CS 302.1 - Automata Theory

Shantanav Chakraborty

Center for Quantum Science and Technology (CQST)

Center for Security, Theory and Algorithms (CSTAR)

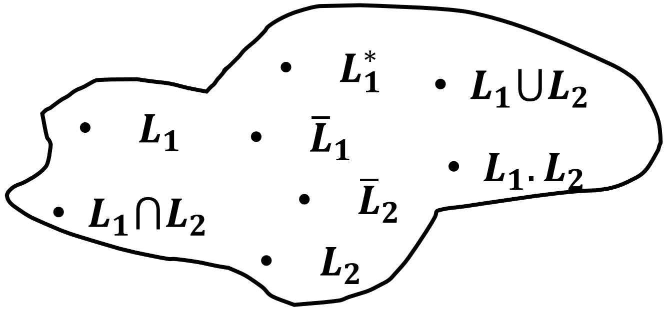
IIIT Hyderabad

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

Quick Recap

- DFAs and NFAs are equivalent
- For every NFA we can obtain a "Remembering DFA" that accepts the same language.
- The language accepted by finite automata are called Regular Languages.
- RL can also be derived from first principles.
- Regular Languages are closed under: Union, Star, Complement, Intersection…
- Regular expressions provide an elegant algebraic framework to represent regular languages.
- We can construct NFAs given a Regular Expression.



Set of all regular Languages

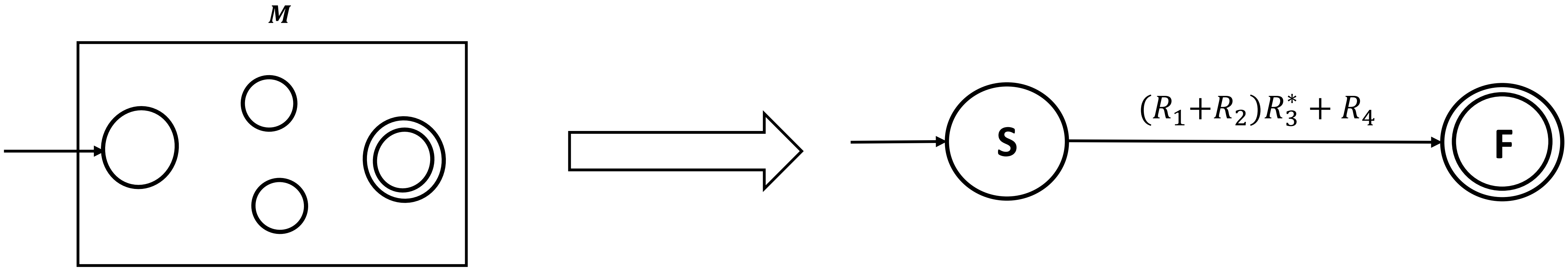
DFA to Regular Expressions

If a language is regular then it accepts a regular expression. We could draw equivalent NFAs for Regular Expressions.

How can we obtain Regular expressions given a DFA?

Given a DFA M, we **recursively** construct a two-state **Generalized NFA** (GNFA) with

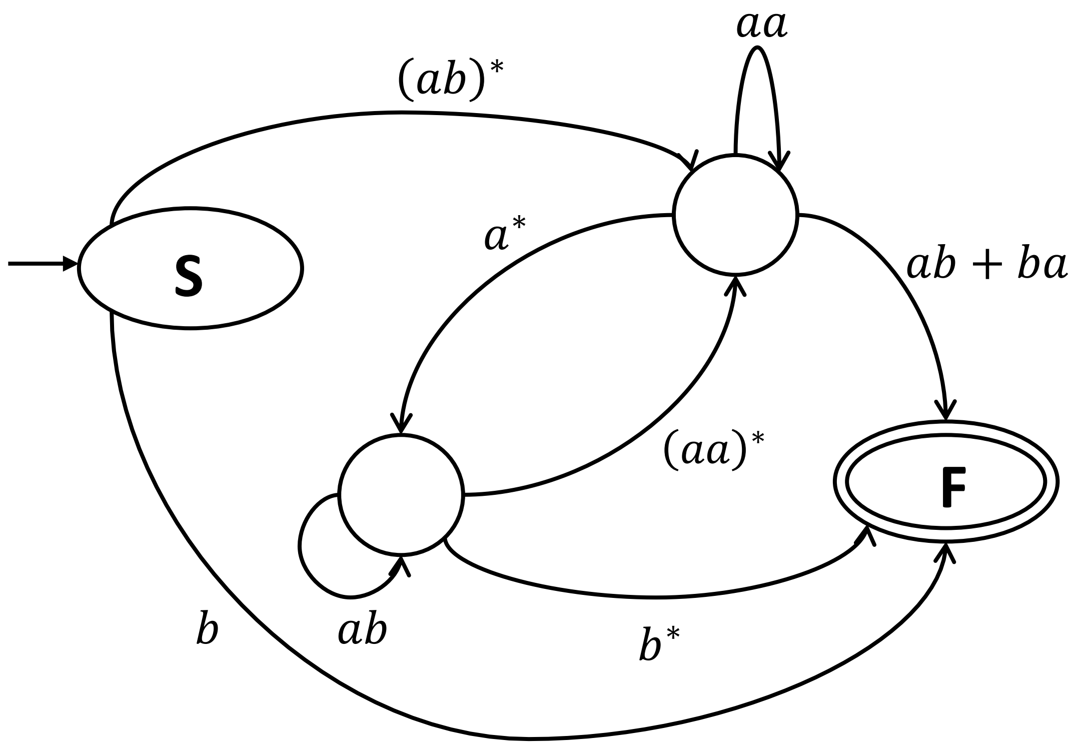
- A start state and a final state
- A single arrow goes from the start state to the final state
- The label of this arrow is the regular expression corresponding to the language accepted by the DFA M.



DFA to Regular Expressions: GNFA

What are GNFAs? They are simply NFAs such that

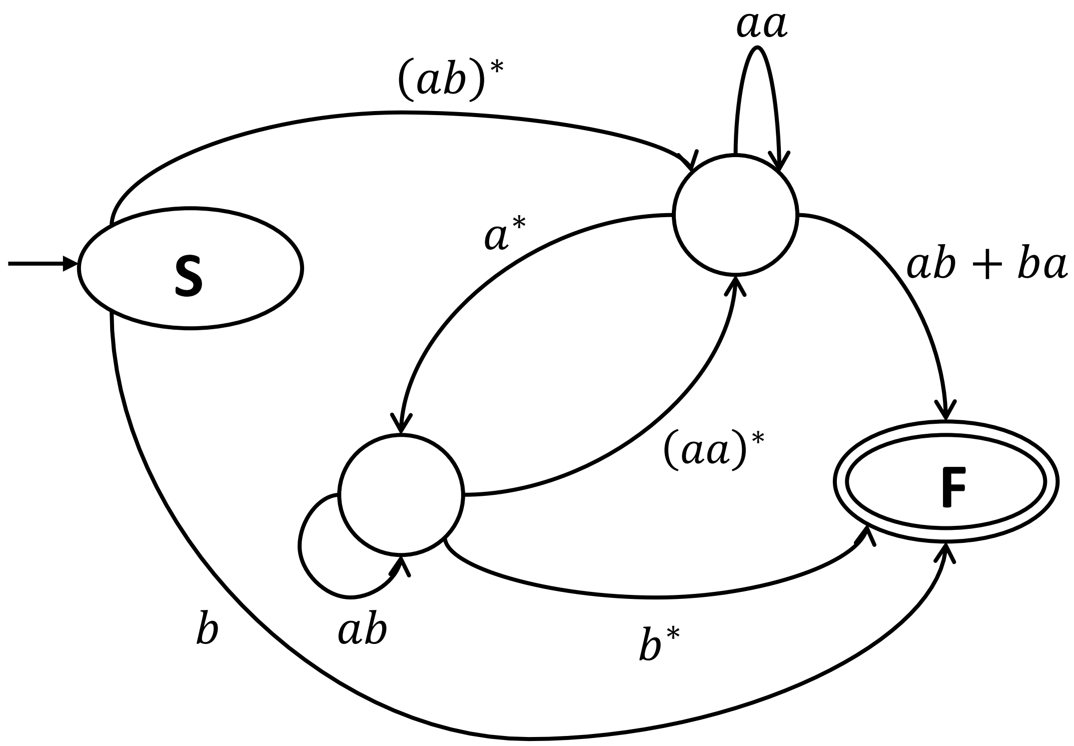
- The transitions may have regular expressions
- A unique start state that has arrows going to other states, but has no incoming arrows
- A unique final state that has arrows incoming from other states, but has no outgoing arrows
- For an input string, **runs** on a GNFA are similar to that of an NFA, except now a block of symbols are read corresponding to the Regular Expressions on the transitions.
- b, $abababab, abaaaba$ are some input strings that have accepting runs for the GNFA on the right



DFA to Regular Expressions: GNFA

What are GNFAs? They are simply NFAs such that

- The transitions may have regular expressions
- A unique start state that has arrows going to other states, but has no incoming arrows
- A unique final state that has arrows incoming from other states, but has no outgoing arrows
- For an input string, **runs** on a GNFA are similar to that of an NFA, except now a block of symbols are read corresponding to the Regular Expressions on the transitions.
- $b,\ abababab, abaaaba$ are some input strings that have accepting runs for the GNFA on the right

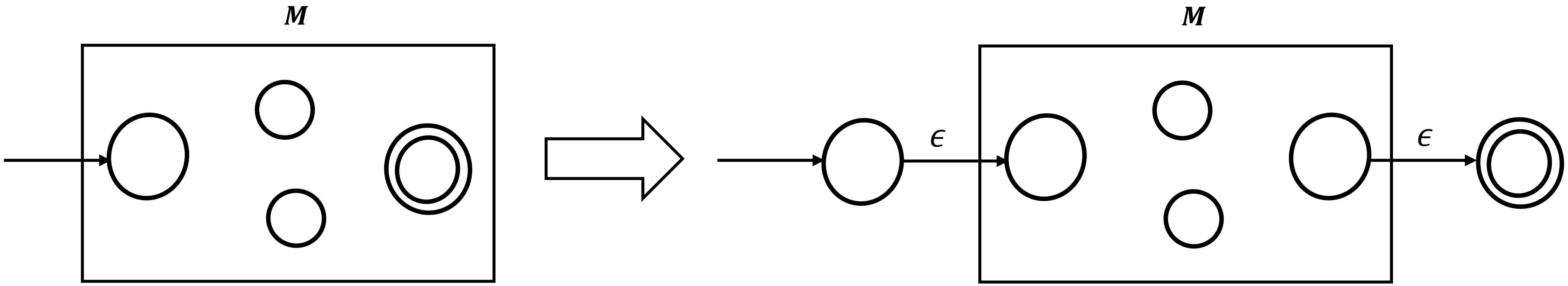


Starting from a DFA we will begin by constructing a GNFA with k states. We then outline a recursive procedure by which at each step, we will construct a GNFA with one less state. This step will be repeated until we obtain the **2-state GNFA**.

DFA to Regular Expressions: GNFA

Starting from the DFA M,

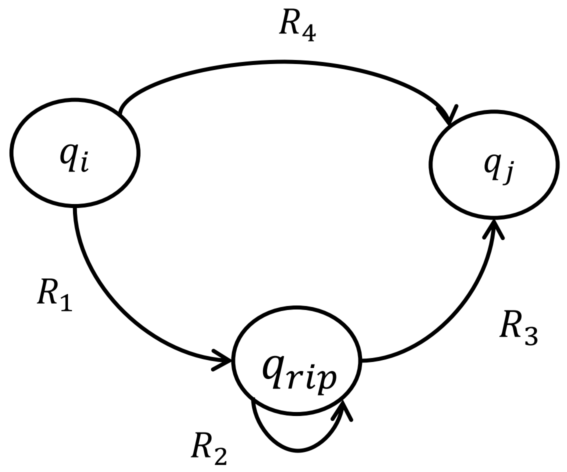
- Add a new start state with an ϵ arrow to the old start state.
- Add a new final state by with an ϵ arrow to the old final state.



DFA to Regular Expressions: GNFA

The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states. This is what we shall show next.

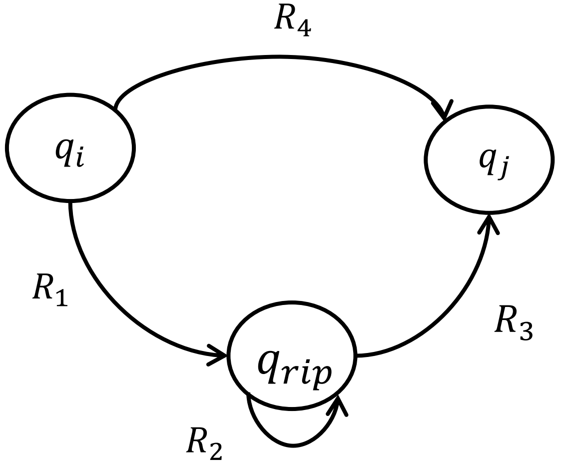
- Start by picking any state of the GNFA (except the new start and final states)
- Let us call this state q_{rip}. We "rip" q_{rip} out of the machine and create a GNFA with $k-1$ states.
- Of course, we need to "repair" the machine by altering the regular expressions that label each of the remaining arrows.
- The new labels compensate for the loss of q_{rip}.

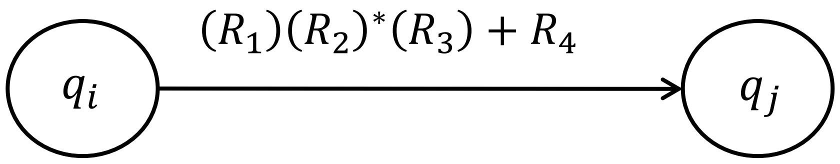


DFA to Regular Expressions: GNFA

The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states. This is what we shall show next.

- Start by picking any state of the GNFA (except the new start and final states)
- Let us call this state q_{rip}. We “rip” q_{rip} out of the machine and create a GNFA with $k-1$ states.
- Of course, we need to “repair” the machine by altering the regular expressions that label each of the remaining arrows.
- The new labels compensate for the loss of q_{rip}.





DFA to Regular Expressions: GNFA

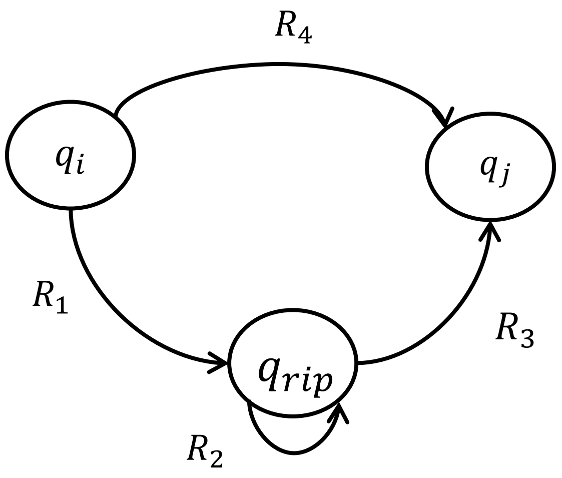
The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states.

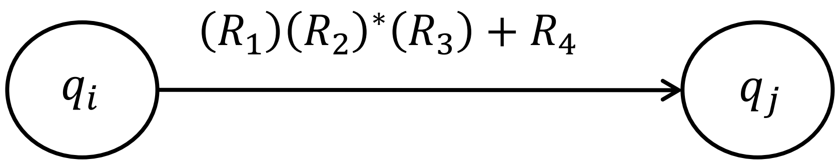
How do we remove q_{rip}? In the old machine if

- q_i goes to q_{rip} with an arrow labelled R_1
- q_{rip} goes to itself with an arrow labelled R_2
- q_{rip} goes to q_j with an arrow labelled R_3
- q_i goes to q_j with an arrow labelled R_4

Repeat this until $\boldsymbol{k = 2}$

then in the new machine, the arrow from q_i to q_j has the label $(R_1)(R_2)^*(R_3) + R_4$

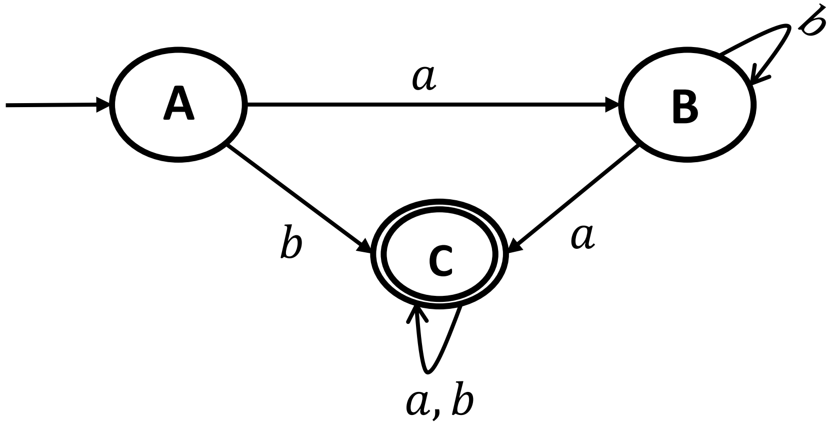




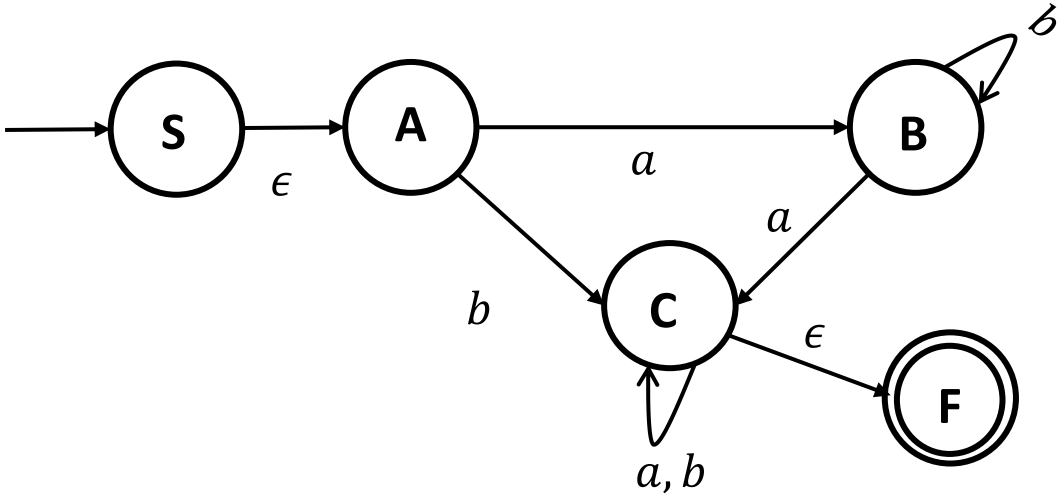
This should be done for **every pair** of arrows outgoing and incoming q_{rip}

DFA to Regular Expressions: GNFA

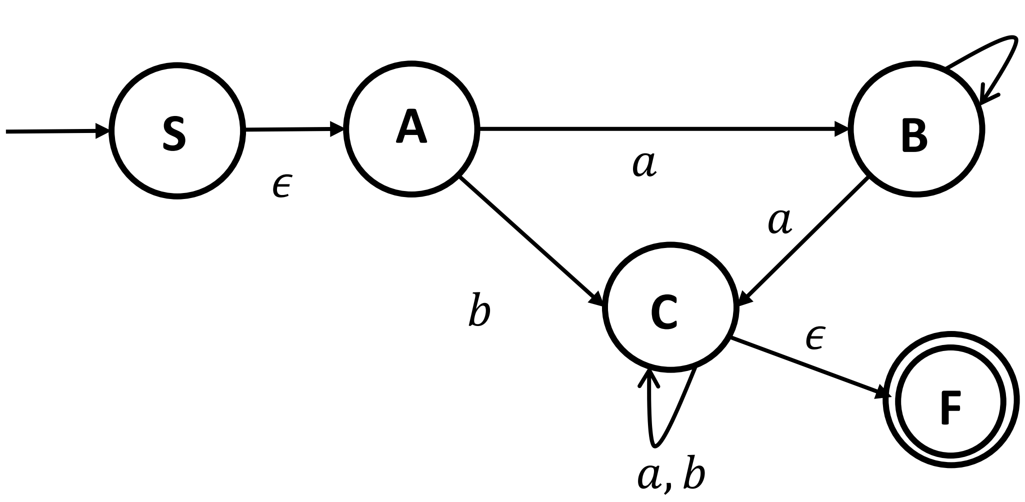
Let us look at an example. Consider the original DFA M below and find the regular expression corresponding to $L(M)$.



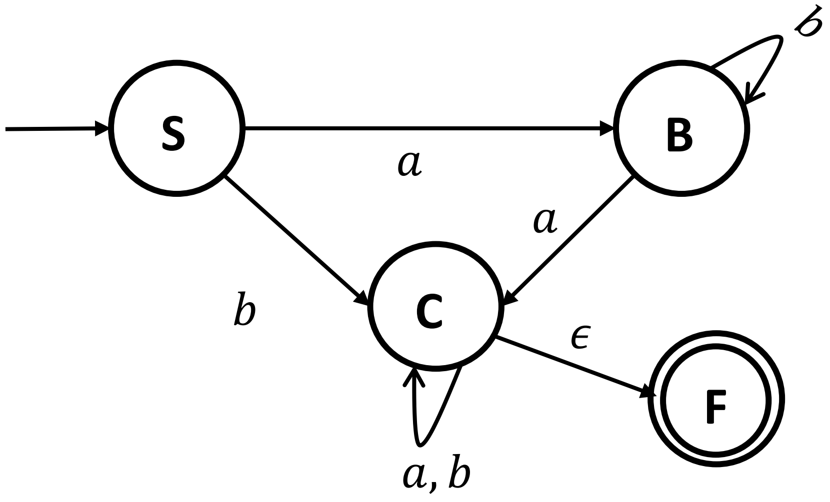
Step 1: Add new start and final states



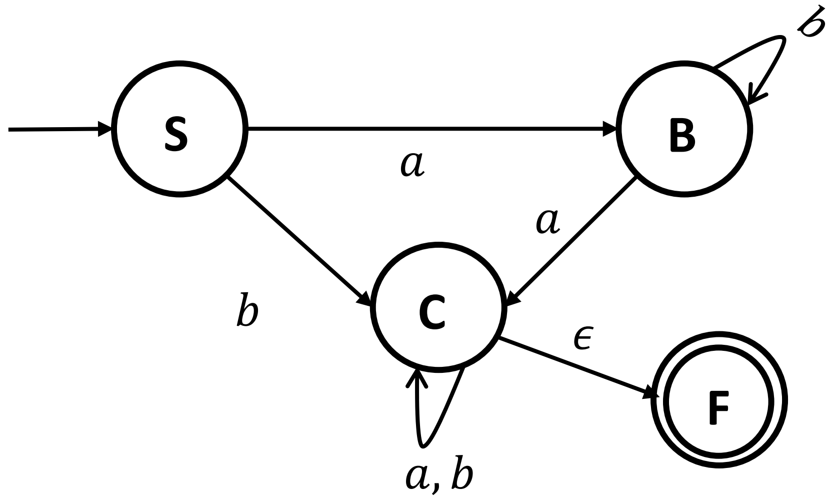
DFA to Regular Expressions: GNFA



Step 2: Eliminate A

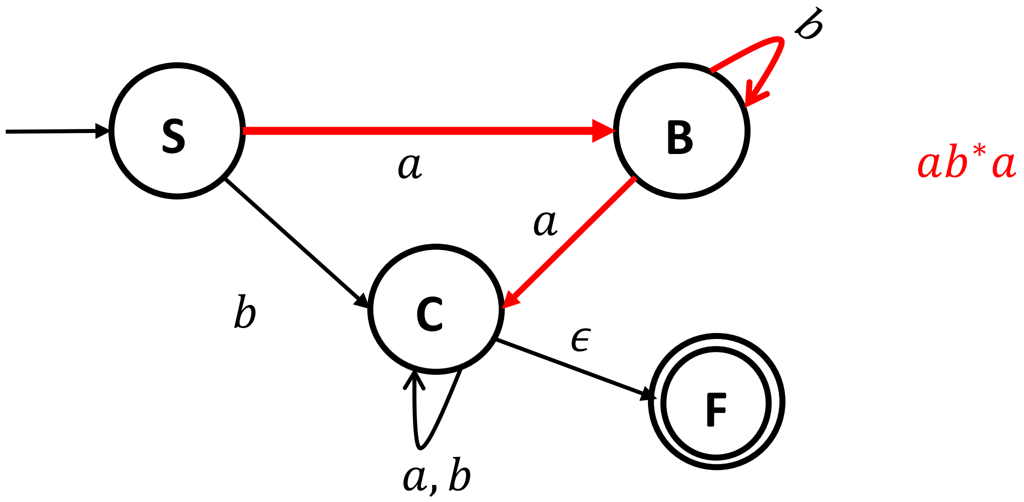


DFA to Regular Expressions: GNFA

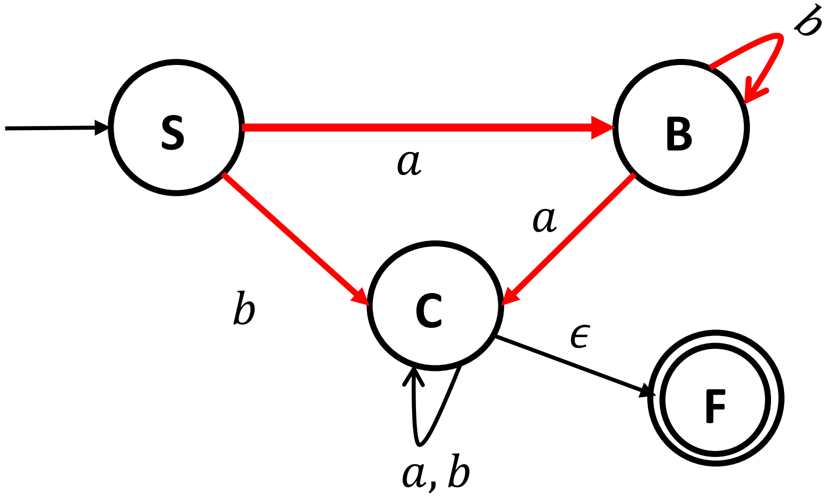


Step 2: Eliminate $\boldsymbol{B}$

$S \rightarrow C$ via B, RE: ab^*a

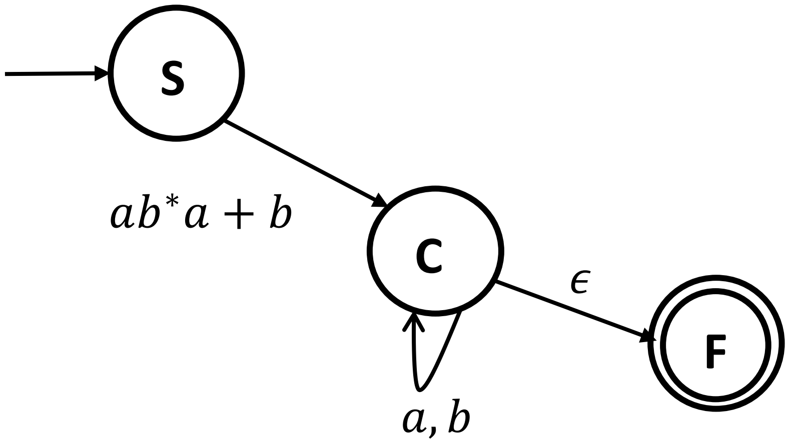


DFA to Regular Expressions: GNFA

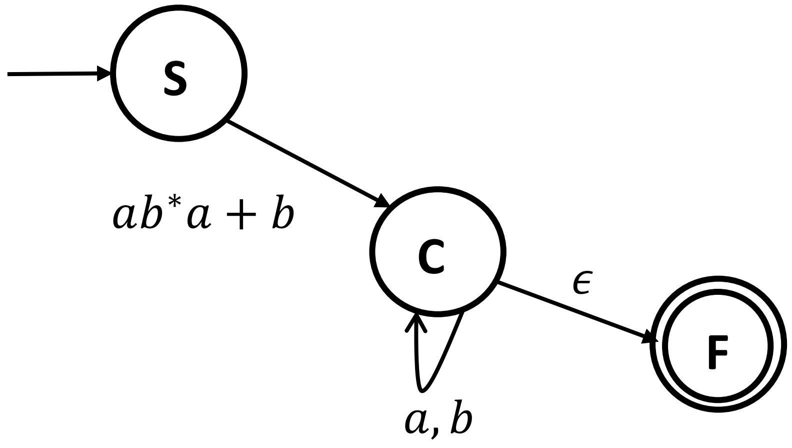


Step 2: Eliminate $\boldsymbol{B}$

$S \rightarrow C$ via B, RE: ab^*a
Overall RE for $S \rightarrow C$: $\boldsymbol{ab^*a + b}$

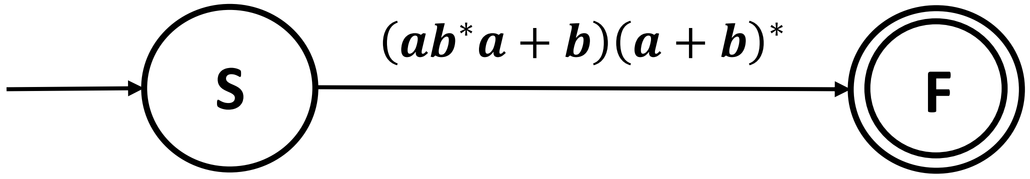


DFA to Regular Expressions: GNFA

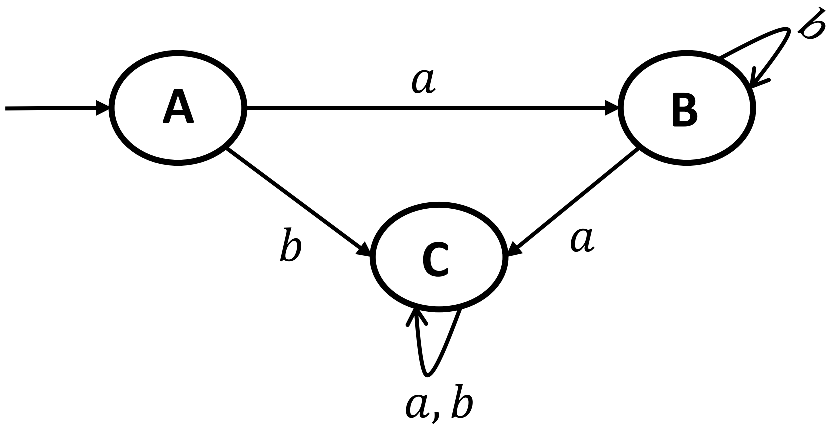


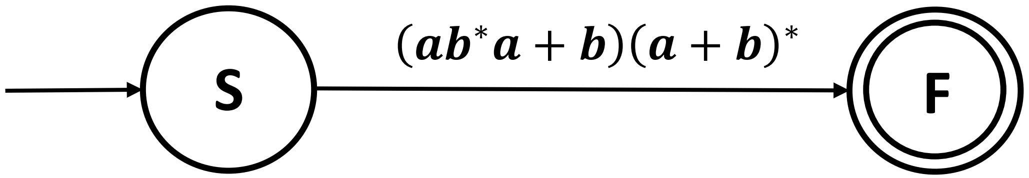
Step 2: Eliminate $\boldsymbol{C}$

$S \to F$ via C, RE: $(\boldsymbol{ab^*a} + \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b})^*$



DFA to Regular Expressions: GNFA





Recursively, we managed to convert the DFA M to a 2-state GNFA such that the label from of the arrow from the start state to the final state of the GNFA is the Regular Expression corresponding to $L(M)$.

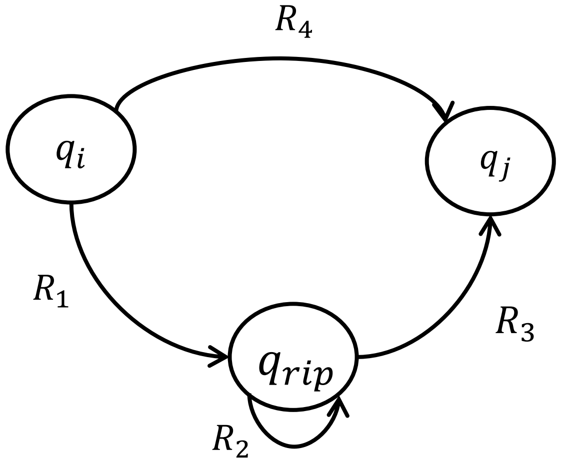
DFA to Regular Expressions: GNFA

Formally, a GNFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is the input alphabet.
- $\delta: Q - \{q_0\} \times Q - \{F\} \mapsto \mathcal{R}$ is the transition function.
- q_0 is the start state.
- F is the final state.

Convert k-state GNFA to a 2-state GNFA:

We provide a recursive algorithm CONVERT(G) for this.



CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then return the label R of the arrow between the start and the final state.
3. If $k > 2$, select any state Q different from q_0 and F and let G' be the GNFA$(Q', \Sigma, \delta', q_0, F)$, where

$$Q' = Q - \{q_{rip}\},$$

and for any $q_i \in Q' - \{q_0\}$ and any $q_j \in Q' - \{q_0\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) + R_4,$$

for $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$ and $R_4 = \delta(q_i, q_j)$

4. Compute CONVERT(G') and return its value.

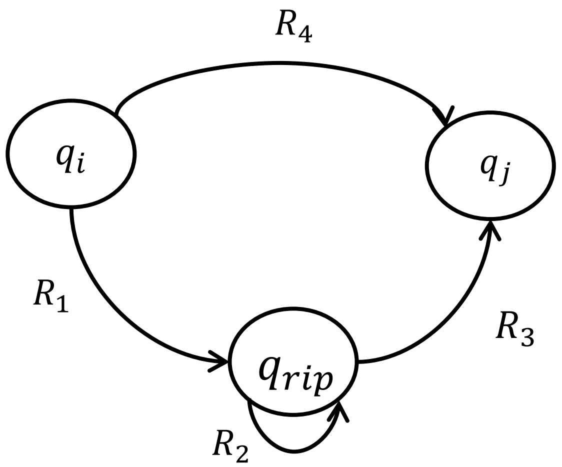
DFA to Regular Expressions: GNFA

Formally, a GNFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is the input alphabet.
- $\delta: Q - \{q_0\} \times Q - \{F\} \mapsto \mathcal{R}$ is the transition function.
- q_0 is the start state.
- F is the final state.

Convert k-state GNFA to a 2-state GNFA:

We provide a recursive algorithm CONVERT(G) for this.



DFA, NFA, Regular Expressions have equal power and all of them correspond to Regular Languages

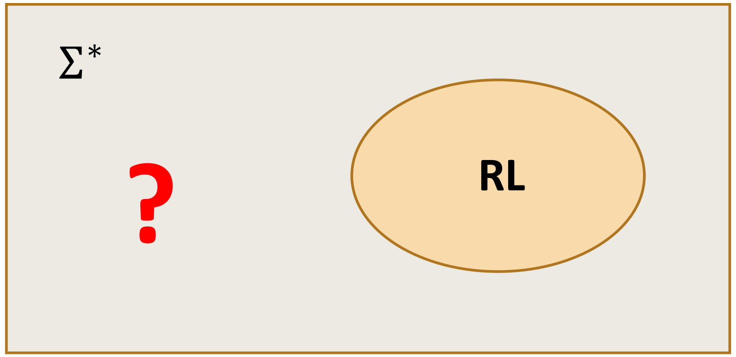
How do Non-regular languages look like?
How can we prove that certain languages are not regular?

Pumping Lemma

Recall that so far, we have proven that the following statements are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

- Not all languages are regular.

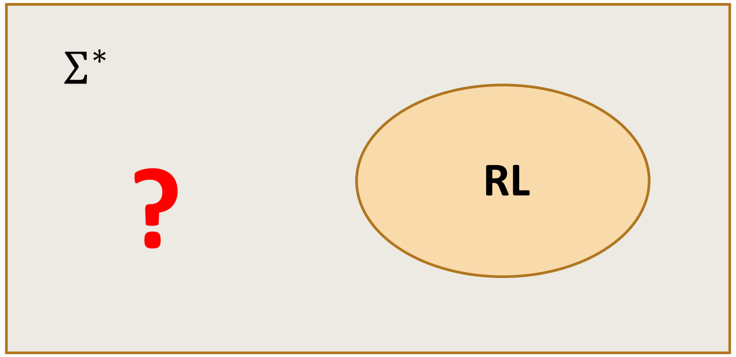


Pumping Lemma

Recall that so far, we have proven that the following statements are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

- Not all languages are regular.



Pumping Lemma

How do we prove that certain languages are non-regular? We start with an example

Let $\Sigma = \{0,1\}$. Consider the language $L = \{0^n 1^n | n \geq 0\}$ and the following conversation between Karl and Mil.

Pumping Lemma

How do we prove that certain languages are non-regular? We start with an example

Let $\Sigma = \{0,1\}$. Consider the language $L = \{0^n 1^n | n \geq 0\}$ and the following conversation between Karl and Mil.

Mil: I have a DFA for L.
Karl: How many states are there?
Mil: n-states (say $n = 10$)

Pumping Lemma

How do we prove that certain languages are non-regular? We start with an example

Let $\Sigma = \{0,1\}$. Consider the language $L = \{0^n 1^n | n \geq 0\}$ and the following conversation between Karl and Mil.

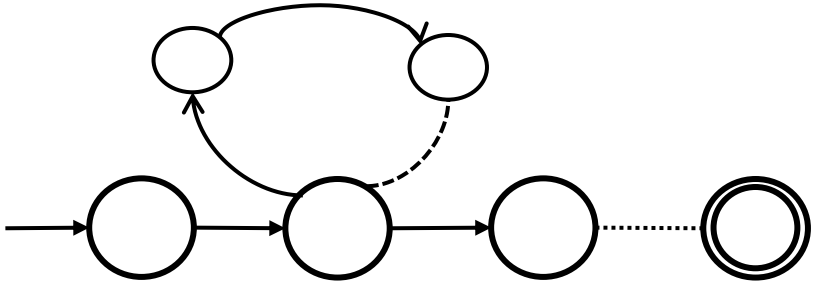
Mil: I have a DFA for L.
Karl: How many states are there?
Mil: n-states (say $n = 10$)
Karl: Then $0^{10}1^{10}$ must be accepted.
By the **pigeonhole principle**, while reading the first $(n = 10)$ symbols, some states need to be revisited. Otherwise $n + 1 = 11$ states would have been present. Hence some loop must be present. How many states are there in the loop?

Pumping Lemma

How do we prove that certain languages are non-regular? We start with an example

Let $\Sigma = \{0,1\}$. Consider the language $L = \{0^n 1^n | n \geq 0\}$ and the following conversation between Karl and Mil.

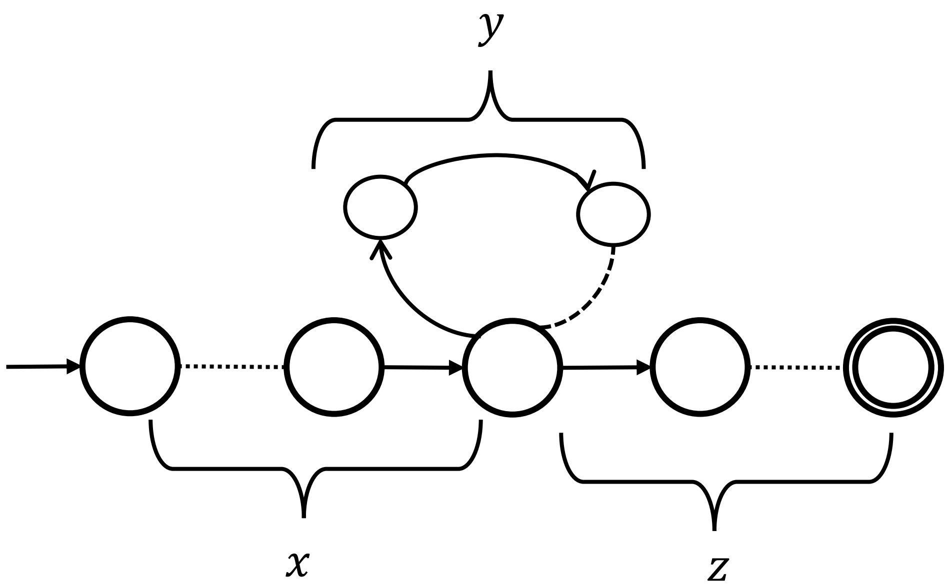
Mil: I have a DFA for L.
Karl: How many states are there?
Mil: n-states (say $n = 10$)
Karl: Then $0^{10}1^{10}$ must be accepted. By the **pigeonhole principle**, while reading the first $(n = 10)$ symbols, some states need to be revisited. Otherwise $n + 1 = 11$ states would have been present. Hence some loop must be present. How many states are there in the loop?
Mil: t-states (say $t = 3$).
Karl: If your DFA accepts $0^n 1^n$, it must also accept $0^{n+t}1^n$. This is because, if we take the loop one extra time, we read t more 0's.



Contradiction as $0^{n+t}1^n \notin L$. So Mil, you never had a DFA for L and in fact, ***L* is not regular.**

Pumping Lemma

If L is a regular language, all strings in the language, larger than a certain length (pumping length) , can be *pumped*: the string contains a certain section that can be repeated *any number of times* and the resulting string still $\in L$.

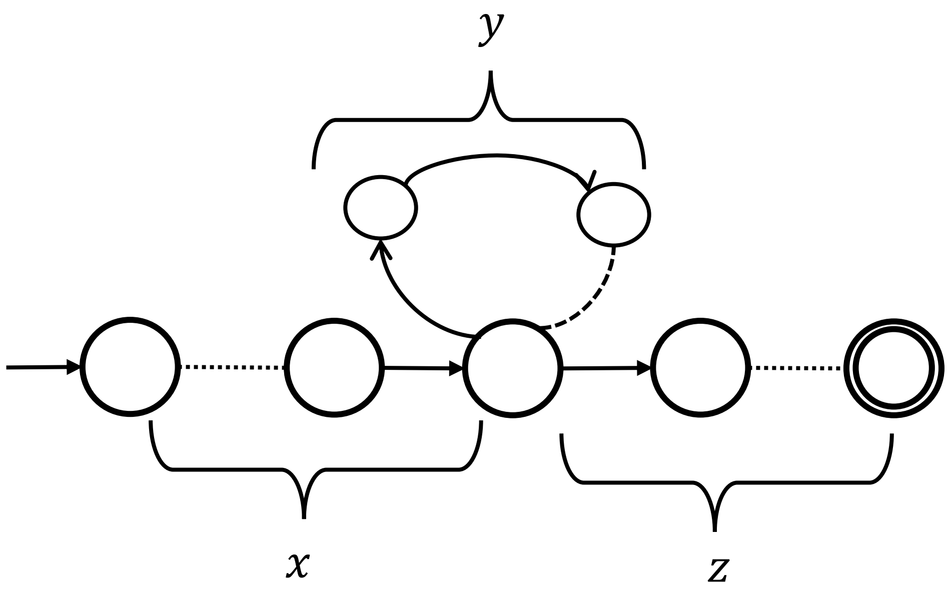


Pumping Lemma

If L is a regular language, all strings in the language, larger than a certain length (pumping length) , can be *pumped*: the string contains a certain section that can be repeated *any number of times* and the resulting string still $\in L$.

(Pumping Lemma) If L is a regular language, then there exists a number p (the pumping length) where for all $s \in L$ of length at least p, there exists x, y, z such that $s = xyz$, such that

1. $|xy| \leq p$.
2. $|y| \geq 1$
3. $\forall i \geq 0, xy^i z \in L$.



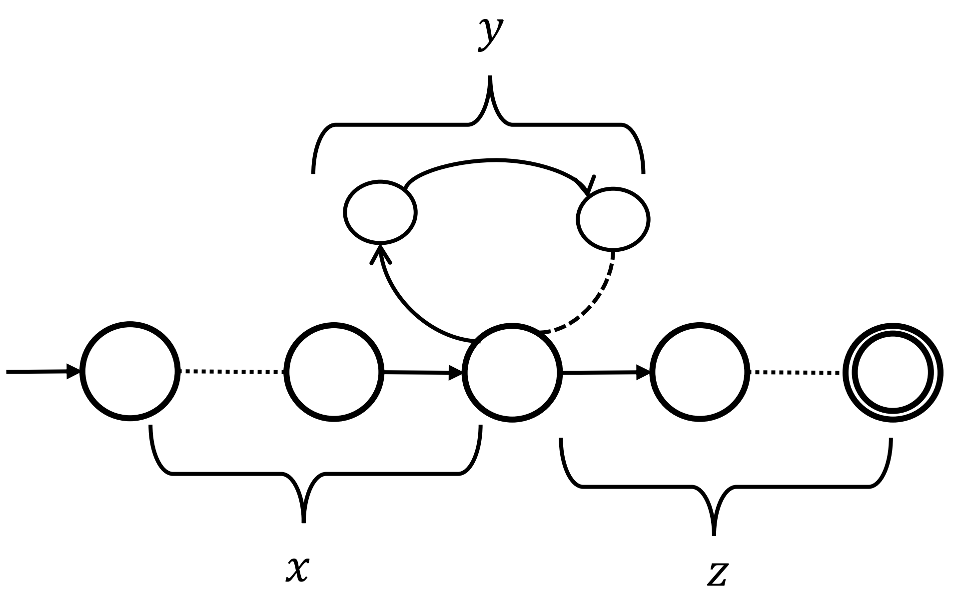
Pumping Lemma

If L is a regular language, all strings in the language, larger than a certain length (pumping length) , can be *pumped*: the string contains a certain section that can be repeated *any number of times* and the resulting string still $\in L$.

(Pumping Lemma) If L is a regular language, then there exists a number p (the pumping length) where for all $s \in L$ of length at least p, there exists x, y, z such that $s = xyz$, such that

1. $|xy| \leq p$.
2. $|y| \geq 1$
3. $\forall i \geq 0, xy^iz \in L$.

Note: $(A \Rightarrow B) \equiv (\neg B) \Rightarrow (\neg A)$



If L is regular then, pumping property is satisfied

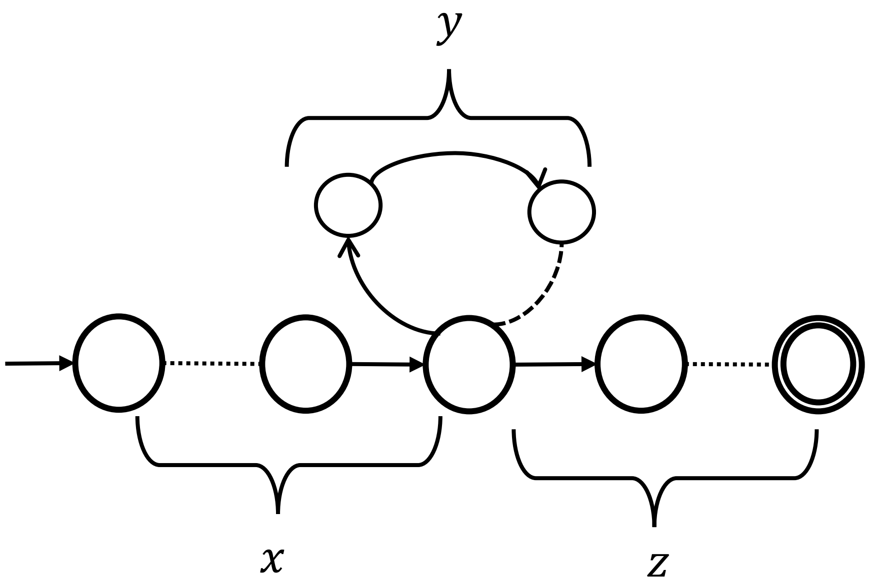
$\equiv$

If pumping property is NOT satisfied, then L is NOT regular.

Pumping Lemma

Proof sketch: Suppose that we have a DFA M of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

This is because of the ***pigeonhole principle***: any such run would encounter $p + 1$ states, but there are p distinct states in the DFA.



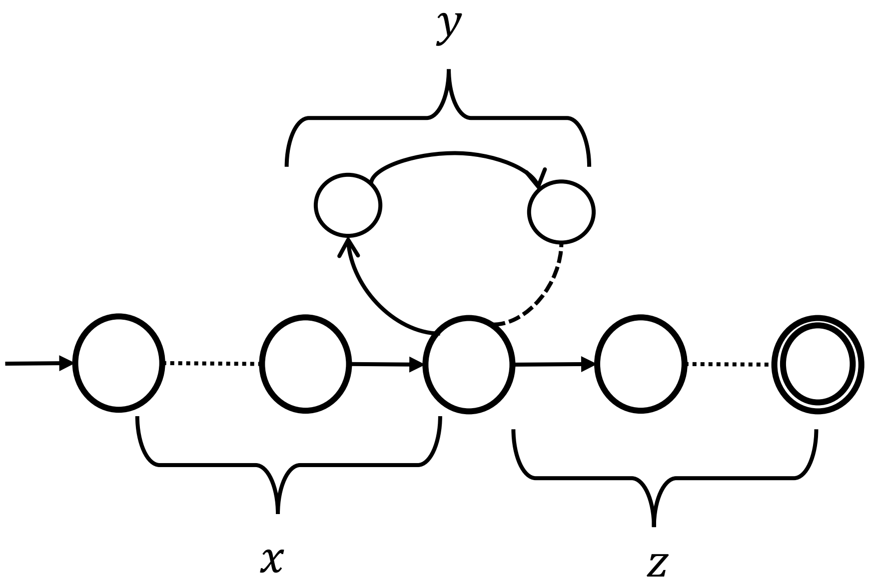
Pumping Lemma

Proof sketch: Suppose that we have a DFA M of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

This is because of the ***pigeonhole principle***: any such run would encounter $p+1$ states, but there are p distinct states in the DFA.

Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length $n\ (\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n + 1 \geq p + 1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.



Pumping Lemma

Proof sketch: Suppose that we have a DFA $M$of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

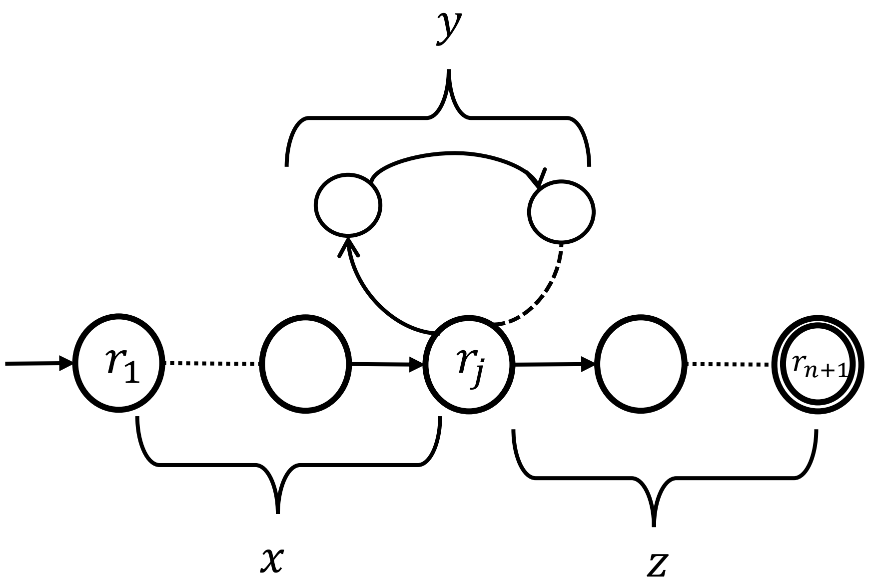
This is because of the ***pigeonhole principle***: any such run would encounter $p+1$ states, but there are p distinct states in the DFA.

Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length $n\ (\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n+1 \geq p+1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.

So we can divide the s into three parts, $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots . s_{l-1}$, $z = s_l \ldots s_n$. For a run on M, due to s

- the x part takes us from r_1 to r_j
- the y part belongs to the loop part (we go from r_j to r_j)
- z takes us from r_j to r_{n+1}, which is a final state if $s \in L$.



Pumping Lemma

Proof sketch: Suppose that we have a DFA $M$of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

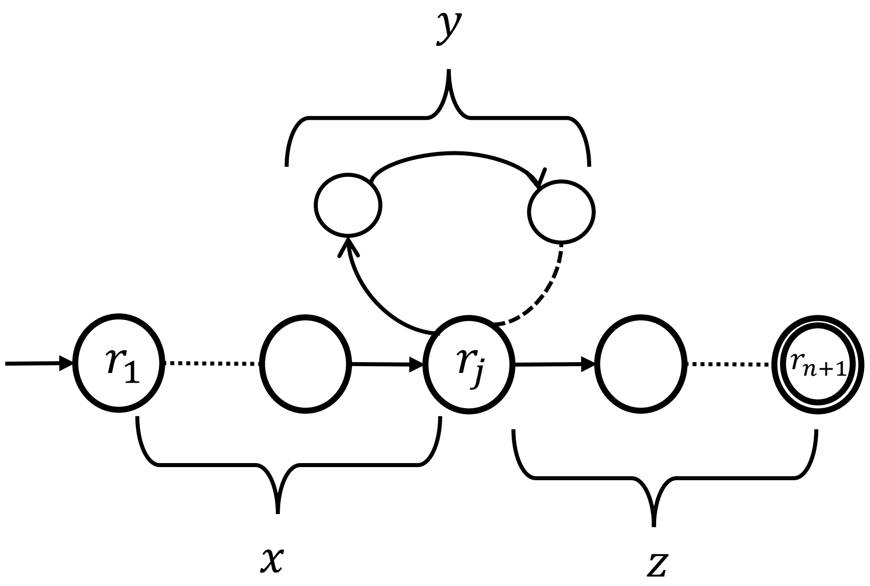
This is because of the ***pigeonhole principle***: any such run would encounter $p + 1$ states, but there are p distinct states in the DFA.

Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length $n\ (\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n + 1 \geq p + 1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.

So we can divide the s into three parts, $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots . s_{l-1}$, $z = s_l \ldots s_n$. For a run on M, due to s

- the x part takes us from r_1 to r_j
- the y part belongs to the loop part (we go from r_j to r_j)
- z takes us from r_j to r_{n+1}, which is a final state if $s \in L$.



- We can traverse the loop bit any number of times and so $\forall i \geq 0, xy^i z \in L$.

Pumping Lemma

Proof sketch: Suppose that we have a DFA $M$of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

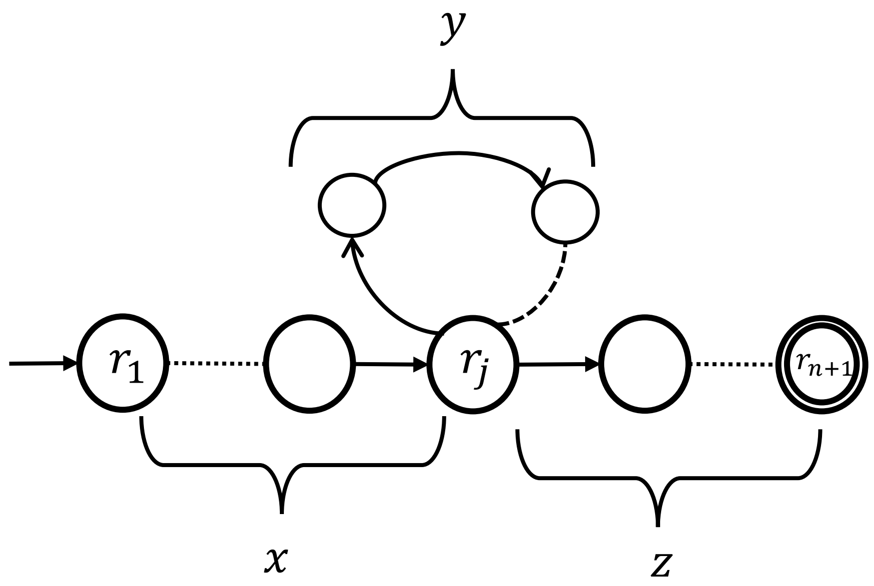
This is because of the ***pigeonhole principle***: any such run would encounter $p+1$ states, but there are p distinct states in the DFA.

Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length $n\ (\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n+1 \geq p+1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.

So we can divide the s into three parts, $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots . s_{l-1}$, $z = s_l \ldots s_n$. For a run on M, due to s

- the x part takes us from r_1 to r_j
- the y part belongs to the loop part (we go from r_j to r_j)
- z takes us from r_j to r_{n+1}, which is a final state if $s \in L$.



- We can traverse the loop bit any number of times and so $\forall i \geq 0, xy^i z \in L$.
- Also, as $j \neq l$, $|\mathrm{y}| \geq 1$
- While reading the input, within the first p symbols of s, some state must be repeated.

Pumping Lemma

Proof sketch: Suppose that we have a DFA $M$of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

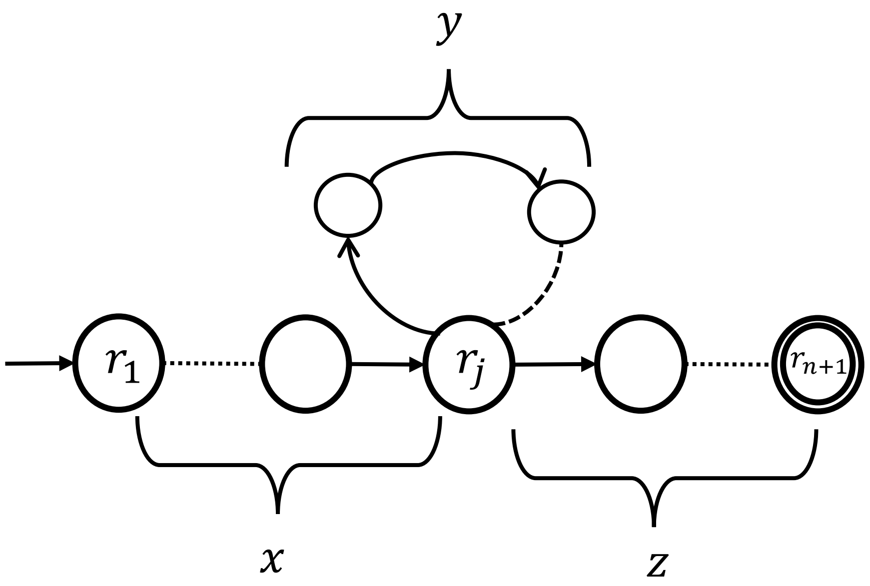
This is because of the ***pigeonhole principle***: any such run would encounter $p + 1$ states, but there are p distinct states in the DFA.

Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length $n \ (\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n + 1 \geq p + 1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.

So we can divide the s into three parts, $x = s_1 \dots s_{j-1}$, $y = s_j \dots . s_{l-1}$, $z = s_l \dots s_n$. For a run on M, due to s

- the x part takes us from r_1 to r_j
- the y part belongs to the loop part (we go from r_j to r_j)
- z takes us from r_j to r_{n+1}, which is a final state if $s \in L$.



- We can traverse the loop bit any number of times and so $\forall i \geq 0, xy^i z \in L$.
- Also, as $j \neq l$, $|y| \geq 1$, and
- The DFA reads $|xy|$ by then and so $|xy| \leq p$.

Pumping Lemma

In order to prove that a language is non-regular,

- Assume that it is regular and obtain a contradiction.

- Find a string in the language of length $\geq p$ (pumping length) that cannot be pumped.

Examples of languages that are NOT regular:

- $\{0^p | \ p \text{ is prime}\}$
- $\{0^n 1^n | n \geq 0\}$
- $\{\omega | \omega$ has equal number of 0's and $1's\}$
- $\{\omega | \omega$ is palindrome$\}$

$\vdots$

$\vdots$

Refer to Sipser (or some other textbook) for proofs using Pumping lemma

The story so far…

- We have built devices (DFAs/NFAs) that *recognize* whether a string belongs to a language
- Regular languages are precisely the ones that are accepted by finite automata.
- For any $L \in RL$, we have DFA/NFA M such that $L(M) = L$.
- Regular expressions describe regular languages algebraically.
- There are languages that are not regular.

DFA $\equiv$ NFA $\equiv$ Regular Expressions

Next up:

- How do we generate the strings in a language?
- **Syntax:** What are the set of legal strings in a language?
- Think of the English language (Rules of **grammar**)

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

- In fact, these concepts have been fundamental in attempts to formalize natural languages.

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

- Consider these rules

$$
\begin{aligned}
&Sentence \rightarrow Subject\ Verb\ Object \\
&Subject \rightarrow Noun.phrase \\
&Object \rightarrow Noun.phrase \\
&Noun.phrase \rightarrow Article\ Noun | Noun \\
&Article \rightarrow \boldsymbol{the} \\
&Noun \rightarrow \boldsymbol{boy} | \boldsymbol{girl} | \boldsymbol{soccer} | \boldsymbol{poetry} \\
&Verb \rightarrow \boldsymbol{loves} | \boldsymbol{plays}
\end{aligned}
$$

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

- Consider these rules

$$\begin{aligned}
&Sentence \rightarrow Subject\ Verb\ Object \\
&Subject \rightarrow Noun.phrase \\
&Object \rightarrow Noun.phrase \\
&Noun.phrase \rightarrow Article\ Noun|Noun \\
&Article \rightarrow \boldsymbol{the} \\
&Noun \rightarrow \boldsymbol{boy}|\boldsymbol{girl}|\boldsymbol{soccer}|\boldsymbol{poetry} \\
&Verb \rightarrow \boldsymbol{loves}|\boldsymbol{plays}
\end{aligned}$$

Terminals consist of strings over the alphabet corresponding to the language that the Grammar generates (Σ^*)

Variables: $\{Sentence, Subject, Verb, Object, Noun, Noun.phrase, Article\}$, **Terminals:** $\{the, girl, loves, plays, soccer, poetry\}$
Start Variable: $Sentence$

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

- Consider these rules

$$
\begin{aligned}
&Sentence \rightarrow Subject\ Verb\ Object \\
&Subject \rightarrow Noun.phrase \\
&Object \rightarrow Noun.phrase \\
&Noun.phrase \rightarrow Article\ Noun | Noun \\
&Article \rightarrow \boldsymbol{the} \\
&Noun \rightarrow \boldsymbol{boy} | \boldsymbol{girl} | \boldsymbol{soccer} | \boldsymbol{poetry} \\
&Verb \rightarrow \boldsymbol{loves} | \boldsymbol{plays}
\end{aligned}
$$

The sentence “**the girl plays soccer**” can be derived from this set of rules.

Variables: $\{Sentence, Subject, Verb, Object, Noun, Noun.phrase, Article\}$, **Terminals:** $\{the, girl, loves, plays, soccer, poetry\}$
Start Variable: $Sentence$

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

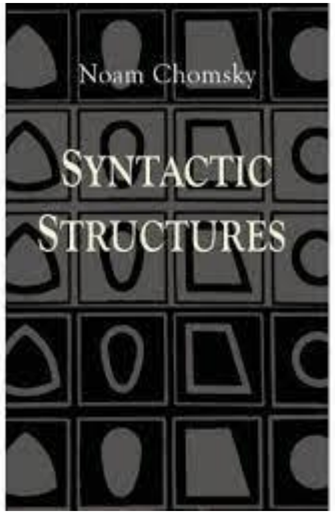
- Consider these rules

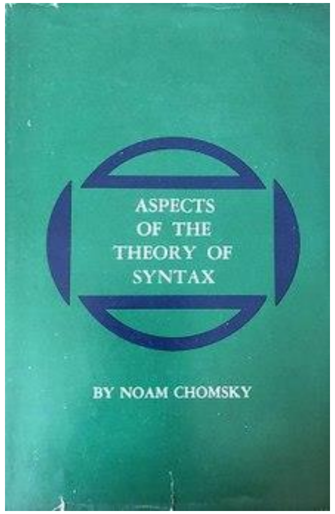
$$
\begin{aligned}
&Sentence \rightarrow Subject\ Verb\ Object \\
&Subject \rightarrow Noun.phrase \\
&Object \rightarrow Noun.phrase \\
&Noun.phrase \rightarrow Article\ Noun|Noun \\
&Article \rightarrow \boldsymbol{the} \\
&Noun \rightarrow \boldsymbol{boy|girl|soccer|poetry} \\
&Verb \rightarrow \boldsymbol{loves|plays}
\end{aligned}
$$

$$
\begin{aligned}
Sentence &\rightarrow Subject\ Verb\ Object \\
&\rightarrow Noun.phrase\ Verb\ Object \\
&\rightarrow Article\ Noun\ Verb\ Object \\
&\rightarrow \boldsymbol{the}\ Noun\ Verb\ Object \\
&\rightarrow \boldsymbol{the\ girl}\ Verb\ Object \\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Object \\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Noun.phrase \\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Noun \\
&\rightarrow \boldsymbol{the\ girl\ plays\ soccer}
\end{aligned}
$$

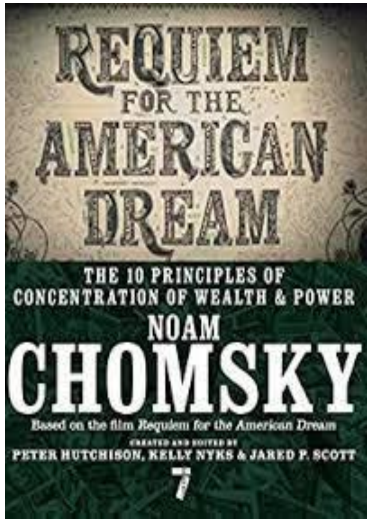
Variables: $\{Sentence, Subject, Verb, Object, Noun, Noun.phrase, Article\}$, **Terminals:** $\{The, girl, loves, plays, soccer, poetry\}$
Start Variable: $Sentence$

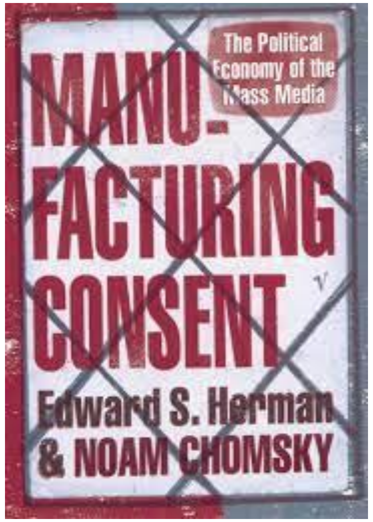
Grammars





Noam Chomsky





- Noam Chomsky did pioneering work on linguistics and formalized many of these concepts.

- Also made great contributions to political economy and has been a champion of anti-imperialist, anti-capitalist, social justice struggles across the globe.

Grammars

(Grammar) Formally, a *Grammar* G is a 5-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals** (disjoint from V)
- P is the set of production **Rules** [$(V \cup \Sigma)^* V (V \cup \Sigma)^* \rightarrow (V \cup \Sigma)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 0X$
$Y \rightarrow 1Y$
$Y \rightarrow \epsilon$

X is the start variable of the Grammar. Variables: $\{X, Y\}$, Terminals: $\{\epsilon$, 0, 1$\}$

Grammars

(Grammar) Formally, a *Grammar* G is a 5-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals** (disjoint from V)
- P is the set of production **Rules** [$(V \cup \Sigma)^* V (V \cup \Sigma)^* \to (V \cup \Sigma)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

Grammars can be used to derive strings.

The sequence of **substitutions** (using the rules of G) required to obtain a certain string is called a **derivation**.

- Begin the **derivation** from the **Start variable.**
- Replace any variable according to a rule. Repeat until only terminals remain.
- The generated string is **derived by the grammar.**

Eg: Consider the grammar G

$X \to 1X$
$X \to 0Y$
$Y \to 1Y$
$Y \to 0X$
$Y \to \epsilon$

X: Start Variable
$\{X, Y\}$: Variables
$\{\epsilon, 0, 1\}$: Terminals

The following is a derivation

$X \to 1X \to 11X \to 110Y \to 1101Y \to \mathbf{1101}$

Grammars

(Grammar) Formally, a *Grammar* G is a 5-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals**
- P is the set of production **Rules** [$(V \cup T)^* V (V \cup T)^* \rightarrow (V \cup T)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$**. The fact that w can be derived using the rules of G, is expressed as $\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w}$.

- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$

Grammars

(Grammar) Formally, a *Grammar* G is a 5-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals**
- P is the set of production **Rules** [$(V \cup T)^* V (V \cup T)^* \rightarrow (V \cup T)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$**. The fact that w can be derived using the rules of G, is expressed as $\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w}$.

- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 1Y$
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$

The string $\boldsymbol{1101} \in \boldsymbol{L}(\boldsymbol{G})$ because there exists the following derivation

$$X \rightarrow 1X \rightarrow 11X \rightarrow 110Y \rightarrow 1101Y \rightarrow 1101$$

Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 1Y$
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$ (indicates that Y is the final state)

Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

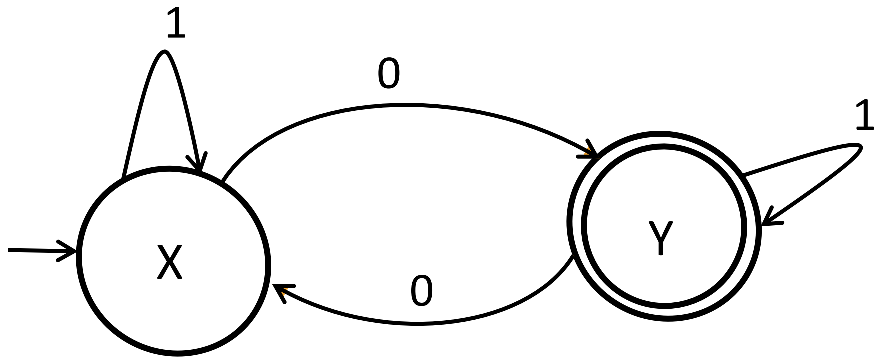
$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 1Y$
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$ (indicates that Y is the final state)



Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

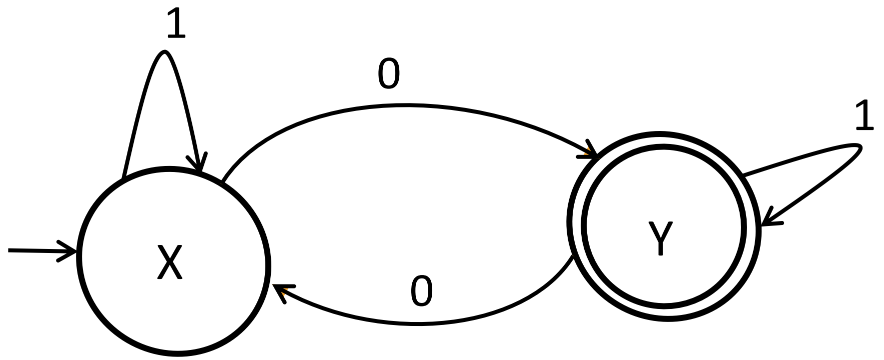
$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 1Y$
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$ (indicates that Y is the final state)



For the string $\mathbf{1101}$:

A **run** in a DFA model is analogous to a **derivation** in a linear grammar.

Derivation: $X \rightarrow 1X \rightarrow 11X \rightarrow 110Y \rightarrow 1101Y \rightarrow 1101$. So $1101 \in L(G)$

Run: $X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{0} Y \xrightarrow{1} Y$ (Accepting Run and so $1101 \in L(M)$).

Grammars for Regular Languages

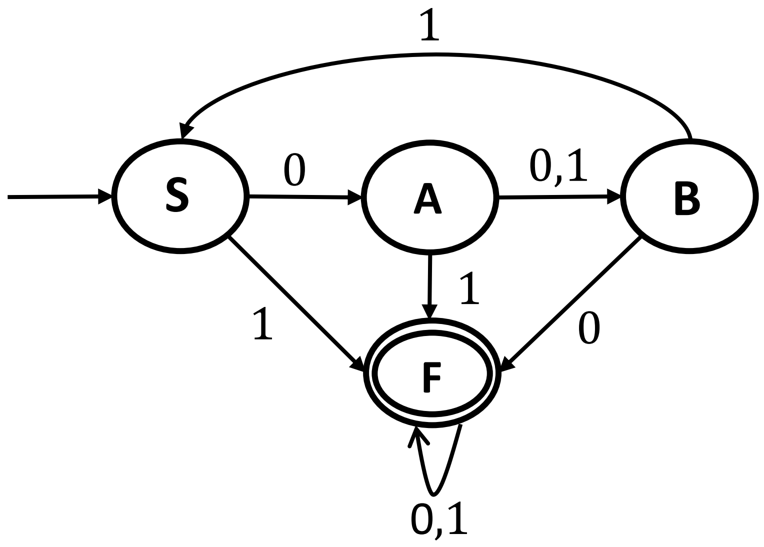
Regular grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \epsilon$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

DFA to Right linear Grammar

Consider the following DFA M



The right-linear grammar G for M

$S \rightarrow 0A$
$A \rightarrow 01B$
$B \rightarrow 1S$
$F \rightarrow 01F$
$A \rightarrow 1F$
$B \rightarrow 0F$
$S \rightarrow 1F$
$F \rightarrow \epsilon$

Grammars for Regular Languages

Right-linear grammar ≡ DFA ≡ NFA ≡ Regular Expressions

Left linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow \boldsymbol{Var}\, Ter$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then such a grammar is called **Left-linear** (all Variables are to the left of terminals in the RHS).

Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here – See Assignment 1)

Grammars for Regular Languages

Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Left linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow \boldsymbol{Var}\, Ter$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then such a grammar is called **Left-linear** (all Variables are to the left of terminals in the RHS).

Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here)

Right-linear grammars and **Left-linear grammars** generate **Regular Languages.**

Note that mixing left-linear grammars and right-linear grammars in the same set of rules **won't generate regular languages**.

Left-linear grammar $\equiv$ Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Thank You!