Probability and Statistics

UG2, Core course, IIIT,H

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1. Cumulative Distribution Function
2. Continuous Random Variable
3. Probability Density Functions
4. Method of Transformation

Outline

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2. Continuous Random Variable
3. Probability Density Functions
4. Method of Transformation

Definition of Cumulative Distribution Function

Definition of CDF

The cumulative distribution function (CDF) of a random variable X is defined as

$$F_X(x) = P(X \leq x), \quad \text{for all } x \in \mathbb{R}.$$

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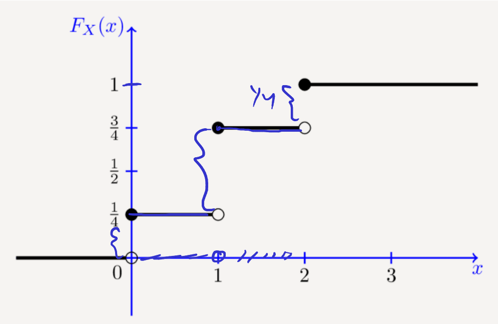

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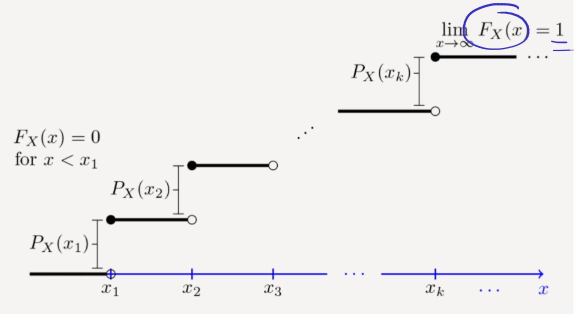

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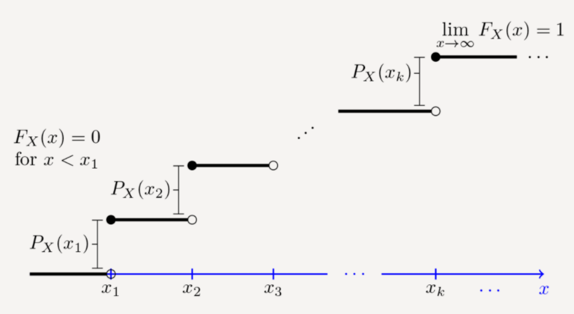

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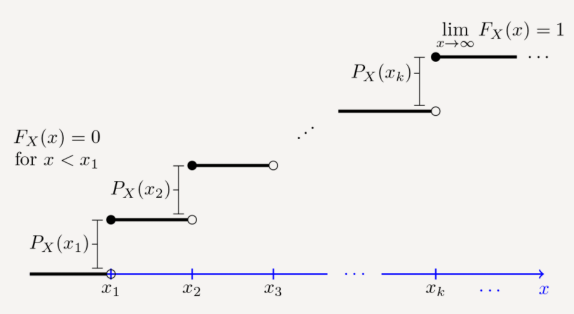

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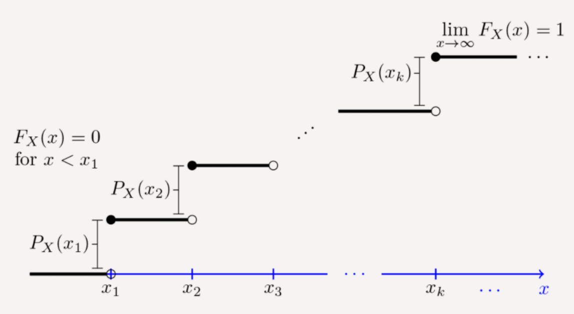

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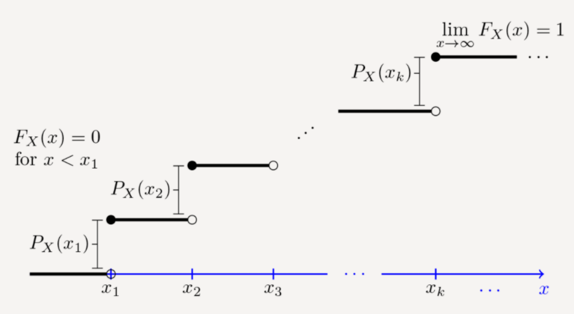

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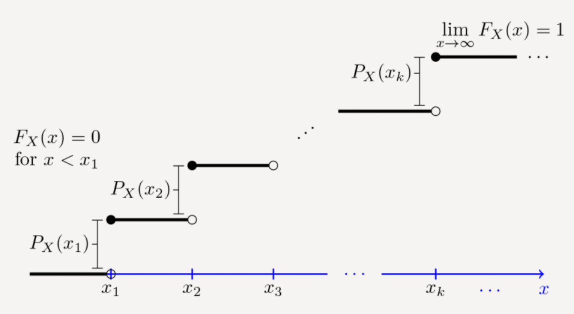

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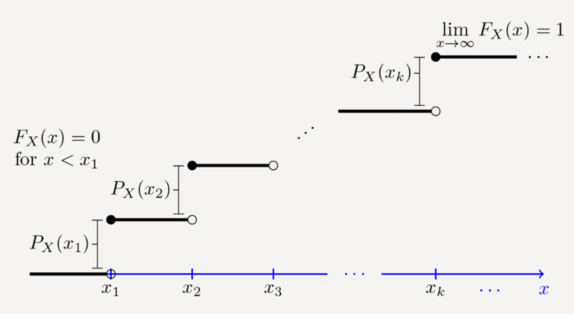

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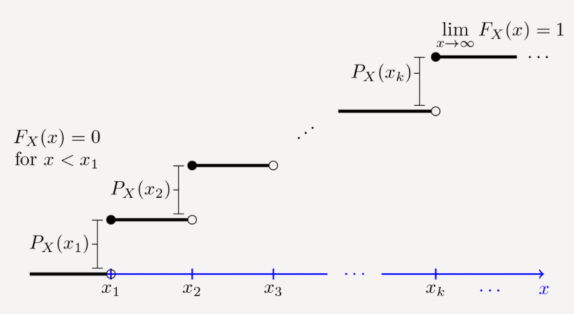

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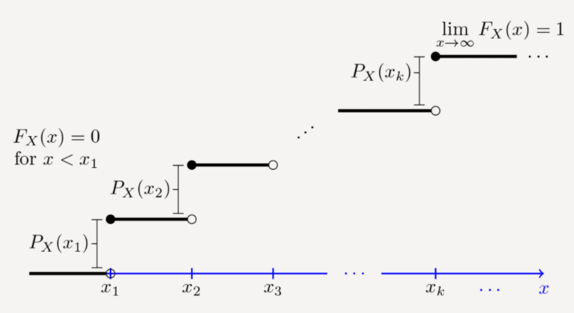

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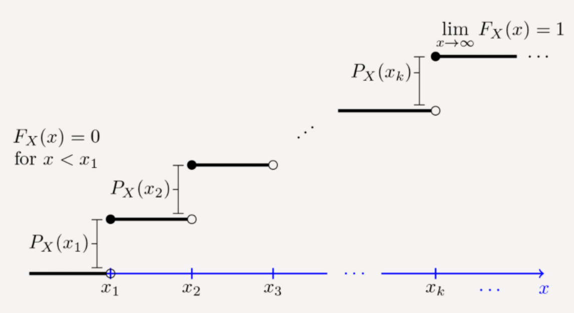

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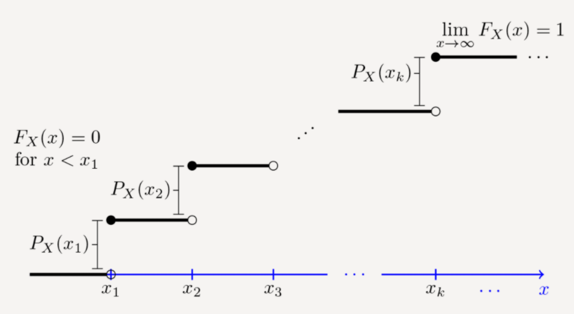

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- if $R_X = \{x_1, x_2, \dots\}$, $\quad F_X(x) = \sum_{x_k \leq x} P_X(x_k)$

Properties of CDF...

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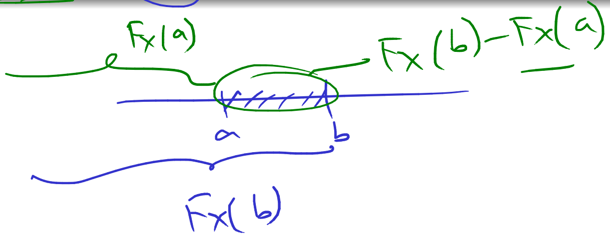
A result

For all $a \le b$, we have

$$P(a < X \le b) = F_X(b) - F_X(a)$$



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2. Find $P(2 < X \leq 5)$
3. Find $P(X > 4)$

- Is this a valid PMF?

Answer to previous problem...

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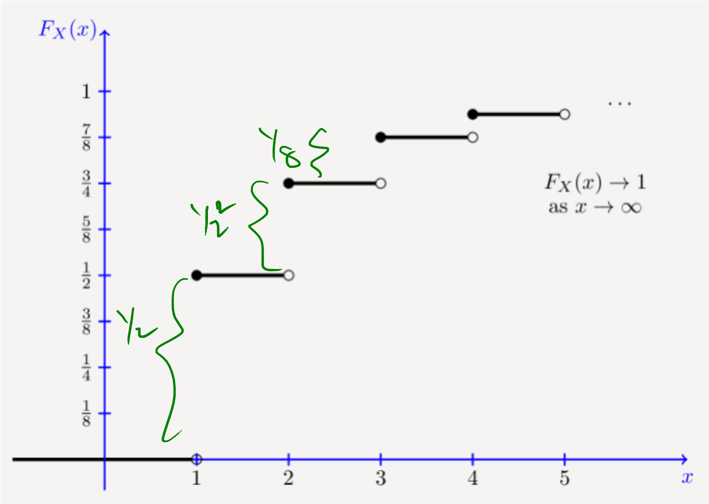

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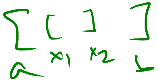

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- Since $P(X \in [a, b]) = 1$, we have

$$P(X \in [x_1, x_2]) = \frac{x_2 - x_1}{b - a}, \quad \text{where } a \leq x_1 \leq x_2 \leq b$$

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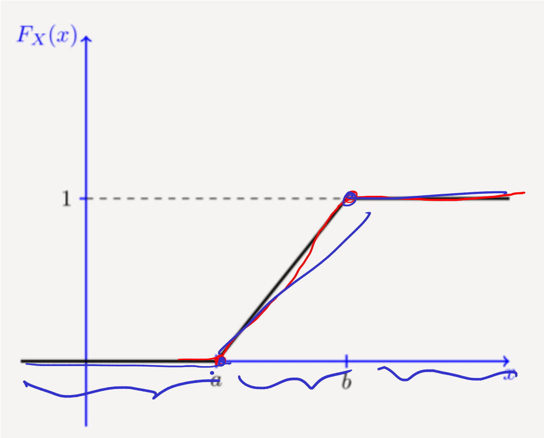

$$\frac{a-x}{b-a}$$

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PMF

PDF $\xleftarrow{\frac{d}{dx}}$ CDF

Probability per unit (length, area, volume)

a x b

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- We will usually assume that the CDF of a continuous R.V. is differentiable
- Although PMF does not makes sense for continuous random variable, we define probability density function

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$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

is called the probability density function of X. We assume that $F_X(x)$ is differentiable.

Example of CDF and PDF for Continuous Random Variable...

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Example: Derive PDF from CDF for Continuous R.V

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$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

Example of CDF and PDF for Continuous Random Variable...

only in discr case

x_1 x_2 x_3

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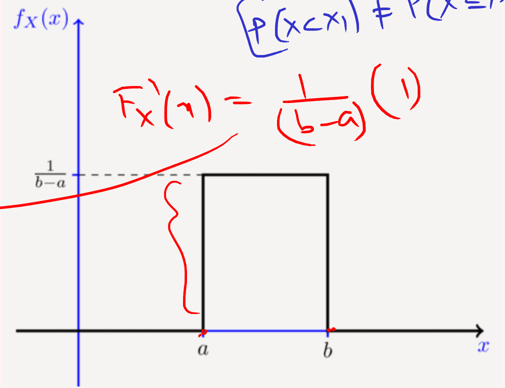
$P(X<x_1) + P(X \le x_1)$

$F_X'(x) = \frac{1}{(b-a)}(1)$

- Does not matter if we use $<$ or $\le$. That is, $P(X < 2) = P(X \le 2)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$



Properties of PDF...

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Recall Calculus

$f_X(x) = \frac{d}{dx} F_X(x)$

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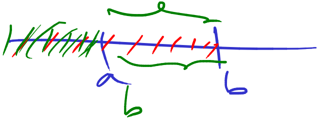
$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$

$$= \int_{-\infty}^{b} f_X(u)\,du - \int_{-\infty}^{a} f_X(u)\,du = \int_{a}^{b} f_X(u)\,du$$



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$\sum p(x_i) = 1$

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Properties of PDF

- $\underbrace{f_X(x) \geq 0}$ for all $x \in \mathbb{R}$

Properties of PDF...

Recall Calculus.

Derivative of non-decr. fn is always non-negative.

- Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

$$\boxed{f_X(x) = F_X'(x)} = \lim_{\Delta \to 0^+} \frac{F(x+\Delta) - F(x)}{\Delta}$$

Properties of PDF

- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

We know that $F_X(x)$ is non-decreasing

$\Rightarrow F(x+\Delta) - F(x) \geq 0 \quad \forall \Delta > 0$

$\Rightarrow \frac{F(x+\Delta) - F(x)}{\Delta} \geq 0$

$\Rightarrow F_X'(x) \geq 0$

Properties of PDF...

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Properties of PDF...

measure thm

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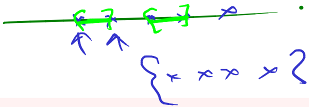

Properties of PDF

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- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
- More generally, for a set A, $\quad P(x \in A) = \int_A f_X(u)\, du$
- If $A = [0, 1] \cup [3, 4]$: $P(X \in A) = \int_0^1 f_X(u)\, du + \int_3^4 f_X(u)\, du$

Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

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1. Find c
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$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

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Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

1. Find c
2. Find the CDF of X, $F_X(x)$
3. Find $P(1 < X \leq 3)$ $= F_X(3) - F_X(1)$

$P(1 < X < 3) = P(1 < X \leq 3) = P(1 \leq X < 3) = P(1 \ldots$

Recall

We know that

$$\int_{-\infty}^{\infty} f_X(x) = 1 \Rightarrow \int_{-\infty}^{\infty} ce^{-x}\,dx = c\left[\frac{e^{-x}}{-1}\right]_{0}^{\infty} = -c\left[e^{-\infty} - e^{-0}\right] = -c[0-1] = \boxed{c = 1}$$

Example: PDF and CDF of Continuous Random Variable

$P(1<X<3) = F_X(3) - F_X(1)$

$= (1-e^{-3}) - (1-e^{-1})$

$= e^{-1} - e^{-3}$ ≥ 0

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

1. ~~Find c~~
2. Find the CDf of X, $F_X(x)$
3. Find $P(1 < X < 3)$

using PDF

$\int_1^3 f_X(x)\,dx = \int_1^3 e^{-x}\,dx = \left[-e^{-x}\right]_1^3$

$= \left[-e^{-3} + e^{-1}\right]$

$= e^{-1} - e^{-3}$

(2) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_0^x f_X(u)\,du = \int_0^x ce^{-u}\,du$

$= c\left[-e^{-u}\right]_0^x = c\left[-e^{-x} + e^{-0}\right] = c\left[1 - e^{-x}\right]$

$= 1 - e^{-x}$

Answer to previous problem...

Answer to previous problem...

Range, Expectation, Variance of Continuous Random Variable...

Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

In discr

$R_x = \{x \mid P_X(x) > 0\}$

Range, Expectation, Variance of Continuous Random Variable...

$$E[X] = \sum_x x\, p_X(x)$$

$\nwarrow$ discr.

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Recall that the expected value of discrete R.V. is

Range, Expectation, Variance of Continuous Random Variable...

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Recall that the expected value of discrete R.V. is

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Range, Expectation, Variance of Continuous Random Variable...

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Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

$$EX = \sum_{x_k \in R_x} x_k P_X(x_k)$$

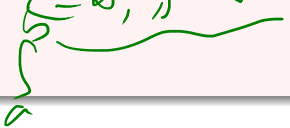
PMF becoms PDF

$\Sigma \to \int$

Replacing sum by integral, and PMF by PDF we have

$$EX = \int_{-\infty}^{\infty} x \underline{f_X(x)}\, dx \;\to$$

$(-\infty, 0)$ $(\;\;)$



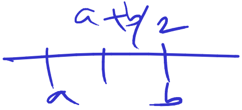
Example of Expected Value of Continuous Random Variable...

Example of Expected Value of Continuous Random Variable...

Example

Let $X \sim \text{Uniform}(a, b)$. Find EX.

We know that



$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2) = \frac{b+a}{2}$$

Expected Value of a Function of Continuous Random Variable...

Expected Value of a Function of Continuous Random Variable...

$$E[g(X)] = \sum_x \underline{\underline{g(x)}}\, p_X(x)$$

Expected Value of a Function of Continuous Random Variable...

Recall that for the discrete random variable we had

Expected Value of a Function of Continuous Random Variable...

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PMF

By changing the sum to integral and changing PMF to PDF we have

PDF

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Expected Value of a Function of Continuous Random Variable...

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We recall that the linearity of $E[\cdot]$ holds:

Expected Value of a Function of Continuous Random Variable...

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By changing the sum to integral and changing PMF to PDF, we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

We recall that the linearity of $E[\,]$ holds.

1 $E[aX + b] = aE[X] + b$ for all $a, b \in \mathbb{R}$

2 $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$

$$E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= aE[x] + b$$

Example of Expected Value for Continuous Random Variable...

Example of Expected Value for Continuous Random Variable...

Example

Let the PDF of a continuous R.V. be given by

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X^n]$, $n \in \mathbb{N}$.

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \int_0^1 x^n \left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^{n+1} + \frac{1}{2} x^n\right) dx = \left[\frac{x^{n+2}}{n+2} + \frac{1}{2}\frac{x^{n+1}}{n+1}\right]_0^1 = \left[\frac{1}{n+2} + \frac{1}{2}\cdot\frac{1}{n+1}\right]$$

$$= \frac{3n+4}{2(n+1)(n+2)}$$

Variance of a Continuous Random Variable...

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Definition: Variance of Continuous Random Variable

Recall that the variance of a random variable is defined as

Variance of a Continuous Random Variable...

Definition: Variance of Continuous Random Variable

Recall that the variance of a random variable is defined as

$$\mathrm{Var}(X) = E[(X - \mu_X)^2] = E[X^2] - E[X]^2.$$

Variance of a Continuous Random Variable...

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So, for a continuous random variable, we have

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Recall that the variance of a random variable is defined as

$$\mathrm{Var}(X) = E[(X-\mu_X)^2] = E[X^2] - E[X]^2.$$

So, for a continuous random variable, we have

$$\begin{aligned}\mathrm{Var}(X) = E[(X-\mu_X)^2] &= \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx \\ = E[X^2] - E[X]^2 &= \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\,dx}_{E[x^2]} - \underbrace{\mu_X^2}_{E[x]^2}\end{aligned}$$

Variance of a Continuous Random Variable...

Definition: Variance of Continuous Random Variable

Recall that the variance of a random variable is defined as

$$\mathrm{Var}(X) = E[(X - \mu_X)^2] = E[X^2] - E[X]^2.$$

So, for a continuous random variable, we have

$$\begin{aligned}\mathrm{Var}(X) = E[(X - \mu_X)^2] &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx \\ = E[X^2] - E[X]^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \mu_X^2\end{aligned}$$

Recall that for $a, b \in \mathbb{R}$, we have

$$\boxed{\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X).}$$

Ex

10^6

10^{10} 10^{1000}

Example: Expected Value and Variance

Example

Consider the following PDF of the continuous random variable X

Example: Expected Value and Variance

$V[X] = E[X^2] - E[X]^2$

Example

Consider the following PDF of the continuous random variable X

$$f_X(x) = \begin{cases} \dfrac{3}{x^4} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x)\, dx = \int_1^{\infty} \cancel{x} \frac{3}{x^{\cancel{4}}} dx = \int \frac{3}{x^3} dx$$

$$= 3 \int_1^{\infty} x^{-3} dx = 3 \left[\frac{x^{-3+1}}{-3+1} \right]_1^{\infty} = -\frac{3}{2} \left[x^{-2} \right]_1^{\infty}$$

$$= -\frac{3}{2} [0 - 1] = +\frac{3}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_1^{\infty} x^2 \cdot \frac{3}{x^4} dx = 3 \left[\frac{x^{-2+1}}{-2+1} \right]_1^{\infty} = -3[0-1] = 3$$

$$\Rightarrow \operatorname{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - 9/4 = \frac{12-9}{4} = 3/4$$

Functions of Continuous Random Variable...

Functions of Continuous Random Variable...

Example of a function of continuous random variable

Let $X \sim \text{Uniform}(0, 1)$, and let $Y = e^X$.

Functions of Continuous Random Variable...

Uniform(a, b)

Example of a function of continuous random variable

Let $X \sim$ Uniform$(0, 1)$, and let $\boxed{Y = e^X.}$

- Find the CDF of Y

Since $X \sim$ Uniform$(0,1)$, we have

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise}. \end{cases}$$

Recall

$$\begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 0 & x < a \\ 1 & x > b \end{cases}$$

PDF

$$\begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{others} \end{cases}$$

$$F_Y(y) = P(Y \le y)$$
$$= P(e^X \le y)$$
$$= P(X \le \ln y)$$

Functions of Continuous Random Variable...

Example of a function of continuous random variable

Let $X \sim$ Uniform$(0, 1)$, and let $Y = e^X$.

1. Find the CDF of Y
2. Find the PDF of Y

② $f_Y(y) = F'_Y(y)$

$$= \begin{cases} 1/y, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

① if $y < 1$, then $\ln y < 0$

$\Rightarrow P(X \le \ln y) = 0$

② If $y \geq e$, then $\ln y \geq 1$

$\Rightarrow P(X \le \ln y) = 1$

③ If $y \in [1, e] \Rightarrow 0 \le \ln y \le 1$

$\Rightarrow P(X \le \ln y)$

$= \ln y$

Summarize

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ \ln y & 0 \le y \le 1 \\ 1 & \text{for } y \geq e \end{cases}$$

CDF of Y

Functions of Continuous Random Variable...

Example of a function of continuous random variable

Let $X \sim$ Uniform$(0, 1)$, and let $Y = e^X$.

- Find the CDF of Y
- Find the PDF of Y
- Find $E[Y]$

$$E[Y] = E[e^X] = \int_{-\infty}^{\infty} e^x f_X(x)\, dx$$

$$= \int_0^1 e^x \cdot 1\, dx = \left[e^x\right]_0^1 = e^1 - e^0 = \underline{\underline{e-1}}$$

Functions of Continuous Random Variable...

$\frac{a+b}{2}$

Example of a function of continuous random variable

e^{2X}

Let $X \sim$ Uniform$(0, 1)$, and let $Y = e^X$.

- Find the CDF of Y
- Find the PDF of Y
- Find $E[Y]$

$E[Y^2]$

Ex

$Var[Y] = E[Y^2] - (E[Y])^2$

Find the mean and variance of $\cancel{X}$ Y.