Noise power spectrum.

$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left| \int_0^T e^{i\omega t} x(t)\, dt \right|^2$$

Wiener Kinchin theorem

$$S_x(\omega) = \mathrm{Re} \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \left\langle x(0)\, x(t) \right\rangle \right|$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos\omega t \left\langle x(0)\, x(t) \right\rangle$$

For Brownian motion

$$\dot{v} = -\gamma v + \xi(t) \qquad \left\langle \xi(t)\, \xi(t') \right\rangle = \Gamma \delta(t-t')$$

power spectrum for white noise

$$S_\xi(t) = \mathrm{Re}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left\langle \xi(0)\, \xi(t) \right\rangle dt$$

$$= \frac{\Gamma}{2\pi}$$

power spectrum of $v(t)$

$$S_v(\omega) = \mathrm{Re}\ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \langle v(0)\, v(t) \rangle dt$$

$$= \mathrm{Re}\ \frac{1}{\pi} \frac{\Gamma}{2\gamma} \int_{0}^{\infty} e^{i\omega t}\, e^{-\gamma t}\, dt$$

$$= \mathrm{Re}\ \frac{\Gamma}{2\pi\gamma} \left. \frac{e^{-\gamma t + i\omega t}}{i\omega - \gamma} \right|_{0}^{\infty}$$

$$= \frac{\Gamma}{2\pi} \frac{1}{\gamma} \frac{\gamma}{\omega^2 + \gamma^2} = \frac{\Gamma}{2\pi} \frac{1}{\omega^2 + \gamma^2}$$

$$= \frac{1}{\omega^2 + \gamma^2} S_\xi(\omega)$$

Inverse fourier transform of $S_v(\omega)$ gives the autocorrelation function

$$\langle v(0)\, v(t) \rangle = \int_{-\infty}^{+\infty} S_v(\omega)\, e^{-i\omega t}\, d\omega.$$

$$\int^{+\infty} \quad e^{-i\omega t}$$

$$= \frac{\Gamma}{2\pi} \int_{-\infty} \frac{e}{\omega^2 + \gamma^2} d\omega .$$

Since the function is symmetric in ω

$$\langle v(0)\, v(t) \rangle = \frac{\Gamma}{2\pi} 2 \int_0^\infty \frac{\cos \omega t}{\omega^2 + \gamma^2} d\omega$$

$$= \frac{\Gamma}{\pi} \frac{\pi}{2\gamma} e^{-\gamma t} = \frac{\Gamma}{2\gamma} e^{-\gamma t}$$

we can get back the power spectrum by taking fourier transform on both sides of the equation

$$\dot{v} + \gamma v = \xi(t)$$

$$v(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \tilde{v}(\omega)\, d\omega$$

$$\Rightarrow v(\omega) = \frac{\xi(\omega)}{-i\omega + \gamma}$$

$$S_v(\omega) = \frac{1}{\quad} S_\xi(\omega)$$

$v(\omega)$

$\omega^2+\gamma^2$

We can calculate the variance in ~~...~~ $\langle v^2 \rangle$ by taking integral over all ω.

$$\langle v^2 \rangle = \int \langle v(\omega)\, v^*(\omega) \rangle\, d\omega = \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2+\gamma^2} \frac{\Gamma}{2\pi}$$

$$= 2\int_{0}^{\infty} \frac{d\omega}{\omega^2+\gamma^2} \frac{\Gamma}{2\pi}$$

$$\langle v^2 \rangle = \frac{\pi}{\gamma} \cdot \frac{\Gamma}{2\pi} = \frac{\Gamma}{2\gamma}$$

For poisson birth and death process

$$\xrightarrow{\alpha} x \xrightarrow{\beta}$$

$$\frac{dx}{dt} = \alpha - \beta x$$

$$\frac{d\Delta x}{dt} = -\beta \Delta x + \eta(t)$$

$$\langle \eta(t)\, \eta(t') \rangle = \frac{2\alpha}{\Omega}\delta(t-t')$$

Taking fourier transform on both sides

$$\Delta x(\omega) = \frac{\eta(\omega)}{-i\omega + \beta}$$

$$S_x(\omega) = \langle |\Delta x(\omega)|^2 \rangle = \frac{1}{\omega^2+\beta^2} S_\eta(\omega)$$

$$= \frac{1}{\omega^2+\beta^2}\left(\frac{2\alpha}{2\pi}\right)\frac{1}{\Omega}$$

$$\langle \Delta x^2 \rangle = \frac{\alpha}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\omega^2+\beta^2}\, d\omega = \frac{\alpha}{\Omega\, 2\pi}\,\frac{\pi}{\beta} = \frac{\alpha}{\beta}\frac{1}{\Omega}$$

$$\langle \Delta x^2 \rangle = \frac{1}{\Omega}\,\frac{\alpha}{\beta}\,;\quad \langle x \rangle = \frac{\alpha}{\beta}$$

For the reaction

$$X_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} X_2$$

$$\frac{dx_2}{dt} = \alpha x_T - (\alpha+\beta) x_2$$

$$\frac{d}{dt} \Delta x_2 = -(\alpha+\beta) x_2 + \eta(t)$$

$$\langle x_2 \rangle = \frac{\alpha}{\alpha+\beta} x_T$$

$$\langle \eta(t)\, \eta(t') \rangle = D\, \delta(t-t')$$

$$D = \langle \alpha x_1 + \alpha x_2 \rangle = \left(\frac{2\alpha\beta}{\alpha+\beta}\right) \frac{x_T}{\Omega}$$

$$\Delta x_2(\omega) = \frac{\eta(\omega)}{-i\omega + (\alpha+\beta)}$$

$$S_{\Delta x_2}(\omega) = |\Delta x_2(\omega)|^2 = \frac{1}{\omega^2 + (\alpha+\beta)^2} S_\eta(\omega)$$

$$\langle \Delta x_2^2 \rangle = \frac{\pi}{\alpha+\beta} \cdot \frac{D}{2\pi} = \frac{D}{2(\alpha+\beta)} \frac{1}{\Omega}$$

$$\langle \ldots \rangle = \frac{\alpha\beta}{\ldots} x_T \frac{1}{\ldots}$$

$$\langle \Delta x_2 \rangle = \qquad (\alpha+\beta)^2 \qquad \Omega$$

For combined process

$$X_1 \underset{k_d}{\overset{k_a}{\rightleftharpoons}} X_2 \xrightarrow{\alpha} Y \xrightarrow{\beta} \emptyset$$

$$\frac{dx_2}{dt} = k_a x_1 - k_d x_2$$

$$\frac{dy}{dt} = \alpha x_2 - \beta y$$

$$\frac{d}{dt} \Delta x_2 = -(k_a + k_d) \Delta x_2 + \eta_1(t)$$

$$\frac{d}{dt} \Delta y = \alpha \Delta x_2 - \beta \Delta y + \eta_2(t)$$

$$\langle x_2 \rangle = \frac{k_a}{k_a + k_d} x_T$$

$$\langle y \rangle = \frac{\alpha}{\beta} \frac{k_a}{k_a + k_d} x_T$$

$$\langle \eta_1(t)\, \eta_1(t') \rangle = D_1\, \delta(t-t')$$

where $D_1 = \dfrac{2 K_a K_d}{K_a + K_d} \dfrac{x_T}{\Omega}$

$$\langle \eta_2(t)\, \eta_2(t') \rangle = D_2\, \delta(t-t')$$

where $D_2 = \dfrac{1}{\Omega}\left(\cancel{\alpha}_{2s} + \beta y_s \right)$

$$= \frac{1}{\Omega}\, 2\alpha \langle x_2 \rangle = 2\alpha \frac{K_a}{K_a+K_d} x_T$$

Taking fourier transform

$$\Delta x_2(\omega) = \frac{\eta_1(\omega)}{-i\omega + (K_a + K_d)}$$

$$\langle \Delta x_2^2 \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2 + (K_a+K_d)^2} \frac{D_1}{2\pi}$$

$$= \frac{D_1}{(K_a+K_d)} = \frac{K_a K_d}{(K_a+K_d)^2} \frac{x_T}{\Omega}$$

Similarly

$$\Delta y(\omega) = \frac{\alpha \, \Delta x_2(\omega)}{-i\omega + \gamma} + \frac{\eta_2(\omega)}{-i\omega + \gamma}$$

$$S_y(\omega) = \frac{\alpha^2}{\omega^2 + \beta^2} S_x(\omega) + \frac{1}{\omega^2 + \beta^2} S_{\eta_2}(\omega)$$

$$= \frac{\alpha^2}{\omega^2 + \beta^2} \frac{S_{\eta_1}(\omega)}{\omega^2 + (K_a + K_d)^2} + \frac{1}{\omega^2 + \beta^2} S_\eta(\omega)$$

$$= \alpha^2 \frac{D_1}{2\pi} \frac{1}{(\omega^2 + \beta^2)(\omega^2 + (K_a + K_d)^2)}$$

$$+ \frac{1}{\omega^2 + \gamma^2} \frac{D_2}{2\pi}$$

$$I \quad \alpha^2$$

$$\langle \Delta y^2 \rangle = \frac{\alpha^2}{2\pi} \frac{2 K_a K_d}{(K_a + K_d)} \frac{x_T}{\Omega} \frac{\pi}{\beta (K_a + K_d)(K_a + K_d + \beta)}$$

$\alpha' \quad 1$

$$+ \quad \frac{\quad}{2\gamma} \frac{\cdot}{\Omega}$$

$$= \frac{\alpha'}{2\gamma\,\Omega}\left(1 + \frac{\alpha\, k_d}{(k_a + k_d)(k_a + k_d + \beta)}\right)$$

Brownian motion with external force

$$m\dot{v} + \gamma v = \eta(t) + F_{ext}(t)$$

$$m\langle\dot{v}\rangle + \gamma\langle v\rangle = F_{ex}(t)$$

$$\langle v(\omega)\rangle = \frac{F_{ex}(t)}{-i\omega + \gamma}$$

$$v(t) = \int \eta(t') e^{\gamma(t'-t)} dt'$$

$$+ \int F_{ext}(t') e^{\gamma(t'-t)} dt'$$

$$\langle v(t)\rangle = \int_0^t F_{ext}(t')\, e^{\gamma(t'-t)} dt'$$

Lets say $F_{ext}(t) = E_0 \sin \omega_0 t$

$$\langle v(t) \rangle = \int_0^t E_0 \sin \omega_0 t' \, e^{\gamma(t'-t)} dt'$$

$$= e^{-\gamma t} E_0 \int_{-\infty}^{t} \sin \omega_0 t' \, e^{\gamma t'} dt'$$

$$= e^{-\gamma t} E_0 \left. \frac{e^{\gamma t'}\left(\gamma \sin(\omega_0 t') - \omega_0 \cos(\omega_0 t')\right)}{\omega^2 + \gamma^2} \right|_{-\infty}^{t}$$

$$= e^{-\gamma t} \frac{E_0}{\sqrt{\omega^2 + \gamma^2}} \sin(\omega_0 t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{\omega_0}{\gamma}\right)$$

Simulation of Master equation

$$\xrightarrow{\alpha} X \xrightarrow{\beta} \phi$$

Two reactions

$$\xrightarrow{\alpha} X \qquad X \xrightarrow{\beta} \phi$$

rates $r_1 = \alpha$ $r_2 = \beta n_x$

The probability that reaction 1 happens ~~of~~ in time interval Δt

$$p_1 = r_1 \Delta t \qquad p_2 = r_2 \Delta t$$

Highly inefficient as nothing will happen in many consecutive time steps. Gillespie algorithm

probability p(nothing happens in time t)

$$= p(\text{not } 1 \ \& \text{ not } 2)$$

$$= p(\text{not } 1)\, p(\text{not } 2)$$

$$p_1 = r_1 \Delta t \qquad p(\text{not } 1) = (1 - r_1 \Delta t)$$

$$\Delta t = \frac{t}{N}$$

$$p(\text{not } 1) = \lim_{N \to \infty} \left(1 - r_1 \frac{t}{N}\right)^N = e^{-r_1 t}$$

$$e^{-r_2 t}.$$

$$p(\text{not } 2) \lim_{N \to \infty} \left(1 - r_2 \frac{t}{N}\right)^N = e^{-r_2 t}.$$

$$p(\text{not } 1 \ \& \ \text{not } 2) = e^{-(r_1 + r_2) t}$$

$$p(\text{not } 1 \ \& \ \text{not } 2 \text{ in time } t \ \& \ 1 \text{ happens in } \Delta t)_{\Delta t}$$

$$= e^{-(r_1 + r_2) t} \, r_1 \, \Delta t.$$

L

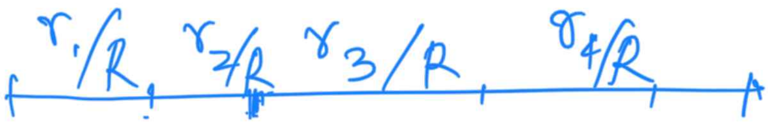
$$p(\quad) = R e^{-(r_1+r_2)t} \frac{r_1}{R}$$

$$r = -\frac{1}{R}\log\left(\frac{1}{d_1}\right)$$

$$\frac{r_1}{R} \qquad \frac{r_2}{R}$$

General case

$$r = \frac{1}{R}\log\left(\frac{1}{d_1}\right) \qquad R = \sum r_i$$



$$\sum_{i=1}^{j} r_i < d_2 < \sum$$