Probability and Statistics

UG2, Core course, IIIT,H

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X, Y

Outline

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Method of Transformation

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We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

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2. $g(x)$ is a strictly increasing function

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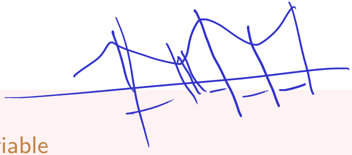
1. $g(x)$ is differentiable
2. $g(x)$ is a strictly increasing function
    - That is, if $x_1 < x_2$, then $g(x_1) < g(x_2)$

Method of Transformation



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   - That is, if $x_1 < x_2$, then $g(x_1) < g(x_2)$

We can directly find the PDF of Y using the following formula

$$f_Y(x) = \begin{cases} \dfrac{f_X(x_1)}{g'(x_1)} = f_X(x_1) \cdot \dfrac{dx_1}{dy} & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have a solution} \end{cases}$$

Proof of Method of Transformation for strictly increasing...

Given that g is strictly incr. $\Rightarrow g^{-1}$ is well-defined, i.e., g is 1-1 & onto.

$\Rightarrow g^{-1}$ exists. For each y, $\exists\ x_1$ s.t.

$g(x_1) = y \Rightarrow \boxed{\dfrac{dy}{dx_1} = g'(x_1)}$ ② &

① $\boxed{x_1 = g^{-1}(y)}$ $\longrightarrow$ ⊛.

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$

$= P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$

To find the PDF of Y, we diff w.r.t. y

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\underbrace{g^{-1}(y)}_{x_1}) = F_X'(x_1) \cdot \frac{dx_1}{dy} = F_X'(x_1) \cdot \frac{1}{g'(x_1)}$$

$$\boxed{f_Y(y) = \frac{F_X'(x_1)}{g'(x_1)}, \quad g(x_1) = y. \qquad = 0 \text{ otherwise}}$$

Recall Chain rule of diff.

$f(x) = u(v(x))$

$$\frac{d}{dx} f(x) = \frac{d}{dx}(u(v(x))) = \frac{du}{dv} \cdot \frac{dv}{dx} = u' \cdot \frac{du}{dx}$$

Proof of Method of Transformation for strictly decreasing...

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$

Since g is strictly decr.,

$P(X \geq g^{-1}(y)) = 1 - P(X \le g^{-1}(y))$

$= 1 - F_X(g^{-1}(y))$

To find PDF, we diff as before.

$f_Y(y) = \frac{d}{dy}\left(1 - F_X(\underbrace{g^{-1}(y)}_{x_1})\right)$

$= -F_X'(x_1) \cdot \frac{dx_1}{dy} = \frac{-F_X'(x_1)}{g'(x_1)}$

$\left[\text{Since } \frac{dy}{dx_1} = g'(x_1) \text{ as before}\right]$

To summarize:

$$f_Y(y) = \begin{cases} \frac{-F_X'(x_1)}{g'(x_1)}, & g(x_1) = y \\ 0, & \text{otherwise} \end{cases}$$

Method of Transformation for Monotonic Function for Monotonic Functions...

Increasing or decreasing.

Method of Transformation for Monotonic Function for Monotonic Functions...

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function.

Method of Transformation for Monotonic Function for Monotonic Functions...

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function. Let $Y = g(X)$.

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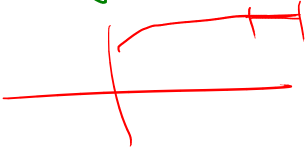
$$f_Y(y) = \begin{cases} \dfrac{f_X(x_1)}{|g'(x_1)|} = f_X(x_1) \cdot \left|\dfrac{dx_1}{dy}\right| & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have a solution} \end{cases}$$

It covers both st. incr. & decr. formulas in one.

Strictly incr: $g'(\cdot) > 0 \Rightarrow |g'(x_1)| = g'(x_1)$

Strictly decr.: $g'() < 0 \Rightarrow |g'(x_1)| = -g'(x_1)$



Example: Using Method of Transformation to Find PDF of Function of Random Variable

Example: Method of Transformation

Consider the PDF of the continuous random variable X

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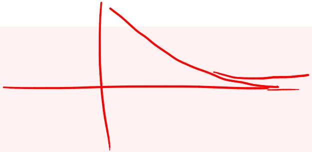
Consider the PDF of the continuous random variable X

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 4x^3 = 1$$

$\Rightarrow$ valid PDF

Example: Using Method of Transformation to Find PDF of Function of Random Variable



Example: Method of Transformation

Consider the PDF of the continuous random variable X

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = \dfrac{1}{X}$. Find $f_Y(y)$.

$Y = g(x) = 1/X$

Recall

$$f_Y(y) = \begin{cases} \dfrac{f_X(x_1)}{|g'(x_1)|}, & g(x_1) = y \\ & \text{oth.} \end{cases}$$

- g is diff. in $(0,1]$
- g is monotonic in $[0,1]$, strictly dec.

$g'(x) = \dfrac{-1}{x^2}$, $R_Y = [1, \infty)$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{4x_1^3}{|-1/x^2|} = 4x_1^5 = \frac{4}{y^5}$$

Summ:

$$f_Y(y) = \begin{cases} 4/y^5, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Method of Transformation for Piecewise Continuous Functions...

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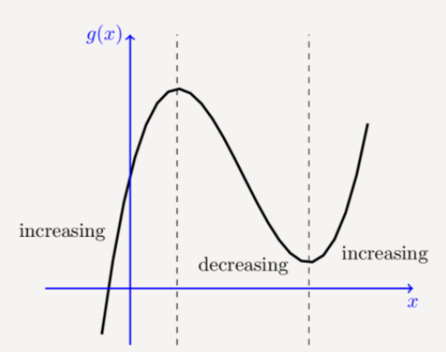


Figure: Partition a function to monotone parts

Method of Transformation for Piecewise Continuous Functions...

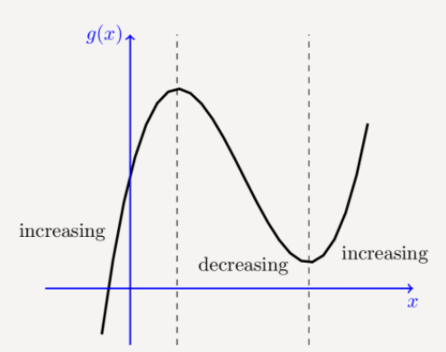
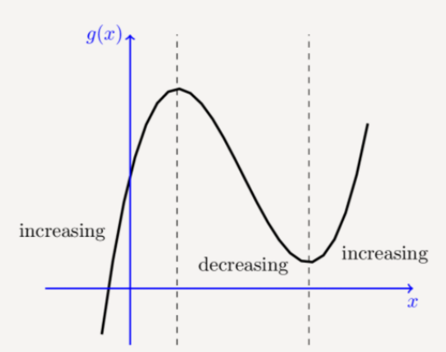


Figure: Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X.

Method of Transformation for Piecewise Continuous Functions...



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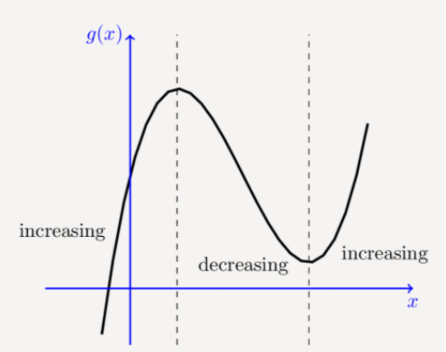

Figure: Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X. Let $Y = g(X)$. Assuming that we can partition R_X into finite number of intervals such that $g(x)$ is strictly monotone and differentiable on each partition.

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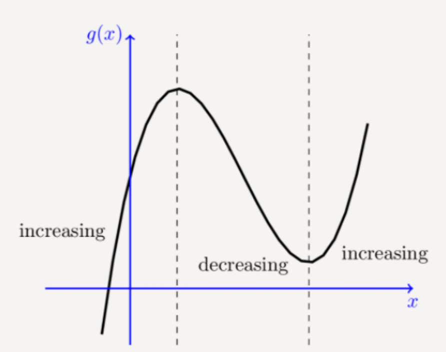

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$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \sum_{i=1}^{n} f_X(x_i) \cdot |\frac{dx_i}{dy}|$$

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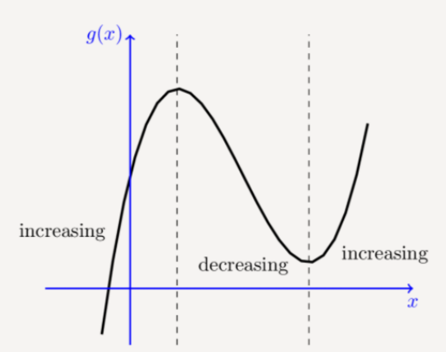

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where $x_1, x_2, \ldots, x_n$ are real solutions to $g(x) = y$.

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Consider the PDF of the random variable X

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Consider the PDF of the random variable X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

and let $Y = X^2$.

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$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

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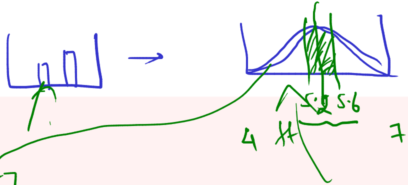
Example: Method of Transformation...

$-(x-\mu)^2$

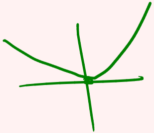


Example

Consider the PDF of the random variable X

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},} \quad \text{for all } x \in \mathbb{R}$$

- Gaussian
- Normal Distr.



and let $Y = X^2$. Find the PDF $f_Y(y)$.

- Does this satisfy the criteria for applying method of transformation?

Neither strictly incr. nor strictly decr.

However, strictly incr. on $(0,\infty)$ & strictly decr. $(-\infty,0)$

Example: Method of Transformation...

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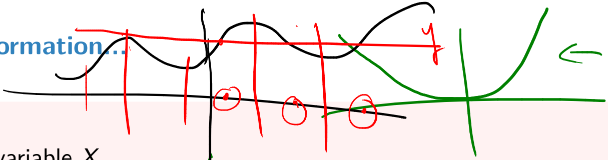

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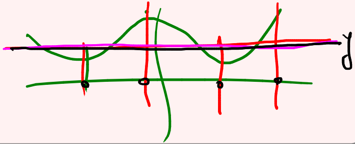

and let $Y = X^2$. Find the PDF $f_Y(y)$.

$Y = g(x) = x^2$

- Does this satisfy the criteria for applying method of transformation?
- Can we partition R_X into intervals such that $g(x)$ is monotone? $(-\infty, 0) \cup (0, \infty)$
- On which intervals $g(x)$ is monotone?

Recall

$$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{g'(x_i)}, \quad R_Y = (0, \infty).$$

For any $y \in R_Y$, we will have two solutions to $y = x^2$; i.e., $x = \sqrt{y}$, $x = -\sqrt{y}$

$[x_1 = \sqrt{y}, \quad x_2 = -\sqrt{y}]$

Solution to Previous Question...

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{2\pi y}} e^{-y/2} + \frac{1}{2\sqrt{2\pi y}} e^{-y/2}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad //$$

Solution to Previous Question...

Outline

1. Continuous Random Variable
2. Method of Transformation
3. Solved Problems
4. Continuous Distributions

Problem-1

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Consider the PDF of the random variable X

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- Find the constant c

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$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find the constant c
- Find $E[X]$ and $\mathrm{Var}[X]$

Problem-1

Problem-1

Consider the PDF of the random variable X

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$-1 \leq x \leq 1$

- Find the constant c
- Find $E[X]$ and $\mathrm{Var}[X]$
- Find $P(X \geq \frac{1}{2})$

Solution to Problem-1

1. To find c, we have

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(u)\, du \\
&= \int_{-1}^{1} cu^2\, du \\
&= \frac{2}{3}c \implies c = \frac{3}{2}.
\end{aligned}
$$

Solution to Problem-1

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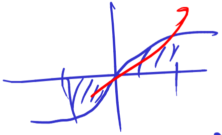
Cancelled -



2 To find $E[X]$, we have

$$
\begin{aligned}
E[X] &= \int_{-1}^{1} u f_X(u)\,du \\
&= \frac{3}{2}\int_{-1}^{1} u^3\,du \\
&= 0
\end{aligned}
$$

- odd fn.
- symmetric about origin

$\int_{-a}^{a} f = 0$

Solution to Problem-1

1 To find c, we have

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2 To find $E[X]$, we have

$$\begin{aligned} E[X] &= \int_{-1}^{1} uf_X(u)\,du \\ &= \frac{3}{2}\int_{-1}^{1} u^3\,du \\ &= 0 \end{aligned}$$

1 We have

0

$$\begin{aligned} \mathrm{Var} &= E[X^2] - E[X]^2 \\ &= \int_{-1}^{1} u^2 f_X(u)\,du \\ &= \frac{3}{2}\int_{-1}^{1} u^4\,du = \frac{3}{5} \end{aligned}$$

Solution to Problem-1

1 To find c, we have

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2 To find $P(X \geq \frac{1}{2})$, we have

$$P(X \geq \frac{1}{2}) = \frac{3}{2}\int_{1/2}^{1} x^2\, dx = \frac{7}{16}.$$

Problem-2

Problem 2

Consider the PDF of continuous random variable X

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad \text{for all } x \in \mathbb{R}$$

If $Y = X^2$, find the CDF of Y.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

Solution to Problem-2

1. We have $R_Y = [0, \infty)$.

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\begin{aligned}
F_Y(y) &= P(Y \le y) = P(X^2 \le y) \\
&= P(-\sqrt{y} \le X \le \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|}\, dx \\
&= \int_{0}^{\sqrt{y}} e^{-x}\, dx = 1 - e^{-\sqrt{y}}
\end{aligned}
$$

Solution to Problem-2

1. We have $R_Y = [0, \infty)$.
2. For $y \in [0, \infty)$,

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) = P(X^2 \le y) \\
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&= \int_0^{\sqrt{y}} e^{-x}\, dx = 1 - e^{-\sqrt{y}}
\end{aligned}
$$

Thus,

$$
F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

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Consider the PDF of the continuous random variable

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq \frac{2}{3} \mid X > \frac{1}{3})$.

Solution to Problem 3

We have

$$
\begin{aligned}
P(X \leq \frac{2}{3} \mid X > \frac{1}{3}) &= \frac{P(1/3 < X \leq 2/3)}{P(X > \frac{1}{3})} \\
&= \frac{\int_{1/3}^{2/3} 4x^3 \, dx}{\int_{1/3}^{1} 4x^3 \, dx} \\
&= 3/16
\end{aligned}
$$

Problem 4

Problem 4

Consider the PDF of random variable X

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Consider the PDF of random variable X

$$f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4

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Consider the PDF of random variable X

$$f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find $\mathrm{Var}(Y)$.

Solution to Problem 4

- We have

$$\mathrm{Var}(Y) = \mathrm{Var}(\frac{2}{X} + 3) = 4\,\mathrm{Var}(\frac{1}{X})$$

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- We have

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- We now find $\text{Var}(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[X])^2$
- We have

$$E[\frac{1}{X}] = \int_0^1 x(2x + \frac{3}{2})\,dx = \frac{17}{12}$$

$$E[\frac{1}{X^2}] = \int_0^1 (2x + \frac{3}{2})\,dx = \frac{5}{2}$$

Solution to Problem 4

- We have

$$\mathrm{Var}(Y) = \mathrm{Var}(\frac{2}{X} + 3) = 4\,\mathrm{Var}(\frac{1}{X})$$

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- We have

$$E[\frac{1}{X}] = \int_0^1 x(2x + \frac{3}{2})\,dx = \frac{17}{12}$$

$$E[\frac{1}{X^2}] = \int_0^1 (2x + \frac{3}{2})\,dx = \frac{5}{2}$$

$$\frac{5}{2} - \left(\frac{17}{12}\right)^2$$

- Hence,

$$\mathrm{Var}(\frac{1}{X}) = \frac{71}{144}$$

Solution to Problem 4

- We have

$$\mathrm{Var}(Y) = \mathrm{Var}(\frac{2}{X} + 3) = 4\,\mathrm{Var}(\frac{1}{X})$$

- We now find $\mathrm{Var}(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[X])^2$
- We have

$$E[\frac{1}{X}] = \int_0^1 x(2x + \frac{3}{2})\,dx = \frac{17}{12}$$

$$E[\frac{1}{X^2}] = \int_0^1 (2x + \frac{3}{2})\,dx = \frac{5}{2}$$

- Hence,

$$\mathrm{Var}(\frac{1}{X}) = \frac{71}{144}$$

- $\mathrm{Var}(Y) = 4\,\mathrm{Var}(\frac{1}{X}) = \frac{71}{36}$ //

Problem 5

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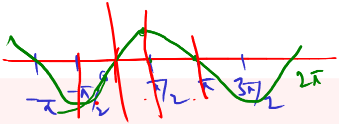

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- Here $Y = g(X)$, where g is a differentiable function
- g is not monotone, but it can be divided to a finite number of regions in which it is monotone
- We have $R_X = [-\pi/2, \pi], \quad R_Y = [-1, 1]$

Assignment

Answer to previous problem 5...

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Outline

Recall Discrete distribution

- Bernoulli
- Binomial
- Geometric
- Poisson

Special Continuous Distributions: Uniform Distribution...

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$$E[X] = \int_{-\infty}^{\infty} \underline{x}\,\underline{f_X(x)}\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{a+b}{2}$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

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$Var = E[X^2] - E[X]^2$

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- Hence the variance is: $\text{Var}(X) = \underline{E[X^2]} - (\underline{E[X]})^2 = \dfrac{(b-a)^2}{12}$ //

Special Distribution: Exponential Distributions...

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Definition of Exponential Distribution

Let X be a continuous random variable. Here X is said to have exponential distribution with parameter $\lambda > 0$ shown as $X \sim \text{Exponential}(\lambda)$, if its PDF is given as follows

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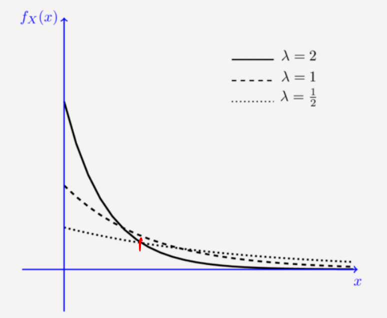

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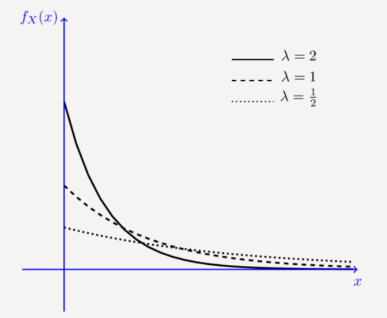

The CDF is given by

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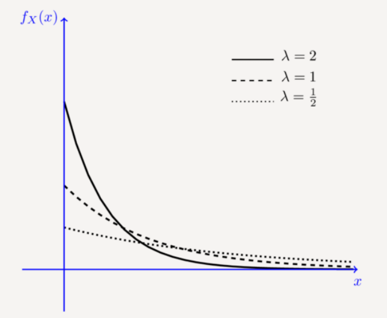

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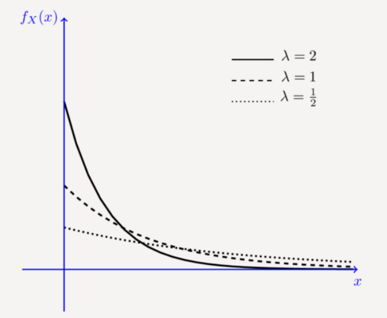

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$$\underline{E[X]} = \int_0^\infty x\lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}\int_0^\infty \underline{ye^{-y}}$$?!

check

$$= \frac{1}{\lambda}[-e^{-y} - ye^{-y}]_0^\infty = \frac{1}{\lambda}$$

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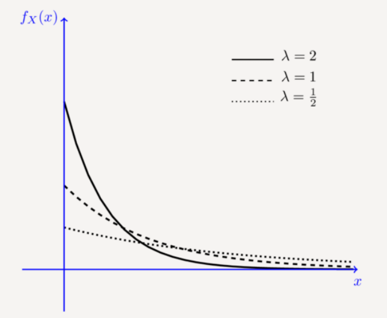

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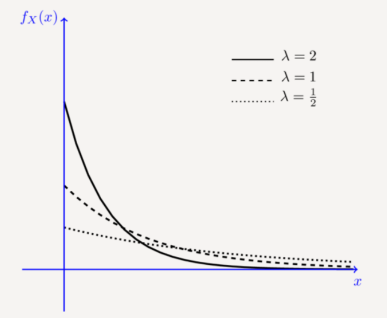

Var(X) is given by:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y}\, dy$$

$$= \frac{1}{\lambda^2} \left[-2e^{-y} - 2ye^{-y} - y^2 e^{-y} \right]_0^\infty = \frac{2}{\lambda^2}$$

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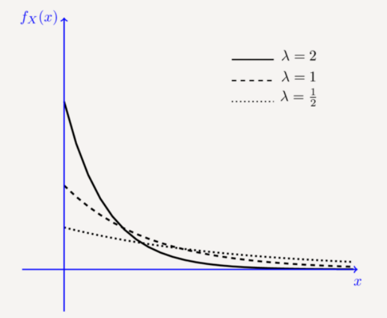

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$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

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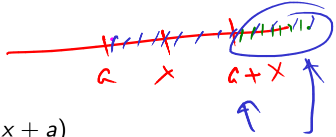

$$\begin{aligned}
P(X > x + a \mid X > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\
&= \frac{P(X > x + a)}{P(X > a)} = \frac{1 - F_X(x + a)}{1 - F_X(a)} \\
&= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} \\
&= P(X > x)
\end{aligned}$$

Check CDF of expondi

intersect

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