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Handout 6: Tools for inference under the presence of nuisance parameters

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References: [4, (Ch. 4)], [5, (Ch. 8)] , [2, (Ch. 10)], [3, (Ch. 4)], [1, (Ch. 22)]

Notation 1. Let $X, X_1, X_2, ..., X_n$ be a sequence of IID random variables (unseen observations) generated from a distribution f_θ labeled by a d-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^d$, and admitting PDF $f(\cdot|\theta)$.

Note 2. Assume the unknown d-dimensional parameter θ is partitioned as $\theta = (\psi, \phi)^\top$, by a d_ψ-dimensional $\psi \in \Psi \subset \mathbb{R}^{d_\psi}$, and d_ϕ-dimensional $\phi \in \Phi \subset \mathbb{R}^{d_\phi}$. Obviously $d = d_\psi + d_\phi$.

Definition 3. Given a statistical model $\left\{X_i \overset{\text{IID}}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, if we are interested in learning the sub-parameter (or parameter function) $\psi = \psi(\theta)$, but we do not care about $\phi = \phi(\theta)$, the sub-parameter (or parameter function) ψ is called the parameter of interest, and the sub-parameter (or parameter function ϕ is called the nuisance parameter.

Example 4. To motivate, consider the LR hypothesis test for comparing between two nested the log-linear models, $[X, YZ]$ and $[XY, YZ]$. Given a statistical model $\left\{X_i \overset{\text{IID}}{\sim} \text{Poi}(\mu(\boldsymbol{\lambda}))\right\}$, what we did was:

$$\begin{cases} \text{H}_0: & [X, YZ] \\ \text{H}_1: & [XY, YZ] \end{cases} \iff \begin{cases} \text{H}_0: & \log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{jk}^{YZ} \\ \text{H}_1: & \log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{jk}^{YZ} + \lambda_{jk}^{XY} \end{cases}$$

$$\iff \begin{cases} \text{H}_0: & \lambda_{jk}^{XY} = 0, \text{ and any } \lambda,\, \lambda_i^X,\, \lambda_j^Y,\, \lambda_k^Z,\, \lambda_{jk}^{YZ} \in \mathbb{R} \\ \text{H}_1: & \lambda_{jk}^{XY} \neq 0, \text{ and any } \lambda,\, \lambda_i^X,\, \lambda_j^Y,\, \lambda_k^Z,\, \lambda_{jk}^{YZ} \in \mathbb{R} \end{cases} \iff \begin{cases} \text{H}_0: & \psi = \psi_*, \text{ and } \forall\phi \\ \text{H}_1: & \psi \neq \psi_*, \text{ and } \forall\phi \end{cases}$$

where $\theta = \boldsymbol{\lambda} = (\psi, \phi)$ is the unknown parameter, $\psi = \psi(\boldsymbol{\lambda}) = (\lambda_{jk}^{XY})_{\forall i,j}$ is the parameter of interest, $\phi = \phi(\boldsymbol{\lambda}) = (\lambda,\, \lambda_i^X,\, \lambda_j^Y,\, \lambda_k^Z,\, \lambda_{jk}^{YZ})_{\forall i,j}$ is the nuisance parameter, and $\psi_* = 0$, is the test value. This LR test does not actually fall in the category of the original likelihood ratio test in (Handout 6) which considers $\text{H}_0 : \theta = \theta_*$ vs $\text{H}_0 : \theta \neq \theta_*$ because we do not infer about parameters $\phi = (\lambda,\, \lambda_i^X,\, \lambda_j^Y,\, \lambda_k^Z,\, \lambda_{jk}^{YZ})_{\forall i,j}$ which just cause inconvenience.

Note 5. To learn ψ from the data $X_i \overset{\text{IID}}{\sim} f(\cdot|\theta = (\psi, \phi))$, as well as consider uncertainty about the unknown ϕ, I need to construct appropriate pivotal quantities $Q(\psi, X_{1:n})$ for ψ as well as compute their sampling distribution which should not depend on the unknown nuisance ϕ. One can derive such statistics by "profiling out" ϕ and constructing corresponding Likelihood ratio, Wald, or Score statistics whose asymptotic distribution can be easily derived.

Definition 6. Given a likelihood $L_n(\theta)$ the profile likelihood $L_{n,p}(\psi)$ of ψ is

$$L_{n,p}(\psi) \quad = \sup_{\forall\phi} L_n(\underbrace{\psi, \phi}_{=\theta}) \quad = \quad L_n(\psi, \hat{\phi}_\psi)$$

where $\hat{\phi}_{\psi}$ denotes the MLE of ϕ as if ψ was a known parameter constant: i.e.

$$\hat{\phi}_{\psi} = \arg \sup_{\forall \phi \in \Phi} L_n(\psi, \phi)$$

Definition 7. The profile log-likelihood $\ell_{n,p}(\psi)$ of ψ, as

$$\ell_{n,p}(\psi) \quad = \quad \log(L_{n,p}(\psi)) \quad = \quad \log(L_n(\psi, \hat{\phi}_{\psi})) \quad = \quad \ell_n(\psi, \hat{\phi}_{\psi})$$

Note 8. Once the profile log-likelihood $L_{n,p}(\psi)$ of ψ is specified, then we can perform inference (point estimation, CI, HT, etc...) as usual but using $L_{n,p}(\psi)$.

1 Point estimation via profile maximum likelihood

Summary 9. The MLE $\hat{\psi} = \hat{\psi}(x_1, ..., x_n)$ of ψ by profiling out ϕ is the

$$\hat{\psi} = \arg \sup_{\forall \psi \in \Psi} \ell_{n,p}(\psi) \quad = \quad \arg \sup_{\forall \psi \in \Psi} \ell_n(\psi, \hat{\phi}_{\psi})$$

It can be found as follows:

1. Pretend that ψ is a known parameter and compute the MLE of ϕ

$$\hat{\phi}_{\psi} = \arg \sup_{\forall \phi \in \Phi} L_n(\psi, \phi)$$

   e.g. as a root of of the ML equations

$$0 = \frac{\mathrm{d}}{\mathrm{d}\phi} \ell_n(\psi, \phi)|_{\phi = \hat{\phi}_{\psi}}$$

2. Compute the profile MLE $\hat{\psi}$ (using the profile likelihood) as

$$\hat{\psi} = \arg \sup_{\forall \psi \in \Psi} \ell_{n,p}(\psi) \quad = \quad \sup_{\forall \psi \in \Psi} \ell_n(\psi, \hat{\phi}_{\psi})$$

   e.g. as a root of the profile ML equations

$$0 = \frac{\mathrm{d}}{\mathrm{d}\psi} \ell_{n,p}(\psi)|_{\psi = \hat{\psi}} \quad \text{or equiv.} \quad 0 = \frac{\mathrm{d}}{\mathrm{d}\psi} \ell_n(\psi, \hat{\phi}_{\psi})|_{\psi = \hat{\psi}}$$

Note 10. It can bee seen that $\left(\hat{\psi}, \hat{\phi}_{\hat{\psi}}\right)$ are the standard MLE: $\left(\hat{\psi}, \hat{\phi}\right) = \arg \sup_{\forall \psi, \phi} L_n(\psi, \phi)$; as

$$\sup_{\forall \psi} L_{n,p}(\psi) = \sup_{\forall \psi} \left(\sup_{\forall \phi} L_n(\psi, \phi) \right) = \sup_{\forall \psi, \phi} L_n(\psi, \phi)$$

Proposition 11. *Assume the assumptions of Cramer Theorem 19 (Handout 4) are satisfied. Let $\theta_0 = (\psi_0, \phi_0)$ be the real value of θ. Then the profile MLE $\hat{\psi}$ is strongly consistent $\hat{\psi} \xrightarrow{as} \psi$, and its asymptotic distribution is such that*

$$\sqrt{n}\left(\hat{\psi} - \psi\right) \xrightarrow{D} N\left(0, \left[\mathcal{I}_{11}(\theta_0) - \mathcal{I}_{21}(\theta_0)\mathcal{I}_{22}^{-1}(\theta_0)\mathcal{I}_{21}^{\top}(\theta_0)\right]^{-1}\right) \tag{1.1}$$

where $\{\mathcal{I}_{11}(\theta_0), \mathcal{I}_{21}(\theta_0), \mathcal{I}_{22}(\theta_0)\}$ is a partition of the Fisher Information matrix as

$$\mathcal{I}(\theta_0) = \begin{bmatrix} \mathcal{I}_{11}(\theta_0) & \mathcal{I}_{21}^{\top}(\theta_0) \\ \mathcal{I}_{21}(\theta_0) & \mathcal{I}_{22}(\theta_0) \end{bmatrix}.$$

Proof. That it is strongly consistent can be proven by considering a projection matric $P = [I, 0]$ and applying Slutsky rules as $P[\hat{\psi}, \hat{\phi}]^{\top} \xrightarrow{as} P[\psi, \phi]^{\top}$. Regarding the asymptotic distribution, from Cramer theorem, it is

$$\sqrt{n}\begin{bmatrix} \hat{\psi} - \psi \\ \hat{\phi} - \phi \end{bmatrix} \xrightarrow{D} \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathcal{I}_{11}(\theta_0) & \mathcal{I}_{21}^{\top}(\theta_0) \\ \mathcal{I}_{21}(\theta_0) & \mathcal{I}_{22}(\theta_0) \end{bmatrix}^{-1}\right)$$

I want the marginal which I get by considering a projection matric $P = [I, 0]$ as $\sqrt{n}\left(\hat{\psi} - \psi\right) = P\sqrt{n}\begin{bmatrix} \hat{\psi} - \psi \\ \hat{\phi} - \phi \end{bmatrix}$ because by Slusky rules I get normal asymp. distribution, zero asympt. mean, and covariance (1,1)- block of $[\mathcal{I}(\theta_0)]^{-1}$ which is as in (1.1). Note that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \left[A - BD^{-1}C\right]^{-1} & -A^{-1}B\left[D - CA^{-1}B\right]^{-1} \\ -D^{-1}C\left[A - BD^{-1}C\right]^{-1} & \left[D - CA^{-1}B\right]^{-1} \end{bmatrix}$$

□

Remark 12. Note that if I applied directly Cramer theorem for $\hat{\psi}$ then

$$\sqrt{n}\left(\hat{\psi} - \psi\right) \xrightarrow{D} \mathrm{N}\left(0, [\mathcal{I}_{11}(\theta_0)]^{-1}\right) \tag{1.2}$$

which would lead to overconfident inference because comparing with the asymptotic variance in (1.1), it is

$$\begin{aligned} &\left[\mathcal{I}_{11}(\theta_0) - \mathcal{I}_{21}(\theta_0)\mathcal{I}_{22}^{-1}(\theta_0)\mathcal{I}_{21}^{\top}(\theta_0)\right]^{-1} - [\mathcal{I}_{11}(\theta_0)]^{-1} \\ &\overset{(*)}{=} \left[\mathcal{I}_{21}(\theta_0)\left[\mathcal{I}_{11}(\theta_0)\right]^{-1}\right]^{\top}\left[D - CA^{-1}B\right]^{-1}\left[\mathcal{I}_{21}(\theta_0)\left[\mathcal{I}_{11}(\theta_0)\right]^{-1}\right] \geq 0 \end{aligned}$$

which is semi- positive definite. Here $(*)$ by Woodbury matrix identity. This is reasonable as in (1.2) I ignored uncertainty about ϕ and the fact I used data to learn ϕ as well.

2 Popular pivotal statistics for CI & HT

Note 13. Due to the presence of nuisance parameter ϕ in the statistical model, we can resort to asymptotic pivotal statistics for ψ by profiling out ϕ from the original Likelihood ratio, Score, and Walds' pivotals.

2.1 The Walds' pivotal statistic

Definition 14. The Wald statistic, is defined as

$$W_{\mathrm{W}}(\psi) = n(\hat{\psi}_n - \psi_0)^T \left[\mathcal{I}_{11}(\theta_0) - \mathcal{I}_{21}(\theta_0)\mathcal{I}_{22}^{-1}(\theta_0)\mathcal{I}_{21}^{\top}(\theta_0)\right](\hat{\psi}_n - \psi_0) \tag{2.1}$$

Definition 15. Other, more tractable variations of the Wald statistic are

$$W'_{\mathrm{W}}(\psi_0) = n(\hat{\psi}_n - \psi_0)^T \left[\mathcal{I}_{11}(\hat{\psi}_n) - \mathcal{I}_{21}(\hat{\psi}_n)\mathcal{I}_{22}^{-1}(\hat{\psi}_n)\mathcal{I}_{21}^{\top}(\hat{\psi}_n)\right](\hat{\psi}_n - \psi_0) \tag{2.2}$$

$$W''_{\mathrm{W}}(\theta_0) = (\hat{\psi}_n - \psi_0)^T \left[\mathcal{J}_{n;11}(\hat{\psi}_n) - \mathcal{J}_{n;21}(\hat{\psi}_n)\mathcal{J}_{n;22}^{-1}(\hat{\psi}_n)\mathcal{J}_{n;21}^{\top}(\hat{\psi}_n)\right](\hat{\psi}_n - \psi_0) \tag{2.3}$$

Proposition 16. *Assume the conditions from the Cramer Theorem 19 (Handout 4) are satisfied. Let $\theta_0 = (\psi_0, \phi_0)$ be the real value of $\theta = (\psi, \phi)$, with $\psi \in \mathbb{R}^{d_\psi}$, $\phi \in \mathbb{R}^{d_\phi}$, and $d = d_\psi + d_\phi$. Then $W_W(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$, $W'_W(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$, and $W''_W(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$ and they are all asymptotically equivalent.*

Proof. The asymptotic equivalence can be proved by showing $W'_{\mathrm{W}}(\psi_0) - W_{\mathrm{W}}(\psi_0) \xrightarrow{p} 0$ for each pair. The Asymptotic distribution can be produced from (1.1) and Slusky rules; or otherwise by using Delta method $\psi = g(\theta) = [I, 0]\theta$. □

Proposition 17. *Given a statistical model $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, the Wald Hypothesis test for*

$$H_0 : \psi = \psi_* \quad vs. \quad H_1 : \psi \neq \psi_*$$

has a rejection area, at sig. level a,

$$RA(X_{1:n}) = \{X_{1:n} : W_{Wald}(\psi_0) \geq \chi^2_{d,1-a}\} \tag{2.4}$$

Similar is the rejection area produced by $\mathrm{W}'_{\mathrm{Wald}}(\psi_0)$ and $W''_{\mathrm{Wald}}(\psi_0)$.

Proposition 18. *Given a statistical model $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, the $(1-a)$ confidence region for ψ is*

$$CI(\psi) = \{\theta \in \Theta : W_{Wald}(\psi) \leq \chi^2_{d,1-a}\} \tag{2.5}$$

produced by inverting the $RA(x_{1:n})$. Similar is the confidence set produced by $\mathrm{W}'_{\mathrm{Wald}}(\psi)$ and $W''_{\mathrm{Wald}}(\psi)$.

3 Score pivotal statistic

Definition. The profile score statistic is defined as

$$U_p(\psi) = \frac{\mathrm{d}}{\mathrm{d}\theta}\ell_{n,p}(\psi) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\ell(\psi, \phi)\right|_{(\psi, \hat{\phi}_\psi)} \tag{3.1}$$

Proposition 19. *[Part of Wilks' Theorem (Appendix...)] The asymptotic distribution of the profile score statistic is*

$$\frac{1}{\sqrt{n}}U_p(\psi) = \frac{1}{\sqrt{n}}\dot{\ell}_{n,p}(\psi) \xrightarrow{D} N\Big(0, \Big[\mathcal{I}_{11}(\theta_0) - \mathcal{I}_{21}(\theta_0)\mathcal{I}_{22}^{-1}(\theta_0)\mathcal{I}_{21}^{\top}(\theta_0)\Big]\Big) \tag{3.2}$$

Proof. The proof is available in [3, (Ch. 4)]. □

Definition 20. The following score pivotal statistic is produced from the score statistic:

$$W_{\text{Score},p}(\psi) = \frac{1}{n}\left[\dot{\ell}_{n,p}(\psi)\right]^{\top}\left[\mathcal{I}_{11}(\theta) - \mathcal{I}_{21}(\theta)\mathcal{I}_{22}^{-1}(\theta)\mathcal{I}_{21}^{\top}(\theta)\right]^{-1}\dot{\ell}_{n,p}(\psi) \tag{3.3}$$

Definition 21. Other, more tractable variations of the Wald statistic are

$$W'_{\text{Score},p}(\psi) = \frac{1}{n}U\left[\dot{\ell}_{n,p}(\psi)\right]^{\top}\left[\mathcal{I}_{11}(\hat{\theta}) - \mathcal{I}_{21}(\hat{\theta})\mathcal{I}_{22}^{-1}(\hat{\theta})\mathcal{I}_{21}^{\top}(\hat{\theta})\right]^{-1}\dot{\ell}_{n,p}(\psi) \tag{3.4}$$

$$W''_{\text{Score},p}(\psi) = \left[\dot{\ell}_{n,p}(\psi)\right]^{\top}\left[\mathcal{J}_{n;11}(\hat{\theta}) - \mathcal{J}_{n;21}(\hat{\theta})\mathcal{J}_{n;22}^{-1}(\hat{\theta})\mathcal{J}_{n;21}^{\top}(\hat{\theta})\right]^{-1}\dot{\ell}_{n,p}(\psi) \tag{3.5}$$

Proposition 22. *Assume the conditions from the Cramer Theorem 19 (Handout 4) are satisfied. Let $\theta_0 = (\psi_0, \phi_0)$ be the real value of $\theta = (\psi, \phi)$, with $\psi \in \mathbb{R}^{d_\psi}$, $\phi \in \mathbb{R}^{d_\phi}$, and $d = d_\psi + d_\phi$. Then $W_{Score,p}(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$, $W'_{Score,p}(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$, and $W''_{Score,p}(\psi_0) \xrightarrow{D} \chi^2_{d_\psi}$ and they are all asymptotically equivalent.*

Proof. The asymptotic equivalence can be proved by showing $W'_{\text{Score},p}(\psi_0) - W_{\text{Score},p}(\psi_0) \xrightarrow{p} 0$ for each pair. The asymptotic distribution can be produced from Proposition 19 and Slusky rules. The proof is available in [3, (Ch. 4)] □

Proposition 23. *Given a statistical model $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, the Score Hypothesis test*

$$H_0 : \psi = \psi_* \quad vs. \quad H_1 : \psi \neq \psi_*$$

has a rejection area, at sig. level a,

$$RA(X_{1:n}) = \{X_{1:n} : W_{Score,p}(\psi_0) \geq \chi^2_{d_\psi, 1-a}\} \tag{3.6}$$

Similar is the rejection area produced by $W'_{\text{Score},p}(\psi_0)$ and $W''_{\text{Score},p}(\psi_0)$.

Proposition 24. *Given a statistical model $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, the $(1-a)$ Score confidence interval for ψ is*

$$CI(\psi) = \{\psi \in \Psi : W_{Score,p}(\psi) \leq \chi^2_{d_\psi, 1-a}\} \tag{3.7}$$

produced by inverting the $RA(X_{1:n})$. Similar is the confidence set based on $\mathrm{W}'_{\text{Score}}(\psi_0)$ and $W''_{\text{Score}}(\psi_0)$.

3.1 Likelihood ratio (LR) pivotal statistic

Note 25. To profile out ϕ from the likelihood ratio statistic, it would be reasonable to modify the original likelihood ratio to use profiled likelihoods suitably

$$W_{\mathrm{LR},p}(\psi_*) = -2\log\left(\frac{L_{n,p}(\psi_*)}{\sup_{\forall\psi\neq\psi_*} L_{n,p}(\psi)}\right) = -2\log\left(\frac{L_n(\psi_*, \hat{\phi}_{\psi_*})}{\sup_{\forall\psi\neq\psi_*,\forall\phi} L_n(\psi,\phi)}\right) = -2(\,\ell_n(\psi_*, \hat{\phi}_{\psi_*}) - \ell_n(\hat{\theta})\,)$$

where $\hat{\theta} = (\hat{\psi}, \hat{\phi})$ is the MLE of $\theta = (\psi, \phi)$.

Definition 26. Given a statistical model $\left\{X_i \overset{\text{IID}}{\sim} f(\cdot|\theta)\right\}$ where $\theta = (\psi, \phi)$, the log likelihood ratio statistic at ψ is

$$W_{\mathrm{LR},p}(\psi) = -2\left(\ell_{n,p}(\psi) - \ell_{n,p}(\hat{\psi}_n)\right) = -2\left(\ell_{n,p}(\psi, \hat{\phi}_{\psi_n}) - \ell_{n,p}(\hat{\psi}_n, \hat{\phi}_{\hat{\psi}_n})\right) \tag{3.8}$$

where $\hat{\psi}_n$ is the profiled MLE of ψ.

Theorem 27. *[Part of Wilks' Theorem (Appendix...)] Assume a statistical model* $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ *with unknown parameter* $\theta = (\psi, \phi)$*, where* $\theta \in \Theta \subset \mathbb{R}^d$*,* $\psi \in \Psi \subset \mathbb{R}^{d_\psi}$*, and* $\phi \in \Phi \subset \mathbb{R}^{d_\phi}$*. Assume the conditions from the Cramer Theorem 19 (Handout 4) are satisfied. Let* $\theta_0 = (\psi_0, \phi_0)$ *be the real value of* $\theta = (\psi, \phi)$*. Then*

$$W_{LR,p}(\psi_0) = -2(\ell_{n,p}(\psi_0) - \ell_{n,p}(\hat{\psi}_n)) \xrightarrow{D} \chi^2_{d_\psi} \tag{3.9}$$

where $\hat{\psi}_n$ *is the profiled MLE of* θ*.*

Proof. The proof is available in [3, (Ch. 4)], [1, (Ch. 22)]. □

Proposition 28. *Given a statistical model* $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ *where* $\theta = (\psi, \phi)$*, the profiled LR hypothesis test for*

$$H_0 : \theta = \theta_* \quad vs. \quad H_1 : \theta \neq \theta_*$$

has rejection area, at sig. level a, is

$$RA(X_{1:n}) = \{X_{1:n} : W_{LR}(\theta_*) \geq \chi^2_{d,1-a}\} \tag{3.10}$$

Proposition 29. *Given a statistical model* $\left\{X_i \overset{IID}{\sim} f(\cdot|\theta)\right\}$ *where* $\theta = (\psi, \phi)$*, the* $(1-a)$ *profiled LR confidence region for* ψ *is*

$$CI(\psi) = \{\psi \in \Psi : W_{LR}(\psi) \leq \chi^2_{d,1-a}\} \tag{3.11}$$

as produced by inverting the $RA(x_{1:n})$

4 Examples

Example 30. Let random sample $x_1, ..., x_n \overset{\text{IID}}{\sim} \mathrm{N}(\mu, \sigma^2)$, where μ and σ^2 are unknown. We are interested in inference on μ.

1. Calculate the profile likelihood for μ
2. Find the likelihood ratio rejection area (at sig. level a) for the hypothesis test

$$\mathrm{H}_0 : \mu = \mu_* \text{ vs. } \mathrm{H}_1 : \mu \neq \mu_*$$

   with respect to the t statistic $t = \sqrt{n}\frac{(\bar{x}-\mu_*)}{s}$, $s = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Solution. Ok, I need to perform inference about the parameter of interest μ under the presence of a nuisance parameter σ^2.

1. The profile likelihood is

$$L_{n,p}(\mu) = \sup_{\forall \sigma^2} L_n(\mu, \sigma^2) = L_n(\mu, \hat{\sigma}^2_\mu)$$

   where $\hat{\sigma}^2_\mu$ is the n MLE of σ^2 for a given μ.

   So first I need to find $\hat{\sigma}^2_\mu$. Okay, then, ...

   The joint likelihood is

$$L_n(\mu, \sigma^2) = (\frac{1}{2\pi\sigma^2})^{\frac{n}{2}} \exp(-\frac{1}{2}\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2)$$

   The joint log likelihood is

$$\ell_n(\mu, \sigma^2) = -\frac{n}{2}\log(\sigma^2) - \frac{1}{2}\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 + \text{const.}$$

   So, to find $\hat{\sigma}^2_\mu$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma^2}\ell_n(\mu, \sigma^2)|_{\sigma^2=\hat{\sigma}^2_\mu} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2}\frac{1}{(\sigma^2)^2}\sum_{i=1}^{n}(x_i - \mu)^2|_{\sigma^2=\hat{\sigma}^2_\mu}$$

   then

$$\hat{\sigma}^2_\mu = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

   Hence the profile likelihood for μ is

$$\begin{aligned} L_{n,p}(\mu) = L_n(\mu, \hat{\sigma}^2_\mu) &= (\frac{1}{2\pi\hat{\sigma}^2_\mu})^{\frac{n}{2}} \exp(-\frac{1}{2}\frac{1}{\hat{\sigma}^2_\mu}\sum_{i=1}^{n}(x_i - \mu)^2) \\ &= (\frac{1}{2\pi}\frac{n}{\sum_{i=1}^{n}(x_i - \mu)^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n) \end{aligned}$$

2. To test

$$\mathrm{H}_0 : \mu = \mu_* \text{ vs. } \mathrm{H}_1 : \mu \neq \mu_*$$

I need to find the log likelihood ratio

$$W_{\mathrm{LR},p}(\mu_*) = -2\log(\frac{\sup_{\mathrm{H}_0} L_{n,p}(\mu)}{\sup_{\mathrm{H}_1} L_{n,p}(\mu)})$$

Under the null hypothesis H_0 it is

$$L_{n,p}(\mu_*) = (\frac{1}{2\pi}\frac{n}{\sum_{i=1}^n (x_i - \mu_*)^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n)$$

Under the alternative hypothesis H_1 it is

$$\begin{aligned} \sup_{\mathrm{H}_1} L_{n,p}(\mu) &= \sup_{\forall \mu}((\frac{1}{2\pi}\frac{n}{\sum_{i=1}^n (x_i - \hat{\mu})^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n)) \\ &= (\frac{1}{2\pi}\frac{n}{\sum_{i=1}^n (x_i - \bar{x})^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n) \end{aligned} \tag{4.1}$$

because the MLE of μ under the H_1 is $\hat{\mu} = \bar{x}$: In fact, under H_1 it is

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma^2}\ell_n(\mu, \sigma^2)|_{\sigma^2 = \hat{\sigma}^2, \mu = \hat{\mu}} \implies \begin{cases} \hat{\mu} = & \bar{x} \\ \hat{\sigma}^2 = & \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 \end{cases}$$

or otherwise you can see that $L_{n,p}(\mu)$ maximizes by minimizing the sum-of-squares term. So

$$\begin{aligned} W_{\mathrm{LR},p}(\mu_*) &= -2\log(\frac{\sup_{\mathrm{H}_0} L_{n,p}(\mu)}{\sup_{\mathrm{H}_1} L_{n,p}(\mu)}) = -2\log(\frac{(\frac{1}{2\pi}\frac{n}{\sum_{i=1}^n (x_i - \mu_*)^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n)}{(\frac{1}{2\pi}\frac{n}{\sum_{i=1}^n (x_i - \bar{x})^2})^{\frac{n}{2}} \exp(-\frac{1}{2}n)}) \\ &= n\log(\frac{\sum_{i=1}^n (x_i - \mu_*)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}) \qquad = n\log(\frac{\sum_{i=1}^n (x_i \pm \bar{x} - \mu_*)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}) \\ &= n\log(\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_*)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}) \\ &= n\log(1 + \frac{1}{n-1}\underbrace{n\frac{(\bar{x} - \mu_*)^2}{\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2}}_{=t^2}) \\ &= n\log(1 + \frac{1}{n-1}t^2) \xrightarrow{D} \chi^2_{\underbrace{2-1}_{=1}} \end{aligned}$$

where $t = \sqrt{n}\frac{(\bar{x} - \mu_*)}{s}$ with $s = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$.

Therefore the rejection area at sig. level a is

$$\mathrm{RA}(x_{1:n}) = \{x_{1:n} : n\log(1 + \frac{1}{n-1}t^2) \geq \chi^2_{1,1-a}\}$$

Exercise sheet

Exercise #33, 34, 35

References

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