1 Handout 1: Basic probability tools in asymptotics

This is out of the scope

Exercise 1. ($\star\star$) From Fatou-Lesbeque Lemma, prove Monotone Convergence theorem. (Hint: Use $Y \equiv 0$, use $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$)

Solution. Since $0 < X_1 \le ... \le \lim_{n\to\infty} X_n = X$ a.s.. Then $\mathrm{E}X_n \le \mathrm{E}X$ or $\limsup_{n\to\infty} \mathrm{E}X_n \le \mathrm{E}X$.

From Fatou-Lesbeque Lemma, it is $\liminf_{n\to\infty} \mathrm{E}X_n \ge \mathrm{E}X$. Also the limit $\lim \mathrm{E}X_n$ exists. Then, it is $\lim \mathrm{E}X_n = \mathrm{E}X$

This is out of the scope

Exercise 2. ($\star\star$) From Fatou-Lesbeque Lemma, prove Lesbeque Dominant Convergence theorem. (Hint: Use that $-Y \le -X_n$ and $-Y \le X_n$, use $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$)

Solution.

Since $|X_n| \le Y$, it is $-Y \le -X_n$, and because $X_n \xrightarrow{a.s.} X$ it is $\liminf_{n\to\infty} \mathrm{E}(-X_n) \ge \mathrm{E}(-Y) \iff \limsup_{n\to\infty} \mathrm{E}(X_n) \le \mathrm{E}(Y)$

Since $|X_n| \le Y$, it is $-Y \le X_n$ and because $X_n \xrightarrow{a.s.} X$ it is $\liminf_{n\to\infty} \mathrm{E}(X_n) \ge \mathrm{E}(Y)$

So $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(Y)$

Exercise 3. ($\star\star$) Let μ be a constant. Show that $X_n \xrightarrow{\mathrm{qm}} \mu$ if and only if $\mathrm{E}X_n \to \mu$ and $\mathrm{Var}(X_n) \to 0$, both in uni-variate and multivariate case.

Solution. It is $\mathrm{E}(X_n - \mu)^2 = \mathrm{Var}(X_n) + (\mathrm{E}X_n - \mu)^2$. Hence, $\mathrm{E}(X_n - \mu)^2 \to 0$. In the multivariate case, it is $\mathrm{E}(X_n - \mu)^T (X_n - \mu) = \mathrm{E}\sum_{i=1}^{d}(X_{n,i} - \mu_i)^2 \to 0$ by treating each element separately.

Exercise 4. (★★) Consider a sequence of discrete r.v. $\{X_n\}$ with probability $P(X_n = k) = \frac{1}{n}$, for $k = 1/n, 2/n, ..., n/n$. Show that $X_n \xrightarrow{D} X$ where $X \sim U(0,1)$. (Hint: Just use the definition.)

Solution. The probability function is $P(X_n \leq x) = k/n$ for $k/n \leq x \leq (k+1)/n$.

Then because $|k/n - x| < 1/n$, we have $\lim_{n \to \infty} P(X_n \leq x) = x$.

Note that $P(X \leq x) = x$ is the distribution function of the Uniform random variable $X \sim U(0,1)$. So $X_n \xrightarrow{D} U(0,1)$.

Exercise 5. (★)

1. Show that
$$E_\pi(X - \theta)^T(X - \theta) = \text{Var}_\pi(X) + (E_\pi(X) - \theta)^T(E_\pi(X) - \theta)$$
, where is a constant point, and X is a random variable $X \sim d\pi(\cdot)$.

2. Show that
$$E_\pi|X - \theta|^2 = \text{Var}_\pi(X) + |E_\pi(X) - \theta|^2$$
, where is a constant point, X is a random variable $X \sim d\pi(\cdot)$, and $|X| = \sqrt{X_1^2 + ...X_d^2}$ is the Euclidean norm.

   Solution.

   (a) It is

$$E_\pi(X - \theta)^T(X - \theta) = E_\pi([X - E_\pi(X)] + [E_\pi(X) - \theta])^T([X - E_\pi(X)] + [E_\pi(X) - \theta]) = ...$$
$$= E_\pi(X - \theta)^T(X - \theta) + (E_\pi(X) - \theta)^T(E_\pi(X) - \theta)$$
$$= \text{Var}_\pi(X) + (E_\pi(X) - \theta)^T(E_\pi(X) - \theta)$$

   (b) It is

$$E_\pi|X - \theta|^2 = E_\pi(X - \theta)^T(X - \theta)$$
$$|E_\pi(X) - \theta|^2 = (E_\pi(X) - \theta)^T(E_\pi(X) - \theta)$$

   from the definition of the is the Euclidean norm $|X| = \sqrt{X_1^2 + ...X_d^2}$. So the result follows from then previous task.

Exercise 6. Show that
$$\log(1 + x) = x + o(x), \text{ as } x \to 0.$$

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Solution. Let $f(x) = \log(1+x)$. Then $\dot{f}(x) = \frac{1}{1+x}$. The 1st order Taylor expansion of $f(x)$ around 0 is

$$f(x) = f(0) + \frac{1}{1!}\dot{f}(0)(x-0) + o(x), \text{ as }, \text{ as } x \to 0$$

where $h = x - 0$.

So

$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$

Exercise 7. Show that

$$\lim_{n\to\infty}(1 + \frac{1}{n}a_n)^n = \exp(\lim_{n\to\infty}a_n)$$

provided that $\frac{1}{n}a_n \to 0$, as $n \to \infty$.

Hint: From Taylor expansion, it is

$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$

Solution.

- It is

$$\begin{aligned}
(1 + \frac{1}{n}a_n)^n &= \exp(n\log(1 + \frac{1}{n}a_n)) \\
&= \exp(n(\frac{1}{n}a_n + o(\frac{1}{n}a_n))) \\
&= \exp(a_n(1 + o(1)))
\end{aligned}$$

- Then provided that a_n increases slower than n, aka $\frac{1}{n}a_n \to 0$ it is

$$\lim_{n\to\infty}(1 + \frac{1}{n}a_n)^n = \exp(\lim_{n\to\infty}a_n)$$

Exercise 8. It is $X_n \xrightarrow{a.s.} X$ if and only if

$$\text{for every } \epsilon > 0, \quad P(|X_k - X| < \epsilon, \forall k \geq n) \to 1, \quad \text{as } n \to \infty,$$

Solution. Let $A_{n,\epsilon} = \{|X_k - X| < \epsilon, \forall k \geq n\}$. Then

$$P(\lim_{n\to\infty}X_n = X) = P\{\forall \epsilon > 0, \exists n > 0, \text{ s.t. } |X_k - X| < \epsilon, \forall k \geq n\} = P\{\cap_{\epsilon>0}\cup_{\forall n}A_{n,\epsilon}\}$$

So $X_n \xrightarrow{a.s.} X$ is equivalent to $P\{\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}\} = 1$. Because sets $\cup_{\forall n} A_{n,\epsilon}$ decrease to $\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}$ as $\epsilon \to 0$, it is

$$P\{\cap_{\epsilon>0}\cup_{\forall n} A_{n,\epsilon}\} = 1 \iff P\{\cup_{\forall n} A_{n,\epsilon}\} = 1, \forall \epsilon > 0$$

Because $A_{n,\epsilon}$ increases to $\cup_{\forall n} A_{n,\epsilon}$ as $n \to \infty$, it is

$$P\{\cup_{\forall n} A_{n,\epsilon}\} = 1 \iff P\{A_{n,\epsilon}\} = 1, \text{ as } n \to \infty, \forall \epsilon > 0$$

Exercise 9. Prove the following relations betweeng different modes of convergence

1. $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$

2. $X_n \xrightarrow{r} X$, for some $r > 0 \implies X_n \xrightarrow{P} X$

3. $(\star\star\star) X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$

Solution.

1. For any $\epsilon > 0$, then

$$P(|X_n - X| > \epsilon) \geq P(|X_k - X| < \epsilon, \forall k \geq n) \to 1, \text{ as, } n \to \infty$$

   from Lemma 6 in the Handout.

2. It is

$$E|X_n - X|^r \geq E(|X_k - X|^r 1(|X_n - X| \geq \epsilon)) \geq \epsilon^r P(|X_n - X| \geq \epsilon) \to 0, \text{ as, } n \to \infty$$

   This is Markov inequality (Prob. I)

3. This is difficult and it can be skipped.

   For any $\epsilon > 0$, $\{X > z + 1\epsilon\}$ and $|X_n - X| < \epsilon$ imply $\{X_n > z\}$. Hence, $\{X_n > z\} \supseteq \{X > z + \epsilon\} \cap \{|X_n - X| < \epsilon\}$. By taking complements, we get $\{X_n \leq z\} \subseteq \{X \leq z + 1\epsilon\} \cup \{|X_n - X| > \epsilon\}$. So I get $P(X_n \leq z) \leq P(X \leq z + \epsilon) + P(|X_n - X| > \epsilon)$.

   In a similar way (by interchanging X and X_n), I get $P(X_n \leq z) \geq P(X \leq z - \epsilon) + P(|X_n - X| > \epsilon)$. [1]

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[1] It is:

(a) $\limsup_{n\to\infty} f_n := \lim_{n\to\infty}(\sup_{\forall m \geq n} f_m)$ and $\liminf_{n\to\infty} f_n := \lim_{n\to\infty}(\inf_{\forall m \geq n} f_m)$

(b) It is $\liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} f_n$ if both exist.

(c) It is $\lim_{n\to\infty} f_n = \liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ if $\lim_{n\to\infty} f_n$ exists

So as $n \to \infty$

$$P(X \leq z - 1\epsilon) \leq \lim_{n\to\infty} \inf P(X_n \leq z) \leq \lim_{n\to\infty} \sup P(X_n \leq z) \leq P(X \leq z + 1\epsilon)$$

As $F_X(x) = P(X \leq x)$ is continuous at z, the two ends should converge to $F_X(z) = P(X \leq z)$ as $\epsilon \to 0$, which implies that $\lim_{n\to\infty} F_{X_n}(z) = F_X(z)$

Exercise 10. ($\star\star$) Prove that:

1. if $Z \sim N(0, I)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^T t)$, where $Z \in \mathbb{R}^d$

2. if $X \sim N(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T \mu - \frac{1}{2}t^T \Sigma t)$, where $X \in \mathbb{R}^d$

Hint: Assume as known that if $Z \sim N(0, 1)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^2)$, where $Z \in \mathbb{R}$

Solution.

1. It is

$$\varphi_Z(t) = E(\exp(it^T Z)) = E(\exp(i\sum_{j=1}^{d}(t_j Z_j))) = E(\prod_{j=1}^{d} \exp(it_j Z_j)) = \prod_{j=1}^{d} E(\exp(it_j Z_j))$$

$$= \prod_{j=1}^{d} \varphi_{Z_j}(t) = \prod_{j=1}^{d} \exp(-\frac{1}{2}t_j^2) = \exp(-\frac{1}{2}\sum_{j=1}^{d} t_j^2) = \exp(-\frac{1}{2}t^T t)$$

2. Assume a matrix L such as $\Sigma = LL^T$. It is $X = \mu + LZ$. Then

$$\varphi_X(t) = \varphi_{\mu+LZ}(t) = e^{it^T \mu} \varphi_Z(L^T t) = e^{it^T \mu} \exp(-\frac{1}{2}(L^T t)^T L^T t)$$

$$= e^{it^T \mu} \exp(-\frac{1}{2}t^T LL^T t) = \exp(it^T \mu - \frac{1}{2}t^T \Sigma t)$$

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2 Handout 2: Basic tools for asymptotics in statistics

Exercise 11. Let $X, X_1, X_2, ...$ be i.i.d. random vectors, and let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

then

1. (Weak law) If $E|X| < \infty$, then $\bar{X}_n \xrightarrow{P} E(X)$

2. (Strong law) $E|X| < \infty$, iff $\bar{X}_n \xrightarrow{as} E(X)$

3. (in qm) $E|X|^2 < \infty$, iff $\bar{X}_n \xrightarrow{qm} E(X)$

4. Let $\varphi_X(t) = E(e^{it^T X})$, and $\mu = E(X)$.

Solution.

1. It is

$$\varphi_{\bar{X}_n}(t) = \varphi_{X_1 + ... + X_n}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} \varphi_{X_j}\left(\frac{t}{n}\right) = \left(\varphi_X\left(\frac{t}{n}\right)\right)^n$$
$$= \left(\varphi_X(0) + \left(\int_0^1 \dot{\varphi}_X(u\frac{t}{n})du\right)\frac{t}{n}\right)^n$$

since by the Mean-Value theorem

$$\varphi_X\left(\frac{t}{n}\right) = \varphi_X(0) + \left(\int_0^1 \dot{\varphi}_X(u\frac{t}{n})du\right)\frac{t}{n}.$$

Because $\varphi_X(0) = 1$, and $\lim_{\epsilon \to 0} \dot{\varphi}_X(\epsilon) = \dot{\varphi}_X(0) = i\mu^T$ it is

$$\lim_{n \to \infty} \varphi_{\bar{X}_n}(t) = \exp\left(\lim_{n \to \infty}\left(\int_0^1 \dot{\varphi}_X(u\frac{t}{n})du\right)t\right) = \exp(i\mu^T t) \tag{1}$$

Here I used that $\lim_{n \to \infty}(1 + a_n)^n = \exp(\lim_{n \to \infty} na_n)$ if $\lim_{n \to \infty} na_n$ exists (Exercise #7).

So (1) says that the characteristic function of $\bar{X}_n$ converges to a characteristic function of the degenerate random variable μ

$$\varphi_{\bar{X}_n}(t) \to \varphi_\mu(t)$$

From the continuity Theorem 24 it is $\bar{X}_n \xrightarrow{D} \mu$. Then from Theorem 7(3) it is $\bar{X}_n \xrightarrow{P} \mu$ because $\mu = E(X)$ is just a constant point.

(a) Proof is out of the scope; for more details see in[?].

(b) It is

$$
\begin{aligned}
\mathrm{E}|\bar{X}_n - \mu|^2 &= \mathrm{E}(\bar{X}_n - \mu)^T(\bar{X}_n - \mu) \\
&= \frac{1}{n^2} \sum_i \sum_j \mathrm{E}(X_i - \mu)^T(X_j - \mu) \\
&\stackrel{\text{simplify}}{=} \frac{1}{n^2} \sum_i \mathrm{E}(X_i - \mu)^T(X_i - \mu) \stackrel{\text{iid}}{=} \frac{1}{n^2} n \mathrm{E}(X - \mu)^T(X - \mu) \\
&= \frac{1}{n}\mathrm{Var}(X) \to 0
\end{aligned}
$$

as the 2nd mode is finite.

Exercise 12. Show

$$
\text{If} \quad h_n \to 0, \quad \text{and} \quad X_n = O_P(h_n) \quad \text{then} \quad X_n = o_P(1).
$$

Solution.

- Deterministic: If $x_n = O(h_n)$ and $h_n \to 0$, then $x_n = o(1)$, because we sandwich $|x_n| \le Kh_n \to 0$.

- Stochastic: If $x_n = O_P(h_n)$ and $h_n \to 0$, then $x_n = o_P(1)$. Because $h_n \to 0$, for sufficiently large $n > 0$ $Kh_n \le \delta$. Also as $x_n = O_P(h_n)$ for any $\epsilon > 0$ I can find a $K > 0$ such that $P(|x_n| \le Kh_n) \ge 1 - \epsilon$. Putting both together, for any $\epsilon > 0$ and any $\delta > 0$, I can get K such that, for sufficiently large $n > 0$, I can get

$$
P(|x_n| \le \delta) \ge P(|x_n| \le Kh_n) \ge 1 - \epsilon
$$

Exercise 13. Let $X_1, X_2, \ldots$ IID random vectors $X_i \in \mathbb{R}^d$ with mean $\mathrm{E}(X_i) = \mu$ and finite covariance matrix $\mathrm{Var}(X_i) < \infty$ for all $i = 1, \ldots$, Then

$$
\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \mathrm{N}(0, \Sigma)
$$

Solution.   We 'll gonna use again the characteristic function, and its property with the IID variables. It is

$$
\sqrt{n}(\bar{X}_n - \mu) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n}(X_j - \mu)
$$

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Hence, for any $t \in \mathbb{R}^d$

$$\varphi_{\sqrt{n}(\bar{X}_n - \mu)}(t) = \varphi_{\frac{1}{\sqrt{n}}\sum_{j=1}^n (X_j - \mu)}(t) = \varphi_{\sum_{j=1}^n (X_j - \mu)}(\frac{t}{\sqrt{n}})$$

$$= \prod_{j=1}^n \varphi_{(X_j - \mu)}(\frac{t}{\sqrt{n}})$$

$$= \left(\varphi_{(X_j - \mu)}(\frac{t}{\sqrt{n}}) \right)^n = \left(\varphi_{(X-\mu)}(\frac{t}{\sqrt{n}}) \right)^n$$

Here, let $\varphi(t) := \varphi_{(X_j - \mu)}(t)$ for notation convenience, as $X_1, X_2, ...$ are IID and hence have the same moments. We use Taylor expansion around 0 as

$$\varphi_{(X-\mu)}(\frac{t}{\sqrt{n}}) = \cancel{\varphi_{(X-\mu)}(0)}^1 + \cancel{\dot{\varphi}_{(X-\mu)}(0)\frac{t}{\sqrt{n}}}^{\mathbf{0}} + t^T \left(\int_0^1 \int_0^1 v\ddot{\varphi}_{(X-\mu)}(0 + vu\frac{t}{n})\mathrm{d}u\mathrm{d}v \right) \frac{t}{n}$$

because $\ddot{\varphi}_X(t)$ is obviously continuous. So

$$\varphi_{\sqrt{n}(\bar{X}_n - \mu)}(t) = \left(\varphi_{(X-\mu)}(\frac{t}{\sqrt{n}}) \right)^n$$

$$= \left(1 + t^T \left(\int_0^1 \int_0^1 v\ddot{\varphi}_{(X-\mu)}(vu\frac{t}{n})\mathrm{d}u\mathrm{d}v \right) \frac{t}{n} \right)^n$$

Because $\lim_{n\to\infty}(1 + a_n)^n = \exp(\lim_{n\to\infty} na_n)$ if $\lim_{n\to\infty} na_n$ exists (Exercise #7), it is

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{X}_n - \mu)}(t) = \lim_{n\to\infty} \left(1 + t^T \left(\int_0^1 \int_0^1 v\ddot{\varphi}_{(X-\mu)}(vu\frac{t}{n})\mathrm{d}u\mathrm{d}v \right) \frac{t}{n} \right)^n$$

$$= \exp \left(\lim_{n\to\infty} t^T \left(\int_0^1 \int_0^1 v\ddot{\varphi}_{(X-\mu)}(vu\frac{t}{n})\mathrm{d}u\mathrm{d}v \right) t \right) \tag{2}$$

$$= \exp \left(t^T \left(\int_0^1 \int_0^1 v(-\Sigma)\mathrm{d}u\mathrm{d}v \right) t \right)$$

$$= \exp(-\frac{1}{2}t^T \Sigma t)$$

This is because $\ddot{\varphi}_{(X-\mu)}(\cdot)$ is continuous so $\lim_{n\to\infty} \ddot{\varphi}_{(X-\mu)}(u\frac{t}{n}) = \ddot{\varphi}_{(X-\mu)}(0) = -\mathrm{E}((X - \mu)^T(X - \mu)) = -\Sigma$.

Since $\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{X}_n - \mu)}(t) = \exp(-\frac{1}{2}t^T \Sigma t)$, aka $\varphi_{\sqrt{n}(\bar{X}_n - \mu)}(t) \to \varphi_Z(t)$ where $Z \sim \mathrm{N}(0, \Sigma)$, it is $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \mathrm{N}(0, \Sigma)$.

Exercise 14. ($\star\star$) Consider that $\sqrt{n}(X_n - \mu) \xrightarrow[D]{} Z$, where $Z \sim \mathrm{N}(0, \Sigma)$ for $\Sigma > 0$ (positive definite). Show that $X_n \xrightarrow{P} \mu$. (Hint: Use the concept 'bounded in probability)'

Solution. I show this result by using 2 ways.

First way: It is $\sqrt{n}(X_n - \mu) \underset{D}{\to} Z$, so

$$\sqrt{n}(X_n - \mu) = O_P(1) \implies (X_n - \mu) = O_P(1)O_P(1/\sqrt{n}) = O_P(1)o_P(1) = o_P(1)$$

So $X_n \overset{P}{\to} \mu$.

Second way: I observe that

$$(X_n - \mu) = \underbrace{\frac{1}{\sqrt{n}}}_{=A_n} \underbrace{\sqrt{n}(X_n - \mu)}_{=B_n}$$

It is $A_n = \sqrt{n}(X_n - \mu) \underset{D}{\to} Z$, and $B_n = \frac{1}{\sqrt{n}} \to 0$. By Slutsky theorem it is

$$(X_n - \mu) = A_n B_n \underset{D}{\to} 0Z = 0$$

So

$$(X_n - \mu) \underset{D}{\to} 0$$

which implies

$$X_n \overset{P}{\to} \mu$$

Exercise 15. ($\star\star$)

1. If $X_1, X_2, ...$ are IID in $\mathbb{R}^2$ with distribution giving probability

$$P(X = x) = \begin{cases} \theta_1 & \text{, if } x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \theta_2 & \text{, if } x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \theta_1 + \theta_2 & \text{, if } x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

there $\theta_1 + \theta_2 \leq 1$.. What is the asymptotic distribution of $\bar{X}_n$ given the CLT?

2. If $X_1, X_2, ...$ are IID from a Poisson distribution $\text{Poi}(\theta)$ distribution as

$$P(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} 1(x \in \{0, 1, 2, ...\})$$

Let Z_n be the proportion of zeros observed $Z_n = \frac{1}{n}\sum_{j=1}^{n} 1(X_j = 0)$. What is the joint asymptotic distribution of $(\bar{X}_n, Z_n)$

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Solution.

1. It is $\mu = \mathrm{E}(X) = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $\mathrm{E}(X^T X) = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$, so $\mathrm{Var}(X) = \mathrm{E}(X - \mathrm{E}(X))^T (X - \mathrm{E}(X)) =$

   $\mathrm{E}(X^T X) - \mu^T \mu = \begin{bmatrix} \theta_1(1 - \theta_1) & -\theta_1 \theta_2 \\ -\theta_1 \theta_2 & \theta_2(1 - \theta_2) \end{bmatrix}$ The CLT says $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \mathrm{N}(0, \Sigma)$

2. It is $E(X) = \theta$, $E(1(X = 0)) = \exp(\theta)$, $\mathrm{Var}(X) = \theta$, $\mathrm{Var}(1(X = 0)) = \exp(-\theta)(1 - \exp(-\theta))$

   and $E(X 1(X = 0)) = 0$, so $\mathrm{cov}(X, 1(X = 0)) = -\theta \exp(-\theta)$. So $\Sigma = \begin{bmatrix} \theta & -\theta \exp(-\theta) \\ -\theta \exp(-\theta) & \exp(-\theta)(1 - \exp(-\theta)) \end{bmatrix}$

   . The CLT says $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \mathrm{N}(0, \Sigma)$

Exercise 16. (⋆⋆⋆⋆Super difficult) (The autoregressive model) Consider that $\{\epsilon_n\}$ are IID, with mean $\mathrm{E}(\epsilon_n) = \mu$, and variance $\mathrm{Var}(\epsilon_n) = \sigma^2$, $\forall n$. A time series $\{X_n\}_{n \geq 1}$ is modeled as $X_n \sim \mathrm{AR}(\beta)$ where $\beta \in (-1, 1)$ if

$$X_n = \beta X_{n-1} + \epsilon_n; \text{ for } n \geq 2$$
$$X_1 = \epsilon_1$$

Show that $\bar{X}_n \xrightarrow{\mathrm{qm}} \mu/(1 - \beta)$

1. Show that $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} \epsilon_j (1 - \beta^{n-j+1})/(1 - \beta)$

2. Find $\lim_{n \to \infty} \mathrm{E}(\bar{X}_n) =$?

3. Show that $\lim_{n \to \infty} \mathrm{Var}(\bar{X}_n) = 0$

4. Show that $\bar{X}_n \xrightarrow{\mathrm{qm}} \mu/(1 - \beta)$

[Hint] (1.) Show that $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} \epsilon_j (1 - \beta^{n-j+1})/(1 - \beta)$ (2) Find $\lim_{n \to \infty} \mathrm{E}(\bar{X}_n) = \mu/(1 - \beta)$;
(3) Show that $\lim_{n \to \infty} \mathrm{Var}(\bar{X}_n) = 0$, (4.) ...

Solution.

1. It is $X_i = \sum_{j=1}^{i} \epsilon_j \beta^{n-j}$. So

$$\bar{X}_n = \frac{1}{n} \frac{\mu}{1 - \beta} \sum_{i=1}^{n} \sum_{j=1}^{i} \epsilon_j \beta^{n-j} = \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \frac{1 - \beta^{n-j+1}}{1 - \beta}$$

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2. It is

$$\mathrm{E}\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}(\epsilon_j) \frac{1 - \beta^{n-j+1}}{1 - \beta} = \frac{1}{n} \frac{\mu}{1 - \beta} \sum_{j=1}^{n} (1 - \beta^{n-j+1})$$

$$= \frac{1}{n} \frac{\mu}{1 - \beta} \sum_{j=1}^{n} (1 - \beta^j)$$

$$= \frac{1}{n} \frac{\mu}{1 - \beta} (n - \frac{\beta(1 - \beta^n)}{1 - \beta})$$

$$= \frac{\mu}{1 - \beta} - \frac{\beta\mu}{n} \frac{(1 - \beta^n)}{(1 - \beta)^2}$$

So $\lim \mathrm{E}\bar{X}_n = \frac{\mu}{1-\beta}$

3. It is

$$\mathrm{Var}(\bar{X}_n) = \sum_{i=1}^{n} \mathrm{Var}(\epsilon_j) (\frac{1}{n} \frac{1 - \beta^{n-j+1}}{1 - \beta})^2 = \sigma^2 \frac{1}{n^2} \sum_{i=1}^{n} \frac{(1 - \beta^{n-j+1})^2}{(1 - \beta)^2}$$

$$\leq \sigma^2 \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{(1 - \beta)^2} \leq \sigma^2 \frac{1}{n}$$

as $\beta \in (0, 1)$. So $\lim \mathrm{Var}(\bar{X}_n) = 0$

4. It is

$$\lim (\mathrm{E}\bar{X}_n - \frac{\mu}{1 - \beta})^2 = \lim (\mathrm{Var}(\bar{X}_n) + (\mathrm{E}\bar{X}_n - \frac{\mu}{1 - \beta})^2)$$

$$= \lim \mathrm{Var}(\bar{X}_n) + (\lim \mathrm{E}\bar{X}_n - \frac{\mu}{1 - \beta})^2$$

$$= 0$$

So $\bar{X}_n \xrightarrow{\mathrm{qm}} \mu/(1 - \beta)$

Exercise 17. ($\star\star$) Let $X_i \overset{\mathrm{IID}}{\sim} F_X$ for $i = 1, ..., n$, and $F_X = P(X \leq x)$. Show that the empirical distribution function $\hat{F}_X(x) = \frac{1}{n} \sum_{i=1}^{n} 1(x \in [x_i, \infty))$ is a strongly consistent estimator of F_X.

Solution. It is $\mathrm{E}(\hat{F}_X(x)) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}(1(x \in (-\infty, x_i])) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{P}(x \in (-\infty, x_i]) \leq \frac{1}{n} \sum_{i=1}^{n} 1 < \infty$ So the strong LLN applies.

Exercise 18. ($\star\star$) Assume X_1, X_2, X_3 independent from Uniform distribution $\mathrm{U}(0, 1)$. Compare the exact, Normal approximation, and Edgeworth approximation.

Hint: The exact result is $P(X_1 + X_2 + x_3 \leq 2) = 0.8333$

Solution.

It is $\mu = 1/2$, $\sigma^2 = 1/12$, $\kappa_3 = 0$. Also, $E(X - 1/2)^4 = \int_0^1 (x - 1/2)^4 dx = 1/80$. So $\kappa_4 = E(X - 1/2)^4/\sigma^4 - 3 = -1.2$.

So

Normal approx. $P(X_1 + X_2 + x_3 \leq 2) = P(\sqrt{3}(\bar{X}_3 - \mu)^2/\sigma \leq (\frac{2}{3} - \frac{1}{2})\sqrt{12}\sqrt{3}) \approx \Phi(1) = 0.8413$

Edgeworth Expansion. $P(X_1 + X_2 + x_3 \leq 2) \approx \Phi(1) + 0 - 1.2(1 - 3)/(24 \times 3)\phi(1) = 0.8332$

The next exercise is from Problem Class 2

3 Handout 3: Asymptotics after transformations

Exercise 19. Consider random variables $X, X_1, X_2, ...$, where $\mu_n = E(X - \mu)^n$, and $\mu = E(X)$

1. Show that,

$$\sqrt{n}(\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix})$$

2. Show that the asymptotic distribution of the coefficient of variation $cv = \frac{s_x}{\bar{X}}$, is

$$\sqrt{n}(\frac{s_x}{\bar{X}} - \frac{\sigma}{\mu}) \xrightarrow{D} N(0, \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4})$$

3. Show that the asymptotic distribution of the 3rd central moment $m_3 = \frac{1}{n}\sum_{i=1}^{n}(X_j - \bar{X})^3$ is

$$\sqrt{n}(m_3 - \mu_3) \xrightarrow{D} N(0, \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6)$$

Solution.

1.

- I observe that

$$\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} \bar{X} - \mu \\ \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 - (\bar{X} - \mu)^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu) \\ \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu))^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} m_1' \\ m_2' - (m_1')^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

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where $m_1' = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$ and $m_2' = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$.

- I will try to find the joint asymptotic distribution of $(m_1', m_2')^T$ by CLT, and then the asymptotic distribution of $(\bar{X}, s_x^2)^T$ by Delta method.

- Let

$$\xi_i = \begin{bmatrix} X_i - \mu \\ (X_i - \mu)^2 \end{bmatrix}$$

IID random vectors. It is

$$\mu_\xi = E(\xi_i) = \begin{bmatrix} E(X_i - \mu) \\ E(X_i - \mu)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

and

$$\Sigma_\xi = \mathrm{Var}(\xi_i) = \begin{bmatrix} \mathrm{Var}(X_i - \mu) & \mathrm{Cov}((X_i - \mu), (X_i - \mu)^2) \\ \mathrm{Cov}((X_i - \mu), (X_i - \mu)^2) & \mathrm{Var}(X_i - \mu)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$$

since

$$
\begin{aligned}
\mathrm{Cov}((X_i - \mu), (X_i - \mu)^2) &= E(((X_i - \mu) - E(X_i - \mu))((X_i - \mu)^2 - E(X_i - \mu)^2)) \\
&= E(((X_i - \mu) - \mu_1)((X_i - \mu)^2 - \mu_2)) \\
&= E((X_i - \mu)^3 - (X_i - \mu)\mu_2 - \mu_1(X_i - \mu)^2 + \mu_1\mu_2) \\
&= E(X_i - \mu)^3 - \underline{E(X_i - \mu)}^{\,0}\mu_2 - \mu_1\underline{E(X_i - \mu)^2}^{\,\mu_2} + \mu_1\mu_2 \\
&= E(X_i - \mu)^3 = \mu_3
\end{aligned}
$$

It is

$$\bar{\xi} = \begin{bmatrix} m_1' \\ m_2' \end{bmatrix}$$

So by CLT, I have,

$$\sqrt{n}\left(\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}\right) \xrightarrow{D} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}\right)$$

- Now, I will calculate the asymptotic distribution of $(\bar{X}, s_x^2)^T$ by Delta method.

Let,

$$g(x, y) = \begin{bmatrix} x \\ y - x^2 \end{bmatrix}$$

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with

$$\dot{g}(x,y) = \frac{\mathrm{d}g(x,y)}{\mathrm{d}(x,y)} = \begin{bmatrix} -1 & 0 \\ -2x & 1 \end{bmatrix}$$

So

$$g(\underbrace{m_1', m_2'}_{=\bar{\xi}}) = \begin{bmatrix} m_1' \\ m_2' - (m_1')^2 \end{bmatrix} = \begin{bmatrix} \bar{X} - \mu \\ s_x^2 \end{bmatrix}; \qquad g(0, \sigma^2) = \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

$$\dot{g}(\underbrace{0, \sigma^2}_{=\mu_\xi}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \Sigma_g = \dot{g}(\underbrace{0, \sigma^2}_{=\mu_\xi})\Sigma_\xi \dot{g}(\underbrace{0, \sigma^2}_{=\mu_\xi})^T = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$$

Then, according to Delta theorem

$$\sqrt{n}(g(\bar{\xi}) - g(\mu_\xi)) \xrightarrow{D} \mathrm{N}(0, \dot{g}(\mu_\xi)\Sigma_\xi \dot{g}(\mu_\xi)^T)$$

$$\sqrt{n}(\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} \mathrm{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix})$$

2. Since I have the asymptotic distribution of $(\bar{X}, s_x^2)^T$, I can use the Delta method.

- Let $h(a,b) = \sqrt{b}/a$, with $\dot{h}(a,b) = (-\frac{\sqrt{b}}{a^2}, \frac{1}{2a\sqrt{b}})$.

- Then

$$h(\bar{X}, s_x^2) = \frac{s_x}{\bar{X}}; \qquad h(\mu, \sigma^2) = \frac{\sigma}{\mu}$$

$$\dot{h}(\mu, \sigma^2) = \left[-\frac{\sigma}{\mu^2}, \quad \frac{1}{2\mu\sigma} \right];$$

$$\Sigma_h = \dot{h}(\mu, \sigma^2)\Sigma_g \dot{h}(\mu, \sigma^2)^T$$

$$= \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4}$$

- Then, according to Delta theorem

$$\sqrt{n}(h(\bar{X}, s_x^2) - h(\mu, \sigma^2)) \xrightarrow{D} \mathrm{N}(0, \dot{h}(\mu, \sigma^2)\Sigma_g \dot{h}(\mu, \sigma^2)^T)$$

$$\sqrt{n}(\frac{s_x}{\bar{X}} - \frac{\sigma}{\mu}) \xrightarrow{D} \mathrm{N}(0, \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4})$$

3. I observe that

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}((\underbrace{X_i - \mu}_{=Z_i}) - (\underbrace{\bar{X} - \mu}_{=\bar{Z}}))^3 =$$

$$= \frac{1}{n}\sum_{i=1}^{n} Z_i^3 - 3\frac{1}{n}\sum_{i=1}^{n} Z_i^2 \bar{Z} + 2\bar{Z}$$

$$= m_3' - 3m_2' m_1' + 2(m_1')^2$$

where $m_j' = \frac{1}{n}\sum_{i=1}^{n} Z_i^j = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^j$, since $Z_i = X_i - \mu$.

- I will use the CLT to calculate the joint asymptotic distribution of $(m_1', m_2', m_3')^T$ and then I will use Delta method to calculate that of m_3.

  I specify

$$\psi_i = \begin{bmatrix} Z_i \\ Z_i^2 \\ Z_i^3 \end{bmatrix} = \begin{bmatrix} X_i - \mu \\ (X_i - \mu)^2 \\ (X_i - \mu)^3 \end{bmatrix};$$

which are IID, with

$$\bar{\psi} = \frac{1}{n}\sum_{i=1}^{n} \psi_i = \begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix}$$

$$\mu_\psi = \mathrm{E}(\psi_i) = \begin{bmatrix} \mathrm{E}(X_i - \mu) \\ \mathrm{E}(X_i - \mu)^2 \\ \mathrm{E}(X_i - \mu)^3 \end{bmatrix} = \begin{bmatrix} \mu \\ \sigma^2 \\ \mu_3 \end{bmatrix};$$

$$\Sigma_\psi = \mathrm{Var}(\psi_i) = \begin{bmatrix} \mathrm{Var}(X_i - \mu) & \mathrm{Cov}((X_i - \mu),(X_i - \mu)^2) & \mathrm{Cov}((X_i - \mu),(X_i - \mu)^3) \\ \mathrm{Cov}((X_i - \mu),(X_i - \mu)^2) & \mathrm{Var}((X_i - \mu)^2) & \mathrm{Cov}((X_i - \mu)^2,(X_i - \mu)^3) \\ \mathrm{Cov}((X_i - \mu),(X_i - \mu)^3) & \mathrm{Cov}((X_i - \mu)^2,(X_i - \mu)^3) & \mathrm{Var}((X_i - \mu)^3) \end{bmatrix};$$

$$= ...\text{calculations...} = \begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix}$$

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For instance, you can compute the covariance terms as

$$\text{Cov}((X_i - \mu)^2, (X_i - \mu)^3) = \text{E}(\,((X_i - \mu)^2 - \text{E}(X_i - \mu)^2)\,((X_i - \mu)^3 - \text{E}(X_i - \mu)^3)\,)$$

$$= \text{E}(\,((X_i - \mu)^2 - \mu_2)\,((X_i - \mu)^3 - \mu_3)\,)$$

$$= \text{E}(\,(X_i - \mu)^5 - \text{E}(X_i - \mu)^2\mu_3 - \mu_2(X_i - \mu)^3 + \mu_2\mu_3\,)$$

$$= \mu_5 - \mu_2\mu_3$$

So by CLT

$$\sqrt{n}(\underbrace{\begin{bmatrix} m'_1 \\ m'_2 \\ m'_3 \end{bmatrix}}_{=\bar{\psi}} - \underbrace{\begin{bmatrix} \mu \\ \sigma^2 \\ \mu_3 \end{bmatrix}}_{=\mu_\psi}) \xrightarrow{D} \text{N}(0\,\underbrace{\begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix}}_{=\Sigma_\psi})$$

- Now, in order to find the asymptotic distribution of $m_3 = m'_3 - 3m'_2 m'_1 + 2(m'_1)^2$, I will use Delta method

Let
$$q(a, b, c) = c - 3ab + 2a^3$$

then
$$\dot{q}(a, b, c) = \frac{\text{d}}{\text{d}(a, b, c)} q(a, b, c) = \begin{bmatrix} -3b + 6a^2, & -3a, & 1 \end{bmatrix}$$

So
$$q(m'_1, m'_2, m'_3) = m'_3 - 3m'_2 m'_1 + 2(m'_1)^2$$

$$q(\mu'_1, \mu'_2, \mu'_3) = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^2 = \mu_3$$

and
$$\dot{q}(\mu, \sigma^2, \mu_3) = \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix}$$

and

$$\dot{q}(\mu_\psi)\Sigma_\psi\dot{q}(\mu_\psi)^T = \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix} \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix}^T$$

$$= \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6$$

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So the asymptotic distribution of m_3 is such that

$$\sqrt{n}(q(\bar{\psi}) - q(\mu_\psi)) \xrightarrow{D} N(0, \dot{q}(\mu_\psi)\Sigma_\psi \dot{q}(\mu_\psi)^T)$$

$$\sqrt{n}(m_3 - \mu_3) \xrightarrow{D} N(0, \mu_6 - \mu_3^2 - 6\sigma^2 \mu_4 + 9\sigma^6)$$

Exercise 20. ($\star\star\star$) Consider an M-way contingency table and consider the quantities obs. cell counts, cell probabilities, cell proportions in their vectorised forms as

$$\underset{\sim}{n} = (n_1, ..., n_N)^T ; \qquad \underset{\sim}{\pi} = (\pi_1, ..., \pi_N)^T ; \qquad \underset{\sim}{p} = (p_1, ..., p_N)^T$$

where $n = \sum_{j=1}^{N} n_j$, and $p_j = n_j/n$.

1. Consider a constant matrix $C \in \mathbb{R}^{k \times N}$, and show that

$$\sqrt{n}(C \log(\underset{\sim}{p}) - C \log(\underset{\sim}{\pi})) \xrightarrow{D} N(0, C\text{diag}(\pi)^{-1}C^T - C11^T C^T) \qquad (3)$$

2. Consider a 3×3 contingency table with probabilities $(\pi_{i,j})$. Find the joint asymptotic distribution of the vector of different log odd ratios

$$\log(\underset{\sim}{\theta}^C) = \begin{bmatrix} \log(\frac{\pi_{11}\pi_{22}}{\pi_{21}\pi_{12}}) \\ \log(\frac{\pi_{22}\pi_{33}}{\pi_{23}\pi_{32}}) \end{bmatrix}$$

Solution.

1. By using CLT (same as in Example in the CLT section in the Handouts), we get

$$\sqrt{n}(p - \pi) \xrightarrow{D} N(0, \text{diag}(\pi) - \pi\pi^T)$$

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Recall from the example:

Denote the i-th observation by $\xi_i = (\xi_{i,1}, ..., \xi_{i,N})^T$, where

$$\xi_{i,j} = \begin{cases} 1 & \text{, if observation } i \text{ falls in cell } j \\ 0 & \text{, if observation } i \text{ does not fall in cell } j \end{cases}$$

Since its observation falls in only one cell, $\sum_j \xi_{i,j} = 1$ and $\xi_{i,j}\xi_{i,k} = 0$ when $j \neq k$.

Therefore p can be considered as the arithmetic mean of $\{\xi_{i,j}\}_{i=1}^n$ IID variables as

$$p = \frac{1}{n} \sum_{i=1}^n \xi_i$$

The moments of $\{\xi_i\}$, are equal to

$$\mathrm{E}(\xi_i) = \pi$$
$$\mathrm{Var}(\xi_i) = \Sigma$$

where

$$[\Sigma]_{j,j} = \mathrm{var}(\xi_{i,j}) = \mathrm{E}(\xi_{i,j}^2) - (\mathrm{E}(\xi_{i,j}))^2 = \pi_j(1 - \pi_j)$$
$$[\Sigma]_{j,k} = \mathrm{cov}(\xi_{i,j}, \xi_{i,k}) = \mathrm{E}(\xi_{i,j}\xi_{i,k}) - \mathrm{E}(\xi_{i,j})\mathrm{E}(\xi_{i,k}) = -\pi_j\pi_k$$

because

$$\mathrm{E}(\xi_{i,j}) = \mathrm{P}(\xi_{i,j} = 1) = \pi_j$$
$$\mathrm{E}(\xi_{i,j}^2) = \mathrm{P}(\xi_{i,j} = 1) = \pi_j$$
$$\mathrm{E}(\xi_{i,j}\xi_{i,k}) = 0, \text{ if } j \neq k$$

Hence

$$\Sigma = \mathrm{diag}(\pi) - \pi\pi^T$$

Therefore, according to the CLT

$$\sqrt{n}(p - \pi) \xrightarrow{D} \mathrm{N}(0, \mathrm{diag}(\pi) - \pi\pi^T) \tag{4}$$

Consider a function

$$g(x) = C\log(x) = C \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

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Calculate

$$\dot{g}(x) = C\text{diag}(\pi)^{-1} = C \begin{bmatrix} 1/\pi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1/\pi_N \end{bmatrix}$$

and notice that it is continuous. So Delta method can be used.

Calculate

$$\dot{g}(\mu)\left(\text{diag}(\pi) - \pi\pi^T\right)\dot{g}(\mu)^T = \dot{g}(\mu)\text{diag}(\pi)\dot{g}(\mu)^T - \dot{g}(\mu)\pi\pi^T\dot{g}(\mu)^T$$

$$= C\text{diag}(\pi)^{-1}\overbrace{\text{diag}(\pi)\text{diag}(\pi)^{-1}}^{I}C^T - C\underbrace{\text{diag}(\pi)^{-1}\pi}_{1}\pi^T\underbrace{\text{diag}(\pi)^{-1}C^T}_{1^T}$$

$$= C\text{diag}(\pi)^{-1}C^T - C11^TC^T$$

Hence from Delta method we get

$$\sqrt{n}(C\log(\underset{\sim}{p}) - C\log(\underset{\sim}{\pi})) \xrightarrow{D} \text{N}(0, C\text{diag}(\pi)^{-1}C^T - C11^TC^T)$$

(a) Let $\underset{\sim}{\pi} = \begin{bmatrix} \pi_{11} & \pi_{21} & \pi_{31} & \pi_{12} & \pi_{22} & \pi_{32} & \pi_{13} & \pi_{23} & \pi_{33} \end{bmatrix}^T$. In fact, the vector of different log odd ratios

$$\log(\underset{\sim}{\theta}^C) = \begin{bmatrix} \log(\frac{\pi_{11}\pi_{22}}{\pi_{21}\pi_{12}}) \\ \log(\frac{\pi_{22}\pi_{33}}{\pi_{23}\pi_{32}}) \end{bmatrix}$$

can be expressed as $\log(\underset{\sim}{\theta}^C) = C\log(\underset{\sim}{\pi})$ with

$$\log(\underset{\sim}{\theta}^C) = \begin{bmatrix} \log(\pi_{11}) - \log(\pi_{12}) - \log(\pi_{21}) + \log(\pi_{22}) \\ \log(\pi_{22}) + \log(\pi_{33}) - \log(\pi_{23}) - \log(\pi_{32}) \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\log(\underset{\sim}{\pi}) = \begin{bmatrix} \log\pi_{11} & \log\pi_{21} & \log\pi_{31} & \log\pi_{12} & \log\pi_{22} & \log\pi_{32} & \log\pi_{13} & \log\pi_{23} & \log\pi_{33} \end{bmatrix}^T$$

so

$$\sqrt{n}(\log(\hat{\underset{\sim}{\theta}}^C) - \log(\underset{\sim}{\theta}^C)) \xrightarrow{D} \text{N}(0, \Sigma)$$

where

$$\Sigma = C\text{diag}(\pi)^{-1}C^T - C11^TC^T = \dots =$$

$$= \begin{bmatrix} \frac{1}{\pi_{11}} + \frac{1}{\pi_{12}} + \frac{1}{\pi_{21}} + \frac{1}{\pi_{22}} & \frac{1}{\pi_{22}} \\ \frac{1}{\pi_{22}} & \frac{1}{\pi_{22}} + \frac{1}{\pi_{32}} + \frac{1}{\pi_{23}} + \frac{1}{\pi_{33}} \end{bmatrix}$$

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Exercise 21. ($\star\star\star$) Consider a random sample $X, X_1, X_2, \ldots$ an IID sample with finite moments $E(X) = 0$, and $\mathrm{E}(X^4) < \infty$.

1. Show that if $m_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$ then

$$\sqrt{n}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}\right) \xrightarrow{D} \mathrm{N}(0, \Sigma)$$

   where $\Sigma = \begin{bmatrix} Var(X) & Cov(X^2, X) \\ Cov(X^2, X) & Var(X^2) \end{bmatrix}$

2. Find an $(1-a)\%$ asymptotic confidence interval for S_n^2.

Solution.

1. Consider $\xi_i = \begin{bmatrix} X_i \\ X_i^2 \end{bmatrix}$, and $\bar{\xi} = \frac{1}{n} \sum_{i=1}^{n} \xi_i$ then $\bar{\xi} = (m_1, m_2)^T$. So from the CLT, we get

$$\sqrt{n}\left(\bar{\xi} - E\begin{bmatrix} X_i \\ X_i^2 \end{bmatrix}\right) \xrightarrow{D} \mathrm{N}\left(0, \begin{bmatrix} Var(X) & Cov(X^2, X) \\ Cov(X^2, X) & Var(X^2) \end{bmatrix}\right)$$

   which is what I want to show

2. I will try to compute the asymptotic distribution of S_n^2 with the Delta Method, and then I ll compute the asymptotic confidence interval.

   - Because $S_n^2 = m_2 - (m_1)^2$, I consider $g((x, y)) = y - x^2$.
   - Because $\frac{\mathrm{d}}{\mathrm{d}(x,y)} g((x, y)) = (-2x, 1)$ and continuous, then the assumptions of Delta method are satisfied, with

$$\begin{aligned} \dot{g}((0, \sigma^2)) \Sigma \dot{g}((0, \sigma^2))^T &= Var(X^2) = E((X^2)^2) - (E(X^2))^2 \\ &= EX^4 - (E(X^2) - (\overset{0}{\cancel{EX}})^2)^2 \\ &= EX^4 - (Var(X))^2 = EX^4 - \sigma^4 \end{aligned}$$

   - So
$$\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{D} \mathrm{N}(0, EX^4 - \sigma^4)$$

   or
$$\sqrt{n} \frac{S_n^2 - \sigma^2}{\sqrt{EX^4 - \sigma^4}} \xrightarrow{D} \mathrm{N}(0, 1)$$

   - By using slusky theorem it is $\frac{EX^4 - \sigma^4}{X^4 - S^4} \xrightarrow{D} 1$

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- and again by using slusky theorem it is

$$\sqrt{n}\frac{S_n^2 - \sigma^2}{\sqrt{\overline{X^4} - S^4}} \xrightarrow{D} N(0, 1)$$

- Hence

$$\{S_n^2 \pm z_{1-\frac{a}{2}}\sqrt{\frac{\overline{X^4} - S^4}{n}}\}$$

The next exercise is from Homework 3

Exercise 22. ($\star\star\star$) Consider an IID sample $X, X_1, X_2, ...$ with $EX = 0$, $EX^4 < \infty$. Consider that

$$\sqrt{n}\frac{S_n^2 - \sigma^2}{\sqrt{EX^4 - \sigma^4}} \xrightarrow{D} N(0, 1) \tag{5}$$

where $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

1. Find the asymptotic distribution of $\log(S_n^2)$.

2. Produce the $1 - a$ asymptotic confidence interval for $\log(\sigma^2)$; by performing suitable calculations, so that the boundaries of the confidence interval do not depend on any unknown moments of the real distribution.

Solution.

Exercise 23. ($\star\star\star$) Let function $g : \mathbb{R} \to \mathbb{R}$ such that $\dot{g}(x)$ and $\ddot{g}(x)$ are continuous in a neighborhood of $\mu \in \mathbb{R}$, and $\dot{g}(\mu) = 0$. Prove the following statement:

- If $X_n \in \mathbb{R}$ is a sequence of random vectors such that $\sqrt{n}(X_n - \mu) \xrightarrow{D} N(0, \sigma^2)$ then

$$n(g(X_n) - g(\mu)) \xrightarrow{D} \frac{\sigma^2 \ddot{g}(\mu)}{2}\chi_1^2$$

Hint-1. Use Taylor expansion of 2nd order.

Hint-2. The Taylor expansion of function $f : \mathbb{R} \to \mathbb{R}$ around point x_0 is:

$$f(x) = \sum_{k=0}^{n}\frac{1}{k!}(x - x_0)f^{(k)}(x_0) + R_n(x)$$

where $R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x - t)^n f^{(n)}(x_0) = o((x - x_0)^n)$ as $x \to x_0$, provided that the n-th derivative $f^{(n)}(x)$ exists in some interval containing x_0.

Solution.

We expand $g(X_n)$ by Taylor (2nd degree) around μ. So

$$g(x) = g(\mu) + \dot{g}(\mu)^{0}(x - \mu) + \frac{\ddot{g}(\mu)}{2}(x - \mu)^2 + o((x - \mu)^2)$$

$$= g(\mu) + \frac{\ddot{g}(\mu)}{2}(x - \mu)^2 + o((x - \mu)^2)$$

So

$$n(g(X_n) - g(\mu)) \approx \frac{\sigma^2 \ddot{g}(\mu)}{2}(\sqrt{n}\frac{X_n - \mu}{\sigma})^2 + o\left((\sqrt{n}(X_n - \mu))^2\right)$$

For the first term, because $\sqrt{n}(X_n - \mu) \xrightarrow{D} \mathrm{N}(0, \sigma^2)$, it is $\sqrt{n}\frac{X_n - \mu}{\sigma} \xrightarrow{D} \mathrm{N}(0, 1)$.

For the second term, because $\sqrt{n}(X_n - \mu) \xrightarrow{D} \mathrm{N}(0, \sigma^2)$ then $\sqrt{n}(X_n - \mu) = O_p(1)$, then $(\sqrt{n}(X_n - \mu))^2 = O_p(1)$. Hence $o\left((\sqrt{n}(X_n - \mu))^2\right) = o(O_p(1)) = o_p(1)$.

Hence by Slutsky rules:

$$n(g(X_n) - g(\mu)) \xrightarrow{D} \frac{\sigma^2 \ddot{g}(\mu)}{2}\chi_1^2$$

The next exercise is from Homework 3

Exercise 24. (★★★) Consider random sample $X, X_1, X_2, ...$ IID from a Bernoulli distribution with probability of success p. Find the variance stabilization transformation for the estimator average $\hat{p}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

Solution.

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4   Handout 4: Estimation by the method of Maximum Likelihood

Exercise 25. Consider random sample $x_1, ..., x_n \overset{IID}{\sim} \mathrm{G}(a, b)$, $a > 0$, $b > 0$ with PDF

$$f(x|a, b) = \frac{1}{\Gamma(a)b^a}x^a e^{-x\frac{1}{b}}1(x > 0)$$

1. Find the moment estimator $\tilde{\theta}$ of $\theta = (a, b)^T$ by using the first raw moment and the first central moment

2. Is the moment estimator $\tilde{\theta}$ consistent and asymptotically Normal?

3. Find the one step estimator by Fisher scoring algorithm.

Hint-1 Digamma function $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$

Hint-2 Trigamma function $\psi_1(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \Gamma(x)$

Hint-3 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solution.

1. The first raw moment is the expected value/mean, and the first central moment is the variance.

   The first raw moment is

   $$\mathrm{E}(x) = \int_0^1 x \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} \mathrm{d}x = \int_0^1 \frac{1}{\frac{1}{a}\Gamma(a+1)\frac{1}{b}b^{a+1}} x^{(a+1)-1} e^{-x/b} \mathrm{d}x = ab$$

   and the sample one

   $$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

   The first central moment is

   $$\mathrm{var}(x) = \mathrm{E}(x^2) - (\mathrm{E}(x))^2$$

   So

   $$\mathrm{E}(x^2) = \int_0^1 x^2 \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} \mathrm{d}x = \int_0^1 \frac{1}{\frac{1}{a(a+1)}\Gamma(a+2)\frac{1}{b^2}b^{a+2}} x^{(a+2)-1} e^{-x/b} \mathrm{d}x = a(a+1)b^2$$

   and hence

   $$\mathrm{var}(x) = \mathrm{E}(x^2) - (\mathrm{E}(x))^2 = ab^2$$

   The sample first central moment is

   $$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

   From the method of moments I get

   $$\begin{cases} \mathrm{E}(x|\tilde{a}, \tilde{b}) = & \bar{x} \\ \mathrm{var}(x|\tilde{a}, \tilde{b}) = & s^2 \end{cases} \implies \begin{cases} \tilde{a} = & \frac{\bar{x}^2}{s^2} \\ \tilde{b} = & \frac{\bar{x}^2}{s^2} \end{cases} \implies \begin{cases} \tilde{a} = & \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} = \frac{\bar{x}^2}{s^2} \\ \tilde{b} = & \frac{\mathrm{var}(x)}{\mathrm{E}(x)} = \frac{\bar{x}^2}{s^2} \end{cases}$$

   So the moment estimator is

   $$\tilde{\theta} = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} \frac{\bar{x}^2}{s^2} \\ \frac{s^2}{\bar{x}} \end{bmatrix} \tag{6}$$

2. It is consistent because $\tilde{\theta} \xrightarrow{as} \theta$. This is because of the following.

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It is

$$\begin{cases} \mathrm{E}(x) = & ab \\ \mathrm{var}(x) = & ab^2 \end{cases} \implies \begin{cases} a = & \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\ b = & \frac{\mathrm{var}(x)}{\mathrm{E}(x)} \end{cases} \implies \begin{cases} a = & \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\ b = & \frac{\mathrm{var}(x)}{\mathrm{E}(x)} \end{cases}$$

From SLLN, $\bar{x} \xrightarrow{as} \mathrm{E}(x)$. From SLLN, $\overline{x^2} \xrightarrow{as} \mathrm{E}(x^2)$. From Slutsky Theorem, $s^2 = \overline{x^2} - (\bar{x})^2 \xrightarrow{as} \mathrm{E}(x^2) - \mathrm{E}(x)^2 = \mathrm{var}(x)$

So From Slutsky theorem

$$\tilde{\theta} = \begin{bmatrix} \frac{\bar{x}^2}{s^2} \\ \frac{s^2}{\bar{x}} \end{bmatrix} \xrightarrow{as} \begin{bmatrix} \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\ \frac{\mathrm{var}(x)}{\mathrm{E}(x)} \end{bmatrix} = \theta$$

It is asymptotically Normal because of the following.

$\bar{x}$ and s^2 are asymptotically Normal by the CLT, as averages of IID quantities. Hence, by Delta method, (16) is asymptotically Normal.

3. Recall the the one-step estimators

$$\text{Newton alg.} \qquad \breve{\theta}_n = \tilde{\theta}_n - \ddot{\ell}_n(\tilde{\theta}_n)^{-1}\dot{\ell}_n(\tilde{\theta}_n) \tag{7}$$

$$\text{Fisher scoring alg.} \qquad \breve{\theta}_n = \tilde{\theta}_n + \frac{1}{n}\mathcal{I}(\tilde{\theta}_n)^{-1}\dot{\ell}_n(\tilde{\theta}_n) \tag{8}$$

For the Fisher algorithm, I need to find $\mathcal{I}(\theta)^{-1}$. It is

$$\log f(x|\theta) = -\log\Gamma(a) - a\log(b) - \frac{1}{b}x + (a-1)\log(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log f(x|\theta) = \begin{bmatrix} -\psi(a) - \log(b) + \log(x) \\ -\frac{a}{b} + \frac{1}{b^2}x \end{bmatrix}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log f(x|\theta) = \begin{bmatrix} -\psi_1(a) & -\frac{1}{b} \\ -\frac{1}{b} & -\frac{2x-ab}{b^3} \end{bmatrix}$$

$$\mathcal{I}(\theta) = \begin{bmatrix} \psi_1(a) & \frac{1}{b} \\ \frac{1}{b} & \frac{a}{b^2} \end{bmatrix}$$

$$\mathcal{I}(\theta)^{-1} = \frac{1}{a\psi_1(a) - 1}\begin{bmatrix} a & -b \\ -b & b^2\psi_1(a) \end{bmatrix}$$

$$\ell_n(\theta) = -n\log\Gamma(a) - na\log(b) - \frac{1}{b}\sum_{i=1}^n x_i + (a-1)\sum_{i=1}^n\log(x_i)$$

$$\dot{\ell}_n(\theta) = \begin{bmatrix} -n\psi(a) - n\log(b) + \sum_{i=1}^n\log(x_i) \\ -n\frac{a}{b} + n\frac{1}{b^2}\bar{x} \end{bmatrix}$$

The Fisher recursion is

$$\breve{\theta}_n = \tilde{\theta}_n + \frac{1}{n}\mathcal{I}(\tilde{\theta}_n)^{-1}\dot{\ell}_n(\tilde{\theta}_n)$$

$$\breve{\theta}_n = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{n}\frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} \tilde{a} & -\tilde{b} \\ -\tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -n\psi(\tilde{a})-n\log(\tilde{b})+\sum_{i=1}^n\log(x_i) \\ -n\frac{\tilde{a}}{\tilde{b}}+n\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} \tilde{a} & -\tilde{b} \\ -\tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -\psi(\tilde{a})-\log(\tilde{b})+\frac{1}{n}\sum_{i=1}^n\log(x_i) \\ -\frac{\tilde{a}}{\tilde{b}}+\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} -\tilde{a}\psi(\tilde{a})-\frac{1}{\tilde{b}}(\bar{x}-\tilde{a}\tilde{b})-\tilde{a}\log(\tilde{b})+\frac{\tilde{a}}{n}\sum_{i=1}^n\log(x_i) \\ \tilde{b}\psi(\tilde{a})-\psi_1(\tilde{a})(\bar{x}-\tilde{a})+\tilde{b}\log(\tilde{b})-\frac{\tilde{b}}{n}\sum_{i=1}^n\log(x_i) \end{bmatrix}$$

So bu substituting

$$\breve{\theta}_n = \begin{bmatrix} \frac{\bar{x}^2}{s^2} \\ \frac{s^2}{\bar{x}} \end{bmatrix} + \frac{1}{\frac{\bar{x}^2}{s^2}\psi_1(\frac{\bar{x}^2}{s^2})-1}\begin{bmatrix} -\frac{\bar{x}^2}{s^2}\psi(\frac{\bar{x}^2}{s^2})-\frac{\bar{x}^2}{s^2}\log(\frac{s^2}{\bar{x}})+\frac{1}{n}\frac{\bar{x}^2}{s^2}\sum_{i=1}^n\log(x_i) \\ \frac{s^2}{\bar{x}}\psi(\frac{\bar{x}^2}{s^2})-\psi_1(\frac{\bar{x}^2}{s^2})(\bar{x}-\frac{\bar{x}^2}{s^2})+\frac{s^2}{\bar{x}}\log(\frac{s^2}{\bar{x}})-\frac{\tilde{b}}{n}\sum_{i=1}^n\log(x_i) \end{bmatrix}$$

Additionally for the Newton recursion I need

$$\ddot{\ell}_n(\theta) = -n\begin{bmatrix} \psi_1(a) & \frac{1}{b} \\ \frac{1}{b} & \frac{2\bar{x}-ab}{b^3} \end{bmatrix}$$

$$(\ddot{\ell}_n(\theta))^{-1} = -\frac{1}{n}\frac{1}{\psi_1(a)\frac{2\bar{x}-ab}{b}-1}\begin{bmatrix} \frac{2\bar{x}-ab}{b} & -b \\ -b & b^2\psi_1(a) \end{bmatrix}$$

The Newton recursion is

$$\breve{\theta}_n = \tilde{\theta}_n - (\ddot{\ell}_n(\theta))^{-1}\dot{\ell}_n(\tilde{\theta}_n)$$

$$\breve{\theta}_n = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{n}\frac{1}{\psi_1(\tilde{a})\frac{2\bar{x}-\tilde{a}\tilde{b}}{\tilde{b}}-1}\begin{bmatrix} 2\bar{x}-\tilde{a}\tilde{b} & \tilde{b} \\ \tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -n\psi(\tilde{a})-n\log(\tilde{b})+\sum_{i=1}^n\log(x_i) \\ -n\frac{\tilde{a}}{\tilde{b}}+n\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\psi_1(\tilde{a})\frac{2\bar{x}-\tilde{a}\tilde{b}}{\tilde{b}}-1}\begin{bmatrix} 2\bar{x}-\tilde{a}\tilde{b} & \tilde{b} \\ \tilde{b} & \tilde{b}^2\psi_1(\tilde{b}) \end{bmatrix}\begin{bmatrix} -\psi(\tilde{a})-\log(\tilde{b})+\frac{1}{n}\sum_{i=1}^n\log(x_i) \\ -\frac{\tilde{a}}{\tilde{b}}+\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= ...\text{calculations}$$

Exercise 26. Prove the Information inequality theorem:

Let $x \in \mathbb{R}^d$ random vector following distribution $\mathrm{d}f_\theta(\cdot)$ labeled by an parameter $\theta \in \Theta \subset \mathbb{R}^r$ and admitting PDF $f(\cdot|\theta)$. Consider an estimator $\hat{\theta}_n := \hat{\theta}_n(x) \in \Theta \subset \mathbb{R}^r$ such that $g(\theta) = \mathrm{E}_{f_\theta}(\hat{\theta}_n)$ exists on Θ. Assume that, $\frac{\mathrm{d}}{\mathrm{d}\theta}f(x|\theta)$ exists ; $\frac{\mathrm{d}}{\mathrm{d}\theta}$ can pass under the integral sign in $\int f(x|\theta)\mathrm{d}x$ and

$\int \hat\theta_n(x) f(x|\theta)\mathrm{d}x$. Then

$$\text{var}_{f_\theta}(\hat\theta_n(x)) \geq \frac{1}{n}\dot g(\theta)\mathcal{I}(\theta)^{-1}\dot g(\theta)^T \tag{9}$$

where $\mathcal{I}(\theta)$ is the Fisher's information matrix.

- The quantity $\frac{1}{n}\dot g(\theta)\mathcal{I}(\theta)^{-1}\dot g(\theta)^T$ is called Cramer-Rao lower bound (CRLB).

Hint-1: Use $0 \leq \text{var}_{f_\theta}(\hat\theta_n - \dot g(\theta)\mathcal{I}(\theta)^{-1}\Psi(x,\theta)) = ...$

Hint-2: Use $\text{var}_{f_\theta}(A + B) = \text{var}_{f_\theta}(A) + \text{var}_{f_\theta}(B) + 2\text{cov}_{f_\theta}(A, B)$

Solution. Let $\Psi(x,\theta) = (\frac{\mathrm{d}}{\mathrm{d}\theta}\log f(x|\theta))^T$.

It is

$$\mathrm{E}_{f_\theta}\Psi(X,\theta) = 0 \quad \text{(you have proved it before)}$$

$$\begin{aligned}
\dot g_n(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\int \hat\theta_n(x)f(x|\theta)\mathrm{d}x = \int \hat\theta_n(x)\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}f(x|\theta)}{f(x|\theta)}f(x|\theta)\mathrm{d}x \\
&= \int \hat\theta_n(x)\frac{\mathrm{d}}{\mathrm{d}\theta}\log f(x|\theta)f(x|\theta)\mathrm{d}x = \mathrm{E}_{f_\theta}(\hat\theta_n(x)(\Psi(x,\theta) - \underbrace{\mathrm{E}_\theta\Psi(X,\theta)}_{= 0})) \\
&= \text{cov}_{f_\theta}(\hat\theta_n(x), \Psi(x,\theta)) \tag{10}
\end{aligned}$$

So

$$\begin{aligned}
0 &\leq \text{var}_{f_\theta}(\hat\theta_n - \dot g(\theta)\mathcal{I}(\theta)^{-1}\Psi(x,\theta)) \\
&= \text{var}_{f_\theta}(\hat\theta_n) - 2\text{cov}_{f_\theta}(\hat\theta_n, \dot g(\theta)\mathcal{I}(\theta)^{-1}\Psi(x,\theta)) + \text{var}_{f_\theta}(\dot g(\theta)\mathcal{I}(\theta)^{-1}\Psi(x,\theta)) \\
&= \text{var}_{f_\theta}(\hat\theta_n) - 2\frac{1}{n}\dot g(\theta)\mathcal{I}(\theta)^{-1}\dot g(\theta)^T + \frac{1}{n}\dot g(\theta)\mathcal{I}(\theta)^{-1}\mathcal{I}(\theta)\mathcal{I}(\theta)^{-1}\dot g(\theta)^T \\
&= \text{var}_{f_\theta}(\hat\theta_n) - \frac{1}{n}\dot g(\theta)\mathcal{I}(\theta)^{-1}\dot g(\theta)^T
\end{aligned}$$

and the proof is done

Exercise 27. Prove the following statement: Given that the assumptions of Cramer Theorem (for the Normality of MLE) are satisfied, and that $\mathcal{I}(\theta)$ and $\mathcal{J}_n(\theta)$ are continuous on θ, then

$$\sqrt{n}\mathcal{I}(\theta_0)^{1/2}(\hat\theta_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{11}$$

$$\sqrt{n}\mathcal{I}(\hat\theta_n)^{1/2}(\hat\theta_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{12}$$

$$\mathcal{J}_n(\hat\theta_n)^{1/2}(\hat\theta_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{13}$$

where $\hat{\theta}_n$ denotes the MLE, θ_0 denotes the true value of θ, and $A^{1/2}$ denotes the lower triangular matrix of the Cholesky decomposition of A; i.e., $A = A^{1/2}(A^{1/2})^T$.

Solution.

- Eq 11 results from Cramer Theorem, and the properties of covariance matrix.

  - Eq. 12 results by using Cramer Theorem and Slutsly theorems. Precisely, because $\hat{\theta}_n \xrightarrow{a.s.} \theta_0$, Slutsky implies $\mathcal{I}(\hat{\theta}_n) \xrightarrow{a.s.} \mathcal{I}(\theta_0)$ which implies $\mathcal{I}(\hat{\theta}_n)^{1/2}\mathcal{I}(\theta_0)^{-1/2} \xrightarrow{a.s.} I$. Therefore, by Slutsky

  $$\underbrace{\mathcal{I}(\hat{\theta}_n)^{1/2}\mathcal{I}(\theta_0)^{-1/2}\sqrt{n}\mathcal{I}(\theta_0)^{1/2}(\hat{\theta}_n - \theta_0)}_{=\sqrt{n}\mathcal{I}(\hat{\theta}_n)^{1/2}(\hat{\theta}_n-\theta_0)} \xrightarrow{D} \underbrace{I \times N(0, I)}_{=N(0,I)}$$

  - Eq. 13 results by using the USLLN and Slutsly theorems. So I just need to show that

  $$\frac{1}{n}\mathcal{J}_n(\hat{\theta}_n) \xrightarrow{a.s.} \mathcal{I}(\theta_0)$$

  Set $U(x, \theta) = -\frac{d^2}{d\theta^2}\log(f(x|\theta))$, and $\mathcal{I}(\theta) = E(U(x, \theta))$. Then

  $$|\frac{1}{n}\sum_{i=1}^{n}\underbrace{\left(-\frac{d^2}{d\theta^2}\log(f(x_i|\hat{\theta}_n))\right)}_{U(x_i,\hat{\theta}_n)} - \mathcal{I}(\theta_0)| \leq |\frac{1}{n}\sum_{i=1}^{n}U(x_i, \hat{\theta}_n) - \mathcal{I}(\hat{\theta}_n)| + |\mathcal{I}(\hat{\theta}_n) - \mathcal{I}(\theta_0)|$$

  $$(14)$$

  $$\leq \sup_{|\hat{\theta}_n - \theta_0| \leq \delta}|\frac{1}{n}\sum_{i=1}^{n}U(x_i, \theta) - \mathcal{I}(\theta)| + |\mathcal{I}(\hat{\theta}_n) - \mathcal{I}(\theta_0)|$$

  $$(15)$$

  The first term converges to zero because the assumptions of the USLLN are satisfied. The second term converges to zero because $\hat{\theta}_n \xrightarrow{a.s.} \theta_0$ and hence $\mathcal{I}(\hat{\theta}_n) \xrightarrow{a.s.} \mathcal{I}(\theta_0)$ by using Slutsky theorem.
  So by Slutsy $(\frac{1}{n}\mathcal{J}_n(\hat{\theta}_n))^{1/2}\mathcal{I}(\theta_0)^{-1/2} \xrightarrow{a.s.} I$, and by Slutsky again

  $$\underbrace{(\frac{1}{n}\mathcal{J}_n(\hat{\theta}_n))^{1/2}\mathcal{I}(\theta_0)^{-1/2}I(\theta_0)^{1/2}(\hat{\theta}_n - \theta_0)}_{=\mathcal{J}_n(\hat{\theta}_n)^{1/2}(\hat{\theta}_n-\theta_0)} \xrightarrow{D} \underbrace{I \times N(0, I)}_{=N(0,I)}$$

Exercise 28. ($\star\star$) (Shannon-Kolmogorov Information Inequality) Prove the Shannon-Kolmogorov Information Inequality. Let f_0 and f_1 (like $f_0(\cdot) = f(\cdot|\theta_0)$ and $f_1(\cdot) = f(\cdot|\theta_1)$) be PDFs of corre-

sponding distributions with respect to x. Then

$$\text{KL}(f_0, f_1) = \text{E}_0 \log \frac{f_0(X)}{f_1(X)} = \int \log \frac{f_0(X)}{f_1(X)} f_0(X) \mathrm{d}X \geq 0$$

with the equality iff $f_0(x) = f_1(x)$ a.s.

Solution. Function $\log(\cdot)$ is convex, then Jensen's inequality[2] implies

$$-K(f_0, f_1) = \text{E}_0 \log \frac{f_1(X)}{f_0(X)} \quad :: \quad \begin{cases} < \log \text{E}_0 \frac{f_1(X)}{f_0(X)} & \text{,if } f_1(x) \neq f_0(x) \\ = \log \text{E}_0 \frac{f_1(X)}{f_0(X)} & \text{,if } f_1(x) = f_0(x) \end{cases}$$

But

$$\text{E}_0 \frac{f_1(x)}{f_0(x)} = \int \frac{f_1(x)}{f_0(x)} f_0(x) \mathrm{d}x = \int_{S_0} f_1(x) \mathrm{d}x \leq 1$$

at $S_0 = \{x : f_0(x) > 0\}$. Hence,

$$\text{KL}(f_0, f_1) : \begin{cases} > 0 & \text{,if } f_1(x) \neq f_0(x) \\ = 0 & \text{,if } f_1(x) = f_0(x) \end{cases}$$

5 Handout 5: Improving sub-efficient estimators

Exercise 29. Consider random sample $x_1, ..., x_n \overset{IID}{\sim} \text{G}(a, b)$, $a > 0$, $b > 0$ with PDF

$$f(x|a, b) = \frac{1}{\Gamma(a)b^a} x^a e^{-x\frac{1}{b}} 1(x > 0)$$

1. Find the moment estimator $\tilde{\theta}$ of $\theta = (a, b)^T$ by using the first raw moment and the first central moment

2. Is the moment estimator $\tilde{\theta}$ consistent and asymptotically Normal?

3. Find the one step estimator by Fisher scoring algorithm.

Hint-1 Digamma function $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$

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[2]Jensen's inequality: Consider a function φ, it is
- $\text{E}(\varphi(x)) \leq \varphi(\text{E}(x))$ if $\varphi(\cdot)$ is convex
- $\text{E}(\varphi(x)) \geq \varphi(\text{E}(x))$ if $\varphi(\cdot)$ is concave
- The equality holds if x is constant a.s.

Hint-2 Trigamma function $\psi_1(x) = \frac{d^2}{dx^2} \log \Gamma(x)$

Hint-3 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solution.

1. The first raw moment is the expected value/mean, and the first central moment is the variance.

   The first raw moment is

   $$E(x) = \int_0^1 x \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} dx = \int_0^1 \frac{1}{\frac{1}{a}\Gamma(a+1)\frac{1}{b}b^{a+1}} x^{(a+1)-1} e^{-x/b} dx = ab$$

   and the sample one

   $$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

   The first central moment is

   $$\text{var}(x) = E(x^2) - (E(x))^2$$

   So

   $$E(x^2) = \int_0^1 x^2 \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} dx = \int_0^1 \frac{1}{\frac{1}{a(a+1)}\Gamma(a+2)\frac{1}{b^2}b^{a+2}} x^{(a+2)-1} e^{-x/b} dx = a(a+1)b^2$$

   and hence

   $$\text{var}(x) = E(x^2) - (E(x))^2 = ab^2$$

   The sample first central moment is

   $$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

   From the method of moments I get

   $$\begin{cases} E(x|\tilde{a}, \tilde{b}) = & \bar{x} \\ \text{var}(x|\tilde{a}, \tilde{b}) = & s^2 \end{cases} \implies \begin{cases} \tilde{a} = & \frac{\bar{x}^2}{s^2} \\ \tilde{b} = & \frac{\bar{x}^2}{s^2} \end{cases} \implies \begin{cases} \tilde{a} = & \frac{(E(x))^2}{\text{var}(x)} = \frac{\bar{x}^2}{s^2} \\ \tilde{b} = & \frac{\text{var}(x)}{E(x)} = \frac{\bar{x}^2}{s^2} \end{cases}$$

   So the moment estimator is

   $$\tilde{\theta} = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} \frac{\bar{x}^2}{s^2} \\ \frac{s^2}{\bar{x}} \end{bmatrix} \tag{16}$$

2. It is consistent because $\tilde{\theta} \xrightarrow{as} \theta$. This is because of the following.

It is

$$
\begin{cases}
\mathrm{E}(x) = & ab \\
\mathrm{var}(x) = & ab^2
\end{cases}
\implies
\begin{cases}
a = & \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\
b = & \frac{\mathrm{var}(x)}{\mathrm{E}(x)}
\end{cases}
\implies
\begin{cases}
a = & \frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\
b = & \frac{\mathrm{var}(x)}{\mathrm{E}(x)}
\end{cases}
$$

From SLLN, $\bar{x} \xrightarrow{as} \mathrm{E}(x)$. From SLLN, $\overline{x^2} \xrightarrow{as} \mathrm{E}(x^2)$. From Slutsky Theorem, $s^2 = \overline{x^2} - (\bar{x})^2 \xrightarrow{as}$ $\mathrm{E}(x^2) - \mathrm{E}(x)^2 = \mathrm{var}(x)$

So From Slutsky theorem

$$
\tilde{\theta} =
\begin{bmatrix}
\frac{\bar{x}^2}{s^2} \\
\frac{s^2}{\bar{x}}
\end{bmatrix}
\xrightarrow{as}
\begin{bmatrix}
\frac{(\mathrm{E}(x))^2}{\mathrm{var}(x)} \\
\frac{\mathrm{var}(x)}{\mathrm{E}(x)}
\end{bmatrix}
= \theta
$$

It is asymptotically Normal because of the following.

$\bar{x}$ and s^2 are asymptotically Normal by the CLT, as averages of IID quantities. Hence, by Delta method, (16) is asymptotically Normal.

3. Recall the the one-step estimators

$$
\text{Newton alg.} \qquad \breve{\theta}_n = \tilde{\theta}_n - \ddot{\ell}_n(\tilde{\theta}_n)^{-1} \dot{\ell}_n(\tilde{\theta}_n) \tag{17}
$$

$$
\text{Fisher scoring alg.} \qquad \breve{\theta}_n = \tilde{\theta}_n + \frac{1}{n} \mathcal{I}(\tilde{\theta}_n)^{-1} \dot{\ell}_n(\tilde{\theta}_n) \tag{18}
$$

For the Fisher algorithm, I need to find $\mathcal{I}(\theta)^{-1}$. It is

$$
\log f(x|\theta) = -\log \Gamma(a) - a \log(b) - \frac{1}{b} x + (a-1) \log(x)
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(x|\theta) =
\begin{bmatrix}
-\psi(a) - \log(b) + \log(x) \\
-\frac{a}{b} + \frac{1}{b^2} x
\end{bmatrix}
$$

$$
\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log f(x|\theta) =
\begin{bmatrix}
-\psi_1(a) & -\frac{1}{b} \\
-\frac{1}{b} & -\frac{2x-ab}{b^3}
\end{bmatrix}
$$

$$
\mathcal{I}(\theta) =
\begin{bmatrix}
\psi_1(a) & \frac{1}{b} \\
\frac{1}{b} & \frac{a}{b^2}
\end{bmatrix}
$$

$$
\mathcal{I}(\theta)^{-1} = \frac{1}{a\psi_1(a) - 1}
\begin{bmatrix}
a & -b \\
-b & b^2\psi_1(a)
\end{bmatrix}
$$

$$
\ell_n(\theta) = -n \log \Gamma(a) - na \log(b) - \frac{1}{b} \sum_{i=1}^{n} x_i + (a-1) \sum_{i=1}^{n} \log(x_i)
$$

$$
\dot{\ell}_n(\theta) =
\begin{bmatrix}
-n\psi(a) - n \log(b) + \sum_{i=1}^{n} \log(x_i) \\
-n\frac{a}{b} + n\frac{1}{b^2}\bar{x}
\end{bmatrix}
$$

The Fisher recursion is

$$\breve{\theta}_n = \tilde{\theta}_n + \frac{1}{n}\mathcal{I}(\tilde{\theta}_n)^{-1}\dot{\ell}_n(\tilde{\theta}_n)$$

$$\breve{\theta}_n = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{n}\frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} \tilde{a} & -\tilde{b} \\ -\tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -n\psi(\tilde{a}) - n\log(\tilde{b}) + \sum_{i=1}^{n}\log(x_i) \\ -n\frac{\tilde{a}}{\tilde{b}} + n\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} \tilde{a} & -\tilde{b} \\ -\tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -\psi(\tilde{a}) - \log(\tilde{b}) + \frac{1}{n}\sum_{i=1}^{n}\log(x_i) \\ -\frac{\tilde{a}}{\tilde{b}} + \frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\tilde{a}\psi_1(\tilde{a})-1}\begin{bmatrix} -\tilde{a}\psi(\tilde{a}) - \frac{1}{\tilde{b}}(\bar{x} - \tilde{a}\tilde{b}) - \tilde{a}\log(\tilde{b}) + \frac{\tilde{a}}{n}\sum_{i=1}^{n}\log(x_i) \\ \tilde{b}\psi(\tilde{a}) - \psi_1(\tilde{a})(\bar{x} - \tilde{a}) + \tilde{b}\log(\tilde{b}) - \frac{\tilde{b}}{n}\sum_{i=1}^{n}\log(x_i) \end{bmatrix}$$

So bu substituting

$$\breve{\theta}_n = \begin{bmatrix} \frac{\bar{x}^2}{s^2} \\ \frac{s^2}{\bar{x}} \end{bmatrix} + \frac{1}{\frac{\bar{x}^2}{s^2}\psi_1(\frac{\bar{x}^2}{s^2})-1}\begin{bmatrix} -\frac{\bar{x}^2}{s^2}\psi(\frac{\bar{x}^2}{s^2}) - \frac{\bar{x}^2}{s^2}\log(\frac{s^2}{\bar{x}}) + \frac{1}{n}\frac{\bar{x}^2}{s^2}\sum_{i=1}^{n}\log(x_i) \\ \frac{s^2}{\bar{x}}\psi(\frac{\bar{x}^2}{s^2}) - \psi_1(\frac{\bar{x}^2}{s^2})(\bar{x} - \frac{\bar{x}^2}{s^2}) + \frac{s^2}{\bar{x}}\log(\frac{s^2}{\bar{x}}) - \frac{\tilde{b}}{n}\sum_{i=1}^{n}\log(x_i) \end{bmatrix}$$

Additionally for the Newton recursion I need

$$\ddot{\ell}_n(\theta) = -n\begin{bmatrix} \psi_1(a) & \frac{1}{b} \\ \frac{1}{b} & \frac{2\bar{x}-ab}{b^3} \end{bmatrix}$$

$$(\ddot{\ell}_n(\theta))^{-1} = -\frac{1}{n}\frac{1}{\psi_1(a)\frac{2\bar{x}-ab}{b}-1}\begin{bmatrix} \frac{2\bar{x}-ab}{b} & -b \\ -b & b^2\psi_1(a) \end{bmatrix}$$

The Newton recursion is

$$\breve{\theta}_n = \tilde{\theta}_n - (\ddot{\ell}_n(\theta))^{-1}\dot{\ell}_n(\tilde{\theta}_n)$$

$$\breve{\theta}_n = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{n}\frac{1}{\psi_1(\tilde{a})\frac{2\bar{x}-\tilde{a}\tilde{b}}{\tilde{b}}-1}\begin{bmatrix} 2\bar{x} - \tilde{a}\tilde{b} & \tilde{b} \\ \tilde{b} & \tilde{b}^2\psi_1(\tilde{a}) \end{bmatrix}\begin{bmatrix} -n\psi(\tilde{a}) - n\log(\tilde{b}) + \sum_{i=1}^{n}\log(x_i) \\ -n\frac{\tilde{a}}{\tilde{b}} + n\frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} + \frac{1}{\psi_1(\tilde{a})\frac{2\bar{x}-\tilde{a}\tilde{b}}{\tilde{b}}-1}\begin{bmatrix} 2\bar{x} - \tilde{a}\tilde{b} & \tilde{b} \\ \tilde{b} & \tilde{b}^2\psi_1(\tilde{b}) \end{bmatrix}\begin{bmatrix} -\psi(\tilde{a}) - \log(\tilde{b}) + \frac{1}{n}\sum_{i=1}^{n}\log(x_i) \\ -\frac{\tilde{a}}{\tilde{b}} + \frac{1}{\tilde{b}^2}\bar{x} \end{bmatrix}$$

$$= ...\text{calculations}$$

The next exercise is from Homework 4

Exercise 30. Let $x_1, ..., x_n \overset{IID}{\sim} f_\theta$ with unknown parameter $\theta \in (0, \infty)$ and PDF

$$f(x|\theta) = \begin{cases} \theta\exp(-x) + (1-\theta)x\exp(-x) & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

1. Calculate the moment estimator $\tilde{\theta}_n$ of θ, (I give you a bit of freedom here)

2. Calculate the asymptotic distribution of the $\tilde{\theta}_n$

3. Find the 1-step estimator $\breve{\theta}_n$ of θ such that it can be asymptotically efficient.

   Hint: Recall that $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x$, and $\Gamma(a) = (a-1)\Gamma(a-1)$

   Solution.

6 Handout 6: Confidence intervals and hypothesis tests

Exercise 31. (Log likelihood ratio statistic)

1. Let $x_1, x_2, ..., x_n$ be IID random variables generated from a distribution f_θ labeled by a d-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^d$, and admitting PDF $f(\cdot|\theta)$. Assume the conditions from the Cramér Theorem are satisfied, and that θ_0 is the true value. Prove that

$$W_{\mathrm{LR}}(\theta_0) = -2(\ell_n(\theta_0) - \ell_n(\hat{\theta}_n)) \xrightarrow{D} \chi_d^2$$

it is where $\hat{\theta}_n$ is the MLE of θ.

   Hint-1 Expand $\ell_n(\theta_0)$ around $\hat{\theta}_n$ by Taylor expansion
   Hint-2 Prove that $W_{\mathrm{LR}}(\theta_0) \xrightarrow{a.s} n(\theta_0 - \hat{\theta}_n)^T \mathcal{I}(\theta_0)(\theta_0 - \hat{\theta}_n)$
   Hint-3 Prove that $W_{\mathrm{LR}}(\theta_0) \xrightarrow{D} \chi_d^2$

2. Calculate the asymptotic distribution of the statistic

$$\tilde{W}_{\mathrm{LR}}(\theta_0) = -2(\ell_n(\theta_0) - \ell_n(\breve{\theta}_n))$$

   where $\breve{\theta}_n$ is the one step estimator produced from the Fisher iterative method using the method of moments estimator as initial step.

Solution.

1. Right, let's expand it,

$$\ell_n(\theta_0) = \ell_n(\hat{\theta}_n) + \dot{\ell}_n(\hat{\theta}_n)(\theta_0 - \hat{\theta}_n) + (\theta_0 - \hat{\theta}_n)^T \int_0^1 \int_0^1 u\,\ddot{\ell}_n(\hat{\theta}_n + uv(\theta_0 - \hat{\theta}_n))\mathrm{d}u\mathrm{d}v\,(\theta_0 - \hat{\theta}_n)$$

$$= \ell_n(\hat{\theta}_n) + \dot{\ell}_n(\hat{\theta}_n)(\theta_0 - \hat{\theta}_n) + (\theta_0 - \hat{\theta}_n)^T\,n\int_0^1 \int_0^1 u\,\frac{1}{n}\ddot{\ell}_n(\hat{\theta}_n + uv(\theta_0 - \hat{\theta}_n))\mathrm{d}u\mathrm{d}v\,(\theta_0 - \hat{\theta}_n)$$

So by rearranging the terms

$$-2(\ell_n(\theta_0)-\ell_n(\hat{\theta}_n)) = -\underbrace{\dot{\ell}_n(\hat{\theta}_n)}_{=0}(\theta_0-\hat{\theta}_n)-n(\theta_0-\hat{\theta}_n)^T \underbrace{\int_0^1\int_0^1 u\,\frac{1}{n}\ddot{\ell}_n(\hat{\theta}_n + uv(\theta_0 - \hat{\theta}_n))\mathrm{d}u\mathrm{d}v}_{\xrightarrow{a.s}-\frac{1}{2}\mathcal{I}(\theta_0)} (\theta_0-\hat{\theta}_n)$$

It is

$$\dot{\ell}_n(\hat{\theta}_n) = 0$$

because $\hat{\theta}_n$ is an MLE.

From Cramer' Theorem $\hat{\theta}_n \xrightarrow{a.s} \theta_0$. Then by Slutsky's theorem, $\hat{\theta}_n + uv(\theta_0 - \hat{\theta}_n) \xrightarrow{a.s} \theta_0$. Then

$$\int_0^1\int_0^1 u\,\frac{1}{n}\ddot{\ell}_n(\theta_0)\mathrm{d}u\mathrm{d}v \xrightarrow{a.s} \int_0^1\int_0^1 u\,\frac{1}{n}\ddot{\ell}_n(\theta_0)\mathrm{d}u\mathrm{d}v = \frac{1}{2}\frac{1}{n}\ddot{\ell}_n(\theta_0)$$

But $\frac{1}{2}\frac{1}{n}\ddot{\ell}_n(\theta_0) \xrightarrow{a.s} -\frac{1}{2}\mathcal{I}(\theta_0)$.

So to sum up

$$-2(\ell_n(\theta_0) - \ell_n(\hat{\theta}_n)) \xrightarrow{a.s} n(\theta_0 - \hat{\theta}_n)^T\mathcal{I}(\theta_0)(\theta_0 - \hat{\theta}_n) \tag{19}$$

From Cramer' Theorem I know that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, \mathcal{I}(\theta_0)^{-1})$$
$$\implies \sqrt{n}\mathcal{I}(\theta_0)^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I)$$
$$\implies n(\hat{\theta}_n - \theta_0)^T\mathcal{I}(\theta_0)(\hat{\theta}_n - \theta_0) \xrightarrow{D} \chi_d^2$$

But $-2(\ell_n(\theta_0) - \ell_n(\hat{\theta}_n))$ is asymptotic equivalent to $n(\hat{\theta}_n - \theta_0)^T\mathcal{I}(\theta_0)(\hat{\theta}_n - \theta_0)$ from (19). So by the Slutsky's theorem

$$-2(\ell_n(\theta_0) - \ell_n(\hat{\theta}_n)) \xrightarrow{D} \chi_d^2$$

2. It is

$$\tilde{W}_{\mathrm{LR}}(\theta_0) = -2(\ell_n(\theta_0) - \ell_n(\breve{\theta}_n)) \xrightarrow{D} \chi_d^2$$

because $\breve{\theta}_n$ and $\hat{\theta}_n$ are asymptotic equivalent.

The next exercise is from Homework 4

Exercise 32. Let

$$y_i \overset{\mathrm{ind}}{\sim} \mathrm{Bin}(n, \pi_i)$$

where $i = 1, ..., N$. Consider that the probability of success is modeled such as

$$\text{logit}(\pi_i) = x_i^T \theta \tag{20}$$

where $\text{logit}(\pi_i) = \log(\frac{\pi_i}{1-\pi_i})$. Here $x_i = (x_{i,1}, ..., x_{i,d})^T$ are known vertors containing the values of the d regessions at the i-th observation, and $\theta \in \mathbb{R}^d$.

1. Show that

$$\pi_i = \frac{e^{x_i^T \theta}}{1 + e^{x_i^T \theta}}$$

2. Assume that the MLE $\hat{\theta}$ of θ is known/calculated. Show that the $(1 - a)$ Wald confidence interval for the unknown parameter θ, by using the observed information matrix, is

$$\text{C.I.} : \{\, \theta \in \mathbb{R}^d \, : \, (\hat{\theta}_n - \theta)^T X^T (\text{diag}_{\forall i}(n\hat{\pi}_i(1 - \hat{\pi}_i))) X(\hat{\theta}_n - \theta) \le \chi^2_{d,1-a} \,\}$$

where

$$\hat{\pi}_i = \frac{e^{x_i^T \hat{\theta}}}{1 + e^{x_i^T \hat{\theta}}}$$

X is the so called design matrix from the regression

$$\begin{bmatrix} \text{logit}(\pi_1) \\ \vdots \\ \text{logit}(\pi_N) \end{bmatrix} = \underbrace{\begin{bmatrix} \longleftarrow x_1^T \longrightarrow \\ \vdots \\ \longleftarrow x_N^T \longrightarrow \end{bmatrix}}_{=X} \theta$$

and $\text{diag}_{\forall i}(\heartsuit_i) = \begin{bmatrix} \heartsuit_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \heartsuit_N \end{bmatrix}$.

3. Find the score statistic rejection area for the hypothesis test $H_0 : \theta = \theta_*$ versus $H_1 : \theta \ne \theta_*$.

   Solution.

---

7 Handout 7: The Profile likelihood (MLE under the presence of nuisance parameters)

Exercise 33. For $i = 1, ..., k$, let $x_{i,1}, ..., x_{i,n} \overset{\text{IID}}{\sim} \text{Poi}(\theta_i)$. Find the asymptotic likelihood ratio rejection area for teasting the hypothesis

$$\text{H}_0 : \theta_1 = ... = \theta_k$$

Hint: It is

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!} 1(x \in \mathbb{N})$$

Solution. Under H_1, the log-likelihood is

$$\ell_1(\theta) = \sum_{i=1}^{k} \sum_{j=1}^{n} (-\theta_i + x_{i,j} \log(\theta_i) - \log(x_{i,j}!))$$

$$\propto -n \sum_{i=1}^{k} \theta_i + \sum_{i=1}^{k} \log(\theta_i) \sum_{j=1}^{n} x_{i,j}$$

The MLE is

$$0 = \frac{\text{d}}{\text{d}\theta_i} \ell_1(\theta)|_{\theta = \hat{\theta}^{(1)}} = -n + \frac{1}{\hat{\theta}_i^{(1)}} \sum_{j=1}^{n} x_{i,j}$$

$$\implies \hat{\theta}_i^{(1)} = \frac{1}{n} \sum_{j=1}^{n} x_{i,j}$$

$$\implies \hat{\theta}^{(1)} = (\bar{x}_{1,\bullet}, ..., \bar{x}_{k,\bullet})^T$$

and there are $d_1 = k$ free parameters for estimation.

Under H_0, it is the log-likelihood is $\theta_1 = ... = \theta_k = \theta$

$$\ell_0(\theta) = \sum_{i=1}^{k} \sum_{j=1}^{n} (-\theta + x_{i,j} \log(\theta) - \log(x_{i,j}!))$$

$$\propto -nk\theta + \log(\theta) \sum_{i=1}^{k} \sum_{j=1}^{n} x_{i,j}$$

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The MLE is

$$0 = \frac{\mathrm{d}}{\mathrm{d}\theta}\ell_0(\theta)|_{\theta=\hat{\theta}^{(0)}} = -nk + \frac{1}{\hat{\theta}^{(0)}}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{i,j}$$

$$\implies \hat{\theta}^{(0)} = \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{i,j}$$

$$\hat{\theta}^{(0)} = \bar{x}_{\bullet,\bullet}$$

and there is $d_0 = 1$ free parameter for estimation.

So

$$-2(\ell_n(\hat{\theta}^{(0)}) - \ell_n(\hat{\theta}^{(1)})) \xrightarrow{D} \chi^2_{k-1}$$

where

$$-2(\ell_n(\hat{\theta}^{(0)}) - \ell_n(\hat{\theta}^{(1)})) = -2(-nk\hat{\theta}^{(0)} + \log(\hat{\theta}^{(0)})\sum_{i=1}^{k}\sum_{j=1}^{n}x_{i,j}$$

$$-n\sum_{i=1}^{k}\hat{\theta}_i^{(1)} + \sum_{i=1}^{k}\log(\hat{\theta}_i^{(1)})\sum_{j=1}^{n}x_{i,j})$$

$$= -2(\cancel{-nk\bar{x}_{\bullet,\bullet}} + \log(\bar{x}_{\bullet,\bullet})\sum_{i=1}^{k}\sum_{j=1}^{n}x_{i,j}$$

$$+n\cancel{\sum_{i=1}^{k}\bar{x}_{i,\bullet}} - \sum_{i=1}^{k}\log(\bar{x}_{i,\bullet})\sum_{j=1}^{n}x_{i,j})$$

$$= 2n\sum_{i=1}^{k}\log(\bar{x}_{i,\bullet})\bar{x}_{i,\bullet} - 2nk\log(\bar{x}_{\bullet,\bullet})\bar{x}_{\bullet,\bullet}$$

So the rejection area is

$$\mathrm{RA} = \{2n\sum_{i=1}^{k}\log(\bar{x}_{i,\bullet})\bar{x}_{i,\bullet} - 2nk\log(\bar{x}_{\bullet,\bullet})\bar{x}_{\bullet,\bullet} \geq \chi^2_{k-1,1-a}\}$$

Exercise 34. Let $x = (x_1, ..., x_c) \sim \mathrm{Mult}(\pi_1, ..., \pi_c)$, with $\pi_i \in (0, \infty)$ and $\sum_{i=1}^{c}\pi_i = 1$. Find the asymptotic likelihood ratio rejection area for teasting the hypothesis

$$\mathrm{H}_0 : \pi_1 = ... = \pi_c = \frac{1}{c}$$

Hint: It is

$$f(x|\theta) = \binom{n}{x_1...x_c} \prod_{i=1}^{c} \pi_i^{x_i}$$

Solution. It is

$$\ell_n(\pi) = \log\binom{n}{x_1...x_c} + \sum_{i=1}^{k} x_i \log(\pi_i)$$

Lagrandge function is

$$\mathcal{L}(\pi, \theta) = \log\binom{n}{x_1...x_c} + \sum_{i=1}^{k} x_i \log(\pi_i) - \theta(\sum_{i=1}^{c} \pi_i - 1)$$

Under H$_1$, the MLE is

$$0 = \frac{\mathrm{d}}{\mathrm{d}\pi_i}\mathcal{L}(\pi, \theta)|_{\pi=\hat\pi, \theta=\hat\theta} \implies \hat\pi_i = \frac{x_i}{\theta}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{L}(\pi, \theta)|_{\pi=\hat\pi, \theta=\hat\theta} \implies \sum_{i=1}^{c} \pi_i = 1$$

$$\implies \hat\pi_i = \frac{x_i}{n}$$

$$\implies \hat\pi^{(1)} = (\frac{x_1}{n}, ..., \frac{x_c}{n})^T$$

So

$$\ell(\hat\pi^{(1)}) = \log\binom{n}{x_1...x_c} + \sum_{i=1}^{k} x_i \log(\frac{x_i}{n})$$

with $d_1 = c - 1$ free parameters.

Under H$_0$,

$$\hat\pi^{(0)} = (\frac{1}{c}, ..., \frac{1}{c})^T$$

So

$$\ell(\hat\pi^{(0)}) = \log\binom{n}{x_1...x_c} + n\bar{x} \log(\frac{1}{c})$$

with $d_0 = 0$ free parameters.

So

$$-2(\ell_n(\hat\pi^{(0)}) - \ell_n(\hat\pi^{(1)})) \xrightarrow{D} \chi^2_{c-1}$$

where

$$-2(\ell_n(\hat{\pi}^{(0)}) - \ell_n(\hat{\pi}^{(1)})) = -2(n\bar{x}\log(\frac{1}{c}) - \sum_{i=1}^{k} x_i \log(\frac{x_i}{n}))$$

$$=2\sum_{i=1}^{c} x_i \log(\frac{cx_i}{n})$$

So the rejection area is

$$\text{RA} = \{2\sum_{i=1}^{c} x_i \log(\frac{cx_i}{n}) \geq \chi^2_{c-1,1-a}\}$$

8   Others

Exercise 35. (Very difficult) Consider a contigency table with N cells. Consider a Multimomial sampling scheme was used to collect n observations. Let $y = (y_1, ..., y_N)^T$ be the observed counts, and $\pi = (\pi_1, ..., \pi_N)^T$ be the expected probabilities in N cells of a contingency table. Let the total number of observations be $n = \sum_{i=1}^{N} y_i$. Assume that

$$y \sim \text{Mult}(n, \pi) \tag{21}$$

where

$$f(y|n, \pi) = \binom{n}{y_1...y_N} \prod_{i=1}^{n} \pi_i^{y_i}$$

Consider a log-linear model

$$\pi_i = \pi_i(\theta) = \frac{\exp(x_i^T \theta)}{\sum_{\forall k} \exp(x_i^T \theta)} \tag{22}$$

$\theta \in \Theta$ is a d-dimensional vector of unknown coefficients, and $x_i = (x_{i,1}, ..., x_{i,d})^T$ are the values of d regressors.

In a matrix form

$$\pi = \frac{\exp(X\theta)}{1_d^T \exp(X\theta)}$$

where

$$X = \begin{bmatrix} \longleftarrow x_1^T \longrightarrow \\ \vdots \\ \longleftarrow x_N^T \longrightarrow \end{bmatrix}$$

Assume that Cramer's Theorem conditions are satisfied. Consider that the MLE $\hat{\theta}_n$ of θ is computed/calculated, and that θ_0 is the unknown true value of θ. Then

1. Show that

$$\frac{\mathrm{d}\pi}{\mathrm{d}\theta} = (\mathrm{diag}(\pi) - \pi\pi^T)X$$

2. Show that the likelihood equations to find the MLE $\hat{\theta}$ of θ are such as

$$X^T y = nX^T \pi(\hat{\theta}_n)$$

   Does it ring a bell?

3. Consider the j-th single observation $\xi_j = (\xi_{j,1}, ..., \xi_{j,N})^T$ where $\xi_{j,i} = 1$ if it falls in cell i and $\xi_{j,i} = 0$ if it does not fall in cell i. Write the probability distribution $f(\xi_i|...) =?$ in the form of the Multinomial distribution.

4. Calculate the asymptotic distribution of the MLE $\hat{\theta}$ of θ.

   Hint: Use the fact that a single observation falls in only one cell, and use its probability.

5. Calculate the asymptotic distribution of cell probability estimators $\hat{\pi}$ of π.

6. Calculate the Wald's $(1-a)$ CI for θ, that results as an ellipsoid easy to compute or plot in 2D on 3D.

Solution.

1. It is

$$
\begin{aligned}
[\frac{\mathrm{d}\pi}{\mathrm{d}\theta}]_{i,j} = \frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j} &= \frac{\mathrm{d}}{\mathrm{d}\theta_j} \frac{\exp(x_i^T\theta)}{\sum_{\forall k} \exp(x_k^T\theta)} \\
&= \frac{\exp(x_i^T\theta)x_{i,j} \sum_{\forall k}\exp(x_k^T\theta) - \exp(x_i^T\theta)\sum_{\forall k}\exp(x_k^T\theta)x_{k,j}}{[\sum_{\forall k}\exp(x_k^T\theta)]^2} \\
&= \frac{\exp(x_i^T\theta)}{\sum_{\forall k}\exp(x_k^T\theta)} \frac{x_{i,j}\sum_{\forall k}\exp(x_k^T\theta) - \sum_{\forall k}\exp(x_k^T\theta)x_{k,j}}{[\sum_{\forall k}\exp(x_k^T\theta)]^2} \\
&= \frac{\exp(x_i^T\theta)}{\sum_{\forall k}\exp(x_k^T\theta)} \left(x_{i,j}\frac{\sum_{\forall k}\exp(x_k^T\theta)}{\sum_{\forall k}\exp(x_k^T\theta)} - \sum_{\forall k}\frac{\exp(x_k^T\theta)}{\sum_{\forall k}\exp(x_k^T\theta)}x_{k,j} \right) \\
&= \pi_i \left(x_{i,j} - \sum_{\forall k}\pi_k x_{k,j} \right) = \pi_i x_{i,j} - \pi_i \sum_{\forall k}\pi_k x_{k,j}
\end{aligned}
$$

   So if I write it in a matrix form

$$
\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}\theta} &= \mathrm{diag}(\pi)X - \pi\pi^T X \\
&= (\mathrm{diag}(\pi) - \pi\pi^T)X
\end{aligned}
$$

2. Well,
$$\ell_n(\theta) = \log\left(\binom{n}{y_1...y_N}\right) + \sum_{i=1}^{N} y_i \log(\pi_i(\theta))$$

It is

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}\ell_n(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta_j}\sum_{i=1}^{N} y_i \log(\pi_i(\theta))$$

$$= \sum_{i=1}^{N} y_i \frac{1}{\pi_i(\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta_j}\pi_i(\theta)$$

$$= \sum_{i=1}^{N} y_i \frac{1}{\pi_i(\theta)} \left(\pi_i(\theta)x_{i,j} - \pi_i(\theta)\sum_{\forall k}\pi_k(\theta)x_{k,j}\right)$$

$$= \sum_{i=1}^{N} y_i x_{i,j} - \underbrace{\sum_{i=1}^{N} y_i}_{=n} \sum_{\forall k}\pi_k(\theta)x_{k,j}$$

$$= \sum_{i=1}^{N} y_i x_{i,j} - n \sum_{\forall k}\pi_k(\theta)x_{k,j}$$

So
$$\dot{\ell}_n(\theta) = X^T y - nX^T\pi(\theta)$$

Hence
$$0 = \ell_n(\theta)|_{\theta=\hat{\theta}} \quad \Longrightarrow \quad X^T y = nX^T\pi(\hat{\theta})$$

It is the same equation as the one for the log-linear model under the Piosson samp[ling scheme, when $\mu(\theta) = n\pi(\theta)$.

3. Based on the Multinomial sampling scheme, each observation can fall in one only cell. Observation ξ_j can fall in i-cell with probability π_i. So

$$\xi_j \overset{\text{IID}}{\sim} \text{Mult}(1, \pi)$$

with

$$f(\xi_j|\pi) = \prod_{i=1}^{N} \pi_i^{\xi_{j,i}}$$

where $\xi_{j,i} \in \{0,1\}^d$ and $\sum_i \xi_{j,i} = 1$.

4. Since Cramer's Theorem conditions are satisfied, I will use Cramer's Theorem. Namely,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \overset{D}{\to} \text{N}(0, \mathcal{I}(\theta_0)^{-1})$$

where $\mathcal{I}(\theta)$ is the Fisher's information matrix, aka the information matrix for 1 observation, Let's say observation ξ. Therefor, I just need to find $\mathcal{I}(\theta)$ with

$$[\mathcal{I}(\theta)]_{j,k} = \mathrm{E}\left(\frac{\mathrm{d}}{\mathrm{d}\theta_j}\log(f(\xi|\pi))\frac{\mathrm{d}}{\mathrm{d}\theta_k}\log(f(\xi|\pi))\right)$$

.

It is

$$\log(f(\xi|\pi)) = \sum_{i=1}^{N}\xi_i\log(\pi_i)$$

It is

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}\log(f(\xi|\pi)) = \sum_{i=1}^{N}\xi_i\frac{1}{\pi_i}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta_k}\log(f(\xi|\pi)) = \sum_{i=1}^{N}\xi_i\frac{1}{\pi_i}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_k}$$

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So

$$
\begin{aligned}
[\mathcal{I}(\theta)]_{j,k} =& \mathrm{E}\left(\left(\sum_{i=1}^{N}\xi_i\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}\right)\left(\sum_{i'=1}^{N}\xi_{i'}\frac{\mathrm{d}\pi_{i'}}{\mathrm{d}\theta_k}\right)\right) \\
=& \mathrm{E}\left(\sum_{i=1}^{N}\sum_{i'=1}^{N}\xi_i\xi_{i'}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}\frac{\mathrm{d}\pi_{i'}}{\mathrm{d}\theta_k}\right) \\
=& \sum_{i=1}^{N}\sum_{i'=1}^{N}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}\frac{\mathrm{d}\pi_{i'}}{\mathrm{d}\theta_k}\underbrace{\mathrm{E}(\xi_i\xi_{i'})} \quad = \begin{cases} \mathrm{E}\left(\xi_i^2\right) = 1^2\pi_i + 0^2(1-\pi_i) = \pi_i & ,\ i = i' \\ 0 & ,\ i \neq i' \end{cases}
\end{aligned}
$$

$=\{$so we care for those where $\xi_i = \xi_{i'} = 1\}$

$$
\begin{aligned}
=& \sum_{i=1}^{N}\pi_i\frac{1}{\pi_i}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}\frac{1}{\pi_i}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_k} \\
=& \sum_{i=1}^{N}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_j}\frac{1}{\pi_i}\frac{\mathrm{d}\pi_i}{\mathrm{d}\theta_k} \\
=& \sum_{i=1}^{N}\left(\pi_i x_{i,j} - \pi_i\sum_{\forall s}\pi_s x_{s,j}\right)\left(\pi_i x_{i,k} - -\pi_i\sum_{\forall s}\pi_s x_{s,k}\right)\frac{1}{\pi_i} \\
=& \sum_{i=1}^{N}\left(x_{i,j} - (\pi^T X_{:,j})\right)\left(\pi_i x_{i,k} - \pi_i(\pi^T X_{:,k})\right) \\
=& \sum_{i=1}^{N}x_{i,j}\pi_i x_{i,k} - \left(\sum_{i=1}^{N}x_{i,j}\pi_i\right)(\pi^T X_{:,k}) - \underbrace{\sum_{i=1}^{N}(\pi^T X_{:,j})(\pi_i x_{i,k}) + (\pi^T X_{:,j})\overbrace{\sum_{i=1}^{N}\pi_i(\pi^T X_{:,k})}^{=1}}_{=0} \\
=& \sum_{i=1}^{N}x_{i,j}\pi_i x_{i,k} - \sum_{i=1}^{N}x_{i,j}\pi_i(\pi^T X_{:,k}) \\
=& X_{:,j}^T\mathrm{diag}(\pi)X_{:,k} - (\pi^T X_{:,j})^T(\pi^T X_{:,k})
\end{aligned}
$$

So in a matrix form, it is

$$
\begin{aligned}
\mathcal{I}(\theta) &= X^T\mathrm{diag}(\pi)X - X^T\pi\pi^T X \\
&= X^T(\mathrm{diag}(\pi) - \pi\pi^T)X
\end{aligned}
$$

So

$$
\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}\left(0, \left(X^T(\mathrm{diag}(\pi_0) - \pi_0\pi_0^T)X\right)^{-1}\right) \tag{23}
$$

where $\pi_0 = \pi(\theta_0)$.

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5. Because $\hat{\pi}$ is a continuous function of θ's and because I know that (23), I can use Delta method in order to find the asymptotic distribution of $\hat{\pi}$.

According to Delta method, it is

$$\sqrt{n}(\hat{\pi} - \pi_0) \xrightarrow{D} \mathrm{N}\left(0, \Sigma_{\pi_0}\right)$$

where $\pi_0 = \pi(\theta_0)$, and

$$\begin{aligned}
\Sigma_{\pi_0} =& \frac{d\pi}{d\theta}|_{\theta=\theta_0} \left(X^T(\mathrm{diag}(\pi_0) - \pi_0\pi_0^T)X\right)^{-1} (\frac{d\pi}{d\theta}|_{\theta=\theta_0})^T \\
=& (\mathrm{diag}(\pi_0) - \pi_0\pi_0^T)X \left(X^T(\mathrm{diag}(\pi_0) - \pi_0\pi_0^T)X\right)^{-1} X^T(\mathrm{diag}(\pi_0) - \pi_0\pi_0^T)
\end{aligned}$$

6. Well, the $(1-a)100\%$ confidence interval for θ which is touch to invert is

$$\begin{aligned}
\mathrm{CI}(\theta) =& \left\{\theta \in \mathbb{R}^d : \sqrt{n}(\hat{\theta} - \theta)\mathcal{I}(\theta)(\hat{\theta} - \theta)^T \leq \chi^2_{d,1-a}\right\} \\
=& \left\{\theta \in \mathbb{R}^d : \sqrt{n}(\hat{\theta} - \theta)\left(X^T(\mathrm{diag}(\pi(\theta)) - \pi(\theta)\pi(\theta)^T)X\right)(\hat{\theta} - \theta)^T \leq \chi^2_{d,1-a}\right\}
\end{aligned}$$

So probably I would go with the asymptotic equivalent one

$$\begin{aligned}
\mathrm{CI}(\theta) =& \left\{\theta \in \mathbb{R}^d : \sqrt{n}(\hat{\theta} - \theta)\mathcal{I}(\hat{\theta})(\hat{\theta} - \theta)^T \leq \chi^2_{d,1-a}\right\} \\
=& \left\{\theta \in \mathbb{R}^d : \sqrt{n}(\hat{\theta} - \theta)\left(X^T(\mathrm{diag}(\hat{\pi}) - \hat{\pi}\hat{\pi}^T)X\right)(\hat{\theta} - \theta)^T \leq \chi^2_{d,1-a}\right\}
\end{aligned}$$

where $\hat{\pi} = \pi(\hat{\theta})$.

The degrees of freedom of the critical values in the CI are d because there are d free parameters in θ.