Bra-Ket and Matrix representation:

Let us consider the eigenkets and eigenvalues of a Hermitian operator A.

We begin with an important theorem.

Theorem: The eigenvalues of a Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal.

(Proof already discussed).

Eigenkets as base kets:

We have seen that the normalized eigenkets of A form a complete orthonormal set.

Given an arbitrary ket $|\alpha\rangle$ in the ket space spanned by the eigenkets of A, we expand it as $|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$.

We have seen that $c_{a'} = \langle a'|\alpha\rangle$.

Therefore we have $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$

This is analogous to an expansion of a vector $\vec{V}$ in (real) Euclidean space.

$$\vec{V} = \sum_i \hat{e}_i (\hat{e}_i \cdot \vec{V})$$

with $\{\hat{e}_i\}$ form an orthonormal basis.

We have already learned that

$$\sum_{a'} |a'\rangle\langle a'| = \mathbb{I} .$$

This relation is known as ~~a~~ the "completeness relation".

This product $|\cdot\rangle\langle\cdot|$ is called the "outer product".

Usefullness of completeness relation can never be overestimated. For example:

$$\langle\alpha|\alpha\rangle = \langle\alpha|\left(\sum_{a'} |a'\rangle\langle a'|\right)|\alpha\rangle$$

$$= \sum_{a'} |\langle a'|\alpha\rangle|^2$$

Incidentally if $|\alpha\rangle$ is normalized,

$$\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1 = \sum_{a'} |C_{a'}|^2$$

Therefore $C_{a'} = \langle a'|\alpha\rangle$ can be understood as probability amplitude.

Let us consider an element of $|\alpha\rangle = \sum_{a'} C_{a'} |a'\rangle$ as

$$C_{a'} |a'\rangle = |a'\rangle\langle a'|\alpha\rangle .$$

We see that $|a'\rangle\langle a'|$ selects a portion of $|\alpha\rangle$, parallel to $|a'\rangle$.

Therefore "$|a'\rangle\langle a'|$" operator is known as projection operator, denoted by

$$\Lambda_{a'} = |a'\rangle\langle a'| .$$

The completeness relation can therefore be written as

$$\sum_{a'} \Lambda_{a'} = \mathbb{I}$$

Matrix representation.

Having spefied the base kets, we now show how to represent an operator 'X' by a square matrix.

Before that, let us understand that since $| \rangle$ - ket represents a vector, it can be represented as a column vector:

$$|\cdot\rangle = \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}$$

It's conjugate bra:

$$\langle\cdot| = (a^*, b^*, \cdots).$$

The complex number inner product:

$$\langle\cdot|\cdot\rangle \Longleftarrow = (a^*, b^* \cdots) \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}.$$

The matrix outer product

$$|\cdot\rangle\langle\cdot| = \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} (a^* \; b^* \cdots).$$

Now let us consider operator X., which can be written as

$$X = \left(\sum_{a''} |a''\rangle\langle a''|\right) X \left(\sum_{a'} |a'\rangle\langle a'|\right).$$

$$= \sum_{a', a''} |a''\rangle \langle a''|X|a'\rangle \langle a'| .$$

There are altogether N^2 numbers of form $\langle a''|X|a'\rangle$, where N is the ~~t~~ dimension of the ket space.

We can arrange them as a $N \times N$ square matrix such that, the elements of the matrix can be understood as

$$\langle a''|X|a'\rangle .$$

$$\swarrow \qquad\qquad \searrow$$

Row $\qquad\qquad$ Column.

Explicits written form.

$$X = \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & & \end{pmatrix}$$

where $\langle a''|X|a'\rangle = \langle a'|X^{\dagger}|a''\rangle^{*}$

Then for X hermitian, we have

$$\langle a''|X|a'\rangle = \langle a'|X|a''\rangle^{*}$$

Let us consider a more specific example from 2 dimension.

Let $|a_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|a_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

$$\langle a_1|a_1\rangle = \langle a_2|a_2\rangle = 1 \quad \text{(Check)}.$$

$$\langle a_1|a_2\rangle = \langle a_2|a_1\rangle = 0 \quad \text{(Check)}.$$

$$|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \text{(Check)}.$$

Now consider an operator

$$\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad \text{(One of the Pauli operators)}.$$

$$\langle a_1|\sigma_z|a_1\rangle = 1 \quad \text{(Check)}.$$

$$\langle a_2|\sigma_z|a_2\rangle = -1 \quad \text{(Check)}.$$

$$\langle a_1|\sigma_z|a_2\rangle = \langle a_2|\sigma_z|a_2\rangle = 0.$$

Therefore, $$\sigma_z = \sum_{\substack{i=1,2\\ j}} |a_i\rangle\langle a_i|\sigma_z|a_j\rangle\langle a_j|.$$

with the elements $\langle a_i|\sigma_z|a_j\rangle$ ~~speci~~ specified earlier.

The previous example is a bit easy, because $|a_1\rangle$, $|a_2\rangle$ are the normalized eigenvectors of σ_z operator. $\sigma_z|a_1\rangle = |a_1\rangle$; $\sigma_z|a_2\rangle = -|a_2\rangle$

Let us therefore consider a different example.

Consider another Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

For ~~for~~ the expansion in the basis $|a_1\rangle, |a_2\rangle$.

we have

$$\langle a_1|\sigma_x|a_1\rangle = (1 \quad 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1 \quad 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle a_2|\sigma_x|a_2\rangle = 0$$

$$\langle a_1|\sigma_x|a_2\rangle = \langle a_2|\sigma_x|a_1\rangle = 1.$$

Now consider a different basis.

$$|+\rangle = \frac{|a_1\rangle + |a_2\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$|-\rangle = \frac{|a_1\rangle - |a_2\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Check $\langle +|+\rangle = \langle -|-\rangle = 1$

$$\langle +|-\rangle = \langle -|+\rangle = 0.$$

$$|+\rangle\langle +| + |-\rangle\langle -| = \mathbb{I}.$$

$$\langle + | \sigma_x | + \rangle = \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}.$$

$$= \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1.$$

$$\langle - | \sigma_x | - \rangle = \left(\tfrac{1}{\sqrt{2}} \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

$$= \left(\tfrac{1}{\sqrt{2}} \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -1.$$

$$\langle + | \sigma_x | - \rangle = \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

$$= \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 0.$$

$\langle - | \sigma_x | + \rangle = 0$ (similarly).

Therefore in the basis $(|+\rangle, |-\rangle)$ the matrix $\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$|+\rangle$ and $|-\rangle$ are the eigenbasis of σ_x.

$$\sigma_x |+\rangle = |+\rangle.$$

$$\sigma_x |-\rangle = -|-\rangle.$$

The way we arrange $\langle a''|X|a'\rangle$ into a square matrix is in conformity with the usual rule of matrix multiplication. To see this just note that the matrix representation of the operator relation

$$Z = XY$$

reads $\langle a''|Z|a'\rangle = \langle a''|XY|a'\rangle$.

$$= \sum_{a'''} \langle a''|X|a'''\rangle \langle a'''|Y|a'\rangle .$$

Here we have inserted the identity operator.

Let us now examine how the ket relation

$$|\gamma\rangle = X|\alpha\rangle$$

can be represented in our base ket representation.

$$|\gamma\rangle = X|\alpha\rangle.$$

$$\langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle$$

$$= \sum_{a''} \langle a'|X|a''\rangle \langle a''|\alpha\rangle$$

These are just the rules of matrix multiplication.

$$|\alpha\rangle = \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \vdots \end{pmatrix} \qquad |\gamma\rangle = \begin{pmatrix} \langle a^{(1)}|\gamma\rangle \\ \langle a^{(2)}|\gamma\rangle \\ \vdots \\ \vdots \end{pmatrix}$$

The inner product :

$$\langle \beta | \alpha \rangle = \sum_{a'} \langle \beta | a' \rangle \langle a' | \alpha \rangle$$

$$= \left(\langle a^{(1)} | \beta \rangle^*, \langle a^{(2)} | \beta \rangle^* \cdots \right) \begin{pmatrix} \langle a^{(1)} | \alpha \rangle \\ \langle a^{(2)} | \alpha \rangle \\ \vdots \\ \vdots \end{pmatrix}$$

The outer product :

$$|\beta\rangle \langle \alpha| = \sum_{a', a''} |a'\rangle \langle a' | \beta \rangle \langle \alpha | a'' \rangle \langle a''| .$$

$$\therefore \; |\beta\rangle \langle \alpha| = \begin{pmatrix} \langle a^{(1)} | \beta \rangle \langle a^{(1)} | \alpha \rangle^* & \langle a^{(1)} | \beta \rangle \langle a^{(2)} | \alpha \rangle^* & \cdots \\ \langle a^{(2)} | \beta \rangle \langle a^{(1)} | \alpha \rangle^* & \langle a^{(2)} | \beta \rangle \langle a^{(2)} | \alpha \rangle^* & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix}$$

Spin - 1/2 Systems.

We are now going to consider one of the most important and elementary systems in quantum mechanics.
The smallest possible system that can be considered in QM is the spin-1/2 system, which can be represented by a 2-dimensional Hilbert space.

The base kets are denoted as $|1\rangle$ and $|0\rangle$.

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the language of Physics, they are also called energy eigen basis.
In the language of Quantum information Science, they are called computational basis.
The identity operator can be written by the completeness relation

$$\mathbb{I} = \sum_{\alpha} |\alpha\rangle\langle\alpha| = |0\rangle\langle 0| + |1\rangle\langle 1| .$$

The Pauli matrix σ_z can be written as

$$\sigma_z = \frac{\hbar}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right).$$

$$= \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenket - eigenvalue relation.

$$\cancel{\sigma_z |\pm\rangle = \pm \frac{\hbar}{2} |}$$

$$\sigma_z |1\rangle = \frac{\hbar}{2} |1\rangle$$

$$\sigma_z |0\rangle = -\frac{\hbar}{2} |0\rangle .$$

It is instructive to look at two other operators,

$$\sigma_+ = \hbar |1\rangle\langle 0| \quad , \quad \sigma_- = \hbar |0\rangle\langle 1| .$$

which are both non-hermitian.

The operator σ_+ acting on $|0\rangle$, turns $|0\rangle$ into $|1\rangle$ and σ_- does the vice-versa

The spin-½ system in a nut-shell can be represented as.

The Pauli operators : σ_x , σ_y , σ_z

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

$$\cancel{\sigma_z = (1}$$

$$\sigma_+ = \sigma_x + i\sigma_y$$

$$\sigma_- = \sigma_x - i\sigma_y .$$

$$(\sigma_+)^\dagger = \sigma_- .$$

Measurements:

Having developed the mathematics of ket space, we are now in a position to discuss the quantum theory of measurement process.

To ~~start,~~ begin, we start with the description of measurement in the words of great P.A.M. Dirac –

" A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured

Let us elaborate this statement.

$$|\alpha\rangle = \sum_{a'} C_{a'}|a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle .$$

be a quantum state.

When the measurement is performed, the system is "thrown into" one of the eigenstates. $|a'\rangle$ of an observable $\hat{A}$.

In other words

$$|\alpha\rangle \xrightarrow{\hat{A}\text{ measurement}} |a'\rangle$$

Example:

Let us consider a projection operator $\hat{P} = |a_1\rangle\langle a_1|$ and a quantum state

$$|\psi\rangle = \sum_i c_{a_i} |a_i\rangle$$

Therefore $$\hat{P}|\psi\rangle = \sum_i c_{a_i} |a_1\rangle\langle a_1|a_i\rangle$$

$$= c_{a_1} |a_1\rangle .$$

i.e. the operator $\hat{P}$ projects $|\psi\rangle$ into the eigen ket $|a_1\rangle$, discarding all the others components.

Now consider a general one.

$$\sigma_z = \frac{\hbar}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) .$$

$$|\psi\rangle = \alpha|1\rangle + \beta|0\rangle .$$

Therefore $$\sigma_z|\psi\rangle = \frac{\hbar}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)(\alpha|1\rangle + \beta|0\rangle)$$

$$= \frac{\hbar}{2}(\alpha|1\rangle - \beta|0\rangle) .$$

We donot know in advance, that into which state it will be thrown into, as a result of measurement. We do know, however, the probability of jumping into a particular state $|a_i\rangle$ as

$$P_{a_i} = |\langle a_i|\psi\rangle|^2 .$$

$|\psi\rangle \rightarrow$ normalized.

The probability interpretation of the squared inner product $|\langle a_i|\alpha\rangle|^2$ is one of the most fundamental postulates of QM. Therefore it cannot be proven.

Expectation value :

We define the expectation value of an operator $\hat{A}$ with respect to state $|\alpha\rangle$ as

$$\langle \hat{A} \rangle_\alpha = \langle \alpha | \hat{A} | \alpha \rangle ,$$

providing $|\alpha\rangle$ is normalized.

However, if $|\alpha\rangle$ is not normalized, then

$$\langle \hat{A} \rangle_\alpha = \frac{\langle \alpha | \hat{A} | \alpha \rangle}{\langle \alpha | \alpha \rangle} .$$

Donot confuse expectation values with eigenvalues.

We will show the difference by considering an example in spin-1/2 system.

Example:

$$\sigma_z = \frac{\hbar}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right).$$

Let us consider $|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$.

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1.$$

Eigenvalues: $\sigma_z|1\rangle = \frac{\hbar}{2}|1\rangle$

Eigenvalues.

$\sigma_z|0\rangle = -\frac{\hbar}{2}|0\rangle$

Expectation value:

$$\langle\sigma_z\rangle_\psi = \langle\psi|\sigma_z|\psi\rangle.$$

$$= \left(\alpha^*\langle 1| + \beta^*\langle 0|\right)\left[\frac{\hbar}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)\right]\left(\alpha|1\rangle + \beta|0\rangle\right)$$

$$= \left(\alpha^*\langle 1| + \beta^*\langle 0|\right)\left(\frac{\hbar}{2}\alpha|1\rangle - \frac{\hbar}{2}\beta|0\rangle\right).$$

$$= |\alpha|^2\frac{\hbar}{2} + |\beta|^2\hbar/2.$$

$$= \frac{\hbar}{2}\left(|\alpha|^2 + |\beta|^2\right) = \hbar/2.$$

General expression:

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle.$$

$$\langle\hat{A}\rangle_\alpha = \langle\alpha|\hat{A}|\alpha\rangle = \sum_{a'a''}\langle\alpha|a'\rangle\langle a'|\hat{A}|a''\rangle\langle a''|\alpha\rangle.$$

$$= \sum_{a'a''} c_{a'}^* c_{a''}\langle a'|\hat{A}|a''\rangle.$$

Spin-1/2 system once again:

Before proceeding into a general discussion of observables, we again consider a spin-1/2 system.

$|1\rangle$ and $|0\rangle$ are the eigenkets of the Pauli operator σ_z.

Let us now consider a quantum state $|+\rangle_x$, such that

$$|\langle 1|+\rangle_x| = |\langle 0|+\rangle_x| = \frac{1}{\sqrt{2}}.$$

i.e. $|+\rangle$ state contains $|1\rangle$ and $|0\rangle$ with equal probabilities of $1/2$.

Let us expan $|+\rangle$ in the basis of $\{|1\rangle, |0\rangle\}$

$$|+\rangle_x = \mathbb{I}.|+\rangle_x = (|0\rangle\langle 0| + |1\rangle\langle 1|)|+\rangle_x$$

$$= \langle 0|+\rangle_x |0\rangle + \langle 1|+\rangle_x |1\rangle .$$

Now since, $|\langle 0|+\rangle_x| = |\langle 1|+\rangle_x| = \frac{1}{\sqrt{2}}$,

we can consider $\langle 1|+\rangle_x = \frac{1}{\sqrt{2}}$ and

$$\langle 0|+\rangle_x = \frac{1}{\sqrt{2}} e^{i\delta_1} .$$

with $\delta_1 \to$ real.

Therefore $|+\rangle_x = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}} e^{i\delta_1}|0\rangle$.

$\delta_1 \to$ is called the phase.

Similarly we can construct another state vector

$$|-\rangle_x = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}} e^{i\delta_1}|0\rangle .$$

It is easy to check that $\{|+\rangle_x, |-\rangle_x\}$ form an orthonormal basis. (Check).

Let us construct the operator $\hat{S}_x$ using the orthonormal basis $\{|+\rangle, |-\rangle\}$ as.

$$\hat{S}_x = \frac{\hbar}{2}\left[|+\rangle_x{}_x\langle +| - |-\rangle_x\langle -|\right].$$

$$= \frac{\hbar}{2}\left[e^{-i\delta_1}|1\rangle\langle 0| + e^{i\delta_1}|0\rangle\langle 1|\right].$$

(Check).

Notice that $\hat{S}_x$ we have constructed is hermitian.

Similarly we can construct another orthonormal set

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}|1\rangle \pm e^{i\delta_2}|0\rangle .$$

and Hermitian operator

$$\hat{S}_y = \frac{\hbar}{2}\left[e^{-i\delta_2}|1\rangle\langle 0| + e^{i\delta_2}|0\rangle\langle 1|\right].$$

Is there any way to determine δ_1, δ_2 ?

Let us consider a particular situation. 1

As we have considered.

~~$|\langle + | 1 \rangle| =$~~

$$|\langle 1 | + \rangle_x| = |\langle 0 | + \rangle_x| = \frac{1}{\sqrt{2}} .$$

and $$|\langle 1 | - \rangle_x| = |\langle 0 | - \rangle_x| = \frac{1}{\sqrt{2}} .$$

Similarly, we have

$$|{}_y\langle \pm | + \rangle_x| = |{}_y\langle \pm | - \rangle_x| = \frac{1}{\sqrt{2}} .$$

The interpretation of this consideration is the same as before.

Inserting $|\pm\rangle_x$ and $|\pm\rangle_y$, we have

$$\frac{1}{2} |1 \pm e^{i(\delta_1 - \delta_2)}| = \frac{1}{\sqrt{2}} \quad \text{(check)} .$$

which is satisfied only if

$$\delta_2 - \delta_1 = \pi/2 \quad \text{or} \quad -\pi/2 .$$

We can therefore consider $\delta_1 = 0$ and $\delta_2 = \pi/2$.

To summerize, we have

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} |1\rangle \pm \frac{1}{\sqrt{2}} |0\rangle .$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}} |1\rangle \pm \frac{i}{\sqrt{2}} |0\rangle .$$

$S_x \neq \frac{\hbar}{2}$

and $\hat{S}_x = \frac{\hbar}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$.

$\hat{S}_y = \frac{\hbar}{2}(-i|1\rangle\langle 0| + i|0\rangle\langle 1|)$.

There are nothing but the Pauli operators, we discussed before.

$$\hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y .$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Commutator → $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

Anti commutator → $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

Relations:

i) $[\hat{S}_i, \hat{S}_j] = i\varepsilon_{ijk}\hbar\hat{S}_k$.

ii) $\{\hat{S}_i, \hat{S}_j\} = \frac{\hbar^2}{2}\delta_{ij}\mathbb{I}$.

If $\hat{S}^2 = \hat{S}\cdot\hat{S} = S_x^2 + S_y^2 + S_z^2$, then

iii) $[\hat{S}^2, S_i] = 0$.

$\varepsilon_{ijk} \rightarrow$ Levi-Civita symbol.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (x,y,z), (y,z,x)\ (z,x,y). \\ -1 & \text{if } (z,y,x), (x,z,y)\ (y,x,z) \\ 0 & \text{if repitition occurs.} \end{cases}$$

$\delta_{ij} \rightarrow$ Kronecker Delta.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j. \\ 0 & \text{if } i \neq j \end{cases}.$$

Example Levi Civita $\rightarrow$

$$\varepsilon_{xyz} = \varepsilon_{yzx} = \varepsilon_{zxy} = 1.$$

$$\varepsilon_{yxz} = -1 = \varepsilon_{yxz}$$

$$\varepsilon_{yyz} = 0.$$