Probability and Statistics

UG2, Core course, IIIT,H

Pawan Kumar

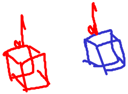
IIIT, Hyderabad

August 27, 2021

0 Computing probabilities...

Experiment: Throw two dice.

0 Computing probabilities...



Experiment: Throw two dice.

Event: Bigger number on red than blue.

0 Computing probabilities...

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

0 Computing probabilities...

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

- Probability space: all outcomes

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

- Probability space: all outcomes
- Event: some favourable outcomes

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

- Probability space: all outcomes
- Event: some favourable outcomes
- Probability: (#favorable)/36

11	12	13	14	15	16
$\boxed{21}$	22	23	24	25	26
$\boxed{31}$	$\boxed{32}$	33	34	35	36
$\boxed{41}$	$\boxed{42}$	$\boxed{43}$	44	45	46
$\boxed{51}$	$\boxed{52}$	$\boxed{53}$	$\boxed{54}$	55	56
$\boxed{61}$	$\boxed{62}$	$\boxed{63}$	$\boxed{64}$	$\boxed{65}$	66

Experiment: Throw two dice.
Event: Bigger number on red than blue.
Quiz: What is the probability of this event?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

Hence, probability, $p = \frac{15}{36}$

- Probability space: all outcomes
- Event: some favourable outcomes
- Probability: (#favorable)/36

11	12	13	14	15	16
$\boxed{21}$	22	23	24	25	26
$\boxed{31}$	$\boxed{32}$	33	34	35	36
$\boxed{41}$	$\boxed{42}$	$\boxed{43}$	44	45	46
$\boxed{51}$	$\boxed{52}$	$\boxed{53}$	$\boxed{54}$	55	56
$\boxed{61}$	$\boxed{62}$	$\boxed{63}$	$\boxed{64}$	$\boxed{65}$	66

Quiz

- Experiment: Toss a coin n times

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits
- Total number of outcomes: 2^n outcomes

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits
- Total number of outcomes: 2^n outcomes
- Assumption: equiprobable

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits
- Total number of outcomes: 2^n outcomes
- Assumption: equiprobable
- Probability of all heads: $1/2^n$

$\{H \ H \ \ldots \ldots \ H\}$

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits
- Total number of outcomes: 2^n outcomes
- Assumption: equiprobable
- Probability of all heads: $1/2^n$
- Consider an event of "first bit" = "last bit"

H T . . . . H T

H H . . . T H

1st bit = last bit

Quiz

- Experiment: Toss a coin n times
- Outcome: sequence of n bits
- Total number of outcomes: 2^n outcomes
- Assumption: equiprobable
- Probability of all heads: $1/2^n$
- Consider an event of "first bit" = "last bit"
- What is the probability of the above event?

0 Answer to Quiz...

Answer to Quiz:

Answer to Quiz:

- Number of possible cases:

1st bit = last bit

$$\begin{array}{cccccc} H & * & \cdots & * & H \\ T & * & \cdots & * & H \\ H & * & \cdots & * & T \\ T & * & \cdots & * & T \end{array}$$

Answer to Quiz:

- Number of possible cases:

$$
\begin{aligned}
&H * \cdots * H \\
&T * \cdots * H \\
&H * \cdots * T \\
&T * \cdots * T
\end{aligned}
$$

- Observation: In half of the cases, first bit does not equal last bit

Answer to Quiz:

- Number of possible cases:

$$
\begin{array}{c}
H * \cdots * H \\
T * \cdots * H \\
H * \cdots * T \\
T * \cdots * T
\end{array}
$$

- Observation: In half of the cases, first bit does not equal last bit
- Hence, probability of the event is $1/2$

0 Remark on Counting. Does probability boils down to counting always?

0 Remark on Counting. Does probability boils down to counting always?

- It is true that in many cases probability boils down to counting

0 Remark on Counting. Does probability boils down to counting always?

- It is true that in many cases probability boils down to counting
- However, there are non-uniform distributions $\leftarrow$

0 Remark on Counting. Does probability boils down to counting always?

- It is true that in many cases probability boils down to counting
- However, there are non-uniform distributions
- We also study continuous distributions

0 Remark on Counting. Does probability boils down to counting always?

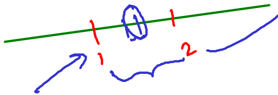
- It is true that in many cases probability boils down to counting
- However, there are non-uniform distributions
- We also study continuous distributions
- In the continuous case, we need to consider some other measure

0 Remark on Counting. Does probability boils down to counting always?

- It is true that in many cases probability boils down to counting
- However, there are non-uniform distributions
- We also study continuous distributions
- In the continuous case, we need to consider some other measure
- What is the probability that a number 1.5 is picked in [1, 2]?

$\frac{1}{\infty} = 0$

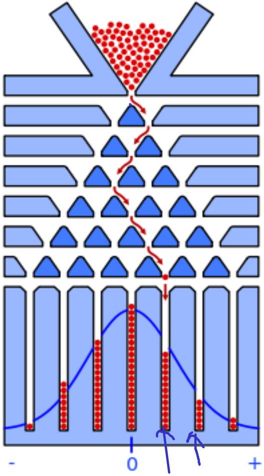
$\frac{\infty}{\infty}$ not defined



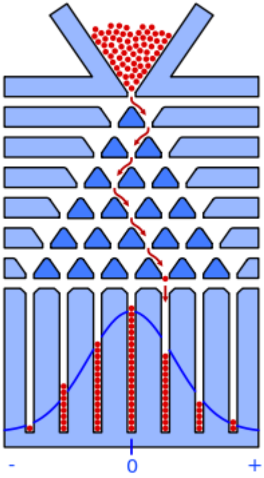
0 Remark on Counting. Does probability boils down to counting always?

- It is true that in many cases probability boils down to counting
- However, there are non-uniform distributions
- We also study continuous distributions
- In the continuous case, we need to consider some other measure
- What is the probability that a number 1.5 is picked in $[1, 2]$? Does this make sense?

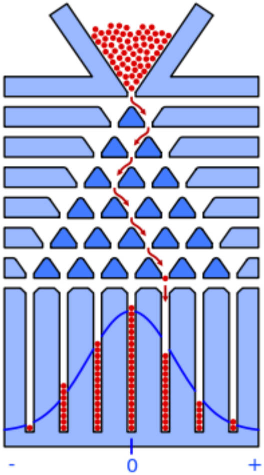
Movie of Galton board here!



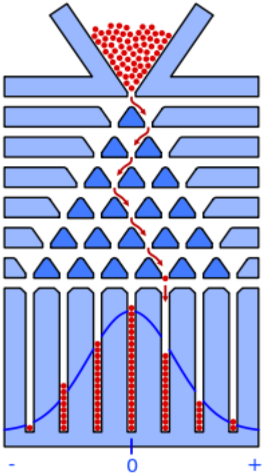
-
0
+



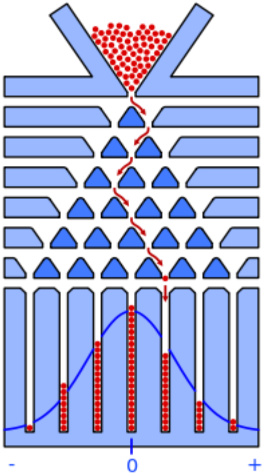
- This is called galton board, Sir Francis Galton (1822-1911)



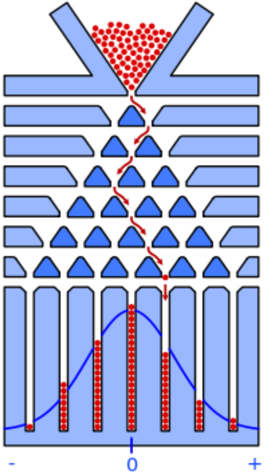
- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall



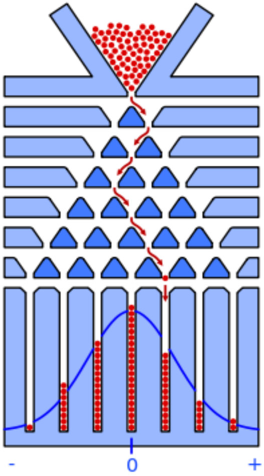
- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall



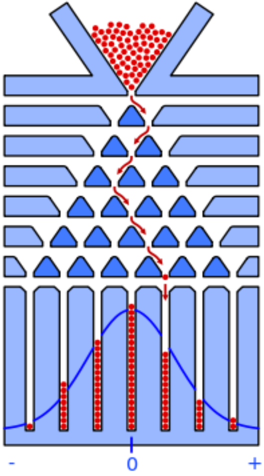
- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall
- Quiz: Why do we have more marbles in the middle?



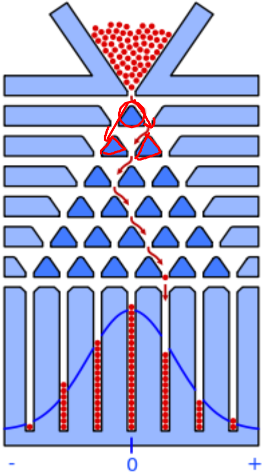
- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall
- Quiz: Why do we have more marbles in the middle?
- Does probability have any role in this distribution?



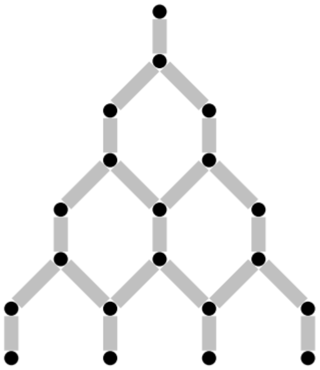
- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall
- Quiz: Why do we have more marbles in the middle?
- Does probability have any role in this distribution?
- Can we prove that the distribution will look like this mathematically?

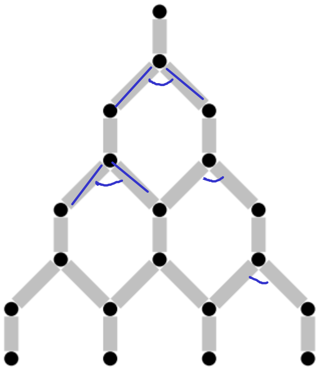


- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall
- Quiz: Why do we have more marbles in the middle?
- Does probability have any role in this distribution?
- Can we prove that the distribution will look like this mathematically?
- Does Pascal triangle has anything to do with this?

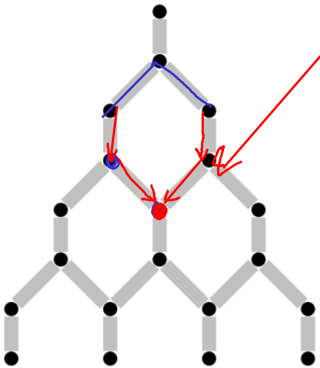


- This is called galton board, Sir Francis Galton (1822-1911)
- The set of red marbles are allowed to fall
- We observe the distribution of the marbles after the fall
- Quiz: Why do we have more marbles in the middle?
- Does probability have any role in this distribution?
- Can we prove that the distribution will look like this mathematically?
- Does Pascal triangle has anything to do with this?
- Let us analyze this in detail ...

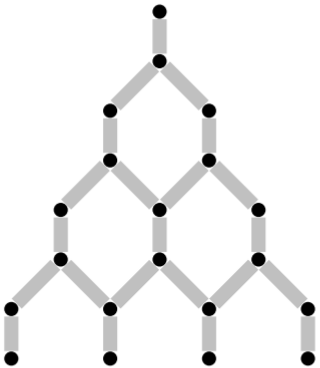




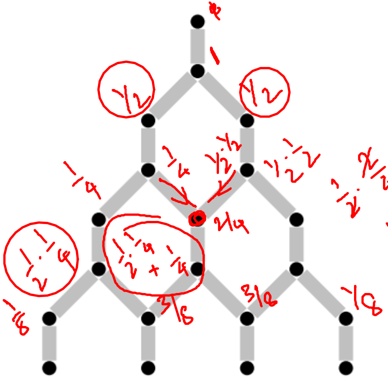
- Assume that at each level the beams are splitted evenly



- Assume that at each level the beams are splitted evenly
- Probability that ball hits current beam is sum of probability of this ball hitting left and right beam above



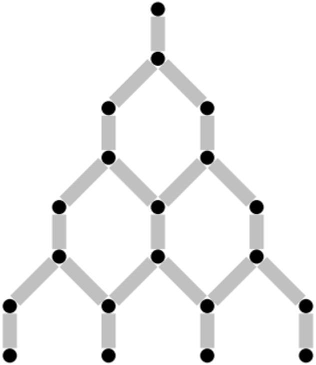
- Assume that at each level the beams are splitted evenly
- Probability that ball hits current beam is sum of probability of this ball hitting left and right beam above
- If x is the top left probability, and y is the top right probability, then probability of hitting the current bin is $(x+y)/2$



- Assume that at each level the beams are splitted evenly
- Probability that ball hits current beam is sum of probability of this ball hitting left and right beam above
- If x is the top left probability, and y is the top right probability, then probability of hitting the current bin is $(x+y)/2$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1/2 & & 1/2 & & \\ & 1/4 & & 2/4 & & 1/4 & \\ 1/8 & & 3/8 & & 3/8 & & 1/8 \end{array}$$

$\binom{n}{k}/2^k$



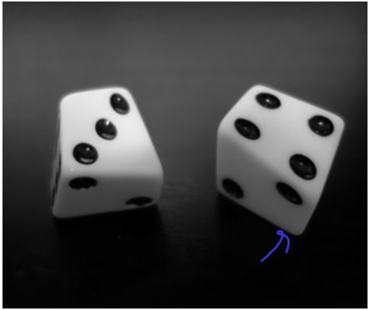
- Assume that at each level the beams are splitted evenly
- Probability that ball hits current beam is sum of probability of this ball hitting left and right beam above
- If x is the top left probability, and y is the top right probability, then probability of hitting the current bin is $(x + y)/2$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1/2 & & 1/2 & & \\
 & 1/4 & & 2/4 & & 1/4 & \\
1/8 & & 3/8 & & 3/8 & & 1/8
\end{array}
$$

- The probability of current bin $= \binom{n}{k}/2^n$

0 Non equiprobable outcomes!

What is wrong with these dice?



What is wrong with these dice?

- So far we assumed equiprobable outcomes

What is wrong with these dice?

- So far we assumed equiprobable outcomes
- On the left, we have skewed dice, i.e., unfair dice

0 Non equiprobable outcomes!

What is wrong with these dice?

- So far we assumed equiprobable outcomes
- On the left, we have skewed dice, i.e., unfair dice
- Will our previous definition of calculating probability work?

What is wrong with these dice?

$p(5) = \frac{1}{\ }$

- So far we assumed equiprobable outcomes
- On the left, we have skewed dice, i.e., unfair dice
- Will our previous definition of calculating probability work?
- Unfortunately not, how to deal with skewed cases?

0 Non equiprobable outcomes!

- Since the dice is skewed, frequency of outcomes are different

- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently

- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently
- Let p_i denote the probability of ith number

0 Non equiprobable outcomes!

- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently
- Let p_i denote the probability of ith number
- If we throw the dice long enough, the frequencies stabilise

10 5 20 50 5 10

1 2 3 4 5 6

0 Non equiprobable outcomes!

- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently
- Let p_i denote the probability of ith number
- If we throw the dice long enough, the frequencies stabilise
- For example, the probability of getting an even number $= p_2 + p_4 + p_6$

- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently
- Let p_i denote the probability of ith number
- If we throw the dice long enough, the frequencies stabilise
- For example, the probability of getting an even number $= p_2 + p_4 + p_6$
- Also, we know that the sum of all the probabilities sum to 1, even though the dice is skewed!
  - That is: $p_1 + p_2 \cdots + p_6 = 1$

0 Example: Dice Throw and Venn Diagram

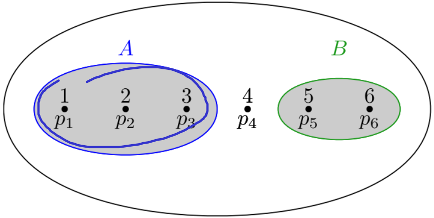
Experiment: Throw a fair dice once

0 Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

0 Example: Dice Throw and Venn Diagram

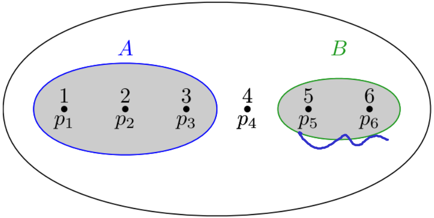
Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



- Consider event $A = \{1, 2, 3\}$

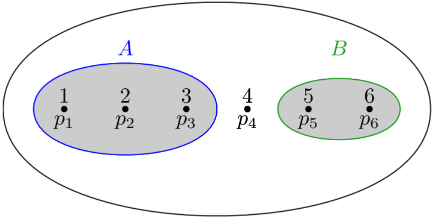
0 Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

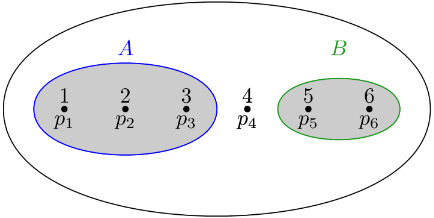
Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$

Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

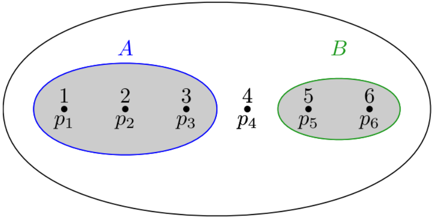


- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$
- $Pr(A) = p_1 + p_2 + p_3$

0 Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

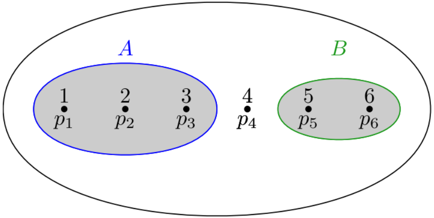


- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$
- $Pr(A) = p_1 + p_2 + p_3$
- $Pr(B) = p_5 + p_6$

Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



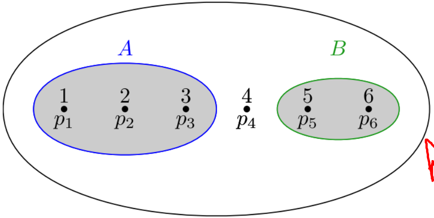
- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$
- $Pr(A) = p_1 + p_2 + p_3$
- $Pr(B) = p_5 + p_6$
- Consider the event A or $B = \{1, 2, 3, 5, 6\}$

0 Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$
- $Pr(A) = p_1 + p_2 + p_3$
- $Pr(B) = p_5 + p_6$
- Consider the event A or $B = \{1, 2, 3, 5, 6\}$
- $Pr(A \text{ or } B) =$

Axiom-3

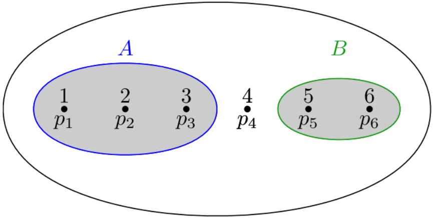
$$Pr(A) + Pr(B) = p_1 + p_2 + p_3 + p_5 + p_6$$

Why $Pr(A \text{ or } B) = Pr(A) + Pr(B)$?

0 Example: Dice Throw and Venn Diagram

Experiment: Throw a fair dice once

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$



- Consider event $A = \{1, 2, 3\}$
- Consider event $B = \{5, 6\}$

- Since the dice is fair, we have $p_1 + p_2 + \cdots + p_6 = 1$
- $Pr(A) = p_1 + p_2 + p_3$
- $Pr(B) = p_5 + p_6$
- Consider the event A or $B = \{1, 2, 3, 5, 6\}$
- $Pr(A \text{ or } B) =$

$$Pr(A) + Pr(B) = p_1 + p_2 + p_3 + p_5 + p_6$$

Why $Pr(A \text{ or } B) = Pr(A) + Pr(B)$?
When are we allowed to add probabilities?

Mutually Exclusive Events

Two events are said to be mutually exclusive if they have disjoint set of outcomes.

Mutually Exclusive Events

Two events are said to be mutually exclusive if they have disjoint set of outcomes. Let A and B be two mutually exclusive events, then we have

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

Mutually Exclusive Events

Two events are said to be mutually exclusive if they have disjoint set of outcomes. Let A and B be two mutually exclusive events, then we have

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

Probability of a complement

Let A be any event, and A^c be its complement.

Mutually Exclusive Events

Two events are said to be mutually exclusive if they have disjoint set of outcomes. Let A and B be two mutually exclusive events, then we have

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

Probability of a complement

Let A be any event, and A^c be its complement. Then clearly $A \cup A^c$ is the complete set of outcomes, i.e., it is a sample space.

Mutually Exclusive Events

Two events are said to be mutually exclusive if they have disjoint set of outcomes. Let A and B be two mutually exclusive events, then we have

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

Probability of a complement

Let A be any event, and A^c be its complement. Then clearly $A \cup A^c$ is the complete set of outcomes, i.e., it is a sample space. Hence, we have

$$Pr(A) + Pr(A^c) = 1 \quad \Longrightarrow \underline{Pr(A^c) = 1 - Pr(A)}$$

Mutually disjoint events

A set of n events denoted by $A_1, A_2, \cdots, A_n$ are called mutually disjoint events if the following holds:

Mutually disjoint events

A set of n events denoted by $A_1, A_2, \cdots, A_n$ are called mutually disjoint events if the following holds:

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n).$$

Mutually disjoint events

A set of n events denoted by $A_1, A_2, \cdots, A_n$ are called mutually disjoint events if the following holds:

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n).$$

From Axiom-3, it holds for countably infinite unions.

0 Quiz: Use concept of disjoint events

Quiz

In a cricket tournament with four teams denoted by $\{A, B, C, D\}$, team A has 20% chance of winning, while team B has a 40% chance of winning. What is the probability that A or B win the tournament?

$p(\text{A wins}) = 20\%$

$p(\text{B wins}) = 40\%$

$p(\text{A or B wins}) = p(\text{A wins}) + p(\text{B wins})$

disjoint events.

$= 60\%$

Quiz

Suppose we know the following:

- there is a 50% chance that it will be hot Today
- there is a 30% chance that it will be hot Tomorrow
- there is 10% chance that it wont be hot Today or Tomorrow

Quiz

Suppose we know the following:

- there is a 50% chance that it will be hot Today
- there is a 30% chance that it will be hot Tomorrow
- there is 10% chance that it wont be hot Today or Tomorrow

Answer the following:

$P(A\cap B) = 0.9 - 0.8 = 0.1$

A : hot Today
B : hot Tomorrow

$P(A) = 50\% = 0.5$
$P(B) = 30\% = 0.3$
$P(B^c) = 0.7$
$P(A^c \cup B^c) = 10\% = 0.1$

Quiz

Suppose we know the following:

- there is a 50% chance that it will be hot Today
- there is a 30% chance that it will be hot Tomorrow
- there is 10% chance that it wont be hot Today or Tomorrow

neither nor and

$P(A \cup B)$
$P(A\cap B)$
$P(A\cap B^c)$

Answer the following:

- probability that it will be hot today or tomorrow
- probability that it will be hot today and tomorrow
- probability that it will be hot today but not tomorrow
- probability that it either will be hot today or tomorrow, but not both

$P((A\cup B)\setminus(A\cap B))$

$P(A^c\cap B^c) = 10\% = 0.1$
$P((A\cup B)^c) = 0.1$
$P(A\cup B) = 0.9$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$= 0.5 + 0.3 - 0.9 = -0.1$

0 Non-Mutually Exclusive Events

- What if the two events A and B are not mutually exclusive?

0 Non-Mutually Exclusive Events

- What if the two events A and B are not mutually exclusive?
- What relation we have

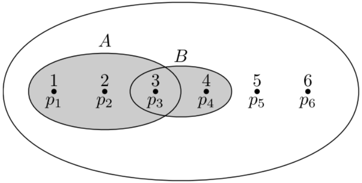
$$Pr(A \cup B) \quad ? \quad Pr(A) + Pr(B) \tag{1}$$

0 Non-Mutually Exclusive Events

- What if the two events A and B are not mutually exclusive?
- What relation we have

$$Pr(A \cup B) \quad ? \quad Pr(A) + Pr(B) \tag{1}$$

- Let us consider an example below with two events A and B shaded

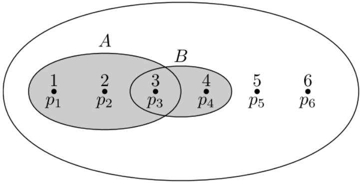


0 Non-Mutually Exclusive Events

- What if the two events A and B are not mutually exclusive?
- What relation we have

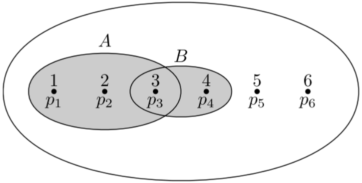
$$Pr(A \cup B) \quad ? \quad Pr(A) + Pr(B) \tag{1}$$

- Let us consider an example below with two events A and B shaded

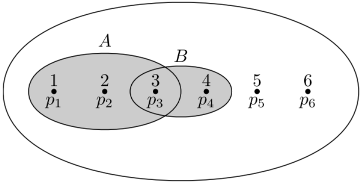


- For the events A and B, which relation (1) holds?

- Consider the example as before



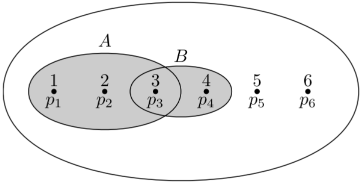
- Consider the example as before



- Let us calculate the probabilities of the two events A and B

$$Pr(A) = p_1 + p_2 + p_3, \quad Pr(B) = p_3 + p_4$$

- Consider the example as before

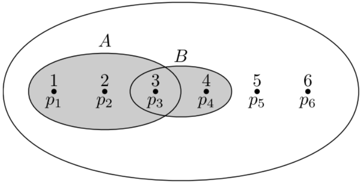


- Let us calculate the probabilities of the two events A and B

$$Pr(A) = p_1 + p_2 + p_3, \quad Pr(B) = p_3 + p_4$$

- Now we calculate probability of $A \cup B$

- Consider the example as before



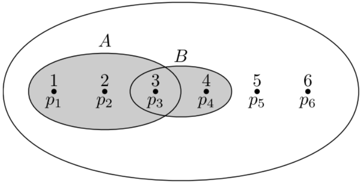
- Let us calculate the probabilities of the two events A and B

$$Pr(A) = p_1 + p_2 + p_3, \quad Pr(B) = p_3 + p_4$$

- Now we calculate probability of $A \cup B$

$$Pr(A \cup B) = Pr(\{1, 2, 3, 4\}) = p_1 + p_2 + p_3 + p_4$$

- Consider the example as before



- Let us calculate the probabilities of the two events A and B

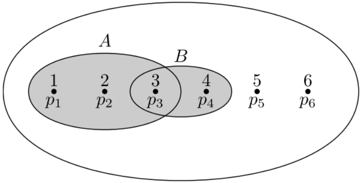
$$Pr(A) = p_1 + p_2 + p_3, \quad Pr(B) = p_3 + p_4$$

- Now we calculate probability of $A \cup B$

$$Pr(A \cup B) = Pr(\{1, 2, 3, 4\}) = p_1 + p_2 + p_3 + p_4$$

- Using above: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

- Consider the example as before



- Let us calculate the probabilities of the two events A and B

$$Pr(A) = p_1 + p_2 + p_3, \quad Pr(B) = p_3 + p_4$$

- Now we calculate probability of $A \cup B$

$$Pr(A \cup B) = Pr(\{1, 2, 3, 4\}) = p_1 + p_2 + p_3 + p_4$$

- Using above: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
- Recall discrete mathematics: inclusion-exclusion principle. (Proof?)

Inclusion-Exclusion Principle

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Inclusion-Exclusion Principle

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(B \cap C) + P(A \cap B \cap C)$

0 Generalized Inclusion-Exclusion Principle

Inclusion-Exclusion Principle

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- In general for n events, we have

Inclusion-Exclusion Principle

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- In general for n events, we have

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$
$$- \cdots (-1)^{n-1} P(\cap_{i=1}^{n} A_i)$$

- Sample space can be of two types: discrete or continuous

- Sample space can be of two types: discrete or continuous
- in discrete probability models we can compute the probability of events by adding all the corresponding outcomes

0 Discrete Probability Models

- Sample space can be of two types: discrete or continuous
- in discrete probability models we can compute the probability of events by adding all the corresponding outcomes
- in continuous probability models we need to use integration instead of summation

- Sample space can be of two types: discrete or continuous
- in discrete probability models we can compute the probability of events by adding all the corresponding outcomes
- in continuous probability models we need to use integration instead of summation

Definition of discrete probability model

If the sample space S is a countable set, then it refers to discrete probability model. Since S is countable: $S = \{s_1, s_2, \cdots,\}$

- Sample space can be of two types: discrete or continuous
- in discrete probability models we can compute the probability of events by adding all the corresponding outcomes
- in continuous probability models we need to use integration instead of summation

Definition of discrete probability model

If the sample space S is a countable set, then it refers to discrete probability model. Since S is countable: $S = \{s_1, s_2, \cdots, \}$

- For an event $A \subset S$, by 3rd axiom

$$P(A) = \sum_{s_j \in A} P(s_j)$$

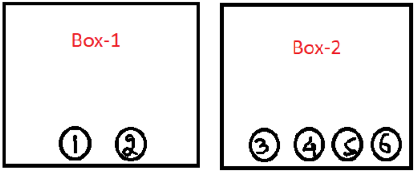
0 Choices and Choice Tree

- Consider 6 balls labelled $\underbrace{1,2,3,4,5,6}$

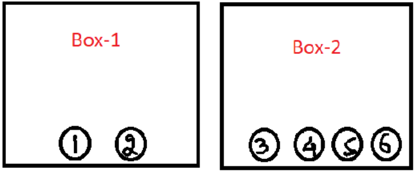
0 Choices and Choice Tree

- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls

- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls

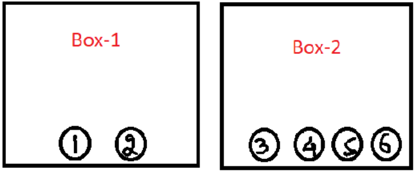


- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls



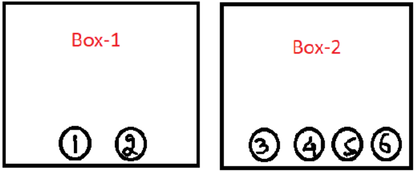
- Assume the following:

- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls



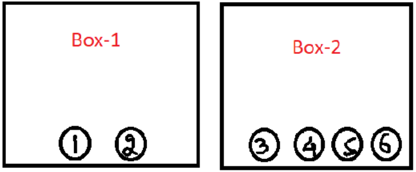
- Assume the following:
  - The two boxes are equiprobable

- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls



- Assume the following:
  - The two boxes are equiprobable
  - All the balls in each box is equiprobable

- Consider 6 balls labelled 1,2,3,4,5,6
- The balls are kept in two boxes. In the first box, we have $1, 2$ balls, and in the second box, we have $3, 4, 5, 6$ balls



- Assume the following:
  - The two boxes are equiprobable
  - All the balls in each box is equiprobable
- Question: What are the probabilities $p_1, p_2, \cdots, p_6$?

$p_1 = \frac{1}{4}$

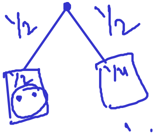
$p_2 = \frac{1}{4}$

$p_3 = \frac{1}{8}$

$p_4 = \frac{1}{8}$

$p_5 = \frac{1}{8}$

$p_6 = \frac{1}{8}$



$\frac{1}{2} \cdot \frac{1}{2}$

$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

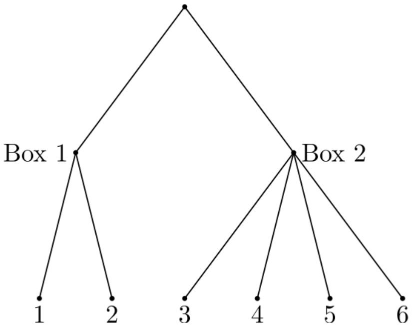


Figure: Consider a choice tree for the problem

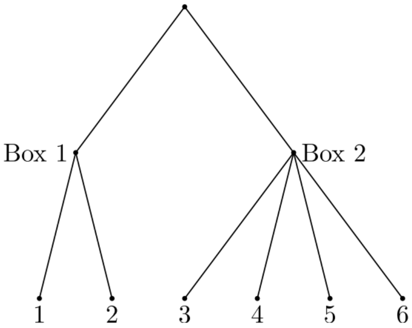


Figure: Consider a choice tree for the problem

- We have the following

$$p_1 + p_2 = 1/2, \quad p_3 + p_4 + p_5 + p_6 = 1/2$$

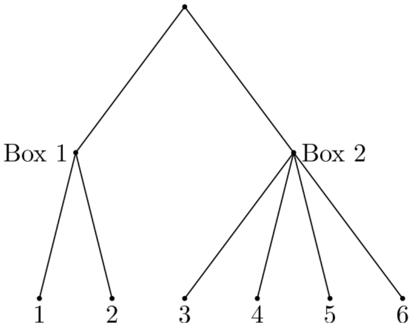


Figure: Consider a choice tree for the problem

- We have the following

$$p_1 + p_2 = 1/2, \quad p_3 + p_4 + p_5 + p_6 = 1/2$$

- Balls in box-1 are equiprobable: $p_1 = p_2$

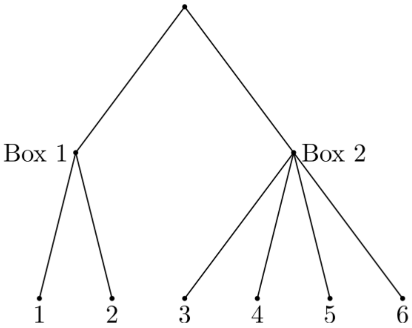


Figure: Consider a choice tree for the problem

- We have the following

$$p_1 + p_2 = 1/2, \quad p_3 + p_4 + p_5 + p_6 = 1/2$$

- Balls in box-1 are equiprobable: $p_1 = p_2$
- Balls in box-2 are equiprobable: $p_3 = p_4 = p_5 = p_6$

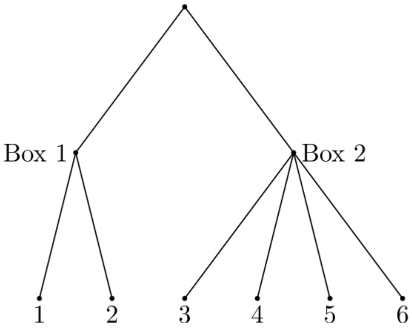


Figure: Consider a choice tree for the problem

- We have the following

$$p_1 + p_2 = 1/2, \quad p_3 + p_4 + p_5 + p_6 = 1/2$$

- Balls in box-1 are equiprobable: $p_1 = p_2$
- Balls in box-2 are equiprobable: $p_3 = p_4 = p_5 = p_6$
- From above, we have

$$p_1 = p_2 = 1/4, \quad p_3 = p_4 = p_5 = p_6 = 1/8$$

Shrewd Prisoner Problem

Long ago a prisoner was to be executed.

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns.

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns,

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked,

0 Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked, after which he had to draw a ball at random from one of these urns.

50 white 50 black

50 W

50 B

$\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 0 = \frac{1}{4}$

1/2

1/2

$\frac{1}{2} = \frac{50}{100}$

0

50W + 50B

$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked, after which he had to draw a ball at random from one of these urns.

Question: How should the prisoner put the balls such that the probability of his release is maximized?

- How did the prisoner arrange to make his chance of success as great as possible?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
    - The probability of choosing a white ball (from either boxes) is $1/2$

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?
- What if he puts all white in one, and all black in other urn?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?
- What if he puts all white in one, and all black in other urn?
  - probability of choosing white from urn-1 is $1/2$, but probability of choosing white from urn-2 is 0. Can he do better?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?
- What if he puts all white in one, and all black in other urn?
  - probability of choosing white from urn-1 is $1/2$, but probability of choosing white from urn-2 is 0. Can he do better?
- Can he do better than this? If yes, then how?

0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is 1/2
  - Can he do better?

2

- What if he puts all white in one, and all black in other urn?
  - probability of choosing white from urn-1 is 1/2, but probability of choosing white from urn-2 is 0. Can he do better?

3

- Can he do better than this? If yes, then how?
- Put 1 white and 0 black in urn-1, and put 49 white and 100 black in urn-2

1W, 0B

49W, ~~100~~ 50B

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{49}{99} > \frac{1}{2}$$

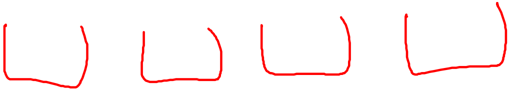
0 Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?
- What if he puts all white in one, and all black in other urn?
  - probability of choosing white from urn-1 is $1/2$, but probability of choosing white from urn-2 is 0. Can he do better?
- Can he do better than this? If yes, then how?
- Put 1 white and 0 black in urn-1, and put 49 white and 100 black in urn-2
- The probability of success is $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{49}{99} \approx 3/4$

100 W
100 B

Quiz-1

What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?



Quiz-1

What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?

Quiz-2

What if the number of balls is increased?

1 A related Problem in Game Theory: Prisoner's dilemma

1 A related Problem in Game Theory: Prisoner's dilemma

- If both of them do not confess, they serve 2 years each

1 A related Problem in Game Theory: Prisoner's dilemma

- If both of them do not confess, they serve 2 years each
- If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years

1 A related Problem in Game Theory: Prisoner's dilemma

- If both of them do not confess, they serve 2 years each
- If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years
- If both confess, then both end up spending 5 years each in jail

1 A related Problem in Game Theory: Prisoner's dilemma

- If both of them do not confess, they serve 2 years each
- If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years
- If both confess, then both end up spending 5 years each in jail
- Given that Wanda and Fred have no reason to trust, what is the good option?


Wanda
Fred
DON'T CONFESS
CONFESS
DON'T CONFESS
2/2
10/0
CONFESS
0/10
5/5

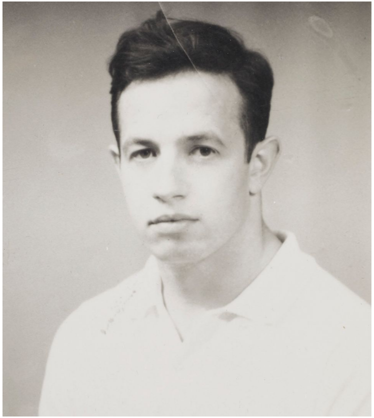
- Indeed, if both don't confess,

- Indeed, if both don't confess, then it is best for them;

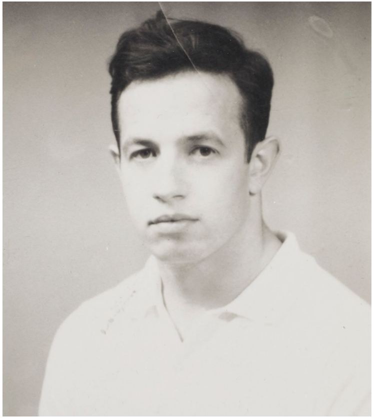
- Indeed, if both don't confess, then it is best for them; 2 years each!

- Indeed, if both don't confess, then it is best for them; 2 years each!
- But another best option is when they both confess. Because in other cases, one of then has to spend 10 years in jail

- Indeed, if both don't confess, then it is best for them; 2 years each!
- But another best option is when they both confess. Because in other cases, one of then has to spend 10 years in jail
- This is part of co-operative games, and 5/5 is called Nash equilibrium
- These topics are part of topic known as game theory (John Nash!)

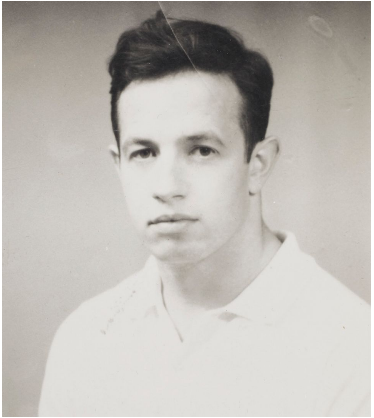


1 About John Nash



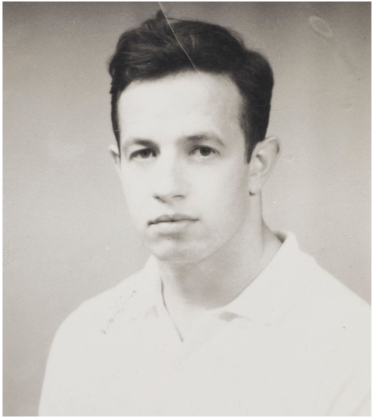
- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations

1 About John Nash

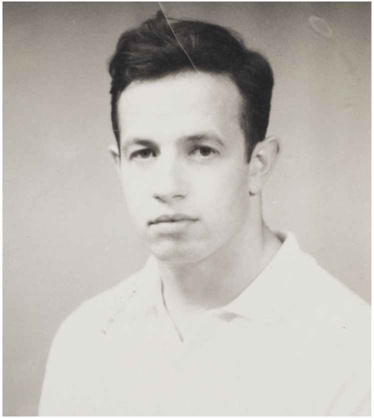


- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994

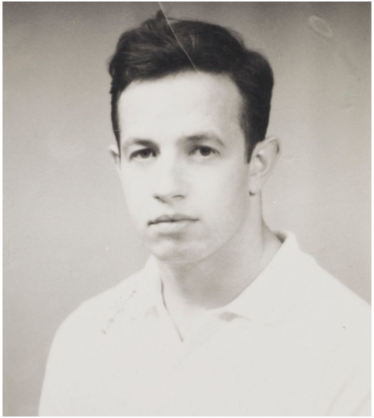
1 About John Nash



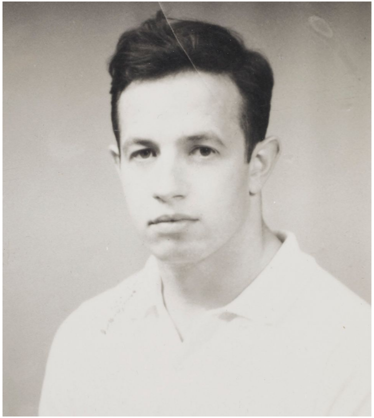
- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015



- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015
- Movie on him: “A Beautiful Mind”



- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015
- Movie on him: "A Beautiful Mind"
- Bar Scene (Game theory part...): https://www.youtube.com/watch?v=LJS7Igvk6ZM



- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015
- Movie on him: "A Beautiful Mind"
- Bar Scene (Game theory part...): https://www.youtube.com/watch?v=LJS7Igvk6ZM

More on this in Topics in Applied Optimization Elective!