Mechatronics System Design

EC4.404 - M2023

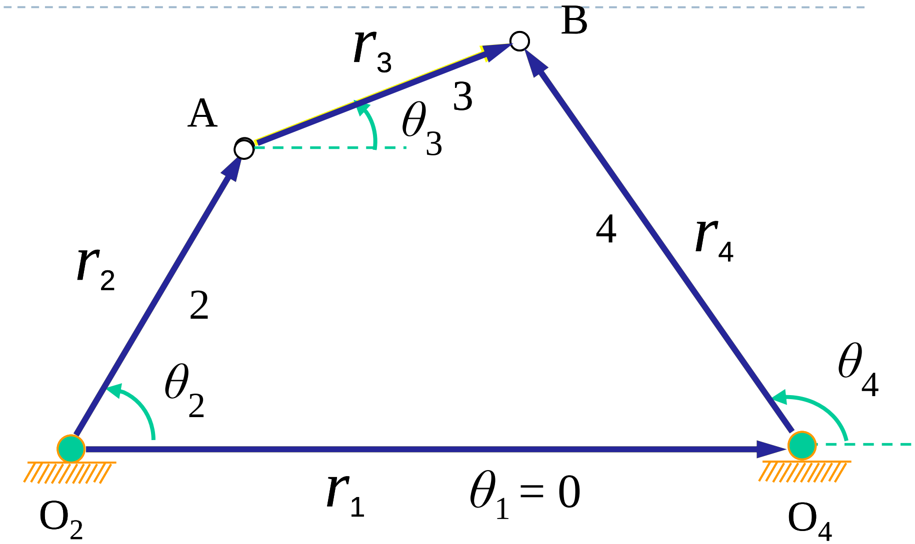
Nagamanikandan Govindan

Robotics Research Center, IIIT Hyderabad.
nagamanikandan.g@iiit.ac.in

Closed Loop Vector Equation – Complex Polar Notation

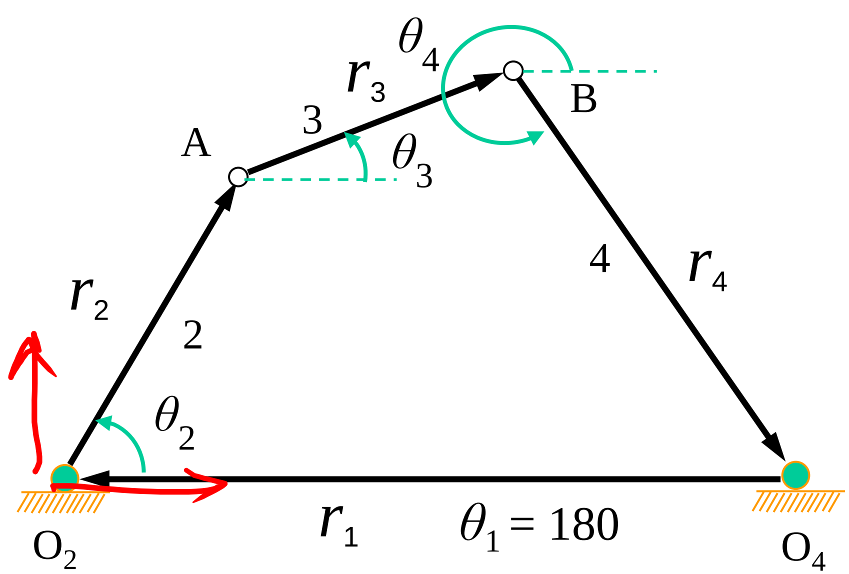
$$\overline{r}_2 + \overline{r}_3 = \overline{r}_1 + \overline{r}_4$$

$$r_2 e^{i\theta_2} + r_3 e^{i\theta_3} = r_1 e^{i\theta_1} + r_4 e^{i\theta_4}$$



Positive sign convention - all angles are measured with respect to the horizontal line in counterclockwise direction.

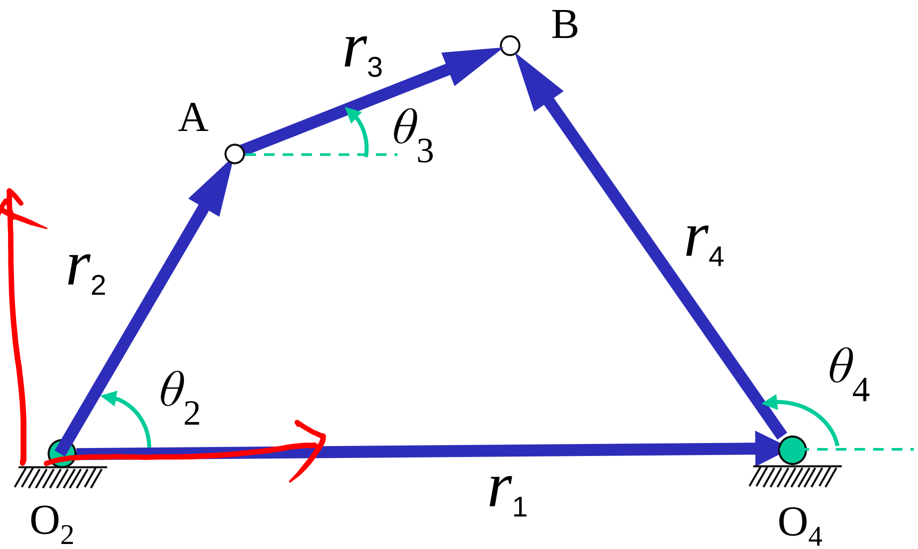
$$\overline{r}_2 + \overline{r}_3 + \overline{r}_4 + \overline{r}_1 = 0$$



Freudenstein's method

$$\bar{r}_2 + \bar{r}_3 = \bar{r}_1 + \bar{r}_4$$

$$-r_1 e^{i\theta_1} + r_2 e^{i\theta_2} + r_3 e^{i\theta_3} - r_4 e^{i\theta_4} = 0$$



Euler equation

$$e^{i\theta} = cos(\theta) + i\, sin(\theta)$$

Real part of the equation

$$-r_1 cos(\theta_1) + r_2 cos(\theta_2) + r_3 cos(\theta_3) - r_4 cos(\theta_4) = 0$$

Imaginary part of the equation

$$-\cancel{r_1 sin(\theta_1)} + r_2 sin(\theta_2) + r_3 sin(\theta_3) - r_4 sin(\theta_4) = 0$$

Analytical Synthesis –Function Generation Mechanism

$\theta_1 = 0$

$$\begin{cases} -r_1 + r_2 cos(\theta_2) + r_3 cos(\theta_3) - r_4 cos(\theta_4) = 0 \\ r_2 sin(\theta_2) + r_3 sin(\theta_3) - r_4 sin(\theta_4) = 0 \end{cases}$$

$$\begin{cases} [r_3 cos(\theta_3)]^2 = [r_1 - r_2 cos(\theta_2) + r_4 cos(\theta_4)]^2 \\ [r_3 sin(\theta_3)]^2 = [- r_2 sin(\theta_2) + r_4 sin(\theta_4)]^2 \end{cases}$$

Add the two equations

$$r_3^{\,2} = [- r_2 sin(\theta_2) + r_4 sin(\theta_4)]^2 + [r_1 - r_2 cos(\theta_2) + r_4 cos(\theta_4)]^2$$

Expand and simplify

$$r_3^{\,2} = (r_1)^2 + (r_2)^2 + (r_4)^2 - 2r_1 r_2 cos(\theta_2) + 2r_1 r_4 cos(\theta_4) - 2r_2 r_4 cos(\theta_2 - \theta_4)$$

Analytical Synthesis –Function Generation Mechanism

$$r_3^{\,2} = (r_1)^2 + (r_2)^2 + (r_4)^2 - 2r_1 r_2 \cos(\theta_2) + 2r_1 r_4 \cos(\theta_4) - 2r_2 r_4 \cos(\theta_2 - \theta_4)$$

Divide the above equation by $2r_2 r_4$

$$\frac{r_1}{r_2}\cos(\theta_4) - \frac{r_1}{r_4}\cos(\theta_2) + \frac{(r_1)^2 + (r_2)^2 - (r_3)^2 + (r_4)^2}{2r_2 r_4} = \cos(\theta_2 - \theta_4)$$

Define K_1 K_2 K_3

$$K_1 = \frac{r_1}{r_2} \qquad K_2 = \frac{r_1}{r_4} \qquad K_3 = \frac{(r_1)^2 + (r_2)^2 - (r_3)^2 + (r_4)^2}{2r_2 r_4}$$

$$\boxed{K_1 \cos(\theta_4) - K_2 \cos(\theta_2) + K_3 = \cos(\theta_2 - \theta_4)}$$

Freudenstein's equation

SHOW MATLAB FILE

Function Generation

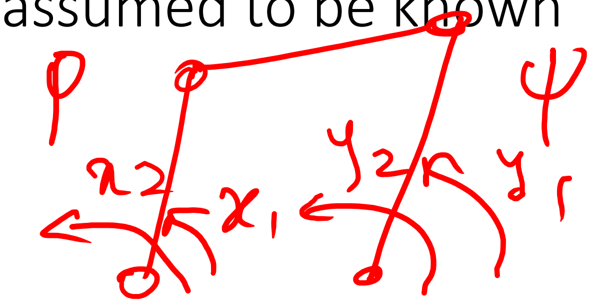
link lengths

Find $r_k, k = 1\ to\ 4$, that allow the linkage to produce the set of m i/p and o/p pairs $\{\varphi_k, \psi_k\}, k = 1\ to\ m$

The algebraic relation between $\varphi_k\ and\ \psi_k$ is assumed to be known in the form of implicit function.

$y = f(x)$

$F(\phi\ \varphi$



$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

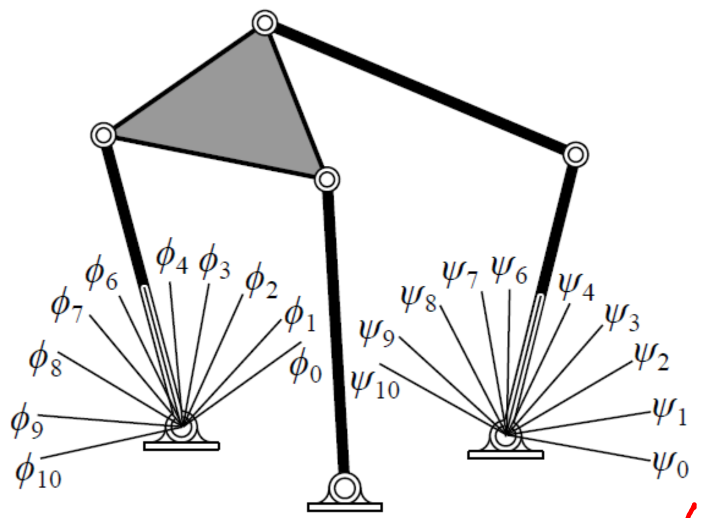
$\vdots$

$x_m \rightarrow y_m$

Function Generation

$\psi \quad \phi$

$$K_1 cos(\theta_4) - K_2 cos(\theta_2) + K_3 = cos(\theta_2 - \theta_4)$$



Freudenstein's equation

$r_1 = 1$

Define

$$K_1 = \frac{r_1}{r_2} \qquad K_2 = \frac{r_1}{r_4} \qquad K_3 = \frac{(r_1)^2 + (r_2)^2 - (r_3)^2 + (r_4)^2}{2 r_2 r_4}$$

$r_2 = \frac{r_1}{K_1} \quad r_4 = \frac{r_1}{K_2} \quad r_3$

K's are the independent algebraic expressions.

Freudenstein's equation holds true for each position of the linkage.

$\psi_k \qquad \phi_k \qquad (\phi_k - \psi_k)$

$$K_1 cos(\theta_4) - K_2\, cos(\theta_2) + K_3 = cos(\theta_2 - \theta_4)$$

$F(\phi_k, \psi_k) = 0$

ϕ_1	ψ_1
ϕ_2	ψ_2

$\cancel{K_1 C \phi_1 = K_2 S}$

$K_1 C\psi_1 - K_2 C\phi_1 + K_3 = C(\phi_1 - \psi_1)$

$K_1 C\psi_2 - K_2 C\phi_2 + K_3 = C(\phi_2 - \psi_2)$

$\vdots$

$$\phi_3 \quad \phi_2 \quad \phi_1$$

$$\begin{bmatrix} c\psi_1 & -c\phi_1 & 1 \\ c\psi_2 & -c\phi_2 & 1 \\ \vdots & & \\ c\psi_m & -c\phi_m & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} c(\phi_1 - \psi_1) \\ \vdots \\ c(\phi_m - \psi_m) \end{bmatrix}$$

$$y = f(x)$$

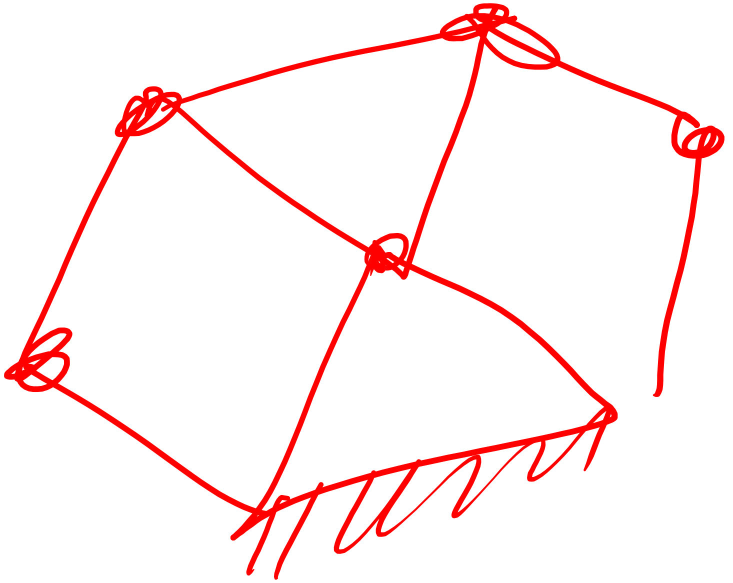
$y_1 \quad x$

$\quad\;\; x_2$

y_2

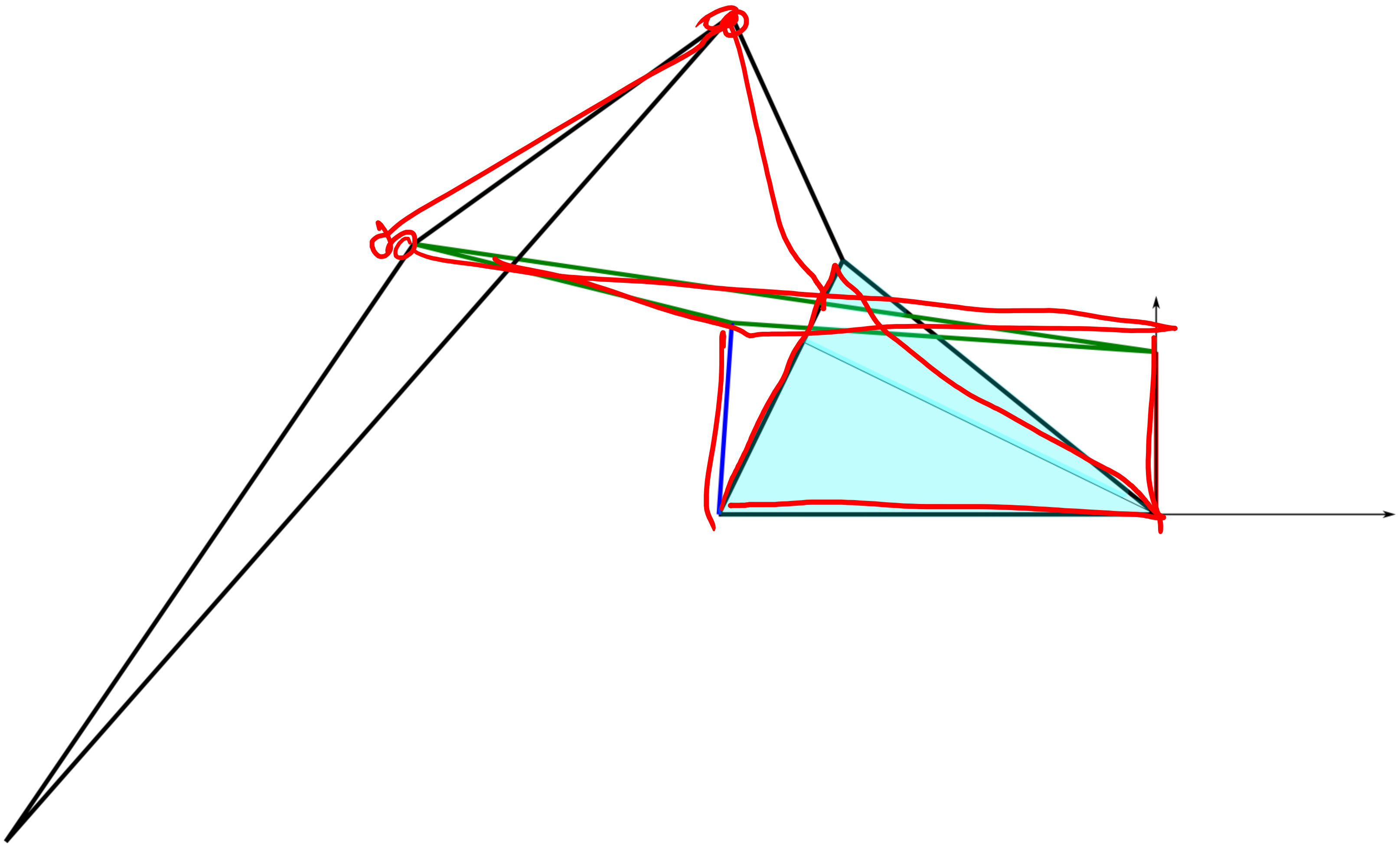
$y_3 \quad x_3$

Multiple Loops

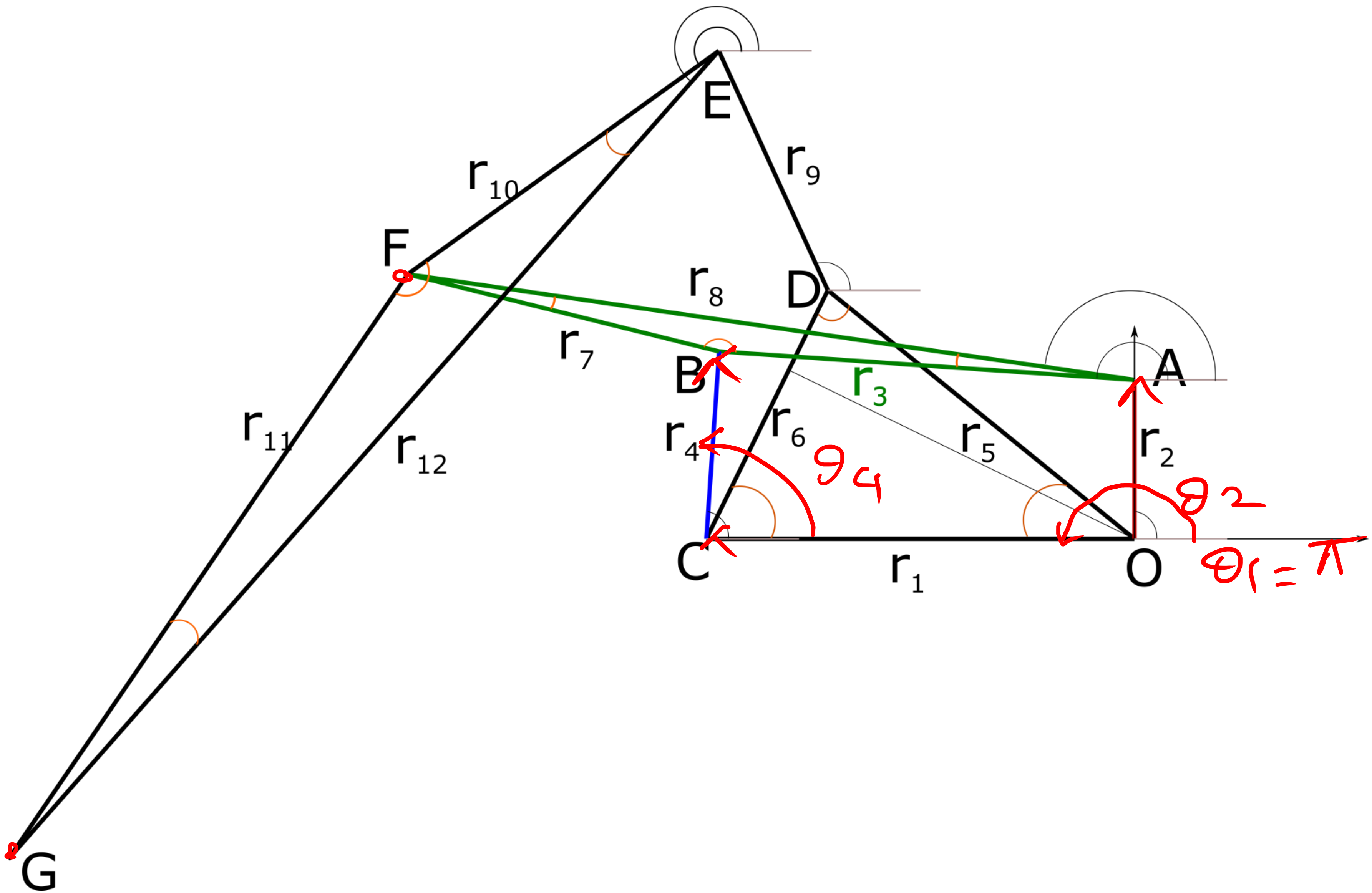


Multiple Loops

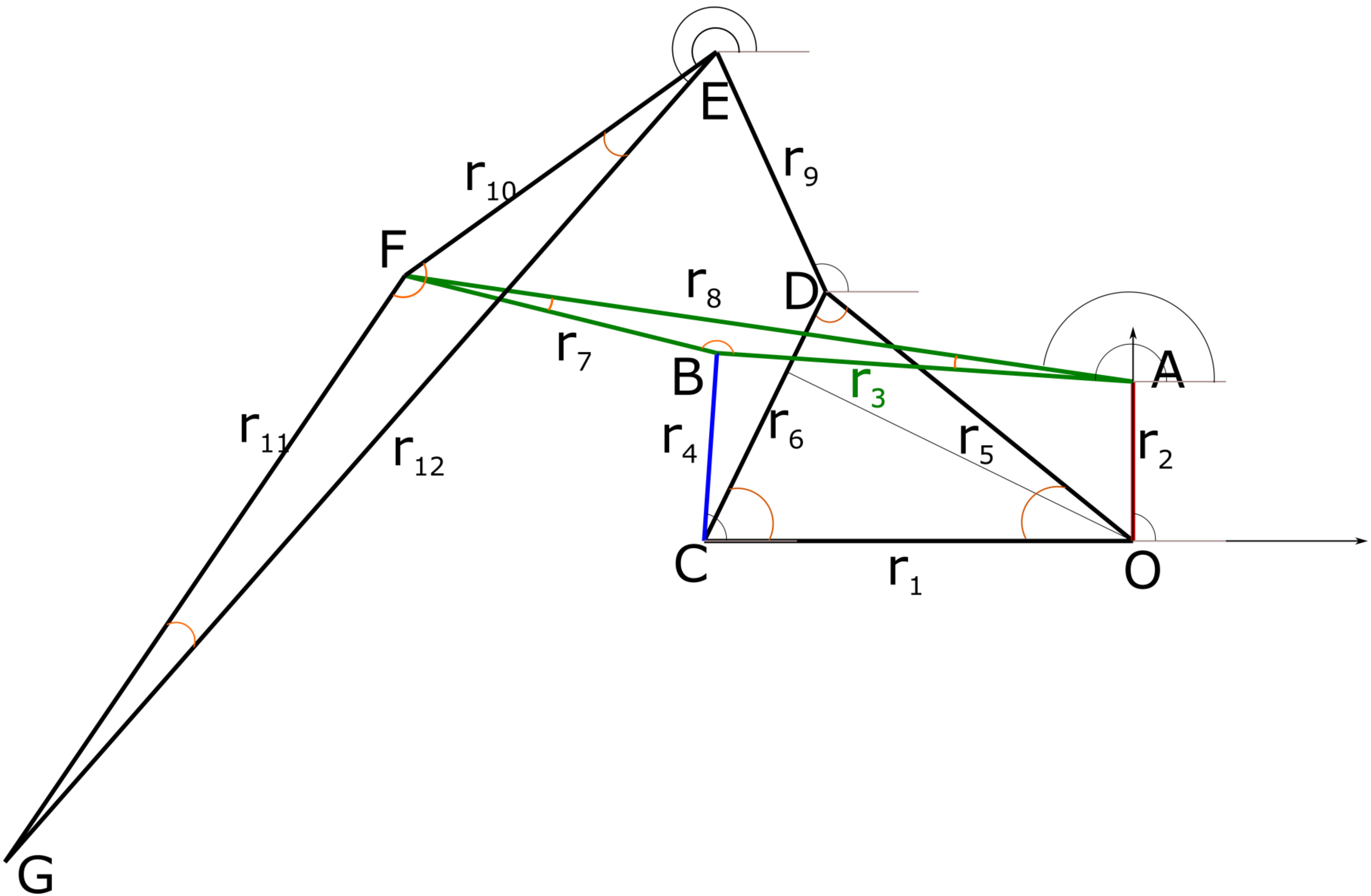
Klann Mechanism



Klann Mechanism



Klann Mechanism

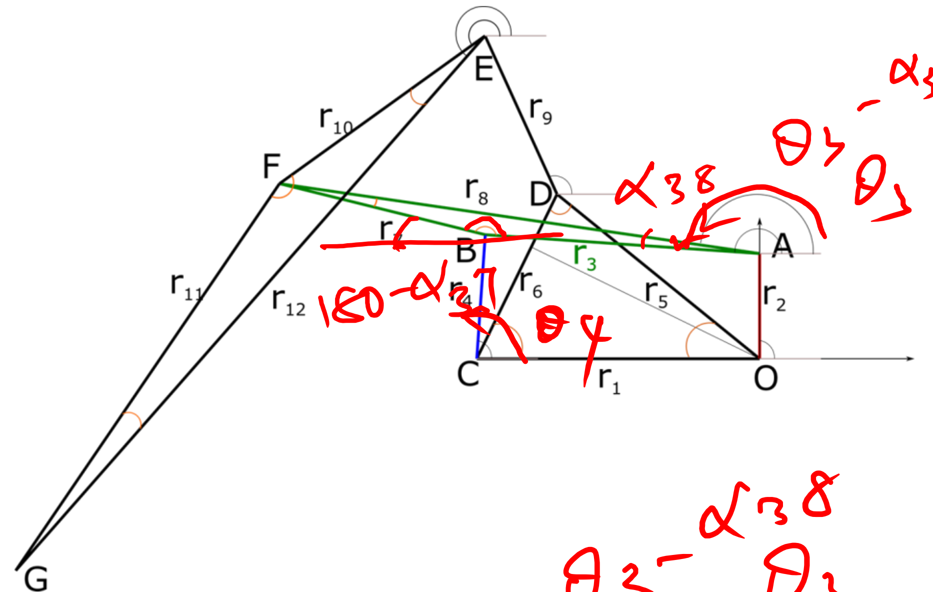


Klann Mechanism

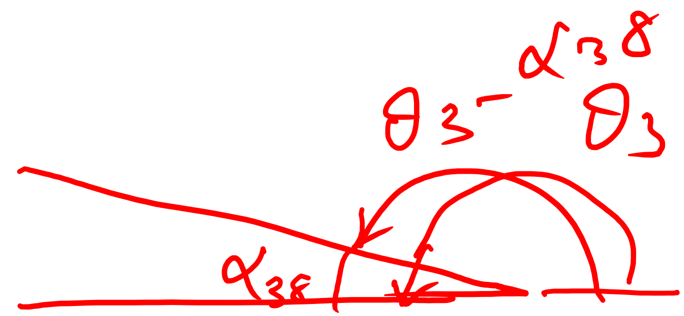
R₁=r₁*{Cos[π],I*Sin[π]};

```
R1=r1*{Cos[π],I*Sin[π]};
R2=r2*{Cos[θ2],I*Sin[θ2]};
R3=r3*{Cos[θ3+π],I*Sin[θ3+π]};
R4=r4*{Cos[θ4],I*Sin[θ4]};
```

θ3



$$-R_1 - R_4 + R_3 + R_2 = 0$$



$$\mathrm{Cos}[\theta_3] \to \frac{r_1 + \mathrm{Cos}[\theta_2]r_2 - \mathrm{Cos}[\theta_4]r_4}{r_3}$$

$$\mathrm{Sin}[\theta_3] \to \frac{\mathrm{Sin}[\theta_2]r_2 - \mathrm{Sin}[\theta_4]r_4}{r_3}$$

Klann Mechanism

$$\frac{r_1^2 + r_2^2 - 2\text{Cos}[\theta_2 - \theta_4]r_2 r_4 + r_4^2 + 2r_1(\text{Cos}[\theta_2]r_2 - \text{Cos}[\theta_4]r_4)}{r_3^2} == 1$$

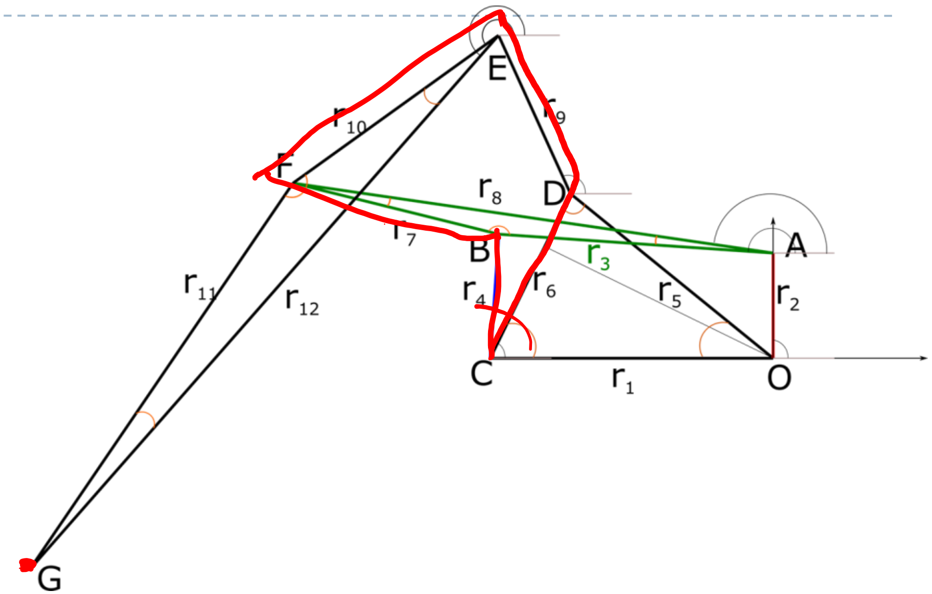
$$\text{K3} = (r_1^2 + r_2^2 + r_4^2 - r_3^2)/(2r_2r_4)\,;$$
$$\text{K2} = (2r_1r_2)/(2r_2r_4)\,;$$
$$\text{K1} = (-2r_1r_4)/(2r_2r_4)\,;$$

$$\text{K1} * \text{Cos}[\theta_4] + \text{K2} * \text{Cos}[\theta_2] + \text{K3} == \text{Cos}[\theta_2 - \theta_4]$$

$$\text{K1} + \text{K3} + (-1 + \text{K2})\text{Cos}[\theta_2] - 2\text{Sin}[\theta_2]t_4 + (-\text{K1} + \text{K3} + (1 + \text{K2})\text{Cos}[\theta_2])t_4^2 == 0$$

Klann Mechanism

```
R7=r7*{Cos[θ7],I*Sin[θ7]};
R4=r4*{Cos[θ4],I*Sin[θ4]};
R6=r6*{Cos[θ6],I*Sin[θ6]};
R9=r9*{Cos[θ9],I*Sin[θ9]};
R10=r10*{Cos[θ10],I*Sin[θ10]};
```



$$+R_7 + R_4 + R_{10} - R_6 - R_9 = 0$$

$$\left\{\mathrm{Cos}[\theta_9] \to \frac{\mathrm{Cos}[\theta_4]r_4 - \mathrm{Cos}[\theta_6]r_6 + \mathrm{Cos}[\theta_7]r_7 + \mathrm{Cos}[\theta_{10}]r_{10}}{r_9}\right\}$$

$$\left\{\mathrm{Sin}[\theta_9] \to \frac{\mathrm{Sin}[\theta_4]r_4 - \mathrm{Sin}[\theta_6]r_6 + \mathrm{Sin}[\theta_7]r_7 + \mathrm{Sin}[\theta_{10}]r_{10}}{r_9}\right\}$$

Klann Mechanism

$$\frac{1}{r_9^2} \times \left(r_4^2 + r_6^2 + r_7^2 + 2\mathrm{Cos}[\theta_7 - \theta_{10}]r_7 r_{10} + r_{10}^2 + r_4(-2\mathrm{Cos}[\theta_4 - \theta_6]r_6 + 2\mathrm{Cos}[\theta_4 - \theta_7]r_7 + 2\mathrm{Cos}[\theta_4 - \theta_{10}]r_{10}) - 2r_6(\mathrm{Cos}[\theta_6 - \theta_7]r_7 + \mathrm{Cos}[\theta_6 - \theta_{10}]r_{10})\right) == 1$$

$$\underline{\mathrm{K4} * \mathrm{Cos}[\theta_{10}] + \mathrm{K5} * \mathrm{Sin}[\theta_{10}] + \mathrm{K6} == 0}$$

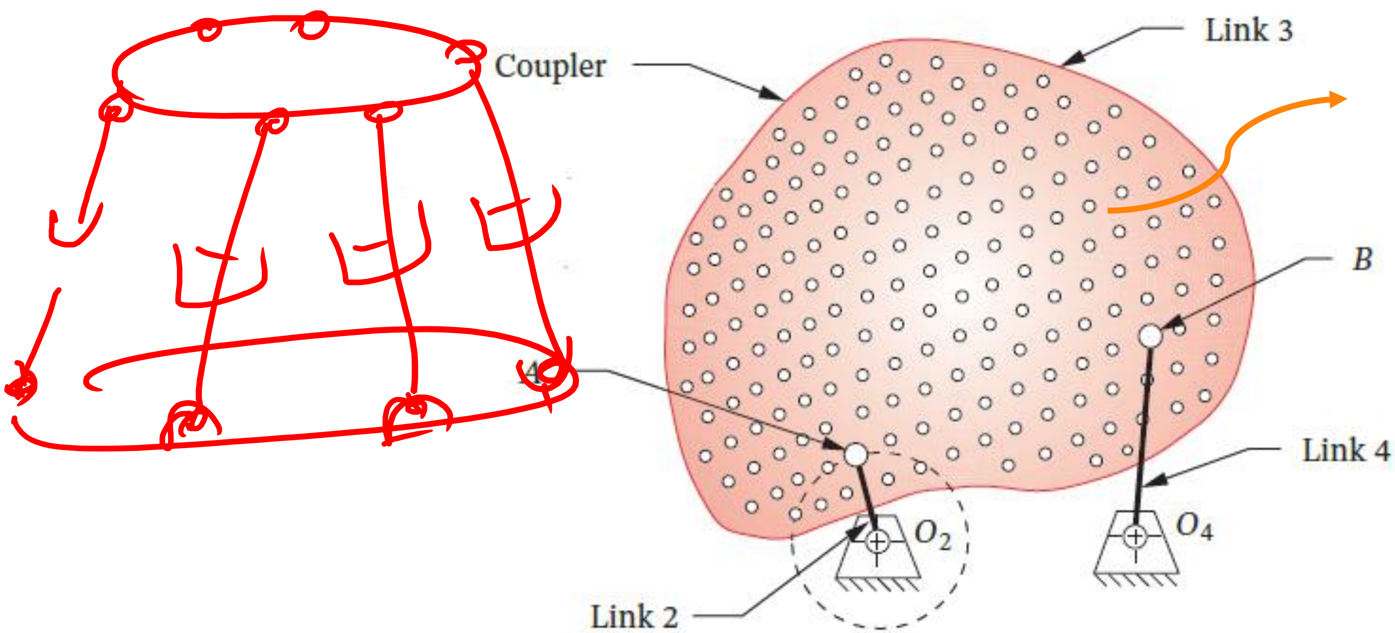
$$\mathrm{K6} = r_4^2 + r_6^2 - 2\mathrm{Cos}[\alpha_{37} - \theta_3 + \theta_6]r_6 r_7 + r_7^2 + r_4(-2\mathrm{Cos}[\theta_4 - \theta_6]r_6 + 2\mathrm{Cos}[\alpha_{37} - \theta_3 + \theta_4]r_7) + r_{10}^2 - r_9^2;$$

$$\mathrm{K5}=2\ (-\mathrm{Sin}[\theta_4]\ r_4+\mathrm{Sin}[\theta_6]\ r_6+\mathrm{Sin}[\alpha_{37}-\theta_3]\ r_7)\ r_{10};$$

$$\mathrm{K4}=-2\ (\mathrm{Cos}[\theta_4]\ r_4-\mathrm{Cos}[\theta_6]\ r_6+\mathrm{Cos}[\alpha_{37}-\theta_3]\ r_7)\ r_{10};$$

Coupler Curves

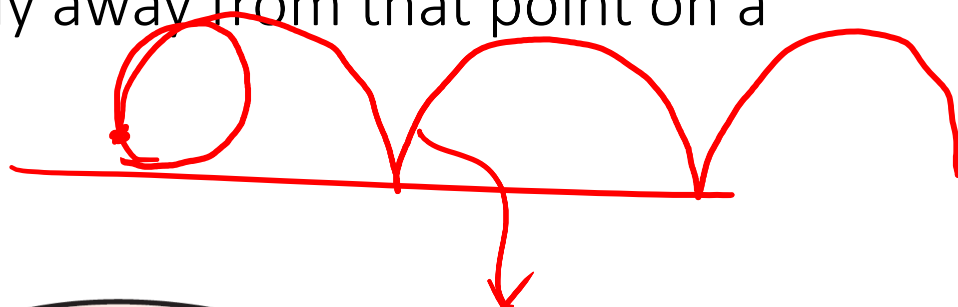
- A coupler is the most interesting link in a fourbar mechanism.
- It has complex motion, i.e., translation and rotation
- Any one of the infinite number points on the coupler will generate a curve that, in general, **tricircular sextic curve** (sixth degree, meaning six intersections with a line and three loops in it)

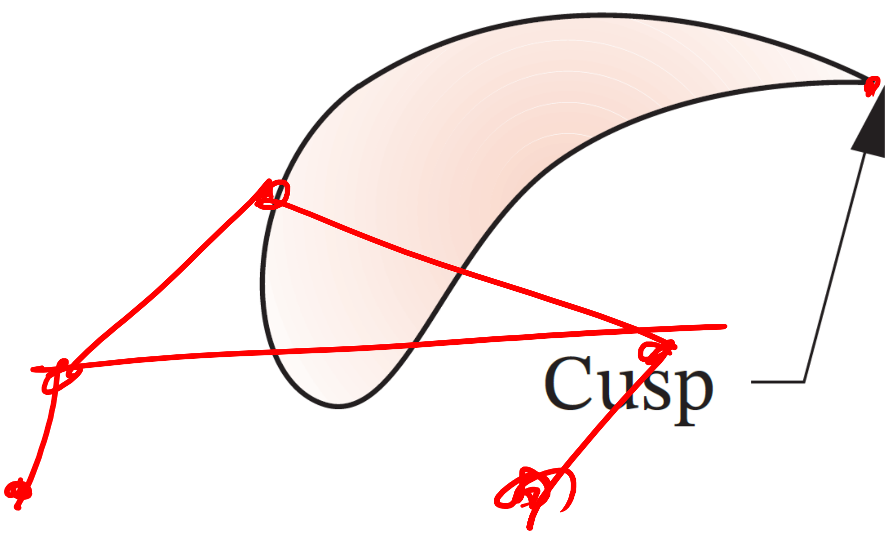


Infinite points on a coupler

Cusp

- simplest example of a curve with a cusp is the cycloid curve which is generated by a point on the rim of a wheel rotating on a flat surface.
- cusp point will come smoothly to a stop along one path and then accelerate smoothly away from that point on a different path

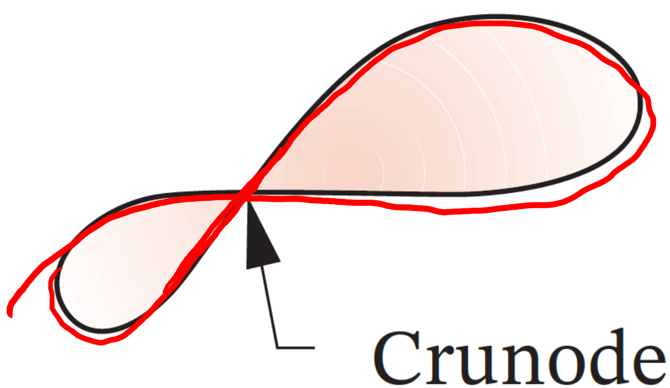




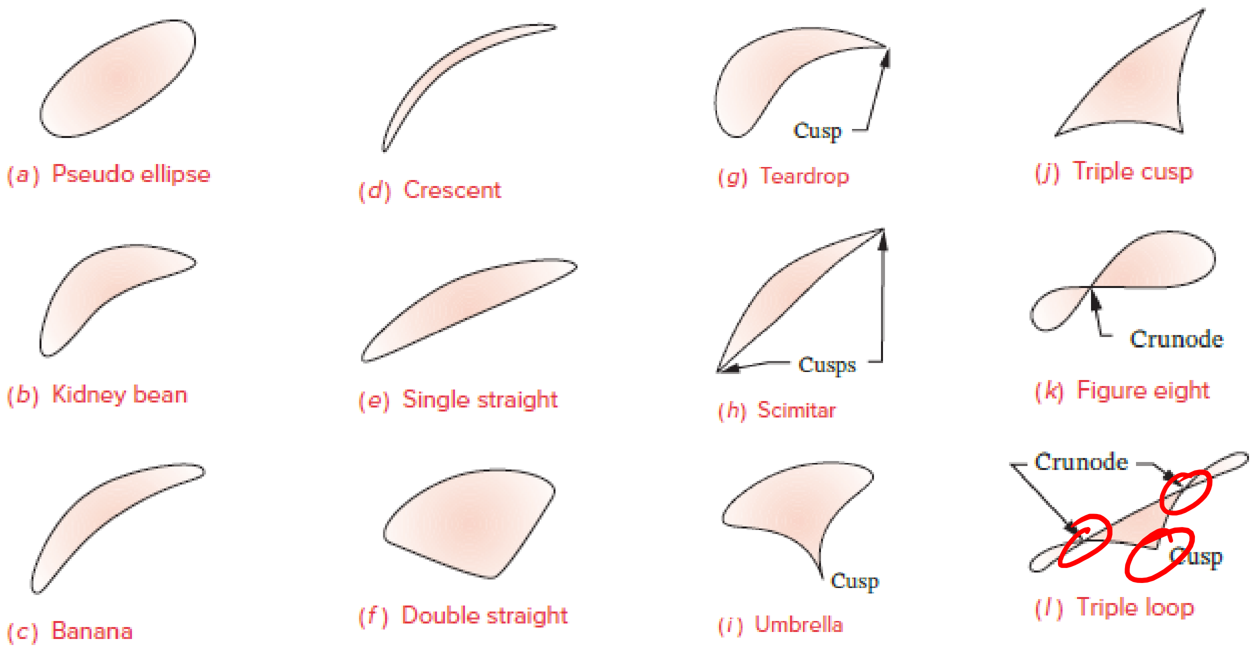
Crunode

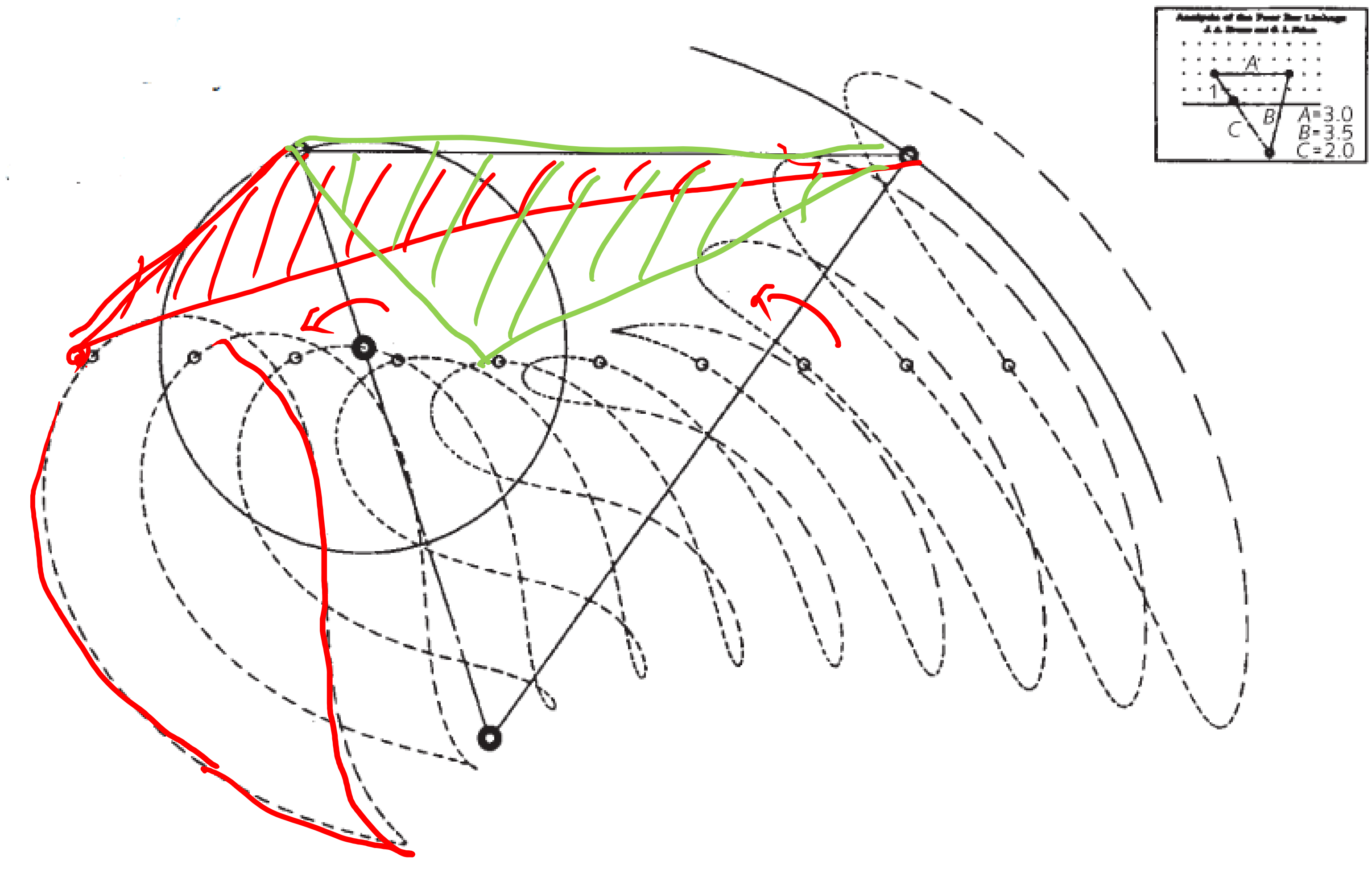
- A double point that occurs where the coupler curve crosses itself creating multiple loops

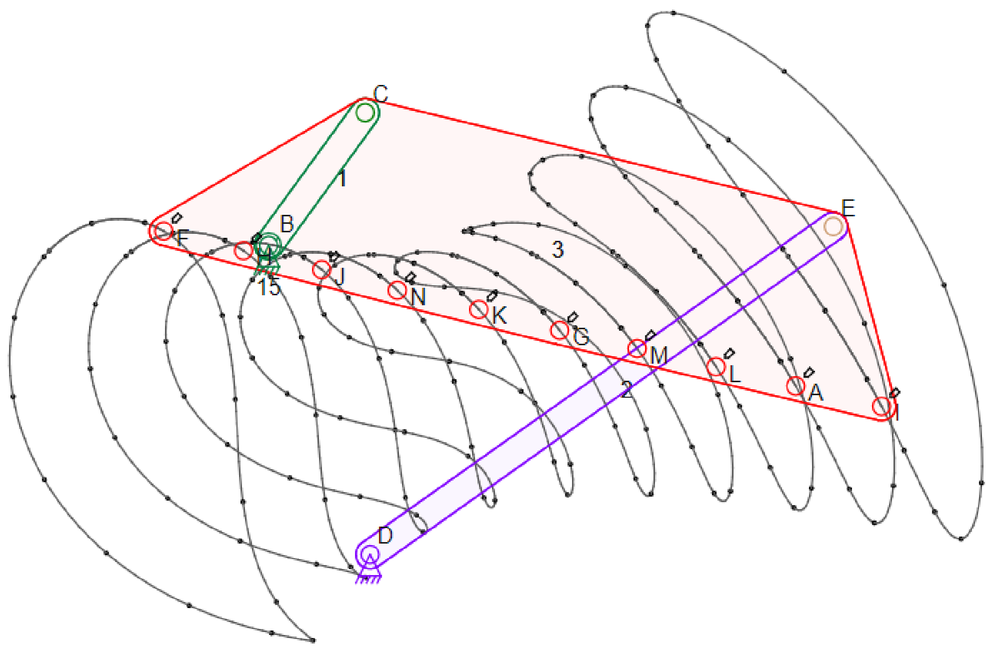
- The two slopes (tangents) at a crunode give the point two different velocities, neither of which is zero in contrast to the cusp.



Possible shapes of curves




A
1
B
C
A=3.0
B=3.5
C=2.0


C
1
B
E
F
H
15
J
N
K
3
G
M
L
A
I
2
D

Coupler Curves

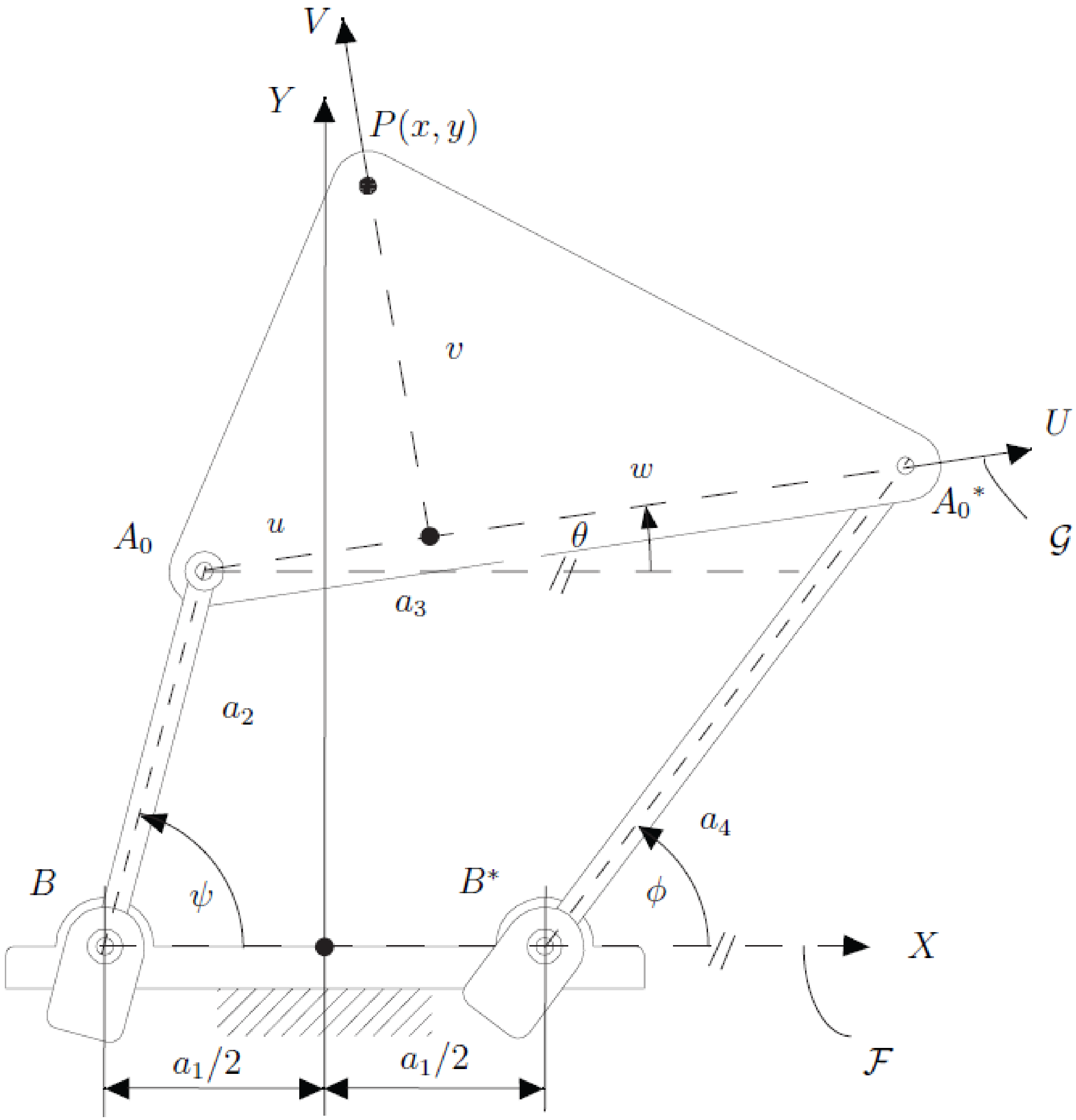
- A coupler is the most interesting link in a fourbar mechanism.
- The curve traced by any point of the coupler link of a
  - planar four-bar linkage is algebraic, of sixth degree
  - fourbar crank-slider has fourth-degree coupler curves

- Expression for the highest degree m possible for a coupler curve of a mechanism of n links connected with only revolute joints.

$$m = 2 * 3^{\left(\frac{n}{2} - 1\right)}$$

- Degree 6, 18, and 54 for fourbar, sixbar, and eight bar linkage couper curves.

- Specific points on coupler may degenerate to lower degree


V
Y
P(x, y)
v
U
w
u
A_0
$A_0{}^*$
$\mathcal{G}$
θ
a_3
a_2
a_4
B
ψ
B^*
ϕ
X
$\mathcal{F}$
$a_1/2$
$a_1/2$

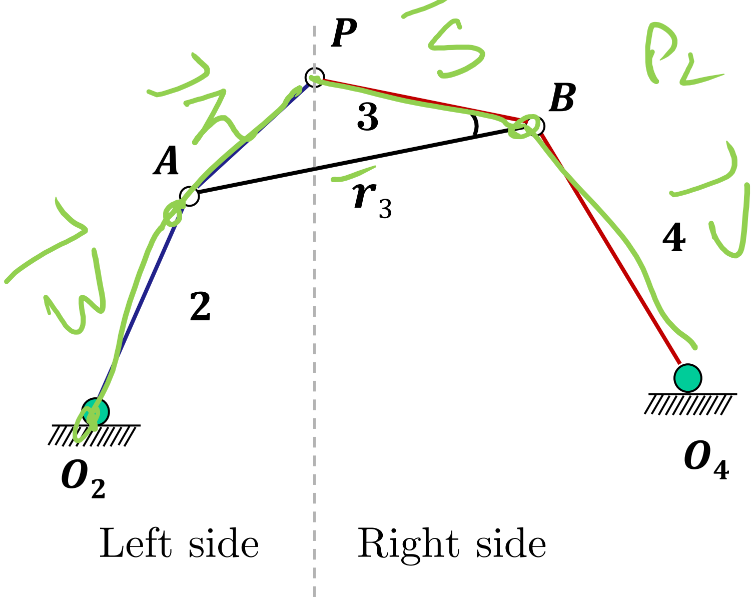
PRECISION POINTS

- The points, or positions, prescribed for successive locations of the output (coupler or rocker) link in the plane are generally referred to as precision points or precision positions.

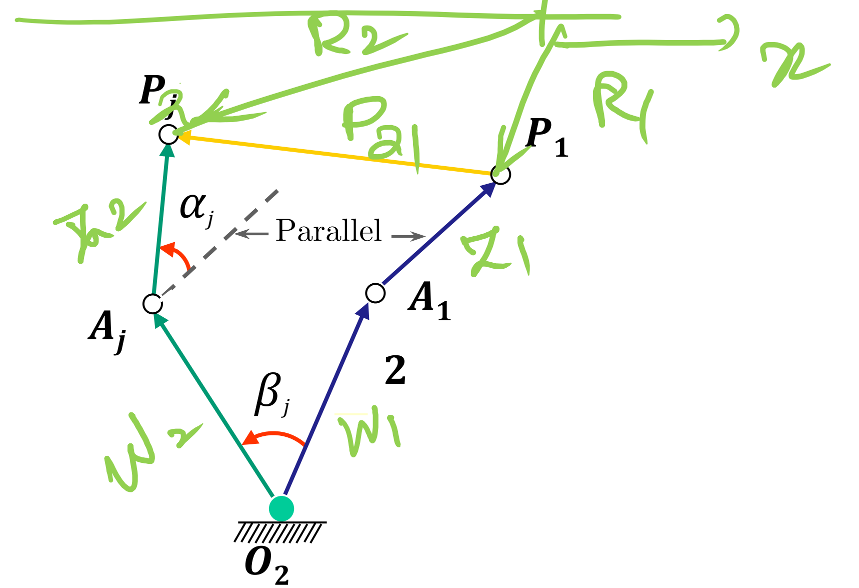
- The fourbar linkage can be synthesized by closed-form methods for
  - up to five precision points for motion or path generation
  - up to seven points for function generation (rocker output)

Analytical Synthesis – Standard Dyad Form

4 Bar mechanism



Left side of the mechanism



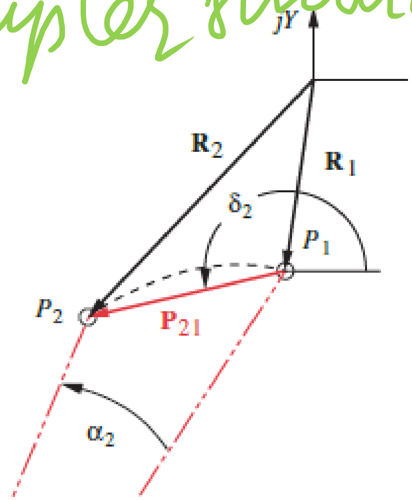
Design the left side of the 4 bar → $\boldsymbol{r}_2 \,\&\, \boldsymbol{r}'_3$

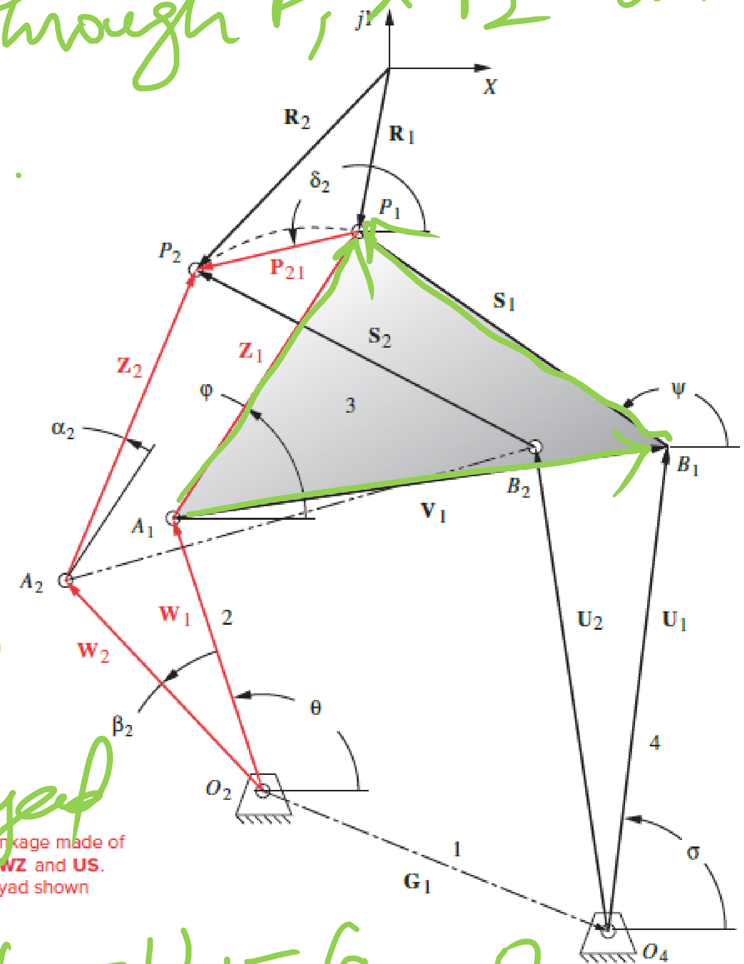
Design the right side of the 4 bar → $\boldsymbol{r}_4 \,\&\, \boldsymbol{r}''_3$

$$\bar{W}_1 + \bar{Z}_1 + \bar{P}_{21} - \bar{Z}_2 - \bar{W}_2 = 0$$

Two-position analytical motion synthesis procedure

Design a mechanism ⇒ a line on the coupler should pass through P_1 & P_2 with some angle α.





(b) Schematic linkage made of two dyads **WZ** and **US**. Left-hand dyad shown

WZ defines left dyad

VS define right dyad

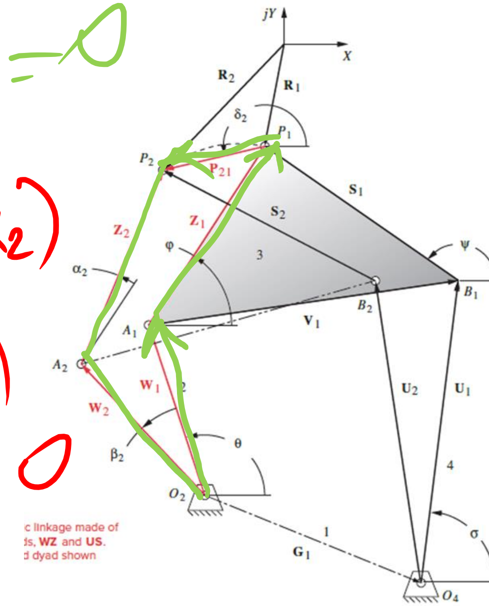
$W_1 + V_1 - U_1 - G_1 = 0$

$V_1 = S_1 - Z_1$

Two-position analytical motion synthesis procedure

$$W_1 + Z_1 + P_{21} - Z_2 - W_2 = 0$$

$$we^{i\theta} + ze^{i\varphi} + p_{21}e^{i\delta_2} - ze^{i(\phi+\alpha_2)} - we^{i(\theta+\beta_2)} = 0$$



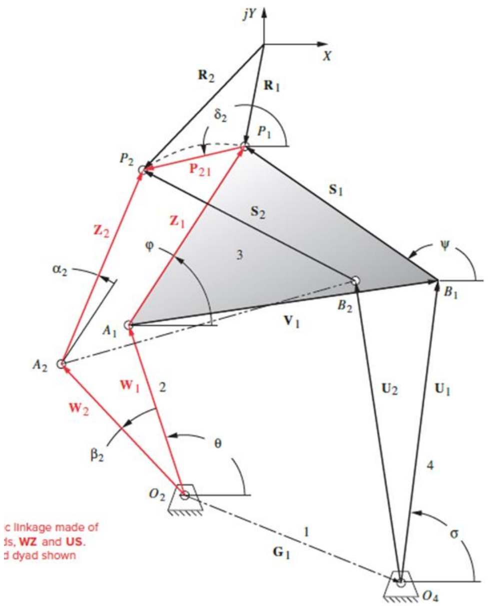
2 scalar equation given

8 variables, 2 are given

∴ 4 free variables

Two-position analytical motion synthesis procedure

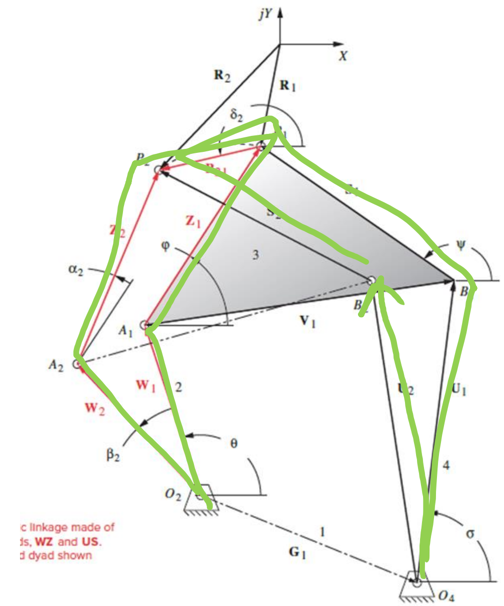
$W, \theta,$,



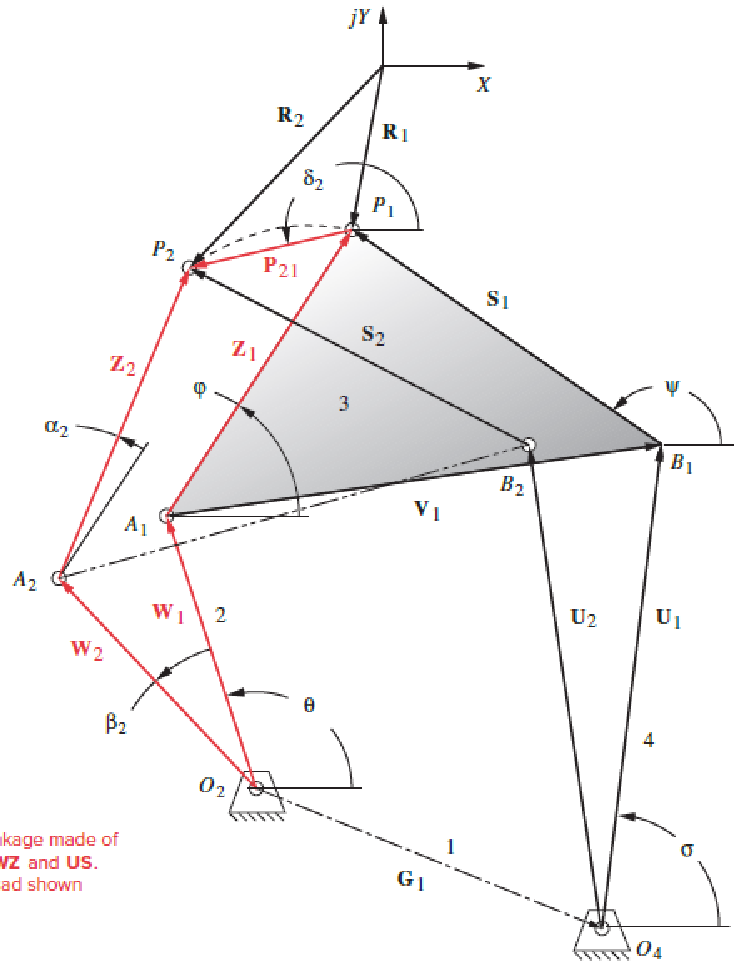
Two-position analytical motion synthesis procedure

$$Aw + Bz = C$$

$$Dw + Ez = F$$

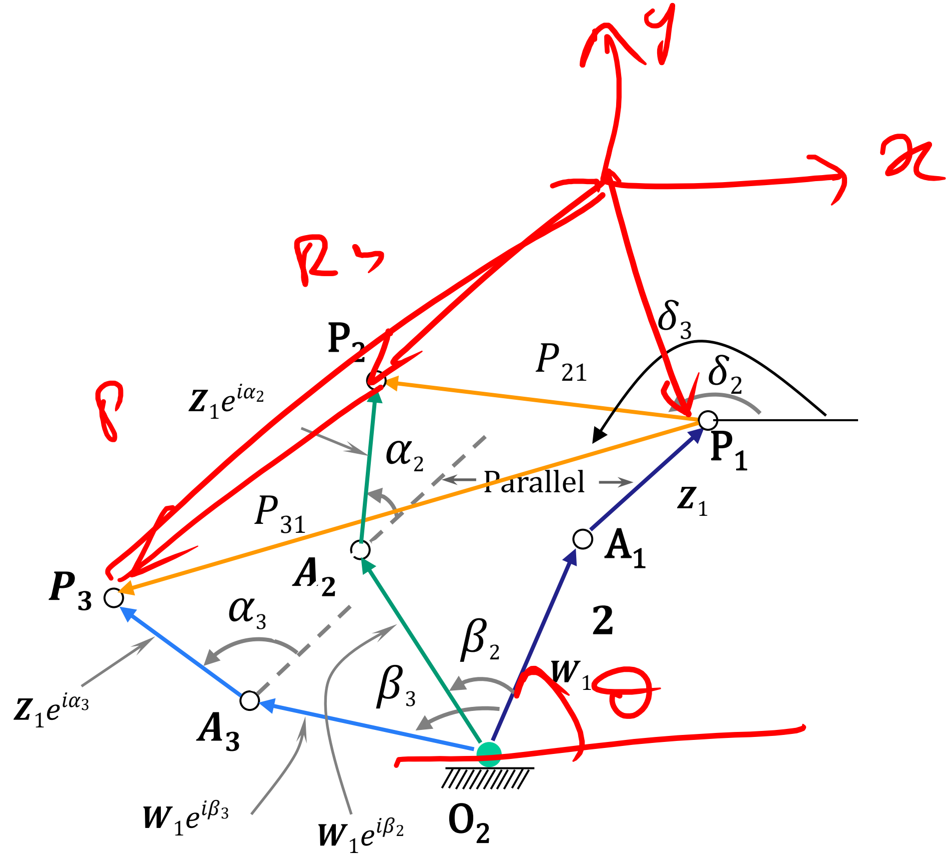


Three-position analytical motion synthesis procedure

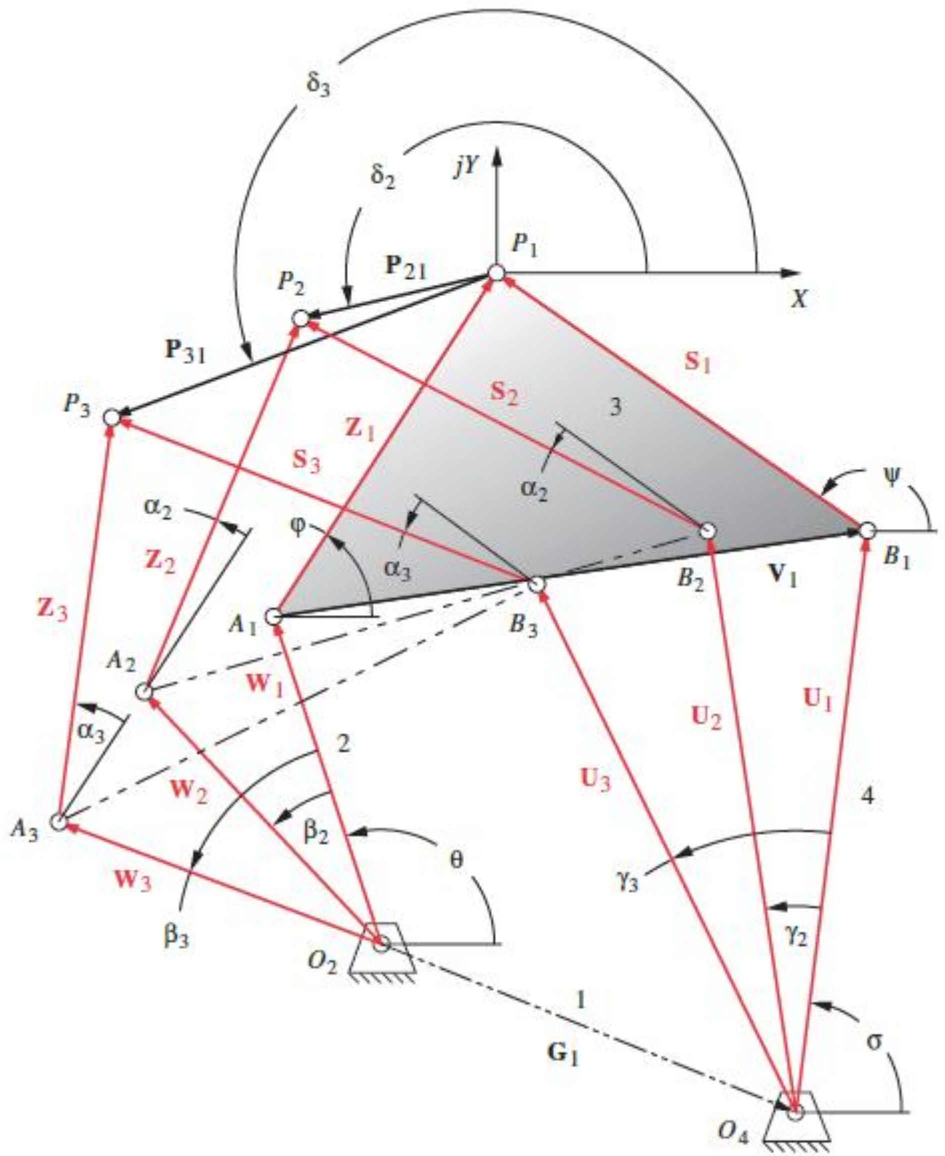


(*b*) Schematic linkage made of two dyads, **WZ** and **US**. Left-hand dyad shown

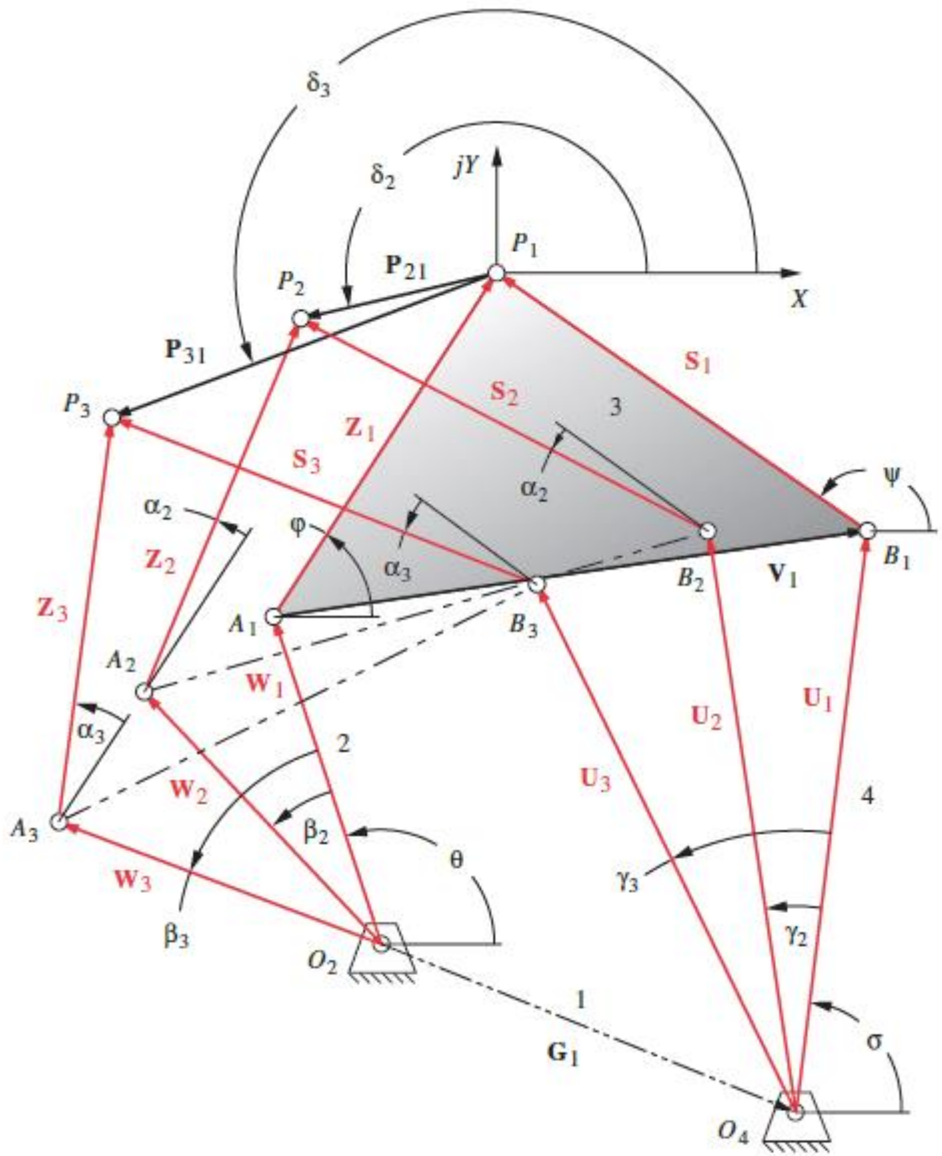
Analytical Synthesis
Three Position Motion & Path Generation Mechanisms



Three-position analytical motion synthesis procedure



Four-position analytical motion synthesis procedure



ANALYTICAL SYNTHESIS OF A PATH GENERATOR

β_{21}, γ_2

β_{31}, δ_3

α_2, α_3

Number of Variables and Free Choices for Analytical Precision-Point Motion and Timed Path Synthesis. [6]

No. of Positions (*n*)	No. of Scalar Variables	No. of Scalar Equations	No. of Prescribed Variables	No. of Free Choices	No. of Available Solutions
2	8	2	3	3	∞^3
3	12	4	6	2	∞^2
4	16	6	9	1	∞^1
5	20	8	12	0	Finite