partial p_x} \quad (21)$$

$$v^z = \frac{dr^z}{dt} = \frac{\partial \mathcal{H}}{\partial p_z} = 2p_z f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) + (p_x^2 + p_z^2) f\left(\frac{p_x}{p_z}, r^x, r^z\right) \frac{\partial f}{\partial p_z} \quad (22)$$

$$\frac{dp_x}{dt} = -\frac{\partial \mathcal{H}}{\partial r^x} = - (p_x^2 + p_z^2) f\left(\frac{p_x}{p_z}, r^x, r^z\right) \frac{\partial f}{\partial r^x} \quad (23)$$

$$\frac{dp_z}{dt} = -\frac{\partial \mathcal{H}}{\partial r^z} = - (p_x^2 + p_z^2) f\left(\frac{p_x}{p_z}, r^x, r^z\right) \frac{\partial f}{\partial r^z} \quad (24)$$