Probability and Statistics

UG2, Core course, IIIT,H

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Outline

Gamma Distribution...

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- Widely used distribution

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- Related to exponential and normal

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Gamma Function: Extension of Factorial Function

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$n! = n \cdot n-1 \cdot \cdot \cdot$

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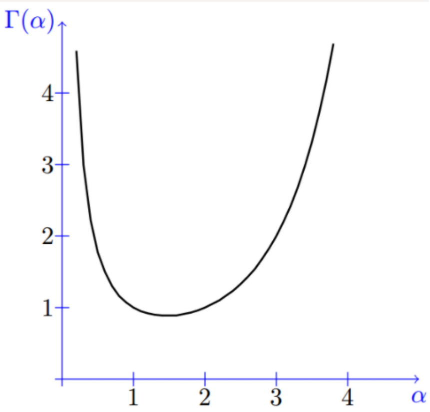

Figure: Gamma function for positive real values

Properties of the Gamma Function...

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3. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
4. $\Gamma(n) = (n-1)!$, for $n = 1, 2, 3, \ldots$
5. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Proof of Properties of Gamma Function...

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Recall $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{\boxed{-x}}\, dx$ (Gamma)

2. $\underbrace{\int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx}_{\text{L.H.S}} = \underbrace{\frac{\Gamma(\alpha)}{\lambda^\alpha}}_{\text{R.H.S}}, \quad \text{for } \lambda > 0$

Make a change of variable: $x = \lambda y$, $dx = \lambda dy$. When $x=0$, $y=0$; when $x=\infty$, $y=\infty$.

$$\Gamma(\alpha) = \int_0^\infty (\lambda y)^{\alpha-1} e^{-\lambda y} \lambda\, dy = \lambda^\alpha \underbrace{\int_0^\infty y^{\alpha-1} e^{-\lambda y}\, dy}_{[\lambda>0 \text{ is required}]} = \lambda^\alpha (\text{L.H.S})$$

$$\Rightarrow \text{L.H.S} = \int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

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3. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
4. $\Gamma(n) = (n-1)!, \quad$ for $n = 1, 2, 3, \ldots$

$\int x^m = \frac{x^{m+1}}{m+1} \qquad 0 \cdot \infty = 0$

2nd 1st f:

We have using ②

$$\Gamma(\alpha) = \lambda^\alpha \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \lambda^\alpha \left[\left[\cancel{e^{-\lambda x}} \frac{x^\alpha}{\alpha} \right]_0^\infty - \int_0^\infty -\lambda e^{-\lambda x} \frac{x^\alpha}{\alpha} dx \right]$$

0

Integration by parts.

Recall $\left[\Gamma(\alpha) = \int x^{\alpha-1} e^{-x} dx\right]$

$$= \lambda^\alpha \left(\frac{\lambda}{\alpha} \int_0^\infty x^\alpha e^{-\lambda x} dx \right) \longrightarrow \frac{\cancel{\lambda^{\alpha+1}}}{\alpha} \frac{\Gamma(\alpha+1)}{\cancel{\lambda^{\alpha+1}}}$$

We have

$$\int_0^\infty x^\alpha e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$$

$$\Rightarrow \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

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  1. First we show a fact from calculus that $d\,xdy = r\,dr\,d\theta$

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Homework: Tutorial

- We show this in three steps:
  1. First we show a fact from calculus that $d\,xdy = r\,dr\,d\theta$ Special
  2. Second we show that the constant in normal distribution is $1/\sqrt{2\pi}$

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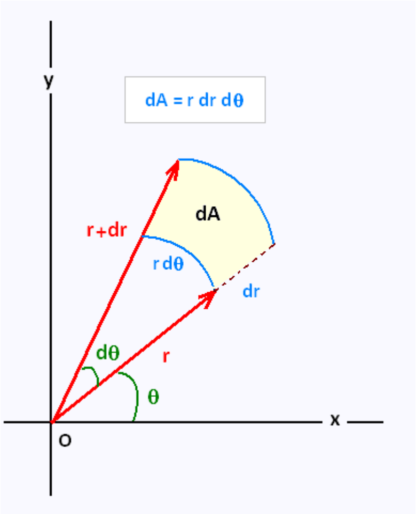
- We show this in three steps:
    1. First we show a fact from calculus that $d\,xdy = r\,dr\,d\theta$
    2. Second we show that the constant in normal distribution is $1/\sqrt{2\pi}$
    3. Finally, using above, we then show the final result stated above

Step-1: **Proof that** $dxdy = r\,dr\,d\theta$



Step-2: Proof that Constant in the Normal Distribution is $1/\sqrt{2\pi}$

Step-3: Proof of $\Gamma(1/2) = \sqrt{\pi}$

Solved Problem on Gamma Function...

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$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

Problem on Gamma Function

- Find $\Gamma(7/2)$
- Find the value of the following integral

$$I = \int_0^{\infty} x^6 e^{-5x}\, dx$$

$\Gamma(7/2) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma(3/2) = \frac{5}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{5}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

We know:

$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}$

$\alpha = 7, \quad \lambda = 5$

$I = \frac{\Gamma(7)}{5^7} = \frac{6!}{5^7}$ //

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For $\alpha = 1$, we obtain

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1. That is, $\text{Gamma}(1, \lambda) = \text{Exponential}(\lambda)$
2. Sum of n independent $\text{Exponential}(\lambda)$ RVs is $\text{Gamma}(n, \lambda)$ RV (proof: try!)

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Let $X \sim \text{Gamma}(n, \lambda), \alpha > 0, \lambda > 0$.

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Properties of Gamma ~~Function~~... distribution.

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Prove the following:

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Prove the following:

1. $\int_0^\infty f_X(x) = 1$

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2. $E[X] = \dfrac{\alpha}{\lambda}$
3. $\text{Var}(X) = \dfrac{\alpha}{\lambda^2}$

Answer to previous problem...

Solved Problem 1

Problem 1

Let $U \sim \text{Uniform}(0,1)$ and $X = -ln(1-U)$. Show that $X \sim \text{Exponential}(1)$.

Solution: $\underline{\xi_x}$

Solved Problem 2

Problem 2

Let $X \sim N(2, 4)$ and $Y = 3 - 2X$.

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Let $X \sim N(2,4)$ and $Y = 3 - 2X$.

- Find $P(X > 1)$ =
- Find $P(-2 < Y < 1)$
- Find $P(X > 2 \mid Y < 1)$

$P(-2 < Y < 1) = F_Y(1) - F_Y(-2)$

$= \Phi\left(\frac{1-(-1)}{4}\right) - \Phi\left(\frac{-2-(-1)}{4}\right)$

$= \Phi(0.5) - \Phi(-0.25)$

Look in the tables.

$\cdot\ X \sim N(2,4)$, $\mu_X = 2$, $\sigma_X = 2$

$P(X>1) = 1 - P(X \le 1) = 1 - F_X(1) = 1 - \Phi\left(\frac{1-2}{2}\right) = 1 - \Phi(-0.5)$

$= \Phi(0.5)$

$\cdot\ P(-2 < Y < 1)$. Since $Y = 3 - 2X$

$\Rightarrow Y \sim N(-1, 16) \longrightarrow$ old on top.

[Recall, if $Y = aX + b$, then $\mu_Y = a\mu_X + b$, $\sigma_Y = a\sigma_X$]

Answer to previous problem...

$$P(X>2 \mid Y<1) = P(X>2 \mid 3-2X<1)$$

$$= P(X>2 \mid X>1) = \frac{P(X>2, X>1)}{P(X>1)}$$

$$= \frac{P(X>2)}{P(X>1)} = \frac{1-P(X\le 2)}{P(X>1)}$$

$$= \frac{1-F_X(2)}{P(X>1)} = \frac{1-\phi\left(\frac{2-2}{2}\right)}{P(X>1)}$$

$$= \frac{1-\phi(0)}{P(X>1)}$$

known from tables

$\leftarrow$ 0.84

$$3-2X<1$$

$$-2X<-2$$

$$X>1$$

Solved Problem 3

$N(0,1) \leftarrow$ std normal

Problem 3

Let $X \sim N(0, \sigma^2)$. Find $E[\,|X|\,]$.

normal but not necessarily std. normal

Solution:

$X = \sigma Z$, where Z is std. normal, $Z \sim N(0,1)$.

$E[|X|] = \sigma E[|Z|]$

$E[|Z|] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t| e^{-t^2/2}\, dt = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t e^{-t^2/2}\, dt$

even fn

$d\left(-\frac{t^2}{2}\right) = -\frac{2t}{2} = -t\,dt$

$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2/2}\, d\left(\frac{t^2}{2}\right) = \sqrt{\frac{2}{\pi}} \left[-e^{-t^2/2}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\left[0 - (-1)\right]$

$\frac{\;\;}{e^{-u}\; du}$

$= \sqrt{2/\pi} \Rightarrow \boxed{E[|X|] = \sigma\sqrt{2/\pi}}$

Solved Problem 4

Problem 4

Show that

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Outline

Mixed Random Variable...

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Example of mixed random variable

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$Y = g(X) = \begin{cases} X & 0 \le X \le \frac{1}{2} \\ \frac{1}{2} & X > \frac{1}{2} \end{cases}$$

Find the CDF of Y.

for $0 < y < 1/2$

$F_Y(y) = P(Y \le y) = P(X \le y)$

$= \int_0^y 2x\,dx = y^2$

$y > 1/2$

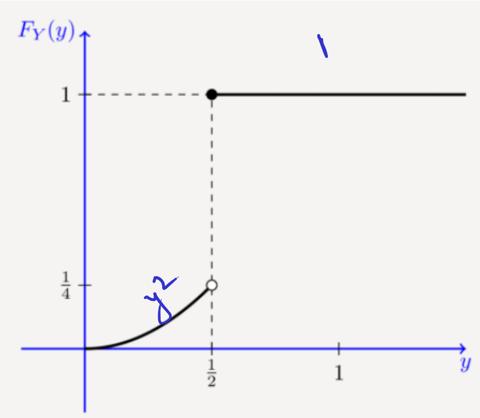
$$F_Y(y) = \begin{cases} 1, & y \ge 1/2 \\ y^2, & 0 \le y < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

$R_X = [0,1]$. For $X \in [0,1] \Rightarrow 0 \le g(X) \le 1/2 \Rightarrow \boxed{R_Y = [0, 1/2]}$

$\Rightarrow F_Y = 0$ for $y < 0$

$= 1$ for $y > 1/2$ $\Rightarrow$ $\boxed{P(Y = \tfrac{1}{2}) = P(X > 1/2) = \int_{1/2}^{1} 2x\,dx = 3/4}$

Plot of the Mixed Random Variable Example



$$F_Y\left(\tfrac{1}{2}\right) = \underline{\underline{F_Y\left(y<\tfrac{1}{2}\right)}} + P_Y\left(Y=\tfrac{1}{2}\right)$$

$$= \frac{1}{4} + \boxed{\frac{3}{4}} = 1$$

$\Rightarrow$ Y is not a continuous R.V because the CDF of Y is not continuous.

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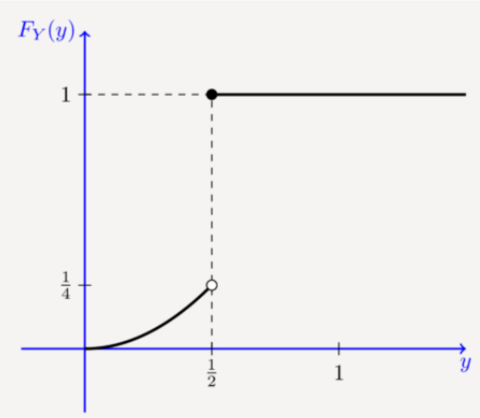

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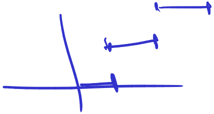

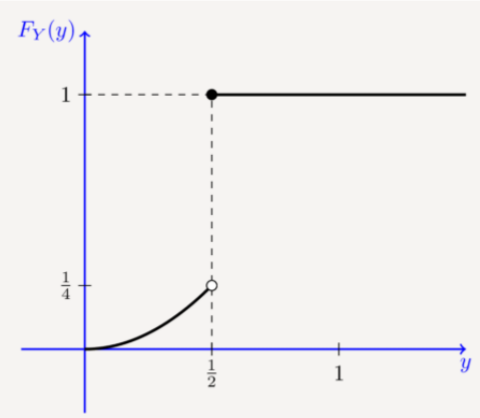

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- the CDF is not in the staircase form, so it is not a discrete random variable either

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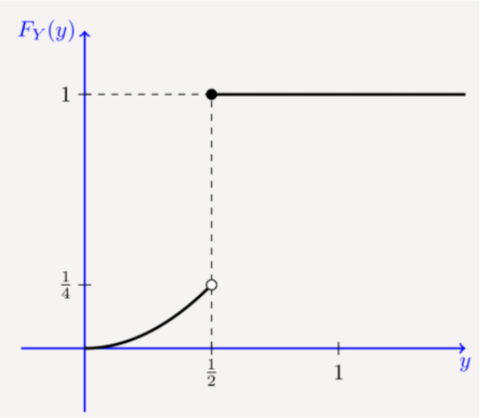

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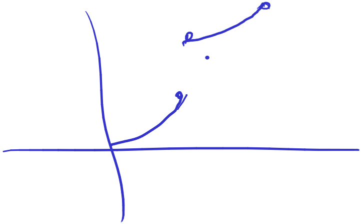

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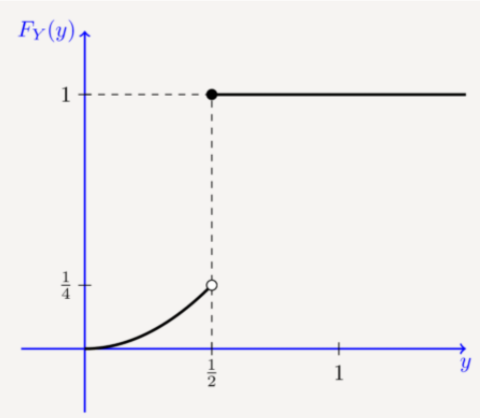

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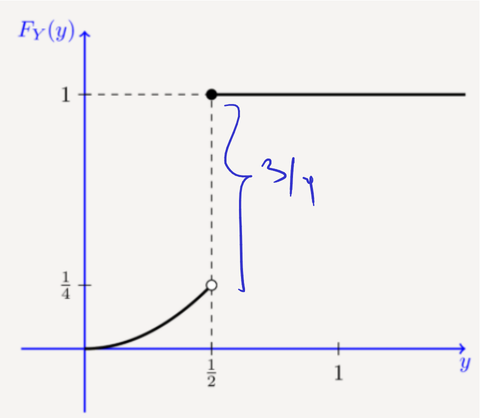

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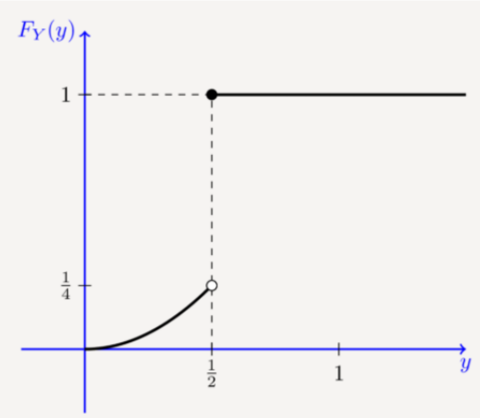

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- amount of jump is 1 - $1/4 = 3/4$
- CDF is continuous at other points

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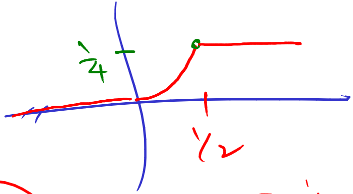
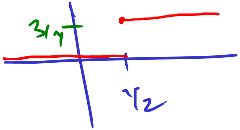


Not a CDF in itself.

and the discrete part is

$$D(y) = \begin{cases} 3/4 & y \geq 1/2 \\ 0 & y < 1/2 \end{cases}$$



CDF of a mixed RV as a sum of Continuous and Discrete CDF...

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CDF of Mixed Random Variable

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The CDF of mixed random variable Y can be written as a sum of continuous and discrete/staircase function

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cnt : $\int f(x) = 1$

Discr = $\sum p(x_i) = 1$

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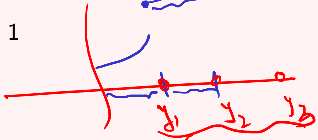

The expected value of Y can be obtained as

$$E[Y] = \int_{-\infty}^{\infty} y\, c(y)\, dy + \sum_{y_k} y_k\, P(Y = y_k)$$

$E[Y^2]$ -

Check the Validity of CDF of Mixed RV...

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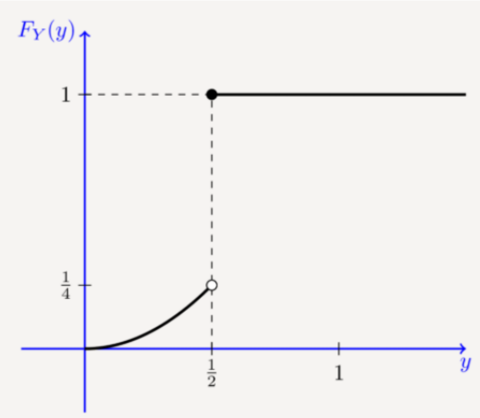

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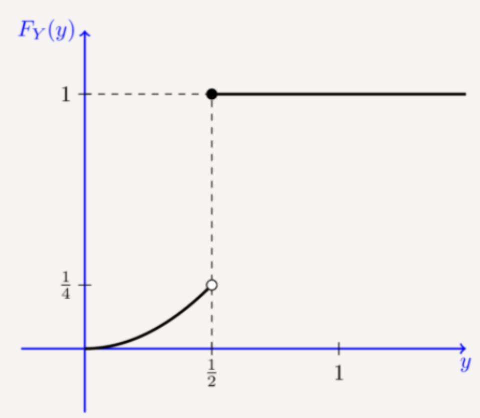

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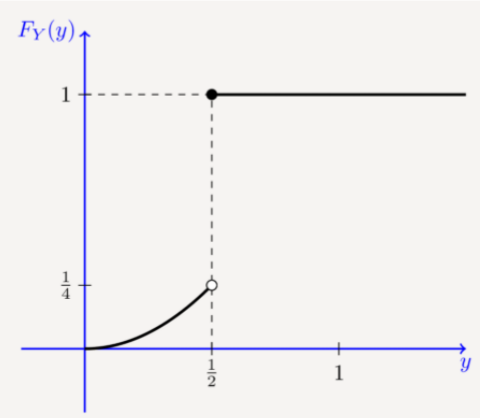

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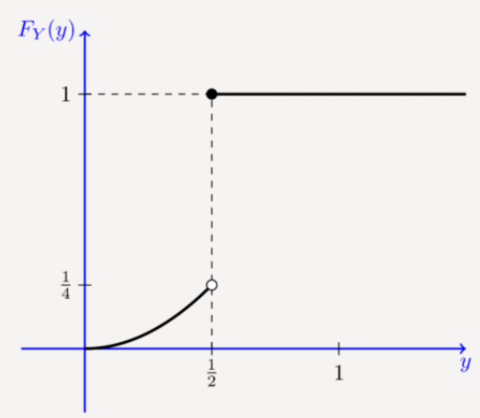

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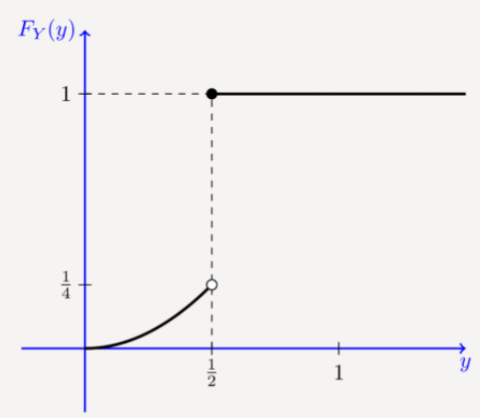

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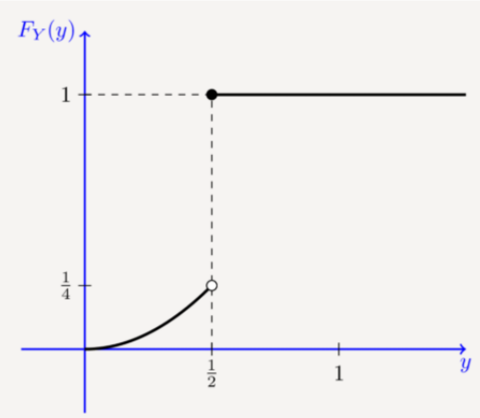
Check the Validity of CDF of Mixed RV...

$$\int_0^\infty = \int_0^{1/2} 2y\,dy + \frac{3}{4} = (y^2)\Big|_0^{1/2} + \frac{3}{4} = \frac{1}{4} + \frac{3}{4} = 1$$



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Check this again

Check that

$$\int_{-\infty}^{\infty} c(y)\, dy + \sum_{y_k} P(Y = y_k) = 1$$