Probability and Statistics

UG2, Core course, IIIT,H

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Outline

Conditional PMF and Conditional CDF...

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Example Motivation for Conditional PMF and CDF

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I roll a fair die.

Conditional PMF and Conditional CDF...

Example Motivation for Conditional PMF and CDF

I roll a fair die. Let X be the observed number.

Example Motivation for Conditional PMF and CDF

I roll a fair die. Let X be the observed number. Find the conditional PMF of X given that we know the observed number was less than 5.

Solution: $R_X = \{1, 2, 3, 4, 5, 6\}$.

$A = \{X < 5\}$

$$P_{X|A}(1) = P(X=1 \mid A) = \frac{P(X=1, A)}{P(A)} = \frac{P(X=1, X<5)}{P(A)}$$

$$= \frac{P(X=1)}{P(X<5)} = \frac{1/6}{4/6} = 1/4$$

$P_{X|A}(2) = P_{X|A}(3) = P_{X|A}(4) = 1/4$

$P_{X|A}(5) = P_{X|A}(6) = 0$

Conditional PMF and Conditional CDF...

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Definition of Conditional PMF and Conditional CDF

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$$\begin{aligned} P_{X|A}(x_i) &= P(X = x_i|A) \\ &= \frac{P(X = x_i \text{ and } A)}{P(A)}, \quad \text{for any } x_i \in R_X \end{aligned}$$

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The conditional CDF of X is given by

$$F_{X|A}(x) = P(X \leq x \mid A).$$

Conditional PMF of X given Y...

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$$P_{X|Y}(x_i, y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)} \leftarrow \text{Joint PMF}$$

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For discrete random variables X and Y, the conditional PMFs of X and Y is defined as follows

$$P_{X|Y}(x_i, y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}$$

$$P_{Y|X}(x_i, y_j) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}$$

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for any $x_i \in R_X$ and $y_j \in R_Y$.

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$$P_{X|Y}(x_i|y_j) = P(X = x_i \mid Y = y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)} = \frac{P_X(x_i)P_Y(y_j)}{P_Y(y_i)} = P_X(x_i)$$

Example of Joint and Marginal PDF and Conditional PMF...

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Example of Joint and Marginal PDF and Conditional PMF...

$x = \{0, 1, -1, 2, -2\}$
$y = \{0, 1, -1, 2, -2\}$

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$$G = \{(x, y) \mid x, y \in \mathbb{Z}, \quad |x| + |y| \leq 2\}.$$

If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

$G = \{(0,0), (0,1), (1,1), (1,-1), \cdots\}$

$|G| = 13$ (En

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If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

1. Find the joint and marginal PMFs of X and Y.
2. Find the conditional PMF of X given $Y = 1$.
3. Are X and Y independent?

① Joint PMF

$$P_{XY}(x, y) = \begin{cases} 1/13, & (x,y) \in G \\ 0, & \text{otherwise} \end{cases}$$

$$P_Y = \sum_{x_i} P_{XY}(x_i, y), \quad P_X = \sum_{y_i} P_{XY}(x, y_i)$$

Answer to previous problem...

$P_x(-2) = P_{XY}(-2,0) = 1/13$

$P_x(-1) = P_{XY}(-1,-1) + P_{XY}(-1,1) + P_{XY}(-1,0) = 3/13$

$\parallel$

$P_x(1)$ — — . — — .

$P_Y(\ )$ — — $P_{XY}(0,-1) + P_{XY}(0,1)$

$P_x(0) = P_{XY}(0,0), P_{XY}(0,-2) + P_{XY}(0,2) + P_{XY}(0,2) = 5/13$

$P_{X|Y}(1,1) = 1/13$

$\neq 1/13 = P_x(1) = 3/13$

② Given $Y=1$

Possible $x = -1, 0, 1$

$P_{X|Y}(i,1) = \dfrac{P_{XY}(i,1)}{P_Y(1)}$

$= \dfrac{1/13}{3/13}$ $[P_Y(1) = P_X(-1)]$

$= 1/3$

$\boxed{P_{X|Y}(i|1)} = \begin{cases} 1/3 & \text{for } i=-1,0,1 \\ 0 & \text{otherwise} \end{cases}$

Answer to previous problem...

(3) Are X, Y ind?

$$P_{X|Y}(i, 1) = P_X(i)$$

Not true

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Let A be any event. Let X and Y be two random variables with ranges R_X and R_Y respectively. Then the conditional expectations are defined as follows

$$E[X \mid A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$$E[X \mid Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i \mid y_j)$$

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Example of Conditional Expectation

When $Y=1$, Possible for
$x = -1, 0, 1$

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Consider the set of points in set G defined as follows

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If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

1. Find $E[X \mid Y = 1]$

we have from before

$$P_{X|Y} = \begin{cases} 1/3, & x = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow E[X|Y=1] = (-1) P_{X|Y}(-1) + 0\, P_{X|Y}(0) + 1 \cdot P_{X|Y}(1)$$

$$= -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$$

Example of Conditional Expectation

$P_{X|A}(-2) = \frac{13}{8} P(X=-2, A)$

If $-1 < Y < 2 \Rightarrow X = -2, -1, 0$ $= \frac{13}{8} P(-2, 0) = \frac{13}{8} \cdot \frac{1}{13} = \frac{1}{8}$

Example

Consider the set of points in set G defined as follows

$$G = \{(x, y) \mid x, y \in \mathbb{Z}, \quad |x| + |y| \leq 2\}.$$

If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

1. Find $E[X \mid Y = 1]$
2. Find $E[X \mid -1 < Y < 2]$ ← $P_{X|Y}$

$-1 < Y < 2 \Rightarrow Y = 0, 1$, $Y \in \{0, 1\} = A$

$P(-1 < Y < 2) = P(A) = P_Y(0) + P_Y(1) = \frac{5}{13} + \frac{3}{13} = 8/13$

$P_{X|A}(k) = \frac{P(X=k, A)}{P(A)} = \frac{13}{8} P(X=k, A)$

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If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

1. Find $E[X \mid Y = 1]$
2. Find $E[X \mid -1 < Y < 2]$
3. Find $E[|X| \mid -1 < Y < 2]$

Answer to previous problem... $R_X = \{-2, 0, 1, 2\}$

$$\boxed{P_{X|A}(-1)} = \frac{13}{8} P(X=-1, A)$$

$$= \frac{13}{8} P_{XY}(-1, 0) + \frac{13}{8} P_{XY}(-1, 1)$$

$$= \frac{\cancel{13}}{8} \cdot \frac{1}{\cancel{13}} + \frac{\cancel{13}}{8} \cdot \frac{1}{\cancel{13}}$$

$$= 1/4.$$

Similarly

$$P_{X|A}(0) = \frac{13}{8} P_{XY}(0,0) + \frac{13}{8} P_{XY}(0,1)$$

$$= 1/4.$$

$$P_{X|A}(2) = 1/8.$$

$$E[X|A]$$

$$= \sum_i x_i P_{X|A}(x_i)$$

$$= -2 \cdot \frac{1}{8} + (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8}$$

$$=$$

Answer to previous problem...

Law of Total Probability...

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$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A \mid Y = y_j) P_Y(y_j), \quad \text{for any set } A$$

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  1. If $B_1, B_2, \dots$ is a partition of sample space S

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$\sum x_i P(x_i)$

- Law of Total Expectation:
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$$\boxed{E[X]} = \sum_i E[X \mid B_i] P(B_i)$$

  2. For a RV X and a discrete RV Y

$$E[Y] = \sum_{y_j \in R_Y} E[X \mid Y = y_j] P_Y(y_j)$$

Solved Example

Solved Example

X

$\sum x_i p(x_i)$

Solved Example 1

Let $X \sim \text{Geometric}(p)$. Find $E[X]$. [Hint: condition on first coin toss.]

Geometric: number of times I toss to get 1st head.

X = total no. of tosses to get 1st H.

$p(H) = p, \quad p(T) = 1-p.$

Two outcomes: $\{H\}$ or $\{T\}$

Using the law of total expectation.

$$E[X] = E[X|H]\,p(H) + E[X|T]\,p(T)$$
$$= p\,E[X|H] + (1-p)\,E[X|T]$$
$$= p \cdot 1 + (1-p)(1 + E[X])$$

$E[X|T] = 1 + E[X]$

Solve for $E[X]$. $\rightarrow 1/p$ (dud)

Solved Example

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Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.

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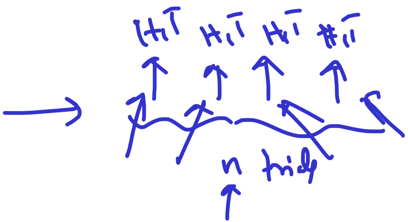
Solved Example

$X \mid N$

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?



→ together

Binomial.

Answer to previous problem...

Given $N=n$, X is a sum of n ind. Bernoulli(p), which means

$$X \mid N=n \sim \text{Binomial}(n,p)$$

$$P_{X|N}(k \mid n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$$E[X \mid N=n] = np.$$

We want: $E[X]$

= exp. no. of customers purchasing drinks.

L.O.T.E

$$= \sum_{n=0}^{\infty} E[X \mid N=n] P_N(n)$$

$$= \sum_{n=0}^{\infty} np \cdot P_N(n)$$

$$= p \sum_{n=0}^{\infty} n \, P_N(n)$$

PMF of Poisson.

$$= p \lambda$$

Exp. of Poisson

PMF and Expectation of Two Random Variables...

PMF and Expectation of Two Random Variables...

PMF and Expectations of Two Random Variables

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PMF and Expectation of Two Random Variables...

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PMF and Expectation of Two Random Variables...

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where $A_z = \{(x_i, y_j) \in R_{XY} : \underbrace{g(x_i, y_j) = z}\}$

more than one
for given z
if g is not 1-1

PMF and Expectation of Two Random Variables...

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$$P_Z(z) = P(g(X, Y) = z) = \sum_{(x_i, y_j) \in A_z} P_{XY}(x_i, y_j),$$

$g(x)P(x)$

where $A_z = \{(x_i, y_j) \in R_{XY} : g(x_i, y_j) = z\}$

- The expectation is given as follows

$$E[g(X, Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j)\, P_{XY}(x_i, y_j)$$

Linearity of Expectation for Two Random Variable...

Linearity of Expectation for Two Random Variable...

Linearity of Expectation for Two RV

Let X, Y be two discrete RVs. Then $E[X + Y] = E[X] + E[Y]$.

PMF of Difference of Two Geometric Distributions...

PMF of Difference

Let $X, Y \sim Geometric(p)$ be two random variables. Let $Z = X - Y$. Find the PMF of Z.

Independent.

$R_X, R_Y = \{1, 2, 3, \ldots\} = \mathbb{N}$

$\Rightarrow R_Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \mathbb{Z}$.

Since X, Y Geometric

$P_X(k) = P_Y(k) = p\,q^{k-1}, \quad k = 1, 2, 3, \ldots$

$P_Z(k) = P(Z = k) = P(X - Y = k) = P(X = Y + k)$

L.O.T.P

$= \sum_{j=1}^{\infty} P(X = Y + k \mid Y = j)\, P(Y = j) = \sum_{j=1}^{\infty} P(X = j + k \mid Y = j)\, P(Y = j)$

X, Y ind

$= \sum_{j=1}^{\infty} P(X = j + k)\, P(Y = j)$

Answer to previous problem...

$$P_Z(k) = \sum_{j=1}^{\infty} \underline{p q^{j+k-1}} \cdot \underline{p q^{j-1}}$$

$\underline{k \geq 0}$

$\underline{\text{For } k < 0}$

$$P_Z(k) = \sum_{j=-k+1}^{\infty} p q^{j+k-1} \cdot p q^{j-1}$$

Answer to previous problem...

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- Here $E[X \mid Y = y]$ is a function of y. That is $g(y) = E[X \mid Y = y]$
- We can think of $g(Y) = E[X \mid Y]$ as a random variable

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$$E[X \mid Y = y] = \sum_{x_i \in R_X} x_i \, P_{X|Y}(x_i \mid y)$$

- Here $E[X \mid Y = y]$ is a function of y. That is $g(y) = E[X \mid Y = y]$
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- Since $E[X \mid Y]$ is a RV, we can find its PMF, CDF, Variance, etc

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5. Find $\text{Var}(Z)$