Probability and Statistics

UG2, Core course, IIIT,H

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Outline

Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

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- P(n heads on n coin flips)
- P(n tails on n coin flips) $P(TT\cdots T)$

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Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

- P(n heads on n coin flips)
- P(n tails on n coin flips)
- P(first k heads, then $n-k$ tails) $P(\underbrace{H \cdots H}_{k}\underbrace{T \cdots T}_{n-k})$

Examples involving independent trials...

$\binom{n}{k} p^k (1-p)^{n-k}$

Problem: coin toss

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- P(n heads on n coin flips)
- P(n tails on n coin flips)
- P(first k heads, then $n-k$ tails)
- P(exactly k heads on n coin flips)

$$= P(H_1 \cap H_2 \cap H_3 \cdots \cap H_n) = P(H_1)P(H_2)\cdots P(H_n)$$

because H_i's are independ.

$$= \underbrace{p \cdot p \cdots p}_{n \text{ times}} = p^n$$

$$P(T_1 \cap T_2 \cap \cdots \cap T_n) = P(T_1)P(T_2)\cdots P(T_n)$$

$$= P(H_1^c)P(H_2^c)\cdots P(H_n^c)$$

$$= (1-P(H_1))(1-P(H_2))\cdots(1-P(H_n))$$

$$= (1-p)^n$$

$$P(H_1 \cap \cdots \cap H_k \cap T_{k+1} \cap \cdots \cap T_n)$$

$$= P(H_1)\cdots P(H_k)P(T_{k+1})\cdots P(T_n)$$

$$= p^k (1-p)^{n-k}$$

Biased Coin, Independence, Infinite Sample Space, Total Probability, Bayes Theorem...

2, 4, 6, 8 . . .

Problem

A biased coin (with probability of obtaining a Head equal to $p > 0$ is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

Solution: Consider the partition of E into $E_1, E_2, \ldots E_k, \ldots$

where E_k = event that 1st head occurs on the $\underbrace{2k^{th}}_{\text{even!}}$ toss

$E = \bigcup_{k=1}^{\infty} E_k$, E_k's are mutually exclusive!

$P(E) = \sum_{k=1}^{\infty} P(E_k)$, $P(E_k) = (1-p)^{2k-1} \cdot p$

$$P(E) = \sum_{k=1}^{\infty} (1-p)^{2k-1} \cdot p = \frac{1-p}{2-p}$$

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2. Consider event H_1 : head on first toss.

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We want to compute $P(E)$.

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We want to compute $P(E)$.
How do we solve problems like this?

Solution...

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Coin Toss Example...

Coin Toss Example...

H | T | H H T T H H H T . . . . T T ← experiment ends!

$n-1$ n

Problem

A coin for which P(Heads) $= p$ is tossed until two successive Tails are obtained. Find the probability that the experiment is completed on the nth toss.

E_1: 1st toss is H

E_2: first two tosses are TH

E_3: first two tosses are TT

F_n: experiment completes on nth toss.

E_1, E_2, E_3 partition Ω (Sample sp.)

H _ _ _ _ _ _

T H

T T

Must have n atleast 2

For $n=2$: $p_2 = P(F_2) = (1-p)(1-p) = (1-p)^2$

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Solution

What are sample space and events?

1. Sample space, S: all possible infinite sequences of tosses
2. event E_1 : first toss is H
3. event E_2 : first two tosses are TH
4. event E_3 : first two tosses are TT
5. event F_n : experiment completed on the nth toss.

Solution to problem in previous slide...part-1

For $n > 2$: $P(F_n | E_1) = P(F_{n-1})$

H | ... TT

as if exp. started from $n = 2$

TH | ... TT

TT ...

$P(F_n | E_2) = P(F_{n-2})$

$P(F_n | E_3) = 0$

Notation: $P(F_n) = p_n$

Using total probab. $n > 2$,

$$P(F_n) = P(F_n | E_1) P(E_1) + P(F_n | E_2) P(E_2) + \cancel{P(F_n | E_3)} P(E_3)$$

$$p_n = p_{n-1} \cdot p + p_{n-2} \cdot (1-p) p + 0$$

$$\boxed{p_n = p \, p_{n-1} + p(1-p) \, p_{n-2},}$$

$$\boxed{\begin{aligned} p_1 &= 0 \\ p_2 &= (1-p)^2 \end{aligned}}$$

Solution to problem in previous slide...part-2

Solution to problem in previous slide...part-3

Properties of conditional probabilities...

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For any events A,B, and E we have the following:

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- $0 \leq P(A \cap E) \leq 1$

Properties of conditional probabilities...

$$P(A|E) = \frac{P(A \cap E)}{P(E)}$$

Properties

For any events A,B,and E we have the following:

- $0 \leq P(A \cap E) \leq 1$
- $P(A \mid E) = 1 - P(A^c \mid E)$

$$P(A^c|E) = \frac{P(A^c \cap E)}{P(E)}$$

$$P(A|E) + P(A^c|E) = \frac{1}{\cancel{P(E)}}\left[\underbrace{P(A \cap E) + P(A^c \cap E)}_{= \cancel{P(E)}}\right]$$

$$E = (A \cap E) \cup (A^c \cap E)$$

mutually exc.

$$P(E) = P(A \cap E) + P(A^c \cap E)$$

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- $P(A \cap B \mid E) = P(B \cap A \mid E)$

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For any events A,B,and E we have the following:

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- $P(A \mid E) = 1 - P(A^c \mid E)$
- $P(A \cap B \mid E) = P(B \cap A \mid E)$
- $P(A \cap B \mid E) = P(B \mid E)P(A \mid B \cap E)$

Properties of conditional probabilities...

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For any events A,B,and E we have the following:

- $0 \leq P(A \cap E) \leq 1$
- $P(A \mid E) = 1 - P(A^c \mid E)$
- $P(A \cap B \mid E) = P(B \cap A \mid E)$
- $P(A \cap B \mid E) = P(B \mid E)P(A \mid B \cap E)$
- $P(A \mid B \cap E) = \dfrac{P(B \mid A \cap E)P(A \mid E)}{P(B \mid E)}$

Scratch Space for Proving Conditional Probabilities...

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Outline

1. Conditional Probability, Bayes Theorem
2. Conditional Independence
3. Random Variables
4. Expectation
5. Saint Petersberg Paradox

Conditional Independence...

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Definition of conditional independence

Two events A and B are conditionally independent given E if

$$P(A \cap B|E) = P(A \mid E)P(B \mid E)$$

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Two events A and B are conditionally independent given E if

$$P(A \cap B|E) = P(A \mid E)P(B \mid E)$$

Fact on Conditional Independence

A and B independent does not mean that A and B are independent given E. That is,

$$\underbrace{P(A \cap B) = P(A)P(B)}_{} \not\Longrightarrow P(A \cap B \mid E) = P(A \mid E)P(B \mid E)$$

Quiz on independence...

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Quiz-1

Two events E and F are independent if

1. Knowing that F happens means that E can't happen

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2. Knowing that F happens doesn't change probability that E happened.

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What is your answer?

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Quiz-1

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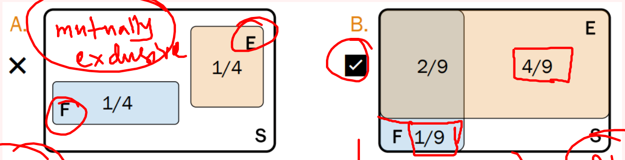
What is your answer?

Quiz-1

Are E and F independent in the following pictures (not to scale)?



mutually exclusive

F, E ind. $\Rightarrow P(E \cap F) = P(E)P(F) \Rightarrow E, F$ not ind.

$0 = \frac{1}{4} \cdot \frac{1}{4}$ ✗

$P(E \cap F) = 2/9$

$P(E) = 6/9, \; P(F) = \frac{3}{9}$

$P(E)P(F)$

$= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

$\Rightarrow E, F$ are ind.

Mutually Exclusive and Independent Events...

Mutually Exclusive and Independent Events...

Quiz

When are two events both mutually exclusive and independent?

More Problems on Independent Trials...

Problem: String-part 1

There are m strings that are hashed unequally into a hash table with n buckets. Each string hashed is an independent trial with probability p_i of getting hashed into bucket i.What is $P(E)$ if

- $E =$ bucket 1 has ≥ 1 string hashed into it?

More Problems on Independent Trials...

Problem: String-part 2

There are m strings that are hashed unequally into a hash table with n buckets. Each string hashed is an independent trial with probability p_i of getting hashed into bucket i.What is $P(E)$ if

- $E =$ at least 1 of buckets 1 to k has ≥ 1 string hashed into it?

Outline

Random Variables

Python $\leftarrow$ JIT

Random Variables

Examples of typed variables in C

In some languages, such as, C/C++. We have the concept of a typed variable:

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In some languages, such as, C/C++. we have the concept of a typed variable:

- int $i = 4$;
- float $x = 10$;
- char $y =$ 'x';

Random Variables

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Examples of random variable

Let X denote the outputs after we roll a die, then

$$X = 3$$

$P(\{X = 3\})$

means that after rolling a die, we obtain 3 as output.

Random Variables

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Examples of random variable

Let X denote the outputs after we roll a die, then

$$X = 3$$

means that after rolling a die, we obtain 3 as output.
Since the number that is going to be assigned to variable X is going to be random, it is called random variable.

Define Random Variable

Define Random Variable

$X : \{1, \ldots\} - 1, 2 \ldots$ $x = 1 \Leftarrow$ $x = 2 \Leftarrow$ $P\{x = -2\} \in [0, 1]$

Definition of Random Variable

A random variable X is a function from the sample space to the real numbers.

$$X : \underline{S} \to \mathbb{R}$$

P

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- The range of a random variable is the set of possible values of X

$X = \{0, 1, 2, 3, ..., 10\}$

Examples of Random Variables...

Find the range of the following random variables:

- I toss a coin 10 times. Let X be the number of heads I observe

Define Random Variable

Q^c Q

Definition of Random Variable

A random variable X is a function from the sample space to the real numbers.

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range of $Y = \mathbb{N}$

countable set

- We usually denote random variables by capital letters: X,Y,...
- Random variable is a function with domain S and co-domain $\mathbb{R}$
- The range of a random variable is the set of possible values of X

$Y = \{1, 2, 3, \ldots\} \subset \mathbb{R}$

Examples of Random Variables...

Find the range of the following random variables:

- I toss a coin 10 times. Let X be the number of heads I observe
- I toss a coin until the first tail appears. Let Y be the total number of coin tosses

Quiz on Random Variable...

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Consider and Experiment: 3 coins are flipped. Let X be the number of tails. Answer the following:

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Consider and Experiment: 3 coins are flipped. Let X be the number of tails. Answer the following:

- What is the value of X for the outcomes?
  - $(H, H, H) \leftarrow$ 0
  - $(T, T, H) \leftarrow$ 2

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Consider and Experiment: 3 coins are flipped. Let X be the number of tails. Answer the following:

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  - (T, T, H)
- What is the event when $X = 2$?

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Consider and Experiment: 3 coins are flipped. Let X be the number of tails. Answer the following:

- What is the value of X for the outcomes?
  - (H, H, H)
  - (T, T, H)
- What is the event when $X = 2$? = event that we get 2 tails when 3 coins are tossed.
- What is $P(X = 2)$?

$$\{X=2\} = \{\underbrace{(TTH), (HTT), (THT)}\}$$

$$2 \cdot 2 \cdot 2 = 8$$

$$P(X=2) = \frac{3}{8}$$

Random Variables are Not Events!

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Remarks on Random variables

- random variables are not events!

Random Variables are Not Events!

Remarks on Random variables

- random variables are not events!
- when a random variable is assigned a value, then it becomes event

$X = x$	Set of Outcomes	$P(X = k)$
$X = 0$	$\{(T,T,T)\}$	1/8
$X = 1$	$\{(H,T,T),(T,H,T),(T,T,H)\}$	3/8
$X = 2$	$\{(H,H,T),(H,T,H),(T,H,T)\}$	3/8
$X = 3$	$\{(H,H,H)\}$	1/8
$X \geq 4$	$\{\ \}$	0

Table: Consider an experiment where 3 coins are flipped, and X denotes number of heads

Discrete Random Variables...

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$\Sigma \quad \longleftrightarrow \quad \int$

Recall: countable sets

A set A is countable if either it is a finite set, or it can be put in 1-1 correspondence with set of natural numbers.

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There are three types of random variables:

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There are three types of random variables:

1. discrete random variables
2. continuous random variables

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3. mixed random variables