Probability and Statistics

UG2, Core course, IIIT,H

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IIIT, Hyderabad

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1. Set Theory
2. Product Rule
3. Tuples
4. Combinations
5. Pascal's Triangle and Combinations...

1 Outline

1. Set Theory
2. Product Rule
3. Tuples
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1 Review of Set theory

Definition of a Set

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- A set which is countable and not finite is called countably infinite

1 Set Theory: Venn Diagram

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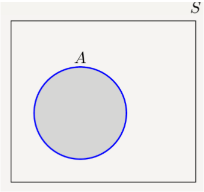

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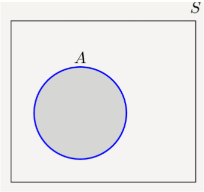
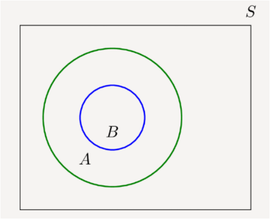


- Venn diagram showing subset relationship



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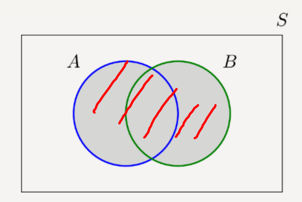

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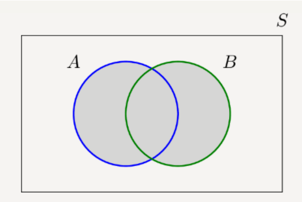

- Similarly, we define union of three or more sets as follows

$$A_1 \cup A_2 \cup \cdots \cup A_k = \cup_{i=1}^{k} A_i$$

1 Quiz

- If A and B are countable, then $A \cup B$ is also countable

1 Quiz

$A_1 \cup A_2$

- Countable union of countable sets is countable

$\bigcup A_i$

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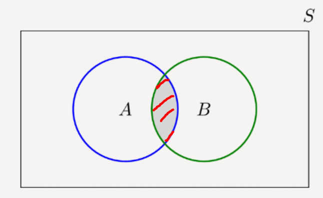

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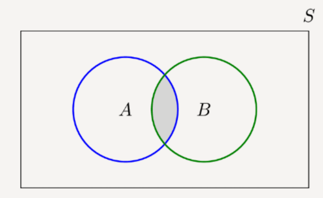

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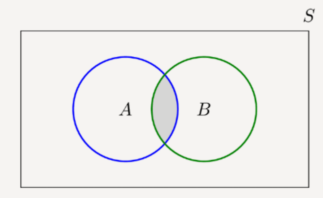

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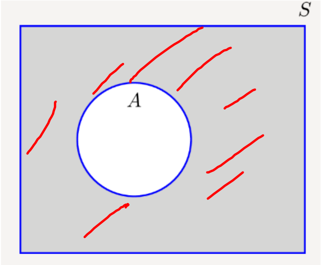

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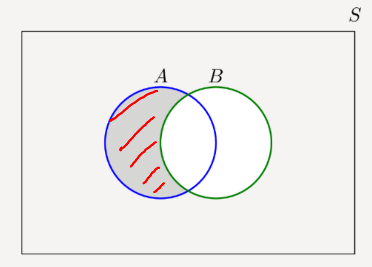

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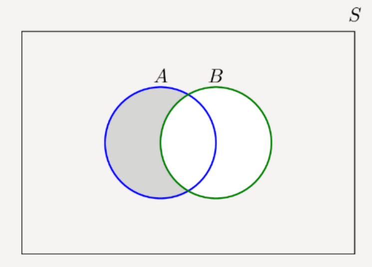

- Two sets A and B are mutually exclusive or disjoint if they have no shared element, i.e., $A \cap B = \phi$

$(a_1, b_1), (a_1, b_2) \cdots (a_1, b_n),$
$(a_2, b_1), (a_2, b_2) \cdots (a_2, b_n)$
$\ddots$

Define Cartesian Product of Sets

Cartesian product of two sets $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$ denoted by $A \times B$ is defined as follows

$$A \times B = \underbrace{\cup_{i,j}\{(a_i, b_j)\}}$$

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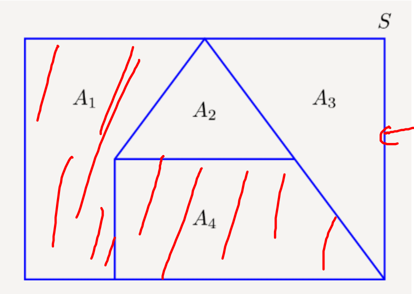

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1 Rule of Sum using Set Language

Rule of Sum

If there is a set A with k elements, a set B with n elements and these sets do not have common elements, then the set $A \cup B$ has $n + k$ elements

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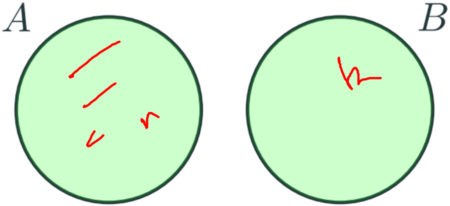

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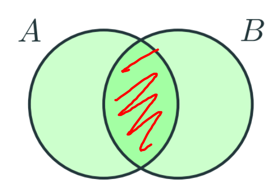

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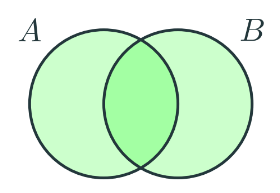

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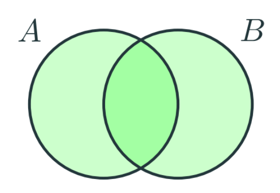

- If we consider $|A| + |B|$ as in sum rule, then we will be wrong
- We will count elements that belong to both A and B twice
- $|A \cup B| = |A| + |B| - |A \cap B|$ (Inclusion-Exclusion Principle)

1 Applications of sum rule

Sum rule

Count all integers from 1 to 10 that are divisible by 2 or by 3

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Sum rule

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let us count all the numbers from 1 to 10:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

- Here 6 is divisible both by 2 and by 3. Hence, rule of sum can't be applied!

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Number of Paths

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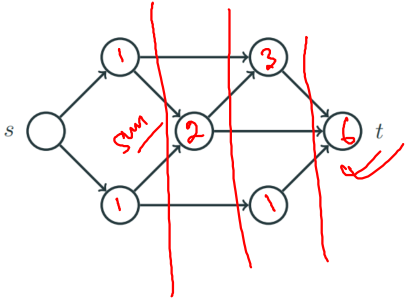

Claim: Apply Sum Rule!

A

B

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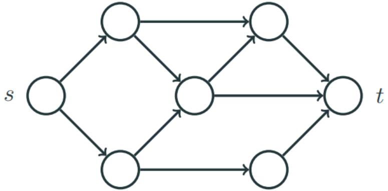

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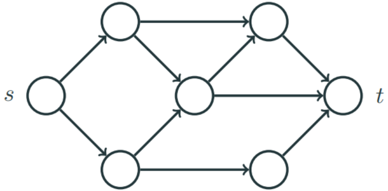

- counting can be done recursively
- for each node count the number of paths from s to this node
    - sum rule will be used

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2 Product Rule

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$|A \times B|$

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2 All possible pairs

2 Rule of Product Using Sets

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Product Rule

If there is a finite set A and a finite set B, then there are $|A| \times |B|$ pairs of objects, the first from A, and the second from B

2 Tuples: Application of Product Rule

a a a b b

Tuples

How many different 5-symbol passwords can we create using lower case Latin letters only? (the size of the alphabet is 26)

$|\{a, b, c, d, \ldots, z\}| = 26$

$\underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26} = 26^5$

$A \times B \times C \times D \times E$

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- What about 2-letters?
- How many different 3-letter words are possible?
- Can you now answer the question above?

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Question

Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols?

- can apply the same argument as above! (product rule!)

$\underbrace{n \cdot n \cdot n \cdots n}_{k \text{ times}} = n^k$

(Like guessing example)

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- similarly, There are n possibilities to pick the second letter, and so on...
- thus, the answer is a product of n by itself k times, that is n^k

3 Number of Number Plates

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Consider the typical vehicle number plate in India

3 Number of Number Plates

CC

26 · 26 10 · 10

26 26 10 10 10 10

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- the first two letters denote state

Consider the typical vehicle number plate in India

- the first two letters denote state
- the following two-digit number stands for district number

3 Number of Number Plates

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Question

How many vehicles are there?

3 Tuples with Restrictions (Combine Sum and Product Rule)

Question

How many integer numbers are there between 0 and 9999 that have exactly one 5 digit?

0001

0005

0051

0015

0052

Product

$9\ \boxed{5}\ 9\ 9 \rightarrow 9^3$

$\boxed{5}\ 9\ 9\ 9 \rightarrow 9^3$

$9\ 9\ \boxed{5}\ 9 \rightarrow 9^3$

$9\ 9\ 9\ \boxed{5} \rightarrow 9^3$

sum $\rightarrow 4 \cdot 9^3$

4 choices to put 5

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- This gives us 4 cases; if we compute the number of sequences in all four cases, we can get the answer by the rule of sum
- If we fix 5 in one place, then there are $5 \times 5 \times 5 = 125$ sequences
- There are 4 ways to arrange 5 among 4 places
- Hence, there are $4 \times 125 = 500$ four digit numbers below 10,000 with exactly one 5

4 Permutations

Definition

Tuples of length k without repetitions are called *$k-$permutations*

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Question

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Question

Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols if we are not allowed to use the same symbol twice?

$$\underbrace{\underline{n}\ \underline{(n-1)}\ \underline{(n-2)}\ \underline{(n-3)}\ \cdots\ \underline{n-(k-1)}}_{k \text{ length.}}$$

$$\frac{n!}{(n-k)!} \longleftarrow n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1))$$

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$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k \\ * & * & * & \cdots & * \\ n & n-1 & n-2 & & \end{array}$$

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- Hence there are

$$n \times (n-k) \times \cdots (n-k+1)$$

$k-$permutations, which is $n!/(n-k)!$

4 Permutation Examples

$n - (n-1)$

Question

In how many ways we can arrange n different books in n different bins on shelf?

$n \cdot n-1 \cdot \ldots \cdot 1 = n!$

Question

In how many ways we can arrange n different books in n different bins on shelf?

Answer

Hint: Use previous result with $k = n$.

5 Outline

1. Set Theory
2. Product Rule
3. Tuples
4. Combinations
5. Pascal's Triangle and Combinations...

5 Combinations

Question

You are organizing a car journey.

Question

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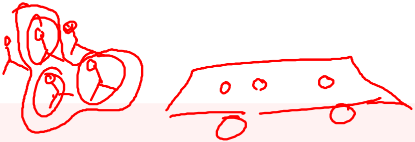
5 Combinations



Question

You are organizing a car journey. You have five friends, but there are only three vacant places in you car.

$\binom{5}{3}$

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Above Question Reformulated

We are essentially asking: What is the number of ways of choosing 3 elements out of a set containing 5 elements?

5 Answer to Previous Question...

Answer

- There are five choices of the first friend,

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Answer

- There are five choices of the first friend, four choices of the second friend, and three choices of the third friend

$5 \cdot 4 \cdot 3$

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M, S, R
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- There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
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1

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- We define combinations in next slide...

5 Combinations: $k-$combination

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For a set S, a $\underbrace{k-\text{combination}}$ is a subset of S of size k

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Definition of $k-$combinations

5 friends

The number of $k-$combinations of an n element set is denoted by $\binom{n}{k}$.

3 friends

Pronounced:"n choose k". Proof by example!

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Number of $k-$combinations

The number of $k-$combinations of an n element set is given by

$$\frac{n!}{(n-k)!}$$

$\binom{5}{3}$

$3!$

abc	abd	abe	acd	ace	ade	bcd	bce	bde	cde
acb	adb	aeb	adc	aec	aed	bdc	bec	bed	ced
bac	bad	bae	cad	cae	dae	cbd	cbe	dbe	dce
bca	bda	bea	cda	cea	dea	cdb	ceb	deb	dec
cba	dba	eba	dca	eca	eda	dcb	ecb	edb	edc
cab	dab	eab	dac	eac	ead	dbc	ebc	ebd	ecd

$$3!\binom{5}{3} = \frac{5!}{(5-3)!}$$

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6 Pascal triangle...

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A result...

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

6 Proof of previous identity...

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Hence recursion for n choose k is...

$$\binom{n}{k} = \underbrace{\binom{n-2}{k-2} + \binom{n-2}{k-1}} + \underbrace{\binom{n-2}{k-1} + \binom{n-2}{k}} = \cdots$$

6 Pascal's Triangle

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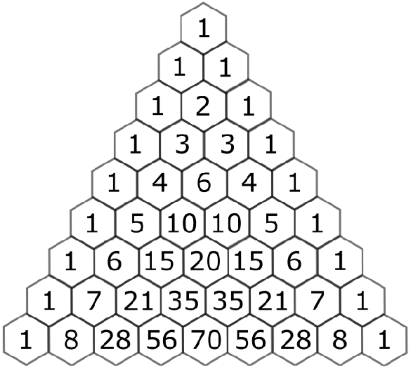

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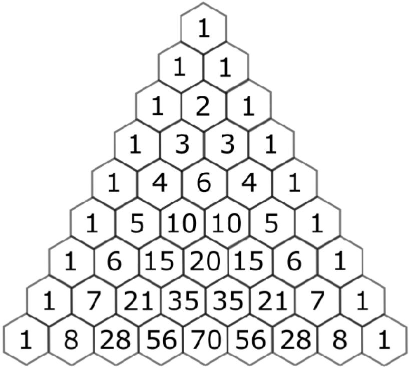

Quiz

Do you know how to grow Pascal's triangle? What is the rule?

6 Pascal's Triangle and Many Relations...

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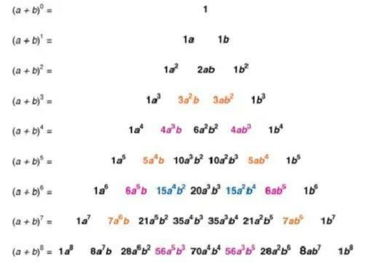
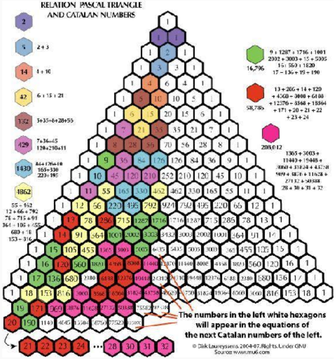
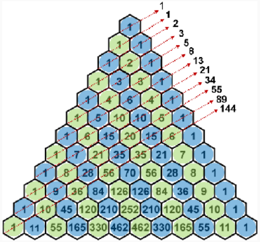
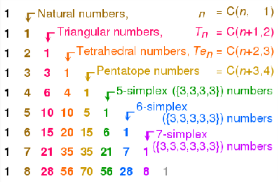
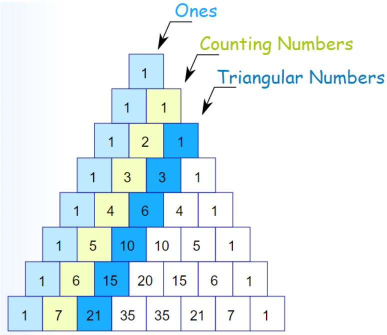








6 Pascal's Triangle and Triangular Numbers



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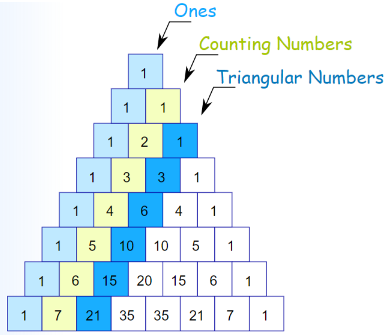
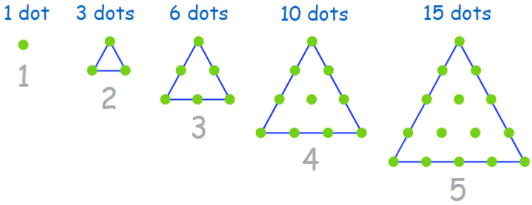


- If we look at the indicated colors, we obtain counting numbers, triangular numbers, etc



- Triangular numbers are the number of dots

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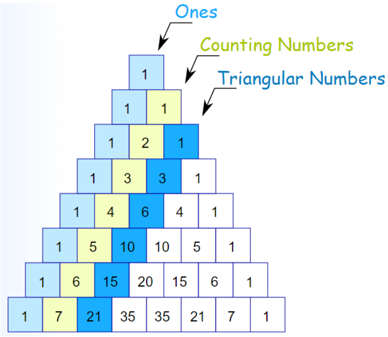

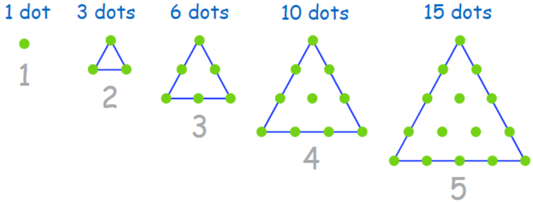

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- Triangular numbers are the number of dots
- Add one more row and dots to get next triangular number

6 Pascal's Triangle: Horizontal Sums and Exponents of 11

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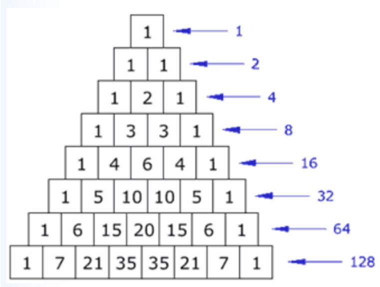

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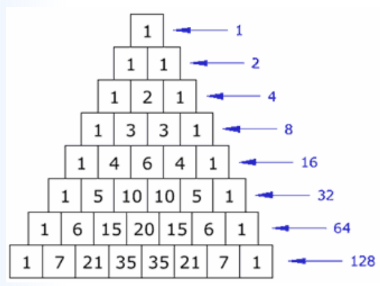

- The horizontal sums are 2^i, i is the ith row

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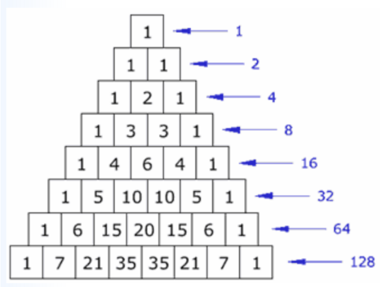

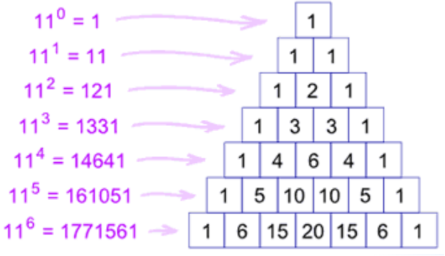

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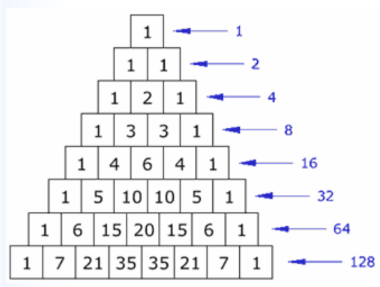
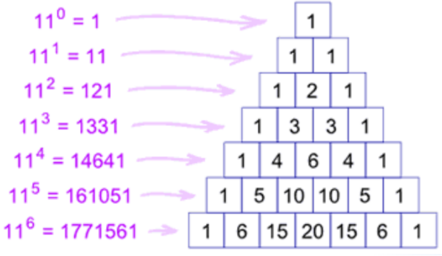
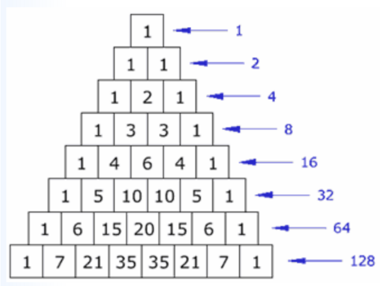




- The row entries are digits of powers of 11
- The horizontal sums are 2^i, i is the ith row

6 Pascal's Triangle: Horizontal Sums and Exponents of 11



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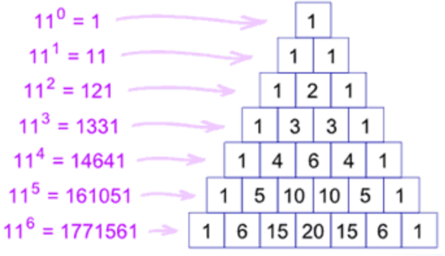

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- The entries of the ith row are digits of 11^i

6 Pascal's Triangle and Symmetry

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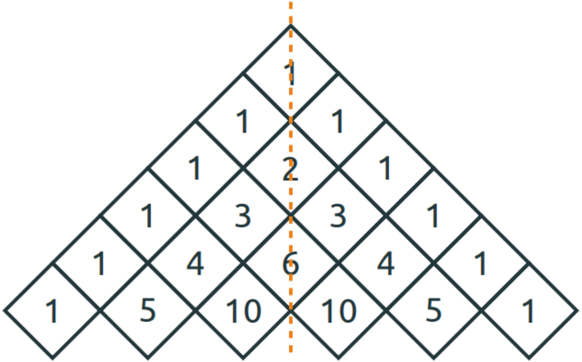

$$\binom{n}{k} = \binom{n}{n-k}$$

6 Proof of symmetry...

Theorem

Prove that

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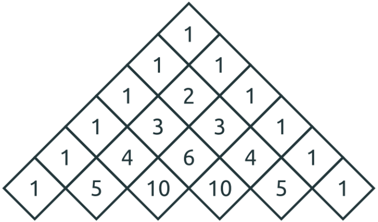
Answer

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

6 Row Sums of Pascal's Triangle...



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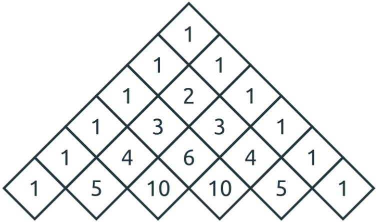

$$
\begin{array}{c}
1 \\
1 + 1 \\
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1
\end{array}
$$

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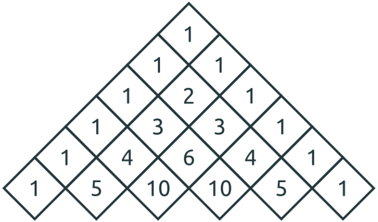

1
1 + 1
1 + 2 + 1
1 + 3 + 3 + 1
1 + 4 + 6 + 4 + 1
1 + 5 + 10 + 10 + 5 + 1

Theorem

The sum of all the numbers in the n-th row of Pascal's triangle is equal to 2^n :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

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- The base case (0-th row) holds

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- $\binom{n}{k}$ is the number of k-subsets of a set of size n
- The sum $\binom{n}{k}$ for all k (from 0 to n) is the number of all subsets of an n element set; this is 2^n by the product rule (how?)

6 Alternating Row Sum in Pascal

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & - & 1 & & & & & & & = 0 \\
 & & & & 1 & - & 2 & + & 1 & & & & & & = 0 \\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & & & = 0 \\
 & & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & & = 0 \\
 & 1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & & = 0
\end{array}
$$

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\begin{array}{ccccccccccccl}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & - & 1 & & & & & = 0 \\
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Theorem

For $n > 0$, $\quad \sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

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 & & & 1 & - & 3 & + & 3 & - & 1 & & & = 0 \\
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1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
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Theorem

For $n > 0, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

- Hint: Number of odd size subsets = Number of even size subsets

6 Counting Problems...

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Question

What is the number of 5-card hands dealt off of a standard 52-card deck?

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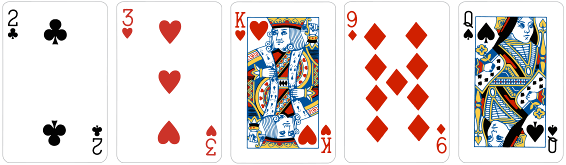

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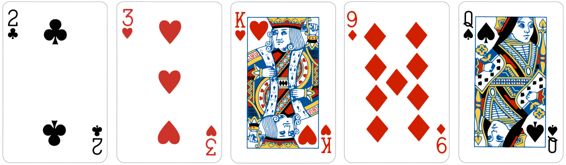

Answer

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 52 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2598960$$

Question

What is the number of 5-card hands with two hearts and three spades?

52

hearts spades diamond clubs

(13)

2

3

_ _ _ _ _

$$\binom{13}{2} \times \binom{13}{3}$$

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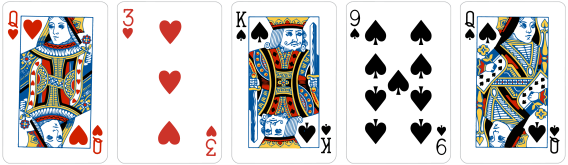

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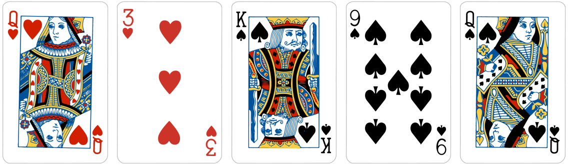

Answer

- Number of ways of picking 2 hearts from 13 hearts

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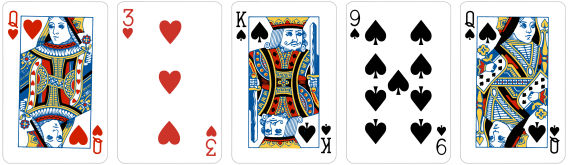

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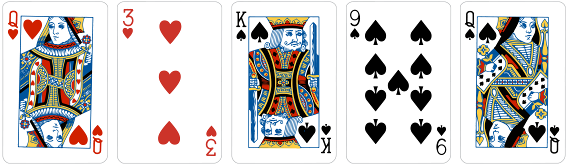

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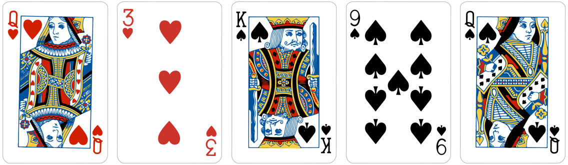

Answer

- Number of ways of picking 2 hearts from 13 hearts
- Number of ways of picking 3 spades from 13 spades
- Now apply product rule!
- The answer is: $\binom{13}{2}\binom{13}{2} = 22308$

$10 \rightarrow 9$

$\neg 7$ (atleast one) = none

Question

What is the number of non-negative integers with at most four digits $\boxed{\text{at least one}}$ of which is equal to 7?

Four digit numbers with no 7 in them = $\underline{9}\ \underline{9}\ \underline{9}\ \underline{9} = 9^4$.

Total no. of 4 digits nos = $\underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 10^4$.

$10^4 - 9^4 =$ number of ~~numbe~~ 4-digit nos with atleast one digit 7. =

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- Total number of 4 digit numbers $= 10^4$
- Total number of 4 digit number that does not contain $7 = 9^4$
- Hence, the answer is $10^4 - 9^4 = 3439$

6 Counting Problems...

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What is the number of non-negative integers with at most four digits whose digits are increasing?

3 5 6 9

[5 4 1 8]

[1 4 5 8]

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- Hence, the answer is $\binom{10}{4} = 210$

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A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?

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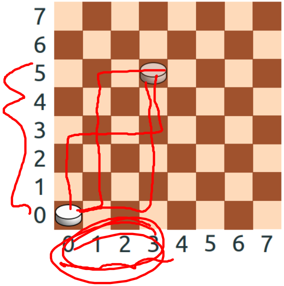

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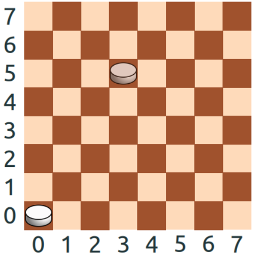

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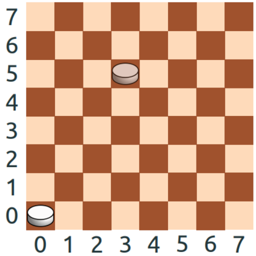

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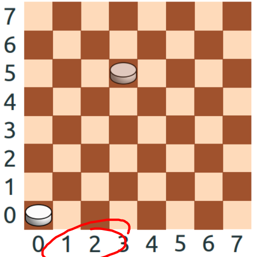

- We want to go to the cell [5,3]. How many ways we can go?
- Any path to [5,3] must involves 3 moves right and 5 moves up!
- Hence, answer is $\binom{8}{3} = 56 \Leftarrow \binom{8}{5}$

$\binom{n}{k} = \binom{n}{n-k}$ $5+3$

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1							
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It is now only a matter of filling the (5,3) cell...

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So far we have considered selections of k items out of n possible options. Consider $n = 2$ and $n = 3$ options: a, b, c

	With repetitions	Without repetitions
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So far we have considered selections of k items out of n possible options. The formulas we have derived are the following:

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- Let us try to find out...

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Question

So, what could be your answer?

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T T C O
T C C O
O C C T

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- We will list all possible salads, then count them
- But we want to do it wisely!

7 Solution continued...

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Our goal: To pick 4 items out of 3 salads (Onions, Bell Peppers, Cucumber)

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- There are 15 possible combinations. Do we see any structure?

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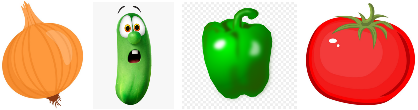

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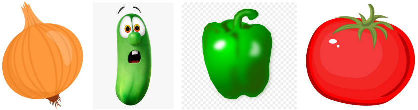

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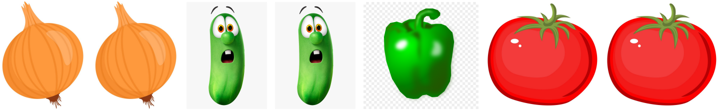

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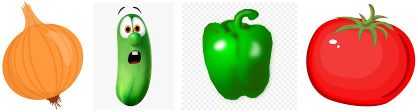

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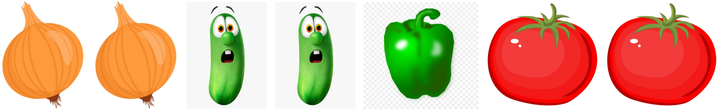

- Do you already see a way to find all possible combinations?

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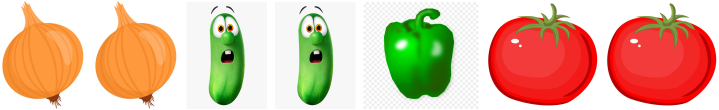

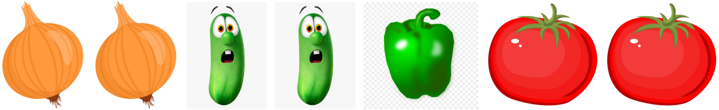

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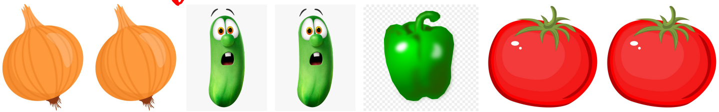

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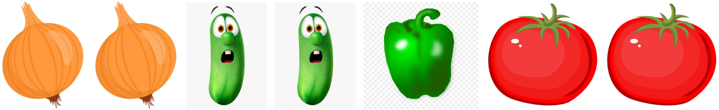

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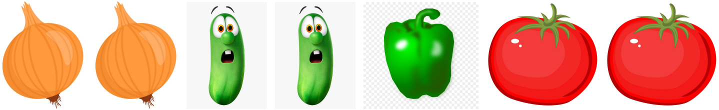
1st Onion

2nd Cucumber

3rd Bell pepper

4th Tomatoes.



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- The number of ways we can put the delimiters determine the number of combinations

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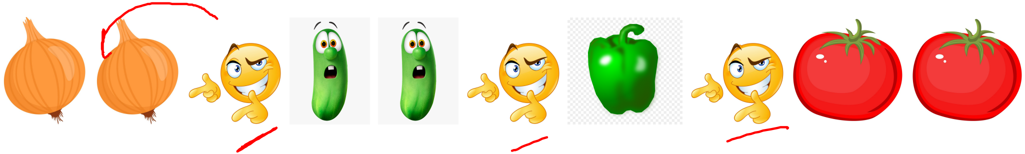

Question

How many delimiters are needed when we want to select 7 items out of 4 choices?

- Form the example above, we need 3=(4-1) delimiters!
- Total number of objects (7 items + 3 delimiters) is 10
- The problem now reduces to arranging 3 delimiters among 10 items! Voila!

7 Combinations with repetitions...

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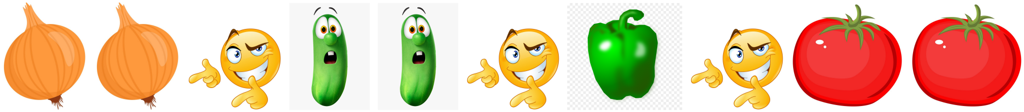

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7 Combinations with repetitions...

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- Hence, the answer is

$$\binom{7+4-1}{4-1} = \binom{10}{3}$$

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	With repetitions	Without repetitions
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Unordered	Combinations+Repet. $\binom{k+n-1}{n-1}$	Combinations $\binom{n}{k}$