Week 6, Lecture 12 on 29 September 2021 CS1.301.M21 Algorithm Analysis and Design

DP Review

- Shortest paths in DAGs
- Longest increasing subsequence
- Edit Distance
- Chain Matrix Multiplication
- Knapsack - NP Hard but in some cases (fractional) can be solved even with a Greedy solution.

> *Coming up with the subproblems is the key step in Dynamic Programming problems*

Shortest Reliable Paths

The Problem

Given a graph G with lengths for each edge, along with two nodes s and t and an integer k, we want the shortest path from s to t that uses at most k edges (a *reliable* path).

The Answer

> *Since identifying the sub problems for the DP approach is the difficult part, once the subproblems are identified we can consider the problem solved*

$i < k, \operatorname{dist}(v, i)$ is the length of the shortest path from s to v that uses i edges.

We get

$$\operatorname{dist}(v, i) = \min_{(u,v) \in E} \{\operatorname{dist}(u, i-1) + l(u, v)\}$$

Solving `dist(v,k)` will give the final answer of the problem.

> *The dynamic programming approach is great. But finding a greedy solution is always better.*

Recursion with memoization is fine. But technically DP is the iteration of a recursion with memoization.

All-pairs Shortest Paths

The Problem

Find the shortest path between all the pairs of nodes in a given graph

The Answer

We can run *Dijkstra's Algorithm* for the shortest pair for a given pair of nodes, for all pairs of nodes. that will be $O(|V|^2|E|)$

The *Floyd-Warshall Algorithm* calculates exactly this in $O(|V|^3)$

> *Does Floyd-Warshall Algorithm work for negative edge weights, given that there are no negative cycles?*

Let $\text{dist}(i, j, k)$ denote the length of shortest path from i to j in which only nodes $\{1, 2, \ldots, k\}$ can be used.

$\text{dist}(i, j, k)$ can be defined as:

$$\text{dist}(i, j, k) = \min\{\text{dist}(i, k, k-1) + \text{dist}(k, j, k-1), \text{dist}(i, j, k-1)\}$$

It turns out that a node k either is present in a shortest path or not. For it to be present we need the path going through it to be a shorter path than without it. This is calculated with the $\min$ function.

The algorithm becomes:

```
for all (i,j):
    initialise dist(i,j,0) = l(i,j)

for all k:
    for all i:
        for all j:
            dist(i,j,k) = min{dist(i,k,k-1) + dist(k,j,k-1), dist(i,j,k-1)}
```

> *Dynamic programming is characterized by the optimum sub structure property. But the speed lent to the paradigm comes only when used on problems that have a large amount of overlap in the subproblems.*

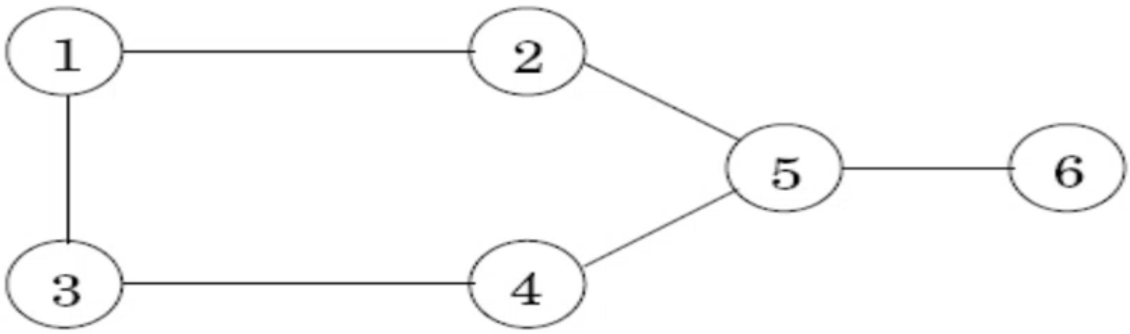
Independent Set in Trees

The Problem

> *The problem is NP hard for non-tree cases*

An independent set is a subset of nodes that have no edges between them.

For example, and independent set in this graph can be $\{1, 4, 6\}$ and is, in fact, of the largest size possible.



Though the shown example is graph, the problem for us is to find the size of the largest independent set in a given **tree**.

The Answer

The largest independent set of a tree should be made of the largest independent set of its subtrees.

Let $I(u)$ be the size of the largest independent set hanging from the node u.

This is the subproblem. And the base case is $I(\text{any leaf}) = 1$

To compute $I(u)$ we use:

$$I(u) = \max\{1 + \sum_{\text{grandchildren w of u}} I(w), \sum_{\text{children w of u}} I(w)\}$$

We compute $I(u)$ in a BFS manner starting from any node. And the final $I(u)$ will be the final answer.
