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Probability and Statistics Assignment - 5

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→ Problem 1

a) $\cancel{f_{XY}(x,y)} =$

$\cancel{f_X(x)} =$

$$f_{XY}(2,0) = \frac{3}{12} \times \frac{2}{11}$$

$$f_{XY}(1,1) = \frac{3}{12} \times \frac{4}{11} + \frac{4}{12} \times \frac{3}{11}$$

$$f_{XY}(0,2) = \frac{4}{12} \times \frac{3}{11}$$

$$f_{XY}(1,0) = \frac{3}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{3}{11}$$

$$f_{XY}(0,1) = \frac{4}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{4}{11}$$

$$f_{XY}(0,0) = \frac{5}{12} \times \frac{4}{11}$$

~~If these~~ and if $x+y > 2$, $f_{XY}(x,y) = 0$

b) $$E[X] = 2 \times \frac{3}{12} \times \frac{2}{11} + 1 \times \left(\frac{3}{12} \times \frac{4}{11} \times 2\right) = \frac{1}{2}$$

→ Problem 2

a) $\Gamma(7/2) = 5/2 \times \Gamma(5/2)$

$= 5/2 \times 3/2 \times 1/2 \times \Gamma(1/2)$

$= 5/2 \times 3/2 \times 1/2 \times \sqrt{\pi}$

b) $\int_0^{\infty} x^7 e^{-5x} \, dx = \frac{\Gamma(8)}{5^8}$

$= \frac{7!}{5^8} = \underline{0.012}$

→ Problem 3

$$f_{Q|X}(q\,|\,1) = \frac{6q(1-q) \times \cancel{6q(1-q)}}{\cancel{6q(1-q)}}$$

$$= 6q(1-q)$$

$$f_{Q|X}(q\,|\,0) = \frac{6q(1-q) \times \cancel{1-6q(1-q)}}{\cancel{1-6q(1-q)}}$$

$$= 6q(1-q)$$

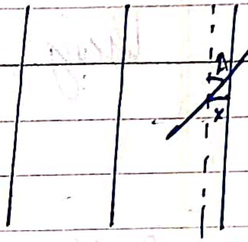
Using $f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

→ Problem 4

Consider the midpoint of needle onto the plane.

Let X be the distance from the nearest line

$X \sim \text{Uniform}(0, d/2)$



Let A be the angle with the vertical.

$A \sim \text{Uniform}(0, \pi/2)$

An intersection occurs if

$X \leq l/2 \sin A$

$$P_{X,A}(x, a) = \frac{2}{\pi} \times \frac{2}{d} = \frac{4}{\pi d}$$

$\therefore$ required probability $= \displaystyle\int_0^{\pi/2} \int_0^{l/2 \sin a} \frac{4}{\pi d} \, dx\, da$

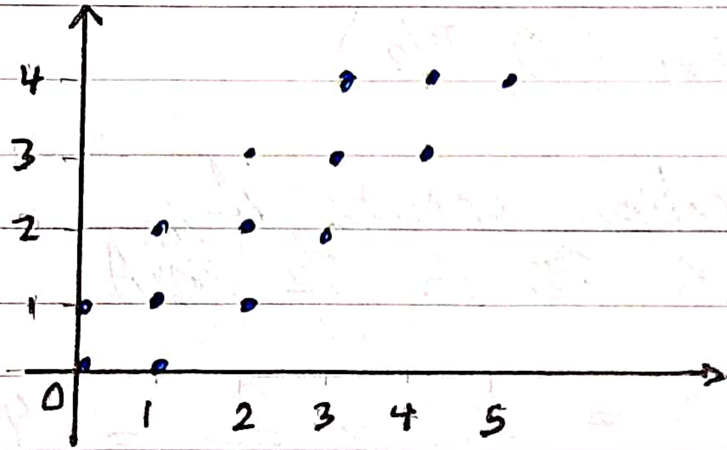
$$= \int_0^{\pi/2} \frac{4\, l/2 \sin(a)}{\pi d} \, da$$

$$= \frac{4l}{2\pi d} \Big[-\cos(a) \Big]_0^{\pi/2}$$

$$= \frac{4l}{2\pi d} \times [-0 - (-1)] = \frac{2l}{\pi d}$$

→ Problem 5

a)



b) $P_X(i) = P_Y(j)$

$$P_X(i) = P_{XY}(i,i) + P_{XY}(i,i-1) \; P_{XY}(i,i+1)$$

$$= \frac{1}{6\cdot 2^{i}} + \frac{1}{6\cdot 2^{i-1}} + \frac{1}{6\cdot 2^{i}}$$

$$= \frac{2}{6\cdot 2^{i}} + \frac{1}{6\cdot 2^{i-1}} = \frac{1}{6\cdot 2^{i-1}} + \frac{1}{6\cdot 2^{i-1}}$$

if $i < 1$, $P_X(i) = \frac{1}{6\cdot 2^{i-1}}$ $\qquad = \frac{1}{6\cdot 2^{i-2}}$ if $i \geq 1$

c) $P(x=y \mid x<2)$

$= \dfrac{P(x=y \cap x<2)}{P(x<2)}$

$P(x<2) = P(x=1) + P(x=0)$

$= \dfrac{1}{6\cdot 2^{-1}} + \dfrac{1}{6\cdot 2^{-1}}$

$= \dfrac{4}{6} = \dfrac{2}{3}$

$P(x=y \cap x<2) = P_{xy}(0,0) + P_{xy}(1,1)$

$= \dfrac{1}{6\cdot 2^{0}} + \dfrac{1}{6\cdot 2^{1}} = \dfrac{1}{6} + \dfrac{1}{12}$

$= \dfrac{1}{4}$

$P(x=y \mid x<2) = \dfrac{1/4}{2/3} = \dfrac{1}{4} \times \dfrac{3}{2} = \dfrac{3}{8}$

e) $P(x=y) = 1/3$

f) $E[X \mid Y=2]$

$= 1\times \dfrac{1}{6\cdot 2^{1}} + 2\times \dfrac{1}{6\cdot 2^{2}} + 3\times \dfrac{1}{6\cdot 2^{2}}$

$= \dfrac{1}{12} + \dfrac{1}{12} + \dfrac{1}{8}$

$= 7/24$

g) $V(X \mid Y=2) =$

$= E[X^2 \mid Y=2] - E[X \mid Y=2]$

$= 1\times \dfrac{1}{6\times 2^{1}} - \dfrac{7}{24} = \dfrac{-1}{12}$

→ Problem 7

$X \sim N(\mu_x, \sigma_x^2)$

$\therefore X = \sigma_x^2 S + \mu_x$ where S is a standard normal random variable

$Y \sim N(\mu_y, \sigma_y^2)$

$\therefore Y = \sigma_y^2 S + \mu_y$

$\Rightarrow X + Y = \sigma_x^2 S + \mu_x + \sigma_y^2 S + \mu_y$

$= (\sigma_x^2 + \sigma_y^2) S + \mu_x + \mu_y$

$\Rightarrow X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

→ Problem 8

a) $Y = 2X_1 + 3X_2$

$= 2[3S + 2] + 3[4S + 1]$

where S is a standard normal random variable

$= 6S + 4 + 12S + 3$

$= 18S + 7$

$\therefore Y \sim N(7, 18)$

b) $Y = X_1 - X_2$

$= 3S + 2 - [4S + 1]$

$= 3S + 2 - 4S - 1 = -S - 1$

$\therefore Y \sim N(-1, -1)$

→ Problem 9

a) $$\int_{y\geq 0}\int^{x^2+y^2\leq 1} c\, dx\, dy = 1$$

$$1 = c \underbrace{\iint^{x^2+y^2\leq 1} dx\,dy}_{\text{area of a unit circle } = \pi}$$

$$= c\pi$$

$$\therefore c = 1/\pi$$

b) $$P_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} c\, dy$$

$$= c\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$

$$= c\left[\sqrt{1-x^2} - (-\sqrt{1-x^2})\right]$$

$$= c \cdot 2\sqrt{1-x^2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi}$$

$$P_Y(y) = \frac{2\sqrt{1-y^2}}{\pi}$$

c) $\because$ y is always $-1 \leq y \leq 1$,

$$P_{X|Y}(x \mid y \text{ such that } -1\leq y\leq 1) = P_X(x)$$

$$= \frac{2\sqrt{1-x^2}}{\pi}$$

d) No.

→ Problem 10

a) $f_X(x) = \int_0^\infty x e^{-x(1+y)} \, dx$

$\frac{d[-x(1+y)]}{dx} = -x$

$f_X(x) = -\int_0^\infty -x \, e^{-x(1+y)} \, dx$

$= -\left[e^{-x(1+y)}\right]_0^\infty$ $\quad \because \int f' e^f = e^f$

$= -\left[e^{-\cancel{x}\infty} - e^{-x}\right]$

$= -\left[0 - e^{-x}\right]$

$= e^{-x}$

~~$f_Y(y)$~~

b) $P(X > 1) = 1 - P(X \le 1)$

$= 1 - \int_0^1 e^{-x} \, dx$

$= 1 - \left[\frac{-e^{-x}}{\cancel{x}}\right]_0^1$

$= 1 + \left[e^{-1} - 1\right]$

$= \cancel{1} \; \frac{1}{e}$