

Noise power Spectrum:

$$S_x(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \left| \int_0^T e^{i\omega t} x(t) dt \right|^2$$

Wiener - Khinchin theorem:

Page 281

$x(t) \rightarrow$ Stationary random process

$S_x(\omega) =$ Fourier transform of its auto-correlation funcⁿ

$$\phi_x(\tau) = \langle x(t) x(t+\tau) \rangle$$

Appendix - H

$$S_x(\omega) = \text{Re} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} \langle x(t) x(t+\tau) \rangle d\tau \right\}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \cos \omega\tau \langle x(t) x(t+\tau) \rangle d\tau \quad (\because \cos(\omega\tau) \text{ is even ; } \sin(\omega\tau) \text{ is odd})$$

$$S_x(\omega) = \frac{1}{\pi} \int_0^{\infty} \cos \omega\tau \langle x(t) x(t+\tau) \rangle d\tau$$

For Brownian motion:

$$\dot{v} = -\gamma v + \xi(t), \text{ where } \langle \xi(t) \xi(0) \rangle = \gamma \delta(t)$$

Power spectrum:

$$S_v(\omega) = \text{Re} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} \langle v(0) v(t+\tau) \rangle d\tau \right\}$$

$$\Rightarrow S_v(\omega) = \text{Re} \left\{ \frac{1}{2\pi} \cdot \frac{\gamma}{\gamma} \int_0^{\infty} e^{i\omega\tau} e^{-\gamma\tau} d\tau \right\}$$

$$\Rightarrow S_v(\omega) = \frac{\gamma}{2\pi\gamma} \cdot \text{Re} \left\{ \frac{e^{-\gamma\tau + i\omega\tau}}{i\omega - \gamma} \Big|_0^{\infty} \right\}$$

$$\Rightarrow S_v(\omega) = \frac{\gamma}{2\pi\gamma} \times \frac{\gamma}{\omega^2 + \gamma^2}$$

$$\Rightarrow S_v(\omega) = \frac{1}{\omega^2 + \gamma^2} \times \frac{\gamma}{2\pi} \quad S_{\xi}(\omega): \text{Power spectrum of white noise}$$

Autocorrelation function from $S_v(\omega)$:

Using inverse Fourier transform.

$$\begin{aligned} \langle v(0) v(t) \rangle &= \int_{-\infty}^{\infty} S_v(\omega) e^{-i\omega t} d\omega \\ &= \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 + \gamma^2} d\omega \\ &= \frac{\gamma}{2\pi} \cdot 2 \int_0^{\infty} \frac{\cos \omega t}{\omega^2 + \gamma^2} d\omega \\ &= \frac{\gamma}{\pi} \times \frac{\pi}{2\gamma} e^{-\gamma t} \end{aligned}$$

$$\Rightarrow \langle v(0) v(t) \rangle = \frac{\gamma}{2\gamma} e^{-\gamma t}$$

Sum Rule:

variance of random process

$$\langle v^2 \rangle = \int_{-\infty}^{\infty} S_v(\omega) d\omega$$

$$\Rightarrow \langle v^2 \rangle = \int_{-\infty}^{\infty} \frac{\gamma}{2\pi} \frac{d\omega}{\omega^2 + \gamma^2} = \frac{\gamma}{2\pi} \times \frac{1}{\gamma} \times \text{Tan}^{-1} \left(\frac{\omega}{\gamma} \right) \Big|_{-\infty}^{\infty}$$

$$\Rightarrow \langle v^2 \rangle = \frac{\gamma}{2\gamma}$$

Poisson Birth & Death Process:

$$\alpha \xrightarrow{\quad} X \xrightarrow{\quad} \beta \quad \frac{dX}{dt} = \alpha - \beta X$$

$e_q^{ns} \{ \}$ comparison

$$\frac{d\Delta X}{dt} = -\beta \Delta X + \eta(t)$$

$$\text{where } \langle \eta(t) \eta(t') \rangle = \frac{2\alpha}{\Omega} \delta(t-t')$$

$$\text{shortcut: } \Gamma = \frac{2\alpha}{\Omega}; \quad Y = \beta$$

$$\langle X \rangle = \frac{\alpha}{\beta}$$

$$\Delta X(w) = \frac{\eta(w)}{-iw + \beta}$$

$$S_x(w) = \frac{1}{w + \beta} \times \frac{2\alpha}{\Omega} \times \frac{1}{2\pi}$$

$$\langle \Delta X^2 \rangle = \frac{2\alpha}{\Omega} \times \frac{1}{2\beta} = \frac{\alpha}{\Omega \beta}$$



$$\frac{dX_2}{dt} = \alpha X_1 - (\alpha + \beta) X_2; \text{ where } X_1 + X_2 = X_T$$

$e_q^{ns} \{ \}$ comparison

$$\frac{d\Delta X_2}{dt} = -(\alpha + \beta) \Delta X_2 + \eta(t)$$

$$\text{where } \langle \eta(t) \eta(t') \rangle = D \delta(t-t')$$

$$\text{shortcut: } \Gamma = D; \quad Y = \beta + \alpha$$

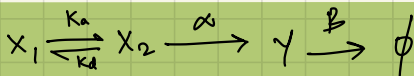
$$\langle X_2 \rangle = \frac{\alpha}{\alpha + \beta} X_T$$

$$\Delta X(w) = \frac{\eta(w)}{-iw + (\beta + \alpha)}$$

$$S_x(w) = \frac{1}{w + (\alpha + \beta)} \times S_E(w)$$

$$\langle \Delta X^2 \rangle = \frac{\pi}{\alpha + \beta} \cdot \frac{D}{2\pi} = \frac{\alpha \beta}{(\alpha + \beta)^2} X_T \times \frac{1}{\Omega}$$

$$D = \langle \alpha X_1 + \alpha X_2 \rangle = \frac{2\alpha \beta}{\alpha + \beta} \frac{X_T}{\Omega}$$



$$\frac{d\Delta X_2}{dt} = -(k_a + k_d) \Delta X_2 + \eta(t)$$

$$\frac{d\Delta Y}{dt} = \alpha \Delta X_2 - \beta \Delta Y + \eta(t)$$

$$\text{shortcut: } \Gamma = D_1; \quad Y = k_a + k_d \quad (\text{for } X_2)$$

$$\text{shortcut: } \Gamma = D_2; \quad Y = \beta \quad (\text{for } Y)$$

$$\frac{dX_2}{dt} = k_a X_1 - (k_a + k_d) X_2; \text{ where } X_1 + X_2 = X_T$$

$$\frac{dY}{dt} = \alpha X_2 - \beta Y$$

$$\langle X_2 \rangle = \frac{k_a}{k_a + k_d} X_T$$

$$\langle Y \rangle = \frac{\alpha}{\beta} \langle X \rangle = \frac{\alpha}{\beta} \frac{k_a}{k_a + k_d} X_T$$

$$\Delta X_2 = \frac{\eta_1(w)}{-iw + (k_a + k_d)} \Rightarrow \langle \Delta X_2^2 \rangle = \frac{k_a k_d}{k_a + k_d} \frac{X_T}{\Omega}$$

ΔY see in slides

$$\frac{2k_a k_d}{k_a + k_d} \frac{X_T}{\Omega}$$

$$\frac{2\alpha k_a}{\Omega k_a + k_d} \frac{X_T}{\Omega}$$

Brownian motion with external force:

$$m\dot{v} + \gamma v = \eta(t) + F_{\text{ext}}(t)$$

$$m\langle \dot{v} \rangle + \gamma \langle v \rangle = F_{\text{ext}}(t)$$

$$\langle v(\omega) \rangle = \frac{F_{\text{ext}}(\omega)}{-i\omega + \gamma}$$

$$v(t) = \int \eta(t') e^{\gamma(t-t')} dt' + \int F_{\text{ext}}(t') e^{\gamma(t-t')} dt'$$

$$\langle v(t) \rangle = \int_{-\infty}^t F_{\text{ext}}(t') e^{\gamma(t-t')} dt'$$

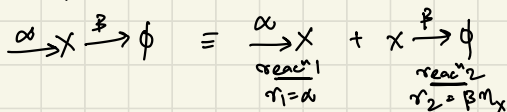
$$F_{\text{ext}} = E_0 \sin \omega t$$

$$\langle v(t) \rangle = \int_{-\infty}^t E_0 \sin \omega t' e^{\gamma(t-t')} dt'$$

$$\begin{aligned} \langle v(t) \rangle &= e^{-\gamma t} E_0 \int_{-\infty}^t \sin \omega t' e^{\gamma t'} dt' \\ &= e^{-\gamma t} E_0 \frac{e^{\gamma t'} (\gamma \sin \omega t' - \omega \cos \omega t')}{\omega^2 + \gamma^2} \Big|_{-\infty}^t \end{aligned}$$

$$\langle v(t) \rangle = e^{-\gamma t} \frac{E_0}{\sqrt{\omega^2 + \gamma^2}} \sin(\omega t - \phi) \quad \phi = \tan^{-1}\left(\frac{\omega}{\gamma}\right)$$

Simulation of Master equation



$$P(\text{reaction 1 happening in time interval } \Delta t) \rightarrow P_1 = r_1 \Delta t$$

$$P(\text{reaction 2 happening in time interval } \Delta t) \rightarrow P_2 = r_2 \Delta t$$

$$P(\text{nothing happens in time } t)$$

$$P(N) = P(\text{not 1 \& not 2})$$

$$= P(\text{not 1}) \cdot P(\text{not 2})$$

$$\begin{aligned} \text{for } \Delta t \rightarrow (1 - r_1 \Delta t) & \quad \text{for } \Delta t \rightarrow (1 - r_2 \Delta t) \\ \text{for } t = N\Delta t \rightarrow \lim_{N \rightarrow \infty} (1 - r_1 \frac{t}{N})^N & \quad \text{for } t = N\Delta t \rightarrow \lim_{N \rightarrow \infty} (1 - r_2 \frac{t}{N})^N \\ \Rightarrow \Delta t = \frac{t}{N} & \quad \Rightarrow \Delta t = \frac{t}{N} \\ = e^{-r_1 t} & \quad = e^{-r_2 t} \\ \Rightarrow P(N) = e^{-r_1 t} \cdot e^{-r_2 t} & = e^{-(r_1 + r_2)t} \end{aligned}$$

What is probability of
1 & 2 not happening for t & 1 happening?
in Δt !

$$P(N \& 1) = P(N) P_1 = e^{-(r_1 + r_2)t} r_1 \Delta t$$