Information Theory.

Four letters. A T C G

	code 1	code 2	code 3	code 4
A	1	00	0	0
T	0	01	10	10
C	10	10	110	110
G	01	00	1110	111

A	T	C	G.
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

code

$\langle l \rangle \quad \frac{1}{2} + \frac{1}{4} \times 2 + \frac{3}{8} + \frac{3}{8}$

$= 75$

code 2

$\langle l \rangle = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$

$= 2$

code 3 $= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{4}{8}$

$= 1 + \frac{7}{8} = 1.91$

A	T	C	G
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

code 4

code 2

code 3

Entropy $\sum_{i=1}^{M} 2^{-l_i} \leq 1$ Kraft inequality

$$f(\{l_i\}) = \sum_i p_i l_i + \lambda (2^{-l_i} - 1)$$

$$\frac{\partial f}{\partial l_i} = p_i - \lambda 2^{-l_i} \ln 2 = 0$$

$$p_i = \lambda 2^{-l_i} \ln 2 \quad = 1$$

$$\lambda = \frac{1}{\ln 2}$$

$$p_i = 2^{-l_i}$$

$$\sum_{i=1}^{M} p_i l_i = \sum_{i=1}^{M} 2^{-l_i} l_i$$

M

$$\langle l \rangle = -\sum_{i=1} p_i \log p_i$$

Joint entropy and conditional entropy.

$$H(x,y) = -\sum_x \sum_y p(x,y) \log p(x,y)$$

$$= -\sum_{xy} p(x,y) \log p(x)\, p(y|x)$$

$$= -\sum_{xy} p(x,y) \log p(x) - \sum_{xy} p(x,y) \log p(y|x)$$

$$= -\sum_{x} p(x) \log p(x) \cancel{-\ p(x,y)}$$

$$- \sum_{xy} p(x)\, p(y|x) \log p(y|x)$$

$$= E(x) + E(y|x)$$

Ex 1.

Y / X	1	2	3	4	
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$	$\log(4)$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$	
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\log\left(\frac{1}{16}\right)$
4	$\frac{1}{4}$	0	0	0	

$$p(x) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

$$p(y) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$H(X|Y) = \sum_{xy} p(y) H(X|Y)$$

$$\begin{aligned} = \; & p(y=1) H(X|y=1) \\ + \; & p(y=2) H(X|y=2) \\ + \; & p(y=3) H(X|y=3) \\ + \; & p(y=4) H(X|y=4) \end{aligned}$$

$$= \frac{1}{4} H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$$

$$+ \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4} H(1, 0, 0, 0)$$

Relation between entropy and mutual information

$$I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

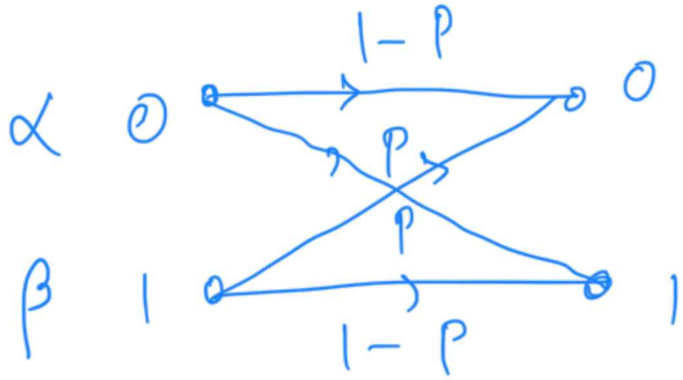
$$= \sum_{x,y} p(x,y) \log \frac{p(x|y)}{p(x)}$$

$$= -\sum p(x,y) \log p(x) + \sum_{x,y} p(x,y) \log p(x|y)$$

$$= -\sum_{x} p(x) \log p(x) - \left(-\sum p(x)\, p(x|y) \log p(x|y)\right)$$

$$= E(X) - E(X|Y)$$

Binary channel.



$p(0|0) = 1-p \qquad p(y=0|x=0) = 1-p$

$p(0|1) = p \qquad p(y=0|x=1) = p$

$p(1|1) = 1-p \qquad p(y=1|x=1) = 1-p$

$p(1|0) = p \qquad p(y=1|x=0) = p$

$$p(y) = \sum_x p(y|x)\, p(x)$$

$$p(y=0) = p(y=0|x=0)\, p(x=0)$$

$$p(y=0|x=1)\, p(x=1)$$

$(1-p)\alpha + p(1-\alpha) = p + \alpha - 2p\alpha$

$$= (1-p)\alpha + p(1-\alpha) = p+\alpha-2p\alpha$$

$$p(y=1) = p(y=1 \mid x=0)\, p(x=0) + p(y=1 \mid x=1)\, p(x=1)$$

$$= 1 - p(y=0)$$

$$H(Y \mid X) = \sum_{x \in \mathcal{X}} p(x)\, H(Y \mid X)$$

$$= p(x=1)\, H(Y \mid x=1) + p(x=0)\, H(Y \mid x=0)$$

$$= -(1-\alpha)\left(p \log p + (1-p)\log(1-p)\right) - \alpha\left(p \log p + (1-p)\log(1-p)\right)$$

$$E(Y) = -(p+\alpha-2p\alpha)\log(p+\alpha-2p\alpha) - \left(1-(p+\alpha-2p\alpha)\right)\log\left(1-(p+\alpha-2p\alpha)\right)$$

$$MI(X;Y) = E(Y) - E(Y|X)$$

At $\alpha = \frac{1}{2}$ Maximum MI is acheived

$$MI(X;Y) = 1 + p\log p + (1-p)\log(1-p)$$

Electric circuit LR

$$L\frac{dI}{dt} + RI = \sqrt{\Gamma}\,\eta(t)$$

$$\frac{dI}{dt} + \frac{R}{L}I = \sqrt{\frac{\Gamma}{L^2}}\,\eta(t)$$

$$\langle I\rangle = 0 \quad \langle I^2\rangle = \frac{\Gamma/L^2}{2R/L} = \frac{\Gamma}{2RL}$$

$$\frac{1}{2}LI^2 = \frac{1}{2}k_BT \quad \sigma_I^2 = \langle I^2\rangle = \frac{k_BT}{L}$$

$$\frac{k_BT}{L} = \frac{\Gamma}{2RL} \qquad \Gamma = 2Rk_BT$$

without external voltage

$$\phi(I) = \frac{1}{\sqrt{2\pi\sigma_I^2}} e^{-\frac{1}{2}\left(I/(k_BT/L)\right)^2}$$

with external voltage V

$$\phi(I|V) = \frac{1}{\sqrt{2\pi\sigma_I^2}} e^{-\frac{1}{2}\left(\frac{I-I_0}{k_BT/L}\right)^2}$$

$$I_0 = \frac{V_0}{R}$$

$$I_1 = \frac{V_1}{R} \qquad I_2 = \frac{V_2}{R}$$

$$\phi(I|V_1) = \frac{1}{\sqrt{2\pi\sigma_I^2}} e^{-\frac{1}{2}\frac{(I-I_1)^2}{\sigma_I^2}}$$

$$\phi(I|V_2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\left(\frac{I-I_2}{\sigma_I}\right)^2}$$

$$\phi(I) = \phi(I|V_1)\,\phi(V_1) + \phi(I|V_2)\,\phi(V_2)$$

$$E(I) = -\int p(I) \log p(I)\, dI$$

$$E(I|V) = p(v_1)\int p(I|v_1) \log p(I, v_1)\, dI$$

$$- p(v_2)\int p(I|v_2) \log p(I|v_2)\, dI$$

$$= \frac{1}{2}\int \log(2\pi\sigma_I^2)\, p(I|v_1)\, dI \cdot \frac{1}{2}$$

$$+ \frac{1}{2}\int \log(2\pi\sigma_I^2)\, p(I|v_2)\, dI \;\frac{1}{2}$$

$$= \frac{1}{2}\log(2\pi\sigma_I^2) + \frac{1}{2}$$

$$E(I) = -\int p(I) \log p(I)\, dI$$

<u>Gaussian approximation</u>

$p(I|V)$  I

$$\sigma_I^2 = \int I^2 p(I, V)\, dI\, dV - \underbrace{\left(\int I\, p(I,V)\, dI\, dV\right)^2}_{\langle I\rangle^2}$$

$$\sigma_{I|V}^2 = \int I^2 p(I|V)\, p(V)\, dI\, dV$$

$$- \int\left(\int I\, p(I|V)\, dI\right)^2 p(V)\, dV$$

$$\int\left(\int I\, p(I|V)\, dI\right) \phi(V)\, dV$$

$$\Rightarrow \int I^2\, p(I|V)\, \phi(V)\, dI\, dV$$

$$= \sigma_{I|V}^2 + \langle I^2 \rangle$$

$$\sigma_I^2 = \sigma_{I|V}^2 + \langle I^2 \rangle - \langle I \rangle^2$$

$$\langle I^2 \rangle = \frac{1}{2}\left(I_1^2 + I_2^2\right)$$

$$\langle I \rangle^2 = \left(\frac{1}{2} I_1 + \frac{1}{2} I_2\right)^2$$

$$\langle I^2 \rangle - \langle I \rangle^2 = \frac{1}{2} I_1^2 + \frac{1}{2} I_2^2 - \frac{1}{4}\left(I_1^2 + I_2^2 + 2 I_1 I_2\right)$$

$$= \frac{1}{4}\left(I_1^2 + I_2^2 - 2 I_1 I_2\right)$$

$$= \frac{1}{4}\left(I_1 - I_2\right)^2 = \frac{1}{4R^2}\left(V_1 - V_2\right)^2$$

signal to noise ration

$$\langle I^2 \rangle \quad (V_1 - V_2)^2$$

$$MI = \log\left(\frac{\sigma_I^2}{\sigma_{I/c}^2}\right) = \log\left(1 + \frac{(V_1-V_2)^2}{4R^2\sigma_{I/V}^2}\right)$$

$$\boxed{MI = \log\left(1 + \frac{L(V_1-V_2)^2}{4R^2\, k_B T}\right)}$$

For

$$p(V_1) = \alpha \qquad p(V_2) = 1-\alpha$$

$$\sigma_{I|V} = \cancel{\alpha}\ \frac{k_B T}{L}$$

$$\langle I^2\rangle = \alpha\left(\frac{V_1}{R}\right)^2 + (1-\alpha)\left(\frac{V_2}{R}\right)^2$$

$$= \frac{1}{R^2}\left(\alpha V_1^2 + (1-\alpha) V_2^2\right)$$

$$\langle I\rangle^2 = \left(\alpha\frac{V_1}{R} + (1-\alpha)\frac{V_2}{R}\right)^2$$

$$= \frac{1}{R^2}\left(\alpha V_1 + (1-\alpha) V_2\right)^2$$

$$\langle I^2\rangle - \langle I\rangle^2 = \frac{1}{R^2}\Big[\alpha V_1^2 + (1-\alpha) V_2^2$$

$$- \alpha^2 V_1^2 - (1-\alpha)^2 V_2^2 - 2\alpha(1-\alpha)V_1V_2\Big]$$

$$- \alpha \, v_1 \quad \ldots \quad \alpha \quad \ldots \, v_2) \Big]$$

$$= \frac{1}{R^2}\Big[\alpha(1-\alpha)v_1^2 + (1-\alpha)v_2^2(1-1+\alpha) - 2\alpha(1-\alpha)v_1v_2\Big]$$

$$= \frac{1}{R^2}\Big[\alpha(1-\alpha)v_1^2 + \alpha(1-\alpha)v_2^2 - 2\alpha(1-\alpha)v_1v_2\Big]$$

$$= \frac{1}{R^2}\alpha(1-\alpha)(v_1-v_2)^2$$

SNR.

$$MI = \log\left(1 + \frac{\alpha(1-\alpha)(v_1-v_2)^2 L}{R^2 k_B T}\right)$$

$$\frac{\partial MI}{\partial \alpha} = \frac{(v_1-v_2)^2 L}{k_B T}(1-2\alpha) \Rightarrow \alpha = \frac{1}{2}.$$

channel capacity

$$\boxed{C = \log\left(1 + \frac{(v_1-v_2)^2 L}{4 k_B T}\right)}$$

Dichotomous process

$$x_1 \underset{\beta}{\overset{\alpha_1 s}{\rightleftharpoons}} x_2.$$

$$p(x_2|s_1) = \left(\frac{\alpha_1 s_1}{\alpha_1 s_1 + \beta}\right)$$

$$p(x_2|s_2) = \left(\frac{\alpha_1 s_2}{\alpha_1 s_2 + \beta}\right)$$

$$E(x_2|s) = \frac{1}{2}\left[\left(\frac{\alpha_1 s_1}{\alpha s_1 + \beta}\right)\log\left(\frac{\alpha_1 s_1}{\alpha_1 s + \beta}\right) + \frac{\alpha_1 s_2}{\alpha_1 s_2 + \beta}\log\left(\frac{\alpha_1 s_2}{\alpha_1 s_2 + \beta}\right)\right]$$

$$p(x_2|s) = \frac{1}{2}\left[\left(\frac{\alpha_1 s_1}{\alpha_1 s_1 + \beta}\right) + \left(\frac{\alpha s_2}{\alpha_1 s_2 + \beta}\right)\right]$$

$$E(x_2) = -\,p(x_2)\log p(x_2)$$

For the reaction

$$X_1 \underset{\beta}{\overset{\alpha s}{\rightleftharpoons}} X_2 \qquad x_1 + x_2 = x_T$$

$$(x) = \left(\frac{\alpha s}{\quad}\right) x_T$$

$$\langle x_2 \rangle = (\alpha s + \beta)$$

$$\langle \sigma_{x_2}^2 \rangle = \frac{\alpha \beta s}{(\alpha s + \beta)^2} x_T$$

$$\sigma_R^2 = \sigma_{R|s}^2 + \langle R^2 \rangle - \langle R \rangle^2$$

$$p(x_2 | s) = \frac{1}{\sqrt{2\pi \sigma_{x_2}^2}} e^{-\frac{(x_2 - \langle x_2 \rangle)^2}{2\sigma_{x_2}^2}}$$

$$\sigma_{R|s}^2 = \frac{x_T}{2} \left[\frac{\alpha \beta s_1}{(\alpha s_1 + \beta)^2} + \frac{\alpha \beta s_2}{(\alpha s_2 + \beta)^2} \right]$$

$$\langle R^2 \rangle = \frac{x_T^2}{2} \left[\left(\frac{\alpha s_1}{\alpha s_1 + \beta} \right)^2 + \left(\frac{\alpha s_2}{\alpha s_2 + \beta} \right)^2 \right]$$

$$\langle R \rangle^2 = \left[\frac{x_T}{2} \left(\frac{\alpha s_1}{\alpha s_1 + \beta} \right) + \left(\frac{\alpha s_2}{\alpha s_2 + \beta} \right) \right]^2$$

$$\langle R^2 \rangle - \langle R \rangle^2 = \frac{x_T^2}{4} \left(\frac{\alpha s_1}{\alpha s_1 + \beta} - \frac{\alpha s_2}{\alpha s_2 + \beta} \right)^2$$

$$\sigma_{R/S}^2 = \frac{x_T}{2}\left[\frac{\alpha\beta S_1 (\alpha S_2+\beta)^2 + \alpha\beta S_2 (\alpha S_1+\beta)^2}{(\alpha S_1+\beta)^2(\alpha S_2+\beta)^2}\right.$$

$$= \frac{x_T}{\alpha}$$

$$MI: \log\left(1 + \frac{x_T}{2\alpha}\frac{\left(\alpha S_1(\alpha S_2+\beta) - \alpha S_2(\alpha S_1+\beta)\right)^2}{\alpha\beta\left(S_1(\alpha S_2+\beta)^2 + S_2(\alpha S_1+\beta)^2\right)}\right)$$

$$= \log\left[1 + \frac{x_T}{2}\frac{\alpha}{\beta}\left(\frac{\beta^2(S_1-S_2)^2}{S_1(\alpha S_2+\beta)^2 + S_2(\alpha S_1+\beta)^2}\right)\right]$$

$$= \log\left[1 + \frac{\alpha\beta}{2}x_T\left(\frac{(S_1 - S_2)^2}{S_1(\alpha S_2+\beta)^2 + S_2(\alpha S_1+\beta)^2}\right)\right]$$

$$S_1\left(\alpha(S_1+\Delta S)+\beta\right)^2$$

$$= S_1\left(\alpha(S_1+\Delta S)^2 + \beta^2 + 2\alpha\beta(S_1+\Delta S)\right)$$

$$= S_1\left(\alpha S_1^2 + 2\alpha S_1\Delta S + \beta^2 + 2\alpha\beta S_1 + \right.$$

$2\alpha\beta\Delta s)$

$$\frac{\partial M_I}{\partial \beta} = \frac{\alpha x_T}{2} \frac{(s_1 - s_2)^2}{s_1(\alpha s_2 + \beta)^2 + s_2(\alpha s_1 + \beta)^2}$$

$$- \frac{\alpha \beta x_T}{2} \frac{(s_1 - s_2)^2 x}{\left(s_1(\alpha s_2 + \beta)^2 + s_2(\alpha s_1 + \beta)^2\right)}$$

K-L divergence

$$\langle l \rangle = \sum_i p_i \log\left(\frac{1}{q_i}\right)$$

$$= \sum p_i \log \frac{p_i}{q_i} - \sum p_i \log p_i$$

$$\langle l \rangle = D(p_i \| q) + H(p)$$

K-L divergence

$$D(p(x) \| q(x)) = \int p(x) \log \frac{p(x)}{q(x)} \, dx$$

$$D'(p(x;\theta) \| p(x;\theta')) = \int p(x;\theta) \log \frac{p(x,\theta)}{p(x,\theta')} dx$$

$$= \int p(x,\theta) \log p(x,\theta)\, dx - \int p(x,\theta) \log p(x;\theta')\, dx$$

$$\frac{\partial D}{\partial \theta'} = -\int \frac{p(x,\theta)}{p(x,\theta')} \frac{\partial p}{\partial \theta'}\, dx.$$

$$\frac{\partial^2 D}{\partial \theta'^2} = \int \frac{p(x,\theta)}{p(x,\theta')^2} \left(\frac{\partial p}{\partial \theta'}\right)^2 dx - \int \frac{p(x,\theta)}{p(x,\theta')} \frac{\partial^2 p}{\partial \theta'^2}\, dx$$

$$= \int \frac{1}{p(x,\theta)} \left(\frac{\partial p}{\partial \theta}\right)^2 dx = \int \frac{1}{p(x;\theta)^2} \left(\frac{\partial p}{\partial \theta}\right)^2 p(x,\theta)\, dx$$

$$= E\left(\frac{\partial}{\partial \theta} \log p(x;\theta)\right)^2$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{p} \frac{\partial p}{\partial \theta}\right) = -\left\langle \frac{1}{p^2}\left(\frac{\partial p}{\partial \theta}\right)^2 \right\rangle + \left\langle \frac{1}{p} \frac{\partial^2 p}{\partial \theta^2} \right\rangle$$

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\log p = -\left\langle \frac{1}{p^2}\left(\frac{\partial p}{\partial\theta}\right)^2\right\rangle$$

$$= -\left(\frac{\partial \log p}{\partial\theta}\right)$$

Fisher information

$$F = \left\langle\left(\frac{\partial}{\partial\theta}\log p(x;\theta)\right)^2\right\rangle$$

For electrical circuits

$$p(I|V) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{I-\langle I\rangle}{\sigma}\right)^2}$$

$$\log p(I|V) = -\frac{1}{2}\left(\frac{I-\langle I\rangle}{\sigma}\right)^2 - \frac{1}{2}\log(2\pi\sigma^2)$$

$$\langle I\rangle = \frac{V}{R}$$

$$\left(\frac{\partial}{\partial V}\log p(I;V)\right)^2 = \left(\frac{(I-\langle I\rangle)}{\sigma^2}\cdot\frac{1}{R}\right)^2$$

$$\left\langle \left(\frac{\partial}{\partial v} \log p(\bar{I}|v) \right)^2 \right\rangle = \frac{1}{\sigma^4 R^2} \left\langle \left(I - \langle I \rangle \right)^2 \right\rangle$$

$$F = \frac{1}{\sigma^2 R^2} = \frac{L}{k_B T R^2}$$

$$MI = \log\left(1 + \frac{(V_1 - V_2)^2 L}{4 k_B T R^2} \right)$$

For the

$$x_1 \underset{\beta}{\overset{\alpha s}{\rightleftharpoons}} x_2$$

$$p(x_2|s) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x_2 - \langle x_2 \rangle}{\sigma}\right)^2}$$

$$\sigma^2 = \frac{\alpha\beta s}{(\alpha s + \beta)^2} \frac{X_T}{\Omega} \quad \mu = \langle x_2 \rangle = \frac{\alpha s}{\alpha s + \beta} X_T$$

$$\log p(x_2|s) = -\frac{1}{2} \frac{(x_2 - \mu)^2}{\sigma^2} - \frac{1}{2} \log(2\pi\sigma^2)$$

$$\log p(x_2|s) \quad 2 \quad \sigma$$

$$\left(\frac{\partial}{\partial s}\log p(x_2|s)\right)^2 = -\frac{(x_2-\mu)}{\sigma^2}\frac{\partial\mu}{\partial s}\;\frac{1}{4\pi\sigma^2}2\sigma\left(\frac{\partial\sigma}{\partial s}\right)$$

$$= \left(\frac{(x_2-\mu)}{\sigma^2} - \frac{1}{2\pi}\;\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial s}\right)\right)^2$$

$$F = \left\langle\frac{(x_2-\mu)^2}{\sigma^4}\right\rangle\left(\frac{\partial\mu}{\partial s}\right)^2$$

$$F = \frac{1}{\sigma^2}\left(\frac{\partial\mu}{\partial s}\right)^2 \qquad x_T \text{ is large.}$$

$$\mu = \frac{\alpha_1 s}{\alpha s+\beta}x_T \qquad \frac{\partial\mu}{\partial s} = \frac{\alpha_1}{\alpha s+\beta} - \frac{\alpha s\cdot\alpha}{(\alpha s+\beta)^2}$$

$$= \frac{\alpha}{(\alpha s+\beta)^2}\left[(\alpha s+\beta) - \alpha s\right]x_T$$

$$\frac{\partial\mu}{\partial s} = \frac{\alpha\beta}{(\alpha s+\beta)^2}x_T$$

$$\frac{1}{\sigma}\left(\frac{\partial \mu}{\partial s}\right)^2 = \frac{(\alpha s+\beta)^2}{\alpha \beta s} \cdot \frac{\alpha^2 \beta^2}{(\alpha s+\beta)^4} X_T \Omega$$

$$F = \frac{\alpha \beta}{s(\alpha s+\beta)^2} X_T \Omega$$

$$MI = \log\left(1 + \frac{\alpha\beta}{2} \frac{(s_1-s_2)^2}{s_1(\alpha s_1+\beta)^2 + s_2(\alpha s_2+\beta)^2}\right)$$

Precision $= \frac{s}{\Delta s}$

$$\mathrm{var}(s) > \frac{1}{F}.$$

Relative error $: \frac{\mathrm{var}(s)}{s} > \frac{1}{sF}$

Precision $\cancel{s}F > \frac{s}{\sqrt{\mathrm{var}(s)}}$

$$P(s) = \frac{s^2}{\mathrm{var}(s)} < s^2 F$$

$\alpha\beta \; s$

$$p(s) = \frac{[illegible]}{(\alpha s+\beta)^2}$$

$$(s+a)^2 = y$$

$$2(s+a)\,ds = dy$$

$$\int_0^{s_\infty} p(s)\,ds = \frac{\cancel{\alpha}\beta}{\alpha}\int \frac{s}{(s+a)^2}\,ds$$

$$= \frac{\beta}{\alpha}\left[\frac{a}{a+s} + \log(a+s)\right]_0^{s_\infty}$$

$$= \frac{\beta}{\alpha}\left[\frac{a}{a+s_\infty} - 1 + \log\left(\frac{a+s_\infty}{a}\right)\right]$$

$$= a\left[-\frac{s_\infty}{a+s_\infty} + \log\left(\frac{a+s_\infty}{a}\right)\right].$$

$$\xrightarrow{\alpha_1} x \xrightarrow{1}$$

$$\xrightarrow{\alpha_2 x} y \xrightarrow{\beta}$$

$$\langle x \rangle = \frac{\alpha_1}{\beta} \qquad \langle \sigma_x^2 \rangle = \frac{\alpha_1}{\beta} \frac{1}{\Omega}$$

$$\frac{d \Delta y}{dt} = \alpha_2 \Delta x - \beta \Delta y + \eta_2(t)$$

$$\Delta y(\omega) = \frac{\alpha_2 \Delta x}{-i\omega + \beta} + \frac{\eta_1(\omega)}{-i\omega + \beta}$$

$$= \frac{\alpha_2 \eta_1(\omega)}{(-i\omega+\beta)(-i\omega+\beta)} + \frac{\eta_1(\omega)}{-i\omega+\beta}$$

$$\langle |\Delta y(\omega)|^2 \rangle = \int \frac{\alpha_2^2 q_1 \, d\omega}{(\omega^2+\beta^2)(\omega^2+\beta^2)} + \int \frac{q_2 \, d\omega}{\omega^2+\beta^2}$$

$$\langle |\Delta y(\omega)|^2 \rangle = \frac{\alpha_2^2 q_1}{2\beta^2 \cdot 2\beta} + \frac{q_2}{2\beta}.$$

$$q_2 = \alpha_2 x + \beta y = 2\alpha_2 \frac{\alpha_1}{\beta}\frac{1}{\Omega}$$

$$q_1 = 2\frac{\alpha_1}{\quad}\frac{1}{\Omega}$$

$$\sigma_y^2 = \langle |\Delta y(t)|^2 \rangle = \frac{2\alpha_2^2\, 2\alpha_1}{4\beta^3\, \Omega} + \frac{\alpha_2\alpha_1}{\beta^2\Omega}$$

$$= \frac{\alpha_2\alpha_1}{\beta^2\Omega}\left[1 + \frac{\alpha_2}{\beta}\right]$$

$$= \frac{\langle y\rangle}{\Omega}\left[1 + \frac{\alpha_2}{\beta}\right]$$

$$\sigma_x^2 = \langle \Delta x^2 \rangle = \frac{\langle x\rangle}{\Omega}$$

$$p(x|s) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{1}{2}\left(\frac{x-\langle x\rangle}{\sigma}\right)^2}$$

$$\sigma_{x|s}^2 = \frac{1}{9}\left(\frac{\alpha_1 s_1}{\beta\,\Omega} + \frac{\alpha_1 s_2}{\beta\,\Omega}\right)$$

$$= \frac{\alpha_1}{\beta\Omega}(s_1 + s_2)$$

$$R_1 = R^2 \, p(R;s) \, \frac{1}{2}\left[\left(\frac{\alpha_1 s_1}{\beta}\right)^2 + \left(\frac{\alpha_1 s_2}{\beta}\right)^2\right]$$

$$R_2 = \left(\frac{1}{2}\frac{\alpha_1 s_1}{\beta} + \frac{1}{2}\frac{\alpha_1 s_2}{\beta}\right)^2$$

$$\sigma_x^2 = \sigma_{x|s}^2 + R_1 - R_2$$

$$= \sigma_{x|s}^2 + \frac{1}{4}\left(\frac{\alpha_1}{\beta}\right)^2 (s_1 - s_2)^2$$

$$MI = \log\left(1 + \frac{\frac{1}{4}(\alpha_1/\beta)^2 (s_1 - s_2)^2 \Omega}{(\alpha_1/\beta)(s_1 + s_2)}\right)$$

$$MI(x;s) = \log\left(1 + \frac{1}{4}\frac{\alpha_1}{\beta_1}\Omega\frac{(s_1 - s_2)^2}{s_1 + s_2}\right)$$

$$\sigma^2_{y|s} = \frac{1}{2}\left(\frac{\alpha_1\alpha_2 s_1}{\beta^2\Omega} + \frac{\alpha_1\alpha_2 s_2}{\beta^2\Omega}\right)\left(1+\frac{\alpha_2}{\beta}\right)$$

$$= \frac{1}{2}\,\frac{\alpha_1\alpha_2}{\beta^2\Omega}\left(1+\frac{\alpha_2}{\beta}\right)\left(s_1+s_2\right)$$

$$R_1 = \frac{1}{2}\left[\left(\frac{\alpha_1\alpha_2 s_1}{\beta^2}\right)^2 + \left(\frac{\alpha_1\alpha_2 s_2}{\beta^2}\right)^2\right]$$

$$R_2 = \left(\frac{1}{2}\,\frac{\alpha_1\alpha_2}{\beta^2} s_1 + \frac{1}{2}\,\frac{\alpha_1\alpha_2 s_2}{\beta^2}\right)^2$$

$$\sigma^2_y = \sigma^2_{y|s} + \frac{1}{4}\left(\frac{\alpha_1\alpha_2}{\beta^2}\right)^2\left(s_1-s_2\right)^2$$

$$MI(y/s) = \log\left(1 + \frac{\frac{\alpha_1\alpha_2}{\beta^2}\left(s_1-s_2\right)^2\Omega}{4\left(1+\alpha_2/\beta\right)\left(s_1+s_2\right)}\right)$$

$$\alpha_1 = \alpha_2 = \beta = 1$$

$$MI(x;s) = \log\left(1 + \frac{\left(s_1-s_2\right)^2}{4\left(s_1+s_2\right)}\right)$$

$$MI(y; S) = \log\left(1 + \frac{1}{2}\frac{(S_1 - S_2)^2}{4(S_1 + S_2)}\right)$$

Fisher information

$$\mu_x = \frac{\alpha_1 S}{\beta}$$

$$\sigma_x^2 = \frac{\alpha_1 S}{\beta \Omega}$$

$$F_x(S) = \frac{1}{\sigma_x^2}\left(\frac{\partial \mu_x}{\partial S}\right)^2$$

Ω ~ large

$$= \frac{\beta \Omega}{\alpha_1 S}\left(\frac{\alpha_1}{\beta}\right)^2 = \Omega\left(\frac{\alpha_1}{\beta}\right)\frac{1}{S}$$

$$\mu_y = \frac{\alpha_1 \alpha_2 S}{\beta^2}$$

$$\sigma_y^2 = \frac{\langle y \rangle}{\Omega}\left(1 + \frac{\alpha_2}{\beta}\right) = \frac{\alpha_1 \alpha_2 S}{\beta^2 \Omega}\left(1 + \frac{\alpha_2}{\beta}\right)$$

$$F_y(S) = \frac{1}{\sigma_y^2}\left(\frac{\partial \mu_y}{\partial S}\right)^2 = \frac{\beta^2 \Omega}{\alpha_1 \alpha_2 S}\frac{1}{1 + \alpha_2/\beta}\left(\frac{\alpha_1 \alpha_2}{\beta^2}\right)^2$$

$$= \frac{\alpha_1 \alpha_2}{\beta} \Omega \frac{1}{1 + \alpha_2/\beta} \frac{1}{s}$$

$\alpha_1 = \alpha_2 = \beta = 1$

$$F_x(s) = \frac{\Omega}{s}$$

$$F_y(s) = \frac{1}{2} \frac{\Omega}{s}$$