Problem class handout 2: Likelihood methods

Lecturer: Georgios Karagiannis georgios.karagiannis@durham.ac.uk

Exercise 1. Consider a 2×2 contingency table where $(n_{i,j})$ is the (i,j)th cell count, and π_{ij} is the (i,j)th cell probability.

1. Show that the marginal distribution of the MLE of the odd ratio $\hat{\theta}$ is such that

$$\sqrt{n}(\log(\hat{\theta})-\log(\theta)) \xrightarrow{D} \mathrm{N}(0,\frac{1}{\pi_{11}}+\frac{1}{\pi_{12}}+\frac{1}{\pi_{21}}+\frac{1}{\pi_{22}}).$$

2. Show that

$$\frac{\log(\hat{\theta})-\log(\theta)}{\sqrt{\frac{1}{n_{11}}+\frac{1}{n_{12}}+\frac{1}{n_{21}}+\frac{1}{n_{22}}}} \xrightarrow{D} \mathrm{N}(0,1).$$

Hint-1: It is $\hat{\theta}=\frac{n_{11}n_{22}}{n_{21}n_{12}}=\frac{p_{11}p_{22}}{p_{21}p_{12}}$, , where $p_{i,j}=n_{i,j}/n$.

Hint-2: From Example 12 (Handout 2), it is:

$$\sqrt{n}(\boldsymbol{p}-\boldsymbol{\pi}) \xrightarrow{D} \mathrm{N}(0,\mathrm{diag}(\boldsymbol{\pi})-\boldsymbol{\pi}\boldsymbol{\pi}^T)$$

where

$$\mathrm{diag}(\boldsymbol{\pi})-\boldsymbol{\pi}\boldsymbol{\pi}^T=\begin{bmatrix}(1-\pi_{11})\pi_{11} & -\pi_{11}\pi_{12} & -\pi_{11}\pi_{21} & -\pi_{11}\pi_{22}\\ -\pi_{11}\pi_{12} & (1-\pi_{12})\pi_{12} & -\pi_{12}\pi_{21} & -\pi_{12}\pi_{22}\\ -\pi_{11}\pi_{21} & -\pi_{12}\pi_{21} & (1-\pi_{21})\pi_{21} & -\pi_{21}\pi_{22}\\ -\pi_{22}\pi_{11} & -\pi_{22}\pi_{12} & -\pi_{22}\pi_{21} & (1-\pi_{22})\pi_{22}\end{bmatrix}$$

Solution.

1.

- In Example 12 (Handout 2), we showed that from the CLT, have

$$\sqrt{n}(\boldsymbol{p}-\boldsymbol{\pi}) \xrightarrow{D} \mathrm{N}(0,\mathrm{diag}(\boldsymbol{\pi})-\boldsymbol{\pi}\boldsymbol{\pi}^T)$$

where

$$\text{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^T = \begin{bmatrix} (1-\pi_{11})\pi_{11} & -\pi_{11}\pi_{12} & -\pi_{11}\pi_{21} & -\pi_{11}\pi_{22} \\ -\pi_{11}\pi_{12} & (1-\pi_{12})\pi_{12} & -\pi_{12}\pi_{21} & -\pi_{12}\pi_{22} \\ -\pi_{11}\pi_{21} & -\pi_{12}\pi_{21} & (1-\pi_{21})\pi_{21} & -\pi_{21}\pi_{22} \\ -\pi_{22}\pi_{11} & -\pi_{22}\pi_{12} & -\pi_{22}\pi_{21} & (1-\pi_{22})\pi_{22} \end{bmatrix}$$

for the whole vectorized quantities $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$, and $\boldsymbol{p} = (p_{11}, ..., p_{22})$.

- It is $\hat{\theta} = \frac{p_{11}p_{22}}{p_{21}p_{12}} \implies \log(\hat{\theta}) = \log(p_{11}) + \log(p_{22}) - \log(p_{12}) - \log(p_{21})$
- So I can specify $g(x) = \log(x_{11}) + \log(x_{22}) - \log(x_{12}) - \log(x_{21})$
- It is

  $$\dot{g}(x) = \frac{\mathrm{d}}{\mathrm{d}x} g(x) = (\frac{1}{x_{11}}, -\frac{1}{x_{12}}, -\frac{1}{x_{21}}, \frac{1}{x_{22}})$$

  and hence $\dot{g}(x)$ is continuous a.s.
- Because all the assumptions of Delta Method are satisfied, it is

  $$\sqrt{n}(\log(\hat{\theta}) - \log(\theta)) \xrightarrow{D} \mathrm{N}(0, \dot{g}(\pi)(\text{diag}(\pi) - \pi\pi^T)\dot{g}(\pi)^T)$$

  with

  $$\begin{aligned} \dot{g}(\boldsymbol{\pi})(\text{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^T)\dot{g}(\boldsymbol{\pi})^T &= \dot{g}(\boldsymbol{\pi})(1, -1, -1, 1)^T \\ &= \frac{1}{\pi_{11}} + \frac{1}{\pi_{12}} + \frac{1}{\pi_{21}} + \frac{1}{\pi_{22}} \end{aligned}$$

2. Using Slutsky theorem, and law of large numbers, similar to Example 19 (Handout 2), we find that

   $$\frac{\sqrt{\frac{1}{p_{11}} + \frac{1}{p_{12}} + \frac{1}{p_{21}} + \frac{1}{p_{22}}}}{\sqrt{\frac{1}{\pi_{11}} + \frac{1}{\pi_{12}} + \frac{1}{\pi_{21}} + \frac{1}{\pi_{22}}}} \xrightarrow{P} 1$$

   and by using Slutsky theorem as in Example 19 (Handout 2), we find

   $$\frac{\log(\hat{\theta}) - \log(\theta)}{\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}} \xrightarrow{D} \mathrm{N}(0, 1).$$

Exercise 2. Consider random variables $X, X_1, X_2, ...$, where $\mu_n = \mathrm{E}(X - \mu)^n$, and $\mu = \mathrm{E}(X)$

1. Show that,

   $$\sqrt{n}(\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} \mathrm{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix})$$

2. Show that the asymptotic distribution of the coefficient of variation $\text{cv} = \frac{s_x}{\bar{X}}$, is

$$\sqrt{n}(\frac{s_x}{\bar{X}} - \frac{\sigma}{\mu}) \xrightarrow{D} \text{N}(0, \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4})$$

3. Show that the asymptotic distribution of the 3rd central moment $m_3 = \frac{1}{n}\sum_{i=1}^{n}(X_j - \bar{X})^3$ is

$$\sqrt{n}(m_3 - \mu_3) \xrightarrow{D} \text{N}(0, \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6)$$

Hint-1: It is:

$$\begin{bmatrix} \text{Var}(X_i - \mu) & \text{Cov}((X_i - \mu), (X_i - \mu)^2) & \text{Cov}((X_i - \mu), (X_i - \mu)^3) \\ \text{Cov}((X_i - \mu), (X_i - \mu)^2) & \text{Var}((X_i - \mu)^2) & \text{Cov}((X_i - \mu)^2, (X_i - \mu)^3) \\ \text{Cov}((X_i - \mu), (X_i - \mu)^3) & \text{Cov}((X_i - \mu)^2, (X_i - \mu)^3) & \text{Var}((X_i - \mu)^3) \end{bmatrix} \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^2 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 \end{bmatrix}$$

Solution.

1.

- I observe that

$$\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} \bar{X} - \mu \\ \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 - (\bar{X} - \mu)^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu) \\ \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu))^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$
$$= \begin{bmatrix} m_1' \\ m_2' - (m_1')^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

  where $m_1' = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$ and $m_2' = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$.

- I will try to find the joint asymptotic distribution of $(m_1', m_2')^T$ by CLT, and then the asymptotic distribution of $(\bar{X}, s_x^2)^T$ by Delta method.

- Let

$$\xi_i = \begin{bmatrix} X_i - \mu \\ (X_i - \mu)^2 \end{bmatrix}$$

  IID random vectors. It is

$$\mu_\xi = \text{E}(\xi_i) = \begin{bmatrix} \text{E}(X_i - \mu) \\ \text{E}(X_i - \mu)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}$$

and

$$
\begin{aligned}
\Sigma_\xi = \mathrm{Var}(\xi_i) &= \begin{bmatrix} \mathrm{Var}(X_i-\mu) & \mathrm{Cov}((X_i-\mu),(X_i-\mu)^2) \\ \mathrm{Cov}((X_i-\mu),(X_i-\mu)^2) & \mathrm{Var}(X_i-\mu)^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4-\sigma^4 \end{bmatrix}
\end{aligned}
$$

since

$$
\begin{aligned}
\mathrm{Cov}((X_i-\mu),(X_i-\mu)^2) &= \mathrm{E}(((X_i-\mu)-\mathrm{E}(X_i-\mu))((X_i-\mu)^2-\mathrm{E}(X_i-\mu)^2)) \\
&= \mathrm{E}(((X_i-\mu)-\mu_1)((X_i-\mu)^2-\mu_2)) \\
&= \mathrm{E}(\,(X_i-\mu)^3-(X_i-\mu)\mu_2-\mu_1(X_i-\mu)^2+\mu_1\mu_2\,) \\
&= \mathrm{E}(X_i-\mu)^3-\cancelto{0}{\mathrm{E}(X_i-\mu)}\mu_2-\mu_1\cancelto{\mu_2}{\mathrm{E}(X_i-\mu)^2}+\mu_1\mu_2 \\
&= \mathrm{E}(X_i-\mu)^3=\mu_3
\end{aligned}
$$

It is

$$
\bar{\xi} = \begin{bmatrix} m_1' \\ m_2' \end{bmatrix}
$$

So by CLT, I have,

$$
\sqrt{n}(\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} \mathrm{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4-\sigma^4 \end{bmatrix})
$$

- Now, I will calculate the asymptotic distribution of $(\bar{X}, s_x^2)^T$ by Delta method.

  Let,

$$
g(x,y) = \begin{bmatrix} x \\ y-x^2 \end{bmatrix}
$$

with

$$
\dot{g}(x,y) = \frac{\mathrm{d}g(x,y)}{\mathrm{d}(x,y)} = \begin{bmatrix} -1 & 0 \\ -2x & 1 \end{bmatrix}
$$

So

$$
g(\underbrace{m_1', m_2'}_{=\bar{\xi}}) = \begin{bmatrix} m_1' \\ m_2'-(m_1')^2 \end{bmatrix} = \begin{bmatrix} \bar{X}-\mu \\ s_x^2 \end{bmatrix}; \qquad g(\underbrace{0,\sigma^2}_{=\mu_\xi}) = \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}
$$

$$
\dot{g}(\underbrace{0,\sigma^2}_{=\mu_\xi}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \Sigma_g = \dot{g}(\underbrace{0,\sigma^2}_{=\mu_\xi})\Sigma_\xi \dot{g}(\underbrace{0,\sigma^2}_{=\mu_\xi})^T = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4-\sigma^4 \end{bmatrix}
$$

Then, according to Delta theorem

$$\sqrt{n}(g(\bar{\xi}) - g(\mu_\xi)) \xrightarrow{D} \mathrm{N}(0, \dot{g}(\mu_\xi)\Sigma_\xi \dot{g}(\mu_\xi)^T)$$
$$\sqrt{n}(\begin{bmatrix} \bar{X} \\ s_x^2 \end{bmatrix} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} \mathrm{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix})$$

2. Since I have the asymptotic distribution of $(\bar{X}, s_x^2)^T$, I can use the Delta method.

- Let $h(a,b) = \sqrt{b}/a$, with $\dot{h}(a,b) = (-\frac{\sqrt{b}}{a^2}, \frac{1}{2a\sqrt{b}})$.
- Then

$$h(\bar{X}, s_x^2) = \frac{s_x}{\bar{X}}; \qquad h(\mu, \sigma^2) = \frac{\sigma}{\mu}$$
$$\dot{h}(\mu, \sigma^2) = \begin{bmatrix} -\frac{\sigma}{\mu^2}, & \frac{1}{2\mu\sigma} \end{bmatrix};$$

$$\begin{aligned} \Sigma_h &= \dot{h}(\mu, \sigma^2)\Sigma_g \dot{h}(\mu, \sigma^2)^T \\ &= \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4} \end{aligned}$$

- Then, according to Delta theorem

$$\sqrt{n}(h(\bar{X}, s_x^2) - h(\mu, \sigma^2)) \xrightarrow{D} \mathrm{N}(0, \dot{h}(\mu, \sigma^2)\Sigma_g \dot{h}(\mu, \sigma^2)^T)$$
$$\sqrt{n}(\frac{s_x}{\bar{X}} - \frac{\sigma}{\mu}) \xrightarrow{D} \mathrm{N}(0, \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4})$$

3. I observe that

$$\begin{aligned} m_3 &= \frac{1}{n}\sum_{i=1}^{n}((\underbrace{X_i - \mu}_{=Z_i}) - (\underbrace{\bar{X} - \mu}_{=\bar{Z}}))^3 = \\ &= \frac{1}{n}\sum_{i=1}^{n} Z_i^3 - 3\frac{1}{n}\sum_{i=1}^{n} Z_i^2 \bar{Z} + 2\bar{Z} \\ &= m_3' - 3m_2' m_1' + 2(m_1')^2 \end{aligned}$$

where $m_j' = \frac{1}{n}\sum_{i=1}^{n} Z_i^j = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^j$, since $Z_i = X_i - \mu$.

- I will use the CLT to calculate the joint asymptotic distribution of $(m_1', m_2', m_3')^T$ and then I will use Delta method to calculate that of m_3.

I specify

$$\psi_i = \begin{bmatrix} Z_i \\ Z_i^2 \\ Z_i^3 \end{bmatrix} = \begin{bmatrix} X_i - \mu \\ (X_i - \mu)^2 \\ (X_i - \mu)^3 \end{bmatrix};$$

which are IID, with

$$\bar{\psi} = \frac{1}{n}\sum_{i=1}^{n} \psi_i = \begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix}$$

$$\mu_\psi = \mathrm{E}(\psi_i) = \begin{bmatrix} \mathrm{E}(X_i - \mu) \\ \mathrm{E}(X_i - \mu)^2 \\ \mathrm{E}(X_i - \mu)^3 \end{bmatrix} = \begin{bmatrix} \mu \\ \sigma^2 \\ \mu_3 \end{bmatrix};$$

$$\Sigma_\psi = \mathrm{Var}(\psi_i) = \begin{bmatrix} \mathrm{Var}(X_i - \mu) & \mathrm{Cov}((X_i - \mu), (X_i - \mu)^2) & \mathrm{Cov}((X_i - \mu), (X_i - \mu)^3) \\ \mathrm{Cov}((X_i - \mu), (X_i - \mu)^2) & \mathrm{Var}((X_i - \mu)^2) & \mathrm{Cov}((X_i - \mu)^2, (X_i - \mu)^3) \\ \mathrm{Cov}((X_i - \mu), (X_i - \mu)^3) & \mathrm{Cov}((X_i - \mu)^2, (X_i - \mu)^3) & \mathrm{Var}((X_i - \mu)^3) \end{bmatrix};$$

$$= \text{...calculations...} = \begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix}$$

For instance, you can compute the covariance terms as

$$\begin{aligned} \mathrm{Cov}((X_i - \mu)^2, (X_i - \mu)^3) &= \mathrm{E}(\,((X_i - \mu)^2 - \mathrm{E}(X_i - \mu)^2)\,((X_i - \mu)^3 - \mathrm{E}(X_i - \mu)^3)\,) \\ &= \mathrm{E}(\,((X_i - \mu)^2 - \mu_2)\,((X_i - \mu)^3 - \mu_3)\,) \\ &= \mathrm{E}(\,(X_i - \mu)^5 - \mathrm{E}(X_i - \mu)^2\mu_3 - \mu_2(X_i - \mu)^3 + \mu_2\mu_3\,) \\ &= \mu_5 - \mu_2\mu_3 \end{aligned}$$

So by CLT

$$\sqrt{n}(\underbrace{\begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix}}_{=\bar{\psi}} - \underbrace{\begin{bmatrix} \mu \\ \sigma^2 \\ \mu_3 \end{bmatrix}}_{=\mu_\psi}) \xrightarrow{D} \mathrm{N}(0\underbrace{\begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix}}_{=\Sigma_\psi})$$

- Now, in order to find the asymptotic distribution of $m_3 = m_3' - 3m_2'm_1' + 2(m_1')^2$, I will use Delta method

Let

$$q(a, b, c) = c - 3ab + 2a^3$$

then

$$\dot{q}(a, b, c) = \frac{\mathrm{d}}{\mathrm{d}(a, b, c)} q(a, b, c) = \begin{bmatrix} -3b + 6a^2, & -3a, & 1 \end{bmatrix}$$

So

$$q(m_1', m_2', m_3') = m_3' - 3m_2'm_1' + 2(m_1')^2$$

$$q(\mu_1', \mu_2', \mu_3') = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^2 = \mu_3$$

and

$$\dot{q}(\mu, \sigma^2, \mu_3) = \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix}$$

and

$$\begin{aligned} \dot{q}(\mu_\psi)\Sigma_\psi\dot{q}(\mu_\psi)^T &= \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \sigma^2 & \mu_5 - \sigma^2\mu_3 \\ \mu_4 & \mu_5 - \sigma^2\mu_3 & \mu_6 - \sigma^2\mu_3 \end{bmatrix} \begin{bmatrix} -3\sigma^2, & \mu, & 1 \end{bmatrix}^T \\ &= \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6 \end{aligned}$$

So the asymptotic distribution of m_3 is such that

$$\sqrt{n}(q(\bar{\psi}) - q(\mu_\psi)) \xrightarrow{D} \mathrm{N}(0, \dot{q}(\mu_\psi)\Sigma_\psi\dot{q}(\mu_\psi)^T)$$
$$\sqrt{n}(m_3 - \mu_3) \xrightarrow{D} \mathrm{N}(0, \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6)$$