Generic geomorphic Hamiltonian

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Summary

The geomorphic Hamiltonian in Stark & Stark (2022) was derived from and equivalent to the Stream-Power Incision Model (SPIM). However, a more generic form for a geomorphic Hamiltonian can be derived using the same techniques. This notebook demonstrates this generality.

Derivation

In N-D

Let's define a general equation for the speed of surface erosion $\xi^{\perp}$ in the normal direction that is a function of the local surface angle(s) $\{\beta_k\}$ and position $\mathbf{r}$:

$$\xi^{\perp}(\{\beta_k\},\mathbf{r}) = f\Big(\{\tan\beta_k\},\{r^i\}\Big) \tag{1}$$

Here the curly brackets $\{\cdot\}$ mean a set of elements (indexed as appropriate), such as the set of local angles $\{\beta_k\}$, indexed by $k \in \{1,2\}$ in 3D and $k \in \{1\}$ in 2D.

Such an erosion model encompasses the SPIM, "granular flow hillslopes" (in the sense of Pauli & Gioia, 2007), and others. It assumes that nonlocal properties such as water volume flux are spatially constant properties that can be parameterized using $\mathbf{r}$ alone. It disallows, for example, changes in drainage network pattern or extent. It is essentially a static model, which is why we can convert it into a static Hamiltonian, and can rewrite it as an eikonal equation (a static Hamilton-Jacobi equation).

We recognize that surface erosion rate direction and magnitude are best expressed using the surface-normal erosion *slowness* covector $\mathbf{p}$. Surface erosion slowness is the reciprocal of surface erosion speed, and since erosion speed scalar is obtained from the erosion velocity vector (in Euclidean space) using the L_2 norm, we also use the L_2 norm (in Euclidean space) to measure the length of the slowness covector:

$$\|\mathbf{p}\|_{L_2} = \frac{1}{\xi^{\perp}} \tag{2}$$

We realize that trigonometric functions of local surface angles can all be written using $\{\tan\beta_i\}$ and that these tangent values can be written as ratios of slowness covector components:

$$\tan\beta_k = p_i/p_j \qquad \text{for} \quad i \neq j,\, i<j \tag{3}$$

for appropriate choices of i, j, k.

Using Okubo's trick where all the surface erosion slowness covector components are scaled by a positive factor $\mathcal{F}_*$,

$$\{p_i\} \to \left\{ \frac{p_i}{\mathcal{F}_*} \right\} \tag{4}$$

substituted into the rate equation, the equation rearranged, and $\mathcal{F}_*$ equated with the fundamental function, we get

$$\frac{\mathcal{F}_*(\mathbf{p}, \mathbf{r})}{\|\mathbf{p}\|_{L_2}} = f\left(\left\{ \frac{p_i}{p_j} \right\}, \{r^k\} \right) \tag{5}$$

where $\mathcal{F}_* = 1$, obviously. So we have the (likely Finsler or pseudo-Finsler) eikonal equation

$$\|\mathbf{p}\|_{L_2} = \frac{1}{f(\cdot, \cdot)} \tag{6}$$

which we could have obtained simply from the definition of $\|\mathbf{p}\|_{L_2}$ (eqn. 2), and which is equivalent to saying that

$$\text{observed surface-normal erosion slowness} = \frac{1}{\text{required surface-normal erosion speed}} \tag{7}$$

From the fundamental function we get the generic geomorphic Hamiltonian

$$\mathcal{H}(\mathbf{p}, \mathbf{r}) := \frac{1}{2} \|\mathbf{p}\|^2_{L_2} f^2\left(\left\{ \frac{p_i}{p_j} \right\}, \{r^k\} \right) \tag{8}$$

Given how the Hamiltonian $\mathcal{H}$ is constructed from $\mathcal{F}_*$ which was obtained by the Okubo substitution, it is inevitable that $\mathcal{H}$ is Euler order-2 homogeneous in $\mathbf{p}$:

$$\begin{aligned}
\mathcal{H}(\lambda \mathbf{p}, \mathbf{r}) &= \frac{1}{2} \|\lambda \mathbf{p}\|^2_{L_2} f^2\left(\left\{ \frac{\lambda p_i}{\lambda p_j} \right\}, \{r^k\} \right) && (9) \\
&= \frac{1}{2} \lambda^2 \|\mathbf{p}\|^2_{L_2} f^2\left(\left\{ \frac{p_i}{p_j} \right\}, \{r^k\} \right) && (10) \\
&= \lambda^2 \mathcal{H}(\mathbf{p}, \mathbf{r}) && (11)
\end{aligned}$$

This order-2 homogeneity is *EXTREMELY IMPORTANT* because it's essentially the origin of all the interesting behaviour of the geomorphic Hamiltonian, such as the close correspondence with geometric optics, the conjugacy of slowness covector and velocity vector $\mathbf{p}(\mathbf{v}) = 1$, the amazing utility of the (semi-)metric tensor g, the parametric form of the geomorphic Lagrangian, the geodesic equation form of the Euler-Lagrange equations, and the consequent geodesic spray.

Hamilton's equations are given in the usual way:

$$\begin{aligned}
v^i = \dot{r}^i &= \frac{\mathrm{d}r^i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i} && (12) \\
\dot{p}_i &= \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial r^i} && (13)
\end{aligned}$$

In 2D

The above explanation is a bit abstract, so to make it more concrete here it is for 2D. Using the same coordinates x-z as Stark & Stark (2022), and having only one local angle to worry about (β), we write the erosion equation as:

$$\xi^{\perp}(\beta, \mathbf{r}) = f(\tan\beta, r^x, r^z) \tag{14}$$

We could just write the eikonal equation directly, given the definition of the surface-normal erosion slowness covector $\mathbf{p}$:

$$\|\mathbf{p}\|_{L_2} = \frac{1}{f(\tan\beta, \mathbf{r})} \tag{15}$$

or we can use the Okubo trick to obtain the fundamental function

$$\frac{\mathcal{F}_*(\mathbf{p}, \mathbf{r})}{\sqrt{p_x^2 + p_z^2}} = f\left(\frac{p_x}{p_z}, r^x, r^z\right) \tag{16}$$

and put $\mathcal{F}_* = 1$. The generic 2D geomorphic Hamiltonian is defined in terms of $\mathcal{F}_*$ as

$$\mathcal{H}(\mathbf{p}, \mathbf{r}) := \frac{1}{2}\left(p_x^2 + p_z^2\right) f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) \tag{17}$$

Euler order-2 homogeneity for $\mathcal{H}$ is inevitable given the manner in which $\mathcal{F}_*$ was obtained

$$\begin{aligned}
\mathcal{H}(\lambda\mathbf{p}, \mathbf{r}) &= \frac{1}{2}(\lambda^2 p_x^2 + \lambda^2 p_z^2)\, f^2\left(\frac{\lambda p_x}{\lambda p_z}, r^x, r^z\right) && (18)\\
&= \frac{1}{2}\lambda^2 (p_x^2 + p_z^2)\, f^2\left(\frac{p_i}{p_j}, r^x, r^z\right) && (19)\\
&= \lambda^2 \mathcal{H}(\mathbf{p}, \mathbf{r}) && (20)
\end{aligned}$$

Hamilton's equations are then:

$$\begin{aligned}
v^x = \frac{\mathrm{d}r^x}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial p_x} = 2p_x f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) + \left(p_x^2 + p_z^2\right) f\left(\frac{p_x}{p_z}, r^x, r^z\right)\frac{\partial f}{\partial p_x} && (21)\\
v^z = \frac{\mathrm{d}r^z}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial p_z} = 2p_z f^2\left(\frac{p_x}{p_z}, r^x, r^z\right) + \left(p_x^2 + p_z^2\right) f\left(\frac{p_x}{p_z}, r^x, r^z\right)\frac{\partial f}{\partial p_z} && (22)\\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial r^x} = -\left(p_x^2 + p_z^2\right)\, f\left(\frac{p_x}{p_z}, r^x, r^z\right)\frac{\partial f}{\partial r^x} && (23)\\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial r^z} = -\left(p_x^2 + p_z^2\right)\, f\left(\frac{p_x}{p_z}, r^x, r^z\right)\frac{\partial f}{\partial r^z} && (24)
\end{aligned}$$