PROBABILITY AND STATISTICS LECTURE ON 3 SEPTEMBER 2021

BAYES THEOREM

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Also proved (refer firgure)

Proof of Bayes Theorem:

$$\begin{gathered} P(F|E) = P(F \cap E)/P(E) \\ \implies P(F|E)P(E) = P(F \cap E) = P(E|F)P(E) \\ \therefore P(F|E)P(E) = P(E|F)(F) \\ \implies p(F|E) = \frac{P(F|E)P(F)}{P(E)} \end{gathered}$$

Question:

- 60% of all email in 2016 is spam
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear", what is the probability that it is spam?

Answer:

Events are

E: "Dear" occurs

F: "Spam" occurs

> Note: Sometimes convention is used such that E means "Evidence" and/or F means "Fact"

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$
$$P(E|F) = 20\%$$
$$P(F) = 60\%$$
$$\text{Using Total Probability:}$$
$$P(E) = P(E|F)P(F) + P(E|F')P(F')$$
$$= 0.2 \times 0.6 + 0.01 \times 0.04 = 0.324$$
$$\text{Then apply Bayes Theorem}$$

Question:

A test is 98% effective at detecting a disease ("true positive"). However, the test has a "false positive" rate of 1%. 0.5% of the US population has the disease. What is the likelihood you have the disease, if you test positive.

Answer:

$$\text{Events are}$$
$$\text{E: you test positive}$$
$$\text{F: you have the disease}$$
$$\text{To find: P(F|E)}$$

$$P(E|F) = 98\%$$
$$P(E|F') = 1\%$$
$$P(E'|F) = 1 - P(E|F) = 2\%$$
$$P(E'|F') = 1 - P(E|F') = 99\%$$
$$\text{Using Bayes Theorem:}$$
$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$
$$= \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F')P(F')}$$
$$= \frac{0.98 \times 0.005}{0.98 \times 0.005 + 0.01 \times 0.995}$$

MONTY HALL PROBLEM

- In a game show there are three doors
- Behind one of the doors, there is a car and in the other two there are goats
- Pick one door and then the host opens another door that definitely has a goat behind it.
- You are given a choice to pick the other unopened door or the door you picked originally
- **Question:** Is it probabilistically wise to change doors.

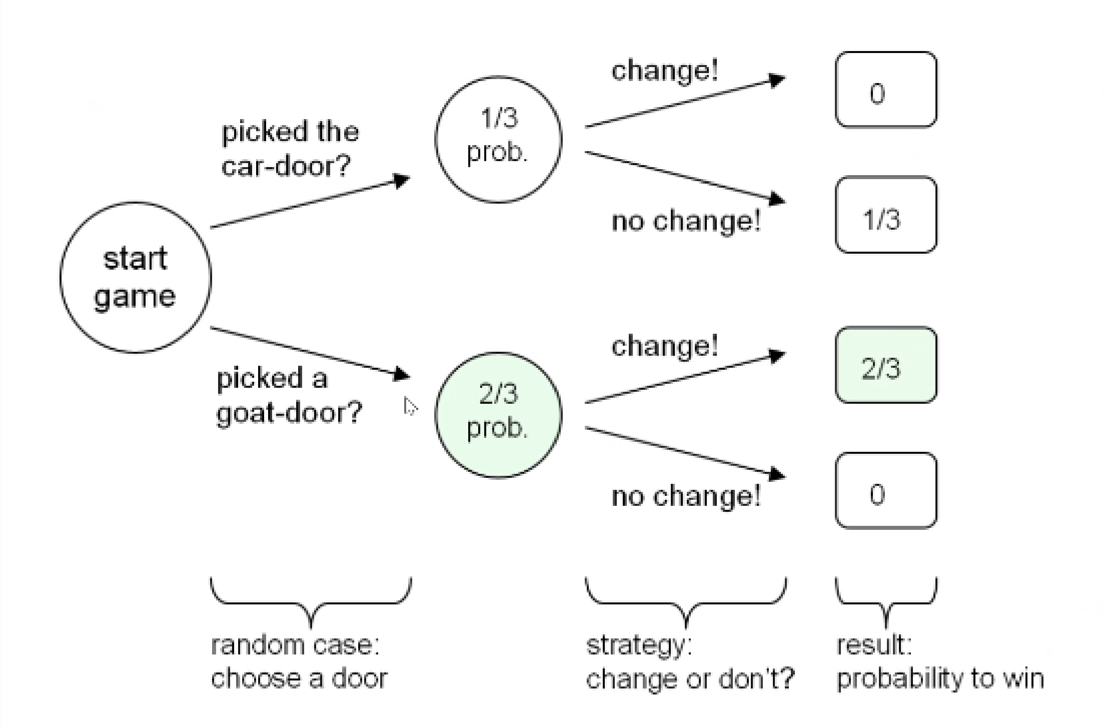
It seems like a 50% chance after the host reveals one goat, since you either stay or change to get either a car or a goat.

The correct strategy is to switch. We have a 66% of picking a goat door first, thus making the switch give you a car since the host will reveal the other goat door.

Door You Choose	Prize in Door	Host Opens	Stay	Switch
1	1	2/3	win	loose
1	2	3	loose	win
1	3	2	loose	win
2	1	3	loose	win
2	2	1/3	win	loose
2	3	1	loose	win
3	1	2	loose	win
3	2	1	loose	win
3	3	1/2	win	loose

Table: Exhaustive list of possibilities

The choices and probabilities to win can be represented with a choice tree:



Proof using Bayes Theorem

Let H be the event that "door 1 has a car behind it" and E be that "a goat was revealed"

P(H | E) will answer the problem.

$$P(H) = 1/3$$
$$P(H') = 1 - 1/3 = 2/3$$
$$P(E|H) = P(\text{a goat was revealed given that door 1 has a car behind it})$$
$$= 1$$
$$P(E|H') = \text{also 1 since a goat door is always revealed}$$
$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$
$$= \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H')P(H')}$$
$$= \frac{1 \times 1/3}{1 \times 1/3 + 1 \times 2/3} = \frac{1}{3}$$