Exercises: Likelihood methods

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This is out of the scope

Exercise 1. ($\star\star$) From Fatou-Lesbeque Lemma, prove Monotone Convergence theorem. (Hint: Use $Y \equiv 0$, use $\limsup_{n\to\infty} f_n$and $\liminf_{n\to\infty} f_n$)

Solution. Since $0 < X_1 \leq ... \leq \lim_{n\to\infty} X_n = X$ a.s.. Then $\mathrm{E}X_n \leq \mathrm{E}X$ or $\limsup_{n\to\infty} \mathrm{E}X_n \leq \mathrm{E}X$.

From Fatou-Lesbeque Lemma, it is $\liminf_{n\to\infty} \mathrm{E}X_n \geq \mathrm{E}X$. Also the limit $\lim \mathrm{E}X_n$ exists. Then, it is $\lim \mathrm{E}X_n = \mathrm{E}X$

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Exercise 2. ($\star\star$) From Fatou-Lesbeque Lemma, prove Lesbeque Dominant Convergence theorem. (Hint: Use that $-Y \leq -X_n$ and $-Y \leq X_n$, use $\limsup_{n\to\infty} f_n$and $\liminf_{n\to\infty} f_n$)

Solution.

Since $|X_n| \leq Y$, it is $-Y \leq -X_n$, and because $X_n \xrightarrow{a.s.} X$ it is $\liminf_{n\to\infty} \mathrm{E}(-X_n) \geq \mathrm{E}(-Y) \Longleftrightarrow \limsup_{n\to\infty} \mathrm{E}(X_n) \leq \mathrm{E}(Y)$

Since $|X_n| \leq Y$, it is $-Y \leq X_n$ and because $X_n \xrightarrow{a.s.} X$ it is $\liminf_{n\to\infty} \mathrm{E}(X_n) \geq \mathrm{E}(Y)$

So $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(Y)$

Exercise 3. ($\star\star$) Let μ be a constant. Show that $X_n \xrightarrow{\text{qm}} \mu$ if and only if $\mathrm{E}X_n \to \mu$ and $\mathrm{Var}(X_n) \to 0$, both in uni-variate and multivariate case.

Solution. It is $\mathrm{E}(X_n - \mu)^2 = \mathrm{Var}(X_n) + (\mathrm{E}X_n - \mu)^2$. Hence, $\mathrm{E}(X_n - \mu)^2 \to 0$. In the multivariate case, it is $\mathrm{E}(X_n - \mu)^T (X_n - \mu) = \mathrm{E} \sum_{i=1}^{d} (X_{n,i} - \mu_i)^2 \to 0$ by treating each element separately.

Exercise 4. ($\star\star$) Consider a sequence of discrete r.v. $\{X_n\}$ with probability $P(X_n = k) = \frac{1}{n}$, for $k = 1/n, 2/n, ..., n/n$. Show that $X_n \xrightarrow{D} X$ where $X \sim \mathrm{U}(0, 1)$. (Hint: Just use the definition.)

Solution. The probability function is $P(X_n \leq x) = k/n$ for $k/n \leq x \leq (k+1)/n$.

Then because $|k/n - x| < 1/n$, we have $\lim_{n\to\infty} P(X_n \leq x) = x$.

Note that $P(X \leq x) = x$ is the distribution function of the Uniform random variable $X \sim \mathrm{U}(0,1)$. So $X_n \xrightarrow{D} \mathrm{U}(0,1)$.

Exercise 5. ($\star$)

1. Show that
$$\mathrm{E}_\pi(X-\theta)^T(X-\theta) = \mathrm{Var}_\pi(X) + (\mathrm{E}_\pi(X)-\theta)^T(\mathrm{E}_\pi(X)-\theta)$$
, where is a constant point, and X is a random variable $X \sim \mathrm{d}\pi(\cdot)$.

2. Show that
$$\mathrm{E}_\pi|X-\theta|^2 = \mathrm{Var}_\pi(X) + |\mathrm{E}_\pi(X)-\theta|^2$$
, where is a constant point, X is a random variable $X \sim \mathrm{d}\pi(\cdot)$, and $|X| = \sqrt{X_1^2 + ...X_d^2}$ is the Euclidean norm.

Solution.

1. It is
$$\begin{aligned}
\mathrm{E}_\pi(X-\theta)^T(X-\theta) &= \mathrm{E}_\pi([X-\mathrm{E}_\pi(X)] + [\mathrm{E}_\pi(X)-\theta])^T([X-\mathrm{E}_\pi(X)] + [\mathrm{E}_\pi(X)-\theta]) = ... \\
&= \mathrm{E}_\pi(X-\theta)^T(X-\theta) + (\mathrm{E}_\pi(X)-\theta)^T(\mathrm{E}_\pi(X)-\theta) \\
&= \mathrm{Var}_\pi(X) + (\mathrm{E}_\pi(X)-\theta)^T(\mathrm{E}_\pi(X)-\theta)
\end{aligned}$$

2. It is
$$\begin{aligned}
\mathrm{E}_\pi|X-\theta|^2 &= \mathrm{E}_\pi(X-\theta)^T(X-\theta) \\
|\mathrm{E}_\pi(X)-\theta|^2 &= (\mathrm{E}_\pi(X)-\theta)^T(\mathrm{E}_\pi(X)-\theta)
\end{aligned}$$
from the definition of the is the Euclidean norm $|X| = \sqrt{X_1^2 + ...X_d^2}$. So the result follows from then previous task.

Exercise 6. Show that
$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$

Solution. Let $f(x) = \log(1+x)$. Then $\dot{f}(x) = \frac{1}{1+x}$. The 1st order Taylor expansion of $f(x)$ around 0 is
$$f(x) = f(0) + \frac{1}{1!}\dot{f}(0)(x-0) + o(x), \text{ as , as } x \to 0$$

where $h = x - 0$.

So

$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$

Exercise 7. Show that

$$\lim_{n\to\infty}(1+\frac{1}{n}a_n)^n = \exp(\lim_{n\to\infty} a_n)$$

provided that $\frac{1}{n}a_n \to 0$, as $n \to \infty$.

Hint: From Taylor expansion, it is

$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$

Solution.

- It is

$$\begin{aligned}(1+\frac{1}{n}a_n)^n =& \exp(n\log(1+\frac{1}{n}a_n))\\ =& \exp(n(\frac{1}{n}a_n + o(\frac{1}{n}a_n)))\\ =& \exp(a_n(1+o(1)))\end{aligned}$$

- Then provided that a_n increases slower than n, aka $\frac{1}{n}a_n \to 0$ it is

$$\lim_{n\to\infty}(1+\frac{1}{n}a_n)^n = \exp(\lim_{n\to\infty} a_n)$$

Exercise 8. It is $X_n \xrightarrow{a.s.} X$ if and only if

$$\text{for every } \epsilon > 0, \quad P(|X_k - X| < \epsilon\,, \forall k \geq n) \to 1, \qquad \text{as } n \to \infty,$$

Solution. Let $A_{n,\epsilon} = \{|X_k - X| < \epsilon\,, \forall k \geq n\}$. Then

$$P(\lim_{n\to\infty} X_n = X) = P\{\forall\epsilon > 0,\ \exists n > 0,\ \text{s.t.}\ |X_k - X| < \epsilon,\ \forall k \geq n\} = P\{\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}\}$$

So $X_n \xrightarrow{a.s.} X$ is equivalent to $P\{\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}\} = 1$. Because sets $\cup_{\forall n}A_{n,\epsilon}$ decrease to $\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}$ as $\epsilon \to 0$, it is

$$P\{\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}\} = 1 \iff P\{\cup_{\forall n}A_{n,\epsilon}\} = 1,\ \forall\epsilon > 0$$

Because $A_{n,\epsilon}$ increases to $\cup_{\forall n} A_{n,\epsilon}$ as $n \to \infty$, it is

$$P\{\cup_{\forall n} A_{n,\epsilon}\} = 1 \Longleftrightarrow P\{A_{n,\epsilon}\} = 1, \text{ as } n \to \infty, \, \forall \epsilon > 0$$

Exercise 9. Prove the following relations betweeng different modes of convergence

1. $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$
2. $X_n \xrightarrow{r} X$, for some $r > 0 \implies X_n \xrightarrow{P} X$
3. $(\star\star\star) X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$

Solution.

1. For any $\epsilon > 0$, then

$$P(|X_n - X| > \epsilon) \geq P(|X_k - X| < \epsilon \,, \forall k \geq n) \to 1, \text{ as, } n \to \infty$$

   from Lemma 6 in the Handout.

2. It is

$$E|X_n - X|^r \geq E(|X_k - X|^r 1(|X_n - X| \geq \epsilon)) \geq \epsilon^r P(|X_n - X| \geq \epsilon) \to 0, \text{ as, } n \to \infty$$

   This is Markov inequality (Prob. I)

3. This is difficult and it can be skipped.

   For any $\epsilon > 0$, $\{X > z + 1\epsilon\}$ and $|X_n - X| < \epsilon$ imply $\{X_n > z\}$. Hence, $\{X_n > z\} \supseteq \{X > z + \epsilon\} \cap \{|X_n - X| < \epsilon\}$. By taking complements, we get $\{X_n \leq z\} \subseteq \{X \leq z + 1\epsilon\} \cup \{|X_n - X| > \epsilon\}$. So I get $P(X_n \leq z) \leq P(X \leq z + \epsilon) + P(|X_n - X| > \epsilon)$.

   In a similar way (by interchanging X and X_n), I get $P(X_n \leq z) \geq P(X \leq z - \epsilon) + P(|X_n - X| > \epsilon)$. [1]

   So as $n \to \infty$

$$P(X \leq z - 1\epsilon) \leq \lim \inf_{n \to \infty} P(X_n \leq z) \leq \lim \sup_{n \to \infty} P(X_n \leq z) \leq P(X \leq z + 1\epsilon)$$

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[1] It is:

(a) $\lim\sup_{n\to\infty} f_n := \lim_{n\to\infty}(\sup_{\forall m \geq n} f_m)$ and $\lim\inf_{n\to\infty} f_n := \lim_{n\to\infty}(\inf_{\forall m\geq n} f_m)$

(b) It is $\lim\inf_{n\to\infty} f_n \leq \lim\sup_{n\to\infty} f_n$ if both exist.

(c) It is $\lim_{n\to\infty} f_n = \lim\inf_{n\to\infty} f_n = \lim\sup_{n\to\infty} f_n$ if $\lim_{n\to\infty} f_n$ exists

As $F_X(x) = P(X \leq x)$ is continuous at z, the two ends should converge to $F_X(z) = P(X \leq z)$ as $\epsilon \to 0$, which implies that $\lim_{n\to\infty} F_{X_n}(z) = F_X(z)$

Exercise 10. ($\star\star$) Prove that:

1. if $Z \sim \mathrm{N}(0, I)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^T t)$, where $Z \in \mathbb{R}^d$
2. if $X \sim \mathrm{N}(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T\mu - \frac{1}{2}t^T\Sigma t)$, where $X \in \mathbb{R}^d$

Hint: Assume as known that if $Z \sim \mathrm{N}(0, 1)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^2)$, where $Z \in \mathbb{R}$

Solution 11.

1. It is

$$\begin{aligned}\varphi_Z(t) &= \mathrm{E}(\exp(it^T Z)) = \mathrm{E}(\exp(i\sum_{j=1}^{d}(t_j Z_j))) = \mathrm{E}(\prod_{j=1}^{d}\exp(it_j Z_j)) = \prod_{j=1}^{d}\mathrm{E}(\exp(it_j Z_j)) \\ &= \prod_{j=1}^{d}\varphi_{Z_j}(t) = \prod_{j=1}^{d}\exp(-\frac{1}{2}t_j^2) = \exp(-\frac{1}{2}\sum_{j=1}^{d}t_j^2) = \exp(-\frac{1}{2}t^T t)\end{aligned}$$

2. Assume a matrix L such as $\Sigma = LL^T$. It is $X = \mu + LZ$. Then

$$\begin{aligned}\varphi_X(t) = \varphi_{\mu+LZ}(t) &= e^{it^T\mu}\varphi_Z(L^T t) = e^{it^T\mu}\exp(-\frac{1}{2}(L^T t)^T L^T t) \\ &= e^{it^T\mu}\exp(-\frac{1}{2}t^T LL^T t) = \exp(it^T\mu - \frac{1}{2}t^T\Sigma t)\end{aligned}$$