Matsumoto's *Slope of a mountain* metric

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Summary

In the first of three 2019 lectures on Finsler spacetime, Javaloyes summarizes the Matsumoto metric like this:

When we go up a mountain we walk slower than when we go down

Given a Riemannian metric g, and a 1-form β

$$M(v) = \frac{g(v,v)}{\sqrt{g(v,v)} - \beta(v)}$$

Sierra Nevada (Granada)

It is strongly convex if $\sqrt{g(v,v)} \geq 2\beta(v)$

Geodesics minimize time in the presence of a slope

Matsumoto's idea – which was inspired by a question raised by Paul Finsler himself – is to measure how far a mountain hiker will walk in a given direction in unit time *if they maintain the same effort regardless of the local slope*.

The physics boils down to this:

- choose a map direction ϕ for the hike
- specify a step interval $\Delta t \ll$ unit time
- take a step in the chosen direction with an fixed impulse
- terminate the step with another fixed impulse
- choose this impulse pair so that *on a horizontal surface* a constant chosen speed c is maintained
- on a sloping surface, the speed will be faster or slower than c depending on the tilt
- include in the termination impulse a lateral component to compensate for any "off-axis" acceleration and to ensure the path continues in the chosen direction
- keep stepping so that a steady pace is established

- measure the distance traveled in unit time (ignoring the vagaries of the first few steps)

Horizontal walking speed in the chosen direction is set by two things:

1. the step-wise accelerations and decelerations induced by the walking impulses
2. the step-wise accelerations induced by the component of gravity resolved in the walking direction

Derivation

Original version: Hiker on a mountain slope

Matsumoto's original paper invokes a hiker who somehow achieves an orientation-dependent "terminal velocity" after a few steps. The physics of this seem a bit vague to me, so in the next section I have adapted it into a more carefully crafted form. First, though, here is the Bao & Robles (2004) version of Matsumoto (1989)

Preliminaries

Next, consider a surface S given by the graph of a smooth function $f(x, y)$. Parametrise S via $(x, y) \mapsto (x, y, f)$. By a slight abuse of notation, set $\partial_x := (1, 0, f_x)$ and $\partial_y := (0, 1, f_y)$, and denote the natural dual of this basis by dx, dy. The Euclidean metric of $\mathbb{R}^3$ induces a Riemannian metric on S:

$$h := \left(1 + f_x^2\right) \, dx \otimes dx + f_x f_y (dx \otimes dy + dy \otimes dx) + \left(1 + f_y^2\right) \, dy \otimes dy.$$

If $Y := u\partial_x + v\partial_y$ is an arbitrary tangent vector on S, we have

$$|Y|^2 := h(Y, Y) = u^2 + v^2 + (uf_x + vf_y)^2.$$

We note for later use that the contravariant description of $df = f_x \, dx + f_y \, dy$ is the vector field

$$(df)^\sharp = \frac{1}{1 + f_x^2 + f_y^2} (f_x \partial_x + f_y \partial_y), \quad \text{with} \quad |(df)^\sharp|^2 = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}.$$

A topographic surface S in 3D space is defined by a function $z = f(x, y)$ and described parametrically by $(x, y, f(x, y))$ [see Matsumoto, 1989, p.22].

The tangent plane at (x,y) has local basis vectors $(1, 0, f_x)$ and $(0, 1, f_y)$ and these are respectively abusively denoted by ∂_x and ∂_y (abusive, because they not simple derivatives). The dual basis is denoted by dx, dy.

We can define a covariant metric tensor g (Bao & Robles denote it as h) for this surface using the mutual Euclidean (3D) lengths of the vectors $(1, 0, f_x)$ and $(0, 1, f_y)$ and the tensor products of the dual basis elements:

$$g = \left((1,0,f_x)\cdot(1,0,f_x)\right)\, dx\otimes dx + \big((1,0,f_x)\cdot(0,1,f_y)\big)\, (dx\otimes dy + dy\otimes dx) + \big((0,1,f_y)\cdot(0,1,f_y)\big)\, dy\otimes dy \tag{1}$$

$$= (1+f_x^2)\, dx\otimes dx + f_x f_y\, (dx\otimes dy + dy\otimes dx) + (1+f_y^2)\, dy\otimes dy \tag{2}$$

$$= \begin{bmatrix} (1+f_x^2) & f_x f_y \\ f_x f_y & (1+f_y^2) \end{bmatrix} \tag{3}$$

The metric tensor provides a means of measuring the length of a tangent vector $Y := u\partial_x + v\partial_y$ on S:

$$\|Y\|_g^2 = g(Y,Y) = g_{ij}Y^iY^j \tag{4}$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} (1+f_x^2) & f_x f_y \\ f_y f_x & (1+f_y^2) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} u(1+f_x^2) + v\, f_x f_y \\ u\, f_y f_x + v(1+f_y^2) \end{bmatrix} \tag{6}$$

$$= u^2(1+f_x^2) + vuf_xf_y + uvf_yf_x + v^2(1+f_y^2) \tag{7}$$

$$= u^2 + v^2 + u^2f_x^2 + 2uvf_xf_y + v^2f_y^2 \tag{8}$$

$$= u^2 + v^2 + (uf_x + vf_y)^2 \tag{9}$$

The dual (contravariant) metric tensor is the inverse of g

$$g^{-1} = \begin{bmatrix} (1+f_x^2) & f_x f_y \\ f_y f_x & (1+f_y^2) \end{bmatrix}^{-1} = \frac{1}{1+f_x^2+f_y^2} \begin{bmatrix} 1+f_y^2 & -f_xf_y \\ -f_yf_x & 1+f_x^2 \end{bmatrix} \tag{10}$$

We can also map the covector differential df to its dual contravariant vector using g^{-1}:

$$(df)^\sharp = g^{-1}df \tag{11}$$

$$= \frac{1}{1+f_x^2+f_y^2} \begin{bmatrix} 1+f_y^2 & -f_xf_y \\ -f_yf_x & 1+f_x^2 \end{bmatrix} \begin{bmatrix} f_x & f_y \end{bmatrix}^T \tag{12}$$

$$= \frac{1}{1+f_x^2+f_y^2} \begin{bmatrix} (1+f_y^2)f_x - f_xf_yf_y \\ -f_yf_xf_x + (1+f_x^2)f_y \end{bmatrix} \tag{13}$$

$$= \frac{1}{1+f_x^2+f_y^2} \begin{bmatrix} f_x \\ f_y \end{bmatrix} \tag{14}$$

$$= \frac{f_x\partial_x + f_y\partial_y}{1+f_x^2+f_y^2} \tag{15}$$

The length of this vector is

$$
\begin{aligned}
\|(df)^\sharp\|_g^2 &= g\left((df)^\sharp, (df)^\sharp\right) && (16)\\
&= gg^{-1}df\, g^{-1}df && (17)\\
&= df\, g^{-1}\, df && (18)\\
&= \frac{1}{1+f_x^2+f_y^2}\begin{bmatrix} f_x & f_y \end{bmatrix}\begin{bmatrix} (1+f_x^2) & -f_x f_y \\ -f_y f_x & (1+f_y^2) \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix} && (19)\\
&= \frac{1}{1+f_x^2+f_y^2}\begin{bmatrix} f_x & f_y \end{bmatrix}\begin{bmatrix} f_x(1+f_x^2) - f_y\, f_x f_y \\ -f_x\, f_y f_x + f_y(1+f_y^2) \end{bmatrix} && (20)\\
&= \frac{1}{1+f_x^2+f_y^2}\begin{bmatrix} f_x & f_y \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix} && (21)\\
&= \frac{f_x^2+f_y^2}{1+f_x^2+f_y^2} && (22)
\end{aligned}
$$

EXAMPLE (MATSUMOTO'S SLOPE-OF-A-MOUNTAIN METRIC). Take the same surface S, but without the wind. View S as the slope of a mountain resting on level ground, with gravity pointing down instead of perpendicular to S. A person who can walk with speed c on level ground navigates this hillside S along a path that makes an angle θ with the steepest downhill direction. The acceleration of gravity (of magnitude g), being perpendicular to level ground, has a component of magnitude $g_{\|} = g\sqrt{(f_x^2+f_y^2)/(1+f_x^2+f_y^2)}$ along the steepest downhill direction. The hiker then experiences an acceleration $g_{\|}\cos\theta$ along her path, and compensates against the $g_{\|}\sin\theta$ which tries to drag her off-course. Under suitable assumptions about frictional forces, the acceleration $g_{\|}\cos\theta$ *rapidly* effects a terminal addition $\frac{1}{2}g_{\|}\cos\theta$ to the pace c generated by her leg muscles. In other words, her speed is effectively of the form $c+a\cos\theta$, where a is independent of θ. Thus the locus of unit time destinations is a limaçon. The unit circle of h, instead of undergoing a rigid translation as in Zermelo navigation, has now experienced a direction-dependent deformation. The norm function F with this limaçon as indicatrix measures travel time on S.

$$\frac{|Y|^2}{c|Y|-(g/2)(uf_x+vf_y)}$$

For simplicity, specialise to the case $c = g/2$. Multiplication by c then converts this norm function to

$$F(x,y;u,v) := \frac{|Y|^2}{|Y|-(uf_x+vf_y)} = |Y|\,\varphi\Big(\frac{(df)(Y)}{|Y|}\Big),$$

with $\varphi(s) := 1/(1-s)$. We see from [Shen 2004] in this volume that metrics of the type $\alpha\varphi(\beta/\alpha)$ are strongly convex whenever the function $\varphi(s)$ satisfies $\varphi(s)>0$, $\varphi(s)-s\varphi'(s)>0$, and $\varphi''(s)\geqslant 0$. For the φ at hand, this is equivalent to $(df)(Y) < \frac{1}{2}|Y|$, which is in turn equivalent to $|(df)^\sharp| < \frac{1}{2}$. (In one direction, set $Y = (df)^\sharp$; the converse follows from a Cauchy–Schwarz inequality.) Using the formula for $|(df)^\sharp|^2$ presented earlier, this criterion produces

$$f_x^2+f_y^2 < \tfrac{1}{3}.$$

Whenever this holds, F defines a Finsler metric. Such is the case for $f(x,y) := \frac{1}{2}x$ but not for $f(x,y) := x$, even though the surface S is an inclined plane in both instances. As for the elliptic paraboloid given by the graph of $f(x,y) := 100-x^2-y^2$, we have strong convexity only in a circular vicinity of the hilltop. $\diamondsuit$

The functions F in these two examples are not absolutely homogeneous (and therefore non-Riemannian) because at any given juncture, the speed with which one could move forward typically depends on the direction of travel.

Modified version: Robot on a mountain slope

Imagine a bipedal robot walking across a smooth plane inclined towards the south at an angle ϕ from the horizontal. Between one step and the next, the robot is surface-parallel accelerated or decelerated depending on the orientation θ of its motion relative to south. As a step ends, the robot applies just the right impulse through its feet that it:

1. compensates for any step-orthogonal motion making it veer away from the intended direction of motion;
2. recovers the step-wise, surface-parallel speed c as it begins the next step.

In this way, it maintains *on average* a constant step-wise, surface-parallel speed (that may be greater or less than c) in the chosen direction of motion.

The question is, how far will it walk in a given direction θ in a fixed time averaged over many steps?

The incremental distance Δr traversed between each step interval Δt will be the result of (a) the speed v asserted when each step is taken, plus (b) the speed boost (or loss) from the component of acceleration due to gravity g_{acc} acting along the generally non-horizontal path.

$$\Delta r = c\Delta t + \tfrac{1}{2} g_{\mathrm{acc}} \Delta t^2 \sin\phi \, \cos\theta \tag{23}$$

$$\frac{\Delta r}{\Delta t} = c + \tfrac{1}{2} g_{\mathrm{acc}} \Delta t \sin\phi \, \cos\theta \tag{24}$$

So the normalized speed is:

$$\tilde{c} = 1 + a\cos\theta \tag{25}$$

where

$$a = \frac{g_{\mathrm{acc}}\Delta t}{2c} \sin\phi \tag{26}$$

As the time interval Δt is reduced (to make motion smoother), c is reduced to ensure a remains constant.

As far as the robot is concerned, assuming it can only sense enough to maintain surface-parallel v at each step, distance appears to vary with orientation. In other words, its distance metric is strongly anisotropic and Finsler: it has a form that cannot be expressed in a simple inner product that applies a metric tensor dependent only on position: if it were, a Riemannian metric would apply. The anisotropy of this metric is encapsulated in its speed-vs-orientation indicatrix, which takes the interesting form of a *limaçon*.

Note that, blithely ignoring units, Bao & Robles assume $c = g_{\mathrm{acc}}/2$ towards the end of their explanation, such that

$$a = \Delta t \sin\phi \tag{27}$$

Presumably they also assume $\Delta t = 1$, and since their (f_x, f_y) is here $(0, \tan\phi)$,

$$f_y^2 = \frac{1}{1-a^2} - 1 = \frac{a^2}{1-a^2} \tag{28}$$

or

$$a^2 = 1 - \frac{1}{1+\tan^2\phi} = \frac{\tan^2\phi}{1+\tan^2\phi} = \frac{f_y^2}{1+f_y^2} \tag{29}$$

If $f_x = 0$, $f_y^2 < \frac{1}{3}$, meaning the surface tilt must be less than $\phi < \pi/6$, then

$$a < \sqrt{\frac{1/3}{1+1/3}} = \frac{1}{2} \tag{30}$$

which means that most of the interesting limaçon-shaped metrics $a \geq 1/2$ are not strongly convex, and thus are not Finsler metrics *sensu stricto*.

Let's simplify the Bao & Robles version of the Matsumoto metric using again the assumption that the surface is planar and tilted only to the south at an angle $f_y = \tan\phi$. Then

$$|Y|^2 = u^2 + v^2 + v^2 f_y^2 = u^2 + v^2(1+f_y^2) \tag{31}$$

and

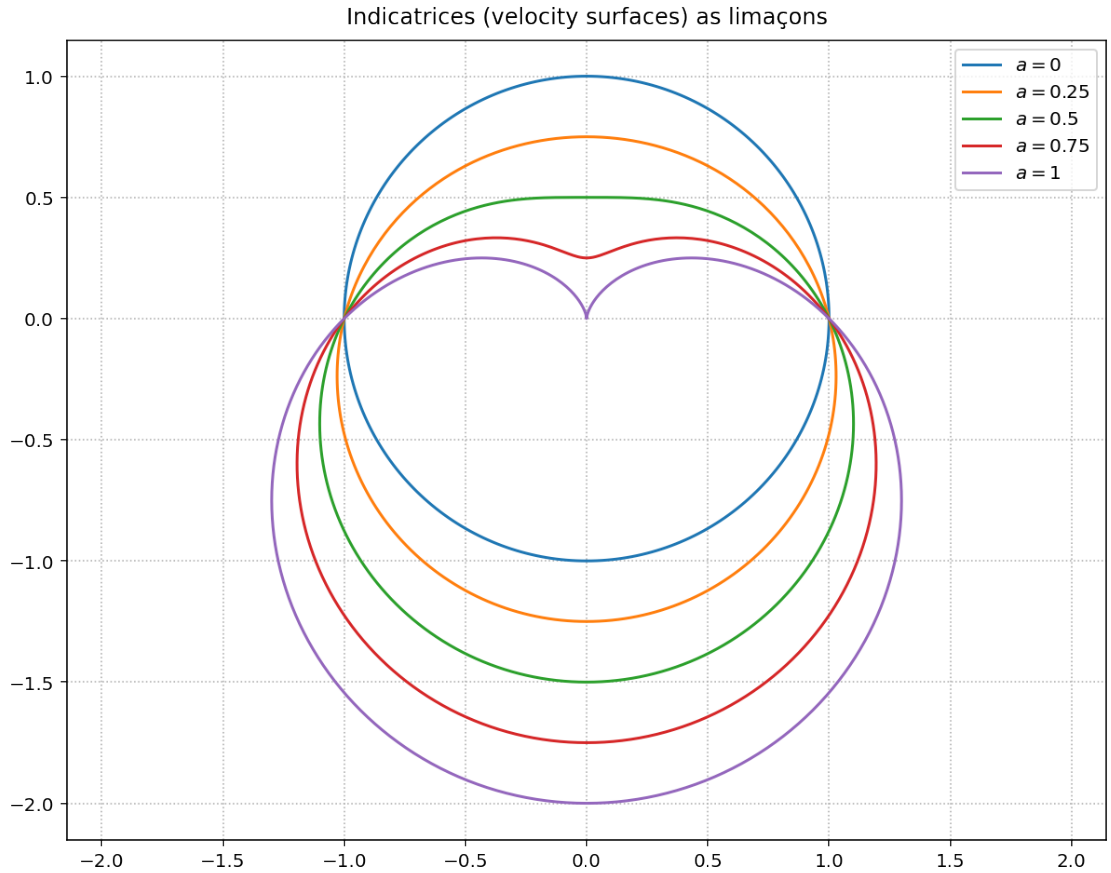
$$|(df)^\#|^2 = \frac{f_y^2}{1+f_y^2} \tag{32}$$

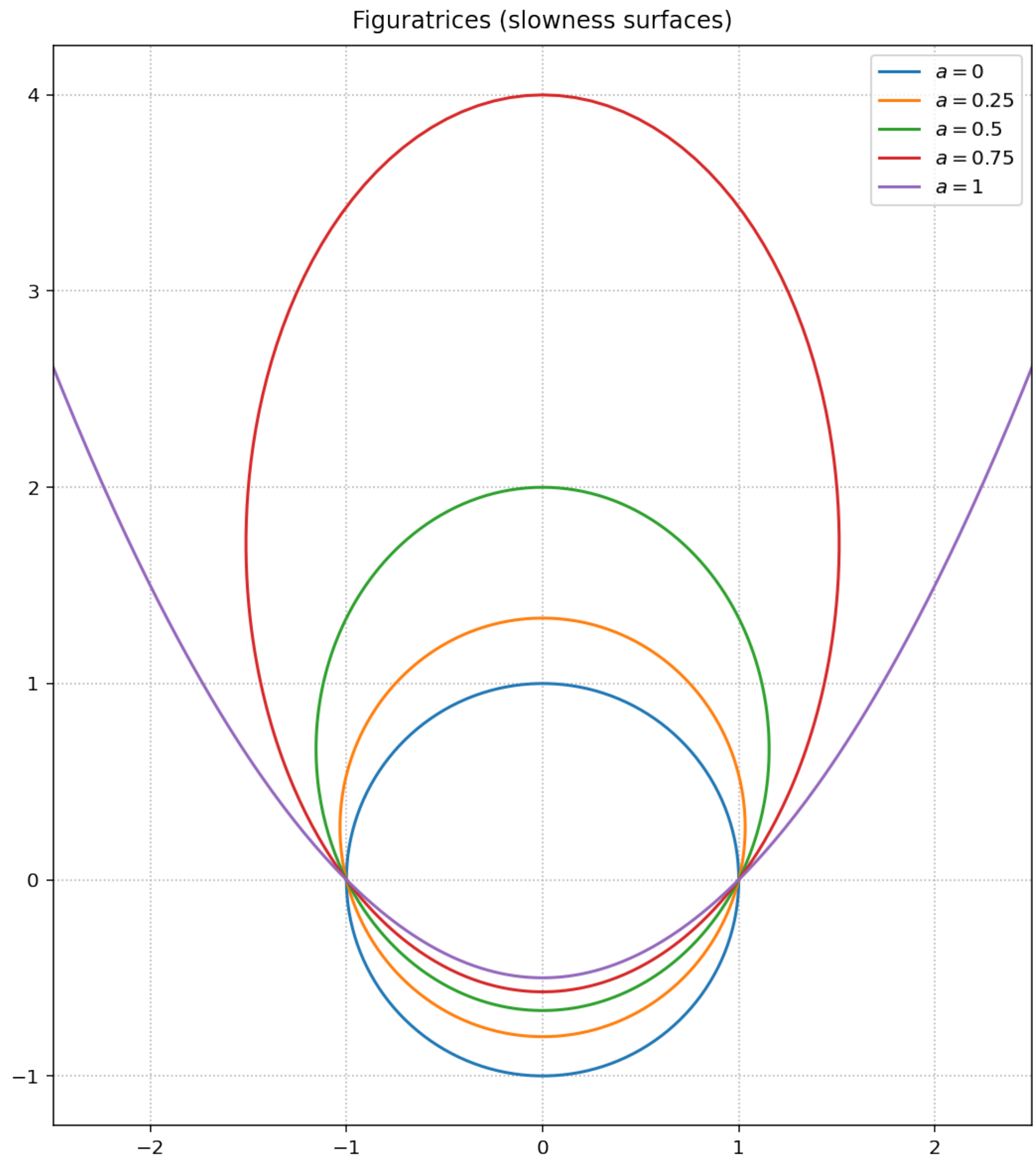
which makes the metric function

$$F = \frac{|Y|^2}{|Y| - vf_y} = \frac{u^2 + v^2(1+f_y^2)}{\sqrt{u^2+v^2(1+f_y^2)} - vf_y} \tag{33}$$

$$F = \frac{|Y|^2}{|Y| - vf_y} = \frac{u^2 + v^2(1+f_y^2)}{\sqrt{u^2+v^2(1+f_y^2)} - vf_y} \tag{34}$$

Plots of the indicatrix and figuratrix





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Appendix

For reference, here is the conservative definition of a Finsler metric from Bao & Robles [2004], pp.199-200. It adopts the requirement that the fundamental tensor g_{ij} be positive definite.

1.1. Finsler metrics

1.1.1. Definition and examples. A Finsler metric is a continuous function

$$F : TM \to [0, \infty)$$

with the following properties:

(i) *Regularity*: F is smooth on uniformized in favor of : $TM \setminus 0 := \{(x, y) \in TM : y \neq 0\}$.

(ii) *Positive homogeneity*: $F(x, cy) = cF(x, y)$ for all $c > 0$.

(iii) *Strong convexity*: the fundamental tensor

$$g_{ij}(x, y) := \left(\tfrac{1}{2}F^2\right)_{y^i y^j}$$

is positive definite for all $(x, y) \in TM \setminus 0$. Here the subscript $_{y^i}$ denotes partial differentiation by y^i.

Strong convexity implies that $\{y \in T_xM : F(x, y) \leqslant 1\}$ is a strictly convex set, but not vice versa; see [Bao et al. 2000].

The function F for a Riemannian metric a is $F(x, y) := \sqrt{a_{ij}(x)y^i y^j}$. In this case, one finds that $g_{ij} := (\frac{1}{2}F^2)_{y^i y^j}$ is simply a_{ij}. Thus the fundamental tensor for general Finsler metrics may be thought of as a direction-dependent Riemannian metric. This viewpoint is treated more carefully in Section 1.1.2.

Many calculations in Finsler geometry are simplified, or magically facilitated, by *Euler's theorem* for homogeneous functions:

Let ϕ be a real valued function on $\mathbb{R}^n$, differentiable at all $y \neq 0$. The following two statements are equivalent.

- *$\phi(cy) = c^r \phi(y)$ for all $c > 0$ (positive homogeneity of degree r).*
- *$y^i \phi_{y^i} = r\phi$; that is, the radial derivative of ϕ is r times ϕ.*

(See, for example, [Bao et al. 2000] for a proof.) This theorem, for instance, lets us invert the defining relation of the fundamental tensor given above to get

$$F^2(x, y) = g_{ij}(x, y)y^i y^j.$$

Consequently, strong convexity implies that F must be positive at all $y \neq 0$. The converse, however, is false; positivity does not in general imply strong convexity. This is because while $g_{ij}(x, y)y^i y^j = F^2(x, y)$ may be positive for $y \neq 0$, the quadratic $g_{ij}(x, y)\tilde{y}^i \tilde{y}^j$ could still be $\leqslant 0$ for some nonzero $\tilde{y}$.

However, Asanov [1985] is more relaxed, noting that by insisting on positive-definiteness we throw out any GR applications, since pseudo-Riemannian manifolds are not positive definite:

Further, it will be assumed that, for any admissible y^i, $F(x, y) > 0$ and

$$\det(\partial^2 F^2(x, y)/\partial y^i \partial y^j) \neq 0. \tag{1.1}$$

Besides this, the function $F(x, y)$ is to be positively homogeneous of degree one with respect to y^i, i.e.,

$$F(x, ky) = kF(x, y) \tag{1.2}$$

for any fixed $k > 0$ and for all $y^i \in M_x^*$. Under these conditions, the triple (M, M_x^*, $F(x, y)$) is called an *N-dimensional Finsler space*, and $F(x, y)$ is called a *Finslerian metric function*. The value of the metric function $F(x, y)$ is treated in Finsler geometry as the length of the tangent vector y^i attached to the point x^i. If a Finsler space allows a coordinate system x^i such that F does not depend on these x^i, the Finsler space and the metric function are called *Minkowskian*.

It will be noted that in mathematical works devoted to Finsler geometry additional conditions are usually imposed on the metric function F which ensure the positive definiteness[2] of the quadratic form $Z^i Z^j \partial^2 F^2(x, y)/\partial y^i \partial y^j$ at any point x^i and for any non-zero vector $y^i \in M_x$. However, it is clear already in the Riemannian formulation of general relativity theory that the metric structure of space-time cannot be positively definite, for the space-time metric tensor must be of the indefinite signature $(+ - - -)$. This reason alone makes one expect that it is indefinite metrics that may be of interest in a Finslerian extension of general relativity. Accordingly, we refrain deliberately from imposing the condition of positive definiteness thereby admitting that the Finsler space under study can be indefinite.

Gallot et al [2004] discuss some of the consequences of allowing for a mixed signature (metric tensor with negative eigenvalues) in the context of Riemannian manifolds (they don't touch on Finsler spaces):

2.D A glance at pseudo-Riemannian manifolds

It is a good question, when studying a result in Riemannian geometry, to ask whether positive-definiteness is really necessary. However, pseudo-Riemannian geometry is not only a free intellectual exercise. It presents many unexpected, counter-intuitive and fascinating features. We will focus on the Lorentzian (signature $(1, n)$) case. It is of course very important because of General Relativity. Moreover, we are not aware of significant results in the higher signature case.

2.D.1 What remains true?

Firstly, if (M, g) is an oriented pseudo-Riemannian manifold, a volume form v_g is assigned to g, by prescribing that $(v_g)_m(e_1, \cdots, e_n) = 1$ for any direct pseudo-orthonormal basis of $T_m M$, exactly like in the Riemannian case (cf. 2.7). If (x_i) is a local coordinate system compatible with the orientation,

$$v_g = \sqrt{|\det(g_{ij}|} dx_1 \wedge \cdots \wedge dx_n.$$

Like in the Riemannian case, orientability is not necessary, and g defines a canonical measure (cf. 3.H).
However, we have to be careful. In the Riemannian case, any submanifold N of M inherits such a measure, since the restriction of g to N is Riemannian. But if g is not positive definite, its restriction to a submanifold may be singular.
Secondly, we have pointed out in B. that the Levi-Civita connection does exist, provided g is everywhere non degenerate. Geodesics are defined in the same way, by the condition $D_{c'} c' = 0$. Therefore, we still have a local existence theorem for geodesics, and an exponential map. For any $m \in M$, it is indeed true that there exists a neighborhood U of $0 \in T_m M$ such that $\exp_m$ is a diffeomorphism on its image. However, *no metric description of U is available*, and this can be rather misleading, see for instance 2.141 below.
A consequence of the existence of the Levi-Civita connection is that pseudo-Riemannian metrics are "rigid" or "finite type" structures (we won't define these terms, whose meaning is fairly intuitive, and refer to [D'A-G] for details).

At the risk of terminal obfuscation, here is a very recent paper by Hohmann et al [2022] and their prescription of a Finsler spacetime: note how, echoing Asanov, they require only that the fundamental tensor be non-degenerate. I don't understand how they deal with light-like (null) curves, but that would seem to be precisely where the determinant of g_{ij} is zero...

A. The notion of Finsler spacetime

Let M be a connected, orientable smooth manifold and TM, its tangent bundle with projection $\pi_{TM}: TM \to M$. We will denote by x^i the coordinates in a local chart on M and by $(x^i, \dot{x}^i)$, the naturally induced local coordinates of points $(x, \dot{x}) \in TM$. Whenever there is no risk of confusion, we will omit the indices, i.e., write $(x, \dot{x})$ instead of $(x^i, \dot{x}^i)$. Commas $_{,i}$ will mean partial differentiation with respect to the base coordinates x^i and dots $_{\cdot i}$ partial differentiation with respect to the fiber coordinates $\dot{x}^i$. Also, by $\overset{\circ}{TM} = TM \backslash \{0\}$, we will mean the tangent bundle of M without its zero section.

A *conic subbundle* of TM is a non-empty open submanifold $\mathscr{Q} \subset TM \backslash \{0\}$, with the following properties:

- $\pi_{TM}(\mathscr{Q}) = M$;
- *conic property:* if $(x, \dot{x}) \in \mathscr{Q}$, then, for any $\lambda > 0$: $(x, \lambda \dot{x}) \in \mathscr{Q}$.

A *pseudo-Finsler space* is,[48], a triple $(M, \mathscr{A}, L)$, where M is a smooth manifold, $\mathscr{A} \subset \overset{\circ}{TM}$ is a conic subbundle and $L : \mathscr{A} \to \mathbb{R}$ is a smooth function obeying the following conditions:

1. positive 2-homogeneity: $L(x, \alpha \dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall \alpha > 0$, $\forall (x, \dot{x}) \in \mathscr{A}$.
2. at any $(x, \dot{x}) \in \mathscr{A}$ and in one (and then, in any) local chart around $(x, \dot{x})$, the Hessian:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$$

is nondegenerate.