Probability and Statistics

UG2, Core course, IIIT,H

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IIIT, Hyderabad

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Outline

CDF of a mixed RV as a sum of Continuous and Discrete CDF...

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CDF of Mixed Random Variable

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$$F_Y(y) = C(y) + D(y).$$

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$$\int_{-\infty}^{\infty} c(y)\, dy + \sum_{y_k} P(Y = y_k) = 1$$

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$$\int_{-\infty}^{\infty} c(y)\, dy + \sum_{y_k} P(Y = y_k) = 1$$

The expected value of Y can be obtained as

$$E[Y] = \int_{-\infty}^{\infty} y\, c(y)\, dy + \sum_{y_k} y_k\, P(Y = y_k)$$

Check the Validity of CDF of Mixed RV...

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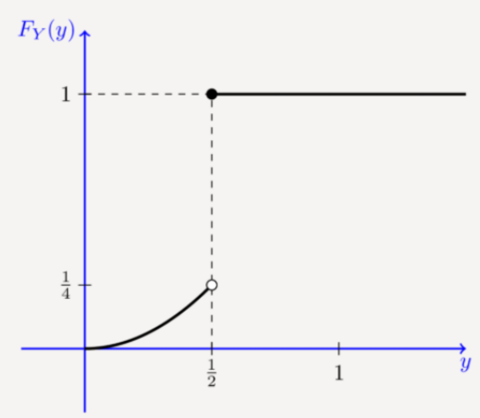

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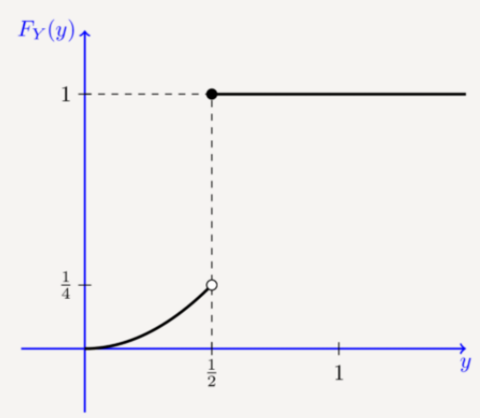

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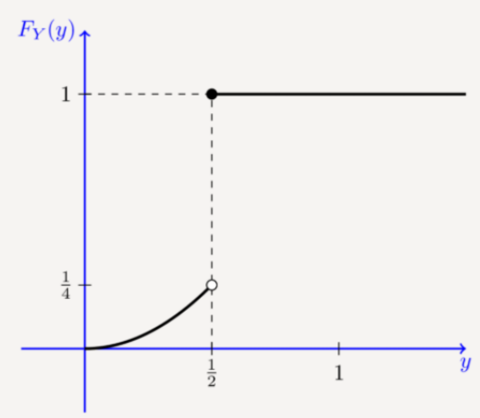

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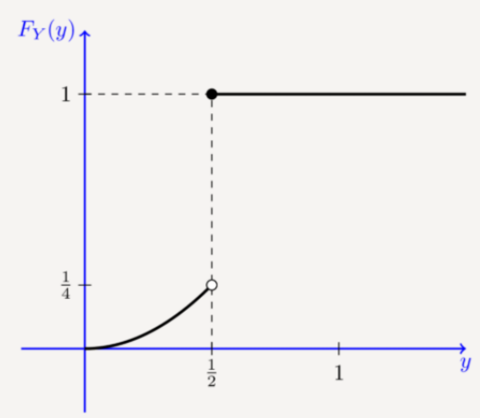

$$F_Y(y) = C(y) + D(y),$$

where the continuous part is

$$C(y) = \begin{cases} 1/4 & y \geq 1/2 \\ y^2 & 0 \leq y < 1/2 \\ 0 & y < 0 \end{cases}$$

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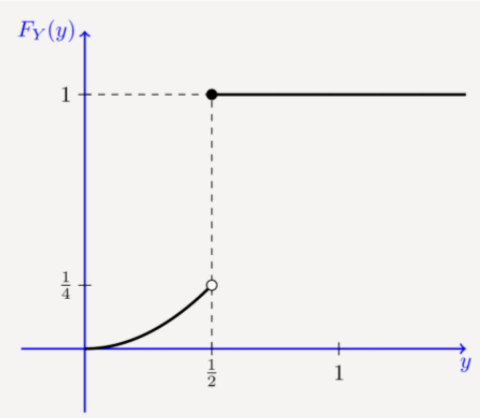

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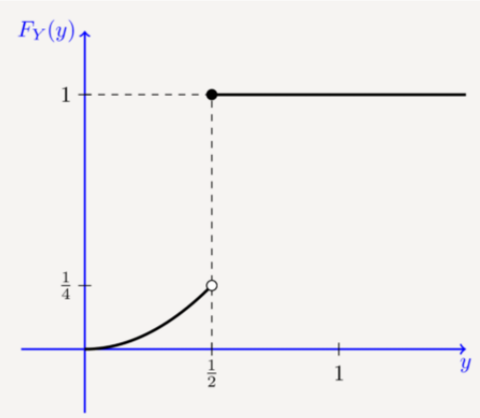
and the discrete part is

Check the Validity of CDF of Mixed RV...

$$F_Y(y) = \begin{cases} 1, & y \geq 1/2 \\ y^2, & 0 \leq y < 1/2 \\ 0, & \text{otherwise.} \end{cases}$$



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and the discrete part is

$$D(y) = \begin{cases} 3/4 & y \geq 1/2 \\ 0 & y < 1/2 \end{cases}$$

PMF

Check that

$$\int_{-\infty}^{\infty} c(y)\, dy + \sum_{y_k} P(Y = y_k) = 1$$

Computing Probabilities of Mixed Random Variable...

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Problem: Computing probabilities of mixed RV

Let X be a continuous random variable with the following PDF

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Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

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Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$Y = g(X) = \begin{cases} X & 0 \le X \le \frac{1}{2} \\ \frac{1}{2} & X > \frac{1}{2} \end{cases}$$

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$P(a \leq X \leq b) = F_X(b) - F_X(a)$

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- Find $P(1/4 \leq Y \leq 3/8)$

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- Find $P(Y \geq 1/4)$

Computing Probabilities of Mixed Random Variable...

If Y was cont.: $\boxed{P(Y \le y) = P(Y < y)}$

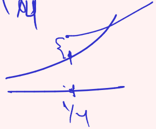
Problem: Computing probabilities of mixed RV

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$F_Y(y) = P(Y \le y)$

$F_Y(1/4) = P(Y \le 1/4)$



Let

$$Y = g(X) = \begin{cases} X & 0 \le X \le \frac{1}{2} \\ \frac{1}{2} & X > \frac{1}{2} \end{cases}$$

- Find $P(1/4 \le Y \le 3/8)$
- Find $P(Y \ge 1/4)$ ←
- Find $E[Y]$ ←

$1 - P(Y < 1/4) = 1 - F_Y(1/4) + P_Y(1/4)$

$= 1 - (1/4)^2 + 0$

$c(y) = \frac{d}{dy} C(y) = \frac{d}{dy} y^2 = 2y$

$\begin{cases} 2y, & 0 \le y \le 1/2 \\ 0, & \text{otherwise.} \end{cases}$

Answer to previous problem...

① $P(1/4 \le Y \le 3/8) = F_Y(3/8) - F_Y(1/4) + \boxed{P_Y(1/4)}$

$0 < 3/8 < 1/2 \qquad 0 < 1/4 < 1/2$

$= \left(3/8\right)^2 - \left(\frac{1}{4}\right)^2 + 0$

$= \_\_\_$ Ans

② done

③ $E[Y] = \int_0^{1/2} y \cdot 2y \, dy + \frac{1}{2} \cdot \underbrace{P(Y = 1/2)}_{\text{jump at } y = 1/2}$

$= 2\left[\frac{y^3}{3}\right]_0^{1/2} + \frac{1}{2} \cdot \frac{3}{4} = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{2} \cdot 3/4$ //

$P(a \le X \le b) = P(a < X < b) + P(a)$

$= P(a < X < b)$

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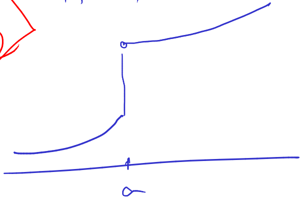
$= P(a < X \le b)$

if X is continuous.



Solved Problem 1

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Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 1 & x \geq 1 \\ \frac{1}{2} + \frac{x}{2} & 0 \leq x < 1 \\ 0 & x < 0 \end{cases}$$

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1. Is the RV X discrete, continuous, or mixed type?

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3. Find $E(e^X)$

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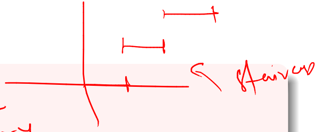

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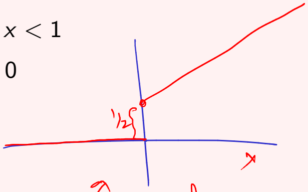
Try



1. Is the RV X discrete, continuous, or mixed type?
2. Find the PDF of X, $f_X(x)$
3. Find $E(e^X)$
4. Find $P(X = 0 \mid X \leq 0.5)$

mixed

Answer to previous problem...

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Solved Problem 2

$$f_X(x) = \begin{cases} \frac{1}{b-a} & -2<x<2 \\ 0 & \end{cases}$$

Problem 2

Let $X \sim \text{Uniform}(-2, 2)$ be a continuous random variable. Let $Y = g(X)$ where

PDF =

$$g(x) = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the CDF of Y.

$\boxed{R_Y = [0, 1]}$

$F_Y(0) = 0$, $y < 0$

$F_Y(1) = 1$, $y \geq 1$

if $0 < y < 1$ corresponds to $0 \leq x < 1$

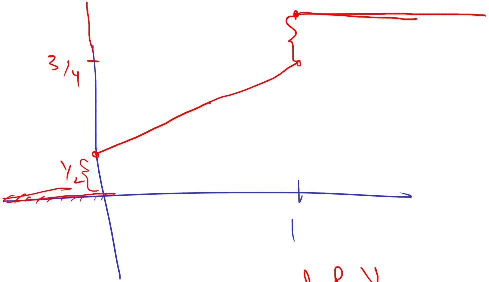
For $0 < y < 1$, $F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y)$

$$= \int_{-2}^{y} \frac{1}{4}\,dx = \frac{1}{4}[x]_{-2}^{y} = \frac{y+2}{4}$$

Answer to previous problem...

Summary

$$F_Y(y) = \begin{cases} \frac{1}{4}(y+2), & 0 < y < 1 \\ 1, & y \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$



Example of mixed R.V

Outline

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- The joint range for X and Y is

$$R_{XY} = \{(x, y) \mid P_{XY} > 0\}$$

$(x_i, y_j) = 0$

- In particular, if $R_X = \{x_1, x_2, \ldots\}$, $R_Y = \{y_1, y_2, \ldots\}$, then

$$R_{XY} \subset R_X \times R_Y = \{(x_i, y_j) \mid x_i \in R_X,\ y_j \in R_Y\}$$

- Sum of joint probabilities must sum to 1: $\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x_i, y_j) = 1$

Computing Probabilities with Joint Distributions...

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For any $A \subset \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \sum_{(x_i, y_j) \in (A \cap R_{XY})} P_{XY}(x_i, y_j)$$

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Marginal PMF: Computing Individual Probabilities from Joint PMF

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P_{XY}

P_X , P_Y

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We call $P_X(x)$ the marginal PMF of X. Similarly, we have

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$$\underline{P_Y(y)} = \sum_{x_i\in R_X} P_{XY}(\underline{x_i}, y), \quad \text{for any } y \in R_Y$$

Solved Example

Example

Let X and Y be two random variables with joint PMF as follows:

Solved Example

Example

Let X and Y be two random variables with joint PMF as follows:

	Y=0	Y = 1	Y = 2
X = 0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
X = 1	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

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1 Find $P(X = 0, Y \leq 1)$

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$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

1. Find $P(X = 0, Y \leq 1)$
2. Find the marginal PMFs of X and Y

Solved Example

$P_Y(1) = \sum_{i=1}^{2} P_{XY}(x_i, 1)$

$= P(0,1) + P(1,1)$

$= 1/6 + 1/6$

$=$

Example

Let X and Y be two random variables with joint PMF as follows:

	Y=0	Y = 1	Y = 2
X = 0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
X = 1	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

x_2

x_1

1. Find $P(X = 0, Y \leq 1)$
2. Find the marginal PMFs of X and Y
3. Find $P(Y = 1 \mid X = 0)$

$P(0,0) + P(0,1) = \frac{1}{6} + \frac{1}{4}$

$P_X(0) = \sum_{i=1}^{3} P_{XY}(0, y_i)$

(3) $\frac{P(Y=1, X=0)}{P(X=0)}$

$= 1/4$

$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2)$

$=$ read from table.

$= P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2)$

$= \boxed{1/6 + 1/4 + 1/8}$

$\frac{1/4}{1/6 + 1/4 + 1/8}$

$=$

Solved Example

$P(A,B) = P(A)\,P(B)$

$\boxed{P(X=x, Y=y) = P(X=x)\,P(Y=y)}$

Example

Let X and Y be two random variables with joint PMF as follows:

	Y=0	Y = 1	Y = 2
X = 0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
X = 1	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

1. Find $P(X = 0, Y \leq 1)$
2. Find the marginal PMFs of X and Y
3. Find $P(Y = 1 \mid X = 0)$
4. Are X and Y independent?

Answer to previous problem...

Answer to previous problem...

Joint Cumulative Distribution Function...

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- $F_{XY}(x, y) = P((X \leq x) \cap (Y \leq y))$

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- Above definition is applicable to discrete and continuous cases
- $0 \leq F_{XY}(x, y) \leq 1$

Joint Cumulative Distribution Function...

Definition of Joint Cumulative Distribution Function

Let X and Y be two random variables. The joint cumulative distribution function of X and Y is given as follows

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

- $F_{XY}(x, y) = P((X \leq x) \cap (Y \leq y))$
- Above definition is applicable to discrete and continuous cases
- $0 \leq F_{XY}(x, y) \leq 1$

Marginal CDF...

Marginal CDF...

Definition of Marginal CDF

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Let X and Y be two random variables with joint CDF $F_{XY}(x, y)$.

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$$F_{XY}(\infty, \infty) = 1,$$

[a, b]

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empty event

$$F_{XY}(\infty, \infty) = 1, \quad F_{XY}(-\infty, y) = 0,$$

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$y \leq -\infty$

empty

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Example of Joint PMF and Joint CDF...

Example of Joint PMF and Joint CDF...

Solved Example on Joint PDF and Joint CDF

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Solved Example on Joint PDF and Joint CDF

Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ be independent, where $0 < p, q < 1$.

Example of Joint PMF and Joint CDF...

p = prob. of Head

Solved Example on Joint PDF and Joint CDF

Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ be independent, where $0 < p, q < 1$. Find the joint PMF and joint CDF for X and Y.

$R_{XY} = \{(0,0), (0,1), (1,0), (1,1)\}$, $0 \equiv$ Head, $1 \equiv$ Tail.

$P_{XY}(i,j) = P_X(i) P_Y(j)$ [X, Y ind.], $i, j = 0, 1$

$P_{XY}(0,0) = P_X(0) P_Y(0) = p \cdot q$

$P_{XY}(0,1) = P_X(0) P_Y(1) = p(1-q)$

$P_{XY}(1,0) = P_X(1) P_Y(0) = (1-p) q$

$P_{XY}(1,1) = P_X(1) P_Y(1) = (1-p)(1-q)$

Joint CDF

$F_{XY}(x,y) = P_{XY}(X \le x, Y \le y)$

1. $F_{XY}(x,y) = 0$, if $x < 0$
2. $F_{XY}(x,y) = 0$, if $y < 0$
3. $F_{XY}(x,y) = 1$, if $x \geq 1, y \geq 1$

Answer to previous problem...

For $0 \le x < 1,\ y \ge 1$

$F_{XY}(x,y) = P_{XY}(X \le x, Y \le y)$

$= P_{XY}(X = 0, Y \le 1)$

$= P_X(X = 0) = p$

For $0 \le y < 1,\ x \ge 1$

$F_{XY}(x,y) = P_{XY}(X \le 1, Y = 0)$

$= P_Y(Y = 0) = q$

Finally for

$0 \le x < 1$ and $0 \le y < 1$

$F_{XY}(x,y) = P(0,0)$

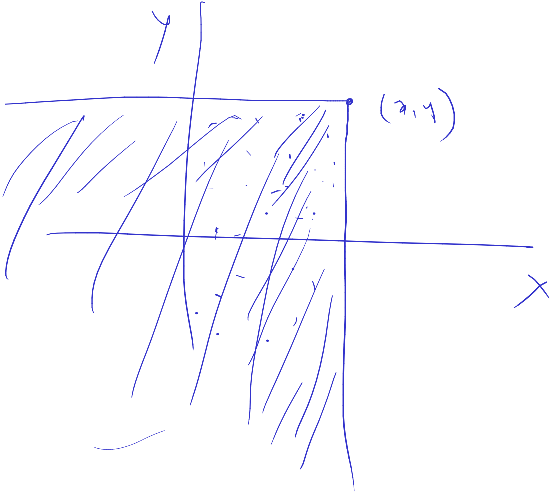
$= p\{ \cdot$

Answer to previous problem...

Plot of Joint CDF

- Figure shows the values of $F_{XY}(x, y)$ in different regions

Plot of Joint CDF

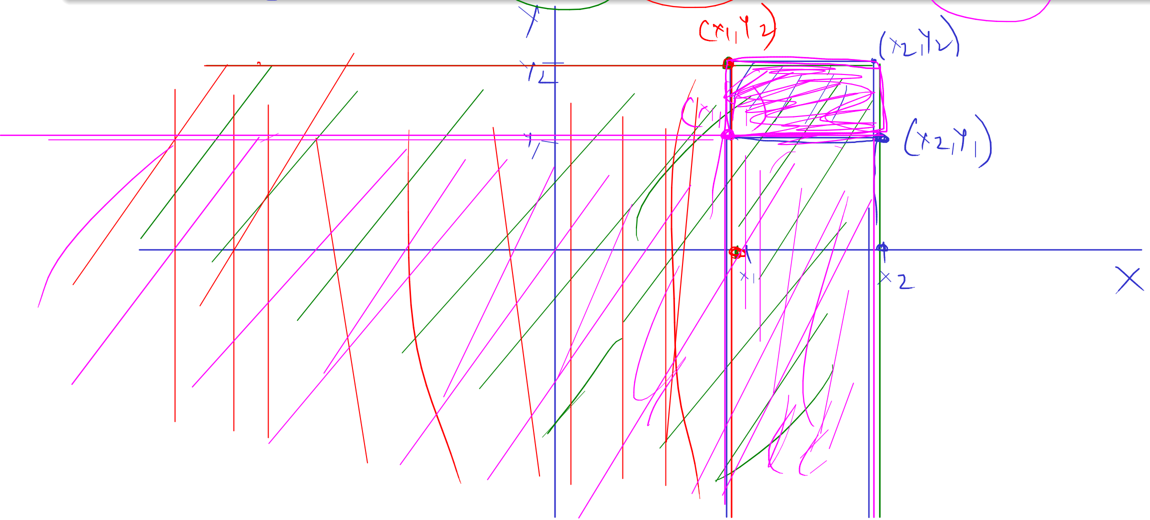


- Figure shows the values of $F_{XY}(x, y)$ in different regions
- Note that in general we need three dimensional graph to show a joint CDF of two random variables

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

A result

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$



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Conditional PMF and Conditional CDF...

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Example Motivation for Conditional PMF and CDF

Conditional PMF and Conditional CDF...

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I roll a fair die.

Conditional PMF and Conditional CDF...

Example Motivation for Conditional PMF and CDF

I roll a fair die. Let X be the observed number.

Conditional PMF and Conditional CDF...

Example Motivation for Conditional PMF and CDF

I roll a fair die. Let X be the observed number. Find the conditional PMF of X given that we know the observed number was less than 5.

Solution:

Conditional PMF and Conditional CDF...

Conditional PMF and Conditional CDF...

Definition of Conditional PMF and Conditional CDF

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$$P_{X|A}(x_i) = P(X = x_i|A)$$

Conditional PMF and Conditional CDF...

Definition of Conditional PMF and Conditional CDF

Let X be a discrete random variable and A be any event. The conditional PMF of X given A is defined as

$$
\begin{aligned}
P_{X|A}(x_i) &= P(X = x_i|A) \\
&= \frac{P(X = x_i \text{ and } A)}{P(A)}, \quad \text{for any } x_i \in R_X
\end{aligned}
$$

Conditional PMF and Conditional CDF...

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$$F_{X|A}(x) = P(X \leq x \mid A).$$

Conditional PMF of X given Y...

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Conditional PMF of X and Y

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For discrete random variables X and Y, the conditional PMFs of X and Y is defined as follows

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For discrete random variables X and Y, the conditional PMFs of X and Y is defined as follows

$$P_{X|Y}(x_i, y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}$$

Conditional PMF of X given Y...

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For discrete random variables X and Y, the conditional PMFs of X and Y is defined as follows

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Conditional PMF of X given Y...

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For discrete random variables X and Y, the conditional PMFs of X and Y is defined as follows

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$$P_{Y|X}(x_i, y_j) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}$$

for any $x_i \in R_X$ and $y_j \in R_Y$.

Independent Random Variables...

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- If X and Y are independent if

$$P_{X|Y}(x_i|y_j) = P(X = x_i \mid Y = y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)} = \frac{P_X(x_i)P_Y(y_j)}{P_Y(y_i)} = P_X(x_i)$$