Probability and Statistics
Assignment 3

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→ Problem 1

from the MGF ~~we can~~, since it is of the form

$$\sum p_x e^{xt}$$

we can infer that

$P_x(-20) = 1/10$, $P_x(-3) = 1/5$, $P_x(4) = 3/10$ and $P_x(5) = 2/5$

$\therefore P(|X| \leq 2) = \underline{0}$

→ Problem 2

① Let $Y = kX$

$$M_y(t) = E[e^{kXt}]$$
$$= E[(e^{Xt})^k]$$
$$= E[e^{X(kt)}]$$
$$= M_X(kt)$$

② Let $Y = X + k$

$$M_y(t) = E[e^{(X+k)t}]$$
$$= E[e^{Xt} \times e^{kt}]$$
$$= e^{kt} E[e^{Xt}]$$
$$= e^{kt} M(t)$$

③ $M_y(t) = E[e^{yt}] = E[e^{(x+k)t}]$

$$= \cancel{\int e^{yt}\, PDF(y)\, dy} = e^{kt} M(t)$$

$$\cancel{\int e^{(x+k)t}}$$

④ $M_y(t) = E[e^{2xt}] = M(2t)$

→ Problem 3

Probability that a person gets their hat back is $\frac{1}{n} = p$

$$P_X(K) = \sum_{K=0}^{n} K \times p^k (1-p)^{n-k} \times \binom{n}{k}$$

$$= \sum \cancel{K} \times \frac{n}{\cancel{K}} \binom{n-1}{K-1} \times p^K (1-p)^{n-k}$$

$$= n \sum \binom{n-1}{K-1} \times p \times p^{K-1} \times (1-p)^{(n-1)-(k-1)}$$

$$= np \sum \binom{n-1}{k-1} \times p^{k-1} (1-p)^{(n-1)-(k-1)}$$

by binomial theorem,

$$= np \, (p + 1 - p)^{n}$$

$$= \underline{np} = \cancel{n} \frac{1}{\cancel{n}} = \underline{1}$$

→ Problem 4

① Expected illumination

$= E[A] + E[B] + E[A]$

$= 0.25 + 0.5 + 0.25$

$= 1$ year.

② Let a random variable X be the number of A type bulbs in ~~until~~ a bulb used excluding the first.

then,

$X \sim \text{Binomial}(a, p)$

and $E[X] = ap$

note that $a = n$ since number of replacements is the same as (number of bulbs used -1).

$$E[X] = np$$

expected total illumination time is

$$= 0.25 + 0.25np + 0.5n(1-p)$$

→ Problem 5

① Probability of a 2-loss round occurring

$$= P_G(-2) \times P_G(-2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

PMF of total rounds played (A)

$$P_A(k) = \left(\frac{8}{9}\right)^{k-1} \frac{1}{9}$$

② Probability of a loss for Mahesh (or Vinay)

$$= P_G(-2) = \frac{1}{3}$$

$$P_Z(k) = \left(\frac{2}{3}\right)^{k-2} \left(\frac{1}{3}\right)^2 \binom{k}{2} \times \frac{1}{3}$$

③ Probability of a win $= 2/3$

Let B be the number of rounds before Mahesh (or Vinay) gets a win

$$P_B(K) = \left(\frac{1}{3}\right)^{K-1} \frac{2}{3}$$

B follows a geometric distribution

$$\therefore E[B] = \frac{1}{P} = \frac{3}{2}$$

E[N] = Max(E[no. of rounds before Mahesh's win], E[no. of rounds before Vinay's win])

these two are the same since they have the same PMF and is equal to $E[B] = 3/2$

$$\therefore E[N] = 3/2$$

→ Problem 6

① Total no. of ways to colour K balls

$$= \binom{254+K}{25\cancel{8}4}$$ (no. of non negative sols to $x_1 + x_2 + \cdots x_{255} = K$)

no. of ways to colour K balls with n colours.

$$= \binom{K-1}{n-1}$$ (no. of positive sols to $x_1 + x_2 + \cdots + x_n = K$)

$$\therefore P_N(n) = \binom{K-1}{n-1} \div \binom{254+K}{254}$$

~~Since K is a fixed value,~~

$$E[N] = \sum_{n=1}^{K} n \times \binom{K-1}{n-1} \div \binom{254+K}{254}$$

$$= \frac{K-1}{2} + 1$$ ($\because \binom{K-1}{n-1}$ are highest at $n = \frac{K-1}{2}+1$)

$$= \frac{K+1}{2}$$

② if a colours are assigned to more than one ball, it can be described by the equation:

$$(x_1+2)+(x_2+2)+(x_3+2)+\cdots+(x_a+2)+(x_{a+1})+x_{a+2}+\cdots+x_{255}=K$$

total no. of ways to assign colours remains

$$= \binom{254+K}{254}$$

$$P_A(a) = \binom{255}{a} \times \frac{\left(\begin{matrix}\text{no. of solutions to:}\\ x_1+x_2+\cdots+x_{255} = K-2a\end{matrix}\right)}{\binom{254+K}{254}}$$

$$= \binom{255}{a} \times \binom{254+K-2a}{254} \div \binom{254+K}{254}$$

$$= \left(\frac{255!}{a!\,(255-a)!} \times \frac{(254+K-2a)!}{254!\times(K-2a)!} \div \frac{254!\times K!}{(254+K)!}\right)$$

$$E[A] = \sum_{a=0}^{255} a\binom{255}{a} \times \binom{254+K-2a}{254} \div \binom{254+K}{254}$$

→ Problem 7

$$\lambda = \frac{500}{500} = 1$$

if X is the number of misprints on a page.

$$P(X=3) = \frac{1^3 e^{-1}}{3!}$$

$$= \frac{1}{3!\,e} = \underline{\underline{0.06}}$$

$$P(x=2) = \cancel{e}\,\frac{1}{2!e} = 0.18$$

$$P(x=1) = \frac{1}{e} = 0.367$$

$$P(x=0) = \frac{1^0 e^{-1}}{\cancel{e}\,0!} = \frac{1}{e} = 0.367$$

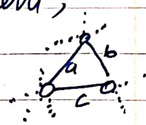
$$\therefore P(x \geq 3) = 1 - P(x \leq 2)$$
$$= 1 - (P(x=2) + P(x=1) + P(x=0))$$
$$= 1 - 0.18 - 0.367 - 0.367$$
$$= \underline{0.086}$$

→ <u>Problem 8</u>

① $P(X_e = 1) = \frac{1}{3} \times \frac{1}{3} \times 3 = \cancel{\frac{8}{9}}\ \frac{1}{3}$

$P(X_e = 0) = \cancel{\frac{8}{9}}\ \frac{2}{3}$

Assuming all X_e are independent,

consider a subgraph as follows: 

$$P_{X_a}(1) \times P_{X_b}(1) \times P_{X_c}(1) = P_{X_a, X_b, X_c}(1,1,1)$$

$$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} =$$

$$\frac{1}{9} \times \frac{1}{3} = \frac{1}{27}$$

but we know that if a b are monochromatic edges the c must be one.

i.e $P_{x_a, x_b, x_c}(1,1,1) = P_{x_a}(1) \times P_{x_b}(1)$

$= 1/9 \neq 1/27$

hence the ~~our~~ assumption is wrong and thus all X_e are not independent.

~~(2) Ex(a) = Let E be the number of~~

(2) $P_Y(a) = \left(\frac{2}{3}\right)^a \left(\frac{1}{3}\right)^{|E|-a}$

$$E[Y] = \sum_{a=0}^{|E|} a \times \left(\frac{2}{3}\right)^a \times \left(\frac{1}{3}\right)^{|E|-a}$$

$$= \sum a \times 2^a \left(\frac{1}{3}\right)^a \times \left(\frac{1}{3}\right)^{|E|-a}$$

$$= \sum a \times 2^a \times \left(\frac{1}{3}\right)^{|E|}$$

$$= \left(\frac{1}{3}\right)^{|E|} \sum_{a=0}^{|E|} a 2^a$$

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→ Problem 9

① Each flip is independent of each other and hence these ~~......~~ trials are also independent.

$\therefore$ P(next two trials give all Tails | previous trial was all tails) = P(next two trials give tails)

$$= \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{22}$$

② For M coins, expected number of trials before getting all same side

$E[A_m]$ = expectation of a geometric dist

$= \frac{1-P}{P}$ where P is the prob of getting all ~~.~~ same in a trial

$$= \frac{1-(1/2)^{m-1}}{(1/2)^{m-1}}$$

$$= \frac{1}{(1/2)^{m-1}} - 1 = 2^{m-1} - 1$$

$$\therefore E[X] = \sum_{m=2}^{M} E[A_m]$$

$$= \sum 2^{m-1} - 1$$

$$= -(M-1) + \sum 2^{m-1}$$

$$= -(M-1) + 2^M - 4$$

$$= 2^M - 4 - M + 1 = 2^M - M - 3$$