MA6.101 (40 marks)

Probability And Statistics **ASSIGNMENT 5** Due by : 11:59 PM, Nov 25

Problem 1

Suppose that 2 batteries are randomly chosen without replacement from the following group of 12 batteries: 3 new, 4 used (working), 5 defective.
Let X denote the number of new batteries chosen.
Let Y denote the number of used batteries chosen.
a) Find $f_{XY}(x, y)$, i.e., the joint probability distribution.

b) Find E[X].

(4 marks)

Problem 2

(a) Find $\Gamma(7/2)$.
(b) Find the value of the following integral:

$$I = \int_0^\infty x^7 e^{-5x} dx$$

(4 marks)

Problem 3

Let Q be a continuous random variable with PDF:

$$f_Q(q) = \begin{cases} 6q(1 - q) & if\ 0 \le q \le 1 \\ 0 & otherwise \end{cases}$$

This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$P(X = 1 \mid Q = q) = q$$

Find $f_{Q|X}(q|x)$ for x $\in$ {0, 1} and all q. (2 marks)

Problem 4

A surface has infinite parallel lines with equal spacing of length d between them. We have a needle of length l which we throw randomly on the surface. What is the probability that the needle intersects a line? Assume $l < d$, and that the needle lies in the same plane as the surface. (4 marks)

Problem 5

Consider two random variables X and Y with the range

$$R_{XY} = \{ (i, j) \in \mathbb{Z}^2 \mid i, j \geq 0, |i - j| \leq 1 \},$$

and joint PMF given by

$$P_{XY}(i, j) = \frac{1}{6 \cdot 2^{min(i,j)}}, \quad for\ (i, j) \in R_{XY},$$

(a) Show R_{XY} in the xy plane graphically.
(b) Find the marginal PMFs $P_X(i)$, $P_Y(j)$.
(c) Find P(X = Y | X < 2).
(d) Find $P(1 \leq X^2 + Y^2 \leq 5)$.
(e) Find P(X = Y).
(f) Find E[X | Y = 2].
(g) Find Var(X | Y = 2).

(7 marks)

Problem 6

Suppose $X \sim Uniform(1, 2)$ and given X = x, Y is an exponential random variable with parameter λ = x.
(a) Find E[Y].
(b) Find Var(Y).

(4 marks)

Problem 7

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then prove:

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

(4 marks)

Problem 8

Let X_1 be a normal random variable with mean 2 and variance 3 and let X_2 be a normal random variable with mean 1 and variance 4. Assume that X_1 and X_2 are independent.

(a) What is the distribution of the linear combination $Y = 2X_1 + 3X_2$?
(b) What is the distribution of the linear combination $Y = X_1 - X_2$?

(4 marks)

Problem 9

Consider the unit disc:

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

Suppose that we choose a point (X,Y) uniformly at random in D. That is, the joint PDF of X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & otherwise \end{cases}$$

(a) Find the constant c.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(c) Find the conditional PDF of X given Y = y, where $-1 \leq y \leq 1$.

(d) Are X and Y independent?

(4 marks)

Problem 10

Two components of a laptop have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f_{XY}(x, y) = \begin{cases} x\, e^{-x(1+y)} & x \geq 0, y \geq 0 \\ 0 & otherwise \end{cases}$$

(a) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
(b) What is the probability that the lifetime of at least one component exceeds 1 year?

(3 marks)