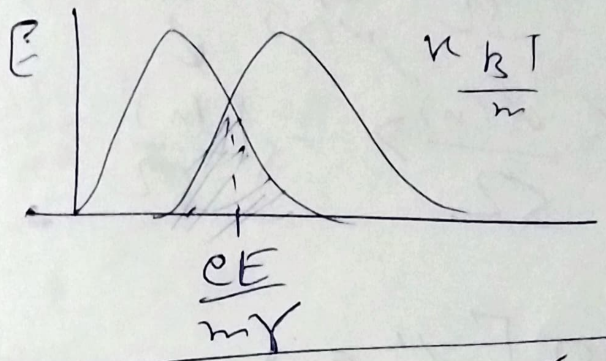
$$\langle v^2 \rangle = \frac{eE}{m\gamma}\langle v \rangle + \frac{k_B T}{m}$$

$$\sigma_v^2 = \langle v^2 \rangle - \langle v \rangle^2 = \frac{k_B T}{m}$$



Instead of considering $\xrightarrow{\alpha} X \xrightarrow{\beta}$

(creation of molecules) $n \longrightarrow n+1$

consider concentration $\longleftarrow$

(Ω : volume)

$$x = \frac{n}{\Omega}$$

Now $\Delta x = \frac{1}{\Omega}$ so for large Ω we have a cont. markov process (with cont. space).

$$\frac{dp(n,t)}{dt} = \alpha(n-1)\, p(n-1,t) + \beta(n+1)\, p(n+1,t) - (\alpha(n) + \beta(n))\, p(t)$$

$\rightarrow$ (for n discrete)

Put $x = \frac{n}{\Omega}$ $\Delta x = \frac{1}{\Omega}$.

$$\frac{dp}{dt}(x,t) = \alpha(x - \Delta x)\, p(x-\Delta x, t) + \beta(x+\Delta x)\, p(x+\Delta x, t) - (\alpha(x) + \beta(x))\, p(x,t)$$

$$\frac{d\langle n\rangle}{dt} = \langle \alpha(n) - \beta(n)\rangle$$

functions of n

$$\frac{d\langle x\rangle}{dt} = \frac{\langle \alpha(n) - \beta(n)\rangle}{\Omega}$$

$\alpha'(x)$ $\quad$ $\beta'(x)$

$$= \left\langle \frac{\alpha(n)}{\Omega}\right\rangle - \left\langle \frac{\beta(n)}{\Omega}\right\rangle$$

$$\frac{dp(x,t)}{dt} = \Omega\Big[\alpha'(x-\Delta x)\, p(x-\Delta x, t)$$

$$+ \beta'(x+\Delta x)\, p(x+\Delta x, t)$$

$$- (\alpha(x) + \beta(x))\, p(x,t)\Big]$$

$$\simeq \Omega\Big[-\Delta x \frac{\partial}{\partial x}\big((\alpha(x) - \beta(x))\, p(x,t)\big)$$

$$+ \frac{\Delta x^2}{2} \frac{\partial^2}{\partial x^2}\big((\alpha'(x) + \beta'(x))\, p(x,t)\big)$$

$$\frac{dp}{dt} = -\frac{\partial}{\partial x}\big(f(x)\, p(x,t)\big)$$

$$+ \frac{\partial^2}{\partial x^2}\left(\frac{g(x)}{\Omega}\, p(x,t)\right)$$

Langenvin equation

Van-Kampen Volume ~~Expansion~~ expansion

$$\dot{x}(t) = f(x) + \sqrt{\frac{g(x)}{\Omega}}\, \eta(t)$$

$f(x) = 0$ at steady state

$x = x_s + \Delta x$ $\quad x = x_s$ small noise approximation.

$$\frac{d}{dt}(\Delta x) = f(x_s + \Delta x) + \sqrt{\frac{g(x_s+\Delta x)}{\Omega}}\,\eta(t)$$

$$= \underbrace{\frac{\partial f}{\partial x}\Big|_{x_s}}_{k}\Delta x + \eta(t)\,\Omega^{-1/2}\left(g(x_s) + \frac{\partial g}{\partial x}\Big|_{x_s}\Delta x\right)^{1/2}$$

$$\frac{d\Delta x}{dt} = k\,\Delta x + \Omega^{-1/2}\sqrt{g(x_s)}\,\eta(t)\left(1 + \frac{1}{g}\frac{\partial g}{\partial x}\Big|_{x_s}\Delta x\right)^{1/2}$$

$$\frac{d\Delta x}{dt} = K\cdot\Delta x + \sqrt{\frac{g(x_s)}{\Omega}}\,\eta(t)$$

$\cancel{x_s = \alpha/\beta}$

$$\frac{d\langle x\rangle}{dt} = \alpha - \beta\langle x\rangle = f(x)$$

$\boxed{x_s = \alpha/\beta}$ $\quad K = \frac{\partial f}{\partial x}\Big|_{x_s} = -\beta$

$$g(x_s) = \alpha(x) + \beta(x)$$

$$g(x_s) = \alpha + \beta x_s = 2\alpha$$

$$\frac{d\Delta x}{dt} = \underbrace{-\beta\,\Delta x}_{f(x)} + \underbrace{\sqrt{\frac{2\alpha}{\Omega}}}_{g(x)}\,\eta(t)$$

$$\frac{d\langle x^2\rangle}{dt} = 2\langle x f(x)\rangle + \langle g(x)\rangle$$

$$\Rightarrow \frac{d\langle \Delta x^2\rangle}{dt} = -2\beta\langle \Delta x^2\rangle + \frac{2\alpha}{\Omega}$$

$$X_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} X_2$$

$n_1 \qquad n_2$

$$\langle n_2\rangle = \sigma_{n_2}^2 = \frac{\alpha\beta}{(\alpha+\beta)^2} N_T$$

$$\frac{dX_2}{dt} = \alpha X_1 - \beta X_2$$

$$= \alpha X_T - (\alpha+\beta) X_2$$

$$X_1 + X_2 = X_T \Rightarrow X_1 = X_T - X_2$$

$$\therefore X_{2s} = \frac{\alpha X_T}{\alpha+\beta}$$

$$g(X_{2s}) = (\alpha X_{1s} + \beta X_{2s})$$

$$= (\alpha X_T + (\beta-\alpha) X_{2s})$$

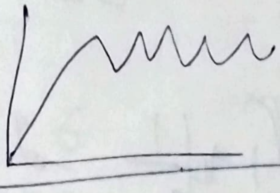
$$= \frac{2\alpha\beta}{\alpha+\beta} X_T$$

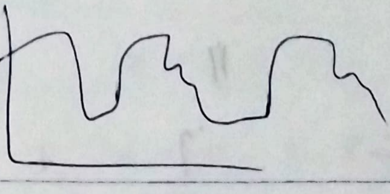
$$\frac{d\Delta x_2}{dt} = -(\alpha+\beta)\Delta x_2 + \sqrt{\frac{g(X_{2s})}{\Omega}}\,\eta(t)$$

$$\frac{d\langle \Delta x_2^2 \rangle}{dt} = -2(\alpha+\beta)\langle \Delta x_2^2\rangle + \frac{g(x)}{\Omega} = 0$$

$$\Rightarrow \langle \Delta x_2^2 \rangle = \frac{g(x_s)}{2(\alpha+\beta)}\frac{1}{\Omega} = \frac{\alpha\beta}{(\alpha+\beta)^2}\frac{X_T}{\Omega}$$

$\Rightarrow$ A.





$X = \frac{n}{\Omega}$ (particles per volume)

→ concentration.

$$\frac{dp(n)}{dt} = \alpha(n-1)\,p(n-1,t) + \beta(n+1)\,p(n+1,t) - [\alpha(n)+\beta(n)]\,p(n,t)$$

↓ continuous time

$$\frac{dp(x,t)}{dt} = \frac{\partial}{\partial x}\left[f(x)\,p(x,t) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{g(x)}{\Omega}\,p(x,t)\right)\right]$$

$\Delta x << x_s$ ↓ small noise

$f(x)=0 \Rightarrow x=x_s$ fixed conc.

$x = x_s + \Delta x$

$\alpha = \left(\frac{\partial f}{\partial x}\right)_{x=x_s}$

$$\frac{dx}{dt} = -\alpha\Delta x + \sqrt{\frac{g(x_s)}{\Omega}}\,\eta(t)$$

$f(x)$

$x = \langle x\rangle + \Delta x$

$$\frac{dx}{dt} = f(x) + \sqrt{\frac{g(x)}{\Omega}}\,\eta(t)$$

$$\frac{d\langle x\rangle}{dt} = \langle f(x)\rangle$$

$$\frac{d}{dt}(x_s+\Delta x) = f(x_s+\Delta x) + \sqrt{\frac{g(x_s+\Delta x)}{\Omega}}\,\eta(t)$$

$$\Delta x = 1/\Omega$$

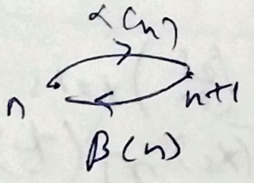
$$= f(x_s) + \left(\frac{\partial f}{\partial x}\right)_{x_s}\Delta x + \sqrt{\frac{g(x_s)+\left(\frac{\partial g}{\partial x}\right)_{x_s}\Delta x}{\Omega}}\,\eta(t)$$

$$= \frac{?}{\Omega} + \frac{g(x_s)}{\Omega}\left[1 + \frac{1}{g}\left(\frac{\partial g}{\partial x}\right)_{x_s}\Delta x\right]^{1/2}\eta(t)$$

$$+\sqrt{\frac{g(x)}{\Omega}}\left(1 + \frac{1}{2}\frac{1}{g}\left(\frac{\partial g}{\partial x}\right)_{x_s}\Delta x\right)\eta(t)$$

$$\left(\sqrt{\frac{g(x)}{\Omega}} + \left(\frac{\partial g}{\partial x}\right)_{x_s}\frac{1}{\Omega^{3/2}}\right)\eta(t)$$

$$\Delta x = 1/\Omega$$



$$x_s = \alpha/\beta$$

$$\langle f(x)\rangle = \frac{d\langle x\rangle}{dt} = \langle \alpha - \beta x\rangle$$

$$\rightarrow \langle f(x)\rangle = \alpha - \beta x$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{x_s} = -\beta$$

$\cancel{\alpha-\beta\langle x\rangle}$

$$x_s = \langle x\rangle = \alpha/\beta.$$

$$g(x) = \alpha(x) + \beta(x)$$

$$g(x_s) = \alpha + \beta x_s = 2\alpha$$

$$\frac{d\Delta x}{dt} = -\beta \Delta x + \sqrt{\frac{2\alpha}{\Omega}}\,\eta(t)$$

$$\frac{d\langle \Delta x^2\rangle}{dt} = -2\beta\langle \Delta x^2\rangle + \frac{2\alpha}{\Omega}$$

$\downarrow$

$$\langle \Delta x^2\rangle_{st} = \frac{\alpha}{\beta}\frac{1}{\Omega}.$$

$$x_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} x_2 \qquad \frac{dx_1}{dt} = \beta x_2 - \alpha x_1$$

$$x_1 + x_2 = x_T$$

$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2.$$

$$\frac{dx_2}{dt} = \underbrace{\alpha x_T - (\alpha+\beta)x_2}_{f(x)}$$

$$g(x) = \alpha x_1 + \beta x_2,$$

$$x_s = \frac{\alpha}{\alpha+\beta}x_T \qquad g(x_s) = \frac{2\alpha\beta}{\alpha+\beta}x_T$$

$$(\partial f/\partial x_2)_{x_s} = -(\alpha+\beta)$$

$$\frac{d\Delta x}{dt} = -(\alpha+\beta) + \sqrt{\frac{2\alpha\beta}{\alpha+\beta}\frac{1}{\Omega}}\,\eta(t).$$

small noise approximation

that is why this is constant

$$\langle \Delta x_2^2\rangle = \frac{2\alpha\beta}{2(\alpha+\beta)^2}\frac{1}{\Omega} = \frac{\alpha\beta}{(\alpha+\beta)^2\Omega}$$

$$x_1 \underset{k_d}{\overset{k_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta}$$

$$\frac{dx_I}{dt} = k_a X_T - (k_a + k_d)X_2$$

$$\Rightarrow (X_2)_s = \frac{k_a}{k_a+k_d}\cdot x_T$$

$$Y_s = \frac{\alpha (X_2)_s}{\beta} = \frac{\alpha}{\beta} \frac{K_a}{K_a + K_d} X_T$$

$$\left(\because \frac{dY}{dt} = \alpha X_2 - \beta Y\right)$$

$$\frac{d\Delta x_2}{dt} = -(K_a + K_d)\Delta x_2 + \eta_1(t)$$

$$\frac{d\Delta y}{dt} = \alpha \Delta x_2 - \beta \Delta y + \eta_2(t)$$

$$\tilde{x} = \begin{pmatrix} \Delta x_1 \\ \Delta y \end{pmatrix} \quad \eta = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} -(K_a + K_d) & 0 \\ \alpha & -\beta \end{pmatrix} \langle \Delta x \, \Delta y \rangle$$

$$\langle \Delta n^2 \rangle = \sqrt{\langle n \rangle}$$

$$\sigma = \begin{pmatrix} \langle \Delta x^2 \rangle & \langle \Delta x \, \Delta y \rangle \\ \langle \Delta x \, \Delta y \rangle & \langle \Delta y^2 \rangle \end{pmatrix}$$

covariance matrix

$$\langle \eta_1(t_1)\, \eta_1(t_2) \rangle = D_1 \delta(t_1 - t_2)$$
$$\langle \eta_2(t_1)\, \eta_2(t_2) \rangle = D_2 \delta(t_1 - t_2)$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x} + \eta(t)$$

$$A\sigma + \sigma A^T - B = 0$$

$$B = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

$$D_1 = \frac{1}{\Omega}\left(k_a x_1 + k_d x_2\right)\Big|_{x_s}$$

$$D_2 = \frac{1}{\Omega}\left(\alpha x_2 + \beta y\right)\Big|_{x_s, y_s}$$

$$D_1 = \frac{1}{\Omega} \frac{2 k_a k_d}{(k_a + k_d)} X_T$$

$$D_2 = \frac{2\alpha x_2}{\Omega} = \frac{2\alpha}{\Omega} \frac{k_a}{k_a + k_d} X_T$$

$$\langle \Delta x_2^2 \rangle = \frac{k_a k_d}{(k_a + k_d)^2} X_T, \quad \langle \Delta y^2 \rangle = \frac{1}{\Omega}\left(\frac{\alpha}{\beta} \frac{k_a}{k_a + k_d} X_T\right)$$

$$\frac{\alpha}{\beta} = \frac{\alpha k_a k_d X_T}{(k_a + k_d)^2 (k_a + k_d + \beta)}$$

$$\frac{1}{\Omega} \frac{\alpha}{\beta} \frac{k_a}{(k_a + k_d)} X_T \quad \frac{\alpha k_d}{(k_a + k_d)(k_a + k_d + \beta)}$$

Langevin eq^n for multiple variable

$$\dot{x} = Ax + \eta(t) \qquad \langle \eta_i(t)\, \eta_j(t') \rangle = B_{ij}\, \delta(t - t')$$

$$\frac{d\sigma}{dt} = A\sigma + \sigma A^T - B = 0$$

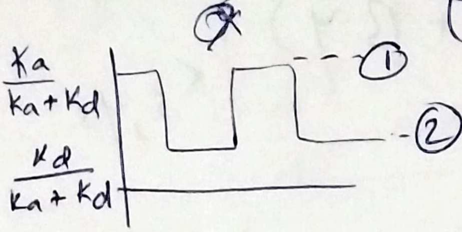
$\sigma \rightarrow$ covariance matrix

$$\sigma_{ij} = \langle x_i x_j \rangle$$

Eg $X_1 \underset{k_d}{\overset{k_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta}$

$$\cancel{\sigma_y^2 = \frac{\langle -y \rangle}{\tau}\left(1 + \frac{\alpha k_d}{(k_a+k_d)(k_a+k_d+\beta)}\right)}$$ ~~if β small (then decay less)~~

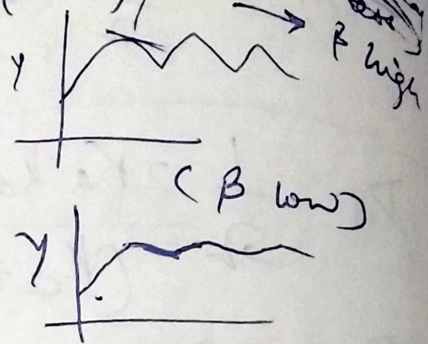
β high



$$p(1) = \frac{k_a}{k_a+k_d}$$

$$p(2) = \frac{k_d}{k_a+k_d}$$

(β low)



$$\sigma_y^2 = \frac{\langle -y \rangle}{\tau}\left(1 + \frac{\alpha k_d}{(k_a+k_d)(k_a+k_d+\beta)}\right)$$

[$(\Delta x_i) \ll \langle x_i \rangle$ for small noise approx.]

You can derive this from prev. result

(Fokker Planck for this case: (multivariable))

$$\frac{dp}{dt} = \sum_{i,j} \frac{d}{dx_i} \underbrace{\left(-A_{ij} x_j p + B_{ij} \frac{\partial p}{\partial x_j}\right)}_{J_i}$$

$$= \sum_i \frac{\partial J_i}{\partial x_i}$$

$$J_i = \sum_j \left(A_{ij} x_j p + B_{ij} \frac{\partial p}{\partial x_j}\right)$$

$\vec{x}$ vector

$$\langle x_k x_l \rangle = \int x_k x_l \, p(\vec{x}, t) \, dx_1 \ldots dx_N$$

$$\frac{d}{dt}\langle x_k x_l \rangle = \int x_k x_l \frac{dp}{dt} \, dx_1 \ldots dx_N$$

$$= \int \underbrace{x_k x_l}_{I} \underbrace{\sum_i \frac{\partial J_i}{\partial x_i}}_{II} dx_1 \cdots dx_n$$

$$= \underbrace{x_k x_l \int \sum_i \underbrace{\frac{\partial J_i}{\partial x_i}}_{=0} dx \cdots dx_N}_{0}$$

(By Part) Here taking derivative wrt x_k & x_l only

$$- \underbrace{\int (J_k x_l + J_l x_k)\, dx_1 \cdots dx_N}$$

$= 0$

$$u' = \frac{\partial}{\partial x_1}(x_k x_l) J_1 + \frac{\partial}{\partial x_2}(x_k x_l) J_2$$

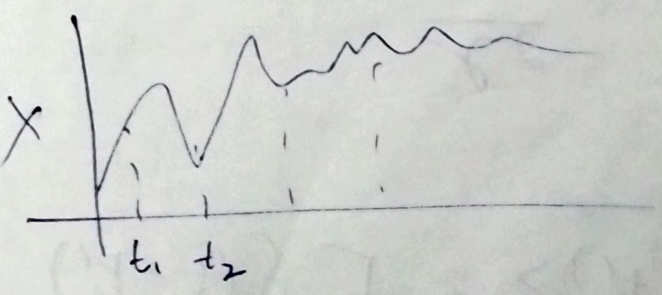
$$+ \cdots \cdots \frac{\partial}{\partial x_n}(x_k x_l) J_N$$

$$= J_k x_l + J_l x_k .$$

$= \cancel{\sigma_{ij}}$? $\quad = - \int \left((J_k x_l) + (J_l x_k) \right) dx_1 \cdots dx_N$

?

Noise Power spectrum



$\langle x(t)\, x(t+\Delta t) \rangle$

$\shortparallel$

$\zeta(\Delta t)$ $\cancel{\sum f(\Delta)}$

Next page

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{2\pi}\frac{1}{T}\left|\sum_{i=1}^{T} e^{i\omega t_i} X(t_i)\right|^2$$

↳ Power spectrum for particular trajectory

$$\langle S_x(\omega)\rangle = \lim_{T\to\infty} \frac{1}{2\pi}\frac{1}{T}\left|\int_0^T e^{i\omega t} X(t)\,dt\right|^2$$

$$= \boxed{\frac{1}{\pi}\int \cos\omega t \,\langle x(0)\, x(t)\rangle\, dt}$$

$$= \frac{1}{2\pi}\int e^{i\omega t} \langle X(0)\, X(t)\rangle\, dt$$

Wiener Kinchin Theorem

Fourier Transform of ~~thing~~ autocorrelation function, we can find the avg. Power spectrum

Power spectrum: Fourier Transform of Trajectory -

Example $\dfrac{dv}{dt} = -\gamma v + \sqrt{\Gamma}\,\eta(t)$

$$\langle v(t)\, v(t')\rangle = \frac{\Gamma}{2\gamma} e^{-\gamma(t-t')}$$

$t > t'$

$|t-t'|$

$$\langle v(0)\, v(t)\rangle = \frac{\Gamma}{2\gamma} e^{-\gamma t}$$

↓

Stationary process

$$\langle \eta(t)\, \eta(t')\rangle = \Gamma\,\delta(t-t')$$

$$\langle \eta(0)\, \eta(t)\rangle = \Gamma\,\delta(t)$$

~~$S_\eta(\omega)$~~

$$|S_\eta(\omega)|^2 = \frac{1}{2\pi}\int e^{i\omega t}\,\Gamma\,\delta(t)\,dt$$

$$\cancel{S_\eta(\omega) = \frac{\Gamma}{2\pi}} \Big\} \rightarrow$$

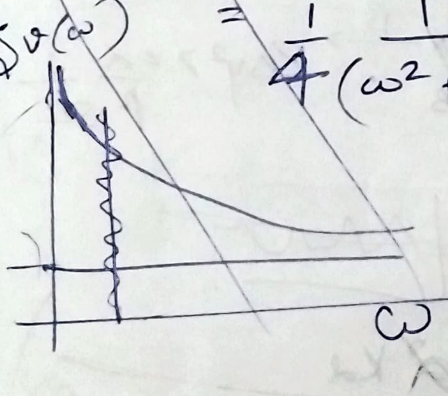
$$|S_\eta(\omega)|^2 = \frac{\Gamma}{2\pi} \Big\} \rightarrow$$ independent of 'ω' thats' why called $\boxed{\text{'White Noise'}}$

$$|S_v(\omega)|^2 = \frac{1}{2\pi}\int_0^\infty \cos\omega t\;\frac{\Gamma}{2\gamma}e^{-\gamma t}\,dt$$

$$= \frac{\Gamma}{4\pi\gamma}\,\frac{1}{-\gamma+i\omega}$$

$$|S_v(\omega)|^2 = \frac{\Gamma}{\gamma^2 4}\,\frac{1}{(2\pi)^2}\,\frac{1}{\omega^2+\gamma^2}$$

$$S_v(\omega) = \frac{1}{4}\,\frac{1}{(\omega^2+\gamma^2)\gamma^2}\,|S_\eta(\omega)|^2$$



$$|S_v(\omega)|^2 = \frac{1}{\pi}\int_0^\infty \left(\frac{\Gamma}{2\gamma}\right)e^{-\gamma t}\cos\omega t\,dt$$

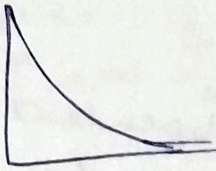
$$= \mathrm{Re}\left(\frac{1}{\pi}\,\frac{\Gamma}{2\gamma}\int_0^\infty e^{t(i\omega t-\gamma)}\,dt\right)$$

$$= \frac{\Gamma}{\pi\, 2\gamma}\,\frac{\gamma}{\omega^2+\gamma^2}$$

$$= \frac{\Gamma}{2\kappa} \frac{1}{\omega^2+\gamma^2} = \frac{1}{\omega^2+\gamma^2} |S_\eta(\omega)|^2$$

$$\approx$$

$S_v(\omega)$



ω

Recap

$$X_1 \underset{K_d}{\overset{K_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta} \qquad \langle Y \rangle = \alpha/\beta$$

$$\cancel{\langle \Delta X_2 \rangle} = \langle X_2 \rangle = \frac{K_a}{K_a+K_d} X_T \qquad X_1 + X_2 = X_T$$

$$\langle \Delta X_2^2 \rangle = \frac{K_a K_d}{(K_a+K_d)^2} X_T \frac{1}{\Omega}$$

$\langle (X_2 - \langle X_2 \rangle)^2 \rangle$ $\qquad$ $X_2 \to$ density of particles (concentration)

Discrete case $\qquad$ $\cancel{X_2 = n_2/\Omega}$

$\downarrow$

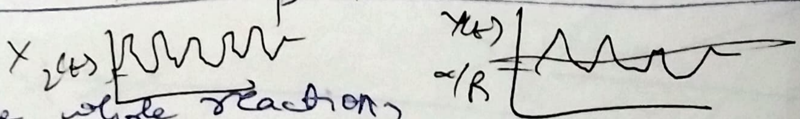
$$\langle \Delta n_2^2 \rangle = \frac{K_a K_d}{(K_a+K_d)^2} \bar{N}_T$$

(n_2: no. of particles)

Just for $\xrightarrow{\alpha} Y \xrightarrow{\beta}$

$\langle n_Y \rangle = \alpha/\beta \qquad \langle Y \rangle = \alpha/\beta \qquad \langle \Delta Y^2 \rangle = \frac{\alpha}{\beta} \frac{1}{\Omega}$

$\langle n_Y^2 \rangle = \alpha/\beta$



Now the whole reaction,

$$\langle \Delta y^2 \rangle = \frac{\langle y \rangle}{\Omega} \left(1 + \frac{\alpha K_d}{(K_a+K_d)(K_a+K_d+\beta)} \right)$$

Intrinsic noise $\qquad$ Propagation of noise

$$\langle Y \rangle = \frac{\alpha \langle X_2 \rangle}{\beta}$$

$$= \frac{\alpha}{\beta} \frac{K_a}{K_a+K_d} X_T$$

Multivariable Langevin eqⁿ $\quad x_1 \underset{k_d}{\overset{k_a}{\rightleftharpoons}} x_2$

$$\dot{X}(t) = AX + \eta(t)$$

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \eta(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix}$$

covariance matrix $\quad \sigma = \begin{pmatrix} \langle x_1^2 \rangle & \langle x_1 x_2 \rangle \\ \langle x_1 x_2 \rangle & \langle x_2^2 \rangle \end{pmatrix}$

$$A\sigma + \sigma A^T - B = 0$$

↳ ~~some constant matrix~~

$$\langle \eta_i(t)\, \eta_j(t') \rangle = B_{ij}\, \delta(t-t') \qquad B_{12} = 0.$$

We want to find σ (for x) knowing σ (for η)

Recap ~~$\dot{X}_2 = k_a X_T - (k_a + k_d) X_2$~~

$$\boxed{\dot{y} = \alpha x_2 - \beta y}$$

$$\boxed{\dot{x}_2 = k_a x_1 - k_d x_2 = k_a x_T - (k_a + k_d) x_2}$$

$$\langle x_2 \rangle = \frac{k_a x_T}{k_a + k_d} \qquad \langle y \rangle = \frac{\alpha \langle x_2 \rangle}{\beta}$$

(Small noise approximatⁿ)
↳ Taylor exp. over noise r.v.

$$\frac{d}{dt} \Delta x_2 = -(k_a + k_d)\Delta x_2 + \eta_1(t)$$

$$\frac{d}{dt} \Delta y = \alpha \Delta x_2 - \beta \Delta y + \eta_2(t).$$

Now write in matrix) $\quad \eta(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix}$

$$X = \begin{pmatrix} \Delta x_2 \\ \Delta y \end{pmatrix}, \qquad \dot{X} = AX + \eta(t)$$

$$A = \begin{pmatrix} -(k_a + k_d) & 0 \\ \alpha & -\beta \end{pmatrix}$$

$$\boxed{\langle \eta_1(t)\, \eta_2(t') \rangle = \Gamma \delta(t-t')}$$

$\langle \Delta x \rangle = 0, \langle \Delta y \rangle = 0$

Revisit

$$\sigma_{22} = \langle \Delta y^2 \rangle = \frac{\langle y \rangle}{\Omega}\left(1 + \frac{\lambda k_d}{(k_a + k_d)(k_a + k_d + \beta)}\right)$$

$$\frac{dp}{dt} = \sum_i \frac{\partial J_i}{\partial x_i}$$

$$\frac{d}{dt}\langle x_k x_l \rangle$$

$$= \int x_k x_l \frac{dp}{dt}\, dx_1 \cdots dx_n$$

$$= \int \underbrace{x_k x_l}\ \underbrace{\sum_i \frac{\partial J_i}{\partial x_i}\, dx_1 \cdots dx_N}$$

By part

$$= x_k x_l \int \sum_i \frac{\partial J}{\partial x_i}\, dx_1 \cdots dx_n$$

$$- \int \left(\frac{\partial}{\partial x_k} x_k x_l\right)\left(\int \left(\sum_i \frac{\partial J}{\partial x_i}\right) dx_i \cdots dx_N\right) dx_1\, dx_n \cdots$$

$$J_i = \sum_j \left(A_{ij} x_j p + \frac{1}{2} B_{ij} x_j \frac{\partial p}{\partial x_j}\right) \sum_i \frac{\partial J_i}{\partial x_i}$$

Skipped one class

$$S_n(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \Big| \underbrace{\int_0^T x(t) e^{i\omega t} dt}_{|X(\omega)|^2} \Big|^2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(0) x(t) e^{i\omega t} dt$$

$S_\eta(\omega) = \frac{\Gamma}{2\pi}$ Brownian Motion

$\dot{v} = -\gamma v + \eta(t)$

$S_v(\omega) = \langle |v(\omega)|^2 \rangle = \frac{1}{\omega^2+\gamma^2} S_\eta(\omega)$

$\cancel{\langle v \rangle}$ $\langle v^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{\omega^2+\gamma^2} \frac{\Gamma}{2\pi} = \frac{\Gamma}{2\gamma}$.

$\xrightarrow{\alpha} x \xrightarrow{\beta}$ $\Gamma = 2\alpha/\Omega$

$S_n(\omega) = \frac{\Gamma}{2\pi} \frac{1}{\omega^2\beta^2}$

$\langle \Delta x^2 \rangle = \frac{\alpha}{\beta} \frac{1}{\Omega}$ $\langle x \rangle = \frac{\alpha}{\beta}$

$X_1 \underset{k_d}{\overset{k_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta}$

$\frac{d\Delta x_2}{dt} = -(k_a+k_d)\Delta x_2 + \eta_1(t)$

$\frac{d\Delta y}{dt} = \alpha \Delta x_2 - \beta \Delta Y + \eta_2(t)$

$\Delta Y(\omega) = \frac{\alpha \Delta X_2(\omega)}{-i\omega+\beta} + \frac{\eta_2(\omega)}{-i\omega+\beta}$

$\Delta X_2(\omega) = \frac{\eta_1(\omega)}{-i\omega + K_a + K_d}$

$\langle |\Delta X_2(\omega)|^2 \rangle = \frac{S_\eta(\omega)}{\omega^2+(K_a+K_d)^2}$

$\langle |\Delta Y(\omega)|^2 \rangle = \frac{\alpha \eta_1(\omega)}{(-i\omega+\beta)(-i\omega+K_a+K_d)} + \frac{\eta_2(\omega)}{-i\omega+\beta}$

If no correlation b/w $\eta_1(t)$ & $\eta_2(t)$

$$\langle |\Delta y(\omega)|^2 \rangle = \int \frac{\alpha^2 S_{\eta_1}(\omega) \, d\omega}{(\omega^2+\beta^2)(\omega^2+(K_a+K_d)^2)}$$

$$+ \int \frac{S_{\eta_2}(\omega)}{\omega^2+\beta^2} \, d\omega$$

$$\boxed{\frac{\pi}{2\beta(K_a+K_d)(K_a+K_d+\beta)} = \int \frac{d\omega}{(\omega^2+\beta^2)(\omega^2+(K_a+K_d)^2)}}$$

$$\therefore \langle |\Delta y|^2 \rangle = \frac{\langle Y \rangle}{\Omega}\left(1 + \frac{\alpha K_d}{(K_a+K_d)(K_a+K_d+\beta)}\right)$$

$$L \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} R$$

$$p_1 = \alpha \Delta t$$

$$p(\text{not } 1) = \left(1 - \frac{\alpha t}{N}\right)^N$$

$e^{-\alpha t}$

① $X_1 \to X_2$ $\quad = e^{-\alpha t}$

~~$p(\text{not } 1)$~~

② $X_2 \xrightarrow{\beta} X_1$ $\quad p(\text{not } 1, \text{not } 2) = \boxed{e^{-(\alpha+\beta)t}}$

$e^{-\beta t}$

Sampling Start at $n_1 = N$, $n_2 = 0$

① choose t by sampling from $e^{-(\alpha+\beta)t}$

② Choose the event → sample another from $U[0,1]$

$$r \begin{cases} < \frac{\alpha}{\alpha+\beta} \to 1 \\ > \to 2 \end{cases}$$

$r_i \in [0,1]$

③ $n_2 = n_2 + 1$ $\quad n_1 = n_1 - 1$

$$T = \frac{1}{\alpha+\beta} \log\left(\frac{1}{r_i}\right)$$

$x_1 \xrightarrow{\alpha} x_2$

$x_2 \xleftarrow{\beta} x_1$ $\quad p_1 =$

$\cancel{\beta =}$ $\quad X_T$

N events $\quad$ M species

$$\left.\begin{array}{ll} x_1 \to x_2 & \text{2 events} \\ x_2 \to x_3 & \text{3 species} \end{array}\right\} \text{eg}$$

$$\cancel{p_1 = \frac{\alpha n_1}{\alpha n_1 + \beta n}} \qquad p(t) = e^{-\sum_{i=1}^{N} r_i t}$$

$$p_i = r_i \Big/ \sum_{i=1}^{N} r_i$$

rate

$$x_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} x_2 \to p_1 = \frac{\alpha n_1}{\alpha n_1 + \beta n_2}, \quad p_2 = \frac{\beta n_2}{\alpha n_1 + \beta n_2}$$

$$x_1 \xrightarrow{\alpha n_1} x_2$$

$$x_2 \xrightarrow{\beta} x_1$$

$$\frac{dv(t)}{dt} = -\gamma v + \eta(t)\sqrt{\Delta t}$$

$$v(t) = v(t - \Delta t) - \gamma v \Delta t + \eta(t)\sqrt{\Delta t}$$

$$\langle v^2 \rangle = D\,\Delta t.$$

$\cancel{\alpha} \xrightarrow{\beta}$ $\xrightarrow{\alpha} X \xrightarrow{\beta}$

$$\frac{dx}{dt} = \alpha - \beta x$$

$$y_1 \underset{\gamma}{\overset{\alpha_2 x}{\rightleftharpoons}} y_2$$

$$\frac{dy_2}{dt} = \alpha_2 x (y_T - y_2) - \gamma y_2$$

$$= \alpha_2 x y_T - (\alpha_2 x + \gamma) y_2$$

$$\frac{d\Delta x}{dt} = \cdot - \beta \Delta x.$$

$$\frac{d\Delta y_2}{dt} = \alpha_2 y_T \Delta x - (\alpha_2 \langle x \rangle + \gamma) \Delta y_2$$

$$- \alpha_2 \langle y_2 \rangle \Delta x,$$

$\langle \ \rangle$ $\Gamma_1 \delta(t)$

$$\boxed{\frac{d\Delta x}{dt} = \left(\frac{\partial f}{\partial x}\right)_{\text{Steady state}} \Delta x + \eta(t)}$$

$\tau_s \ll \tau_\Delta$

$$\frac{d\Delta y}{dt} = \left\{ \left(\frac{\partial f}{\partial y}\right) \Delta y + \left(\frac{\partial f}{\partial x}\right) \Delta x \right\}_{\text{Steady state}} + \eta(t)$$

$$\left[\frac{dx}{dt} = \alpha - \beta x \right]$$

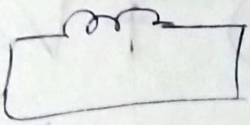
$$\boxed{\alpha + \beta \langle x \rangle / \Omega}$$

Just add them.

$$\Rightarrow \Gamma_1 \text{ (for } \eta_1) = \frac{\langle \alpha(x) \rangle + \beta \langle x \rangle}{\Omega}$$

& $\frac{dy_2}{dt} = \alpha_2 x(y_1) - \gamma y_2$

$\Rightarrow \Gamma_2 = \frac{\alpha_2 \langle x\rangle\langle y_1\rangle + \gamma\langle y_2\rangle}{\Omega}$



$L\frac{dI}{dt} + RI = \eta(t)$

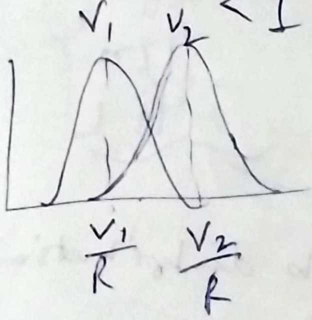
$V_{ext} = 0$

$\hookrightarrow \begin{cases} \langle I\rangle = 0 \\ \langle I^2\rangle = \frac{k_B T}{L} \end{cases}$

$V_{ext} = V$

$L\frac{dI}{dt} + RI = \eta(t) + V$

$\langle I\rangle = V/R.$



$\frac{1}{2}LI^2 = \frac{1}{2}k_B T$

equipartition theorem

$\dot{v} = -\gamma v + \eta(t) + E_0$

$v_0 = \frac{E_0}{\gamma}$

$\overset{\parallel}{\langle v\rangle}$

(Variance)

$\sigma_{v^2} = \frac{k_B T}{m}$

$\downarrow$

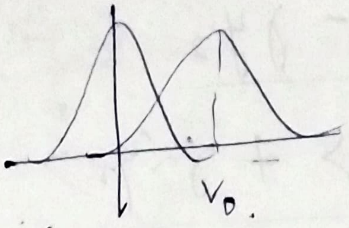
$\langle v^2 - v_0^2\rangle$

If $E_0 = 0$

~~$\langle v_0\rangle = 0$~~

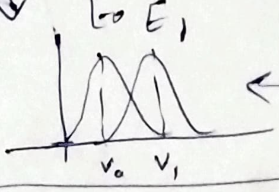
$\langle v\rangle = 0$

$\langle v^2\rangle = k_B T/m$



For a voltage system

$$\frac{L\,dI}{dt} + RI = \eta(t) \cancel{+} V$$



Graph of $\left\{\begin{array}{l} p(v|E_0) \\ p(v|E_1) \end{array}\right.$

Shannon's Information Theory

	Code 1	Code 2	Code 3
A	00	0	0
T	01	10	10
C	10	110	110
G	11	1110	111

Depend on Prob distribution over ATCG

Eg —

A	T	C	G
1/2	1/4	1/8	1/8

→ code 3 will be better.

1/4	1/4	1/4	1/4

→ code 1 will be better

Krafts' Inequality

$$\sum_i 2^{-l_i} \leq 1$$

l : length of code

for each Alphabet) eg → here for A, T, C, G

$$\langle l \rangle = \sum_i p_i l_i$$

we want to find a code that min $\langle l \rangle$

If decodable ⇄ Inequality satisfies

$$f = \sum p_i l_i + \lambda(2^{-l_i} - 1) \quad \text{(convex opt.)}$$

λ → lagrange multiplier

$$\left\{ \frac{\partial f}{\partial l_i} = p_i + (-\lambda 2^{-l_i}) \ln 2 = 0 \right.$$

~~$\Rightarrow \langle l \rangle = -\sum p_i \log p_i$~~

$$p_i = \lambda 2^{-l_i} \ln 2 \qquad \cancel{\lambda = \frac{p_i}{2^{-l_i} \ln 2}}$$

$$\frac{p_1}{2^{-l_1}} = \frac{p_2}{2^{-l_2}}$$

~~$\langle l \rangle = \sum p_1 \log$~~

Entropy

$$\langle l \rangle = - \sum p_i \log p_i$$

$X \longrightarrow Y$ encoding from x to y.

$p(x,y) = p(x|y)\,p(y)$

$E(X, Y)$

$= -\sum_{x,y} p(x,y) \log p(x,y)$

$= -\sum_{x,y} p(x,y) \left[\log p(x|y) + \log p(y)\right]$

$= -\left[\sum_{x,y} p(x,y) \log p(y)\right]$

$- \left[\sum_{x,y} p(x,y) \log p(x|y)\right]$

$= -\sum p(y) \log p(y) - \sum p(y)\, p(x|y) \log (x|y)$

$= E[Y] + E[X|Y]$

Y \ X	1	2	3	4	
1	1/8	1/16	1/32	1/32	1/4
2	1/16	1/8	1/32	1/32	1/4
3	1/16	1/16	1/16	1/16	1/4
4	1/4	0	0	0	1/4
	1/2	1/4	1/8	1/8	

$p(x) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$

$p(y) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$E(X|Y) \cancel{= \sum E(x|y}$

$\cancel{= \sum_{x \in X} E(x)}$ $= \sum_{x,y} E(x|y)\,p(y)$

$= \sum_{i=1}^{4} p(y=i)\, E(x|y=i)$

$= \frac{1}{4}\left[E\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + E\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right) + E\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + E(1, 0, 0, 0)\right]$

$= \frac{11}{8}$ bits.

$$\boxed{\begin{aligned} E[X,Y] &= E[Y] + E[X|Y] \\ &= E[X] + E[Y|X] \end{aligned}}$$

<u>Mutual Information</u>

$MI(X:Y) = E(Y) - E(Y|X)$

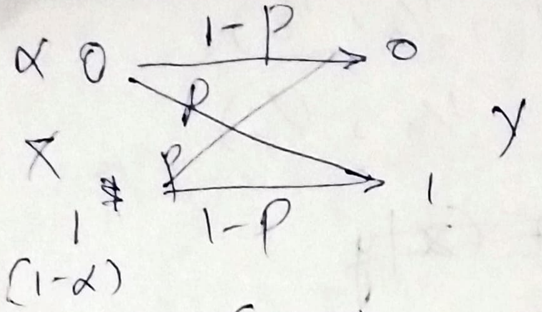
$= E(X) - E(X|Y)$

KL divergence

$= +\sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$

distance

X Y



{symmetric binary channel}

$$MI(X:Y) = f(\alpha)$$

$$\max_{\alpha} MI(X;Y) = C \rightarrow \text{channel capacity}$$

$$p(Y=\cancel{0} \mid X=0) = p$$
$$p(Y=0 \mid X=1) = p$$
$$p(Y=0 \mid X=0) = p(Y=1 \mid X=1) = 1-p$$

$$p(Y=1) = \alpha p + (1-p)(1-\alpha) = \boxed{2p\alpha + 1 - (p+\alpha)}$$
$$p(Y=0) = \cancel{(1-\alpha)(1-} + \alpha(1-p) + (1-\alpha)p$$
$$= \boxed{(p+\alpha) - 2p\alpha}$$

$\cancel{p(x,y)} =$

$$E(y) = -[2p\alpha + 1 - (p+\alpha)]\log[2p\alpha + 1 - (p+\alpha)]$$
$$- \cancel{4}\,[p+\alpha-2p\alpha]\log(p+\alpha-2p\alpha)$$

$\cancel{E(y \mid x)}$

$$E(Y \mid X) =$$
$$-\alpha(p\log p + (1-p)\log(1-p))$$
$$- (1-\alpha)(p\log p + (1-p)\log(1-p))$$
$$= -(p\log p + (1-p)\log(1-p))$$

$$MI = E(Y) - E(Y|X)]$$

~~max MI = B~~

~~$\alpha = 1/2$~~

~~$\Rightarrow \alpha = 1/2$~~

$$\max_{\alpha} E(Y) = E(Y)\Big|_{\alpha = 1/2}$$
$$= 1$$

$$\boxed{\max MI = 1 + p\log p + (1-p)\log(1-p)}$$

$$L\frac{dI}{dt} + RI = \sqrt{\Gamma}\,\eta(t) + V$$

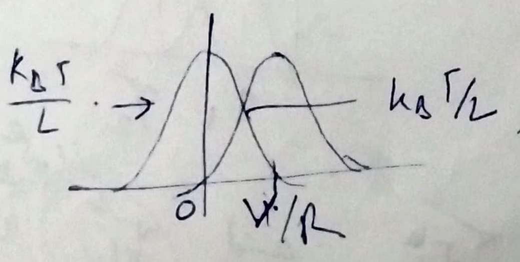
$$\frac{dI}{dt} + \frac{R}{L} I = \sqrt{\frac{\Gamma}{L^2}}\,\eta(t)$$

$$\langle I^2 \rangle = \frac{\Gamma/L^2}{2R/L} = \frac{\Gamma}{2RL}$$

$$\frac{1}{2}L\langle I^2\rangle = \frac{1}{2}k_B T$$

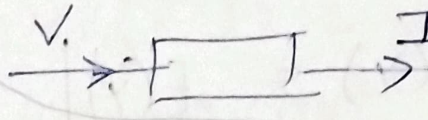
$$\Gamma = 2Rk_B T \Leftarrow \frac{\Gamma}{2R} = k_B T$$

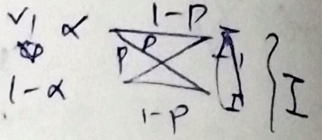
(Johnson's noise)



$$P(I|V) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\frac{(I-\langle I\rangle)^2}{\sigma^2}}$$

$\sigma^2 = \frac{k_B T}{L}$, $\quad \langle I\rangle = \frac{V}{K}$





Entropy of this distribution

$$-\int p(I|V)\log p(I|V)\,dI$$

$$= -\int p(I|V)\left[-\frac{1}{2}\frac{(I-\langle I\rangle)^2}{\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right]$$

entropy

$$E(I|V) = \frac{1}{2} + \frac{1}{2}\log(2\pi\sigma^2)$$

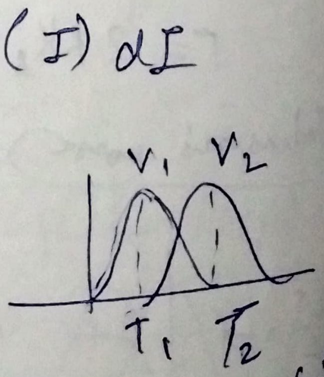
$$= \frac{1}{2} + \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(\sigma^2)$$

$$P(I) = p(I|V_1)\,p(V_1) + p(I|V_2)\,p(V_2)$$

$$= \alpha\, p(I|V_1) - (1-\alpha)\, p(I|V_2)$$

$$E(I) = \int p(I)\log p(I)\,dI$$

MI =



variance of this dist^n $\Rightarrow \sigma_I^2$ and $\sigma^2_{I|V}$ (we already know)

$$\sigma^2_{I|V} = \frac{k_B T}{L}$$

$$\boxed{MI = \log\left(\frac{\sigma_I^2}{\sigma^2_{I|V}}\right)}$$

~~$p(I|V)$~~ $p(I,V)$

$$\sigma_I^2 = \int I^2 \, p(I,V) \, dI \, dV$$

$$- \underbrace{\left(\int I \, p(I,V) \, dI \, dV\right)^2}_{R_1}$$

dI

$$\sigma^2_{I|V} = \left[\int I^2 \, p(I|V) \, dI \right.$$

$$\left. - \underbrace{\left(\int I \, p(I|V) \, dI\right)^2}_{R_2}\right] p(V) \, dV$$

$$\sigma_I^2 = \sigma^2_{I|V} + (R_2 - R_1)$$

~~$R_1 = (\int I \, p(I|V$~~

$$R_1 = \left(\int I \, p(I,V) \, dI \, dV\right)^2$$

$$= \int I \, p(I|V_1) \, p(V_1) \, dI \, dV$$

$$+ \int I \, p(I|V_2) \, p(V_2) \, dI \, dV$$

$$= \left(\alpha \frac{V_1}{R} + (1-\alpha) \frac{V_2}{R}\right)^2$$

$$R_2 = \int \left(\int I \, p(I|V) \, dI \right)^2 p(V) \, dV$$

$$= \alpha \left(\frac{V_1}{R} \right)^2 + (1-\alpha) \left(\frac{V_2}{R} \right)^2$$

$$\therefore \sigma_I^2 = \sigma_{I|V}^2 + \alpha \left(\frac{V_1}{R} \right)^2 + (1-\alpha) \left(\frac{V_2}{R} \right)^2 - \left(\alpha \frac{V_1}{R} + (1-\alpha) \frac{V_2}{R} \right)^2$$

$$= \sigma_{I|V}^2 + \frac{\alpha(1-\alpha)(V_1 - V_2)^2}{R^2}$$

$$\therefore MI = \overbrace{\log(\sigma_I^2)}^{E(I)} - \overbrace{\log(\sigma_{I|V}^2)}^{E(I|V)}$$

$$MI(I,V) = \log \left(1 + \frac{\alpha(1-\alpha)(V_1 - V_2)^2 L}{R^2 k_B T} \right)$$

Take derivative (to find max value)

$$\frac{\partial MI}{\partial \alpha} = \frac{(1-2\alpha) \frac{(V_1 - V_2)^2 L}{R^2 k_B T}}{1 + \frac{\alpha(1-\alpha)(V_1 - V_2)^2 L}{R^2 k_B T}} = 0 \Rightarrow \alpha = 1/2$$

$V_1 \longrightarrow$

$V_2 \longrightarrow$

(symmetric binary channel)

$$\therefore \underset{(=C)}{MI_{max}} = \log\left(1 + \frac{(V_1 - V_2)^2 L}{4R^2 k_B T}\right)$$

channel capacity.

$$\langle l \rangle = \sum_i p_i \log(1/p_i) \iff l_i = \log \frac{1}{p_i}$$

{ We are getting some info but we don't know the actual distⁿ (p_i). so our guess is (q_i). }

p_i

$$l_i = \log\left(\frac{1}{p_i}\right)$$

q_i

$$l_i = \log\left(\frac{1}{q_i}\right)$$

$$\langle l \rangle = \sum_i p_i \log\left(\frac{1}{q_i}\right)$$

$$= \sum_i p_i \log \frac{p_i}{q_i} - \sum p_i \log p_i$$

$D(p|q)$

KL div. (distance b/w p & q).

$H(p)$

$$D(p|q) = \int p(x) \log \frac{p(x)}{q(x)} dx.$$

$$D(p(I|V_1) | p(I|V_2))$$

$$p(I|v_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(I-I_1)^2}{2\sigma^2}}$$

$$\log \frac{p(I|v_1)}{p(I|v_2)} = \log \left(e^{-\frac{1}{2\sigma^2}\left[(I-I_1)^2 - (I-I_2)^2\right]}\right)$$

$$= -\frac{1}{2\sigma^2} \int \left[(I-I_1)^2 - (I-I_2)^2\right] p(I|v_1)\, dI$$

$$\cancel{= -\frac{1}{2\sigma^2} \int \left[\sigma^2 - I^2 - 2I I_2 + I_2^2\right.}$$

$$\cancel{= -\frac{1}{2\sigma^2} \left[+I_1^2 - 2I I_1 - I_2^2 + 2I I_2\right] p(I|v_1)\, dI}$$

$$\sigma^2 = \langle I^2 \rangle - I_1^2$$

$$\sigma^2 + I_1^2 = \langle I^2 \rangle$$

$$= \frac{1}{2\sigma^2} \left[\sigma^2 - (I^2 - 2I I_2 + I_2^2)\right] p(I|v_1)\, dI$$

$$= \frac{1}{2\sigma^2} (I_1 - I_2)^2$$

$$D\left(p(x,y) \,\|\, p(x)p(y)\right)$$

$$= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

$$= \cancel{MI(\alpha} \quad MI(X, Y)$$

Fischer's information

$$D\big(p(I|V) \,|\, p(I|V+\Delta V)\big)$$

$$= D\big(p(I|V) \,|\, p(I|V)\big)$$

$$+ \frac{\partial D}{\partial V}\Delta V + \frac{\partial^2 D}{\partial V^2}\frac{(\Delta V)^2}{2}$$

$$\therefore \cancel{F(V) = \frac{\partial^2 D}{\partial V^2}}$$

$$\boxed{F(V) = \frac{\partial^2 D}{\partial V^2}}$$

$$\xrightarrow{x} \boxed{\quad} \longrightarrow y$$

$$\frac{\partial D}{\partial \theta'}\big(p(x;\theta) \,|\, p(x,\theta')\big)\Big|_{\theta=\theta'}$$

$$D = \int p(x,\theta) \log \frac{p(x,\theta)}{p(x,\theta')}\, dx$$

$$\cancel{\frac{\partial D}{\partial \theta'} = +\int \frac{p(x,\theta)\; p(x,\theta')}{p(x,\theta')} \frac{\partial p}{\partial \theta'}\, dx}$$

$$= \frac{\partial}{\partial \theta'}\int p \, dx$$

$$=$$

$$\left.\frac{\partial D}{\partial \theta'}\right|_{\theta=\theta'} = -\int p(x,\theta) \frac{1}{p(x\,\theta')} \frac{\partial p}{\partial \theta'} dx$$

$$\cancel{\theta=\theta'} \quad = \frac{\partial}{\partial \theta} \int p \, dx$$

$$\frac{\partial^2 D}{\partial \theta'^2} = \int p(x,\theta) \frac{1}{p(x\,\theta')^2} \left(\frac{\partial p}{\partial \theta'}\right)^2 dx$$

$$- \int p(x,\theta) \frac{1}{p(x\,\theta')} \frac{\partial^2 p}{\partial \theta'^2} dx$$

$$= \int p(x,\theta) \left(\frac{\partial}{\partial \theta'} \log p(x,\theta')\right)^2 dx$$

$$\left.\frac{\partial^2 D}{\partial \theta'^2}\right|_{\theta'=\theta} \quad = E\left\langle \left(\frac{\partial(\log p)}{\partial \theta}\right)^2 \right\rangle$$

$$= -E\left[\frac{\partial^2}{\partial \theta^2} \log p\right]$$

$$p(I|\nu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(I-\langle I\rangle)^2}{2\sigma^2}}$$

$$\log \cancel{p(I|\nu)} \; p(I,\nu) = \frac{(I-\langle I\rangle)^2}{2\sigma^2} - \frac{1}{2}\log(\sigma^2)$$

$$\frac{\partial}{\partial V} \log P = \frac{1}{\sigma^2}\left(I - \frac{V}{R}\right)\frac{1}{R}$$

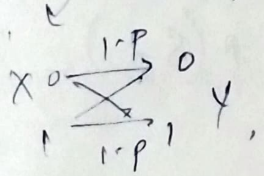
Information Transmission through a noisy channel

$$X \rightarrow \boxed{}^{C} \rightarrow Y.$$

$$p(Y) = p(Y|X)\, p(X)$$

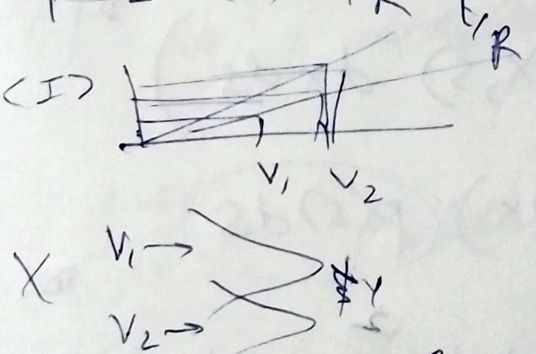
$$MI(X,Y) = E(Y) - E(Y|X)$$

$$MI(X,Y) = 1 + p\log p + (1-p)\log(1-p)$$



$$\xrightarrow{V} \boxed{}^{C} \rightarrow I \qquad MI(I,X) = \log\left(1 + \frac{(V_1 - V_2)^2 L}{4R^2 k_B T}\right)$$

~~p(I~~ $\langle I\rangle = V/R$



$$L\frac{dI}{dt} + RI = \sqrt{\Gamma}\,\eta(t)$$

$$p(I|V_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(I - \langle I_1\rangle)^2}{\sigma^2}}$$

$$\sigma_I^2 = k_B T / L$$

$$\langle I\rangle = V/R$$

$$\xrightarrow{\alpha, S} X \xrightarrow{\beta_1} \qquad \text{find MI b/w } S \text{ and } X$$

$$\xrightarrow{\alpha, X} Y \xrightarrow{\beta_2} \qquad \text{and } S \,\&\, Y.$$

$$P(X|S) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{1}{2}\left(\frac{X - \langle X\rangle}{\sigma}\right)^2}$$

$$H(x|s) = -\int p(x|s) \log p(x|s)\, dx$$
$$= \frac{1}{2}\log(2\pi\sigma_x^2) + \frac{1}{2}$$

$$\langle x\rangle = \frac{\alpha_1 s_1}{\beta} \quad \sigma^2_{x|s_1} = \frac{\alpha_1 s_1}{\beta}\frac{1}{\Omega}$$

$$p(x) = p(x|s_1)p(s_1) + p(x|s_2)p(s_2)$$
$$= \frac{1}{2}\left(p(x|s_1) + p(x|s_2)\right)$$
Assuming $p(s_1) = p(s_2) = \frac{1}{2}$

$$\cancel{\sigma_x^2 = \int x^2 p(x)}$$

$$\sigma_x^2 = \int x^2 p(x,s)\, dx\, ds - \underbrace{\left(\int x\, p(x,s)\, dx\, ds\right)^2}_{R_2}$$

$$\sigma^2_{x|s} = \int \left(x^2 p(x|s)\, dx\right) p(s)\, ds - \underbrace{\left(\int \left(x\, p(x|s)\, dx\right)^2 p(s)\, ds\right)}_{R_1}$$

$$\sigma_x^2 = \sigma^2_{x|s} + R_1 - R_2$$

$$R_1 = \frac{1}{2}\left(\left(\frac{\alpha_1 s_1}{\beta_1}\right)^2 + \left(\frac{\alpha_1 s_2}{\beta_1}\right)^2\right)$$

$$R_2 = \left(\frac{1}{2}\left(\frac{\alpha_1 s_1}{\beta_1} + \frac{\alpha_1 s_2}{\beta_1}\right)\right)^2$$

$$\therefore \sigma_x^2 = \sigma_{x|s}^2 + \frac{1}{4}\left(\frac{\alpha_1}{\beta_1}\right)^2 (S_1 - S_2)^2$$

$$\sigma_{x|s}^2 = \frac{1}{2\Omega}\left(\frac{\alpha_1 S_1}{\beta_1} + \frac{\alpha_1 S_2}{\beta_1}\right)$$

$$\left(\because \sigma_{x|S_1}^2 = \frac{\alpha_1 S_1}{\beta_1}\frac{1}{\Omega}\right)$$

$$MI = \log\left(\frac{\sigma_x^2}{\sigma_{x|s}^2}\right)$$

$$MI(X, S) = \log\left(1 + \frac{1}{2}\frac{(\alpha_1/\beta_1)\,\Omega \cdot (S_1 - S_2)^2}{(S_1 + S_2)}\right)$$

$MI(Y, S)$

$$\frac{dy}{dt} = \alpha_2 X - \beta_2 y$$

$$\langle y \rangle = \cancel{\alpha_1 \alpha_2 S} \quad \frac{\alpha_1 \alpha_2 S}{\beta_1 \beta_2}$$

$$\frac{d\Delta y}{dt} = \alpha_2 \Delta x - \beta_2 \Delta y + \eta_2(t)$$

$$\Delta y(\omega) = \alpha_2 \frac{\Delta x(\omega)}{-i\omega + \beta_2} + \frac{\eta_2(\omega)}{-i\omega + \beta_2}$$

$$\Delta x(\omega) = \frac{\eta_1(\omega)}{-i\omega + \beta_1}$$

$$\cancel{\langle |\Delta y|^2 \rangle = \frac{\alpha_2^2}{(\omega^2 + \beta_2^2)} S_{\Delta x}(\omega) + \frac{S_{\eta_2}(\omega)}{\omega^2 + \beta_2^2}}$$

$$= \cancel{\alpha_1^2}$$

$$\langle y^2 \rangle = \frac{\alpha_2^2 D_1}{2\beta_1\beta_2(\beta_1+\beta_2)} + \frac{\alpha_2 \langle x \rangle}{\beta_2} \frac{1}{\Omega}$$

$$D_1 = 2\alpha_1 s$$

$\beta_1 = \beta_2 = \beta$ $\quad p(s_1) = p(s_2) = 1/2$

$$\sigma^2 = \langle \Delta y^2 \rangle = \frac{\langle y \rangle}{\Omega}\left(1 + \frac{\alpha_2}{2\beta}\right)$$

$$\langle y \rangle = \frac{\alpha_1 \alpha_2 s}{\beta^2}$$

$$MI(y, s) = \log\left(1 + \frac{1}{2} \frac{\frac{\alpha_2}{\beta}(s_1 - s_2)^2}{\left(1+\frac{\alpha_2}{2\beta}\right)(s_1 + s_2)}\right)$$

$$X_1 \underset{\beta}{\overset{\alpha S}{\rightleftharpoons}} X_2$$

$MI(x_2, S)$,

$$\cancel{\sigma_{x_2}^2 = \frac{\alpha_1 \alpha_2 s}{\alpha_1 \beta s +}}$$

$$\sigma^2_{x_2|s} = \frac{\alpha\beta s}{(\alpha s + \beta)^2} \frac{X_T}{\Omega}.$$

$$\langle x_2 \rangle = \frac{\alpha s}{\alpha s + \beta} X_T.$$

$\left(\int x_2 p(x_2|S)\, dx_2\ p(\cdot)\, dS\right)^2$

$\left(\int x_2 p(x_2|S)\, dx_2\ p(S)\, dS\right)^2$

$\sigma_x^2 = \sigma_{x|s}^2 + R_1 - R_2$

$$R_1 = \frac{1}{2}\left[\left(\frac{\alpha S_1}{\alpha S_1+\beta}\right)^2 + \left(\frac{\alpha S_2}{\alpha S_2+\beta}\right)^2\right]$$

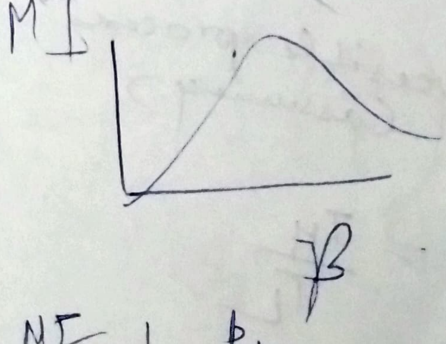
$$R_2 = \frac{1}{4}\left(\frac{\alpha S_1}{\alpha S_1+\beta} + \frac{\alpha S_2}{\alpha S_2+\beta}\right)^2$$

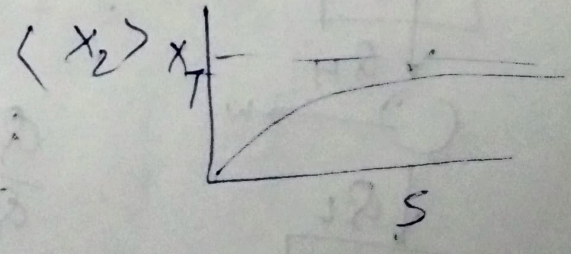
$$R_2 = \left(\int x_2 P(x_2 S)\, dX\, dS\right)^2$$

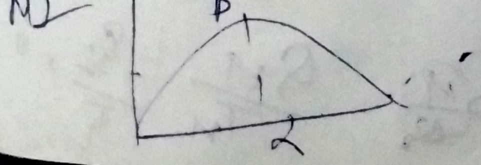
$$R_1 = \int \left(\int x_2\, p(x_2|S)\, dx_2\right)^2 p(S)\, dS$$

$MI(x_2\ S)$

$$= \log\left(1 + \frac{\frac{\alpha\beta}{2}(S_1-S_2)^2\, \Omega}{S_1(\alpha S_2+\beta)^2 + S_2(\alpha S_1+\beta)^2}\right)$$



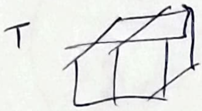




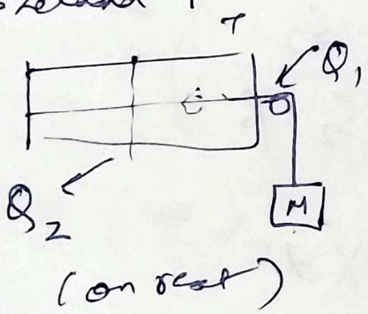
MI$(X_2 S)$ (also)

$$= \log\left(1 + \frac{\beta/\alpha\,(S_1 - S_2)^2}{S_1\left(S_2 + \frac{\beta}{\alpha}\right)^2 + S_2\left(S_1 + \frac{\beta}{\alpha}\right)^2}\right)$$

Maxwell's demon



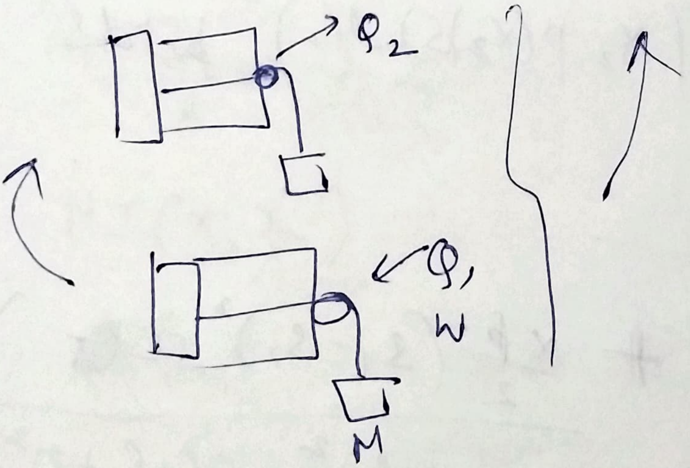
Szeland 1

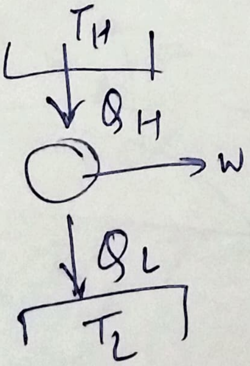


$Q = W$

$k_B T \log 2$

(1 bit of info converted to $k_B T$ work)





Reversible process (assuming)

$$\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$$

$$\frac{Q_H}{T_H} - \frac{Q_L}{T_L} = 0.$$

η_R (efficiency)

$$= \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$$

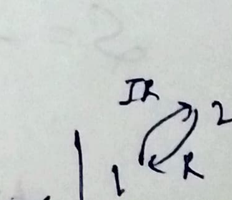
$$\frac{Q_L^{IR}}{Q_H^{IR}} > \frac{Q_L^{R}}{Q_H^{R}} = \frac{T_L}{T_H}$$

$\eta_R > \eta_{IR}$ (IR: Irreversible, R: Reversible)

$$1 - \frac{Q_L^R}{Q_H^R} > 1 - \frac{Q_L^{IR}}{Q_H^{IR}}$$

$$\left(\frac{Q_H}{T_H} - \frac{Q_L}{T_L}\right)_{IR} < 0$$

$$\oint \frac{dQ}{T} = 0 \quad \text{(for R. process)} \quad \text{(all steps rev)}$$



$$\oint \frac{dQ}{T} = \int_1^2 \left(\frac{dQ}{T}\right)^{IR} + \int_2^1 \left(\frac{dQ}{T}\right)^{R} < 0$$

$$= \int_1^2 \left(\frac{dQ}{T}\right)^{IR} + S_1 - S_2 < 0.$$

$$S_2 - S_1 > \int \frac{dQ}{T}^{IR}$$

$$\Delta S > \int \frac{dQ}{T}^{IR}$$

$$\boxed{\Delta s = \int \frac{dQ}{T}\Big|_{IR} + \Delta S_g}$$

entropy change

entropy generated in the system

$$\underset{\text{At steady state}}{0} = \frac{d}{dt}(\Delta s) = \frac{d}{dt}\Delta S_I + \frac{d}{dt}(\Delta S_g)$$

Noise Propagatⁿ & Info. Transmission

Master Eqⁿ

$$\frac{dp(n,t)}{dt} = \sum_{n'} \left(w_{n'n}\, p(n',t) - w_{nn'}\, p(n,t) \right)$$

$$S = -\sum_n p(n,t) \log p(n,t) - \cancel{\sum_n \frac{d}{dt}}$$

$$\frac{dS}{dt} = \sum_n \frac{dp(n,t)}{dt} \log p(n,t) - \cancel{\sum_n \frac{d}{dt} p(n,t)}^{\,0}$$

$$\Downarrow$$

$$= \sum_n \left(\sum_{n'} w_{nn'}\, p(n',t) - w_{n'n}\, p(n,t) \right) \log p(n,t)$$

$$= \sum_{nn'} \left(w_{n'n}\, p(n t) - w_{nn'}\, p(n' t) \right) \log p(n t)$$

(replacing variables)