MATH3431 Machine Learning and Neural Networks III Epiphany term 2024

Exercise sheet

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Part 1. **Elements of convex learning problems**

Exercise 1. ($\star$)Let $f:\mathbb{R}^{d}\rightarrow\mathbb{R}$ such that $f\left(w\right)=g\left(<w,x>+y\right)$ or some $x\in\mathbb{R}^{d}$, $y\in\mathbb{R}$. Show that: If g is convex function then $f$is convex function.

Solution. Let $u,v\in\mathbb{R}^{d}$ and $a\in[0,1]$. It is

$$\begin{aligned}
f\left(\alpha u+\left(1-\alpha\right)v\right)= & g\left(<\alpha u+\left(1-\alpha\right)v,x>+y\right)\\
= & g\left(<\alpha u,x>+<\left(1-\alpha\right)v,x>+y\right)\\
= & g\left(\alpha\left(<u,x>+y\right)+\left(1-\alpha\right)\left(<v,x>+y\right)\right) & & y=\alpha y+\left(1-\alpha\right)y\\
\le & \alpha g\left(<u,x>+y\right)+\left(1-\alpha\right)g\left(<v,x>+y\right) & & (g\text{ is convex})\\
= & \alpha f\left(u\right)+\left(1-\alpha\right)f\left(v\right)
\end{aligned}$$

Exercise 2. ($\star$)Let functions g_{1} be ρ_{1}-Lipschitz and g_{2} be ρ_{2}-Lipschitz. Then, show that, f with $f\left(x\right)=g_{1}\left(g_{2}\left(x\right)\right)$ is $\rho_{1}\rho_{2}$-Lipschitz.

Solution.

$$\begin{aligned}
\left|f\left(w_{1}\right)-f\left(w_{2}\right)\right|= & \left|g_{1}\left(g_{2}\left(w_{1}\right)\right)-g_{1}\left(g_{2}\left(w_{2}\right)\right)\right|\\
\le & \rho_{1}\left|g_{2}\left(w_{1}\right)-g_{2}\left(w_{2}\right)\right|\\
\le & \rho_{1}\rho_{2}\left|w_{1}-w_{2}\right|
\end{aligned}$$

Exercise 3. ($\star$)Let $f:\mathbb{R}^{d}\rightarrow\mathbb{R}$ with $f\left(w\right)=g\left(\langle w,x\rangle+y\right)$ $x\in\mathbb{R}^{d}$ and $y\in\mathbb{R}$. Let $g:\mathbb{R}\rightarrow\mathbb{R}$ be a β-smooth function. Then show that f is a $\left(\beta\left\Vert x\right\Vert ^{2}\right)$-smooth.

Hint:: You may use Cauchy-Schwarz inequality $\langle y,x\rangle\le\left\Vert y\right\Vert \left\Vert x\right\Vert$

$$
\begin{aligned}
f(v) =& g(\langle w, x\rangle + y) \\
\le& g(\langle w, x\rangle + y) + g'(\langle w, x\rangle + y)\langle v - w, x\rangle + \frac{\beta}{2}(\langle v - w, x\rangle)^2 && (g \text{ is smooth}) \\
\le& g(\langle w, x\rangle + y) + g'(\langle w, x\rangle + y)\langle v - w, x\rangle + \frac{\beta}{2}(\|v - w\| \|x\|)^2 && \text{(Cauchy-Schwatz inequality)} \\
=& f(w) + \langle \nabla f(w), v - w\rangle + \frac{\beta \|x\|^2}{2}\|v - w\|^2
\end{aligned}
$$

Exercise 4. ($\star$)Show that $f : S \to \mathbb{R}$ is ρ-Lipschitz over an open convex set S if and only if for all $w \in S$ and $v \in \partial f(w)$ it is $\|v\| \le \rho$.

Hint:: You may use Cauchy-Schwarz inequality $\langle y, x\rangle \le \|y\| \|x\|$

Solution. $\Longrightarrow$ Let $f : S \to \mathbb{R}$ be ρ-Lipschitz over convex set S, $w \in S$ and $v \in \partial f(w)$.

- Since S is open we get that there exist $\epsilon > 0$ such as $u := w + \epsilon \frac{v}{\|v\|}$ where $u \in S$. So $\langle u - w, v\rangle = \epsilon \|v\|$ and $\|u - w\| = \epsilon$.
- From the subgradient definition we get

$$f(u) - f(w) \ge \langle u - w, v\rangle = \epsilon \|v\|$$

- From the Lipschitzness of $f(\cdot)$ we get

$$f(u) - f(w) \le \rho \|u - w\| = \rho\epsilon$$

  Therefore $\|v\| \le \rho$.

  $\Longleftarrow$ It is for all $w \in S$ and $v \in \partial f(w)$ it is $\|v\| \le \rho$.
- For any $u \in S$, it is

$$
\begin{aligned}
f(w) - f(u) \le& \langle v, w - u\rangle && (\text{ because } v \in \partial f(w)\ ) \\
\le& \|v\| \|w - u\| && \text{by Cauchy-Schwarz inequality} \\
\le& \rho \|w - u\| && \text{because } \|v\| \le \rho
\end{aligned}
\tag{1}
$$

- Similarly it results $u, w \in S$

$$f(w) - f(u) \le \langle v, u - w\rangle \|v\| \le \|v\| \|u - w\| \le \rho \|u - w\|$$

  from (1) because w, u can be swaped in (1) as they both are any values in S.

Exercise 5. ($\star$)Let $g_1(w), ..., g_r(w)$ be r convex functions, and let $f(\cdot) = \max_{\forall j}(g_j(\cdot))$. Show that for some w it is $\nabla g_k(w) \in \partial f(w)$ where $k = \arg\max_j(g_j(w))$ is the index of function $g_j(\cdot)$ presenting the greatest value at w.

Solution. Since g_k is convex, for all u

$$g_k(u) \geq g_k(w) + \langle u - w, \nabla g_k(w) \rangle$$

However $f(u) = \max_{\forall j}(g_j(u)) \geq g_k(u)$ for any j, and $f(w) = g_k(w)$ at w. Then

$$\begin{aligned} f(u) \geq & g_k(u) \\ \geq & g_k(w) + \langle u - w, \nabla g_k(w) \rangle \\ = & f(w) + \langle u - w, \nabla g_k(w) \rangle \end{aligned}$$

Then by the definition of the sub-gradient $\nabla g_k(w) \in \partial f(w)$

Exercise 6. ($\star$)Consider the regression learning problem $(\mathcal{H}, \mathcal{Z}, \ell)$ with predictor rule $h(x) = \langle w, x \rangle$ labeled by some unknown parameter $w \in \mathcal{W}$, loss function $\ell(w, (x, y)) = (\langle w, x \rangle - y)^2$, feature $x \in \mathcal{X}$, and target $y \in \mathbb{R}$. Let $\mathcal{W} = \mathcal{X} = \left\{\omega \in \mathbb{R}^d : |\omega| \leq \rho\right\}$ for some $\rho > 0$.

(1) Show that the resulting learning problem is Convex-Lipschitz-Bounded learning problem.
(2) Specify the parameters of Lipschitnzess.

Solution. According to the definitions given in the lecture:

- Convex-Lipschitz-Bounded Learning Problem $(\mathcal{H}, \mathcal{Z}, \ell)$ with parameters ρ, and B, is called the learning problem whose the hypothesis class $\mathcal{H}$ is a convex set, for all $w \in \mathcal{H}$ it is $\|w\| \leq B$, and the loss function $\ell(\cdot, z)$ is convex and ρ-Lipschitz function for all $z \in \mathcal{Z}$.

I have:

Convexity: The function $g : \mathbb{R} \to \mathbb{R}$, defined by $g(a) = a^2$ is convex convex. Eg. $\frac{\mathrm{d}^2}{\mathrm{d}a^2} g(a) = 1 \geq 0$ is non-negative. The convexity of $\ell(w, z = (x, y))$ for all z follows as a composition of g with a linear function.

Lipschitzness: The function $g(a) = a^2$ is 1-Lipschitz since It is

$$|g(a_2) - g(a_1)| = \left|a_2^2 - a_1^2\right| = |(a_2 + a_1)(a_2 - a_1)| \leq 2\rho(a_2 - a_1) = 2\rho |a_2 - a_1|$$

Hence because $|x| \leq \rho$, $g(a)$ is $2\rho^2$-Lipschitz as a composition.

Boundness: The norm of each hypothesis w is bounded by ρ according to the assumptions.

Therefore,

(1) the learning problem under consideration is a Convex-Lipschitz-Bounded learning problem.
(2) the parameter of Lipschitzness is $2\rho^2$.

Exercise 7. ($\star$)If $f$is λ-strongly convex and u is a minimizer of f then for any w

$$f(w) - f(u) \geq \frac{\lambda}{2} \|w - u\|^2$$

Hint:: Use the definition, and set $\alpha \to 0$.

Solution.

The following is given as a homework (Formative assessment 1)

Exercise 8. ($\star$) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex and β-smooth function.

(1) Show that for $v, w \in \mathbb{R}^d$

$$f(v) - f(w) \in \left(\langle \nabla f(w), v - w \rangle, \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2 \right)$$

(2) Show that for $v, w \in \mathbb{R}^d$ such that $v = w - \frac{1}{\beta} \nabla f(w)$, it is

$$\frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - f(v)$$

(3) Additionally assume that $f(x) > 0$ for all $x \in \mathbb{R}^d$. Show that for $w \in \mathbb{R}^d$,

$$\|\nabla f(w)\| \leq \sqrt{2\beta f(w)}$$

Solution.

(1) If $f : \mathbb{R}^d \to \mathbb{R}$ is β-smooth then it is

$$f(v) \leq f(w) + \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2$$
$$f(v) - f(w) \leq \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2$$

If it is convex then it is

$$f(v) \geq f(w) + \langle \nabla f(w), v - w \rangle$$
$$f(v) - f(w) \geq \langle \nabla f(w), v - w \rangle$$

Together these conditions imply upper and lower bounds

$$f(v) - f(w) \in \left(\langle \nabla f(w), v - w \rangle, \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2 \right)$$

(2) For $v, w \in \mathbb{R}^d$ such that $v = w - \frac{1}{\beta} \nabla f(w)$, it is

$$f(v) \leq f(w) + \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|_2^2 \quad \text{(due to smoothness)}$$
$$\iff f(w) - f(v) \leq f(w) - f(v)$$
$$\iff \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|_2^2 \leq f(w) - f(v)$$
$$\iff \left\langle \nabla f(w), \frac{1}{\beta} \nabla f(w) \right\rangle + \frac{\beta}{2} \left\| \frac{1}{\beta} \nabla f(w) \right\|_2^2 \leq f(w) - f(v)$$
$$\iff \frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - f(v)$$
$$\|\nabla f(w)\|^2 \leq 2\beta (f(w) - f(v))$$

as $f(\cdot) \geq 0$

$$\|\nabla f(w)\|^2 \leq 2\beta f(w)$$

(3) From part 2, this is obvious because $f(x) > 0$ for all $x \in \mathbb{R}^d$, as

$$\|\nabla f(w)\|^2 \le 2\beta f(w) \Leftrightarrow \|\nabla f(w)\| \le \sqrt{2\beta f(w)}$$

The following is given as a homework (Formative assessment 1)

Exercise 9. ($\star$)Let $f : \mathbb{R}^d \to \mathbb{R}$ be a λ-strongly convex function. Assume that w^* is a minimizer of f i.e.

$$w^* = \arg\min_w \{f(w)\}$$

Show that for any $w \in \mathbb{R}^d$ it holds

$$f(w) - f(w^*) \ge \frac{\lambda}{2}\|w - w^*\|^2$$

Hint: Use the definition of λ-strongly convex function, properly rearrange it, and ...

Solution. We use the definition of λ-strongly convex function; i.e. for all w, u, and $\alpha \in (0,1)$ we have

$$f(aw + (1-\alpha)u) \le af(w) + (1-\alpha)f(u) - \frac{\lambda}{2}\alpha(1-\alpha)\|w-u\|^2 \Leftrightarrow$$
$$\frac{f(aw + (1-\alpha)u) - f(u)}{\alpha} \le f(w) + f(u) - \frac{\lambda}{2}(1-\alpha)\|w-u\|^2$$

For $u = w^*$ it is

$$\frac{f(aw + (1-\alpha)w^*) - f(w^*)}{\alpha} \le f(w) + f(w^*) - \frac{\lambda}{2}(1-\alpha)\|w-w^*\|^2$$

When $a \to 0$

$$\frac{\lambda}{2}\alpha(1-\alpha)\|w-w^*\|^2 \to 0$$

I know that w^* is the minimizer of f. So 0 is the minimizer of g with $g(a) = f(aw + (1-\alpha)w^*)$ hencewhen $a \to 0$

$$\frac{f(aw + (1-\alpha)w^*) - f(w^*)}{\alpha} \to \left.\frac{\mathrm{d}}{\mathrm{d}a}g(a)\right|_{a=0}$$

So

$$0 \le f(w) + f(w^*) - \frac{\lambda}{2}\|w-w^*\|^2$$

which concludes the proof.

Exercise 10. ($\star$)Show that the function $J(x;\lambda) = \lambda\|x\|^2$ is 2λ-strongly convex

Solution. We just need to check that for all w, u, and $\alpha \in (0,1)$ we have

$$J(aw + (1-\alpha)u;\lambda) \le aJ(w;\lambda) + (1-\alpha)J(u;\lambda) - \frac{2\lambda}{2}\alpha(1-\alpha)\|w-u\|^2 \Longleftrightarrow$$
$$\|aw + (1-\alpha)u\|_2^2 \le a\|w\|_2^2 + (1-\alpha)\|u\|_2^2 - a(1-\alpha)\|w-u\|_2^2 \Longleftrightarrow 0 \le 0$$

Part 2. **Stochastic learning**

Exercise 11. ($\star$) Assume a Bayesian model

$$\begin{cases} z_i|w & \overset{\text{ind}}{\sim} f(z_i|w), \ i=1,...,n \\ w & \sim f(w) \end{cases}$$

and consider that our objective is the discovery of MAP estimate w^* i.e.

$$w^* = \arg\min_{\forall w \in \Theta} \left(-\log\left(L_n(w)\right) - f(w)\right) = \arg\min_{\forall w \in \Theta} \left(-\sum_{i=1}^{n} \log\left(f(z_i|w)\right) - \log\left(f(w)\right)\right)$$

by using SGD with update

$$w^{(t+1)} = w^{(t)} + \eta_t \left(\frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right) + \nabla_w \log\left(f\left(w^{(t)}\right)\right)\right)$$

for some randomly selected set $\mathcal{J}^{(t)} \subseteq \{1,...,n\}^m$ of m integers from 1 to n via simple random sampling (SRS) with replacement. Show that

$$\mathrm{E}_{\mathcal{J}^{(t)} \sim \text{simple-random-sampling}} \left(\frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right)\right) = \sum_{i=1}^{n} \nabla_w \log\left(f\left(z_i|w^{(t)}\right)\right)$$

Solution. It is

$$\begin{aligned}
\mathrm{E}_{\mathcal{J}^{(t)} \sim \mathrm{SRS}} \left(\frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right)\right) &= \frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \mathrm{E}_{\mathcal{J}^{(t)} \sim \mathrm{SRS}} \left(\nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right)\right) \\
&= \frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \mathrm{E}_{\mathcal{J}^{(t)} \sim \mathrm{SRS}} \left(\nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right)\right) \\
&= \frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \frac{1}{n} \sum_{i=1}^{n} \nabla_w \log\left(f\left(z_i|w^{(t)}\right)\right) \\
&= \sum_{i=1}^{n} \nabla_w \log\left(f\left(z_i|w^{(t)}\right)\right)
\end{aligned}$$

It is $\mathrm{E}_{\mathcal{J}^{(t)} \sim \mathrm{SRS}} \left(\nabla_w \log\left(f\left(z_j|w^{(t)}\right)\right)\right) = \frac{1}{n} \sum_{i=1}^{n} \nabla_w \log\left(f\left(z_i|w^{(t)}\right)\right)$ because the expectation is under the probability I get randomly an integer and for the jth on the probability is $1/n$ due to the random scheme. Also $\left|\mathcal{J}^{(t)}\right| = m$.

Exercise 12. ($\star$) Let $\{v_t; t=1,...,T\}$ be a sequence of vectors. Consider an algorithm producing $\left\{w^{(t)}; t=1,2,3,...\right\}$ with

$$w^{(1)} = 0$$
$$w^{(t+1)} = w^{(t)} - \eta v_t$$

Show that

(1) it is

$$\langle w^{(t)} - w^*, v_t \rangle = \frac{1}{2\eta}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|^2\right) + \frac{\eta}{2}\|v_t\|^2$$

(2) it is

$$\sum_{t=1}^{T}\langle w^{(t)} - w^*, v_t \rangle = \frac{1}{2\eta}\sum_{t=1}^{T}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|^2\right) + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2$$

(3) (continue) it is

$$\sum_{t=1}^{T}\langle w^{(t)} - w^*, v_t \rangle \leq \frac{\|w^*\|^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2$$

Solution.

(1) It is

$$\begin{aligned}
\left\langle w^{(t)} - w^*, v_t \right\rangle &= \frac{1}{\eta}\left\langle w^{(t)} - w^*, \eta v_t \right\rangle \\
&= \frac{1}{2\eta}\left(-\left\|w^{(t)} - w^* - \eta v_t\right\|^2 + \left\|w^{(t)} - w^*\right\| + \eta^2\|v_t\|^2\right) \\
&= \frac{1}{2\eta}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\| + \eta^2\|v_t\|^2\right) \\
&= \frac{1}{2\eta}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|\right) + \frac{\eta}{2}\|v_t\|^2
\end{aligned}$$

(2) So

$$\begin{aligned}
\sum_{t=1}^{T}\left\langle w^{(t)} - w^*, v_t \right\rangle &= \frac{1}{2\eta}\sum_{t=1}^{T}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|\right) + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2 \\
&= \frac{1}{2\eta}\left(\left\|w^{(1)} - w^*\right\|^2 - \left\|w^{(T+1)} - w^*\right\|^2\right) + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2
\end{aligned}$$

(3) So

$$\begin{aligned}
\sum_{t=1}^{T}\left\langle w^{(t)} - w^*, v_t \right\rangle &= \frac{1}{2\eta}\left(\left\|w^{(1)} - w^*\right\|^2 - \left\|w^{(T+1)} - w^*\right\|^2\right) + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2 \\
&\leq \frac{1}{2\eta}\left\|w^{(1)} - w^*\right\|^2 + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2 \\
&= \frac{1}{2\eta}\|w^*\|^2 + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2
\end{aligned}$$

Exercise 13. ($\star$) Let $\{v_t; t = 1, ..., T\}$ be a sequence of vectors. Consider an algorithm producing $\left\{w^{(t)}; t = 1, 2, 3, ...\right\}$ with

$$w^{(1)} = 0$$

$$w^{\left(t+\frac{1}{2}\right)} = w^{(t)} - \eta v_t$$

$$w^{(t+1)} = \arg\min_{w\in\mathcal{H}} \left(\left\| w - w^{\left(t+\frac{1}{2}\right)} \right\|\right)$$

for $t = 1, ..., T$.

Hint: You can use the following Lemma

(Projection Lemma): Let $\mathcal{H}$ be a closed convex set and let v be the projection of w onto $\mathcal{H}$,i.e.

$$v = \arg\min_{x\in\mathcal{H}} \|x - w\|^2$$

then for every $u \in \mathcal{H}$ it is

$$\|v - u\|^2 \leq \|w - u\|^2$$

Show that

(1) it is

$$\langle w^{(t)} - w^*, v_t\rangle \leq \frac{1}{2\eta}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|^2\right) + \frac{\eta}{2}\|v_t\|^2$$

(2) it is

$$\sum_{t=1}^{T}\langle w^{(t)} - w^*, v_t\rangle \leq \frac{1}{2\eta}\sum_{t=1}^{T}\left(-\left\|w^{(t+1)} - w^*\right\|^2 + \left\|w^{(t)} - w^*\right\|^2\right) + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2$$

(3) (continue) it is

$$\sum_{t=1}^{T}\langle w^{(t)} - w^*, v_t\rangle \leq \frac{\|w^*\|^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\|v_t\|^2$$

Comment: Above we show that Lemma **??** from "Handout **??**: Gradient descent" holds even when a projection step is included. Hence, even if a projection step is included after the update step of the recursion of GD algorithm or the SGD algorithm the analysis in Section **??** in "Handout **??**: Gradient descent" holds. Hence, even if a projection step is included after the update step of the recursion of SGD algorithm or the SGD algorithm the analysis in Section **??** in "Handout **??**: Stochastic gradient descent" holds.

Solution.

(1) It is

$$
\begin{aligned}
\left\langle w^{(t)} - w^{*}, v_t \right\rangle &= \frac{1}{\eta} \left\langle w^{(t)} - w^{*}, \eta v_t \right\rangle \\
&= \frac{1}{2\eta} \left(- \left\| w^{(t)} - w^{*} - \eta v_t \right\|^2 + \left\| w^{(t)} - w^{*} \right\| + \eta^2 \left\| v_t \right\|^2 \right) \\
&= \frac{1}{2\eta} \left(- \left\| w^{\left(t+\frac{1}{2}\right)} - w^{*} \right\|^2 + \left\| w^{(t)} - w^{*} \right\| + \eta^2 \left\| v_t \right\|^2 \right) \\
&= \frac{1}{2\eta} \left(- \left\| w^{\left(t+\frac{1}{2}\right)} - w^{*} \right\|^2 + \left\| w^{(t)} - w^{*} \right\| \right) + \frac{\eta}{2} \left\| v_t \right\|^2 \\
&\leq \frac{1}{2\eta} \left(- \left\| w^{(t+1)} - w^{*} \right\|^2 + \left\| w^{(t)} - w^{*} \right\| \right) + \frac{\eta}{2} \left\| v_t \right\|^2
\end{aligned}
$$

because from the Projection Lemma

$$
\left\| w^{(t+1)} - w^{*} \right\|^2 \leq \left\| w^{\left(t+\frac{1}{2}\right)} - w^{*} \right\|^2
$$

(2) So

$$
\begin{aligned}
\sum_{t=1}^{T} \left\langle w^{(t)} - w^{*}, v_t \right\rangle &\leq \frac{1}{2\eta} \sum_{t=1}^{T} \left(- \left\| w^{(t+1)} - w^{*} \right\|^2 + \left\| w^{(t)} - w^{*} \right\| \right) + \frac{\eta}{2} \sum_{t=1}^{T} \left\| v_t \right\|^2 \\
&= \frac{1}{2\eta} \left(\left\| w^{(1)} - w^{*} \right\|^2 - \left\| w^{(T+1)} - w^{*} \right\|^2 \right) + \frac{\eta}{2} \sum_{t=1}^{T} \left\| v_t \right\|^2
\end{aligned}
$$

(3) So

$$
\begin{aligned}
\sum_{t=1}^{T} \left\langle w^{(t)} - w^{*}, v_t \right\rangle &\leq \frac{1}{2\eta} \left(\left\| w^{(1)} - w^{*} \right\|^2 - \left\| w^{(T+1)} - w^{*} \right\|^2 \right) + \frac{\eta}{2} \sum_{t=1}^{T} \left\| v_t \right\|^2 \\
&\leq \frac{1}{2\eta} \left\| w^{(1)} - w^{*} \right\|^2 + \frac{\eta}{2} \sum_{t=1}^{T} \left\| v_t \right\|^2 \\
&= \frac{1}{2\eta} \left\| w^{*} \right\|^2 + \frac{\eta}{2} \sum_{t=1}^{T} \left\| v_t \right\|^2
\end{aligned}
$$

The following is given as a homework (Formative assessment 2)

Exercise 14. ($\star$) [1]Consider the binary classification problem with inputs $x \in \mathcal{X}$ where $\mathcal{X} := \left\{ x \in \mathbb{R}^d : \|x\|_2 \leq L \right\}$ for some given value $L > 0$, target $y \in \mathcal{Y}$ where $\mathcal{Y} := \{-1, +1\}$, and prediction

[1]We use standard notation

$$
\operatorname{sign}(\xi) = \begin{cases} -1, & \text{if}\, \xi < 0 \\ +1, & \text{if}\, \xi > 0 \end{cases}
$$

± 1 means either -1 or $+1$, $\mathbb{R}_{+} := (0, +\infty)$, and $\|x\|_2 := \sqrt{\sum_{\forall j} (x_j)^2}$ for the Euclidean distance.

rule $h_w : \mathbb{R}^d \to \{-1, +1\}$ with

$$h_w(x) = \operatorname{sign}\left(w^\top x\right) \tag{2}$$

$$= \operatorname{sign}\left(\sum_{j=1}^{d} w_j x_j\right) \tag{3}$$

Let the hypothesis class is

$$\mathcal{H} = \left\{x \to w^\top x : \forall w \in \mathbb{R}^d\right\} \tag{4}$$

In other words, the hypothesis $h_w \in \mathcal{H}$ is parametrized by $w \in \mathbb{R}^d$, it receives an input vector $x \in \mathcal{X} := \mathbb{R}^d$ and it returns the label $y = \operatorname{sign}\left(w^\top x\right) \in \mathcal{Y} := \{\pm 1\}$ where

$$\operatorname{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0 \\ +1, & \text{if } \xi > 0 \end{cases}$$

Consider a loss function $\ell : \mathbb{R}^d \to \mathbb{R}_+$ with

$$\ell(w, z = (x, y)) = \max\left(0, 1 - y w^\top x\right) + \lambda \|w\|_2^2 \tag{5}$$

for some given value $\lambda > 0$.

Assume there is available a dataset of examples $S_n = \{z_i = (x_i, y_i)\,;\, i = 1, ..., n\}$ of size n.

Do the following:

(1) Show that the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1 - x)$ is convex in $\mathbb{R}$; and show that the loss (5) is convex.

Hint:: You may use Proposition **??** from Handout **??**: Elements of convex learning problems.

(2) Show that the loss $\ell(w, z)$ for $\lambda = 0$ (5) is L-Lipschitz (with respect to w) when $x \in \mathcal{X}$ where $\mathcal{X} := \left\{x \in \mathbb{R}^d : \|x\|_2 \le L\right\}$.

Hint:: You may use the definition of Lipschitz function. Without loss of generality, you can consider any $w_1 \in \mathbb{R}^d$ and $w_2 \in \mathbb{R}^d$ such that $1 - y w_2^\top x \le 1 - y w_1^\top x$, and then take cases $1 - y w_2^\top x >$ or < 0 and $1 - y w_1^\top x >$ or < 0 to deal with the max.

(3) Construct the set of sub-gradients $\partial f(x)$ for $x \in \mathbb{R}$ of the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1 - x)$. Show that the vector v with

$$v = \begin{cases} 2\lambda w, & y w^\top x > 1 \\ 2\lambda w, & y w^\top x = 1 \\ -yx + 2\lambda w, & y w^\top x < 1 \end{cases}$$

is $v \in \partial_w \ell(w, z = (x, y))$, aka a sub-gradient of $\ell(w, z = (x, y))$ at w, for any $w \in \mathbb{R}^d$.

(4) Write down the algorithm of online AdaGrad (Adaptive Stochastic Gradient Descent) with learning rate $\eta_t > 0$, batch size m, and termination criterion $t > T_{\max}$ for some $T_{\max} > 0$ in

order to discover w^* such as

$$w^* = \arg \min_{\forall w : h_w \in \mathcal{H}} \left(\mathrm{E}_{z \sim g} \left(\ell \left(w, z = (x, y) \right) \right) \right) \tag{6}$$

The formulas in your algorithm shoud be implemented for the above learning problem and tailored to 2, 4, and 5.

(5) Use the R code given below in order to generate the dataset of observed examples $S_n = \{z_i = (x_i, y_i)\}_{i=1}^{n}$ that contains $n = 10^6$ examples with inputs x of dimension $d = 2$. Consider $\lambda = 0$. Use a seed $w^{(0)} = (0, 0)^\top$.

(a) By using appropriate values for m, η_t and $T_{\max}$, code in R the algorithm you designed in part 4, and run it.

(b) Plot the trace plots for each of the dimensions of the generated chain $\left\{ w^{(t)} \right\}$ against the iteration t.

(c) Report the value of the output w^*_{adaGrad} (any type) of the algorithm as the solution to (6).

(d) To which cluster y (i.e., -1 or 1) $x_{\text{new}} = (1, 0)^\top$ belongs?

```
# R code. Run it before you run anything else
#
data_generating_model <- function(n,w) {
z <- rep( NaN, times=n*3 )
z <- matrix(z, nrow = n, ncol = 3)
z[,1] <- rep(1,times=n)
z[,2] <- runif(n, min = -10, max = 10)
p <- w[1]*z[,1] + w[2]*z[,2] p <- exp(p) / (1+exp(p))
z[,3] <- rbinom(n, size = 1, prob = p)
ind <- (z[,3]==0)
z[ind,3] <- -1
x <- z[,1:2]
y <- z[,3]
return(list(z=z, x=x, y=y))
}
n_obs <- 1000000
w_true <- c(-3,4)
set.seed(2023)
out <- data_generating_model(n = n_obs, w = w_true)
set.seed(0)
z_obs <- out$z #z=(x,y)
x <- out$x
y <- out$y
#z_obs2=z_obs
#z_obs2[z_obs[,3]==-1,3]=0
#w_true <- as.numeric(glm(z_obs2[,3]~ 1+ z_obs2[,2],family = "binomial"
)$coefficients)
```

Solution.
