Date
Page

Stochastic Processes
Assignment-1

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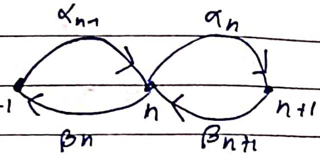
1.

$\cancel{\frac{d}{dt}} \langle n(t) \rangle = \sum n p(n,t)$



Consider,

$$\frac{d\langle n^2(t)\rangle}{dt} = \sum_n n^2 \frac{dp(n,t)}{dt}$$

The master equation is,

$$\frac{dP(n,t)}{dt} = \alpha_{n-1} P(n-1,t) + \beta_{n+1} P(n+1,t) - \alpha_n p(n,t) - \beta_n P(n,t)$$

Substituting in ①,

$$d\langle n^2(t)\rangle = \sum n^2 \left[\alpha_{n-1} P(n-1,t) + \beta_{n+1} P(n+1,t) - \alpha_n P(n,t) - \beta_n P(n,t) \right]$$

Using $n-1 \to n$, $n+1 \to n$,

$$= \sum (n+1)^2 \alpha_n p(n,t) + \sum (n-1)^2 \beta_n P(n,t) - \sum n^2 (\alpha_n + \beta_n) p(n,t)$$

$$= \sum \alpha_n \left[(n+1)^2 - n^2 \right] p(n,t) + \sum \beta_n \left[(n-1)^2 - n^2 \right] P(n,t)$$

$$= \sum \alpha_n (2n+1) P(n,t) + \sum \beta_n (1-2n) P(n,t)$$

$$= \sum 2n (\alpha_n - \beta_n) p(n,t) + \sum (\alpha_n + \beta_n) P(n,t)$$

$$\frac{d\langle n^2(t)\rangle}{dt} = \langle 2n(\alpha_n - \beta_n) \rangle + \langle \alpha_n + \beta_n \rangle$$

4\. For $\frac{d\,P(t)}{dt} = W[P(t)]$

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

When W_{ij} satisfies $i=j$,

$$W_{ij} = -\sum_i W_{ij} \quad \text{where } i \neq j$$

$$\therefore W = \begin{bmatrix} -(W_{21}+W_{31}) & W_{12} & W_{13} \\ W_{21} & -(W_{12}+W_{32}) & W_{23} \\ W_{31} & W_{32} & -(W_{13}+W_{23}) \end{bmatrix}$$

2\. ~~For~~ In a balanced condition,

$$W_{kl}\, P^{st}(l) = W_{lk}\, P^{st}(k) \longrightarrow 1$$

P^{st} is ~~the~~ P under a time independent stationary condition

$$\frac{d\,P^{st}}{dt} = WP$$

$$= \sum^{N} W_{jk}\, P^{st}(k) = \sum^{N} W_{kj}\, P^{st}(j)$$

where $j \neq k$.

From 1, $\frac{P^{st}(l)}{P^{st}(k)} = \frac{W_{lk}}{W_{kl}}$

$\cancel{p^{st} = p^{eq}}$ $\cancel{\sum^{l} p^{eq}(l) = 1}$

Denoting p^{eq} as the equilibrium dist.

Consider the normalization condition,

$$\sum^{l} p^{eq}(l) = 1$$

Dividing by $p^{eq}(K)$,

$$\sum^{l} \frac{p^{eq}(l)}{p^{eq}(K)} = \frac{1}{p^{eq}(k)}$$

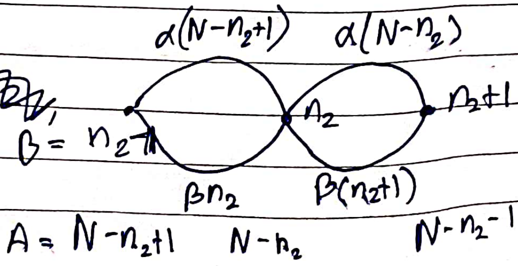
$$\frac{p^{eq}(K)}{p^{eq}(k)} + \sum_{l \neq K} \frac{W_{lK}}{W_{Kl}} = \frac{1}{p^{eq}(k)}$$

$$p^{eq}(K) = \left[1 + \sum_{l \neq K} \frac{W(l|k)}{W(K|l)} \right]^{-1}$$

3.

Master equation: $\underset{n_1}{A} \underset{\alpha}{\overset{\beta}{\rightleftharpoons}} \underset{n_2}{B}$

$$\frac{d\,p(st)}{dt} = \alpha\, p(N-n, t) - \beta\, P(N, t)$$



The Master equation:

$$\frac{d\,P(n_2,t)}{dt} = \alpha (N-n_2+1)\,P(n_2-1,t) + \beta(n_2+1)\,p(n_2+1,t) - (\beta n_2 + \alpha(N-n_2))\,p(n_2,t)$$

Given generating function

$$f(z,t) = \sum_{n_2} z^{n_2}\,p(n_2,t)$$

$$\frac{\partial f}{\partial t} = \sum_{n_2} z^{n_2}\,\frac{\partial P(n_2,t)}{\partial t}$$

$$= \sum_{n_2}\Big[z^{n_2}\alpha(N-(n_2-1))\,p(n_2-1,t) + z^{n_2}\beta(n_2+1)\,p(n_2+1,t) - z^{n_2}\alpha N\,p(n_2,t) - z^{n_2}(\beta-\alpha)n_2\,p(n_2,t)$$

Now,

$$\frac{\partial f(z,t)}{\partial t} = \alpha N z f(z,t) - z^2\alpha\frac{\partial f(z,t)}{\partial z} + \beta\frac{\partial f(z,t)}{\partial z} - \alpha N f(z,t) + z(\alpha-\beta)\frac{\partial f(z,t)}{\partial z}$$