classmate

Date ______
Page ______

Stochastic Assignment 3

George Paul
2021121006

3. $\frac{dx}{dt} = \alpha_1 - \gamma x$

$\frac{dy}{dt} = \alpha_2 x - \gamma y$

$\frac{d\Delta x}{dt} = \gamma \Delta x + \eta_1(t)$

where $\langle \eta_1(0), \eta(t) \rangle = \delta(t)$

$\frac{d\Delta y}{dt} = \alpha_2 \Delta x - \gamma \Delta y + \eta_2(t)$

$\eta = \langle \frac{\alpha_1 + \gamma x}{\Omega} \rangle$

$\eta_1 = \langle \frac{\alpha_1 + \gamma x}{\Omega} \rangle$ and $\langle x \rangle = \frac{\alpha_1}{\gamma}$

$= \frac{2\alpha_1}{\Omega}$

$\eta_2 = \langle \alpha_2 x + \gamma y \rangle / \Omega$

$= \alpha_2 \langle x \rangle + \gamma \langle y \rangle / \Omega$

$= \frac{2\alpha_2 \alpha_1}{\gamma \Omega}$

$\frac{d\Delta x}{dt} = -\gamma \Delta x + \eta_1(t)$

$-i\omega \Delta x(\omega) = \eta_1(\omega) - \gamma \Delta x(\omega)$

classmate

Date
Page

$$\Delta x(\omega) = \frac{\eta_1(\omega)}{\gamma - i\omega}$$

$$S_{\Delta x}(\omega) = \frac{S_{\eta_1}(\omega)}{\gamma^2 + \omega^2}$$

$$= \frac{\eta_1}{2\pi(\gamma^2+\omega^2)}$$

$$S_{\Delta x}(\omega) = \frac{\alpha_1}{\pi\Omega(\gamma^2+\omega^2)}$$

$$\frac{d\Delta y}{dt} = \alpha_2 \Delta x - \gamma \Delta y + \eta_2(t)$$

$$-i\omega \Delta y(\omega) = \alpha_2 \Delta x(\omega) - \gamma \Delta y(\omega) + \eta_2(\omega)$$

$$\Delta y(\omega) = \frac{\alpha_2 \Delta x(\omega)}{\gamma - i\omega} + \frac{\eta_2(\omega)}{\gamma - i\omega}$$

$$S_{\Delta y}(\omega) = \frac{\alpha_2^2 S_{\Delta x}(\omega)}{\gamma^2+\omega^2} + \frac{S_{\eta_2}(\omega)}{\gamma^2+\omega^2}$$

$$= \frac{\alpha_2^2 S_{\Delta x}(\omega)}{\gamma^2+\omega^2} + \frac{\eta_2}{2\pi(\gamma^2+\omega^2)}$$

$$= \frac{\alpha_2^2 \alpha_1}{\Omega\pi(\gamma^2+\omega^2)^2} + \frac{\alpha_2\alpha_1}{\pi\Omega\gamma(\gamma^2+\omega^2)}$$

$$\langle \Delta y^2 \rangle = \int_{-\infty}^{\infty} S_{\Delta y}(\omega) \cdot d\omega$$

classmate

Date ______

Page ______

$$= \frac{\alpha_2^2 \alpha_1}{\Omega \pi} \int_{-\infty}^{\infty} \frac{1}{(\gamma^2+\omega^2)^2} d\omega + \frac{\alpha_2 \alpha_1}{\pi \Omega \cdot \gamma} \int_{-\infty}^{\infty} \frac{1}{\gamma^2+\omega^2} d\omega$$

$$= \frac{\alpha_2^2 \alpha_1}{\Omega \pi} \frac{\pi}{2\gamma^2} + \frac{\alpha_2 \alpha_1}{\pi \Omega \gamma^2} \pi$$

$$= \frac{\alpha_2 \alpha_1}{\Omega \gamma^2} \left(\frac{\alpha_2}{2\gamma} + 1 \right)$$

x —————— x

2.

$$\frac{dv}{dt} = -\gamma v + \eta(t)$$

and $\langle \eta(t) \eta(t+\tau) \rangle = \frac{K_b T}{m} \delta(\tau)$

$$-i\omega v(\omega) = -\gamma v(\omega) + \eta(\omega)$$

$$v(\omega) = \frac{\eta(\omega)}{\gamma - i\omega}$$

$$S_v(\omega) = \frac{S_\eta(\omega)}{\gamma^2+\omega^2}$$

$$S_\eta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \eta(0) \eta(t) \rangle e^{i\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K_b T}{m} \delta(t) e^{i\omega t} dt$$

Date

Page

$$S_\eta(\omega) = \frac{k_b T}{2\pi m}$$

$$\therefore \; S_v(\omega) = \frac{k_b T}{2\pi m(\gamma^2 + \omega^2)}$$

sinusoidal force,

$$m\frac{dv}{dt} = -m\gamma v + m\eta(t) + A\sin(\omega t)$$

$$\frac{dv}{dt} + \gamma v = \eta(t) + \frac{A}{m}\sin(\omega t)$$

$$e^{\gamma t}\frac{dv}{dt} + e^{\gamma t}\gamma v = e^{\gamma t}\eta(t) + e^{\gamma t}\frac{A}{m}\sin(\omega t)$$

$$\frac{d}{dt}(e^{\gamma t}v) = e^{\gamma t}\eta(t) + e^{\gamma t}\frac{A}{m}\sin(\omega t)$$

$$v(t) = \int_0^t \eta(t')e^{\gamma(t'-t)}dt' + \int_0^t \frac{A}{m}\sin(\omega t')\, e^{\gamma(t'-t)}\, dt$$

$$\langle v(t)\rangle = 0 + \int_0^t \frac{A}{m}\sin(\omega t)e^{\gamma(t'-t)}dt$$

$$= \frac{A}{m}e^{-\gamma t}\left[e^{\gamma t'}\frac{(\gamma\sin(\omega t') - \omega\cos(\omega t'))}{\omega^2 + \gamma^2}\right]_0^t$$

$$= \frac{A}{m}\frac{\sin(\omega t - \phi)}{\sqrt{\omega^2 + \gamma^2}}$$

classmate
Date
Page

1. $\frac{dx}{dt} = \alpha_1 \gamma x$ $\qquad \langle x_s \rangle = \frac{\alpha_1}{\gamma}$

$\frac{d\Delta x}{dt} = -\gamma \Delta x + \eta_1(t)$

and $\langle \eta_1(0)\, \eta_1(t) \rangle = \eta_1 \mathcal{g}\, \delta(t)$

$\eta_1 = \frac{\langle \alpha_1 + \gamma x \rangle}{\Omega} = \frac{2\alpha_1}{\Omega}$

$\therefore \frac{2\alpha_1}{\Omega} \delta(t) = \langle \eta_1(0)\, \eta_1(t) \rangle$

$\frac{dy_2}{dt} = \alpha_2 x y_1 - \beta y_2$
$= \alpha_2 x(y + y_2) - y_2(\alpha_2 x + \beta)$

$\langle y_{1s} \rangle = \frac{\alpha_2 \alpha_1 (y + y_2)}{\alpha_2 \alpha_1 + \beta\gamma}$ $\qquad \langle y_{2s} \rangle = \frac{\beta\gamma(y + y_2)}{\alpha_2\alpha_1 + \beta\gamma}$

$\frac{d\Delta y_2}{dt} = \alpha_2 \Delta x (y + y_2) - \langle y_2 s \rangle \alpha_2 \Delta x$
$- \Delta y_2 (\alpha_2 \langle x_s \rangle + \beta) + \eta_2(t)$

$\eta_2 = \frac{\langle \alpha_2 x y_1 + \beta y_2 \rangle}{\Omega}$

$= \frac{\alpha \langle x_1 \mathcal{g} \rangle \langle y_1 s \rangle + \beta \langle y_{2s} \rangle}{\Omega}$

$= \frac{2\alpha_2 \alpha_1 \beta (y + y_2)}{(\gamma\beta + \alpha_1 \alpha_2)\, \Omega}$

$\frac{d\Delta x}{dt} = -\gamma \Delta x + \eta_1(t)$

classmate

Date

Page

$$\Delta x(w) = \frac{\eta(w)}{\gamma - iw}$$

$$S_{\Delta x}(w) = \frac{S_{\eta_1}(w)}{\gamma^2 + w^2}$$

$$= \frac{\alpha_1}{\pi\Omega(\gamma^2 + w^2)}$$

$$\frac{d\Delta y_2}{dt} = \alpha_2 \Delta x (y + y_2) - \frac{\alpha_2^2 \alpha_1 \Delta x}{\alpha_2 \alpha_1 + \beta\gamma} - \frac{\alpha_2 \alpha_1 + \gamma\beta}{\gamma}(\Delta y) + \eta_2(t)$$

$$i w \Delta y_2(w) = \left(\alpha_2 y - \frac{\alpha_2^2 \alpha_1}{\alpha_1 \alpha_2 + \beta\gamma}\right) \Delta x(w) - \frac{\alpha_1 \alpha_2 + \beta w}{\gamma} \overset{\Delta y(w)}{\wedge} + \eta_2(w)$$

$$y_2(w)\left(\frac{\alpha_2 \alpha_1 + \beta\gamma - iw\gamma}{\gamma}\right) = \left(\alpha_2 (y + y_2) - \frac{\alpha_2^2 \alpha_1 (y + y_2)}{\alpha_2 \alpha_1 + \beta\gamma}\right) \Delta x(w) + \eta_2(w)$$

on mod. squaring,

$$S_{y_2}(w) = \left(\frac{\gamma\beta\alpha_2(y + y_2)}{\frac{\alpha_2 \alpha_1 + \beta\gamma}{\gamma}}\right)^2 \frac{S_{\Delta x}(w)}{\frac{\alpha_2 \alpha_1 + \beta\gamma}{\gamma} + w^2} + \frac{S_{\eta_2}(w)}{\left(\frac{\alpha_2 \alpha_1 + \beta\gamma}{\gamma}\right)^2 + w^2}$$

$$\langle \Delta y_2^2 \rangle = \int_{-\infty}^{\infty} S_{y_2}(\omega)\, d\omega$$

$$= \frac{\gamma^2 \beta^2 \alpha_2^2 (y+y_2)^2 \alpha_1}{\left(\frac{\alpha_1 \alpha_2 + \beta\gamma}{\gamma^2}\right)^2 \cancel{\pi} \cancel{\beta}\, \Omega} \frac{\pi}{\gamma} + \frac{\alpha_2 \alpha_1 (y+y_2) \beta\gamma \cancel{\pi}}{\cancel{\pi} \frac{\alpha_2 \alpha_1 + \beta\gamma}{\gamma^2} \Omega}$$

$$= \frac{\alpha_2 \alpha_1 \beta\gamma}{\left(\frac{\alpha_2 \alpha_1 + \beta\gamma}{\gamma^2}\right)^2 \Omega} \left(1 + \frac{\alpha_2 \beta (y+y_2)}{\frac{\alpha_2 \alpha_1 \beta \gamma}{\gamma} + \gamma}\right)$$