Stoch. processes – Take-home

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1. $$\frac{dX_1}{dt} = -K_a S X_1 + K_d X_2 \quad \text{— (1)}$$

$$\frac{dX_2}{dt} = K_a S X_1 - K_d X_2 \quad \text{— (2)}$$

$$\frac{dY}{dt} = \alpha_0 X_1 + \alpha_1 X_2 - \beta_0 Y - \beta_1 Y \quad \text{— (3)}$$

At steady state,

(1):

$$0 = -K_a S X_1 + K_d X_T - K_d X_1$$

$$X_1 = \frac{K_d X_T}{K_a S + K_d}$$

(2): $$0 = K_a S X_1 - K_d X_2$$

$$X_2 = \frac{K_d X_T}{K_a S + K_d}$$

(3): $$0 = \alpha_0 X_1 + \alpha_1 X_2 - \beta_0 Y - \beta_1 Y$$

$$Y = \frac{\alpha_0 X_1 + \alpha_1 X_2}{\beta_0 + \beta_1}$$

$$= \frac{K_d X_T (\alpha_0 + \alpha_1)}{(\beta_0 + \beta_1)(K_a S + K_d)}$$

2. $$\frac{d\Delta X_2}{dt} = -(K_a S + K_d)\, \Delta X_2 + n_1(t)$$

$$\langle X_2 \rangle = \frac{K_a S}{K_a S + K_d} X_T$$

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$$\langle \eta_1(t)\, \eta_1(t')\rangle = \cancel{D_1}\, \delta(t-t') \times \underbrace{\frac{2 K_a S K_d}{K_a S + K_d}}_{D_1} \frac{X_T}{\Omega}$$

on Fourier Transforming,

$$\Delta X_2(\omega) = \frac{\eta_1(\omega)}{-i\omega + (K_a S + K_d)}$$

$$S_{x_2}(\omega) = S_{x_1}(\omega)$$

$$= \frac{1}{\omega^2 + (K_a S + K_d)^2} \times \frac{2 K_a S K_d}{K_a S + K_d} \frac{X_T}{\Omega}$$

$$\frac{d\Delta Y}{dt} = \alpha_0 \Delta X_1 + \alpha_1 \Delta X_2 - \beta_0 \Delta Y - \beta_1 \Delta Y + \eta_2(t)$$

$$\langle Y\rangle = \frac{K_d X_T(\alpha_0 + \alpha_1)}{(\beta_0 + \beta_1)(K_a S + K_d)}$$

$$\langle \eta_2(0)\, \eta_2(t)\rangle = \frac{D_2}{\Omega}\, \delta(t-t')$$

where $D_2 = X_T 2\dfrac{(\alpha_1 K_a S + \alpha_0 K_d)}{K_a S + K_d}$

$$\Delta Y(\omega) = \frac{\alpha_0 \Delta X_1(\omega) + \alpha_1 \Delta X_2(\omega) + \eta_2(\omega)}{-i\omega + \beta_0 + \beta_1}$$

$$S_{\Delta Y}(\omega) = S_{\Delta X_1}(\omega) \frac{\alpha_0^2 + \alpha_1^2}{\omega^2 + (\beta_0 + \beta_1)^2} + \frac{S_{\eta_2}(\omega)}{\omega^2 + (\beta_0 + \beta_1)^2}$$

$$= \frac{D_1(\alpha_0^2 + \alpha_1^2)}{[\omega^2 + (\beta_0 + \beta_1)^2][\omega^2 + (K_a S + K_d)^2]} + \frac{D_2}{2\pi\Omega[\omega^2 + (\beta_0 + \beta_1)^2]}$$

3. $$\sigma^2 = D_1(\alpha_0^2+\alpha_1^2)\int_{-\infty}^{+\infty}\frac{1\,d\omega}{\left[(\beta_0+\beta_1)^2+\omega^2\right]\left[(K_d+K_aS)^2+\omega^2\right]} + \frac{D_2}{2\pi\Omega}\int_{-\infty}^{+\infty}\frac{1}{\omega^2+(\beta_0+\beta_1)^2}\,d\omega$$

$$= \frac{D_1(\alpha_0^2+\alpha_1^2)\pi}{(\beta_0+\beta_1)(K_d+K_aS)(\beta_0+\beta_1+K_d+K_aS)} + \frac{D_2}{2\pi\Omega}\times\frac{1}{(\beta_0+\beta_1)}$$

4. $$F = \left(\frac{\partial}{\partial S}\langle Y\rangle\right)^2\frac{1}{\sigma^2}$$

$$= \frac{\left(\frac{K_a(\alpha_1 K_aS+\alpha_0 K_d)X_T}{(K_aS+K_d)^2(\beta_0+\beta_1)} + \frac{X_T(K_aS+K_d)(\alpha_1 K_d)}{(K_aS+K_d)^2(\beta_0+\beta_1)}\right)^2}{\sigma^2}$$

5. $$\cancel{\frac{d\,\Delta S_T}{dt} + \frac{d\,\Delta S_0}{dt} \neq 0}$$

$$\frac{dS}{dt} = (\alpha_0\langle X_1\rangle - \beta_0\langle Y\rangle)\log\frac{\alpha_0}{\beta_0} + (\alpha_1\langle X_2\rangle - \beta_1\langle Y\rangle)\log\frac{\alpha_1}{\beta_1}$$

$\frac{dy}{dt}$ at steady state $= 0$

$$0 = \alpha_0 X_1 + \alpha_1 X_2 - \beta_0 Y + \beta_1 Y$$

$$\therefore \alpha_0 X_1 - \beta_0 Y = -(\alpha_1 X_2 - \beta_1 Y)$$

$$\Rightarrow \frac{dS}{dt} = (\alpha_0 \langle X_1 \rangle - \beta_0 \langle Y \rangle) \log \frac{\alpha_0}{\beta_0}$$

$$- (\alpha_0 \langle X_1 \rangle - \beta_0 \langle Y \rangle) \log \frac{\alpha_1}{\beta_1}$$

$$= (\alpha_0 \langle X_1 \rangle - \beta_0 \langle Y \rangle) \left(\log \frac{\alpha_0}{\beta_0} - \log \frac{\alpha_1}{\beta_1} \right)$$

$$= (\alpha_0 \langle X_1 \rangle - \beta_0 \langle Y \rangle) \log \left(\frac{\alpha_0 \beta_1}{\beta_0 \alpha_1} \right)$$

6. Python code to run the simulation in q.6 is attached within the Moodle submission.