Probability and Statistics

UG2, Core course, IIIT,H

Pawan Kumar

IIIT, Hyderabad

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Outline

1. Joint Distributions: Two Random Variables
2. Joint Continuous Random Variables

Fact…

Fact...

Fact

Let X, Y be two RVs and g, h be two functions of X and Y respectively. Show that

$$E[g(X)h(Y) \mid X] = g(X)E[h(Y) \mid X]$$

Solution

$$E[g(X)h(Y) \mid \boxed{X=x}] = g(x)E[h(Y) \mid X=x]$$

becomes a cont

instantiated with x

Iterated Expectations...

Iterated Expectations...

L.O.I.E

Law of Iterated Expectations

Let X, Y be two RVs, then we have

$$E[X] = E[E[X \mid Y]]$$

Proof

Recall $g(y) = E[X \mid Y]$ [Saw in last class]

Applying Law of Total Probability: [L.O.T.P]

$$E[X] = \sum_{y_j \in R_Y} \underbrace{E[X \mid Y = y_j]}_{\leftarrow \text{fn of } y} P(Y = y_j)$$

$$= \sum g(y_j) P_Y(y_j) = E[g(Y)] = E[\underbrace{E[X \mid Y]}_{g(Y)}]$$

Solved Example on Application of Iterated Expectation...

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Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.

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Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers.

 $\boxed{\text{L.O.I.E}}$

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?

Solution

Recall: X = # customers who purchase drinks with prob p

N = # customers visiting the restaurant. $N \sim \text{Poisson}(\lambda)$

Recall: $X|N \sim \text{Binomial}(n, p)$

$$E[X] \overset{\text{L.O.I.E}}{=} E\left[E[X|N]\right] = E[Np] = pE[N] = \boxed{p\lambda}$$

Expectation for Independent RV...

Expectation for Independent RV...

Expectation for Independent RVs

Let X, Y be two independent RVs. Then we have the following

1. $E[X \mid Y] = E[X]$
2. $E[g(X) \mid Y] = E[g(X)]$
3. $E[XY] = E[X]E[Y]$
4. $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

① $E[X \mid Y = y] = \sum_{x \in R_X} x \, \underline{P_{X|Y}(x \mid y)} \longrightarrow$ ⊛

Since X, Y ind. $\Rightarrow$ $P_{X|Y}(x \mid y) = P_X(x)$

$\underline{\text{From}}$ ⊛ :

$$\sum x \, P_X(x) \underline{\underline{\text{def}}} \; E[X] \quad //$$

Answer to previous problem...

Answer to previous problem...

Conditional Variance...

Conditional Variance...

- $X|Y$ ~ R.V
- $P_{X|Y}$, $F_{X|Y}$
- $E[X|Y]$

Definition of Conditional Variance

Let X, Y be two RVs.

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Conditional Variance...

$V(X) = E[X^2] - E[X]^2$

Definition of Conditional Variance

Let X, Y be two RVs. By $\mathrm{Var}(X \mid Y = y)$ the conditional variance of X given $Y = y$. Let $\mu_{X|Y}(y) = E[X \mid Y = y]$. Then

$$\mathrm{Var}(X \mid Y = y) = E[X^2 \mid Y = y] - \mu_{X|Y}(y)^2$$

("probabilitycourse.com")

Assign

Proof

Solved Example ...

Solved Example ...

$X|Y=1$

Recall: $X, Y \sim \text{Bernoulli}(2/5)$

and $X|Y=0 \sim \text{Bernoulli}(2/3)$

$\text{Var}(X) = p(1-p)$, if X is Bernoulli(p).

Solved Example

Let X, Y be RV with joint PMF given as follows

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
$X=1$	$\frac{2}{5}$	0

Let $Z = E[X \mid Y]$ and $V = \text{Var}(X \mid Y)$.

1. Find the PMF of V
2. Find $E[V]$
3. Verify that $\text{Var}(X) = E[V] + \text{Var}(Z)$

Note that V is a fn of Y

$$V = \text{Var}(X|Y) = \begin{cases} \text{Var}(X|Y=0) & Y=0 \\ \text{Var}(X|Y=1) & Y=1 \end{cases}$$

$$= \begin{cases} \text{Var}(X|Y=0), & \text{with prob } 3/5 \\ \text{Var}(X|Y=1), & \text{with prob } 2/5 \end{cases}$$

$2/9$

0

Since $Y \sim \text{Bernoulli}(2/5)$

$\text{Var}(X|Y=0) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

$\boxed{\text{Var}(X|Y=1) = 0 \quad \text{check}}$

Answer to previous problem...

$V = \mathrm{Var}(X|Y) = \begin{cases} 2/9 & \text{with prob } 3/5 \\ 0 & \text{with prob } 2/5 \end{cases}$

$\frac{18}{75}$ is $\frac{5}{25}$

(a) $P_V(v) = \begin{cases} 3/5 & \text{if } v = 2/9 \\ 2/5 & \text{if } v = 0 \\ 0 & \text{otherwise} \end{cases}$

(b) $E[V] = \frac{2}{9} \cdot 3/5 + 0 \cdot 2/5$

$= \frac{6}{45}$

(c) $\mathrm{Var}(X) = \frac{2}{5} \cdot \frac{3}{5}$

$= \frac{6}{25}$

$E[V] = 6/45 = \frac{2}{15}$

$\boxed{\mathrm{Var}[Z] = 8/75}$ [Check]

$\frac{6}{25} = \frac{2}{15} + \frac{8}{75}$?

$\cancel{\frac{18}{75}}$ $= \frac{10+8}{75}$

(Check this in Probabilitycourse.com) 6 A TA

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Law of Total Variance...

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Let X, Y be two RVs. The law of total variance says that

$$\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$$

Proof:

See this website

Solved Problem 1

Solved Problem 1

Let X, Y be two independent RVs with the same CDFs F_X and F_Y. Let

$$Z = \max(X, Y)$$
$$W = \min(X, Y)$$

Find the CDFs of Z and W.

Answer to previous problem...

Answer to previous problem...

Outline

Joint Probability Density Function...

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Joint Continuous Density Functions

Two RVs X, Y are jointly continuous if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X, Y) \in A) = \int \int_A f_{XY}(x, y)\, dxdy$$

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$$R_{XY} = \{(x,y) \mid f_{X,Y}(x,y) > 0\}$$

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$$R_{XY} = \{(x,y) \mid f_{X,Y}(x,y) > 0\}$$

2 If we indeed choose $A = \mathbb{R}^2$, then we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dxdy = 1$$

Solved Example...

Solved Example...

Solved Example

Let X, Y be two jointly continuous RVs with joint *PDF*

Solved Example...

Solved Example

Let X, Y be two jointly continuous RVs with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Solved Example...

$$\iint f_{XY}(x,y) = 1$$

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Let X, Y be two jointly continuous RVs with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1 Find the constant c

We have

$$\int_0^1\int_0^1 (x + cy^2)\,dx\,dy = 1$$

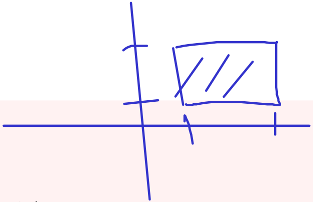
$$\text{LHS} = \int_0^1 \left[\frac{x^2}{2} + cy^2 x\right]_0^1 dy = \int_0^1 \left(\frac{1}{2} + cy^2\right) dy$$

$$= \left[\frac{1}{2}y + \frac{cy^3}{3}\right]_0^1 = \frac{1}{2} + \frac{c}{3} = 1$$

$$\frac{c}{3} = \frac{1}{2}$$

$$\Rightarrow c = 3/2$$

Solved Example...



Solved Example

Let X, Y be two jointly continuous RVs with joint *PDF*

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the constant c
2. Find $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$

$$P(0 \le X \le 1/2, 0 \le Y \le 1/2) = \int_0^{1/2}\int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx\, dy$$

Answer to previous problem...

Answer to previous problem...

Marginal Continuous PDF...

Marginal Continuous PDF...

$$P_X(x) = \sum_y P_{XY}(x,y)$$

Marginal Continuous PDF

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$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy, \quad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx, \quad \text{for all } y.$$

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$$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1 Find the PDFs $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} \left(x + \frac{3}{2}y^2\right) dy = \int_0^1 \left(x + \frac{3}{2}y^2\right) dy$$

Try

Answer to previous problem...

Solved Example...

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Let X, Y be two jointly continuous RVs with joint PDFs

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Let X, Y be two jointly continuous RVs with joint PDFs

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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Let X, Y be two jointly continuous RVs with joint PDFs

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find R_{XY} and plot it

Solved Example...

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4. Find $P\left(Y \leq \dfrac{X}{2}\right)$

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2. Find the constant c
3. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$
4. Find $P\left(Y \le \frac{X}{2}\right)$
5. Find $P\left(\cancel{Y \le \frac{X}{2}} \mid Y \le \frac{X}{2}\right)$

Solution to previous problem...

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Looking at the joint PDF, we have

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The plot is the following

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$y \le x/2$

The plot is the following

Plot $y = x/2$

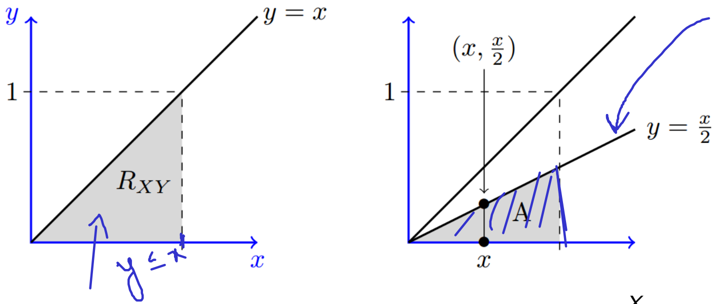

Figure: Figure showing R_{XY} and integration region $P(Y \le \frac{X}{2})$

Answer to previous problem...

(b)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1\int_0^x c x^2 y\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 \left[\frac{c x^2 y^2}{2}\right]_0^x dx = 1$$

$$\Rightarrow \frac{c}{2}\int_0^1 \left[x^4 - 0\right] dx = 1$$

$$\Rightarrow \frac{c}{2}\left[\frac{x^5}{5}\right]_0^1 = 1$$

$$\Rightarrow \frac{c}{10} = 1$$

$$\Rightarrow c = 10$$

Answer to previous problem...

(c) To find the marginal

$R_X, R_Y = [0,1]$

For $0 \le x \le 1$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^x 10x^2 y\,dy = \left[10x^2\frac{y^2}{2}\right]_0^x$$

$$= 5x^4$$

(d) $P(Y \le X/2)$

$$= \int_{-\infty}^{\infty}\int_0^{x/2} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^{1}\int_0^{x/2} 10x^2 y\,dx\,dy$$

$= \underline{\underline{\text{easy}}}$ $\underline{\text{try}}$

Joint Cumulative Distribution...

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3. $F_{XY}(\infty, \infty) = 1$

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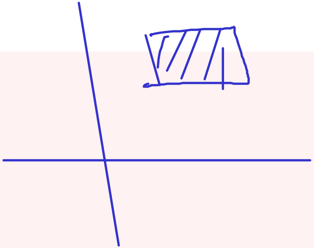
Joint Cumulative Distribution...



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3. $F_{XY}(\infty, \infty) = 1$
4. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
5. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$

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4. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
5. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$
6. If X, Y are independent, then $F_{XY} = F_X(x)F_Y(y)$

Solved Example

Solved Example

Let X, Y be independent two random variables with Uniform(0,1) distribution. Find $F_{XY}(x, y)$.

Since X, Y are uniform(0,1) we have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}, \quad F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Since X, Y are independent $\Rightarrow$ $F_{XY}(x,y) = F_X(x) F_Y(y)$.

$$F_{XY}(x,y) = \begin{cases} 0 & \text{for } x < 0 \text{ \& } y < 0 \\ xy & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \\ y & \text{for } x > 1,\ 0 \le y \le 1 \\ x & \text{for } y > 1,\ 0 \le x \le 1 \\ 1 & \text{for } y > 1,\ x > 1 \end{cases}$$

Answer to previous problem...

Figure for Solved Example...

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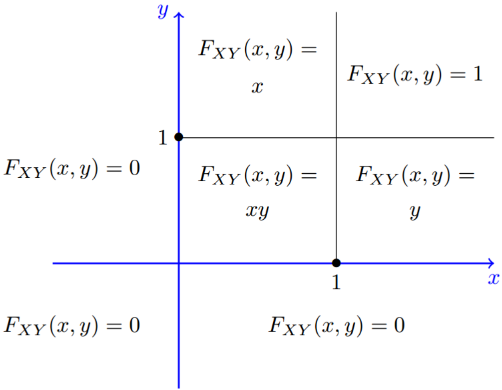

Figure: Plot of joint CDF

Relationship Between CDF and PDF...

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Recall that for single RV we have

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Recall that for single RV we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

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$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Similarly, for two RVs we have

Relationship Between CDF and PDF...

Relationship between CDF and PDF

Recall that for single RV we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Similarly, for two RVs we have

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v)\,du\,dv$$

$$f_{XY} = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Example of Joint CDF...

Example of Joint CDF...

Solved Example

Let X, Y be two jointly continuous RVs with joint *PDF*

Example of Joint CDF...

Solved Example

Let X, Y be two jointly continuous RVs with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Example of Joint CDF...

Solved Example

Let X, Y be two jointly continuous RVs with joint *PDF*

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$3/2$

1 Find the joint CDF of X and Y

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv = \int_{-\infty}^{x}\int_{-\infty}^{y}\left(u+\frac{3}{2}v^2\right)du\,dv$$

$$= \int_{-\infty}^{y}\left[\frac{u^2}{2}+\frac{3}{2}v^2u\right]_0^x dv = \int_{-\infty}^{y}\left[\frac{x^2}{2}+\frac{3}{2}v^2x\right]dv = \left[\frac{x^2}{2}v+\frac{3}{2}\frac{v^3}{3}\cdot x\right]_0^y$$

$$= \frac{x^2y}{2}+\frac{y^3x}{2}$$

Answer to previous problem...

Answer to previous problem...

Definition of Conditional PDF and Conditional CDF

Let X be a continuous RV and A be an event that $a < X < b$ (where possibly $b = \infty$ or $a = -\infty$), then

$$f_{X|A}(x) = \begin{cases} 1 & x > b \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \leq x < b \\ 0 & x < a \end{cases}$$

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(A)} & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

Answer to previous problem...

Answer to previous problem...

Conditional Expectation and Variance...

Conditional Expectation and Variance...

Definition of Conditional Expectation and Variance

For a random variable X and event A, we have

Conditional Expectation and Variance...

Definition of Conditional Expectation and Variance

For a random variable X and event A, we have

$$E[X \mid A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$$

$$E[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$$

$$\mathrm{Var}(X \mid A) = E[X^2 \mid A] - (E[X \mid A])^2$$

Solved Example: Conditional PDF and CDF...

Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$.

Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$

Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$
2. Find $E[X \mid X > 1]$

Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$
2. Find $E[X \mid X > 1]$ $\leftarrow$
3. Find $\text{Var}(X \mid X > 1)$

(a) Let A be the event that $X > 1$

$$P(A) = \int_1^\infty e^{-x}\,dx = 1/e$$

$$f_{X|X>1}(x) = \frac{P(X=x,\ X>1)}{P(X>1)} = \frac{e^{-x}}{1/e} = e^{-x+1}$$

$$F_{X|X>1}(x) = \int_1^x f_{X|A}(u)\,du$$

$$= \int_1^x e^{-u+1}\,du = \left[\frac{e^{-u+1}}{-1}\right]_1^x$$

$$= \frac{e^{-x+1}}{-1} + e^{-1+1} = 1 - e^{-x+1}$$

Answer to previous problem...

$$E[X|X>1] = \int_{-\infty}^{\infty} x\, f_{X|X>1}(x)\, dx$$

$$= \int_{1}^{\infty} x\, e^{-x+1}\, dx = \underline{\text{try}}$$