Topics in statistics III/IV (MATH3361/4071) Michaelmas term, 2020-2021

Handout 1: Basic probability tools in asymptotics

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References: [2, 1]

1 Modes of convergence and their relations

Set-up and notation:

Consider a probability triplet $(\Omega, \mathcal{F}, P)$.

Consider random variable $X : \Omega \to \mathbb{R}^d$, where for simplicity we will denote the d-dimensional random vector as $X := X(\omega)$, $\forall \omega \in \Omega$.

Likewise, we define a sequence of random variables $X_n : \Omega \to \mathbb{R}^d$, and for simplicity denote $X_n := X_n(\omega)$, for $n = 1, 2, ...$, and $\forall \omega \in \Omega$.

The distribution function of r.v. X is denoted as

$$F_X(x) = P(X \le x) = P(X_1 \le x_1, ..., X_d \le x_d).$$

Hereafter, the norm $|\cdot|$ refers to the Euclidean norm; i.e. $|X| = \sqrt{\sum_{j=1}^{d} X_j^2}$, however the results can be generalized.

Definitions of modes of convergence:

Some modes of convergence are defined below.

Definition 1. X_n converges in distribution to X, symb. as $X_n \xrightarrow{D} X$, iff

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all points $x \in \mathbb{R}^d$ at which $F_X(x)$ is continuous.

- Other names: converges in law, and weak convergence

Definition 2. X_n converges in probability to X iff for every $\epsilon > 0$

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0 \tag{1.1}$$

It is symbolized as $X_n \xrightarrow{P} X$.

- It means: for any $\epsilon > 0$, and for any $\delta > 0$, there exists $N_{\epsilon,\delta} > 0$, where $P(|X_n - X| < \epsilon) < \delta$

Definition 3. X_n converges in almost surely to X iff for every $\epsilon > 0$

$$P(\lim_{n\to\infty} X_n = X) = 1 \tag{1.2}$$

It is symbolized as $X_n \xrightarrow{a.s.} X$.

- Other names: converges with probability 1, and strong convergence

Definition 4. X_n converges in the r-th mean to X iff for every $\epsilon > 0$

$$\lim_{n\to\infty} \mathrm{E}|X_n - X|^r = 0$$

where $r \in \{1, 2, ...\}$. It is symbolized as $X_n \xrightarrow{r} X$.

Definition 5. X_n converges in quadratic mean to X iff

$$\lim_{n\to\infty} \mathrm{E}|X_n - X|^2 = 0 \tag{1.3}$$

It is symbolized as $X_n \xrightarrow{\text{qm}} X$

Convergence in probability versus almost surely:

To better understand the difference/connection between the $\xrightarrow{P}$ and $\xrightarrow{a.s.}$, we restate the definitions in words.

convergence in probability $\xrightarrow{P}$: it requires that for every $\epsilon > 0$ the probability that X_n is within ϵ of X to tend to 1 as n tends to infinity

convergence almost surely $\xrightarrow{a.s.}$: it requires that for every $\epsilon > 0$ the probability that X_k STAYS within ϵ of X for all $k \geq n$ to tend to 1 as n tends to infinity

The following Lemma shows the distinction between $\xrightarrow{P}$ and the $\xrightarrow{a.s.}$.

Lemma 6. $X_n \xrightarrow{a.s.} X$ *iff for every* $\epsilon > 0$

$$P(|X_k - X| < \epsilon\,,\, \forall k \geq n) \to 1, \qquad \text{as } n \to \infty$$

Proof. Given as Exercise 8 in the Exercise sheet. □

Relations between convergence modes:

Theorem 7. *Relations between/among different modes of convergence*

1. $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$
2. $X_n \xrightarrow{r} X$, *for some* $r > 0 \implies X_n \xrightarrow{P} X$

3. $X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$

Proof. Given as Exercise 9 in the Exercise sheet. □

Example. ($\star$) Consider $Z \sim U(0,1)$, and $X_n = 2^n 1_{[0,1/n)}(Z)$. Check if $X_n \xrightarrow{r} 0$, $X_n \xrightarrow{a.s.} 0$, or $X_n \xrightarrow{P} 0$

Solution. It is $\mathrm{E}|X_n|^r = \frac{1}{n} 2^{nr} \to \infty$, so $X_n \not\xrightarrow{r} 0$. It is $\mathrm{P}(\{\lim X_n = 0\}) = \mathrm{P}(\{Z > 0\}) = 1$, so $X_n \xrightarrow{as} 0$. It is $\mathrm{P}(\{|X_n| \geq \epsilon\}) = \mathrm{P}(\{X_n = 2^n\}) = \mathrm{P}(Z \in [0,1/n)) = 1/n \to 0$, so $X_n \xrightarrow{P} 0$.

Definition 8. Consider a constant vector $c \in \mathbb{R}^d$. We say that X is a degenerate random variable/vector identically equal to $c \in \mathbb{R}^d$, iff $X(\omega) = c$, $\forall \omega \in \Omega$ (for every element of the sampling space).

Note 9. Mostly, we will use the symbol $c \in \mathbb{R}^d$ to denote the constant point c, as well as the degenerate random vector identically equal to c.

Proposition 10. *The distribution function of a degenerate random variable X equal to c is*

$$F_X(x) = \begin{cases} 1 & , x \geq c \\ 0 & , \mathit{else} \end{cases}$$

Note 11. The Theorem 12, together with Theorem 7, implies that $X_n \xrightarrow{D} c \iff X_n \xrightarrow{P} c$, if c is constant.

Theorem 12. *If $c \in \mathbb{R}^d$ is a constant, then $X_n \xrightarrow{D} c \implies X_n \xrightarrow{P} c$*

Proof. Given as Exercise **??** in the Exercise sheet. □

Exercise sheet (for practice)

Exercises: 8 ; 9 ; **??** ; 3 ; 4 ; 5

2 Taylor expansion

We revise the Taylor expansion in many dimensions. For more details see [1].

Notation 13. Derivative notation:

- If $f : \mathbb{R}^d \to \mathbb{R}^k$, then
$$\dot{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \nabla_x f(x)$$
is a $d \times k$ matrix whose (i,j)th element is $\frac{\mathrm{d}}{\mathrm{d}x_j} f_i(x)$.

- If $f : \mathbb{R}^d \to \mathbb{R}$, then

$$\ddot{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\dot{f}(x)^T$$

is a $d \times d$ matrix whose (i,j)th element is

$$[\ddot{f}(x)]_{i,j} = \frac{\mathrm{d}^2}{\mathrm{d}x_i \mathrm{d}x_j} f(x)$$

Fact 14. *If $f : \mathbb{R}^d \to \mathbb{R}^s$, $g : \mathbb{R}^s \to \mathbb{R}^k$, and $h(x) = g(f(x))$ then*

$$\dot{h}(x) = \dot{g}(f(x))\dot{f}(x) \tag{2.1}$$

Fact 15. *If $f : \mathbb{R}^d \to \mathbb{R}^k$, $g : \mathbb{R}^s \to \mathbb{R}^k$, and $h(x) = f^T(x)g(x)$ then*

$$\dot{h}(x) = g(x)^T \dot{f}(x) + f(x)^T \dot{g}(x)$$

Theorem 16. *[The Mean Value Theorem] If $f : \mathbb{R}^d \to \mathbb{R}^k$ and if $\dot{f}(x)$ is continuous in the ball $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$, then for $|t| < r$,*

$$f(\underbrace{x_0 + t}_{x}) = f(x_0) + \left(\int_0^1 \dot{f}(x_0 + ut)\,du\right) t$$

Proof. Let $h(u) = f(x_0 + ut)$, so that $\dot{h}(u) = \dot{f}(x_0 + ut)t$ (from (2.1)). Then,

$$\int_0^1 \dot{f}(x_0 + ut)t\mathrm{d}u = \int_0^1 h(u)\mathrm{d}u = h(1) - h(0) = f(x_0 + t) - f(x_0)$$

□

Theorem 17. *[The Taylor's formula (2nd order)] Let $f : \mathbb{R}^d \to \mathbb{R}$ and if $\ddot{f}(x)$ is continuous in the ball $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$. Then for $x = x_0 + t$ where $|t| < r$:*

$$f(x) = f(x_0) + \dot{f}(x_0)t + t^T \left(\int_0^1 \int_0^1 u\ddot{f}(x_0 + uvt)\,du dv\right) t$$

Proof. [FYI:] Same trick as above by using $g(v) = t^T \left(\int_0^1 \dot{f}(x_0 + uvt)\mathrm{d}u\right)$... □

Notation 18. If $f : \mathbb{R}^d \to \mathbb{R}$, then we denote the partial derivatives

$$\partial^{(k)}_{\underbrace{i_1 \cdots i_k}_{\#k}} f(x_0) = \frac{\mathrm{d}^k}{\mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}} f(x)\Bigg|_{x=x_0}$$

Notation 19. If $f : \mathbb{R}^d \to \mathbb{R}$ and $t = (t_1, ..., t_d) \in \mathbb{R}^d$, we denote as $f^{(k)}(x; h)$:

$$f^{(k)}(x; h) = \underbrace{\sum_{i_1=1}^{d} \cdots \sum_{i_k=1}^{d}}_{\#k} \partial^{(k)}_{i_1 \cdots i_k} f(x) \underbrace{h_{i_1} \cdots h_{i_k}}_{\#k}$$

E.g.: $\partial^{(2)}_{i,j} f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x_1 \mathrm{d}x_2} f(x)\Big|_{x=x_0}$ and $f^{(k)}(x; h) = \sum_{i=1}^{d} \sum_{j=1}^{d} \partial^{(2)}_{i,j} f(x) h_i h_j$.

Theorem 20. *[The Taylor's formula] Let function $f : \mathbb{R}^d \to \mathbb{R}$ with continuous partial derivatives of order n in the ball $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$. The $n-1$ order Taylor expansion of $f(x)$ around x_0 where $x = x_0 + h$ when $|h| < r$ is* Appendix A

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(x_0; h) + R_n(x_0)$$

where the remainder is given in Lagrange's form by

$$R_n(x_0) = \frac{1}{n!} f^{(n)}(x_0 + th; h), \ \ \textit{for some } t \in (0, 1)$$

or equivalently in the integral form by

$$R_n(x_0) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} f^{(n)}(x_0 + th; h)\, dt$$

Remark 21. Regarding Theorem 20, if $\partial^{(n)}_{i_1 \cdots i_n} f(x) \le M$ for $x \in B_r(x_0)$ and some finite $M > 0$ it is

$$R_n(x_0) \le \frac{M}{n!} \|h\|^n$$

and hence the remainder is of order $R_n(x_0) = O(\|h\|^n)$ or $R_n(x_0) = o\left(\|h\|^{n-1}\right)$. NB: M should not depend on h.

Exercise sheet (for practice)
Exercises: # 6, 7

3 Characteristic functions & other transformations

Characteristic functions provide an alternative way to the probability function for describing a random variable. In fact, it completely determines (see Theorem 23(8)) the behavior and properties of the probability distribution of the random variable X.

Definition 22. The characteristic function of a d dimensional random variable X is

$$\varphi_X(t) = \mathrm{E}(e^{it^T X})$$

for $t \in \mathbb{R}^d$, where $e^{it^T X} = \cos(t^T X) + i \sin(t^T X)$.

Theorem 23. *Some properties of characteristic functions*

1. $\varphi_X(t)$ *exists for all* $t \in \mathbb{R}^d$ *and is continuous*
2. $\varphi_X(0) = 1$ *and* $|\varphi_X(t)| \le 1$ *for all* $t \in \mathbb{R}^d$

3. *$\varphi_{a+BX}(t) = e^{it^T a}\varphi_X(B^T t)$ if $X \in \mathbb{R}^d$ is a random variable, and if $a \in \mathbb{R}^{k\times 1}$ and $B \in \mathbb{R}^{k\times d}$ are constants*

4. *$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$ iff X and Y are independent random variables*

5. *if $E|X| < \infty$, then $\dot{\varphi}_X(t)$ exists, it is continuous, and $\dot{\varphi}_X(0) = iE(X)^T$*

6. *if $E|X|^2 < \infty$, then $\ddot{\varphi}_X(t)$ exists, it is continuous, and $\ddot{\varphi}_X(0) = -E(X^T X)$*

7. *if X is degenerate at $c \in \mathbb{R}^d$ then $\varphi_X(t) = e^{it^T c}$*

8. *$F_Y(t) = F_X(t) \Longleftrightarrow \varphi_Y(t) = \varphi_X(t)$, for any $t \in \mathbb{R}^d$*

9. *if $X \sim N(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T\mu - \frac{1}{2}t^T\Sigma t)$*

Proof. Straightforward from the Definition 22. □

Theorem 24. *[Continuity theorem] Let $X, X_1, X_2, ...$ random vectors*

$$X_n \xrightarrow{D} X \Longleftrightarrow \varphi_{X_n}(t) \to \varphi_X(t), \text{ for any } t \in \mathbb{R}^d$$

Example 25. ($\star$) Show that if $X \sim \text{Ex}(\lambda)$ then $\varphi_X(t) = \frac{\lambda}{\lambda - it}$.

Solution. It is

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itX} \underbrace{\lambda e^{-\lambda x} 1(X > 0)}_{=f_{\text{Ex}}(x|\lambda)} \mathrm{d}x = \lambda \int_{-\infty}^{\infty} e^{-x(\lambda - itX)} \mathrm{d}x = \frac{\lambda}{\lambda - it}$$

Example 26. ($\star$)

1. Find $\varphi_X(t)$ if $X \sim \text{Br}(p)$.
2. Find $\varphi_Y(t)$ if $Y \sim \text{Bin}(n, p)$

Solution.

1. It is
$$\varphi_X(t) = \sum_{x=0,1} e^{itX} P(X = x) = e^{it0}(1-p) + e^{it1}p = (1-p) + pe^{it}$$

2. Because Binomial r.v. results as a summation of n IID Bernoulli r.v., it is $Y = \sum_{i=1}^n X_i$, where $X_i \sim \text{Br}(p)$ $i = 1, ..., n$ and IID. Then
$$\varphi_Y(t) = \varphi_{\sum X_i}(t) = \prod_{i=1}^{n} \varphi_{X_i}(t) = ((1-p) + pe^{it})^n$$

Other Integral transforms

Definition 27. The moment generation function of a d dimensional random variable X is

$$M_X(t) = \mathrm{E}(e^{t^T X})$$

for $t \in \mathbb{R}^d$.

Remark 28. It is $M_X(t) = \phi_X(-it)$. Hence. its properties can be easily derived. E.g., $M_{X+Y}(t) = M_X(t)M_Y(t)$ iff X, Y are independent.

Definition 29. The Cumulant generating function of a d dimensional random variable X is the natural logarithm of the moment-generating function

$$K_X(t) = \log\left(M_X(t)\right) = \log\left(\mathrm{E}(e^{t^T X})\right)$$

for $t \in \mathbb{R}^d$.

Remark 30. Properties of the Cumulant generating functions can be easily derived, e.g. $K_{X+Y}(t) = K_x(t) + K_Y(t)$ iff X and Y are independent, etc...

Note 31. Some books refer to the Cumulant generating function as the log of the Characteristic function– we do not do this here.

Exercise sheet (for practice)

Exercises: #10.
For more practice see the examples from

- `https://www.statlect.com/fundamentals-of-probability/characteristic-function`
- `https://www.statlect.com/fundamentals-of-probability/joint-characteristic-function`

References

[1] Tom M Apostol. *Mathematical analysis; 2nd ed.* Addison-Wesley Series in Mathematics. Addison-Wesley, Reading, MA, 1974. URL `https://cds.cern.ch/record/105425`.

[2] Robert J Serfling. *Approximation theorems of mathematical statistics*, volume 162. John Wiley & Sons, 2009.

Appendix

A The messy but clear form of the Taylor formula

Theorem 32. *[The Taylor's formula] Let function $f : \mathbb{R}^d \to \mathbb{R}$ with continuous partial derivatives of order n in the ball $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$. The $n-1$ order Taylor expansion of $f(x)$ around x_0 where $h = x - x_0$ when $|h| < r$ is* Appendix A

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \left[\frac{1}{k!} \sum_{i_1=1}^{d} \cdots \sum_{i_k=1}^{d} \frac{d^k}{dz_{i_1} \cdots dz_{i_k}} f(z_{i_1}, \cdots z_{i_d}) \bigg|_{z=x_0} \prod_{j=1}^{k} (x_{i_j} - x_{0,i_j}) \right] + R_n(x_0)$$

where the remainder is given in Lagrange's form by

$$R_n(x_0) = \frac{1}{n!} \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \frac{d^n}{dz_{i_1} \cdots dz_{i_n}} f(z_{i_1}, \cdots z_{i_d}) \bigg|_{z=\xi} \prod_{j=1}^{n} (x_{i_j} - x_{0,i_j}),$$

for $\xi = x_0 + th$ for $t \in (0, 1)$, or the integral form by

$$R_n(x_0) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \frac{d^n}{dz_{i_1} \cdots dz_{i_n}} f(z_{i_1}, \cdots z_{i_n}) \bigg|_{z=\xi} \prod_{j=1}^{n} (x_{i_j} - x_{0,i_j}) dt$$