Noise power Spectrum:

$$\boxed{S_x(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \left| \int_0^T e^{i\omega t} x(t)\,dt \right|^2}$$

Wiener - Khinchin theorem:

page 281

Appendix - H

$x(t) \rightarrow$ stationary random process

$S_x(\omega)$ = fourier transform of its auto-correlation func^n $\phi_x(t) = \langle x(0)\,x(t)\rangle$

$$\boxed{S_x(\omega) = \mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\langle x(t)\,x(0)\rangle dt\right\}}$$

$$= \cancel{2}\times\frac{1}{\cancel{2}\pi}\int_0^{\infty} \cos\omega t \,\langle x(t)\,x(0)\rangle\, dt \quad (\because \cos(\omega t) \text{ is even ; } \sin(\omega t) \text{ is odd})$$

$$S_x(\omega) = \frac{1}{\pi}\int_0^{\infty} \cos\omega t\,\langle x(t)\,x(0)\rangle\,dt$$

For Brownian motion:

$$\dot{v} = -\gamma v + \varepsilon(t), \quad \text{where } \langle \varepsilon(t)\,\varepsilon(0)\rangle = \Gamma\,\delta(t)$$

Power Sprectrum:

$$S_v(\omega) = \mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\langle v(0)\,v(t)\rangle dt\right\}$$

$$\Rightarrow S_v(\omega) = \mathrm{Re}\left\{\frac{1}{2\pi}\frac{\Gamma}{\gamma}\int_0^{\infty} e^{i\omega t} e^{-\gamma t} dt\right\}$$

$$\Rightarrow S_v(\omega) = \frac{\Gamma}{2\pi\gamma}\cdot \mathrm{Re}\left\{\left.\frac{e^{-\gamma t + i\omega t}}{i\omega - \gamma}\right|_0^{\infty}\right\}$$

$$\Rightarrow S_v(\omega) = \frac{\Gamma}{2\pi\cancel{\gamma}} \times \frac{\cancel{\gamma}}{\omega^2+\gamma^2}$$

$$\Rightarrow \boxed{S_v(\omega) = \frac{1}{\omega^2+\gamma^2}\,\frac{\Gamma}{2\pi}}$$

$S_\varepsilon(\omega)$: Power spectrum of white noise

Autocorelation function from $S_v(\omega)$:

Using inverse fourier transform.

$$\langle v(0)\,v(t)\rangle = \int_{-\infty}^{\infty} S_v(\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{\Gamma}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2+\gamma^2}\, d\omega$$

$$= \frac{\Gamma}{\cancel{2}\pi}\,\cancel{2}\int_0^{\infty} \frac{\cos\omega t}{\omega^2+\gamma^2}\, d\omega$$

$$= \frac{\Gamma}{\pi}\times\frac{\pi}{2\gamma}\, e^{-\gamma t}$$

$$\Rightarrow \boxed{\langle v(0)\,v(t)\rangle = \frac{\Gamma}{2\gamma}\, e^{-\gamma t}}$$

Sum Rule:

variance of random process

$$\boxed{\langle v^2\rangle = \int_{-\infty}^{\infty} S_v(\omega)\, d\omega}$$

$$\Rightarrow \langle v^2\rangle = \int_{-\infty}^{\infty}\frac{\Gamma}{2\pi}\,\frac{d\omega}{\omega^2+\gamma^2} = \frac{\Gamma}{2\pi}\times\frac{1}{\gamma}\times \left.\mathrm{Tan}^{-1}\left(\frac{\omega}{\gamma}\right)\right|_{-\infty}^{\infty}$$

$$\Rightarrow \boxed{\langle v^2\rangle = \frac{\Gamma}{2\gamma}}$$

Poison Birth & Death Process :

$$\xrightarrow{\alpha} X \xrightarrow{\beta} \qquad \frac{dX}{dt} = \alpha - \beta X$$

eq^{ns} & comparision

$$\frac{d\Delta X}{dt} = -\underbrace{\beta}_{\to \gamma} \Delta x + \eta(t)$$

where $\langle \eta(t)\eta(t')\rangle = \underbrace{\frac{2\alpha}{\Omega}}_{\to \Gamma}\delta(t-t')$

shortcut : $\Gamma = \frac{2\alpha}{\Omega}$; $\gamma = \beta$

* $\langle x \rangle = \frac{\alpha}{\beta}$
* $\Delta x(\omega) = \frac{\eta(\omega)}{-i\omega + \beta}$
* $S_x(\omega) = \frac{1}{\omega^2+\beta^2} \times \frac{2\alpha}{\Omega} \times \frac{1}{2\pi}$
* $\langle \Delta x^2 \rangle = \frac{2\alpha}{\Omega} \times \frac{1}{2\beta} = \frac{\alpha}{\Omega\beta}$

$$X_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} X_2 \qquad \frac{dx_2}{dt} = \alpha X_T - (\alpha+\beta) X_2 \text{ ; where } X_1 + X_2 = X_T$$

eq^{ns} & comparision

$$\frac{d\Delta X_2}{dt} = -\underbrace{(\alpha+\beta)}_{\to \gamma} X_2 + \eta(t)$$

where $\langle \eta(t)\eta(t')\rangle = \underbrace{D}_{\to \Gamma}\delta(t-t')$

shortcut : $\Gamma = D$; $\gamma = \beta + \alpha$

$D = \langle \alpha X_1 + \alpha X_2 \rangle = \frac{2\alpha\beta}{\alpha+\beta}\frac{X_T}{\Omega}$

* $\langle x_2 \rangle = \frac{\alpha}{\alpha+\beta} X_T$
* $\Delta x(\omega) = \frac{\eta(\omega)}{-i\omega + (\beta+\alpha)}$
* $S_x(\omega) = \frac{1}{\omega^2+(\alpha+\beta)^2} \times S_\varepsilon(\omega)$
* $\langle \Delta x^2 \rangle = \frac{\pi}{\alpha+\beta} \cdot \frac{D}{2\pi} = \frac{\alpha\beta}{(\alpha+\beta)^2} X_T \times \frac{1}{\Omega}$

$$X_1 \underset{K_d}{\overset{K_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta} \phi$$

$$\frac{dx_2}{dt} = K_a X_T - (K_a + K_d) X_2 \text{; where } X_1 + X_2 = X_T$$

$$\frac{dY}{dt} = \alpha X_2 - \beta Y$$

$$\frac{d\Delta x_2}{dt} = -(K_a + K_d)\Delta x_2 + \eta_1(t)$$

$$\frac{d\Delta Y}{dt} = \alpha \Delta X_2 - \beta \Delta y + \eta_2(t)$$

shortcut (for x_2) : $\Gamma = D_1$; $\gamma = K_a + K_d$

$D_1 = \frac{2 K_a K_d}{K_a + K_d}\frac{X_T}{\Omega}$

shortcut (for Y) : $\Gamma = D_2$; $\gamma = \gamma$

$D_2 = \frac{2\alpha}{\Omega}\frac{K_a}{K_a+K_d} X_T$

* $\langle x_2 \rangle = \frac{K_a}{K_a + K_d} X_T$
* $\langle y \rangle = \frac{\alpha}{\beta}\langle x \rangle = \frac{\alpha}{\beta}\frac{K_a}{K_a+K_d} X_T$
* $\Delta x_2 = \frac{\eta_1(\omega)}{-i\omega + (K_a + K_d)} \Rightarrow \langle \Delta x_2^2 \rangle = \frac{K_a K_d}{(K_a+K_d)^2}\frac{X_T}{\Omega}$
* ΔY see in slides

Brownian motion with external force:

$$m\dot{v} + \gamma v = \eta(t) + F_{ext}(t)$$

$$m\langle \dot{v} \rangle + \gamma \langle v \rangle = F_{ext}(t)$$

$$\langle v(\omega) \rangle = \frac{F_{ext}(t)}{-i\omega + \gamma}$$

$$v(t) = \int \eta(t') e^{\gamma(t-t')} dt' + \int F_{ext}(t') \gamma^{(t'-t)} dt'$$

$$\langle v(t) \rangle = \int_{-\infty}^{t} F_{ext}(t') e^{\gamma(t'-t)} dt'$$

$$F_{ext} = E_0 \sin\omega_0 t$$

$$\langle v(t) \rangle = \int_{-\infty}^{t} E_0 \sin\omega_0 t' e^{\gamma t'} e^{-\gamma t} dt'$$

$$\langle v(t) \rangle = e^{-\gamma t} E_0 \int_{-\infty}^{t} \sin\omega_0 t' e^{\gamma t'} dt'$$

$$= e^{-\gamma t} E_0 \frac{e^{\gamma t'}(\gamma \sin\omega_0 t' - \omega_0 \cos\omega_0 t')}{\omega_0^2 + \gamma^2}\Bigg|_{-\infty}^{t}$$

$$\langle v(t) \rangle = e^{-\gamma t} \frac{E_0}{\sqrt{\omega_0^2 + \gamma^2}} \sin(\omega_0 t - \phi) \qquad \phi = \tan^{-1}\left(\frac{\omega_0}{\gamma}\right)$$

Simulation of Master equation

$$\xrightarrow{\alpha} X \xrightarrow{\beta} \phi \;\equiv\; \underbrace{\xrightarrow{\alpha} X}_{\substack{\text{reac}^n 1 \\ r_1 = \alpha}} + \underbrace{X \xrightarrow{\beta} \phi}_{\substack{\text{reac}^n 2 \\ r_2 = \beta n_X}}$$

$P\left(\begin{matrix}\text{reaction 1 happening} \\ \text{in time interval } \Delta t\end{matrix}\right) \to P_1 = r_1 \Delta t$

$P\left(\begin{matrix}\text{reaction 2 happening} \\ \text{in time interval } \Delta t\end{matrix}\right) \to P_2 = r_2 \Delta t$

P (nothing happens in time t)

$$P(N) = p(\text{not 1 \& not 2})$$

$$= \underbrace{p(\text{not 1})}_{} \cdot \underbrace{p(\text{not 2})}_{}$$

for $\Delta t \to (1 - r_1 \Delta t)$

for $t = N\Delta t \to \lim_{N\to\infty}\left(1 - r_1 \frac{t}{N}\right)^N$

$\Rightarrow \Delta t = \frac{t}{N}$

$= e^{-r_1 t}$

for $\Delta t \to (1 - r_2 \Delta t)$

for $t = N\Delta t \to \lim_{N\to\infty}\left(1 - r_2 \frac{t}{N}\right)^N$

$\Rightarrow \Delta t = \frac{t}{N}$

$= e^{-r_2 t}$

$$\Rightarrow p(N) = e^{-r_1 t} \cdot e^{-r_2 t} = e^{-(r_1 + r_2)t}$$

what is probablity of 1 & 2 not happening for t & 1 happening in Δt?

$$P(N \,\&\, 1) = P(N) P_1 = e^{-(r_1 + r_2)t} r_1 \Delta t$$