Probability and Statistics
Assgn - 2

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→ Problem 1

(1) $P_x(y) = (0.2)^y (0.8)^{10-y} \times {}^{10}C_y$

(2) $E(X) = \sum_{y=0}^{10} P_x(y) \times y$

$= \sum_{y=0}^{10} {}^{10}C_y (0.2)^y (0.8)^{10-y} \times y$

$= 2$

(3) $Var(X) = E((X-2)^2)$

$= \cancel{\sum_{y=0}^{10} {}^{10}C_{(y-2)} (0.2)^{(y-2)} (0.8)^{(8-y)^2} \times (y-2)^2}$

$= \sum_{y=0}^{10} {}^{10}C_y (0.2)^y (0.8)^{10-y} (y-2)^2$

$= 1.6$

(3) $E(Y) = \sum_{x=0}^{10} P_x(x)(2x-3) = 2E(X) - 3$

$= 1$

$Var(Y) = \sum_{x=0}^{10} P_x(x)(2x-5)^2$

$= 7.4$

(4) $E(Z) = \sum_{x=0}^{10} P_x(x)\, x^2 = 5.6$

→ Problem 2

① range of $X = \{0.2, 0.4, 0.5, 0.8, 1\}$

② $P(X \leq 0.5)$

$= P_X(0.2) + P_X(0.4) + P_X(0.5)$

$= 0.1 + 0.2 + 0.2 = \underline{0.5}$

③ $P(0.25 < X < 0.75)$

$= P_X(0.4) + P_X(0.5)$

$= \underline{0.4}$

④ $E(X) = \sum_{y} P_X(y) \times y$

$= 0.1 \times 0.2 + 0.2 \times 0.4 + 0.2 \times 0.5 + 0.3 \times 0.8 + 0.2 \times 1$

$= 0.64$

$Var(X) = \sum_{y} P_X(y) \times (y - \mu)^2$

$= \underline{0.0684}$

→ Problem 3

Using $Y = X(X-1)(X-2)$:

x	y
0	0
1	0
2	0
3	6

$\therefore$

$P_Y(0) = P_X(0) + P_X(1) + P_X(2)$

$= 0.2 + 0.2 + 0.3$

$= 0.7$

$P_Y(6) = 0.3$

$P_Y(\text{otherwise}) = 0$

→ Problem 4

① ~~Let X be the number of cards dealt before a king is dealt.~~

~~With c cards dealt from a deck and all kings in it the probability of drawing a king is~~ $p = \frac{4}{52-c}$ and $p^c = \frac{52-c-4}{52-c}$

The probability of a 13-card deals in which the first card is a king is $4/52$.
Since there are an equivalent number of deals in which the cards are reversed, the probability of being dealt a king as the 13th card is equivalent.
i.e. required probability is $\underline{1/13}$

② no. of deals of length 12 without a king $= {}^{48}P_{12}$

no. of deals of length 13 with a king at the end $= {}^{48}P_{12} \times 4$

total no of 13 card deals $= {}^{52}P_{13}$

$\therefore$ required probability $= \frac{{}^{48}P_{12} \times 4}{{}^{52}P_{13}} \approx \underline{0.0337}$

→ Problem 5

1pm to 3pm is an interval of 2 hours
$\therefore$ expected no. of customers is $\lambda = 20 \times 2 = 40$

since $X \sim \text{Poisson}(\lambda)$

$$P_X(y) = \frac{\lambda^y e^{-\lambda}}{y!}$$

and so

$$P_X(15 < X < 25) = \sum_{y=16}^{24} P_X(y)$$

$$= \sum_{y=16}^{24} \frac{40^y e^{-40}}{y!}$$

$$= \underline{4.47 \times 10^{-3}}$$

→ Problem 6

The variable X ~~is~~ follows a geometric progression

i.e for $k \in \mathbb{Z}^+$, $P_X(k) = p(1-p)^{k-1}$

$$E(X) = \sum_{j=1}^{\infty} y \times p(1-p)^{y-1}$$

$$= 1p + 2p(1-p) + 3p(1-p)^2 + \cdots \rightarrow (1)$$

$$(1-p)E(X) = 1p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + \cdots \rightarrow (2)$$

$$(1) - (2) = E(X) - (1-p)E(X)$$

$$= p + p(1-p) + p(1-p)^2 + \cdots$$

$$= p[1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots]$$

$$= p \frac{1}{p} = 1$$

$$E(x) - [E(x) - pE(x)] = 1$$

$$\cancel{E(x)} - \cancel{E(x)} + pE(x) = 1$$

$$E(x) = \frac{1}{p}$$

$$Var(x) = E(x^2) - E(x)^2$$

$$E(x(x-1) + x) - \left(\frac{1}{p}\right)^2$$

$$= \sum_{x=1}^{\infty} x(x-1)p(1-p)^{x-1} + \frac{1}{p} - \frac{1}{p^2}$$

$$= [0 + 2p(1-p) + 6p(1-p)^2 + \cdots] + \frac{1}{p} - \frac{1}{p^2}$$

$$= 2p(1-p)[1 + 3(1-p) + 6(1-p)^2 + \cdots] + \frac{1}{p} - \frac{1}{p^2}$$

(Using Binomial theorem)

$$= 2p(1-p)(1-(1-p))^{-3} + \frac{1}{p} - \frac{1}{p^2}$$

$$= 2p(1-p)\,p^{-3} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2\cancel{p}(1-p)\cancel{p^{3}} + p - 1}{p^2} = \frac{2 - 2p + p - 1}{p^2}$$

$$= \cancel{\frac{2p^{-2} - 2p^{-1} + p - 1}{p^2}} = \frac{1-p}{p^2}$$

→ ~~Problem~~ Problem 7

$$P_x(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad P_y(k) = \frac{\mu^k e^{-\mu}}{k!}$$

$Z = X + Y$

$P(Z) = P(X+Y=k)$

$$= \sum_{i=0}^{k} P_x(i)\, P_y(k-i)$$

$$= \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \frac{\mu^{k-i} e^{-\mu}}{(k-i)!}$$

$$= \frac{e^{-\lambda} e^{-\mu}}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!\,i!} \lambda^i \mu^{k-i}$$

$$= \frac{e^{-\lambda} e^{-\mu}}{k!} (\mu+\lambda)^k$$

∴ $Z = X + Y$ follows a Poisson dist with $\lambda = (\mu + \lambda)$

→ Problem 8

Probability of getting any one grade in k papers is the expected value of a geometric distribution i.e. 1/6

With the first paper we will always get a new grade.

with every subsequent paper the expected number of papers to get a new grade is $\frac{6}{\text{no. of unattained grades}}$

$\therefore$ total expected papers before getting every grade is

$$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = \underline{14.7}$$

→ Problem 9

Every second there is a chance of a mosquito landing at 0.5.
And it bites 0.2 of the time.
Hence every second there is a $0.5 \times 0.2 = 0.1$ chance of a ~~b~~ bite.

Let X be the no. of seconds before your next bite.
then

$$P_X(K) = p(1-p)^{K-1}, \quad \text{where } p = 0.1$$ which is a geometric distribution.

The expected value of X is $1/p$
hence expected time between bites is 10 secs.

The variance of X is $1-p/p^2$.
hence variance of time between bites is 90

Problem 10 :

$$E(x) = \mu_x$$

$$\sigma_x^2 = (\mu_x)^2 + \mu_{x^2}$$

where $\mu_{x^2} = \sigma_x^2 - (\mu_x)^2$

$$Z = 3X + 4Y$$

$$E(Z) = E(3X + 4Y)$$
$$= E(3X) + E(4Y)$$
$$= 3\mu_x + 4\mu_y$$

$$Var(Z) = Var(3X + 4Y)$$
$$= Var(3X) + Var(4Y)$$
$$= 9\sigma_x^2 + 16\sigma_y^2$$