

**Homework 1**

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As formative assessment, submit the solutions to all the Exercises

**Exercise 1.** (★) Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a convex and  $\beta$ -smooth function.(1) Show that for  $v, w \in \mathbb{R}^d$ 

$$f(v) - f(w) \in \left( \langle \nabla f(w), v - w \rangle, \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2 \right)$$

(2) Show that for  $v, w \in \mathbb{R}^d$  such that  $v = w - \frac{1}{\beta} \nabla f(w)$ , it is

$$\frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - f(v)$$

(3) Additionally assume that  $f(x) > 0$  for all  $x \in \mathbb{R}^d$ . Show that for  $w \in \mathbb{R}^d$ ,

$$\|\nabla f(w)\| \leq \sqrt{2\beta f(w)}$$

**Solution.**(1) If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\beta$ -smooth then it is

$$f(v) \leq f(w) + \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2$$

$$f(v) - f(w) \leq \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2$$

If it is convex then it is

$$f(v) \geq f(w) + \langle \nabla f(w), v - w \rangle$$

$$f(v) - f(w) \geq \langle \nabla f(w), v - w \rangle$$

Together these conditions imply upper and lower bounds

$$f(v) - f(w) \in \left( \langle \nabla f(w), v - w \rangle, \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2 \right)$$

(2) For  $v, w \in \mathbb{R}^d$  such that  $v = w - \frac{1}{\beta} \nabla f(w)$ , it is

$$\begin{aligned}
f(v) &\leq f(w) + \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|_2^2 \quad (\text{due to smoothness}) \\
&\iff f(w) - f(v) \leq f(w) - f(v) \\
&\iff \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|_2^2 \leq f(w) - f(v) \\
&\iff \left\langle \nabla f(w), \frac{1}{\beta} \nabla f(w) \right\rangle + \frac{\beta}{2} \left\| \frac{1}{\beta} \nabla f(w) \right\|_2^2 \leq f(w) - f(v) \\
&\iff \frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - f(v) \\
&\iff \|\nabla f(w)\|^2 \leq 2\beta (f(w) - f(v))
\end{aligned}$$

as  $f(\cdot) \geq 0$

$$\|\nabla f(w)\|^2 \leq 2\beta f(w)$$

(3) From part 2, this is obvious because  $f(x) > 0$  for all  $x \in \mathbb{R}^d$ , as

$$\|\nabla f(w)\|^2 \leq 2\beta f(w) \Leftrightarrow \|\nabla f(w)\| \leq \sqrt{2\beta f(w)}$$

**Exercise 2.** (\*\*) Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\lambda$ -strongly convex function. Assume that  $w^*$  is a minimizer of  $f$  i.e.

$$w^* = \arg \min_w \{f(w)\}$$

Show that for any  $w \in \mathbb{R}^d$  it holds

$$f(w) - f(w^*) \geq \frac{\lambda}{2} \|w - w^*\|^2$$

**Hint:** Use the definition of  $\lambda$ -strongly convex function, properly rearrange it, and ...

**Solution.** We use the definition of  $\lambda$ -strongly convex function; i.e. for all  $w, u$ , and  $\alpha \in (0, 1)$  we have

$$\begin{aligned}
f(aw + (1 - \alpha)u) &\leq af(w) + (1 - \alpha)f(u) - \frac{\lambda}{2}\alpha(1 - \alpha)\|w - u\|^2 \Leftrightarrow \\
\frac{f(aw + (1 - \alpha)u) - f(u)}{\alpha} &\leq f(w) + f(u) - \frac{\lambda}{2}(1 - \alpha)\|w - u\|^2
\end{aligned}$$

For  $u = w^*$  it is

$$\frac{f(aw + (1 - \alpha)w^*) - f(w^*)}{\alpha} \leq f(w) + f(w^*) - \frac{\lambda}{2}(1 - \alpha)\|w - w^*\|^2$$

When  $a \rightarrow 0$

$$\frac{\lambda}{2}\alpha(1 - \alpha)\|w - w^*\|^2 \rightarrow 0$$

I know that  $w^*$  is the minimizer of  $f$ . So 0 is the minimizer of  $g$  with  $g(a) = f(aw + (1 - \alpha)w^*)$  hence when  $a \rightarrow 0$

$$\frac{f(aw + (1 - \alpha)w^*) - f(w^*)}{\alpha} \rightarrow \left. \frac{d}{da} g(a) \right|_{a=0}$$

So

$$0 \leq f(w) + f(w^*) - \frac{\lambda}{2} \|w - w^*\|^2$$

which concludes the proof.