Probability and Statistics

UG2, Core course, IIIT,H

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Outline

Definition of Covariance...

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Let X and Y be two random variables.

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Remarks

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - (E[X])(E[Y])$$

Derivation

$$E\left[XY - X\underbrace{E[Y]} - \underbrace{E[X]}Y + \underbrace{E[X]E[Y]}\right]$$

Using Lin.

$$= E[XY] - E[Y]E[X] - E[X]\cancel{E[Y]} + E[X]\cancel{E[Y]}$$

$$= E[XY] - E[X]E[Y]$$

Solved Example

Solved Example

Recall $X \sim U(a,b) \Rightarrow E[X] = \frac{a+b}{2}$

Example (Solved Example)

Suppose $X \sim \text{Uniform}(1,2)$ and given $X = x$, Y is exponential with parameter $\lambda = x$. Find $\text{Cov}(X,Y)$.

$\frac{1}{b-a} = \frac{1}{2-1} = 1$

$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$X \sim \text{Uniform}(1,2) \Rightarrow E[X] = 3/2$

$E[Y] = E[\underbrace{E[Y|X]}]$ (by L.O.T.E)

$= E[1/X] = \int_1^2 \frac{1}{x} \cdot 1 \, dx = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2.$

$E[XY] = E[E[XY|X]]$ [L.O.T.E]

$= E[X\,E[Y|X]] = E[X \cdot \frac{1}{X}] = E[1] = 1$

$\boxed{\text{Cov}(X,Y) = 1 - 3/2 \ln 2}$

Properties of Covariance...

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1. $\text{Cov}(X, X) = \text{Var}(X)$

Properties of Covariance...

$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$= E[X]E[Y]$ (if X, Y ind).

Properties of Covariance

1. $\text{Cov}(X, X) = \text{Var}(X)$
2. If X and Y are independent, then $\text{Cov}(X, Y) = 0$. $\boxed{\text{[Note: converse is not true!]}}$

(Look for example)

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Properties of Covariance...

$\mathrm{Cov}(X, Y+c) = \mathrm{Cov}(X, Y)$

$\| \quad \mathrm{Cov}(Y+c, X) = \mathrm{Cov}(Y, X) = \mathrm{Cov}(X, Y)$

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$$\text{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \,\text{Cov}(X_i, Y_j)$$

Use 3, 4, 6 ✓

Solved Example...

Solved Example...

$Var(X) = 1 = E[X^2] - E[X]^2$ $X \sim N(0,1)$ $N(0, 1)$

Example (Solved example)

Let X and Y be two independent random variables following standard normal distribution, and

$$Z = 1 + X + XY^2$$

$$W = 1 + X$$

Find $\text{Cov}(Z, W)$.

$$\text{Cov}(Z, W) = \text{Cov}(1+X+XY^2, 1+X) = \text{Cov}(X+XY^2, X)$$

$$= \text{Cov}(X, X) + \text{Cov}(XY^2, X) = Var(X) + \text{Cov}(XY^2, X)$$

$X \sim N(0,1)$

$$= 1 + E[X^2Y^2] - E[XY^2]\,E[X]$$

$$= 1 + E[X^2]\,E[Y^2] - E[X]\,E[Y^2]\,E[X]$$

$$= 1 + 1 \cdot 1 - 0 = 2$$

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$$\text{Var}(Z) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X, Y) \leftarrow \text{Ex}$$

Solution

$$\text{Var}(Z) = \text{Cov}(Z, Z) = \text{Cov}(X+Y, X+Y)$$

$$= \text{Cov}(X, X) + \text{Cov}(X, Y) + \text{Cov}(Y, X) + \text{Cov}(Y, Y)$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y)$$

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$$= \text{Cov}\left(\frac{X}{\rho_X}, \frac{Y}{\rho_Y}\right) = \frac{\text{Cov}(X, Y)}{\rho_X \rho_Y}$$

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1. $-1 \leq \rho(X, Y) \leq 1$ $\leftarrow$ Try (C-S)

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2. If $\rho(X, Y) = 1$,

Properties of Correlation Coefficient...

$Y = f(X) = X^2 + X^3$

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2. If $\rho(X, Y) = 1$, then $\boxed{Y = aX + b}$, where $a > 0$

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3. If $\rho(X, Y) = 1$, then $Y = aX + b$, where $a < 0$
4. $\rho(aX + b, cY + d) = \rho(X, Y)$ for $a, c > 0$

Cov( )

Answer to previous problem...

Positive Correlation, Negative Correlation, Uncorrelation...

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Definition of positive, negative correlation

Let X and Y be two RVs.

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0

Pairwise uncorrelation and Variance

$2\,\mathrm{Cov}(X,Y)$

If X and Y are uncorrelated, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

More generally, if $X_1, X_2, \ldots, X_n$, are pairwise uncorrelated, i.e., $\rho(X_i, X_j) = 0$ when $i \neq j$, then

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$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) + \cancel{2\mathrm{Cov}(X_i, X_j)} \to 0$$

Solved Example

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Consider a point (X, Y) chosen uniformly at random from the following disc

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$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

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Are X and Y uncorrelated?

Skipped

Answer to previous problem...

Bivariate Normal Distribution...

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Example (Sum of Two Normal Distribution May Not be Normal)

Let RVs $X \sim N(0,1)$ and $W \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ be two independent RVs.

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Find the PDF of Y

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5. If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are jointly normal, then

$$X + Y \sim N(\mu_X + \mu_Y, \quad \sigma_X^2 + \sigma_Y^2 + 2\rho(X, Y)\sigma_X\sigma_Y)$$

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$$X + Y \sim N(\mu_X + \mu_Y, \quad \sigma_X^2 + \sigma_Y^2 + 2\rho(X, Y)\sigma_X\sigma_Y)$$

6. Can we provide a simple way to generate jointly normal random variables?

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3. If X and Y are bivariate normal, then by letting $a = 0, b = 1$, Y must be normal
4. If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma^2)$ are independent, then they are jointly normal
5. If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are jointly normal, then

$$X + Y \sim N(\mu_X + \mu_Y, \quad \sigma_X^2 + \sigma_Y^2 + 2\rho(X, Y)\sigma_X\sigma_Y)$$

6. Can we provide a simple way to generate jointly normal random variables?
7. We first introduce standard bivariate normal distribution

Example of Bivariate Normal...

Example of Bivariate Normal...

Example (Bivariate Normal Variable)

Let Z_1 and Z_2 be two independent $N(0, 1)$ RVs.

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Given that z_1, z_2 are ind.

$$f_{Z_1 Z_1}(z_1, z_2) = \underline{f_{Z_1}}(z_1) f_{Z_2}(z_2)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z_1^2+z_2^2)}{2}}$$

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1. Is X and Y bivariate normal?
2. What is the joint PDF of X and Y?
3. Find $\rho(X, Y)$

Answer to previous problem...

Claim : $ax+by$ is normal

$$ax+by = aZ_1 + b\left(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2\right)$$

$$= (a+b\rho)Z_1 + b\sqrt{1-\rho^2}\, Z_2 .$$

is a linear combination of two standard normal $\Rightarrow$ $ax+by$ is normal.

$\Rightarrow$ $ax+by$ is bivariate normal

Recall

$aZ_1 + bZ_2$ is normal

$Z_1 \sim N(0,1)$

$Z_2 \sim N(0,1)$

Answer to previous problem...

Standard Bivariate Normal Distribution...

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Standard Bivariate Normal Distribution...

$\frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$ ← $\boxed{X+Y}$

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where $\rho \in (-1,1)$. If $\rho = 0$, then we call X and Y to have standard normal bivariate normal distribution.

Definition of Bivariate Normal Distribution...

Definition of Bivariate Normal Distribution...

Std. Normal

Norm

Definition of Bivariate Normal Distribution

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$$f_{XY} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\},$$

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where $\mu_X, \mu_Y \in \mathbb{R}, \sigma_X, \sigma_Y > 0$ and $\rho \in (-1, 1)$ are all constant.

Creating a Jointly Normal Distribution...

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we start with two independent standard normal RVs Z_1 and Z_2 and define

$$\begin{aligned} X &= \sigma_X Z_1 + \mu_X \\ Y &= \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y \end{aligned}$$

and follow the above procedure: solve for Z_1, Z_2, and apply method of transformation

Bivariate Normal and Uncorrelation implies Independence...

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Theorem

If X and Y are bivariate normal and uncorrelated, then they are independent.

Try!

Solved Problem 1

Example (Solved Problem 1)

Let X and Y be two jointly continuous random variables with joint PDF

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Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} 2 & y + x \leq 1,\ x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

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Find $\text{Cov}(X,Y)$ and $\rho(X,Y)$.

Answer to previous problem...

Answer to previous problem...

Solved Problem 2...

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Example (Solved Problem 2)

I roll a fair die n times.

Solved Problem 2...

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Answer to previous problem...

Outline

Union bound and extension...

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Union Bound

Recall the inclusion exclusion principle:

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$$P((\cup_{i=1}^{n} A_i)) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P\left(\cap_{i=1}^{n} A_i\right)$$

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A_i, A_j

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Generalized Union Bounds and Bonferroni Inequalities...

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The above inequality is called Chebyshev inequality.

- Chebyshev inequality states that the difference between X and $E[X]$ is bounded by $\mathrm{Var}(X)$

$$E[Y] = E[(X - E[X])^2] \overset{\text{def of var}}{=} \mathrm{Var}(X)$$

$$\Rightarrow P(Y \geq b^2) \leq \frac{\mathrm{Var}(X)}{b^2}$$

$$\Rightarrow P((X - E[X])^2 \geq b^2) \leq \frac{\mathrm{Var}(X)}{b^2}$$

$Y = (X - E[X])^2$ is non-neg.

$$P(Y \geq b^2) \leq \frac{E[Y]}{b^2} \quad [\text{Markov}]$$

$$\Rightarrow P(|X - E[X]| \geq b) \leq \frac{\mathrm{Var}(X)}{b^2}$$

Answer to previous problem...

Example of Markov Inequality...

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Example (Markov Inequality)

Let $X \sim \text{Binomial}(n, p)$.

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Let $X \sim \text{Binomial}(n, p)$. Using Markov inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$.

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Let $X \sim \text{Binomial}(n, p)$. Using Markov inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Verify this bound for $p = 1/2$ and $\alpha = 3/4$.

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$$P(X \geq a) \leq e^{-sa} M_X(s), \quad \text{for all } s > 0$$

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Answer to previous problem...

Cauchy Schwarz Inequality...

Cauchy Schwarz Inequality...

Linear Alg

$\langle u, v \rangle \le \|u\| \, \|v\|$

Cauchy Schwarz Inequality

For any two random variables X and Y we have

$$E[XY] \leq \sqrt{E[X^2]E[Y^2]}$$

where equality holds if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$.

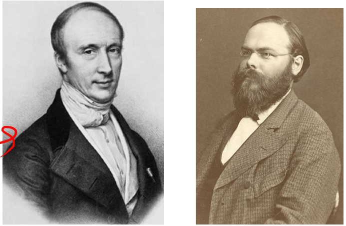

Figure: Left: Cauchy, Right: Schwarz

Answer to previous problem...

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Example of Cauchy Schwarz...

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Example (Application of Cauchy Schwarz)

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$$-1 \leq \rho(X, Y) \leq 1$$

using Cauchy Schwarz inequality.

Can you try?

Example of Cauchy Schwarz...

Example (Application of Cauchy Schwarz)

For any two random variables X and Y, show that

$$|\rho(X, Y)| \leq 1$$

using Cauchy Schwarz inequality. Furthermore, show that $|\rho(X, Y)| = 1$ if and only if $Y = aX + b$ for some constants $a, b \in \mathbb{R}$.

Answer to previous problem...

Convex Functions and Jensen's Inequality...

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Definition of convex function

A function $g : I \to \mathbb{R}$ is convex if for any two points x and y in I and any $\alpha \in [0, 1]$, we have

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If the above inequality is $\geq$, then the function g is concave.

- Here $\alpha x + (1 - \alpha)y$ is the weighted average of x and y

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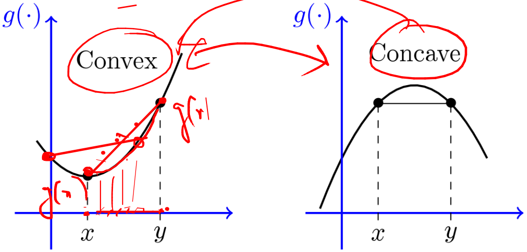

not convex

$tx + (1-t)y$

line segment

betw x, y.

Convex Functions and Jensen's Inequality...

Definition of convex function

A function $g : I \to \mathbb{R}$ is convex if for any two points x and y in I and any $\alpha \in [0, 1]$, we have

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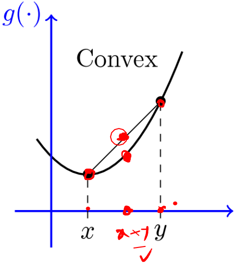

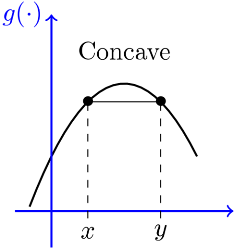

- From the definition of convexity on left, we conclude

$$E[g(X)] \geq g(E[X])$$

Jensen's Inequality...

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Jensen's Inequality...

$E[x^2] \geq (E[x])^2$ $\boxed{g = x^2}$

$g''(x) \geq 0$

$g''(x) = 2 \geq 0$

$\Rightarrow g$ is convex

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- For example, $g(x) = x^2$ is convex in $\mathbb{R}$

$E[-\log x] \geq -\log(E[x])$

$\log x \rightarrow$ concave

$-\log x \rightarrow$ convex

Application of Jensen's Inequality...

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Example (Jensen's Inequality)

Consider a random variable X with $E[X] = 10$, and X being positive.

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Outline

Definition of Sample Mean

i.i.d = independent + Identical

Same mean

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$$\underbrace{\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \epsilon) = 0}$$

$\epsilon = 10^{-10}$

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Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with expected value $E[X_i] = \mu < \infty$ and variance $0 < \mathrm{Var}(X_i) = \sigma^2 < \infty$. Then, the random variable

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- It does not matter what the distribution of X_i is
- The X_i can be discrete, continuous, or mixed random variables

1. Let X_i be Bernoulli(p)
2. Then $E[X_i] = p, \mathrm{Var}(X_i) = p(1-p)$
3. $Y_n = X_1 + X_2 + \cdots + X_n$ has Binomial((n, p))
4. Hence,

$$Z_n = \frac{Y_n - np}{\sqrt{np(1-p)}}$$

5. The figure on the right shows PMF of Z_n for different values of n
6. As we observe, the shape of PMF gets closer to a normal PDF

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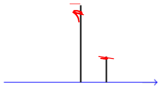
$$Z_1 = \frac{X_1 - p}{\sqrt{p(1-p)}}$$

PMF of Z_1

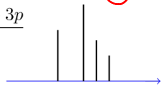


$$Z_2 = \frac{X_1 + X_2 - 2p}{\sqrt{2p(1-p)}}$$

PMF of Z_2

$$Z_3 = \frac{X_1 + X_2 + X_3 - 3p}{\sqrt{3p(1-p)}}$$

PMF of Z_3



$e^{-x^2/2}$

$$Z_{30} = \frac{\sum_{i=1}^{30} X_i - 30p}{\sqrt{30p(1-p)}}$$

PMF of Z_{30}