Probability and Statistics

UG2, Core course, IIIT,H

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October 20, 2021

Outline

Higher Order Moments...

Define nth moment

The nth moment about the mean or nth central moment of a real valued random variable X is defined as follows

$$\mu_n = E[(X - E[X])^n],$$

where E is the expectation operator.

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$$\mu_n = E[(X - \underbrace{E[X]}_{0})^n],$$

$$E[(X-0)^1]$$

$$= E[X]$$

where E is the expectation operator.

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  - First central moment about origin is indeed expectation!

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- The second central moment μ_2 is called variance, denoted by σ^2

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Generating Moments...

Is there a quick way to generate moments?

Moment Generating Function...

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$E[g(X)] = \sum_x g(x) p_X(x)$

Moment Generating Function

The moment generating function $M_X(t)$ is the expectation value

$g(x) = e^{tx}$

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} p_X(x)$$

$e^{cx} = c e^{cx}$

Lemma

- $M_X(0) = 1$
- $E[X] = M'_X(0)$, where ' is the derivative w.r.t. t

$M_X(t) = \sum_x e^{tx} p_X(x)$

$M'_X(t) = \sum_x x e^{tx} p_X(x)$

$M'_X(0) = \sum x p_X(x) = E[X]$

$M''_X(t) = \sum_x x^2 e^{tx} p_X(x)$

$M''_X(0) = \sum x^2 p_X(x)$

$= E[X^2]$

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Binom(n, p)

- The moment generating function is given by

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx} \\
&= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \\
&= (e^t p + 1 - p).
\end{aligned}
$$

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$$E[X] = M'_X(t)\Big|_{t=0}$$

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Differentiating w.r.t. t, we have

$$M_X'(t) = n(e^t p + 1 - p)^{n-1} p e^t.$$

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- Setting $t = 0$, $\quad M_X'(0) = np = E[X]$

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&= \sum_{x=0}^{\infty} e^{-\lambda}\,\frac{(\lambda e^t)^x}{x!} \\
&= e^{\lambda(e^t-1)}
\end{aligned}
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$$M'_X(t) = e^{\lambda(e^t - 1)} \lambda e^t$$

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Variance Using Moment Generating Function...

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$$\mathrm{Var}(X) = \underline{M_X''(0)} - \underline{M_X'(0)}^2$$

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right] = E[X^2] - \left(E[X]\right)^2$$

$$= M_X''(0) - \left(M_X'(0)\right)^2$$

Computing Variance Using Moment Generating Function...

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Computing Variance using moment generating function

Let X be a discrete random variable whose PMF is a binomial distribution with parameters n and p. It has mean $\mu = np$ and the moment generating function is

$$M_X(t) = (e^t p + 1 - p)^n$$

$\cdot\ E[(X - E[X])^n]$

Computing Variance using moment generating function

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Find the Variance using $M_X(t)$.

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$$\underbrace{M_X(t) = e^{\lambda(e^t - 1)}}$$

Find the Variance using $M_X(t)$.

$$Var(X) = M_X''(0) - \left(M_X'(0)\right)^2$$

$$M_X'(t) = \lambda e^{\lambda(e^t - 1)} \Rightarrow M_X'(0) = \lambda$$

$$M_X''(t) = \lambda^2 e^{\lambda(e^t - 1)} \Rightarrow M_X''(0) = \lambda^2$$

$$Var(X) = \lambda^2 - \lambda^2 = 0$$

Outline

Problem 1

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Let X be a discrete R.V. with PMF

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$$P_X(x) = \begin{cases} 0.1 & \text{for } x = 0.2 \\ 0.2 & \text{for } x = 0.4 \\ 0.2 & \text{for } x = 0.5 \\ 0.3 & \text{for } x = 0.8 \\ 0.2 & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

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Answer the following:

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Answer the following:

1. Find R_X

$R_x = \{0.2, 0.4, 0.5, 0.8, 1\}$

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Answer the following:

1. Find R_X
2. Find $P(X \leq 0.5)$ $= P(X=0.2) + P(X=0.4) + P(X=0.5)$

$= 0.1 + 0.2 + 0.2 = 0.5$

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Answer the following:

1. Find R_X
2. Find $P(X \leq 0.5)$
3. Find $P(0.25 < X < 0.75)$ $= P(X = 0.4) + P(X = 0.5)$

$= 0.2 + 0.2 = 0.4$

Problem 1

$P(X=x \mid Y=y) = \dfrac{P(X=x, Y=y)}{P(Y=y)}$ Recall

$P(A \mid B) = \dfrac{P(A,B)}{P(B)}$

Problem

Let X be a discrete R.V. with PMF

$$P_X(x) = \begin{cases} 0.1 & \text{for } x = 0.2 \\ 0.2 & \text{for } x = 0.4 \\ 0.2 & \text{for } x = 0.5 \\ 0.3 & \text{for } x = 0.8 \\ 0.2 & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following:

1. Find R_X
2. Find $P(X \leq 0.5)$
3. Find $P(0.25 < X < 0.75)$
4. Find $P(X = 0.2 \mid X < 0.6)$ =

$$\frac{P(X=0.2, \cancel{X<0.6})}{P(X<0.6)} = \frac{P(X=0.2)}{P(X<0.6)} = \frac{0.1}{0.5} = 1/5$$

Answer to previous problem...

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Let X and Y be two random variables that denote the outcome of the roll of two dice. Answer the following:

1. Find R_X, R_Y and the PMF of X and Y

$R_X = \{1, 2, 3, 4, 5, 6\}$

$R_Y = \{1, 2, 3, 4, 5, 6\}$

$$P_X(x) = \begin{cases} 1/6 , & x = 1, 2, 3, 4, 5, 6 \\ 0 , & \text{otherwise} \end{cases}$$

same for Y

Problem 2

Problem

Let X and Y be two random variables that denote the outcome of the roll of two dice. Answer the following:

1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$ = Since X, Y are independent

$$P(X=2) \cdot P(Y=6)$$

$$= 1/6 \cdot 1/6 = 1/36$$

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Let X and Y be two random variables that denote the outcome of the roll of two dice. Answer the following:

1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid \underline{Y = 2})$ $= \dfrac{P(X>3, Y=2)}{P(Y=2)}$

$1, 2, 3, \boxed{4, 5, 6}$

$$= \frac{P(X>3)\,P(Y=2)}{P(Y=2)} = \frac{\frac{3}{6} \cdot \frac{1}{6}}{1/6}$$

$$= \frac{1}{2}$$

Problem 2

$R_X = \{1, 2, \ldots 6\}$

$R_Y = \{1, 2, \ldots 6\}$

Problem

Let X and Y be two random variables that denote the outcome of the roll of two dice. Answer the following:

1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of Z

$R_Z = \{2, \ldots, 12\}$

$P_Z(2) = P(X=1, Y=1) = P(X=1)P(Y=1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$

$P_Z(3) = P(X=1, Y=2) + P(X=2, Y=1)$

$= P(X=1)P(Y=2) + P(X=2)P(Y=1)$

$\downarrow \quad = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = 2 \cdot \frac{1}{36} = \frac{1}{18}$

Problem 2

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Let X and Y be two random variables that denote the outcome of the roll of two dice. Answer the following:

1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of Z
5. Find $P(X = 4 \mid Z = 8)$

$$\frac{P(X=4, Z=8)}{P(Z=8)} \overset{?}{=} P(X=4) \cdot P(Z=8) \; ? \quad \rightarrow \quad \frac{P(X=4) \cdot P(Y=4)}{P(Z=8)}$$

$X = 4$

$Z = X + Y = 4 + Y$

Answer to previous problem...

Answer to previous problem...

Problem 3

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Consider an exam that contains 20 MCQs.

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1. What is PMF of X?

Problem 3

Bernoulli, Binomial, Geometric, Poisson

0, 1 — 1+1 — S, F

0, 1 — H, T — S, F

Problem

Consider an exam that contains 20 MCQs. Each question has 4 choices. Assume that you can answer 10 questions correctly, but you are not sure about other 10 questions, so you answer them randomly. Let the random variable X denote the score, which is equal to the total number of correct answers. Answer the following:

$$R_X(X) = \{10, 11, \cdots, 20\}$$

1. What is PMF of X?
2. What is $P(X > 15)$?

$$P_X(x) =$$

$X = 10 + Y$, Y = # correct answers to remaining 10 questions

For each question, success prob. is $\frac{1}{4}$.

Answer to previous problem...

For each question, outcome is "correct" or "incorrect". Choosing a correct choice has prob. 1/4. This follows Bernoulli(1/4). There are 10 such questions $\Rightarrow$ 10 independent Bernoulli(1/4)

$\Rightarrow$ This is Binom(10, 1/4).

That is $Y \sim \text{Binom}(10, 1/4)$

So,

$$P_Y(y) = \begin{cases} \binom{10}{y} \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{10-y} & \text{for } y = 0, \dots, 10 \\ 0, & \text{otherwise.} \end{cases}$$

We want: PMF of X

$X = Y + 10$

$R_X = \{10, 11, \dots, 20\}$

$P_X(X=10) = P_X(Y+10=10)$

$= P_X(Y=0) = \binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10}$

$= \left(\frac{3}{4}\right)^{10}$

Answer to previous problem...

$$P(X=11) = P(Y+10=11)$$

$$= P(Y=1) = \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9$$

$$P(X=k) = P(Y+10=k)$$

$$= P(Y=k-10)$$

$$= \binom{10}{k-10}\left(\frac{1}{4}\right)^{k-10}\left(\frac{3}{4}\right)^{20-k}$$

$10-(k-10)$

$20-k$

$k=10, 11, 12, \ldots, 20$

Summarize

$$P_X(x) = \begin{cases} \binom{10}{k-10}\left(\frac{1}{4}\right)^{k-10}\left(\frac{3}{4}\right)^{20-k} & k=10,\ldots,20 \\ 0, & \text{otherwise} \end{cases}$$

$$P_X(X>15) = P_X(X=16) + P(X=17) + \cdots + P(X=20)$$

Problem 4

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Problem

Average number of customers arriving at a grocery store per hour is 10.

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Average number of customers arriving at a grocery store per hour is 10. Let X denote the number of customers arriving from $10AM$ to $11:30AM$.

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Problem

Average number of customers arriving at a grocery store per hour is 10. Let X denote the number of customers arriving from $10AM$ to $11:30AM$. What is $P(10 < X \leq 15)$?

λ

In question, interval length is 1.5 hours.
But λ is given for 1 hour interval.

No. of customers arriving in next 1.5 hours

$$X \sim \text{Poisson}(1.5 \times 10) = \text{Poisson}(15)$$

$$P(10 < X \leq 15) = \sum_{k=11}^{15} p_X(k) = \sum_{k=11}^{15} \frac{15^k e^{-15}}{k!}$$

Problem 5

PMF of Sum of Poisson Random Variables

Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two independent Poisson random variables.

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Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two independent Poisson random variables. Let $Z = X + Y$ be a new random variable.

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$P(A|B) = P(A)$ if A, B are ind.

PMF of Sum of Poisson Random Variables

Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two $\boxed{\text{independent}}$ Poisson random variables. Let $Z = X + Y$ be a new random variable. Find the PMF of Z.

$R_X = \{0, 1, 2, \ldots\}$, $R_Y = \{0, 1, 2, \ldots\}$, $R_Z = \{0, 1, 2, \ldots\}$

$$P_Z(Z = k) = P_Z(X + Y = k)$$

$$= \sum_{i=0}^{k} P(X+Y=k \mid X=i)\, P(X=i)$$

[Law of total probability]

$$= \sum_{i} P(\underline{Y = k-i \mid X = i})\, P(X=i) = \sum_{i} \underbrace{P(Y=k-i)\, P(X=i)}$$

Answer to previous problem...

$$= \sum_i \frac{e^{-\beta}\beta^{k-i}}{(k-i)!} \cdot \frac{e^{-\alpha}\alpha^i}{i!}$$

$$= e^{-(\alpha+\beta)} \sum_i \frac{\alpha^i \beta^{k-i}}{i!\,(k-i)!}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} \sum \frac{k!}{i!\,(k-i)!} \alpha^i \beta^{k-i}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} \sum \binom{k}{i} \alpha^i \beta^{k-i}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} (\alpha+\beta)^k$$

$$= \text{Poisson}(\alpha+\beta)$$