MAG.101 Probability and Statistics ... George Paul
2021121006

Assignment 1

Problem 1:

$A = \{(1,2), (2,1)\}$ $|A| = 2$

$B = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$
$|B| = 5$

$C = \{(2,1), (1,2), (2,2), (2,3), (3,2), (4,2), (2,4), (5,2), (2,5), (6,2), (2,6)\}$
$|C| = 11$
$|S| = 36$

1. $P(A|C) = \frac{P(A \cap C)}{P(C)}$

$= \frac{2/36}{11/36} = \underline{\underline{2/11}}$

2. $P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{2/36}{11/36} = \underline{\underline{2/11}}$

3. $P(A|C) = 2/11$
$P(A) = 2/36$
$P(A|C) \neq P(A)$ $\therefore$ A and C are not independent

$P(B|C) = 2/11$
$P(B) = 5/36$
$P(B|C) \neq P(B)$ $\therefore$ B and C are not independent

Problem 2:

Let

B: Alex is a boy
G: Alex is a girl
S: Alex has a sister

The event that alex has a sister is independent of the events that he/she is a girl or a boy.

$\therefore P(S) = P(S|G) = P(S|B)$

Biologically, the probability of a child being born a female is 50%.
So, $P(S) = 50\%$

1. $P(S|G) = 50\%$
2. $P(S|B) = 50\%$

Problem 3:

Arbitrarily assign the jars as jar 1 and jar 2.
Put exactly one blue candy in jar 1 and the rest of the 99 blue and 100 red candies in jar 2.
The probability to be maximised is:

$P(B) = 50\% \times P(B \text{ from } J1) + 50\% \times P(B \text{ from } J2)$

$P(B \text{ from } J1)$ is 100% since there is only one blue candy in

jar 1
P(B from J2) is 99/199 since there are 99 blue candies and 199 total candies.

$\therefore P(B) = 50\% \times 1 + 50\% \times 99/199$
$\approx 74.87\%$

Moving ~~Adding~~ any more blue candies ~~from~~ to Jar 1 will not increase P(B from J1) but will decrease P(B from J2). even one
Moving any more red candies ~~from~~ to Jar 1 will reduce P(B from J1) by more than 50% but will only increase by around 0.002.

hence the maximal P(B) is 74.87%.

Problem 4:

1. P(HTHT)

$$= \frac{1}{3} \times 0.2 \times 0.8 \times 0.2 \times 0.8 + \frac{1}{3} \times 0.6 \times 0.4 \times 0.6 \times 0.4 + \frac{1}{3} \times 0.4 \times 0.6 \times 0.4 \times 0.6$$

$$= 0.0085\bar{3} + 0.0192 + 0.0192$$
$$= \underline{0.0469\bar{3}}$$

2. Let A : coin with prob of heads 0.

$$P(A|HTHT) = \frac{P(HTHT|A) \times P(A)}{P(HTHT)}$$

$$P(HTHT|A) = 0.2 \times 0.8 \times 0.2 \times 0.8$$
$$= 0.256$$

$$P(A) = 1/3$$
$$P(HTHT) = 0.0469\overline{3}$$

$$\therefore P(A|HTHT) = \frac{0.256 \times 1/3}{0.0469\overline{3}}$$
$$\approx 0.684$$

Problem 6:

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$ (by PIE)

$2/3 = P(A) + P(C) - P(A)P(C)$ ($\because$ A and C are independent)

① ← $P(A)P(C) = P(A) + P(C) - 2/3$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$ (by PIE)

$3/4 = P(B) + P(C) - P(B)P(C)$ ($\because$ B and C are independent)

② ← $P(B)P(C) = P(B) + P(C) - 3/4$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P(A \cap B \cap C)$

$11/12 = P(A) + P(B) + P(C) - 0 - P(A)P(C) - P(B)P(C) - 0$

($\because$ A and B are disjoint)

Using eq. ① and ②

$$11/12 = P(A) + P(B) + P(C) - [P(A) + P(C) - 2/3] - [P(B) + P(C) - 3/4]$$

$$11/12 = \cancel{P(A)} + \cancel{P(B)} + \cancel{P(C)} - \cancel{P(A)} - \cancel{P(C)} + 2/3 - \cancel{P(B)} - P(C) + 3/4$$

$$P(C) = 2/3 + 3/4 - 11/12$$

$$= \frac{8}{12} + \frac{9}{12} - \frac{11}{12} = \frac{1}{2}$$

$$P(C) = 1/2$$

$$② = \frac{3}{4} = P(B) + \frac{1}{2} - \frac{P(B)}{2}$$

$$= \frac{P(B)}{2} + \frac{1}{2}$$

$$\frac{3}{2} = P(B) + 1$$

$$P(B) = 1/2$$

$$① = \frac{2}{3} = P(A) + \frac{1}{2} - \frac{P(A)}{2}$$

$$= \frac{P(A)}{2} + \frac{1}{2}$$

$$\frac{4}{3} = P(A) + 1$$

$$P(A) = 1/3$$

Problem 7:

$P(H) = p$

$\therefore P(T) = 1-p$

Let H_n : first head on nth toss

$P(H_2) = (1-p) \times p$

$P(H_4) = (1-p)^3 \times p$

$P(H_n) = (1-p)^{n-1} \times p$

H_E : first head occurs on an even numbered toss.

$$P(H_E) = P(H_E \mid H_1) P(H_1) + P(H_E \mid H_1') P(H_1')$$

$$= 0 + P(H_E') (1-p)$$

$$= [1 - P(H_E)](1-p)$$

$$\frac{1}{1-p} = \frac{1}{P(H_E)} - 1$$

$$\frac{1}{1-p} + 1 = \frac{1}{P(H_E)}$$

$$\frac{1 + 1-p}{1-p} = \frac{1}{P(H_E)}$$

$$P(H_E) = \frac{1-p}{2-p}$$

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Problem 8:

$$P(A\cap B) = P(A\cap B|C_1)P(C_1) + P(A\cap B|C_2)P(C_2) + \cdots + P(A\cap B|C_M)P(C_M)$$

$$= \sum_{i=1}^{M} P(A|C_i)\, P(B|C_i)\, P(C_i)$$

$\because$ A and B are conditionally independent

$$= P(B)\sum_{i=1}^{M} P(A|C_i)\, P(C_i)$$

$\because$ B is independent of all C_i's

$$= P(B)\, P(A) \quad \text{by total probability}$$

$P(A\cap B) = P(A)P(B)$

$\therefore$ A and B are independent

Problem 9:

D : has the disease

$P(D) = 0.1\%$

$P(+|D)$

: test correctly detects infection

$= 99\%$

$P(+|D)$

false positive : $+|D'$

$0.5\% = P(+|D')$

$+$: tests positive

required probability - $P(D|+)$

$$P(D|+) = \frac{P(+|D)\, P(D)}{P(+)} \quad \text{(Bayes' theorem)}$$

$$P(+) = P(+|D)\,P(D) + P(+|D')\,P(D')$$
(total probability)

$$= 99\% \times 0.1\% + 0.5\% \times 99.9\%$$

$$\cancel{= \frac{1}{100}(99 \times 0.1 + 0.5 \times 99.9)}$$

$$\cancel{= \frac{1}{100}(9.9 + 5 \times 9.99)}$$

$$\cancel{= 59.85\%}$$

$$= \underline{0.005985}$$

$$\therefore P(D|+) = \frac{99/100 \times 0.1/100}{0.005985}$$

$$\approx \underline{\underline{0.16541}}$$

Problem 10:

2 : die shows 2 $\quad P(2) = 1/6$

RC : reports correctly $\quad P(RC) = 3/5$

Rn : reports a n

required probability = $P(2|R2)$

$$P(2|R2) = \frac{P(R2|2)\,P(2)}{P(R2)}$$

$$P(R2) = P(R2 \mid 2)\, P(2) + P(R2 \mid 2')\, P(2')$$

$$= P(RC)\ 1/6 + P(RC')\ 5/6$$

$$= \frac{3}{5} \times \frac{1}{6} + \frac{2}{5} \times \frac{5}{6}$$

$$= 0.4\bar{3}$$

$$P(2 \mid R2) = \frac{3/5 \times 1/6}{0.4\bar{3}}$$

$$\approx \underline{0.2307}$$

<u>Problem 5:</u>

$$P(A^c \cap B^c) = P((A \cup B)^c) = 0.45$$

$$P(A \cup B) = 0.55 = 1 - 0.45$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.55 = 0.45 + P(B) - 0.15$$

$$0.7 - 0.45 = P(B) = \underline{0.25}$$