Discrete probability distribution:

heads p tails q ; N tosses

Sample space $\{0 \ldots \ldots ; N\}$ Number of n heads

Probability of getting $\cancel{N}$ n heads.

$$p(n) = {}^N C_n \, p^n q^{N-n}.$$ Binomial distribution

$$\sum_{n=0}^{N} p(n) = 1$$ Bernoulli trials

Average value $\langle n \rangle$ $\quad Var(n) = \langle n^2 \rangle - \langle n \rangle^2$

$$= \langle (n - \langle n \rangle)^2 \rangle$$

$$SD(n) = \sqrt{var(n)} = \Delta n$$

$\Delta n / \langle n \rangle$ = relative fluctuation.

$$\langle n \rangle = \sum_{n=0}^{N} n \, p_n \qquad \langle n^k \rangle = \sum_{n=0}^{N} n^k p(n)$$

$\sum^{N}$ ... $^{N-n}$ ... $N!$ ... $^{N-n}$

$$\sum_{n=0} n \, {}^N C_n \, p^n (1-p) \quad = \sum n \frac{\cdot}{n!(N-n)!} p (1-p)$$

$$= \frac{N(N-1)!}{(n-1)!(N-n)!} p^n (1-p)^{N-n}$$

$$= N p \sum_{n=1}^{N} \binom{N-1}{n-1} p^{n-1} (1-p)^{N-n}$$

$$\sum n(n-1) \frac{N!}{(N-n)!\, n!} p^n (1-p)^{N-n}.$$

$$= \sum \frac{N!}{(N-n)!(n-2)!} p^n (1-p)^{N-n}.$$

$$= N(N-1)p^2 \sum \frac{(N-2)!}{(N-n)!(n-2)!} p^{n-2} (1-p)^{N-n}$$

$$= N(N-1)p^2 \quad , \quad \langle n(n-1) \rangle = \langle n^2 - n \rangle$$

$$\langle n^2 \rangle = N(N-1)p^2 + \langle n \rangle$$

$$\langle n^2 \rangle - \langle n \rangle^2 = N(N-1)p^2 + Np - N^2 p^2$$

$$= -Np^2 + Np = Np(1-p)$$

Generating function:

$$f(z) = \sum_{n=0}^{N} p_n z^n = \sum_{n=0}^{N} {}^{N}C_n \, p^n (1-p)^{N-n} z^n$$

$$= (pz + 1-p)^N$$

$$\frac{\partial f}{\partial z} = \sum n \, p_n \, z^{n-1} \qquad \left.\frac{\partial f}{\partial z}\right|_{z=1} = \sum n p_n$$

$$\frac{\partial f}{\partial z} = N (pz + 1-p)^{N-1} \cdot p \cdot = Np$$

$$\frac{\partial^2 f}{\partial z^2} = \sum n(n-1) \, p_n \, z^{n-2}$$

$$\frac{\partial^2 f}{\partial z^2} = \langle n(n-1) \rangle$$

$$\frac{\partial^2 f}{\partial z^2} = N(N-1)(p\cancel{z} + 1-p)^{N-2} \cdot p^2$$

$$= N(N-1) p^2$$

$$\langle n(n-1)\rangle = N(N-1)p$$

$$\langle n^2\rangle - \langle n\rangle^2 = N p(1-p)$$

$$\Delta n = \sqrt{N p(1-p)}$$

$$\frac{\Delta n}{n} = \frac{\sqrt{N p(1-p)}}{N p} = \frac{1}{\sqrt{N}}\sqrt{\frac{1-p}{p}}$$

$$= \frac{1}{\sqrt{N}}\left(\frac{1}{p} - 1\right)^{1/2}$$

Physical example.

What is the probability that n particles in v

$$p(n) = {}^{N}C_{n}\left(\frac{v}{V}\right)^{n}\left(1-\frac{v}{V}\right)^{N-n}$$

$$\langle n\rangle = N\frac{v}{V} \qquad \Delta n = N\frac{v}{V}\left(1-\frac{v}{V}\right)$$

$$\frac{\Delta n}{n} = \frac{1}{\sqrt{N}}\left(\frac{V}{v} - 1\right)^{1/2}. \qquad \langle n\rangle = \rho v$$

$\overline{\langle n \rangle}$ $-$ $\sqrt{N}\,(\overline{v}-1)$ $\langle n \rangle$

Geometric distribution

$p(n)$: prob of head first time at $(n+1)$ th toss

$p(n) = (1-p)^n \, p$. sample space

$$f(z) = \sum_{n=0}^{\infty} p(n)\, z^n$$

$$= \cancel{p}\, (qz)^n$$

$$= p\, \frac{1}{1-zq}$$

$$\frac{\partial f}{\partial z} = p \frac{q}{(1-qz)^2} = \frac{q}{p}$$

$$\frac{\partial^2 f}{\partial z^2} = \langle n(n-1) \rangle = p\, \frac{2q^2}{(1-qz)^3}$$

$$= \frac{2q^2}{p^2}$$

$2q^2$

$$\langle n^2 - n\rangle = \frac{2q^2}{p^2}$$

$$\langle n^2\rangle - \langle n\rangle^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p}\left(\frac{q}{p}+1\right)$$

$$= \frac{1-p}{p}\left(\frac{1-p}{p}+1\right) = \frac{1-p}{p^2}$$

$$\Delta n = \sqrt{\frac{1-p}{p^2}}$$

$$\frac{\Delta n}{n} = \sqrt{\frac{1-p}{p^2} \times \frac{p}{1-p}} = \frac{1}{\sqrt{1-p}} > 1$$

photon gas distribution - black body radiation

Binomial distrib

$p \to 0 \qquad Np \to \mu$

$N \to \infty$

$$\binom{N}{n} p^n q^{N-n} \longrightarrow$$

$$\frac{N!}{(N-n)!\,n!} p^n (1-p)^{N-n}$$

Sterling $N! \simeq N^N e^{-N}\sqrt{2\pi N}$

$\overrightarrow{p \to 0}$
$N \to \infty$
$Np \to \infty$

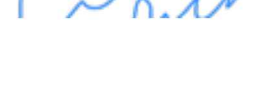
$$\left(\frac{\mu}{N}\right)^n \left(1-\frac{\mu}{N}\right)^{N-n} \left\{1+\frac{1}{12N} + O\left(\frac{1}{N^2}\right)\right\}$$

$$\to \frac{\mu^n e^{-\mu}}{n!}$$ poisson distribution

$$f(z) = \sum_{n=0}^{\infty} p(n)\, z^n = \sum_{n=0}^{\infty} \frac{(\mu z)^n e^{-\mu}}{n!}$$

$$= e^{-\mu} e^{\mu z} = e^{\mu(z-1)}$$

Nuclear decay



Stochastic processes. 

$x_1 \dots\dots x_k$ . . sample space

$t_1 \dots\dots t_n$. sampling epoch.

$f(x_1, t_1) \quad f(x_2 t_2 ; x_1 t_1) \dots f(x_1 t_1 ; x_2 t_2 \dots x_n t_n)$

$f_n(x_n t_n ; x_{n-1} t_{n-1} \dots ; x_1 t_1)$

$= f_n(x_n t_n | x_{n-1} t_{n-1} \dots x_1 t_1) \, f_{n-1}(x_{n-1} t_{n-1} \dots x_1 t_1)$

Markov processes.

$f(x_n t_n | x_{n-1} t_{n-1}, \dots x_1 t_1) = f(x_n t_n | x_{n-1} t_{n-1})$

$= \prod_{r=1} f(x_r t_r | x_{r-1} t_{r-1}) \, f(x_1 t_1)$

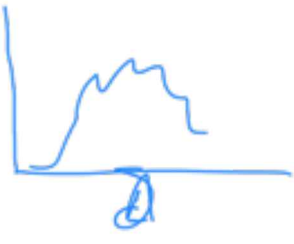
$\tau \sim 1$

Stationarity, $p(K, t \mid J, t')$

$$= p(K, t-t' \mid J, 0)$$

Einstein's derivation

prob of finding the particle $\phi(\Delta)$



at x and $x+\Delta x$ at $t+\tau$

$$p(x, t+\tau) = \int_{-\infty}^{+\infty} d\Delta \, p(x-\Delta, t)\, \phi(\Delta)$$

$$p(x,t) + \tau \frac{\partial p}{\partial t} + \ldots = p(x,t) \int \phi(\Delta)\, d\Delta + \frac{\partial p}{\partial x} \int \Delta\, \phi(\Delta)\, d\Delta + \frac{\partial^2 p}{\partial x^2} \int \frac{\Delta^2}{2} \phi(\Delta)\, d\Delta$$

Markov processes

$$p(x_n, t_n; x_{n-1}, t_{n-1}; \ldots x_1, t_1)$$

$$p(x_2 t_2; x_1 t_1) = p(x_2 t_2 | x_1 t_1)\, p(x_1, t_1)$$

$$p(x_3 t_3; x_2 t_2; x_1 t_1) = p(x_3 t_3 | x_2 t_2; x_1 t_1)$$

$$p(x_2 t_2 x_1 t_1)$$

$$= p(x_3 t_3 | x_1 t_1; x_2 t_2)\, p(x_2 t_2; x_1 t_1)$$

$$= p(x_3 t_3 | x_1 t_1; x_2 t_2)\, p(x_2 t_2 | x_1 t_1)\, p(x_1, t_1)$$

Markow approx : $p(x_3 t_3 | x_2 t_2)\, p(x_2 t_2 | x_1 t_1)\, p(x_1, t_1)$

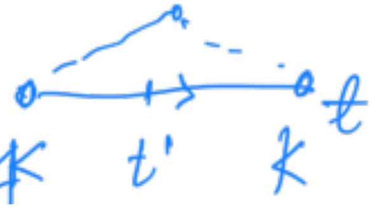
$$p(x_n t_n; x_{n-1} t_{n-1} \ldots x_1 t_1)$$

$$= \prod_{m=1}^{n} p(x_m t_m | x_{m-1} t_{m-1})\, p(x_0, t_0).$$

Stationarity. $p(k\, t_1 | j, t_0) = p(k, t_1 - t_0 | j)$

$$\prod_{m=1}^{n} p(x_m\; t_m - t_{m-1} | x_{m-1})\, p(x_0 t_0)$$

Chapman Kolgomorov condition.



$p(k\, t | j) = \quad p(l | j$

$$= \sum_{\ell} p(k, t-t' | \ell)\, p(\ell\ t' | j)$$

$t - t' = \delta t \quad t' = t - \delta t$

$$p(k\ t | j) = \sum_{\ell} p(k\ \delta t | \ell)\, p(\ell\ t-\delta t | j)$$

$$= \sum_{\ell} \omega_{k\ell}\, p(\ell\ t-\delta t | j)\, \delta t$$

$p(k\ t|j) - p(k\ t-\delta t | j) \quad \sum \omega_{\ell k} \delta t = 1$

$$= \sum_{\ell} \omega_{k\ell}\, p(\ell\ t-\delta t | j) \delta t - \sum_{\ell} \omega_{\ell k} p(k\ t-\delta t | j) \delta t$$

$$= \sum_{\ell \neq k} \omega_{k\ell}\, p(\ell\ t-\delta t | j) \delta t - (\omega_{\ell k} - \omega_{kk}) p(k\ t-\delta t | j)$$

$$\frac{dp(\ell\ t|j)}{dt} = \sum_{\ell \neq k} \Big(\underbrace{\omega_{k\ell}\, p(\ell\ t|j)}_{\text{gain}} - \underbrace{\omega_{\ell k}\, p(k\ t|j)}_{\text{loss}} \Big)$$

steady state $\frac{dp}{dt} = 0$

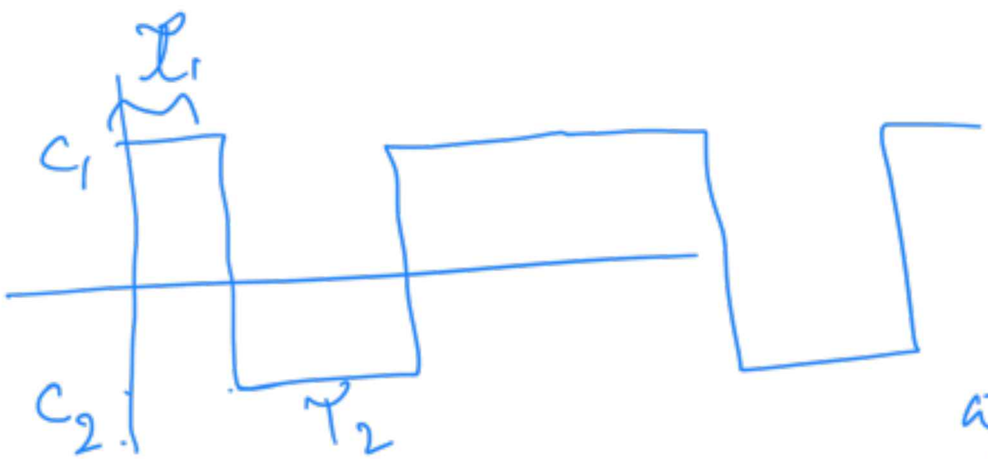
details balance

$$\omega_{kl}\, p(l) - \omega_{lk}\, p(k) = 0$$

$$p(k) = \frac{1}{1 + \sum_{l \neq k} \frac{\omega_{lk}}{\omega_{kl}}}.$$

Examples

dichotomous markov processes.



$$\frac{d}{dt} p(1) = \omega_{12}\, p(2) - \omega_{21}\, p(1)$$

$$\frac{d}{dt} p(2) = \omega_{21}\, p(1) - \omega_{12}\, p(2)$$

$$p(1) + p(2) = 1$$

$$p(2) = \frac{\omega_{21}}{\omega_{12}} p(1)$$

$$\left(\frac{\omega_{21}}{\omega_{12}} + 1\right) p(1) = 1 \qquad p(1) = \frac{\omega_{12}}{\omega_{12}+\omega_{21}}$$

$$p(2) = \frac{\omega_{21}}{\omega_{12}+\omega_{21}}$$

Poisson process. $n(t)$

$$\frac{d}{dt} p(n,t) = \lambda p(n-1,t) - \lambda p(n,t)$$

$$n-1 \xrightarrow{\lambda} n \xrightarrow{\lambda} n+1$$

$$f(z,t) = \sum_{n=0}^{\infty} p(n,t) z^n .$$

$$\sum^{\infty} \frac{d}{dt} p(n,t) z^n = \sum^{\infty} \left(\lambda\, p(n-1,t) z^n\right.$$

$\sum_{n=0} \quad n \geq 0 \quad - \lambda p(n,t) z)$

$$\frac{d}{dt} f(z) = \lambda z \sum_{n=0}^{\infty} p(n-1,t)\, z^{n-1} - \lambda \sum_{n=0}^{\infty} z^n p(n,t)$$

$$= \lambda (z-1) f(z,t)$$

$$f(z) = e^{\lambda t (z-1)} = \sum_{n=0} z^n p(n,t)$$

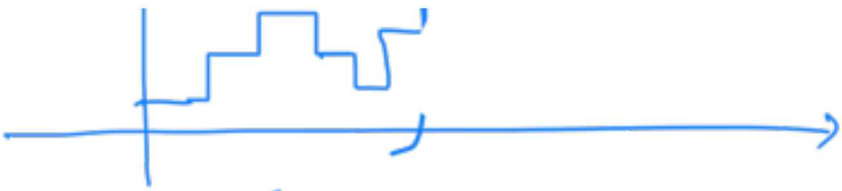
$$= \sum e^{-\lambda t} \frac{(\lambda t)^n}{n!} z^n$$

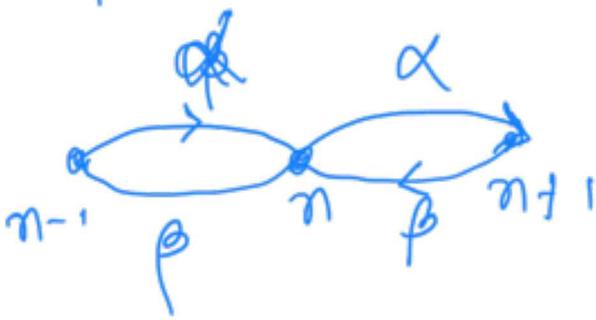
$$\Rightarrow \quad p(n,t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$\langle n(t) \rangle : \lambda t$$

$$\text{var } n(t) : \lambda t.$$

Random walk on a lattice





$$\frac{d}{dt} p(n,t) = \alpha\, p(n-1,t) - \alpha\, p(n,t) + \beta\, p(n+1,t) - \beta\, p(n,t)$$

$$= \alpha\, p(n-1,t) + \beta\, p(n+1,t) - (\alpha+\beta)\, p(n,t)$$

$$f(z) = \sum z^n\, p(n,t)$$

$$\frac{d}{dt} f(z) = \sum \alpha\, z^n\, p(n-1,t) + \beta \sum z^n\, p(n+1,t) - (\alpha+\beta) \sum z^n\, p(n,t)$$

$$\frac{d}{dt} f(z) = \left[\alpha z + \frac{\beta}{z} - (\alpha+\beta)\right] f(z)$$

$$f(z) = e^{\left[\alpha z + \beta/z - (\alpha+\beta)\right]t} \qquad \alpha+\beta=\lambda$$

$$= e^{-\lambda t}\, e^{ut} \qquad u = \alpha z + \frac{\beta}{z}$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(ut)^n}{n!} \qquad \frac{\partial u}{\partial z} = \alpha - \frac{\beta}{z^2}$$

$$\frac{\partial f}{\partial z} = e^{-\lambda t} e^{ut}\, t \frac{\partial u}{\partial z}.$$

$$= e^{-\lambda t} e^{ut}\, t\left(\alpha - \frac{\beta}{z^2}\right)$$

$$\left.\frac{\partial f}{\partial z}\right|_{z=1} = e^{-\lambda t} e^{\lambda t} (\alpha-\beta)\, t = (\alpha-\beta) t$$

$$\Rightarrow \langle n(t)\rangle = (\alpha-\beta)\, t$$

$$\frac{\partial^2 f}{\partial z^2} = e^{-\lambda t}\left[e^{ut}\left(t\frac{\partial u}{\partial z}\right)^2 + e^{ut}\, t \frac{\partial^2 u}{\partial z^2}\right]$$

$$\left.\frac{\partial^2 f}{\partial z^2}\right|_{z=1} = t\left[2\beta + (\alpha-\beta)^2 t\right] = \langle n^2\rangle - \langle n\rangle$$

$$\langle n^2\rangle - \langle n\rangle^2 = t\left[2\beta + (\alpha-\beta)^2 t\right] + (\alpha-\beta)t - (\alpha-\beta)^2 t^2$$

$$\sigma_n^2 = (2\beta + \alpha - \beta)t = (\alpha+\beta)t.$$

$$\sigma_n = \sqrt{(\alpha+\beta)}\sqrt{t}$$

Birth and decay of a molecule

$$\xrightarrow{\alpha} n \xrightarrow{\beta}$$

$$\begin{tikzcd}
n-1 \arrow[r, bend left, "\alpha"] & n \arrow[l, bend left, "\beta n"] \arrow[r, bend left, "\alpha"] & n+1 \arrow[l, bend left, "\beta(n+1)"]
\end{tikzcd}$$

$$\frac{dp(n,t)}{dt} = \alpha\, p(n-1,t) - \beta n\, p(n,t) - \alpha\, p(n,t) + \beta(n+1)\, p(n+1,t)$$

$$= \alpha\, p(n-1,t) + \beta(n+1)\, p(n+1,t)$$

$$-(\alpha+\beta n) f(n,\ell)$$

$$f(z,\ell) = \sum z^n f(n,\ell) \qquad \frac{\partial f}{\partial z} = \sum n z^{n-1} f(n,\ell) = \sum (n+1) z^n f(n+1,\ell)$$

$$\frac{\partial f}{\partial t} = \sum_n \alpha z^n f(n-1,\ell) + \beta \sum_n z^n (n+1) f(n+1,\ell) - \sum (\alpha+\beta n) z^n f(n,\ell)$$

$$= \alpha z f(z) + \beta \frac{\partial f}{\partial z} - \alpha f(z) - \beta z \frac{\partial f}{\partial z}$$

$$= \alpha(z-1) f(z) + \beta(1-z) \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial z} = \frac{\alpha(z-1)}{\beta(z-1)} f(z); \quad \frac{\partial f}{\partial z} = \frac{\alpha}{\beta} \quad \langle n \rangle = \frac{\alpha}{\beta}$$

$$\frac{\partial f}{\partial z} = \frac{\alpha}{\beta} f(z)$$

$$\frac{\partial^2 f}{} \quad \frac{\alpha}{} \frac{\partial f}{} \quad \left(\frac{\alpha}{}\right)^2 = \langle n^2 \rangle - \langle n \rangle$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\alpha}{\beta}\frac{\partial f}{\partial z} = \left(\frac{\alpha}{\beta}\right)^2 = \langle n^2\rangle - \langle n\rangle$$

$$\langle n^2\rangle - \langle n\rangle^2 = \left(\frac{\alpha}{\beta}\right) \Rightarrow \sigma_n^2 = \left(\frac{\alpha}{\beta}\right)$$

$$\frac{\partial f}{\partial z} = \frac{\alpha}{\beta} f(z) \qquad f(z) = A e^{\alpha/\beta\, z}$$

at $z=1$ $f(z)=1$ $\Rightarrow A = e^{-\alpha/\beta}$

$$f(z) = e^{-\alpha/\beta} e^{\alpha/\beta\, z}$$

$$= \sum_n e^{-\alpha/\beta} \frac{(\alpha/\beta)^n}{n!} z^n.$$

$$p(n) = e^{-\alpha/\beta} \frac{(\alpha/\beta)^n}{n!}$$ poisson distribution

$$\Rightarrow \langle n\rangle = \alpha/\beta \quad \sigma_n^2 = \alpha/\beta$$

General case of α_n and β_n

$$\frac{d}{dt}\langle n(t)\rangle = \sum n \frac{dp}{dt}$$

$$= \sum_n n \left[\alpha_{n-1} p(n-1,t) + \beta_{n+1} p(n+1,t) - \alpha_n p(n,t) - \beta_n p(n,t) \right]$$

$$= \sum_n (n+1)\, \alpha_n\, p(n,t) + \sum \beta_n\, p(n,t)(n-1) - \sum \alpha_n\, n\, p(n,t) - \sum \beta_n\, n\, p(n,t)$$

$$= \langle \alpha_n \rangle - \langle \beta_n \rangle = \langle \alpha_n - \beta_n \rangle$$

$$+ \sum \alpha_{n+1}\, p(n,t) + \sum n \beta_n\, p(n,t) - \sum \beta_n\, p(n,t) - \sum \beta_n\, n\, p(n,t)$$

$=$

variance

$$\frac{d}{dt} \langle n^2(t) \rangle = \sum n^2\, \alpha_{n-1}\, p(n-1,t) + \sum n^2\, \beta_{n+1}\, p(n+1,t)$$

$$\sum (\alpha_n + \beta_n)\, n^2\, p(n,t)$$

$$-\sum (\alpha_n + \beta_n) n^2 p(n,t)$$

$$= \sum (n+1)^2 \alpha_n p(n,t) + \sum (n-1)^2 \beta_n p(n,t)$$

$$- \sum n^2 \alpha_n p(n,t) - \sum n^2 \beta_n p(n,t)$$

$$= 2\sum n \alpha_n p(n,t) + \sum \alpha_n p(n,t) - \sum 2 n \beta_n p(n,t)$$

$$+ \sum \beta_n p(n,t)$$

$$= \langle 2n(\alpha_n - \beta_n) \rangle + \langle \alpha_n + \beta_n \rangle$$

$$\frac{d}{dt}\sigma^2(t) = \frac{d}{dt}\left[\langle n^2(t)\rangle - \langle n(t)\rangle^2\right]$$

$$= \frac{d}{dt}\langle n^2(t)\rangle - 2\langle n(t)\rangle \frac{d\langle n(t)\rangle}{dt}$$

$$= \langle 2n(\alpha_n - \beta_n)\rangle + \langle \alpha_n + \beta_n \rangle - 2\langle n(t)\rangle$$

$$\left[\langle \alpha_n - \beta_n \rangle \right]$$

$$= \langle \alpha_n + \beta_n \rangle + 2 \langle n (\alpha_n - \beta_n) \rangle - \langle n \rangle \langle \alpha_n + \beta_n \rangle$$

For birth and decay

$$\frac{d}{dt} \langle n \rangle = \alpha - \beta \langle n \rangle$$

$$\langle n(t) \rangle = \frac{\alpha}{\beta} \left(1 - e^{-\beta t} \right)$$

$$\frac{d}{dt} \langle n^2 \rangle = \langle 2n (\alpha_n - \beta_n) \rangle + \langle \alpha_n + \beta_n \rangle$$

$$= 2\alpha \langle n \rangle - 2\beta \langle n^2 \rangle + \alpha + \beta \langle n \rangle$$

$$\frac{d}{dt} \langle n^2 \rangle = (2\alpha + \beta) \langle n \rangle - 2\beta \langle n^2 \rangle + \alpha .$$

$$\frac{d}{dt}\langle n^2\rangle + 2\beta\langle n^2\rangle = (2\alpha+\beta)\langle n\rangle + \alpha$$

$$\frac{d}{dt}\left(e^{2\beta t}\langle n^2\rangle\right) = e^{2\beta t}\left[(2\alpha+\beta)\left[(\alpha/\beta)(1-e^{-\beta t})\right] + \alpha\right]$$

$$e^{2\beta t}\langle n^2\rangle = \int (2\alpha+\beta)\frac{\alpha}{\beta}\left(e^{2\beta t} - e^{\beta t}\right)dt + \int \alpha e^{2\beta t}\, dt$$

$$= (2\alpha+\beta)\,\alpha/\beta\left(\frac{1}{2\beta}e^{2\beta t} - \frac{1}{\beta}e^{\beta t}\right) + \frac{\alpha}{2\beta}e^{2\beta t} + c$$

$$\langle n^2\rangle = \frac{(2\alpha+\beta)\alpha}{\beta^2}\left[\frac{1}{2} - e^{-\beta t}\right] + \frac{\alpha}{2\beta}.$$

$$= \left\{2\left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta}\right\}\left[\frac{1}{2} - e^{-\beta t}\right] + \frac{\alpha}{2\beta}.$$

$$= \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right) - \left[2\left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta}\right]e^{-\beta t} + \left(\frac{\alpha}{\beta}\right)^2 e^{-2\beta t}$$

$=$

$$\langle n^2\rangle - \langle n\rangle^2 = \left(\frac{\alpha}{\beta}\right)^2 + (\alpha/\beta) - \left[2\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right)\right]e^{-\beta t}$$
$$+ (\alpha/\beta)^2 e^{-\beta t} - \left(\frac{\alpha}{\beta}\right)^2\left(1-e^{-\beta t}\right)^2$$

$$= \cancel{\left(\frac{\alpha^2}{\beta^2}\right)} + \left(\frac{\alpha}{\beta}\right) - 2\cancel{\left(\frac{\alpha}{\beta}\right)^2} e^{-\beta t} - \left(\frac{\alpha}{\beta}\right)e^{-\beta t}$$
$$+ \left(\frac{\alpha}{\beta}\right)^2 e^{-\beta t} - \cancel{\left(\frac{\alpha}{\beta}\right)^2} - \left(\frac{\alpha}{\beta}\right)^2 e^{-2\beta t}$$
$$+ 2\cancel{\left(\frac{\alpha}{\beta}\right)^2} e^{-\beta t}$$

$$\sigma^2(t) = \left(\frac{\alpha}{\beta}\right)\left(1-e^{-\beta t}\right) + \left(\frac{\alpha}{\beta}\right)^2 e^{-\beta t}\left(1-e^{-\beta t}\right)$$

<u>Random walK</u>

$$\frac{d}{dt}\langle n(t)\rangle = \alpha - \beta$$

$$\frac{d}{dt}\langle n^2\rangle = \langle n(\alpha-\beta)\rangle + (\alpha+\beta)$$
$$= (\alpha-\beta)^2 t + (\alpha+\beta)$$

$$\langle n^2 \rangle = (\alpha-\beta)^2 t^2 + (\alpha+\beta) t$$

$$\sigma_n^2 = (\alpha+\beta) t$$

<u>Sedimentation</u>

$$\begin{tikzcd} n-1 \arrow[r, bend left, "\alpha"] & n \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & n+1 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

$$\frac{d}{dt} p(n,t) = \alpha\, p(n-1,t) + \beta\, p(n+1,t) - (\alpha+\beta)\, p(n,t)$$

$$\frac{d}{dt} p(0,t) = \beta\, p(1,t) - \alpha\, p(0,t)$$

$$p(1) = \frac{\alpha}{\beta} p(0)$$

$$\frac{d}{dt} p(1,t) = \alpha\, p(0,t) + \beta\, p(2,t) - (\alpha+\beta)\, p(1,t)$$

$$\alpha\, p(0) + \beta\, p(2) - (\alpha+\beta) \frac{\alpha}{\beta} p(0) = 0$$

$$p(2) = \left(\frac{\alpha}{\beta}\right)^2 p(0)$$

$$p(n) = \left(\frac{\alpha}{\beta}\right)^n p(0)$$

Normalization

$$\sum p(n) = 1 \Rightarrow \sum_n \left(\frac{\alpha}{\beta}\right)^n p(0) = 1$$

$$\frac{1}{1-\alpha/\beta}\, p(0) = 1 \qquad p(0) = 1-\alpha/\beta$$

$$p(n) = \left(1-\frac{\alpha}{\beta}\right)\left(\alpha/\beta\right)^n = (1-r)\, r^n$$

$$\langle n \rangle = \sum n\, p(n)$$

$$\sum r^n = \frac{1}{1-r}. \qquad \frac{\partial}{\partial r}\sum r^n = +\frac{1}{(1-r)^2}$$

$$\sum n r^{n-1} = +\frac{1}{(1-r)^2}$$

$$\langle n \rangle = +\, p_0 \frac{r}{(1-r)^2} = \frac{r}{1-r} = \frac{\alpha/\beta}{1-\alpha/\beta}$$

$$= \frac{\alpha}{\beta-\alpha}$$

Population growth

$$\begin{tikzcd} A \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & {} \end{tikzcd}$$

$$\begin{tikzcd} n-1 \arrow[r, bend left, "\alpha"] & n \arrow[l, "\beta"] \arrow[r, bend left, "\alpha"] & n+1 \arrow[l, "\beta"] \end{tikzcd}$$

$$\frac{dp(n,t)}{dt} = \alpha(n-1)\, p(n-1,t) + \beta(n+1)\, p(n+1,t) - (\alpha+\beta)\, n\, p(n,t)$$

$$\frac{df}{dt} = \alpha \sum n\, p(n,t)\, z^{n+1} + \beta \sum z^{n-1} p(n,t) - \sum (\alpha+\beta)\, n\, z^{n}\, p(n,t)$$

$$= \left[\alpha z^2 + \beta - (\alpha+\beta) z\right] \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial t} = (\alpha z - \beta)(z-1) \frac{\partial f}{\partial z}$$

$$\frac{d}{dt} \frac{\partial f}{\partial z} = \alpha (z-1) \frac{\partial f}{\partial z} + (\alpha z - \beta) \frac{\partial f}{\partial z} + (\alpha z - \beta)(z-1) \frac{\partial^2 f}{\partial z^2}$$

$$\frac{d}{dt} \langle n(t) \rangle = (\alpha - \beta) \langle n(t) \rangle$$

$$\langle n(t) \rangle = n(0)\, e^{(\alpha-\beta)t}$$

$$\frac{d}{dt}\langle n^2(t)\rangle_n = \left\langle 2n(\alpha_n - \beta_n)\right\rangle + \langle \alpha_n + \beta_n\rangle$$

$$= 2\langle n^2\rangle(\alpha-\beta) + \langle n\rangle(\alpha+\beta)$$

$$\frac{d}{dt}\langle n^2\rangle - 2(\alpha-\beta)\langle n^2\rangle = (\alpha+\beta)\, n(0)\, e^{(\alpha-\beta)t}$$

$$\frac{d}{dt}\left(e^{-2(\alpha-\beta)t}\langle n^2\rangle\right) = (\alpha+\beta)\, n(0)\, e^{-(\alpha-\beta)t}$$

$$e^{-2(\alpha-\beta)t}\langle n^2\rangle = (\alpha+\beta)\, n(0)\int_0^t e^{-(\alpha-\beta)t}\,dt$$

$$= (\alpha+\beta)\, n(0)\left[\frac{e^{-(\alpha-\beta)t}-1}{-(\alpha-\beta)}\right]$$

$$= \frac{(\alpha+\beta)\, n(0)}{(\alpha-\beta)}\left[1 - e^{-(\alpha-\beta)t}\right]$$

$$\langle n^2\rangle = \frac{\alpha+\beta}{\alpha-\beta}\, n(0)\left[e^{2(\alpha-\beta)t} - e^{(\alpha-\beta)t}\right]$$

$$\langle n\rangle^2 = n_0^2\, e^{2(\alpha-\beta)t}.$$

$$\beta = 0 \qquad \langle n^2\rangle = n(0)\left[e^{2\alpha t} - e^{\alpha t}\right]$$

$\langle n^2 \rangle^{\sim} = \tilde{n}(0)\, e^{2\alpha t}.$

$$\delta n^{2} = n(0)\, e^{2\alpha t} - e^{\alpha t} - \tilde{n}(0) e^{2\alpha t}$$

<u>Greshgorin theorem</u>

$$\frac{dp(k,t|j)}{dt} = \sum_{l \neq k} \omega_{kl}\, p(l,t|j) - \omega_{lk}\, p(k,t|j)$$

$$P = \begin{pmatrix} p(1) \\ p(2) \\ \vdots \\ p(N) \end{pmatrix} \qquad W_{kl} = \omega_{kl}$$

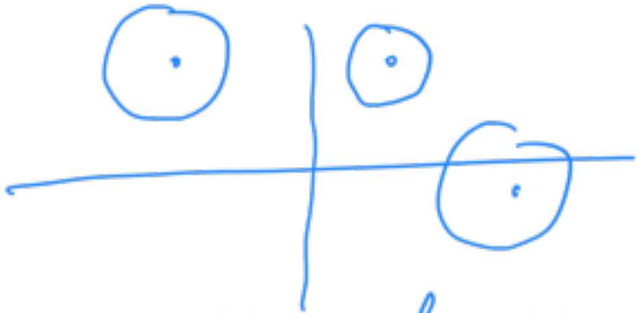
$$W_{kk} = -\sum_{l \neq k} \omega_{lk}$$

$$\frac{dP}{dt} = W P \qquad P(t) = e^{Wt} P(0)$$

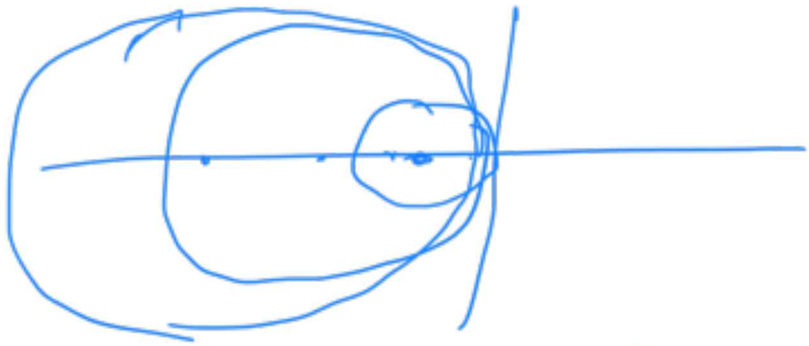
$\det W = 0$ $\quad \lambda = 0$ is an eigenvalue

A steady state distribution exists and p' is always non negative

Greshgorin theorem The eigenvalue lie ~~oo~~ inside the Greshgorin disc.



For this case the diagonal element are real and negative



Hence the eigenvalus cannot be possible. Hence p's cannot become infinity.

$$P(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

dP . h. $P(t) = 0$

$$\sum_{\substack{l=1\\ l\neq k}}^{N} \left(w_{kl}\, p(l) - w_{lk}\, p(k) \right) = 0$$

detailed balance

$$w_{kl}\, p(l) = w_{lk}\, p(k)$$

$$\sum p(k) = 1 \qquad p(k) = \frac{w_{kl}}{w_{lk}}\, p(l)$$

$$p(k) = \frac{w_{k1}}{w_{1k}}\, p(1)$$

$$\sum_k \cancel{k} \frac{w_{kl}}{w_{lk}}\, p(l) = 1$$

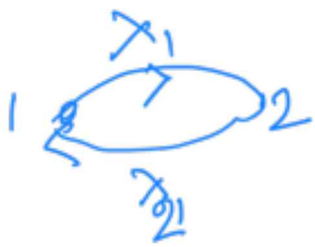
$$\cancel{p(l)} =$$

$$p(l) + \sum_{k} \frac{w_{kl}}{w_{lk}}\, p(l) = 1$$

$$p(l) = \frac{1}{1 + \sum_{k \neq l} \frac{w_{kl}}{w_{lk}}}$$

Random telegraphic process

$$P(t) = e^{Wt}\, P(0)$$



w_{12}

$$W = \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix}$$

$$W^2 = \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix}$$

$$= \begin{matrix} \lambda_1^2 + \lambda_1\lambda_2 & -\lambda_1\lambda_2 - \lambda_2^2 \\ -\lambda_1^2 - \lambda_1\lambda_2 & \lambda_1\lambda_2 + \lambda_2^2 \end{matrix}$$

$$= -\begin{pmatrix} -\lambda_1(\lambda_1+\lambda_2) & \lambda_2(\lambda_1+\lambda_2) \\ \lambda_1(\lambda_1+\lambda_2) & -\lambda_2(\lambda_1+\lambda_2) \end{pmatrix}$$

$$= -(\lambda_1+\lambda_2) W \qquad W^3 = -(\lambda_1+\lambda_2) W^2$$

$$= -2\lambda W$$

$$e^{Wt} = I + Wt + \frac{t^2}{2}(-2\lambda) W + \frac{t^3}{3!}(-2\lambda)^2 W + \dots$$

$$= I + \frac{W}{-2\lambda}\left[-2\lambda t + \frac{(-2\lambda t)^2}{2!} + \dots \right]$$

$$= I + \frac{W}{-2\lambda}\left[e^{-2\lambda t} - 1\right]$$

$$= I + \frac{W}{2\lambda}\left(1 - e^{-2\lambda t}\right)$$

$$P(t) = \left[I + \frac{W}{2\lambda}\left(1 - e^{-\lambda t}\right)\right] P(0)$$

$t \to \infty$ $$P = \left[I + \frac{W}{2\lambda}\right] P(0)$$

$$W = \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(I + \frac{W}{2\lambda}\right) P = \begin{pmatrix} 1 - \frac{\lambda_1}{2\lambda} & \frac{\lambda_2}{2\lambda} \\ \frac{\lambda_1}{2\lambda} & 1 - \frac{\lambda_2}{2\lambda} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P(t \to \infty) = \begin{pmatrix} 1 - \frac{\lambda_1}{2\lambda} \\ \frac{\lambda_1}{2\lambda} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2}{\lambda_1 + \lambda_2} \\ \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{pmatrix}$$

$$\langle x \rangle = c_1 P(1) + c_2 (P_2)$$

$$= c\left(P(1) - P(2)\right) = c\,\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}$$

$$\langle x^2 \rangle = c^2 - c^2 \frac{(\lambda_2 - \lambda_1)^2}{(\lambda_1 + \lambda_2)^2}$$

$$= \frac{4\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} c^2 \qquad \frac{\lambda^2}{2} c^2$$

$$\left\langle \delta x(0)\, \delta x(t) \right\rangle = \sum_i \sum_j x_i\, x_j\, P(i\, t | j)\, p(j)$$

$$= x(1)\, x(1)\, P(1\, t | 1)\, P(1)$$

$$+ x(2)\, x(1)\, P(2\, t | 1)\, P(1)$$

$$= c^2$$

$$P(t) = \left[I + \frac{W}{2\lambda}\left(1 - e^{-2\lambda t}\right) \right] P(0)$$

$$p(t|1) = \left[I + \begin{pmatrix} -\lambda_p & \lambda_2 \end{pmatrix} (1 - e^{-2\lambda t}) \right]$$

$$p(1,t|1) = 1 - \frac{\lambda_1}{2\lambda}\left(1-e^{-2\lambda t}\right)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

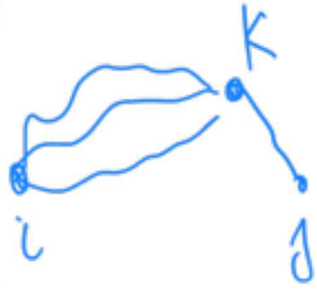
$$p(2,t|1) = \lambda_1\left(1-e^{-2\lambda t}\right)$$

$$\begin{aligned} E(t) &= c^2\left(1-\frac{\lambda_1}{2\lambda}\left(1-e^{-2\lambda t}\right)\right) \\ &\quad - c^2\frac{\lambda_1}{2\lambda}\left(1-e^{-2\lambda t}\right) \\ &= c^2\left(1-\frac{\lambda_1}{\lambda}\left(1-e^{-2\lambda t}\right)\right)\frac{\lambda_2}{2\lambda} \\ &\quad + c^2 \end{aligned}$$

<u>Markov chain</u> $t \neq K$

$$P_{ij} = p\left(x_n = j \mid x_{n-1} = i\right)$$

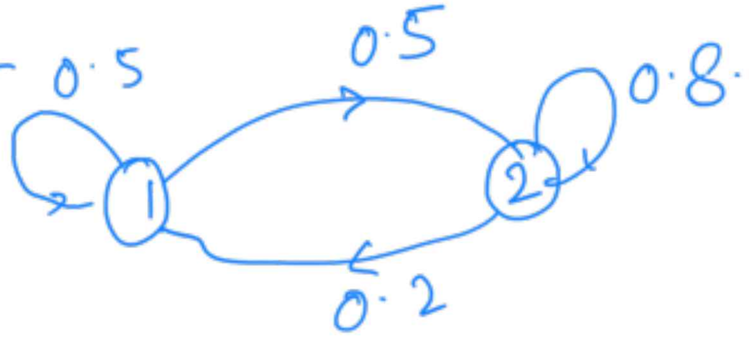
$$r_{ij}(n) = p(x_n = j \mid x_0 = i)$$



Recursion relation

$$r_{ij}(n) = \sum_{K} r_{iK}(n-1)\, p_{Kj}$$

Example



$$r_{11}(n) = r_{11}(n-1) \times 0.5 + r_{12}(n-1) \times 0.2$$

$$r_{12}(n) = r_{11}(n-1) \times 0.5 + r_{12}(n-1) \times 0.8$$

$$r_{21} = r_{22}(n-1) \times 0.8 + r_{21}(n-1) \times 0.5$$

$$r_{22} = 1 - r_{21}$$

	$n=0$	$n=1$	$n=2$	
$r_{11}(n)$	1	$0.5 \xrightarrow{0.5}$	$0.35 \xrightarrow{0.5}$ ..	$\frac{2}{7}$

$r_{12}(n)$	0	0.5	0.65	5/7
$r_{21}(n)$	0	$0.2 \xrightarrow{0.5}$	0.26	2/7
		0.2		5/7