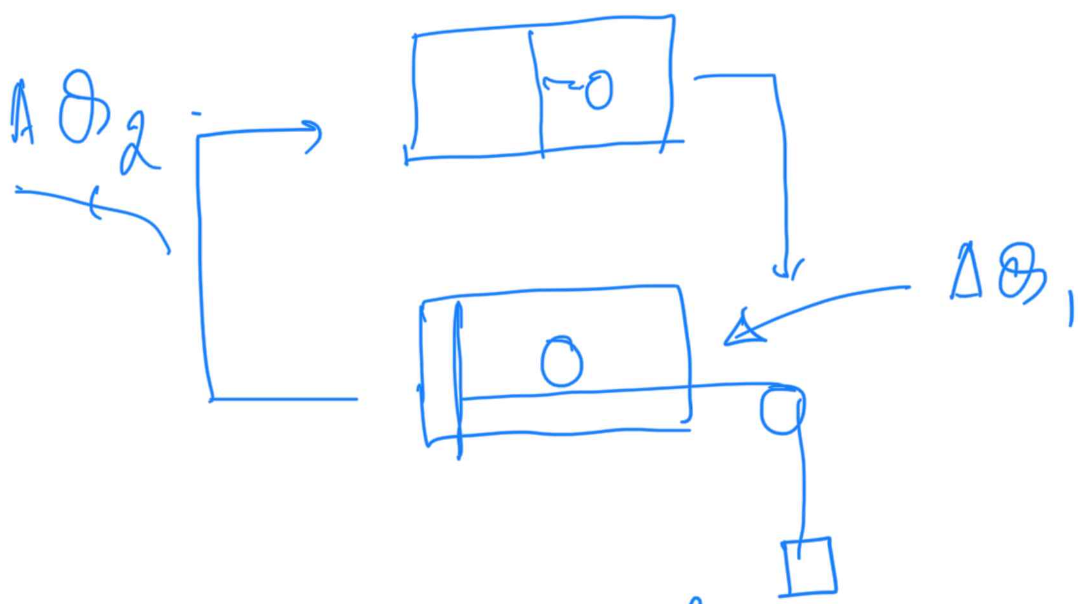
Information and thermodynamics

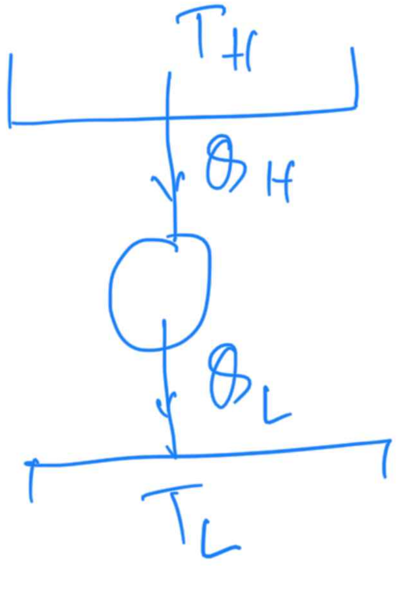
Maxwell's demon

Szilard engine



$$\Delta Q_1 = \Delta W = kT \log 2 .$$

Carnot engine and Clausius inequality



$$\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$$

$$\frac{Q_H}{T_H} - \frac{Q_L}{T_L} = 0$$

$$\oint \frac{dQ}{T} = \frac{Q_H}{T_H} - \frac{Q_L}{T_L} = 0$$

$$\Rightarrow \oint \frac{dQ}{T} = \oint ds$$ S is a state variable.

For reversible process η_R and irreversible process η_{IR}

$$\eta_R > \eta_{IR}$$

$$\Rightarrow 1 - \frac{Q_H}{Q_L} > 1 - \frac{Q_H^{IR}}{Q_L^{IR}}$$

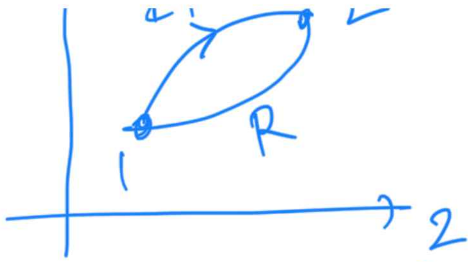
$$\frac{Q_H}{Q_L} > \frac{Q_H^{IR}}{Q_L^{IR}}$$

$$\frac{T_H}{T_L} > \frac{Q_H^{IR}}{Q_L^{IR}} \quad \left(\frac{Q_H}{T_H} - \frac{Q_L}{T_L}\right) < 0$$

$$\oint \frac{dQ}{T} < 0$$

For both cases $\oint \frac{dQ}{T} \leq 0$

Let us consider an cycle process



$$\oint \frac{dQ}{T} \leq 0 \qquad \int_1^2 \frac{dQ}{T} + \int_2^1 \frac{dQ}{T} < 0$$

$$\int_1^2 \frac{dQ}{T} + S_1 - S_2 \leq 0$$

$$S_2 - S_1 \geq \int_1^2 \frac{dQ}{T}$$

$$\Delta S \geq \int_1^2 \frac{dQ}{T}$$

$$\Delta S = \int_1^2 \frac{dQ}{T} + \Delta S_g.$$

$$\Delta S = \Delta S_I + \Delta S_g$$

For Szilard engin

$$\Delta S = \left(\frac{\Delta Q_1 - \Delta Q_2}{T} \right) + \Delta S_g.$$

Entropy production :

$$\Delta S = \Delta S_I + \Delta S_g$$

$$\frac{d}{dt}\Delta S = \frac{d}{dt}\Delta S_I + \frac{d}{dt}\Delta S_g.$$

At steady stat

$$\frac{d}{dt}\Delta S_I + \frac{d}{dt}\Delta S_g = 0$$

For a stochastic process

$$\frac{dp(n,t)}{dt} = \sum_{n'}\left(W_{nn'}\,p(n',t) - W_{n'n}\,p(n,t)\right)$$

$$S = -\sum_{n} p(n,t)\log p(n,t)$$

$$\frac{dS}{dt} = -\sum \frac{dp}{dt}\log p(n,t)$$

$$n$$

$$= -\sum_{n}\sum_{n'}\left(w_{nn'}\,p(n',t) - w_{n'n}\,p(n,t)\right)\log p_n$$

$$= -\sum_{n}\sum_{n'}\left(\omega_{n'n}\,p(n,t) - w_{nn'}\,p(n',t)\right)\log p(n',t)$$

$$\frac{ds}{dt} / \frac{ds}{dt} = \sum \omega_{nn'}\,p(n',t) - w_{n'n}\,p(n,t)\,\log p(n',t)$$

$$\frac{ds}{dt} = \frac{1}{2}\sum_{n}\sum_{n'}\left(\omega_{nn'}\,p(n',t) - \omega_{n'n}\,p(n,t)\right)\log\frac{p_{n'}}{p_n}$$

$$\overbrace{= \frac{1}{2}\sum_{nn'}\left(\omega_{nn'}\,p(n',t) - \omega_{n'n}\,p(n,t)\right)\log\frac{\omega_{nn'}\,p(n',t)}{\omega_{n'n}\,p(n,t)}}^{\frac{dS_g}{dt}}$$

$$\overbrace{- \frac{1}{2}\sum_{nn'}\left(\omega_{nn'}\,p(n',t) - \omega_{n'n}\,p(n,t)\right)\log\frac{\omega_{nn'}}{\omega_{n'n}}}^{\frac{dS_I}{dt}}$$

At steady state

Flux.

$$\frac{dS_g}{dt} = \frac{dS_I}{dt} = \frac{1}{2}\overbrace{\left(\omega_{nn'}\, p(n') - \omega_{nn'}\, p(n)\right)}\log\frac{\omega_{nn'}}{\omega_{n'n}}.$$

$$X_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} X_2$$

$$\frac{dS_g}{dt} = \frac{1}{2}\left(\alpha\, p_1 - \beta\, p_2\right)\log\left(\frac{\alpha}{\beta}\right)$$

$$p_1 = \frac{\beta}{\alpha+\beta} \qquad p_2 = \frac{\alpha}{\alpha+\beta}$$

$$\langle n_1\rangle = \frac{\beta}{\alpha+\beta} N \qquad p_1 = \frac{\langle n_1\rangle}{N}$$

$$\frac{dS_g}{dt} = \frac{1}{2}\left(\alpha\,\langle X_1\rangle - \beta\langle X_2\rangle\right)\log\frac{\alpha}{\beta}$$

Breaking of detailed balance

$$\begin{tikzcd} X_1 \arrow[r, bend left=60, "\alpha"] \arrow[r, shift left, "\alpha_0"] & X_2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

$$\frac{dS_g}{dt} = (\alpha x_1 - \beta x_2)\log\frac{\alpha}{\beta} + (\alpha_0 x_1 - \beta_0 x_2)\log\frac{\alpha_0}{\beta_0}$$

$$\frac{dx_2}{dt} = (\alpha + \alpha_0)x_1 - (\beta_{+} + \beta_{2})x_2$$

$$\langle x_2 \rangle = \frac{\alpha + \alpha_0}{\alpha + \alpha_0 + \beta + \beta_0} X_T$$

$$\langle x_1 \rangle = \frac{\beta + \beta_0}{\alpha + \alpha_0 + \beta} X_T$$

$$\alpha x_1 - \beta x_2 = \left(\frac{\alpha(\beta + \beta_0)}{\alpha + \alpha_0 + \beta} - \frac{\beta(\alpha + \alpha_0)}{\alpha + \alpha_0 + \beta}\right) X_T$$

$$= \frac{\alpha\beta_0 - \alpha_0\beta}{\alpha + \alpha_0 + \beta + \beta_0}$$

$$\frac{dS_g}{dt} = X_T \frac{\alpha\beta_0 - \alpha_0\beta}{\alpha + \alpha_0 + \beta + \beta}\log\left(\frac{\alpha\beta_0}{\alpha_0\beta}\right)$$

$$\alpha + \alpha_0 + \beta + \beta_0$$

$$\simeq \frac{x_T \beta \alpha}{\alpha + 2\beta} \log\left(\frac{\alpha}{\alpha_0}\right)$$

The Fisher information

$$p(x_2|s) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_2 - \langle x_2 \rangle)^2}{2\sigma^2}}$$

$$\sigma^2 = \frac{\alpha' \beta' s}{(\alpha' s + \beta')^2} x_T \qquad \alpha' = \alpha + \alpha_0$$

$$\beta' = \beta + \beta_0$$

$$\langle x_2 \rangle = \frac{\alpha' s}{\alpha' s + \beta'} x_T \frac{1}{\Omega}$$

For large Ω

$$F(s) = \frac{1}{\sigma^2} \left(\frac{\partial x_2}{\partial s}\right)^2$$

$$\frac{\partial x_2}{\partial s} = \left[\frac{\alpha'}{\alpha' s + \beta'} - \frac{\alpha' s}{(\alpha' s + \beta')^2} \cdot \alpha'\right] x_T$$

$$= \frac{\alpha'}{(\alpha' s + \beta')^2} \left[\alpha' s + \beta' - \alpha' s\right]$$

$$(\alpha' s+\beta')^2$$

$$= \frac{\alpha'\beta'}{(\alpha' s+\beta')^2} x_T$$

$$F(s) = \frac{(\alpha' s+\beta)^2 \Omega}{\alpha'\beta' s \, x_T} \frac{(\alpha'\beta')^2 x_T^2}{(\alpha' s+\beta')^4}$$

$$= \frac{\alpha'\beta'}{s(\alpha' s+\beta')^2} x_T \Omega .$$

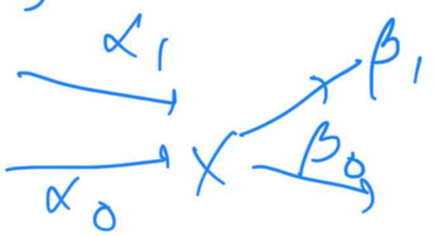
$$pr(s) = \frac{\alpha'\beta' s}{(\alpha' s+\beta')^2} x_T \Omega .$$

$$= \frac{\beta'}{\alpha'} \frac{s}{(s+\beta'/\alpha')^2} x_T \Omega$$

For the poisson process

$$\xrightarrow{\alpha_0} X \xrightarrow{\beta_0}$$

Breaking detail balance



$$\frac{d\phi(n,t)}{dt} = \left(\alpha_1 \phi(n-1) - \beta \phi(n)\right)$$

$$- \left(\alpha_1 \phi(n) - \beta \phi(n+1)\right)$$

$$+ \left(\alpha_0 \phi(n-1) - \beta \phi(n)\right)$$

$$- \left(\alpha_0 \phi(n) - \beta \phi(n)\right)$$

$$\frac{ds}{dt} = \sum_n \left(\alpha_1 \phi(n) - \beta(n+1)\phi(n+1)\right) \log \frac{\alpha_1}{\beta(n+1)}$$

$$- \sum_n \alpha_0 \phi(n) - \beta_0(n+1)\phi(n+1) \log \frac{\alpha_1}{\beta(n+1)}$$

$$\phi(n) : \frac{\mu^n}{\;} e^{-\mu}$$

$$n!$$

$$\mu = \frac{\alpha_0 + \alpha_1}{\beta_0 + \beta_1}$$

$$\frac{ds}{dt} = \sum_n J_n \log \frac{\alpha_1}{\beta(n+1)}$$

$$J_n = \left(\alpha_1 \, p(n) - \beta_1 (n+1)\, p(n+1) \right)$$

$$= \left(\alpha_1 \frac{\mu^n}{n!} - \beta_1 (n+1) \frac{\mu^{n+1}}{(n+1)!} \right) e^{-\mu}$$

$$= \frac{\mu^n}{n!} \left(\alpha_1 - \beta_1 \mu \right) e^{-\mu}.$$

$$\frac{ds}{dt} = \sum \frac{\mu^n}{n!} e^{-\mu} \left(\alpha_1 - \beta_1 \mu \right) e^{-\mu} \log \frac{\alpha_1}{\beta_{\cancel{1}}(n+1)}$$

$$\sum \mu^n \quad e^{-\mu} \quad e^{-\mu} \log \alpha_0$$

$$-\sum_{n!} \frac{\mu^{n}}{n!} e^{-\mu} (\alpha_1 - \beta_1 \mu) e \log \frac{\cdot}{\beta_0 (n+1)}$$

$$\alpha_0 - \beta_0 \frac{\alpha_1 + \alpha_0}{\beta_1 + \beta_0}$$

$$= \sum_{n} \frac{\mu^{n}}{n!} e^{-\mu} (\alpha_1 - \beta_1 \mu) \log \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1}$$

$$= (\alpha_1 \beta_0 - \beta_1 \alpha_0) \log \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1}$$

$$\simeq \beta \alpha_1 \log\left(\frac{\alpha_1}{\alpha_0}\right) \qquad \beta \simeq \beta_0 \quad \alpha_1 >> \alpha_0$$

Fisher information