Contents

1 Financial Markets

Throughout modern history, the value of *anything* has been of great importance. For any two members taking opposite sides of a desired object[1], an exchange takes place if they both agree on the current worth of the object to exchange. For this to happen, both members must be able to find one another. It may also be the case for any of them two to find a third whose worth of the object at play favors any one of them more.

A **security**, is the epitome of a tradable object in financial markets. We define a security as an instrument that either represents ownership or that derives its value from a commodity. We say that a security is **fungible** if any unit of the instrument is economically indistinguishable from all other units. That is, whomever buys (or sells) the instrument can pay (or receive) any unit at the same price at a given point in time.

The problem of searching for the price and the party whose view on the value of a security reflect ours is called **trading**. A **market** is the place where buyers meet sellers. Markets can be both a physical place (called a trading floor), or an electronic system.

Not all financial products are traded in formal markets. Over the counter (OTC) trades happen between two parties directly or via a broker, whom helps them settle the trade.

1.1 The Participants

In a financial market, participants have various reasons to trade. On the one hand there is the **buy-side**. They trade in order to solve a financial

[1] In other words, one member wants to buy (obtain) and the other member to sell (get rid off)

problem originated outside of the market. This group sees the market as a means to an end, and thus, they rely on the resources available in the market.

Trader Type	Examples	Why They Trade	Instruments
Investor	Individual Corporate pension funds Insurance funds Charitable and legal trusts Endowments Mutual Funds Money Managers	To move wealth from the present to the future for themselves or for their clients	Stocks Bonds
Borrowers	Homeowners Students Corporations	To move wealth from the future to the present	Mortgages Bonds Notes
Hedgers	Farmers Manufacturers Miners Shippers Financial Institutions	To reduce business operating risk	Futures contracts Forward contracts Swaps
Asset Exchangers	International corporations Manufacturers Travelers	To acquire an asset that they value mire than the asset that they tender	Currencies Commodities
Gamblers	Individuals	To entertain themselves	Various

Table 1.1: Buy-side of the market. Harris (2002)

On the other hand are the participants who offer exchange services to the buy-side. Their trading purpose is to satisfy the needs of the buy-side by taking the opposite side of the trade. This group sells exchange services to the buy-side, consequently they are known as the **sell-side**.

We can classify the buy-side and the sell-side as those who require liquid-

Trader Type	Examples	Why They Trade
Dealer	Market Maker Specialist Floor Trader Locals Day traders Scalpers	To earn trading profits by supplying liquidity
Brokers	Retail brokers Discount brokers Full-service brokers Institutional Brokers Block brokers Futures commission merchants	To earn commissions by arranging trades for clients
Broker-dealers	Wirehouses	To earn profits and trading commissions

Table 1.2: Sell-side of the market. Harris (2002)

ity and as those who provide it. **Liquidity** is a critical concept in financial markets. Although the term itself can be regarded in many ways, we will denote it as the likelihood of trading constraint to units available of the asset to buy (or sell), price offered, and time.

1.2 The Instruments

Financial instruments comprise a wide array of products. Each one of these serve a purpose to either the buy-side or the sell-side. We will review briefly the instruments that will serve for the development of this work.

1.2.1 Bonds

According to the Securities and Exchange Commission (SEC),

> “A bond is a debt obligation, like an IOU. Investors who buy corporate bonds are lending money to the company issuing the bond. In return, the company makes a legal commitment to

> pay interest on the principal and, in most cases, to return the principal when the bond comes due, or matures."

A bond can be issued either by a corporation or by a government. Its main function is to raise capital for the issuer of the bond, known as the **debtor**, who promises a payment to the buyer of the bond, or as the **creditor** or the **bondholder**. The amount of money to be paid at the time of maturity is the **nominal value** or the **principal** of the bond.

In addition, bonds may or may not pay an interest on the nominal of the bond. In the former case, this is known as a **coupon bearing bond**, while the latter is referred to as the **zero coupon bond**.

1.2.2 Stocks

A stock is a security that represents ownership on a fraction of a corporation; it is a proportional division of a company's assets and distributed through what is known as a **dividend**. Future dividends are generally not known in advance. This contrast a big difference between bonds and stocks, while the former has a predefined number of payments, the latter is uncertain in amount and frequency.

1.2.3 Foreign Exchange Currencies

Kozikowski (2013) defines a foreign currency as "one country's currency freely convertible in the foreign exchange market".

The foreign exchange market is an OTC market where buyers and sellers get together to buy and sell foreign currencies or financial contracts on said currencies. The foreign exchange market is also known as the *FX market*.

1.2.4 Derivatives

Derivatives are the main topic in this work. Hull (2014) defines derivatives as "as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables". This definition leaves room for many possible products that can depend on more than one factor.

Derivatives are mainly used to hedge and/or arbitrage. We can categorize them into two main categories: listed and OTC contracts. Listed contracts can be bought or sold in an organized market. These are standardized products that can be traded in financial markets. OTC contracts, on the other hand, represent a contract between two parties without the need of a market for the two parties to meet.

1.3 Risk: Hedging and Arbitrage

Risk is an inherent aspect of financial markets. For bonds, for example, risk is represented as the plausibility of default for the government or the company which issued the asset. For stocks, risk is the uncertainty of future prices, whether the company goes bankrupt, or the amount of the future dividend to be payed.

From a financial institution's perspective interested in selling derivative contracts, their intention may not be to speculate with the products they sell, as to where the price of the contracts may go, but rather to sell it a service. This demands the financial institution to eliminate all possible risk associated with the contract. Furthermore, the price they sell this contract for must be such that no one else may take a riskless profit from it.

Two essential concepts emerge for a financial institution interested in covering the needs of the sell-side. That of **hedging** and **arbitrage**. Although we will define it more rigourously in upcoming chapters, the following will motivate the idea of what we want to achieve

Consider the following example. Suppose we are bank willing to sell a stock at a fixed price three months from now. Since we do not know the price the stock will take three months from now, and considering we are not interested in taking risk selling this *derivative*, we wrongly decide to price this derivative as the current price of the stock plus an arbitrary fee, thus allowing us to deliver the stock three months from now and get a profit from it.

One clever trader realizes that he can borrow, for three months, the current price of the stock, buy the stock, and pay the interest owed plus nominal three months from now. If the value of the nominal plus interest is greater than the price for which the bank is selling this derivative, the trader can short-sell the stock, invest the amount received and enter the contract.

Three months from now the trader must pay the price of the contract, which he can do since the amount he invested in the bank is greater than the price of the contract. Also, he can return the stock he shorted since the bank will give it to him, thus achieving a riskless profit.

Although the bank decided cover its position in order to fulfill its future obligation, the price it gave the contract was such that the trader made a profit without taking any risk from it. Strictly speaking, the bank **hedged** its position and the trader made an **arbitrage**.

This simple example presents the key ideas to take account whenever we try to price any derivative. By replicating what the trader did, the bank would be hedged and would not incur in anyone making an arbitrage. Composing an effective hedge may not be so easy for every contract.

Let us consider another example: a derivative that provides to the buyer the option to buy a stock at a future date and at a predetermined price. What we would like is to know the present value of this derivative, such that it allows for an effective hedge and prevent anyone from making an arbitrage.

This type of derivative is known as a European Option and, at the time of maturity, the payoff provided by this contract is the following

$$\max\{S - K, 0\} \tag{1.1}$$

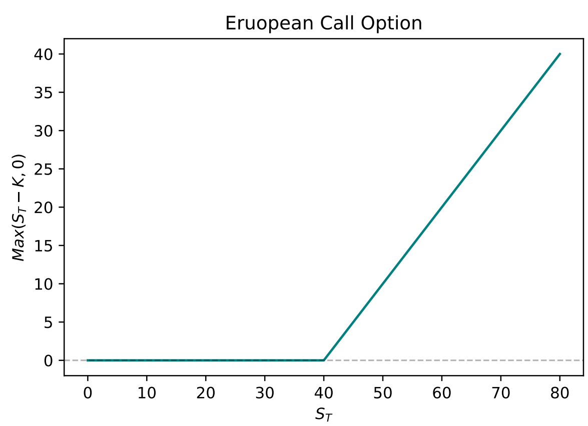

Figure 1.1: European Call Payoff

Note that the payoff of this derivative is *stochastic* in the sense we cannot be certain of the price the stock will reach some future time from now. Then, we cannot be sure of the future payoff of the stock.

The knowledge and tools we currently have can take us only so far if we desire to compute the proper price for an option, such that hedges the position of the seller. Under this constraint, we require to properly take account of the randomness of the stock, in pursuance of the *right* way to price this derivative.

2 | Probability Theory

In order to explain the likelihood of certain random events, we must first define what we mean by *randomness*, specify which events are being referred to, and to express, in mathematical sense, the possibility of any such event. With this in mind, we dedicate this chapter to gradually build the tools that will later be required to *make sense* of random events and further develop a theory to model the dynamics of assets such as stocks and currencies.

We begin by motivating the need of *measurable spaces*, and their fundamental importance throughout this work.

2.1 Measurable Spaces

Assume we had a fair coin and decided to represent the possible outcomes that bring about throwing it twice in a row. Let us represent with the letter H the scenario in which *head* lands; on the contrary, we will denote with the letter T the scenario in which *tail* lands. Evidently, the possible scenarios are the following:

$$\{HH\},\ \{HT\},\ \{TH\},\ \{TT\}.$$

By throwing the fair coin twice, we are effectively performing an **experiment**. In this context, the outcome of any possible single event in the experiment is known as the **universe**. We denote the universe with the letter greek Omega (Ω).

Although Ω contains every element of possible occurrence, its members alone are not enough to answer any possible question we may have about the outcome of the experiment. Suppose, for example, we were interested in knowing which elements of Ω constitute the outcome of the experiment in which at least one head (H) appears. Clearly, this subset of Ω, which contains the elements $\{TH, HT, HH\}$, is not currently expressed. Likewise, the subset of Ω in which only two coins of the same side appear ($\{HH, TT\}$), is not currently expressed.

To address this problem, we define a set of subsets of Ω, called an **algebra**, that captures the feasible questions to ask, as to the occurrences that may happen in the experiment.

Definition 2.1 (**Algebra**). *Let Ω be a set of points ω. A system $\mathscr{A}$ of subsets of Ω is an Algebra if:*

1. *$\Omega \in \mathscr{A}$;*
2. *If $A, B \in \mathscr{A} \Rightarrow A \cap B \in \mathscr{A}$; and*
3. *If $A \in \mathscr{A} \Rightarrow A^{\mathsf{c}} \in \mathscr{F}$.*

With definition 2.1 in mind, we can now consider the *algebra* for $\Omega = \{HH, HT, TH, HH\}$ in which we toss two fair coins:

```
{{}, {HH}, {HT}, {TH}, {TT}, {HH, HT}, {HH, TH},
{HH, TT}, {HT, TH}, {HT, TT}, {TH, TT},
{HH, HT, TH}, {HH, HT, TT}, {HH, TH, TT}, {HT, TH, TT},
{HH, HT, TH, TT}}
```

Universes are not necessarily countably finite. Some may be countably infinite (e.g., the natural numbers), or even uncountably infinite (e.g., the subset of real numbers between 0 and 1). Concerning a more abstract space Ω, we then need to impose an additional property to an algebra called the *σ-additivity.*

Definition 2.2 (**σ-algebra**). *A system $\mathscr{F}$ of subsets of Ω is a σ-algebra if it is an algebra and it satisfies the σ-additive property:*

$$\text{If } \{A_n\}_{n\geq 1} \in \mathscr{F} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathscr{F}.$$

Remark. *For $\{A_n\}_{n\geq 1} \in \mathscr{F}$ then, definition 2.2 is equivalent to say that*

$$\text{If } \{A_n\}_{n\geq 1} \in \mathscr{F} \Rightarrow \bigcap_{k=1}^{\infty} A_k \in \mathscr{F}.$$

Proof. By the generalized form of De Morgan's laws, $(\bigcap_k A_k)^{\mathsf{c}} \equiv \bigcup_k A_k^{\mathsf{c}}$. Since $A_k \in \mathscr{F}\ \forall\ k$ then, by definition, $A_k^{\mathsf{c}} \in \mathscr{F}\ \forall\ k$ which implies $\bigcup_k A_k^{\mathsf{c}}$ and this complement is also in $\mathscr{F}$ $\square$

Definition 2.3. *let $\emptyset \neq A \subseteq \Omega$. The σ-algebra generated by A, denoted by $\sigma(A)$, is the set*

$$\sigma(A) := \bigcap\{\mathscr{F} : \mathscr{F} \text{ is sigma algebra, and } A \subseteq \mathscr{F}\}. \tag{2.1}$$

Defintion 2.3 is the smallest σ-algebra that cointains A. This definition is useful in the construction of one particular σ-algebra of subsets of $\mathbb{R}$.

Consider the collection of open intervals $(a, b) \in \mathbb{R}$. For every $a \leq b$, the smallest σ-algebra generated by this collection is known as the Borel σ-algebra of subsets of $\mathbb{R}$.

Definition 2.4 (**The Borel σ-algebra**).

$$\mathscr{B}(\mathbb{R}) := \sigma\{(a, b) \subseteq \mathbb{R} : a \leq b\}. \tag{2.2}$$

Definition 2.4 is the smallest σ-algebra that contains all open intervals in the real line. Furthermore, it can be proven that $[a, b]$, (a, ∞), $(-\infty, b)$, $[a, b)$, $(a, b]$ are all elements in $\mathscr{B}(\mathbb{R})$.

Concerning the set of elements Ω, equipped with a σ-algebra $\mathscr{F}$, yields a crucial definition that will formalize an abstract space in which we can further develop a theory.

Definition 2.5 (**Measurable Space**). *If $\mathscr{F}$ is a σ-algebra of subsets of Ω, $(\Omega, \mathscr{F})$ is said to be a measurable space. If A is a set in $\mathscr{F}$, we say that A is $\mathscr{F}$-measurable —or measurable with respect to (w.r.t.) $\mathscr{F}$—*

Definition 2.5 urges an auxiliary way to make sense of the information provided by $(\Omega, \mathscr{F})$. Concretely, we now define a way to fathom any abstract measurable space.

Definition 2.6 (**Measure**). *Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real line and let $(\Omega, \mathscr{F})$ be a measurable space. It is said that a function $\mu : \mathscr{F} \to \bar{\mathbb{R}}$ is a measure over $(\Omega, \mathscr{F})$ if*

1. *$\mu(\emptyset) = 0$;*

2. *$\mu(A) \geq 0 \ \forall \ A \in \mathscr{F}$; and*

3. *μ is σ-additive: If $\{A_k\}_{k=1}^{\infty}$ is a set of disjoint events in $\mathscr{F}$ ($A_i \cap A_j = \emptyset \ \forall \ i \neq j$),*

$$\bigcup_{k=1}^{\infty} \mu(A_k) = \mu(\sum_{k=1}^{\infty} A_k).$$

If the last definition holds true for a given μ inside $(\Omega, \mathscr{F})$, then $(\Omega, \mathscr{F}, \mu)$ is said to be a **measure space**. Thus, for any $A \in \mathscr{F}$, $\mu(A)$ is the measure of A.

Furthermore, if $\mu(\Omega) < \infty$ then, μ is said to be a finite measure. Lastly, if $\mu(\Omega) = 1$, then μ is said to be a probability, which we denote $\mathbb{P} \equiv \mu$. Consequently, $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a **probability space**. We formalize this last definition as follows:

Definition 2.7 (**Kolmogorov's Axiom System**). *An ordered triple* $(\Omega, \mathscr{F}, \mathbb{P})$ *where*

1. Ω *is a set of points* ω*;*
2. $\mathscr{F}$ *is a* σ*-algebra of subsets of* Ω*; and*
3. $\mathbb{P}$ *is a probability on* $\mathscr{F}$*,*

is called a ***probability space*** *or* ***model*** *(of an experiment)*

As a result of definition 2.7, if $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $A \in \mathscr{F}$, it is said that $\mathbb{P}(A)$ is the *probability of the event* A.

2.2 Random Variables

For any given probability space $(\Omega, \mathscr{F}, \mathbb{P})$, Ω is defined to be any abstract space. To overcome the problem of abstraction, it is desirable to find a mapping from elements in Ω to the more convenient $\mathbb{R}$, in the sense that, measurements on the real line, if mapped correctly, may become easier to interpret. In view of the coveted property of interpretability, we define a mapping $\Omega \to \mathbb{R}$ that is consistent with the theory developed so far.

Definition 2.8. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space. The function* $X : \Omega \to \mathbb{R}$ *it is defined an* $\mathscr{F}$*-measurable function or a* ***random variable*** *if*

$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathscr{F} \quad \forall \quad B \subseteq \mathbb{R}. \tag{2.3}$$

With definition 2.8, measurements over Ω dovetail with an element of order. Consequently, it becomes natural to define the following:

Definition 2.9. *Consider the event* $\{X \leq x\} := X^{-1}(-\infty, x]$*. The function* $F_X : \mathbb{R} \to [0, 1]$ *defined as:*

$$F_X(x) := \mathbb{P}\{X \leq x\}, \tag{2.4}$$

is known as the ***distribution function*** *of the random variable* X*.*

Furthermore, a probability distribution function has the following properties:

1. For $x \leq y$ then, $F_X(x) \leq F_X(y)$;
2. $\lim_{x \to \infty} F_X(x) = 1$ and $\lim_{x \to -\infty} F_X(x) = 0$; and
3. F_X is right-continuous, meaning that $\lim_{y \to x^+} F_X(y) = F_X(x)$ for every $x \in \mathbb{R}$.

Definition 2.10. *A random variable X is said to be **absolutely continuous** is there exists a nonnegative borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$F_X(x) = \int_{-\infty}^{x} f(t)dt. \tag{2.5}$$

The function f is known as the **probability density function** (p.d.f.) of the random variable X. Perhaps one of the most known probability density functions, and crucial tool for this work, is the p.d.f. of the normal distribution:

$$\phi(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{2.6}$$

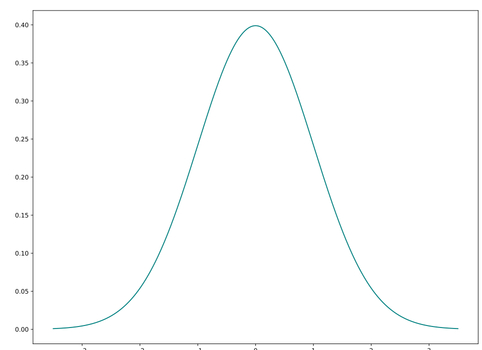

Figure 2.1: Probability Density Function of a Normal Distribution $\phi(x|0, 1)$.

We denote the normal distribution as $\mathcal{N}(\mu, \sigma^2)$. Also, if X normally distributed with parameters μ and. σ^2, we write $X \sim N(\mu, \sigma^2)$.

2.3 Lebesgue Integrals and Expectations

We now present the key ideas behind Lebesgue integration. As we shall later see, random movements over time (e.g. that of a stock or FX exchange rate) can be modeled as the sum of infinitesimal random changes. A through understanding of the ideas behind integration allow us to more easily understand the machinery for much more complex, stochastic models. In addition, Lebesgue integrals pave the way to the mathematical expectation, a core idea in the theory of probability.

We start the motivation of an Lebesgue integral by defining the most basic class of measurable functions.

Definition 2.11. *Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $X : \Omega \to \mathbb{R}$ an $\mathscr{F}$-measurable function. Denote $\mathbb{1}_A(\omega)$ the indicator function of ω in a set A. That is,*

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

*To simplify the notation, we will also denote $\mathbb{1}_A(\omega)$ as $\mathbb{1}_A$. We define the **Lebesgue integral** of $\mathbb{1}_A$ w.r.t. μ as*

$$\int_\Omega \mathbb{1}_A(\omega) d\mu(\omega) := \mu(A). \tag{2.7}$$

Definition 2.12. *Let $(\Omega, \mathscr{F})$ a measurable space and X an $\mathscr{F}$-measurable function. It is said that X is a **simple function** if it takes only a unique finite number of values $\{x_i\}_{i=1}^n \in \mathbb{R}$. Then, X can be written as*

$$X(\omega) = \sum_{i=1}^n x_i \mathbb{1}_{A_i} \tag{2.8}$$

Where,

$$A_i := \{\omega \in \Omega | X(\omega) = x_i\} \ \forall \ i = 1, \dots, n$$

Definition 2.13. *let X be a simple function, we define the integral of X w.r.t. μ as:*

$$\int_\Omega X(\omega) d\mu(\omega) := \sum_{i=1}^n x_i \mu(A_i) \tag{2.9}$$

Theorem 2.14. *If $f : \mathbb{R} \to [0, \infty]$ is measurable, then there exist a collection of real-valued simple functions $\{f_i\}_{i \geq 0}$ on f such that $0 \leq f_1 \leq \dots \leq f$ and $f_n \to f$ pointwise.*

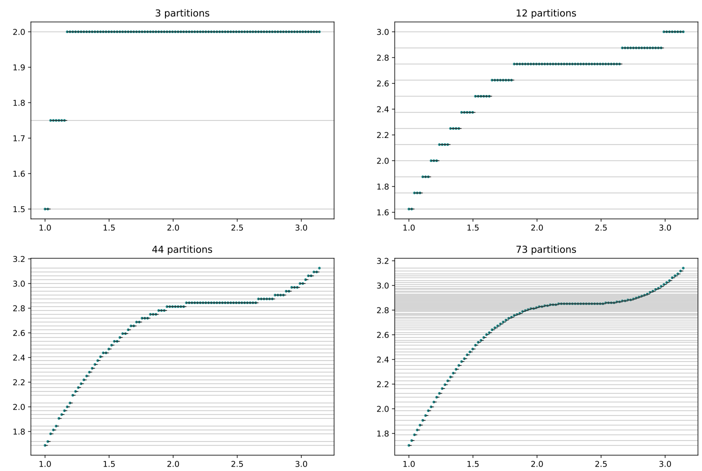

Figure 2.2: Approximation to the Borel-measurable function $\sin^2(x) + x$ via a sequence of simple functions

Proposition 2.15. *Let X be a simple function defined on a measurable space $(\Omega, \mathscr{F}, \mu)$ in which $\mu(\Omega) < \infty$. Then,*

$$X \geq 0 \implies \int_\Omega X d\mu \geq 0 \tag{2.10}$$

Proof. Since X is a simple function,

$$\int_\Omega X d\mu = \sum_{i=1}^{n} x_i \mu(A_1),$$

where $x_i \geq 0 \ \forall \ i \in \{1, \ldots, n\}$. Also, $\mu(A_i) \geq 0 \ \forall \ A_i \in \mathscr{F}$. $\square$

Theorem 2.16. *If X and Y are two non-negative simple functions and $\alpha, \beta \geq 0$,*

$$\int_\Omega (\alpha X + \beta Y) d\mu = \alpha \int_\Omega X d\mu + \beta \int_\Omega Y d\mu; \tag{2.11}$$

If $X \leq Y$ then,

$$\int_\Omega X \leq \int_\Omega Y \tag{2.12}$$

Proof. Let, $X = \sum_{i=i}^{n} x_i \mathbb{1}_{A_i}$ and $Y = \sum_{j=1}^{m} y_i \mathbb{1} B_i$. Note that $\bigcup_i A_i = \bigcup_j B_j = \Omega$. Also, $\mathbb{1}_A = \sum_{i=1}^{n} \mathbb{1}_{A_i}$ since $\{A_i\}$, by definition of the lebesgue integral, is a set of disjoin elements. Then, X and Y can be written as

$$\begin{aligned}
X &= \sum_{i=1}^{n} \mathbb{1}_{A_i \cap \Omega} \\
&= \sum_{i=1}^{n} \mathbb{1}_{A_i \cap (\cup B_j)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \mathbb{1}_{A_i \bigcap B_j}.
\end{aligned}$$

Similarly for Y,

$$\begin{aligned}
Y &= \sum_{j=1}^{m} \sum_{i=1}^{n} y_i \mathbb{1}_{B_j \bigcap A_i} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} y_i \mathbb{1}_{A_i \bigcap B_j}.
\end{aligned}$$

Then,

$$\begin{aligned}
\alpha X + \beta Y &= \alpha \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \mathbb{1}_{A_i \bigcap B_j} + \beta \sum_{i=1}^{n} \sum_{j=1}^{m} y_i \mathbb{1}_{A_i \bigcap B_j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha x_i + \beta y_i) \mathbb{1}_{A_i \bigcap B_j}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_\Omega \alpha X + \beta Y &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha x_i + \beta y_i) \mu(A_i \cap B_j) \\
&= \alpha \sum_{i=i}^{n} \left(\sum_{j=1}^{m} x_i \mu(A_i \cap B_j) \right) + \beta \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_i \mu(A_i \cap B_j) \right) \\
&= \alpha \sum_{i=i}^{n} x_i \mu(A_i) + \beta \sum_{j=1}^{m} y_i \mu(B_i) \\
&= \alpha \int_\Omega X d\mu + \beta \int_\Omega Y d\mu
\end{aligned}$$

We now set out to prove the monotonicity property for nonnegative simple functions. Let $X \leq Y$ then, $Y - X \geq 0$ is an $\mathscr{F}$-measurable function. By proposition 2.15,

$$Y - X \geq 0 \implies \int_{\Omega} (Y - X) d\mu \geq 0 \tag{2.13}$$

$\square$

Definition 2.17. *For any nonnegative function X, we define the integral of X w.r.t. a measure μ as*

$$\int_{\Omega} X d\mu := \sup \left\{ \int_{\Omega} h d\mu \mid 0 \leq h \leq X, \, h \text{ is simple} \right\} \tag{2.14}$$

If X is any measurable function, we define the positive and negative parts of X as:

$$X^{+} := \max\{X(\omega), 0\} \tag{2.15}$$

$$X^{-} := \max\{-X(\omega), 0\} \tag{2.16}$$

Thus, $X = X^{+} - X^{-}$, and $|X| = X^{+} + X^{-}$. Since both parts of X are positive, their integrals are well defined.

If both $\int_{\Omega} X^{+} d\mu$ and $\int_{\Omega} X^{-} d\mu$ are finite, we say that X is integrable w.r.t. μ. By linearity,

$$\int_{\Omega} X d\mu = \int_{\Omega} X^{+} d\mu - \int_{\Omega} X^{-} d\mu. \tag{2.17}$$

And,

$$\int_{\Omega} |X| d\mu = \int_{\Omega} X^{+} d\mu + \int_{\Omega} X^{-} d\mu. \tag{2.18}$$

We will denote by $L_1 \equiv L_1(\Omega, \mathscr{F}, \mu)$ the family of integrable functions w.r.t. μ. Note that $X \in L_1$ if and only if $|X| \in L_1$. i.e.,

$$\int_{\Omega} |X| d\mu = \int_{\Omega} X^{+} d\mu + \int_{\Omega} X^{-} d\mu < \infty \tag{2.19}$$

If $A \in \mathscr{F}$, we define

$$\int_{A} X d\mu := \int_{\Omega} X \mathbb{1}_{A} d\mu \tag{2.20}$$

Proposition 2.18. *For any X, Y $\mathscr{F}$-measurable functions in L_1; A, B members of $\mathscr{F}$, it can be shown that:*

$$X \geq 0 \implies \int_\Omega X d\mu \geq 0 \tag{2.21}$$

If $X \leq Y$ (monotonicity),

$$\int_\Omega X d\mu \leq \int_\Omega Y d\mu \tag{2.22}$$

If $A \subseteq B$ and $X \geq 0$,

$$\int_A X d\mu \leq \int_B X d\mu \tag{2.23}$$

If α, $\beta \in \mathbb{R}$ (linearity),

$$\int_\Omega (\alpha X + \beta Y) d\mu = \alpha \int_\Omega X d\mu + \beta \int_\Omega Y d\mu \tag{2.24}$$

Theorem 2.19. *Let $(\Omega, \mathscr{F}, \mu)$ be a measurable space, and X a nonnegative $\mathscr{F}$-measurable function. If $X : \Omega \to \mathbb{R}$, the mapping from $\Omega \to [0, \infty]$ given by $A \to \int_A f d\mu$ is σ-additive, i.e.*

$$\int_A X d\mu = \sum_{i=1}^{\infty} \int_{A_i} X d\mu \tag{2.25}$$

For disjoint sets $\{A_i\}_{i \geq 1}$

For a proof of theorem 2.19 see Applebaum (2016). Consider 2.14, 2.21, together with theorem 2.19. It is then clear, that the Lebesgue integral for random variables $X \in (\Omega, \mathscr{F}, \mathbb{P})$ such that $X(\omega) \geq 0 \ \forall \ \omega \in \Omega$ is then a measure defined over $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. This last is followed by a corollary that will aid in the result of one of the fundamental theorems of integration and convergence.

Corollary 2.20. *Let $X : \Omega \to \mathbb{R}$ a nonnegative measurable function and $\{E_n\}_{n \geq 1}$ a sequence of sets in Ω such that $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$. Denote $E = \bigcup_{n=1}^{\infty} E_n$. Then,*

$$\int_E X d\mu = \lim_{n \to \infty} \int_{E_n} X d\mu \tag{2.26}$$

Proof. Denote $A_1 = E_1$, and $A_n = E_n - E_{n-1}$ $\forall\ n \geq 2$. The set $\{A_n\}_n$ is a sequence of disjoint sets. Note that, $E = \bigcup_{i=1}^{\infty} A_i$ and $E_n = \bigcup_{i=1}^{n} A_i$ $\forall\ n \in \mathbb{N}$.

By theorem 2.19,

$$\begin{aligned}
\int_E X d\mu &= \int_{\bigcup_{i=1}^{\infty} A_i} X d\mu \\
&= \sum_{i=1}^{\infty} \int_{A_i} X d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} X d\mu \\
&= \lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} A_i} X d\mu \\
&= \lim_{n\to\infty} \int_{E_n} X d\mu
\end{aligned}$$

□

Theorem 2.21 (**Monotone Convergence Theorem**). *Let $\{X_n\}_{n\geq 1}$ be a sequence of nonnegative functions where $X_i : \Omega \to \mathbb{R}$ such that $X_n \leq X_{n+1}$ $\forall\ n \in \mathbb{N}$. Let X be another random variable such that $X_n \to X$ then,*

$$\int_\Omega X d\mu = \lim_{n\to\infty} \int_\Omega X_n d\mu \tag{2.27}$$

Proof. Since $X = \sup_{n\in\mathbb{N}} X_n$, by monotonicity,

$$\int_\Omega X_1 d\mu \leq \int_\Omega X_1 d\mu \leq \ldots \leq \int_\Omega X d\mu$$

Thus,

$$\lim_{n\to\infty} \int_\Omega X_n d\mu \leq \int_\Omega X d\mu$$

To show the reverse inequality, let $0 \leq h \leq X$ a a simple function and let $c \in (0,1)$. Denote $A_n = \{\omega \in \Omega \mid X_n(\omega) \geq c \cdot h(\omega)\}$.

Since $\{X_n\}_{n\geq 1}$ is a sequence of nondecreasing functions, $A_n \subseteq A_{n+1}$ $\forall$ $n \in \mathbb{N}$. One final property of $\{A_n\}$ is the fact that $\bigcup_n A_n = \Omega$. This last property can be shown by noticing that each A_n has two possible outcomes to determine whether any $\omega \in \Omega$ is inside A_n:

Consider $h(\omega) = 0$. In that case, any $A_k = \{\omega \in \Omega \mid X_k(\omega) \geq 0\}$ will capture every element ω. Consider now $h(\omega) \neq 0$. In this case, we do not know for which $k \in \mathbb{N}$ will $c \cdot h \leq X_k$, nonetheless, since $X_n \to X$, for some $k \in \mathbb{N}$, $X_k(\omega) \leq c \cdot h(\omega)$.

By proposition 2.18,

$$\lim_{n\to\infty} \int_\Omega X_n d\mu \geq \int_\Omega X_n d\mu \geq \int_{A_n} X_n d\mu \geq \int_{A_n} c \cdot h d\mu \tag{2.28}$$

As A_n is an increasing sequence,

$$\lim_{n\to\infty} \int_\Omega X_n d\mu \geq \lim_{n\to\infty} \int_{A_n} c \cdot h d\mu \tag{2.29}$$

Taking account of the linearity property for lebesgue integrals and corollary 2.20,

$$\lim_{n\to\infty} \int_{A_n} c \cdot h d\mu = c \int_\Omega \cdot h d\mu \tag{2.30}$$

Choosing an arbitrary $c \in (0,1)$, set $c = 1 - \frac{1}{k}$ and let $k \to \infty$,

$$\lim_{n\to\infty} \int_\Omega X_n d\mu \geq \lim_{k\to\infty} \left(1 - \frac{1}{k}\right) \int_\Omega h d\mu = \int_\Omega \cdot h d\mu \tag{2.31}$$

Choosing and arbitrary h, we find that

$$\lim_{n\to\infty} \int_\Omega X_n d\mu \geq \sup \left\{ \int_\Omega h d\mu \right\} = \int_\Omega X d\mu \tag{2.32}$$

$\square$

Proposition 2.22. *If $f_n \to f$ a.e. then $f_n \to f$ in measure*

Theorem 2.23 (**Bounded Convergence Theorem**). *If the sequence $\{f_n\}_{n\geq 1}$ of measurable functions is uniformly bounded and if $f_n \to f$ in measure then,*

$$\lim_{n\to\infty} \int_\Omega f_n = \int_\Omega \lim_{n\to\infty} f_n d\mu = \int_\Omega f d\mu \tag{2.33}$$

Proof. First we note that

$$|\int f_n d\mu - \int f d\mu| = |\int (f_n - f) d\mu| \leq \int |(f_n - f)| d\mu$$

We now denote $g_n := f_n - f$. Thus, we are left to show that if $g_n \to 0$ in measure and $|g_n| \leq M$ then $\int |g_n| d\mu \to 0$. To prove this, consider the following partition

$$\int |g_n| d\mu = \int_{|g_n| \leq \epsilon} |g_n| d\mu + \int_{|g_n| > \epsilon} |g_n| d\mu \ \forall\ \epsilon > 0.$$

Where,

$$\int_{|g_n| \leq \epsilon} |g_n| d\mu \leq \epsilon,$$

and

$$\int_{|g_n| > \epsilon} f_n d\mu \leq M \int_{|g_n| > \epsilon} d\mu = M \cdot \mu(\{\omega : |g_n(\omega)| > \epsilon\}).$$

Then,

$$\int g_n d\mu \leq \epsilon + M \cdot \mu(\{\omega : |g_n(\omega)| > \epsilon\}). \tag{2.34}$$

Taking limits we find that that

$$\lim_{n \to \infty} |g_n| d\mu \leq \epsilon$$

since $\{\omega : |g_n(\omega)| > \epsilon\}$ converges to an empty set as $n \to \infty$. This completes the proof since $\epsilon > 0$ is as small as desired. □

Theorem 2.24 (Fatou's Lemma). *If $\{f_n\}_{n \geq 1}$ is a sequence of nonnegative measurable functions such that $f_n \to f$ in measure. Then,*

$$\int \lim_{n \to \infty} \inf f_n d\mu = \int f d\mu \leq \lim_{n \to \infty} \inf \int f_n d\mu \tag{2.35}$$

Proof. Let $0 \leq g \leq f$, and define $h_n := \min\{f_n, g\}$ then,

$$\lim_{n \to \infty} h_n = g,$$

since h_n tends to g a.e., by proposition 2.22, $h_n \to g$ in measure. It follows by the bounded convergence theorem that

$$\lim \int h_n d\mu = \int \lim h_n d\mu = \int g d\mu \tag{2.36}$$

Now,

$$\int h_n d\mu = \int \min\{g, f_n\} d\mu \leq \int f_n d\mu \ \forall\ n.$$

It follows that

$$\int h_n d\mu \leq \inf \int f_n d\mu.$$

Finally, taking limits,

$$\lim_{n\to\infty} \int h_n d\mu = \int g d\mu \leq \lim_{n\to\infty} \inf \int f_n d\mu$$

As $0 \leq g \leq f$ is arbitrary, choosing $g = f$ concretes the proof. □

Theorem 2.25 (**Dominated Convergence Theorem**). *If the sequence $\{f_n\}_{n\geq 1}$ is such that $f_n \to f$ in measure, and $|f_n| \leq g$ for measurable g, and all n, ω then,*

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu = \int f d\mu \tag{2.37}$$

Proof. Let $\{f_n\}$ the sequence of measurable simple functions that converge to f in measure. Consider measurable g such that $0 \leq f_n \leq g$. Then, $g + f_n \to g + f$, $g - f_n \to g + f$ in measure. By Fatous's Lemma (2.24),

$$\int (g + f) d\mu \leq \lim_{n\to\infty} \inf \int (g + f_n) d\mu$$

As g is integrable, and considering the linearity of the integral, we can write

$$\int f d\mu \leq \lim_{n\to\infty} \inf \int f_n d\mu.$$

Similarly, for $g - f_n$ we write

$$\int (g - f) d\mu \leq \lim_{n\to\infty} \inf \int (g - f_n) d\mu.$$

Then,

$$\int f d\mu \geq \lim_{n\to\infty} \inf \int f_n d\mu$$

□

We now state, without proof, a theorem that allows to work with different measures under a given measure space.

Theorem 2.26 (**Radon Nikodym Theorem**). *Let μ and λ be σ-finite positive measures defined on $(\Omega, \mathscr{F})$ such that, for every $A \in \mathscr{F}$, $\lambda(A) = 0 \implies \mu(A) = 0$. Then, there exists a function $f : \Omega \to [0, \infty]$ such that*

$$\mu(A) = \int_A f d\lambda.$$

Where the function f is defined up to sets with measure zero. f is sometimes called the Radon-Nikodym derivative and it can be written as $\frac{d\mu}{d\lambda}$

2.4 Expectations

We the tools surveyed so far, we can now introduce a new concept, that of the expectation of a random variable. Aridly put, we will define an expectation as the integral of a random variable w.r.t. its probability measure. Nonetheless, the concept of an expectation under a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ serves a practical purpose that will be of much use throughout the rest of this work.

Definition 2.27 (**Expectation**). *Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. The integral of X w.r.t. the measure $\mathbb{P}$ is known as the expectation of X w.r.t. $\mathbb{P}$.*

$$\mathbb{E}_{\mathbb{P}}[X] := \int_{\Omega} X d\mathbb{P} \tag{2.38}$$

Theorem 2.28. *Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and $h : \mathbb{R} \to \mathbb{R}$ a Borel-measurable function. Let F_X be the distribution of X and $P_X(B) := \mathbb{P}[X^{-1}(B)]$ for every $B \in \mathscr{B}(\mathbb{R})$ the probability measure induced by X on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ then,*

$$h \in L_1(\Omega, \mathscr{F}, \mathbb{P}) \iff h \in L_1(\mathbb{R}, \mathscr{B}(\mathbb{R}), P_X). \tag{2.39}$$

Measuring elements is often times clearer on Borel-measurable sets as opposed to on some abstract σ-algebra. Theorem 2.28 then assures us that if h is Borel-measurable and belongs to an integrable abstract space $L_1(\Omega, \mathscr{F}, \mathbb{P})$, we can find an equivalent probability space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), P_X)$ for which $h \in L_1(\mathbb{R}, \mathscr{B}(\mathbb{R}), P_X)$.

As a synthesis for the following proof, we will make use of the *standard machine* approach. In it, we will prove the theorem using indicator functions; then, by linearity, it is true for simple function; finally, we conclude using the monotone convergence theorem.

Proof. Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and let h be the indicator function for some set $B \in \mathscr{B}(\mathbb{R})$. Then, for every $\omega \in \Omega$, $\mathbb{1}_B(X(\omega)) = 1$ iff $\omega \in X^{-1}(B) := \{\omega \in \Omega | X(\omega) \in B\}$. This is because for $\mathbb{1}_B(X(\omega))$ to be 1, the value $X(\omega)$ in $\mathscr{B}(\mathbb{R})$, say x, is an element in B if and only if, the inverse image of X over B i.e. the set of elements ω that comprise B through a mapping from X, has ω as an element in it.

From this on,

$$\int_{\Omega} \mathbb{1}_B(X(\omega))\, d\mathbb{P}(\omega) = \int_{\Omega} \mathbb{1}_{X^{-1}(B)}(\omega) d\mathbb{P}(\omega) = \int_{X^{-1}(B)} d\mathbb{P}(\omega)$$

Then, by definition,

$$\int_{X^{-1}(B)} d\mathbb{P}(\omega) = \mathbb{P}\left[X^{-1}(B)\right] = P_X(B) \tag{2.40}$$

And it follows,

$$P_X(B) = \int_{\mathbb{R}} \mathbb{1}_B(x) dP_X = \int_{\mathbb{R}} h(x) dP_X(x)$$

Assume h is now a simple function, $h(\cdot) =: h_n(\cdot) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}(\cdot)$. Then,

$$\int_{\Omega} h_n(X(\omega)) d\mathbb{P} = \int_{\Omega} \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}(X(\omega)) d\mathbb{P}$$

by linearity,

$$\begin{aligned}
&= \sum_{i=1}^{n} \alpha_i \int_{\Omega} \mathbb{1}_{B_i}(X(\omega)) d\mathbb{P} \\
&= \sum_{i=1}^{n} \alpha_i \int_{\mathbb{R}} \mathbb{1}_{B_i}(x) P_X \\
&= \int_{\mathbb{R}} \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}(x) dP_X \\
&= \int_{\mathbb{R}} h_n(x) dP_x.
\end{aligned}$$

Finally, let h be a nonnegative Borel-Measurable function. Then, there exists a nondecreasing sequence of simple functions such that $h_n \to h$ as $n \to \infty$,

$$\int_{\Omega} h((X(\omega)) d\mathbb{P} = \int_{\Omega} \lim_{n\to\infty} h_n(X(\omega)) d\mathbb{P}.$$

By the monotone convergence theorem,

$$\begin{aligned}
&= \lim_{n\to\infty} \int_{\Omega} h_n(X(\omega)) d\mathbb{P} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} h_n(x) dP_x \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} h_n(x) dP_x \\
&= \int_{\mathbb{R}} h(x) dP_x.
\end{aligned}$$

$\square$

Now, recall that if X is any random variable with p.d.f. $f(x)$ then, for any $B \in \mathscr{B}(\mathbb{R})$,

$$P_X(B) = \int_{\mathbb{R}} f(x)dx.$$

The same argument as theorem 2.28 results in the following corollary,

Corollary 2.29. *Let X be a random variable with density function $f(x)$, and let h be a Borel-measurable function h such that. $\mathbb{E}[h(X)] < \infty$ then,*

$$\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(x)f(x)dx. \tag{2.41}$$

The Radon-Nikodym theorem (2.26) paves way for the following definition.

Definition 2.30 (**Conditional Expectation w.r.t. a σ-algebra**)**.** *The conditional expectation of a nonnegative random variable X with respect to the σ-algebra $\mathscr{G}$ is a nonnegative random variable denoted $\mathbb{E}[X|\mathscr{G}]$ or $\mathbb{E}[X|\mathscr{G}](\omega)$ such that,*

1. *$\mathbb{E}[X|\mathscr{G}]$ is $\mathscr{G}$-measurable; and*
2. *For every $A \in \mathscr{G}$,*

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}[X|\mathscr{G}]d\mathbb{P}$$

.

Definition 2.31 (**Conditional Probability**)**.** *Let $B \in \mathscr{F}$. The conditional expectation $\mathbb{E}[B|\mathscr{G}] := \mathbb{P}(B|\mathscr{G})$ is defined as the conditional probability of B w.r.t. the σ-algebra $\mathscr{G} \subseteq \mathscr{F}$*

Definition 2.32. *Let X be a random variable and consider the σ-algebra generated by some random variable Y. Then,*

$$\mathbb{E}[X|\sigma\{Y\}] =: \mathbb{E}[X|Y]. \tag{2.42}$$

For $B \in \mathscr{F}$,

$$\mathbb{P}[X|\sigma\{Y\}] =: \mathbb{P}[X|Y]. \tag{2.43}$$

We now note some important properties of the conditional expectation.

Proposition 2.33. 1. *If $a, b \in \mathbb{R}$, $\mathbb{E}[\alpha X + \beta Y|\mathscr{G}] = \alpha\mathbb{E}[X|\mathscr{G}] + \beta\mathbb{E}[Y|\mathscr{G}]$;*

2. *$\mathbb{E}[\mathbb{E}[X|\mathscr{F}]] = \mathbb{E}[X]$;*

3. *$\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_1]$ if $\mathscr{F}_1 \subseteq \mathscr{F}_2$;*

4. *$\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_2]$ if $\mathscr{F}_2 \subseteq \mathscr{F}_1$; and*

5. *Let X be a random variable such that $X \perp \mathscr{G}$ then, $\mathbb{E}[X|\mathscr{G}] = \mathbb{E}[X]$. Consequently, for any borel-measurable function h, $\mathbb{E}[h(X)|\mathscr{G}] = \mathbb{E}[h(X)]$.*

2.5 Stochastic Processes, Filtrations & Martingales

Hitherto the random variables for which we have been taking measures from constitute a single outcome of an experiment. In pursuance of a model that describes the dynamics of financial assets over time, we require a way to pinpoint both place in time, and distribution of a random variable. In other words, we would like to represent the evolution of a distribution over time in which random variables take values. With this idea, we can formulate a comprehensive mathematical definition for the movement of any random variable indexed by time.

Definition 2.34 (**Stochastic Process**). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A stochastic process is a set of random variables that takes values in some set S called the **state space**, and are indexed by some set T.*

$$\{X_t \mid t \in T\}. \tag{2.44}$$

For S to make sense, it must be measurable with respect to some σ-algebra. In particular, we will consider $S \subseteq \mathbb{R}$ where

$$S \cap \mathscr{B}(\mathbb{R}) \neq \emptyset.$$

Note that T can be any arbitrary set, nonetheless, for the sake of simplicity, we will only work with any of the following sets:

1. If $T = \{0, 1, \dots, \}$, we say that the process is a **discrete time process**. For notation purposes, we will denote a discrete time process as

$$\{X_n\}_{n\geq 0} := \{X_n \mid n \in \{0, 1, \dots, \}\}; \text{ and} \tag{2.45}$$

2. if $T = \{t \mid t \in \mathbb{R}^+\}$, the process is said to be a **continuous time process**. In this case, we will denote this process as the one indexed by the letter t,

$$\{X_t\}_{t\geq 0} := \{X_t \mid t \in \mathbb{R}^+\}. \tag{2.46}$$

With this in mind, we can consider a stochastic process as a two variable function

$$X : T \times \Omega \to S$$

Where,

1. For for fixed $t \in T$, $\omega \to X_t(\omega)$ is random variable; and
2. for fixed $\omega \in \Omega$, $t \to X_t(\omega)$ is the path of the process.

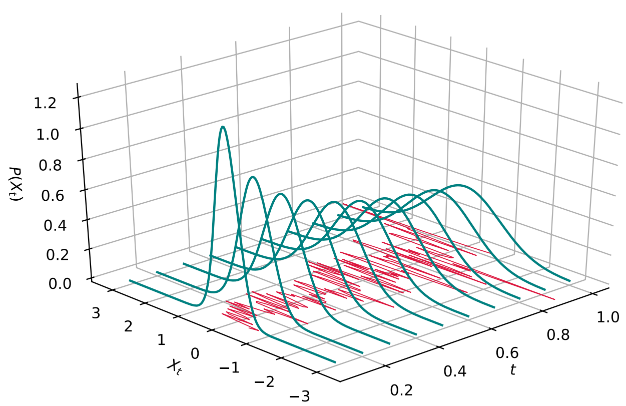

Figure 2.3: Distribution and Path of an Stochastic Process.

Consider figure 2.3, where we graph a single path of the stochastic process $\{X_t\}_{t\geq 0}$ in which $X_t \sim \mathcal{N}(0, t)$ (see eq. 2.6). Indeed, $\{X_t\}_{t\geq 0}$ can be seen as either a function indexed by T, in which case we graph the path of a single trajectory (red colored); or, for fixed $t \geq 0$, as a single random variable with probability density function $\phi(x|0, t)$.

With stochastic processes well defined, the next step is to measure the outcome of an event at a given $t \geq 0$. Unlike this last example, stochastic processes need not be $\mathscr{B}(\mathbb{R})$-measurable for every t. The values that any $\{X_t\}_{t\geq 0}$ can take at a certain t may vary. For this reason, it is sensible to equip a stochastic process, with σ-algebras of subsets of $\mathscr{F}$ that give us clue as to the subsets that are *reasonable* to measure at a given t.

Definition 2.35 (**Filtration**). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space. A filtration is a non-decreasing sequence of sub-σ-algebras of $\mathscr{F}$, $\{\mathscr{F}_n\}_{n\geq 0}$ such that*

$$\mathscr{F}_t \leq \mathscr{F}_s \ \forall\ 0 \leq t \leq s.$$

The stochastic process $\{X_t\}_{t\geq 0}$ is said to be **adapted** to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ if X_t is $\mathscr{F}_t$-measurable for every $t \geq 0$.

The minimal σ-algebra generated by $\{X_t\}_{t\geq 0}$ at time τ is known as the **natural filtration** of $\{X_t\}_{t\geq 0}$ at time τ. Note that $\{X_t\}_{t\geq 0}$ is always adapted to its natural filtration at time t. That is X_t is $\sigma(\{X_s | s \leq t\})$-measurable.

In a discrete time process, we denote the minimal σ-algebra generated by $\{X_n\}_{n\geq 0}$ up to time k as $\{X_1, \ldots, X_k\} := \sigma(\{X_1, \ldots, X_k\})$.

We conclude this chapter by defining a particular subset of the family of stochastic processes. We will limit ourselves to define it here and develop its intuition in the following chapter.

Definition 2.36 (**Martingale**). *An stochastic process $\{X_t\}_{t\geq 0}$ defined over $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a martingale w.r.t. a filtration $\{\mathscr{F}_n\}_{n\geq 0}$ if, for all $0 \geq s \leq t$,*

1. *X_t is $\mathscr{F}_t$-measurable;*
2. *X_t is in L_1 (i.e. is is measurable for all t); and*
3. *$\mathbb{E}[X_t | \mathscr{F}_s] = X_s$.*

Theorem 2.37 (**Doob's Martingale Inequality**). *If. $\{X_t\}_{t\geq 0}$ is a continuous-time martingale, then*

$$\mathbb{P}(\sup_{t\in[0,T]} |X_t| \geq \lambda) \leq \frac{1}{\lambda^p} \mathbb{E}\left[|X_t|^p\right]. \tag{2.47}$$

3 A Primer on Option Pricing

As we saw in the first chapter, a financial institution selling financial derivatives must take into account certain factors in order to avoid the occurrence of arbitrage against it. One must then inquire about the *correct* price to charge for any of these derivatives. Consider the valuation of the forward seen on chapter 1. One may price it as the expected value of the asset at maturity, discounted at some rate r. Simply put, assuming we can sell many units of a given product, by **Kolmogorov's Strong Law of Large Numbers (LLN)**, we would expect, in the long run, to break even:

Theorem 3.1. *Let $\{X_n\}_{n\geq 1}$ be a collection of i.i.d. random variables with mean μ. Denote $S_n = \frac{1}{n}\sum_{i=1}^{n} X_n$. Then, with probability one,*

$$S_n \to \mu \tag{3.1}$$

Feasible as it may seem, as pointed out by Baxter(2003), this approach could lead to disaster for the financial institution. Nonetheless, by following no-arbitrage arguments, it can be shown that the the price of a forward is the discounted value of the current price of the stock.

In this chapter, we aim to develop an intuition behind option valuation, and give a hint on what is to expect from a more mathematically-robust model. To achieve the latter we present the binomial tree model, and relax the mathematical-rigor used in the previous chapter.

We develop this chapter twofold: we first lay the foundations required to value an option using a single binomial tree; the second chapter comprises the generalization of the single binomial tree and an example of its usage.

3.1 One-Step Binomial Models

In order to generalize an arbitrage-free approach to price a European option on a stock, it is desirable to construct a model that truly reflects the market (unlike the LLN approach). In its simplest form, this market should consist of a cash bond and a stock. We will assume, that the market moves in discrete units of time.

The Stock

Between any two units of time (e.g. from $t = 0$ to $t = 1$) the stock can either go up with probability p, or down with probability $1 - p$; it will have an initial value S_0; and, with any of these movements, the stock goes up by a factor u, or a down by a factor d. Evidently, $0 < d < 1 < u$. Finally, we will assume that unlimited amounts of the stock can be bought at any time and there is no cost incurred in the transaction.

$$
\begin{tikzcd}
 & S_0u \\
S_0 \arrow[ur, "\mathrm{p}"] \arrow[dr, "1-p"'] & \\
 & S_0d
\end{tikzcd}
$$

Figure 3.1: Possible path of the stock after one unit of time.

The Bond

The bank account represents the time value of money. We will assume a constant, risk-free, continuously compounding interest rate r. For $\$B_0$ invested at $t = 0$, said rate guarantees, at the end of T periods, B_0e^{rT}. Also, we will assume that we can lend or borrow at the same interest rate r.

The Derivative

This simple market carries within the possibility of a third that depends on the price of the stock. That is, there could be a payoff with price f_0 today that derives its value from the possible directions the stock may take: either f_u if it goes up, or f_d otherwise

$$
\begin{tikzcd}
 & f_u \\
f_0 \arrow[ur, "\mathrm{S_0u}"] \arrow[dr, "S_0d"'] & \\
 & f_d
\end{tikzcd}
$$

3.1.1 Pricing

We can now ask whether we can construct a portfolio that replicates f from a suitable strategy, thus paying the promised amount. What we are looking for is to guarantee the value of the derivative at the moment of settlement, thus hedging the risk away.

Consider the portfolio (ϕ, ψ) with ϕ number of shares, and ψ monetary units in the bank. At $t = 0$, this portfolio is worth

$$\phi S_0 + \psi B_0$$

At $t = \delta_t$, this portfolio will be worth either $\phi S_0 u + \psi B_0 e^{r\delta_t}$, or $\phi S_0 d + \psi B_0 e^{r\delta_t}$.

With this in mind, we now turn to answer the initial question. In order to hedge the risk, we would like to find values for (ϕ, ψ) in order to replicate the payoff:

$$\begin{aligned} \phi S_0 u + \psi B_0 e^{r\delta_t} &= f_u \\ \phi S_0 d + \psi B_0 e^{r\delta_t} &= f_d \end{aligned}$$

We can now solve for ϕ and ψ:

$$\phi S_0 (u - d) = f_u - f_d$$

$$\phi = \frac{f_u - f_d}{S_0(u - d)} \tag{3.2}$$

$$\begin{aligned} \psi &= e^{-r\delta_t}[f_u - \phi u] \\ &= e^{-r\delta_t}\left[f_u - \frac{f_u - f_d}{S_0(u - d)} S_0 u\right] \\ &= e^{-r\delta_t}\frac{u f_d - d f_u}{u - d} \end{aligned}$$

Thus, buying the portfolio (ϕ, ψ) guarantees the payoff at $t = \delta_t$. Denote $\mathcal{V}$ the value of the portfolio at $t = 0$. The value of $\mathcal{V}$ is:

$$
\begin{aligned}
\mathcal{V} &= \phi S_0 + \psi \\
&= \left[\frac{f_u - f_d}{S_0(u-d)} S_0\right] + \left[e^{-r\delta_t}\right] \\
&= e^{-r\delta_t}\left[\frac{e^{r\delta_t}(f_u - f_d + uf_d - df_u)}{u-d}\right] \\
&= e^{-r\delta_t}\left[\frac{e^{r\delta_t} f_u - df_u}{u-d} + \frac{uf_d - e^{r\delta_t} f_d}{u-d}\right].
\end{aligned}
$$

Therefore,

$$
\mathcal{V} = e^{-r\delta_t}\left[f_u \frac{e^{r\delta_t} - d}{u-d} + f_d \frac{u - e^{r\delta_t}}{u-d}\right]. \tag{3.3}
$$

$\mathcal{V}$ is the cost of a risk-free strategy that guarantees the payoff, and assures the impossibility to commit arbitrage. Consider any other price $\mathcal{P} < \mathcal{V}$. Buying the portfolio $\mathcal{P}$ and selling $\mathcal{V}$ at $t = 0$ guarantees, at $t = \delta_t$, enough money to settle and earn $\mathcal{V} - \mathcal{P}$ risk-free. The same argument can be done for a value $\mathcal{P} > \mathcal{V}$.

What, then, would have happened, had we decided to replicate this payoff using only the bank account and the probabilities p and $1 - p$? Accordingly, we would've priced the derivative at the present value of the expected payoff under the $\mathbb{P}$ measure. Denote $\hat{\mathcal{V}}$ this alternate value, then:

$$
\hat{\mathcal{V}} = \exp(-r\delta_t)\mathbb{E}[f] = \exp(-r\delta_t)[f_u \cdot p + f_d \cdot (1-p)] \tag{3.4}
$$

Pricing $\hat{\mathcal{V}}$ for f does not guarantee $\hat{\mathcal{V}} = \mathcal{V}$, which could lead to arbitrage otherwise. Assuming $\hat{\mathcal{V}} \neq \mathcal{V}$, the expected payoff of f would not be $\hat{\mathcal{V}}$, market participants would be inclined to make a profit, thereby breaking the assumption that every position taken on the derivative is independent of one another; we cannot expect to break even in the long run.

3.1.2 The Q Measure

Consider (3.3) with payoff $S_T - K$.

$$
\mathcal{V} = e^{-r\delta_t}\left[S_0 u \frac{e^{r\delta_t} - d}{u-d} + S_0 d \frac{u - e^{r\delta_t}}{u-d}\right]
$$

Denote

$$
q := \frac{\exp r\delta_t - d}{u-d} \Longrightarrow 1 - q = \frac{u - \exp(r\delta_t)}{u-d} \tag{3.5}
$$

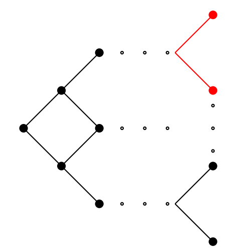

Figure 3.2: The n-period model

We can argue that $d < \exp(r\delta_t) < u$ implies no arbitrage. To see why,

1. Suppose $\exp(r\delta_t) < d < u$.
   At t=0 we can borrow $\$S_0$ and buy the stock. At $t = \delta_t$, we collect either S_0u or S_0d, thereby making a risk-free profit of $S_0(u - \exp(r\delta_t))$ or $S_0(d - \exp(r\delta_t))$

2. Suppose $d < u < \exp(r\delta_t)$
   At $t = 0$, lend $\$S_0$ and short the stock. At $t = \delta_t$, collect $S_0\exp(r\delta_t)$ and buy the stock at either S_0u or S_0d. Again, this guarantees a risk-free profit of either $S_0(\exp(r\delta_t) - u)$ or $S_0(\exp(r\delta_t) - d)$

Being $d < \exp(r\delta_t) < u$ implies $0 < q < 1$. We can rewrite 3.3 as

$$\mathcal{V} = \exp(-r\delta_t)[S_0u \cdot q + S_0d \cdot (1 - q)] \tag{3.6}$$

$\mathcal{V}$ is the discounted expectation under a probability measure $\mathbb{Q}$; it is not the expected value of the derivative, but a change in the measure of the tree such that guarantees no arbitrage. Theoretically, this is backed up by the Radon-Nikodym derivative (see theorem 2.26).

3.2 Generalized Binomial Tree Model

In order to generalize our model to n periods we only need to consider each node as the root of another binomial tree. Consequently, we get a tree with n periods and 2^n nodes at $t = n$.

To motivate this generalized model, we will now present an example of a 3-period binomial model.

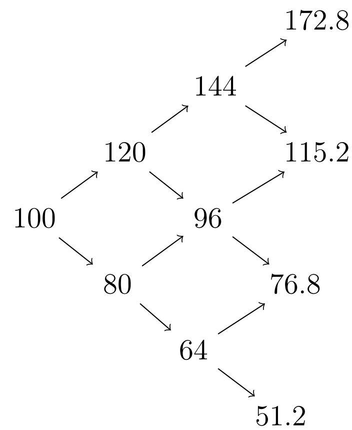

Figure 3.3: The Stock Process

3.2.1 An Example

Suppose that at $t = 0$ we have a stock worth $S_0 = 100$ that can go up by a factor of 1.20, or down by a factor of 0.80. Assume further that the $\mathbb{P}$ measure probability for an up move is $\frac{3}{4}$, finally, suppose $r = 0$. Being consistent with the notation used, $u = 1.20$, $d = 0.80$, $p = \frac{3}{4}$ and $1 - p = \frac{1}{4}$.

We set up to price a call option on S_0 with strike price $K = 110$, three periods from now. Denote f_t the value of the derivative at time t. The payoff at maturity ($t = 3$) is

$$f_3 = \max\{S_3 - K, 0\} =: [S_3 - K]^+ \tag{3.7}$$

Since S can either move up or down with known factors u and d respectively, we know all the values the stock can take up to $t = 3$. For example, from $t = 0$ to $t = 1$, the stock can either move to $100 \cdot 1.20$, or down to $100 \cdot 0.80$.

Given f and the stock-tree process we can compute, for example, the amount to be paid if the stock moves up at every step. Under this scenario, $S_3 = 172.8$. Then $f(S_3) = [172.8 - 110]^+ = 62.80$, which means that the buyer of the option would receive \$62.80. Computing the payoff at every last node yields a set of payoffs at maturity.

With the set of payoffs at maturity, we proceed to find the value of the

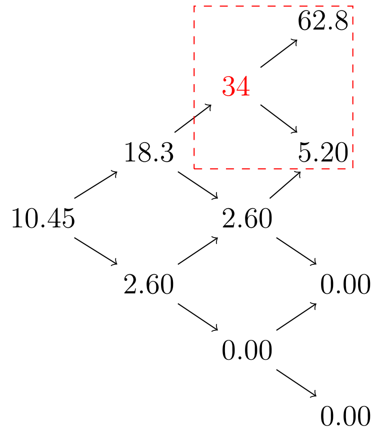

Figure 3.4: The Claim Tree

option at every other node in the tree solving iteratively from $t = 2$ to $t = 0$.

For example, the upper node at $t = 2$ is computed by taking the discounted expectation under the $\mathbb{Q}$ measure as shown in 3.6. Were the stock to climb up two times then, the payoff at $t = 3$ can either be 62.8 or 5.2. Denote f_2^{uu} the value the derivative at $t = 2$ for the upper node, its value is then

$$f_2^{uu} = \exp(-0)[62.8q + 5.2(1-q)] = 34; \; q = \frac{0 - 0.80}{1.20 - 0.80}$$

Loosely speaking, every node in the claim-tree represents the value of the option at every step. Evidently, at $t = 3$, the value of the option can only be its current value since there is nowhere to move elsewhere in the tree. At $t = 0$ we have the risk-free value for the option that guarantees no arbitrage.

To see why this is true, suppose we were to sell this option at \$10.45. For the sake of the argument, suppose the stock moves up two times and down at the last time-tick. If 10.45 is truly the arbitrage-free price option, we should be able to replicate the movement of the stock at each time interval and arrive at the payoff without incurring any other cost.

According to equation 3.2, at $t = 0$ we would need to buy 0.3925 units of S and borrow 28.8 to cover the expenses; if the price goes up at $t = 1$, around 0.65 units of the stock are required, which would lead to a purchase of 0.26 extra units of the stock, thus incrementing the amount borrowed to 60.2. At $t = 2$, 1 unit of S is required, increasing the amount borrowed to 110. Finally, at $t = 3$, the stock takes a down move and \$5.2 is the amount owed to the option buyer.

The final value in the position of the stock is $S_0 u^2 d \cdot \phi = 115.20$, minus the \$110 owed, brings the total value of the portfolio to \$5.20, exacltly the amount required to fulfill the obligation.

4 L_2 Spaces and the Brownian Motion

In chapter one we posed the question about the theoretical value of a financial call option that starts today. Specifically, we want to answer the following question: what is today's value for a derivative that pays $\max\{S_T - K, 0\}$ at maturity? Note that, at time T, the payoff of the option is a function of S_T, since all else is given. Consequently, preceding the valuation of any option we must first take into account the dynamics of the asset we are referring to, i.e., we must come up with a model for S_t for all $t \in [0, \infty)$.

As saw in chapter 3, the basic assumption about the financial asset we want to value is that is is always positive, it starts at a known value and has a component that allow us to determine the possible paths it may take.

In the following chapter, we aim to attain a model that explains the random changes of financial assets in continuous time. Since we assume that the stock follows both a stochastic component and a deterministic one, we intent to describe the "noise" of an asset over time in continuous time.

We begin by motivating the need for such process in a mathematically-informal way; we continue laying the mathematical foundations for what is the final process to model the *noise*: the Brownian motion; finally, we conclude the chapter by proving remarkable properties about the latter.

4.1 Random Walks

Suppose that two players (A and B) toss an even coin n times, one toss for every unit of time[1]. Whenever tails comes up, A will pay B \$1. On

[1] Note that this unit of time can be one second, one day, 5 minutes, etc.

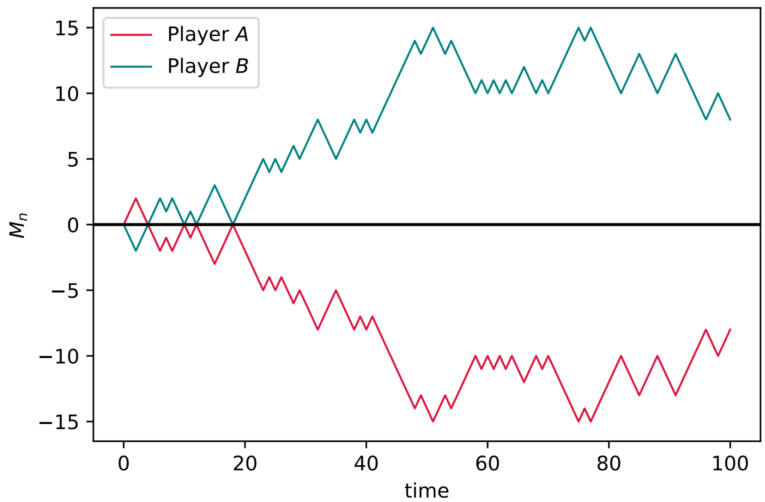

Figure 4.1: Path of a Symmetric Random Walk

the contrary, A will receive \$1 from B. Suppose further that we decide to model how much did A won (or lost) after n tosses. To do so, we define the following:

Definition 4.1. *Let $\{X_j\}_{j\geq 0}$ be a set of independent and identically distributed (i.i.d.) random variables for which $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$; $i \geq 1$. Define $M_0 = 0$ and*

$$M_n := \sum_{k=1}^{n} X_k \;\forall\; n \in \mathbb{N}. \tag{4.1}$$

The process $\{M_d\}_{d\geq 0}$ is known as a ***symmetric random walk***

Proposition 4.2. *The expected value of a symmetric random walk is 0.*

Proof. Let $n \in \mathbb{N}$ then,

$$\begin{aligned}\mathbb{E}(M_n) &= \mathbb{E}\left[X_0 + \sum_{i=1}^{d} X_i\right] \\ &= 0 + n\left(1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}\right).\end{aligned}$$

□

Proposition 4.2 shows that the definition of a symmetric random walk (4.1) captures what we intended to model. From the outset, both players start with with \$0 and, as the game progresses, whichever amount A wins, B looses it. The type of games, in which the amount player A wins, player B losses it are called **zero-sum games**.

After a large number of times played we expect to break even. Nobody would win or loose once played a large number of times[2]

A symmetric random walk also happens to be known as a fair-game. This is because what one would expect to win at any given future point in time is dependent on the amount that one has today. In other words, if both of the players start with a P&L of \$0, then they would not expect, after a large amount of games[3] to earn any more money than what they started with: \$0.

To see why, consider the following proposition

Proposition 4.3. *The simple random walk is a martingale for $k < \ell$*

Proof.

$$\begin{aligned}
\mathbb{E}[M_\ell|\mathscr{F}_k] &= \mathbb{E}[(M_\ell - M_k) + M_k|\mathscr{F}_k] \\
&= \mathbb{E}[(M_\ell - M_k)|\mathscr{F}_k] + \mathbb{E}[M_k|\mathscr{F}_k] \\
&= \mathbb{E}[M_\ell - M_k] + M_k \\
&= M_k
\end{aligned}$$

□

No matter the point in time in the future that we refer to, the value that one would expect to have in the future the one that we have today. As a consequence, the only *useful* information we have today about the future value of the game, is the one we have today.

4.2 Scaled Random Walk

The simple random walk can altered such that between any two units of time the process takes n steps.

[2]In fact, according to Law of Large Number, we would expect to break even after a number so large that it approaches to infinity.

[3]Mathematically speaking, the expectation of the game is zero in the sense that, as the number of games tends to infinity, the average P&L for any of the players turn to zero

Definition 4.4. *We define the* ***scaled random walk*** $W^n(t)$ *as the following transformation of the simple random walk:*

$$W^n(t) := \frac{1}{\sqrt{n}} M_{nt}$$

Remark. *The scaled random walk has the following properties:*

- *The increments are independent:*

$$(W^n(t_{i+1}) - W^n(t_i)) \perp (W^n(t_{j+1}) - W^n(t_j)) \;\; \forall \; i \neq j \;\; i, j \geq 1$$

- *For* $s < t$, *and* ns, $nt \in \mathbb{W}$,
  - $\mathbb{E}[W^n(t_i) - W^n(t_j)] = 0$*; and*
  - $\mathbb{V}[W^n(t_i) - W^n(t_j)] = t - s$.
- *The Scaled random walk is a martingale: for all* $k < \ell$

$$\mathbb{E}[W^n(\ell)|\mathscr{F}_k] = W^n(k)$$

.

One complication arises in modeling a financial asset, that of choosing the partition between any two units of time. Consider fig.4.2, it shows sample paths for different partitions of time. As the number of partition increases, the number of possible path increases too. Conversely, if there is only one partition, the process will reach either 1 or -1. With this in mind, the question of whether there exists a *correct* partition for a particular asset may seem partial and confined.

To further develop this theory we will start assuming that the market moves in continuous time. This, as we will later see, allow us to use more sophisticated methods to work with the process while generalizing this to any market we desire. This assumptions leads to an important theorem that will help us keep building the model for the financial asset.

Theorem 4.5. *Let* $t \geq 0$. *As* $n \to \infty$, *the distribution of the scaled random walk* $W^n(t)$ *converges to a normal distribution with mean* 0 *and variance* t.

Proof. We will show that the moment generating function for a symmetric random walk tends to the moment generation function of a normal with $\mu = 0$ and $\sigma^2 = t$.

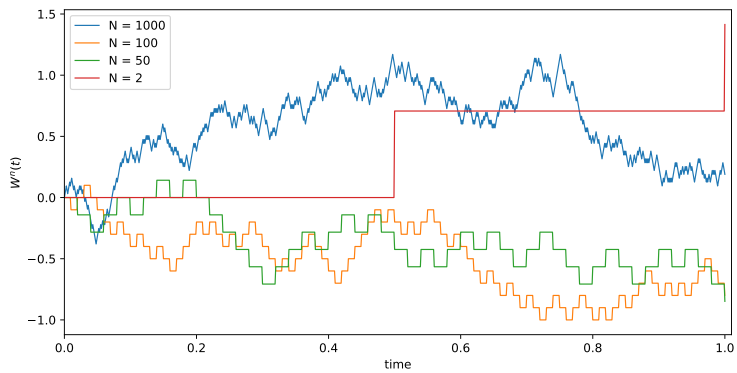

Figure 4.2: Scaled Random Walk

Denote $\varphi_n(u)$ and $\varphi(u)$ the moment generating functions of the symmetric random walk and the normal distribution respectively. We will then show that $\varphi_n(u) \to \varphi(u)$. Where $\varphi_n(u) = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)^{nt}$ and $\varphi(u) = e^{\frac{1}{2}u^2 t}$ (Refer to appendix A.1 and A.2 for the derivation of both moment generating functions).

Since,

$$\varphi_n(u) = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)^{nt}.$$

Then,

$$\ln \varphi_n(u) = nt \cdot \ln\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right). \tag{4.2}$$

Let $x = \frac{1}{\sqrt{n}}$, and take the limit as n tends to infinity,

$$\begin{aligned}\lim_{n\to\infty} \ln \varphi_n(u) &= \ln \lim_{n\to\infty} \varphi_n(u) && (\varphi_n \text{ is cont. at } n > 0)\\ &= \lim_{x\to 0} \frac{t \ln\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}\end{aligned}$$

Applying l'Hôpital's rule

$$= \frac{t}{2} \lim_{x\to 0} \frac{\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}}{x}$$

Applying l'Hôpital's rule again

$$= \frac{t}{2} \lim_{x \to 0} \left(\frac{u^2}{2} e^{ux} + \frac{u^2}{2} e^{-ux} \right).$$

We conclude, $\lim_{n\to\infty} \varphi_n(u) = e^{\frac{1}{2}u^2 t}$. □

Theorem 4.5 shows the distribution that follows at every time t if we assume that the market moves continuously. The process that emerges from this assumption is paramount in what follows. In fact, this process has a name of its own, which we will denote the **Brownian motion**.

In what follows of this chapter, we formalize the **Brownian motion**

4.3 L_2 Spaces

Definition 4.6. *let V a linear space, we define a **norm** over v as the function $\|\cdot\| : v \in V \to \mathbb{R}$ such that, for every $v \in V$, $\alpha \in \mathbb{R}$*

1. $\|v\| \geq 0$;
2. $\|v\| = 0 \iff v = 0$;
3. $\|\alpha x\| = |\alpha| \, \|x\|$; *and*
4. $\|v + w\| = \|v\| + \|w\|$.

We say that $(V, \|\cdot\|)$ is a complete normed spaced (or a Banach space) if, for $\{v_n\}_{n\geq 0}$, and $\{v_m\}_{m\geq 0}$; $\|v_n - v_m\| \to 0$ as $n, m \to \infty$. The sequence $\{\|v_m, v_n\|\}_{n,m\geq 0}$ is known as a **Cauchy sequence**. This last fact imposes the existence of $v \in V$ such that $\lim_{n\to\infty} v_n = v$.

Definition 4.7. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a measurable space, we define the L_2 space $L_2(\Omega, \mathscr{F}, \mathbb{P})$ as the family of random variables such that*

$$\|X\|_2 := \left(\mathbb{E}[|X|^2]\right)^{1/2} < \infty. \tag{4.3}$$

It can be shown that if X, Y are two random variables such that $\mathbb{P}(X = Y) = 1$, then the linear normed space $(L_2, \|\cdot\|_2)$ is a Banach space.

Definition 4.8. *An **inner product** over a linear space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that, for every x,y,$z \in V$ and α,$\beta \in \mathbb{R}$,*

1. $\langle x, y \rangle = \langle y, x \rangle$;

2. $\langle x, x\rangle \geq 0$;

3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$; *and*

4. $\langle x, x\rangle = 0 \iff x = 0_n$.

Definition 4.9. *Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ two normed vector spaces with inner product $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ respectively. It is said that a linear function $I : V_1 \to V_2$ is an* ***isometry*** *if, for every, $v \in V_1$,*

$$\|I(v)\|_2 = \|v\|_1 . \tag{4.4}$$

Remark. *From definition 4.9 it follows that*

$$\langle I(v), I(u)\rangle_2 = \langle v, u\rangle_1 \ \forall\ v, u \in V_1. \tag{4.5}$$

Definition 4.10 (Dense Space)**.** *Let $(V, \|\cdot\|)$ a Banach space, we say that $U \subseteq V$ is dense on V if for every $v \in V$, then there exists $\{u_k\}_{k\geq 1} \subseteq U$ such that*

$$\lim_{k\to\infty} \|v - u_k\| = 0 \tag{4.6}$$

Proposition 4.11. *Let $(V, \|\cdot\|_V)$ a complete normed space and $(W, \|\cdot\|_W)$ a normed space. Let $U \subseteq V$ dense, and $I : U \to W$ and isometry. Then,*

1. *$I(\cdot)$ can be extended to all V ($I : V \to W$); and*

2. *$I(\cdot)$ is also an isometry for V ($\|I(v)\|_W = \|I(v)\|_V$).*

Proposition 4.12. *Let $\langle\cdot,\cdot\rangle$ an inner product defined over V and define $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ then,*

1. *$|\langle x, y\rangle| \leq \|x\| \|y\|$;*

2. *$\|x\|$ defines a norm over V; and*

3. *if $x_n \to x$ and $y_m \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Remark. *If the norm over V as defined in 4.12 is complete, then $(V, \langle\cdot,\cdot\rangle)$ is said to be a* ***Hilbert space****.*

For any two random variables X,$Y \in L_2(\Omega, \mathscr{F}, \mathbb{P})$, we will define their inner product

$$\langle X, Y\rangle := \mathbb{E}[XY] \tag{4.7}$$

As a consequence,

$$\|X\|_2 = \sqrt{\mathbb{E}[X^2]}; \text{ and} \tag{4.8}$$

$$\mathbb{E}[XY] \leq \|X\|_2 \|Y\|_2 = \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}. \tag{4.9}$$

Proposition 4.13. *Let $X_n \to X$ and $Y_n \to Y$ in L_2 then*

$$E[X_n Y_m] \to E[XY] \tag{4.10}$$

Definition 4.14 (L_2 **convergence**)**.** *Let $\{X_n\}_{n\geq 0}, X \in L_2$. We say that $X_n \to X$ in L_2 iff $\|X_n - X\|_2 \to 0$, i.e., $\mathbb{E}[(X_n - X)^2] \to 0$.*

Proposition 4.15 (**criteria for the existence of a limit in** L_2)**.** *Let $\{X_t\}_{t\geq 0}$ a stochastic process in L_2 ($\|X_t\| < \infty$ for every $t \in T$) then,*

1. *There exists a random variable $X \in L_2$ such that $X_t \to X$ in L_2 whenever $t \to t_0$; and*
2. *There exists $\ell \in \mathbb{R}$ such that $\mathbb{E}[X_{t_n} X_{t'_m}] \to \ell$ as n,$m \to \infty$ for any two sequences $\{t_n\}$ and $\{t'_m\}$ where $t_n \to t_0$ and $t'_m \to t_0$*

Definition 4.16 (L_2 **continuity**)**.** *We say that a stochastic $\{X_t\}_{t\geq 0}$ process is L_2 continuous if*

$$X_{t+h} \to X_t \text{ as } h \to 0 \ \forall \ t \in T \tag{4.11}$$

Remark. *By Jensen's inequality,*

$$\mathbb{E}[(X_{t+h} - X_t)^2] \geq (\mathbb{E}[X_{t+h} - X_t])^2 = (\mathbb{E}[X_{t+h}] - \mathbb{E}[X_t])^2 \tag{4.12}$$

Note that if X_t is L_2 continuous, $\mathbb{E}[(\cdot)^2] < \infty$ and is bounded from above. If follows then that $\mathbb{E}[X_t]$ is L_2 continuous since the latter implies that $\mathbb{E}[\cdot]$ is well defined.

Theorem 4.17. *Suppose $\mathbb{E}[X_t]$ is L_2 continuous, then X_t is L_2 continuous iff $Cov(X_k, X_s) := \mathbb{E}[(X_k - \mathbb{E}[X_k])(X_s - \mathbb{E}[X_s])]$ is continuous at (t,t).*

Proof. First we note that X is L_2 continuous iff $X' := \{X - \mathbb{E}[X]\}$ is L_2 continuous.

Also, X'_t has $\mathbb{E}[X'_t] = 0$ and

$$\begin{aligned} Cov(X'_t, X'_t) &= \mathbb{E}[(X'_t)^2] \\ &= \mathbb{E}[(X'_t)^2] \\ &= \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] \\ &= Cov(X_t, X_t) \end{aligned}$$

Implying that

$$\mathbb{E}[X_t] = 0 \tag{4.13}$$

($\Rightarrow$ Since X_t is L_2 continuous, there exists h, h' such that $X_{t+h} \to X_t$ and $X_{t+h'} \to X_t$. It follows by proposition 4.13 and (4.13)

$$\begin{aligned} Cov(X_{t+h}, X_{t+h'}) &= \mathbb{E}[(X_{t+h} - \mathbb{E}[X_{t+h}])(X_{t+h'} - \mathbb{E}[X_{t+h'}])] \\ &= \mathbb{E}[X_{t+h}X_{t+h'}] \\ &\to \mathbb{E}[(X_t)^2] \text{ as } h \to 0. \end{aligned}$$

$\Leftarrow$) Suppose $Cov(X_t, X_t)$ is continuous,

$$\begin{aligned} \mathbb{E}[(X_{t+h} - X_t)^2] &= \mathbb{E}[X_{t+h}^2] + \mathbb{E}[X_t^2] - 2\mathbb{E}[X_t X_{t+h}] \\ &\to 2\mathbb{E}[X_t^2] - 2\mathbb{E}[X_t^2] \\ &= 0. \end{aligned}$$

Implying the convergence of X_t in L_2. □

Definition 4.18 (L_2 differentiability). *Let $\{X_t\}_{t\geq 0}$ a stochastic process defined on an L_2. We say that X is L_2 differentiability at some point $t \in T$ if there exists another random variable X' such that.*

$$\frac{X_{t+h} - X_t}{h} \to X_t' \text{ as } h \to 0 \tag{4.14}$$

4.4 Brownian Motion

Definition 4.19. *A* ***Brownian Motion*** *with parameter σ^2 is a stochastic process $\{W_t \in \mathbb{R} : t \geq 0\}$ that satisfies the following properties:*

- $W_0 = 0$
- *The trajectories are continuous*
- *The process has independent increments*
- $\forall\ 0 \leq s \leq t \Longrightarrow (W_t - W_s) \sim N(0, \sigma^2(s-t))$

Remark. *A Brownian Motion with $\sigma^2 = 1$ is known as a Standard Brownian Motion.*

The Brownian Motion is a first desirable result since it allow us to work in continuous time. Furthermore, it allow us to measure a probability at every $t \geq 0$ with tools already known for continuous random variables.

Although it is computationally impossible to properly simulate a Brownian Motion, it can be approximated via the scaled random walk for big n.

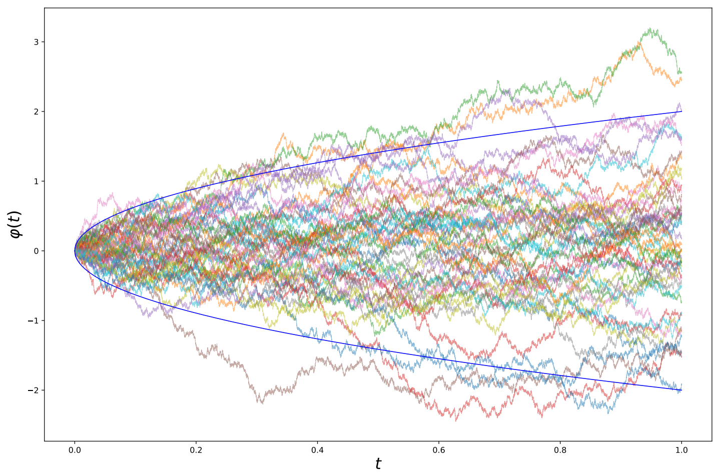

Figure 4.3: Sample Paths of a Brownian Motion

Figure 4.4 shows this idea approximating a brownian motion with $n = 9000$. In pursuance of pricing a European call option, we will see that it is possible to arrive at a closed form solution for these types of option. Nevertheless, more complex products may or may not have a closed form solution

Proposition 4.20 (Covariance of brownian motion)**.** *Let W_t a one dimensional brownian motion process, then*

$$Cov(W_t, W_s) = \min\{t, s\}$$

Proof. Let W_t a brownian motion process, and

$$Cov(W_t, W_s) = \mathbb{E}[(W_s - \mathbb{E}[W_s])(W_t - \mathbb{E}[W_t])] \tag{4.15}$$

Consider, by definition of the brownian motion, that $\mathbb{E}[W_s] = \mathbb{E}[W_s - W_0] = 0$, and let $t > s$. It follows that

$$\begin{aligned}
Cov(W_t, W_s) &= \mathbb{E}[W_s W_t] \\
&= \mathbb{E}[W_s W_t - W_s W_s + W_s W_s] \\
&= \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= \mathbb{E}[(W_s - W_0)]\mathbb{E}[(W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= t.
\end{aligned}$$

If $t < s$, it follows that $Cov(W_t, W_s) = s$. $\square$

Remark. *Since $\mathbb{E}[W_t] = 0$ and $Cov(W_t, W_s) \geq 0$ for every $t, s \geq 0$ A consequence of proposition 4.20*

Proposition 4.21. *Let W_t a one dimensional Brownian motion process. Then W_t is nowhere L_2 differentiable.*

Proof. Let W_t a one dimensional brownian motion process, and consider $W_{t+h} - W_t \sim N(0, h\sigma^2)$. Then,

$$\begin{aligned} \mathbb{E}[\frac{1}{h}(W_{t+h} - W_t)^2] &= \frac{1}{h^2}\mathbb{E}[(W_{t+h} - W_t)^2] \\ &= \frac{1}{h^2}\mathbb{V}[(W_{t+h} - W_t)^2] \\ &= \frac{1}{h^2}(h\sigma^2) \end{aligned} \tag{4.16}$$

Evidently, as $h \to 0$, (4.16) tends to infinity.

We now suppose the Brownian motion is defined, i.e., there exists W_t' such that

$$\lim_{h\to 0} \mathbb{E}\left[\frac{1}{h}(W_{t+h} - W_t) - W_t'\right]^2 = 0. \tag{4.17}$$

Then

$$\lim_{h\to 0} \mathbb{E}\left[\left(\frac{1}{h}(W_{t+h} - W_t\right)^2 - 2\left(\frac{1}{h}(W_{t+h} - W_t\right)W_t' + \left(W_t'\right)^2\right] = 0 \tag{4.18}$$

Which implies,

$$\mathbb{E}\left[W_t'\right]^2 = \lim_{h\to 0}\left(\mathbb{E}\left[\left(\frac{1}{h}(W_{t+h} - W_t\right)^2\right] - 2\mathbb{E}\left[\left(\frac{1}{h}(W_{t+h} - W_t\right)W_t'\right]\right)$$

Considering equation (4.16), we conclude that

$$\mathbb{E}\left[W_t'\right]^2 = \lim_{h\to 0}\mathbb{E}\left[\left(\frac{1}{h}(W_{t+h} - W_t\right)^2\right] = \infty$$

Therefore, the brownian motion is L_2 nowhere differentiable. $\square$

To further develop the properties of the Brownian Motion, let us define the following:

Definition 4.22. *Let f be a real-valued function defined over $[0, T]$. Consider a partition $\Pi = \{t_i\}_{i=0}^{n}$ over $[0, T]$ where $t_0 = 0$, $t_n = T$, and $||\Pi|| = \max_j\{t_{j+1} - t_j\}$. We define the **Quadratic Variation** of f up to time T as*

$$[f, f]_T := \lim_{||\Pi|| \to 0} \sum_{t=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

The quadratic variation for a function f represents the amount of total quadratic oscillation inside $D = [0, T]$. It can be shown (see Shreve (2004)) that for $f \in C^1(D)$, $[f, f]_T = 0$.

Asking how much does $W(t)$ oscillates between D posses the question of whether $W(t) \in C^1(D)$. The answer is no, since the Brownian Motion is a continuous, nowhere-differentiable function. A consequence of this last remark is the following.

Theorem 4.23. *Let $W(t)$ be a Standard Brownian Motion, then $[W, W]_T = T \;\forall\; T \geq 0$ almost surely.*

Proof. Let $\Pi = \{t_j\}_{j=0}^{n}$ be a partition of $[0, T]$. Define the sample quadratic variation corresponding to this partition to be

$$Q_\Pi = \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2$$

To show that $[W, W]_T = T$, is equivalent to show that, as $n \to \infty$, $\mathbb{E}[Q_\Pi] = T$ and $\mathbb{V}(Q_\Pi) = 0$ almost surely.

First note that, by definition of the Standard Brownian Motion,

$$\mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2] = \mathbb{V}[W_{t_{j+1}} - W_{t_j}] = t_{j+1} - t_j \tag{4.19}$$

$$\mathbb{V}([W_{t_{j+1}} - W_{t_j}]^2) = 2(t_{j+1} - t_j)^2 \tag{4.20}$$

Refer to appendix A.3 for a derivation of eq. 4.20.

Consider now

$$\begin{aligned}
\mathbb{E}[Q_\Pi] &= \mathbb{E}[\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j)}^2] \\
&= \sum_{j=0}^{n-1}\mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2] \\
&= \sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= T
\end{aligned}$$

Now,

$$\mathbb{V}(Q_T) = \mathbb{V}(\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2)$$

By independence,

$$\begin{aligned}
&= \sum_{j=0}^{n-1}\mathbb{V}((W_{t_{j+1}} - W_{t_j})^2) \\
&= \sum_{j=0}^{n-1}2(t_{j+1} - t_j)^2 \\
&\leq 2\sum_{j=0}^{n-1}||\Pi|| \cdot (t_{j+1} - t_j)
\end{aligned} \tag{4.21}$$

Finally, note that

$$\lim_{||\Pi||\to 0} 2\sum_{j=0}^{n-1}||\Pi|| \cdot (t_{j+1} - t_j) = \lim_{||\Pi||\to 0} 2||\Pi||T = 0 \tag{4.22}$$

Therefore, $\lim_{||\Pi||\to 0} Q_\Pi = T$ almost surely. □

We state theorem 4.23 informally by writing

$$dW^2 = dt \tag{4.23}$$

Two final consequences of 4.23 are the following

$$\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) = 0 \tag{4.24}$$

$$\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 = 0 \tag{4.25}$$

Equations 4.24 and 4.25 are informally expressed as $dW_t dt = 0$ and $dt^2 = 0$

5 Stochastic Calculus

5.1 Itô's Integral

We want to make sense of

$$\int_a^b f(t,\omega) dW_t(\omega) \tag{5.1}$$

Where W_t is a standard Brownian motion and f belongs to a special family of functions denoted by $\mathcal{N}[a,b]$

Definition 5.1. *Let $0 \leq a \leq b$. Denote $\mathcal{N}[a,b]$ the family of stochastic process such that*

1. $(t,\omega) \to f(t,\omega) : [0,\infty] \times \Omega \to \mathbb{R}$ *is* $\mathscr{B}(\mathbb{R}) \times \mathscr{F}$ *measurable.*
2. $f(t,\omega)$ *is* $\mathscr{F}_t$*-adapted*
3. $f \in L_2([a,b] \times \Omega)$ *i.e.* $\int_a^b \int_\Omega f^2 d\mathbb{P} dt = \int_a^b \mathbb{E}\left[f^2\right] dt = \mathbb{E}\left[\int_a^b f^2 dt\right] < \infty$

Definition 5.2 (Simple Process). *We define the process $f \in \mathcal{N}[a,b]$ as simple if there exists a partition $\{t_i\}_{i=0}^n$ where $t_0 = a$, $t_n = b$, and $t_i < t_j \ \forall \ i < j$, such that f can be represented as a lineal combination of the form*

$$f(t) = \sum_{i=0}^{n-1} f(t_i) \mathbb{1}_{[t_i, t_{i+1}]}(t) \tag{5.2}$$

Definition 5.3. *We define the family of processes f in $\mathcal{M}[a,b]$ that are processes in $\mathcal{N}[a,b]$ and can be written as simple processes.*

Definition 5.4. *Let $f \in \mathcal{M}[a,b]$, we define the Itô integral of f as*

$$\int_a^b f(t) dW_t := \sum_{i=0}^{n-1} \Delta W_i. \tag{5.3}$$

With $\Delta W_i := W_{t_{i+1}} - W_{t_i}$.
We will also denote Itô's integral as $I(f)$ *ot* $\int f$

Proposition 5.5. *Let* $f,\ g \in \mathcal{M}[0,t]$ *then*

1. $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$
2. $\mathbb{E}[I(f)] = 0$
3. $\mathbb{V}[I(f)] = \mathbb{E}\left[\left(\int_0^t f(s)dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t f^2(t)dt\right]$ *(Itô Isometry)*

Proof. 1). Let $f, g \in \mathcal{M}[0,t]$, and $\alpha, \beta \in \mathbb{R}$. Then, we can write f and g as a simple process. Namely,

$$f(t) = \sum_{i=0}^{n-1} f(t_i)\mathbb{1}_i(t) \tag{5.4}$$

$$g(t) = \sum_{j=0}^{m-1} g(t_j)\mathbb{1}_j(t) \tag{5.5}$$

Where $\mathbb{1}_i(t) := \mathbb{1}_{t_i, t_{i+1}}(t)$. We now consider the following

$$\begin{aligned}
\int_0^t (\alpha f(s) + \beta g(s))dW_s &= \int_0^t \left(\alpha \sum_{i=0}^{n-1} f(t_i)\mathbb{1}_i(t) + \beta \sum_{j=0}^{m-1} g(t_j)\mathbb{1}_j(t)\right) dW_s \\
&= \sum_{k=0}^{n-1} \left(\alpha \sum_{i=0}^{n-1} f(t_i)\mathbb{1}_i(t) + \beta \sum_{j=0}^{m-1} g(t_j)\mathbb{1}_j(t)\right) \Delta W_j \\
&= \alpha \sum_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} f(t_i)\mathbb{1}_i(t)\right) \Delta W_j + \\
&\quad \beta \sum_{k=0}^{n-1} \left(\sum_{j=0}^{m-1} g(t_j)\mathbb{1}_j(t)\right) \Delta W_j \\
&= \alpha \int_0^t f(s)dW_s + \beta \int_0^t g(s)dW_s
\end{aligned}$$

2) For this next proof, we first note that ΔW_i is independent of its σ-algebra. We can then write

$$\mathbb{E}[\Delta W_i | \mathscr{F}_{t_i}] = \mathbb{E}[\Delta W_i] = 0 \tag{5.6}$$

$$\mathbb{E}[\Delta W_i^2 | \mathscr{F}_{t_i}] = \mathbb{E}[\Delta W_i^2] = \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = t_{i+1} - t_i \tag{5.7}$$

Consider now

$$\begin{aligned}
\mathbb{E}\left[f(t_i)\Delta W_i\right] &= \mathbb{E}\left[\mathbb{E}\left[f(t_i)\Delta W_i|\mathscr{F}_{t_i}\right]\right] \\
&= \mathbb{E}\left[f(t_i)\mathbb{E}\left[\Delta W_i|\mathscr{F}_{t_i}\right]\right] \\
&= 0
\end{aligned}$$

It follows,

$$\begin{aligned}
\mathbb{E}\left[I(f)\right] &= \mathbb{E}\left[\sum_{i=0}^{n-1} f(t_i)\Delta W_i\right] \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[f(t_i)\Delta W_i\right] \\
&= 0
\end{aligned}$$

Finally, let us consider

$$\begin{aligned}
\mathbb{E}\left[I(f)^2\right] &= \mathbb{E}\left[\left(\sum_{i=0}^{n-1} f(t_i)\Delta W_i\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{n-1} f^2(t_i)\left(\Delta W_i\right)^2 + 2\sum_{i<j} f(t_i)f(t_j)\Delta W_i \Delta W_j\right] \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[f^2(t_i)\left(\Delta W_i\right)^2\right] + \qquad (5.8) \\
&\quad 2\sum_{i<j} \mathbb{E}\left[f(t_i)f(t_j)\Delta W_i \Delta W_j\right] \qquad (5.9)
\end{aligned}$$

Consider first (5.9):

$$\begin{aligned}
\mathbb{E}\left[f(t_i)f(t_j)\Delta W_i \Delta W_j\right] &= \mathbb{E}\left[\mathbb{E}\left[f(t_i)f(t_j)\Delta W_i \Delta W_j|\mathscr{F}_{t_j}\right]\right] \qquad (5.10) \\
&= \mathbb{E}\left[f(t_i)f(t_j)\Delta W_i \mathbb{E}\left[\Delta W_j|\mathscr{F}_{t_j}\right]\right] \qquad (5.11) \\
&= 0 \qquad (5.12)
\end{aligned}$$

□

5.2 Itô's Formula

5.3 Stochastic Differential Equations

6 The Black-Scholes-Merton Formula

7 | Pricing Under Real Market Conditions

Appendices

A | Aditional Proofs

A.1 MGF for the symmetric random walk

Proposition A.1. *The moment generating function of an scaled random walk $W^n(t)$ is $\varphi_{W^n}(u) = (\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}})^{nt}$*

Proof. By definition, the moment generating function for a scaled random walk is

$$\begin{aligned}\mathbb{E}[e^{uW^n(t)}] &= \mathbb{E}(e^{\frac{u}{\sqrt{n}}M_{nt}}) \\ &= \mathbb{E}\left[\exp\left(\frac{u}{\sqrt{n}}\sum_{k=0}^{nt} X_k\right)\right]\end{aligned}$$

Since $X_i \perp X_j \ \forall \ i \neq j$

$$\begin{aligned}&= \prod_{k=1}^{nt} \mathbb{E}\left[\exp\left(\frac{u}{\sqrt{n}}X_k\right)\right] \\ &= \left[\frac{1}{2}\exp\left(\frac{u}{\sqrt{n}}\cdot 1\right) + \frac{1}{2}\exp\left(\frac{u}{\sqrt{n}}\cdot -1\right)\right]^{nt}\end{aligned}$$

□

A.2 MGF for the the normal distribution

Proposition A.2. *The moment generating function for a random variable $X \sim N(0,t)$ is $\varphi_x(u) = e^{\frac{1}{2}u^2 t}$*

Proof. Recall, for $X \sim N(0,t)$,

$$f_X(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

Now,

$$\mathbb{E}[e^{ux}] = \int_{-\infty}^{\infty} e^{ux} f(x)dx$$
$$= \frac{1}{\sqrt{2\pi t}} e^{ux - \frac{x^2}{2t}} \tag{A.1}$$

Note that

$$\begin{aligned} ux - \frac{x^2}{2t} &= \frac{2utx - x^2}{2t} \\ &= \frac{-1}{2t}\left[x^2 - 2(ut)x + (ut)^2 - (ut)^2\right] \\ &= \frac{-1}{2t}\left[x - ut\right]^2 + \frac{1}{2}u^2 t \end{aligned}$$

Thus, A.1 becomes

$$e^{\frac{1}{2}u^2 t} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-ut)^2}{2t}} = e^{\frac{1}{2}u^2 t} \tag{A.2}$$

□

A.3 Derivation of equation 4.20

Proposition A.3. *Let W_t be a standard Brownian Motion,*

$$\mathbb{V}(\left[W_{t_j+1} - W_{t_j}\right]^2) = 2\left(t_{j+1} - t_j\right)^2 \ \ \forall\ j \geq 0$$

Proof.

$$\begin{aligned} \mathbb{V}(\left[W_{t_j+1} - W_{t_j}\right]^2) &= \mathbb{E}\left[\left(W_{t_j+1} - W_{t_j}\right)^2 - \mathbb{E}\left[W_{t_j+1} - W_{t_j}\right]^2\right] \\ &= \mathbb{E}\left[\left(W_{t_j+1} - W_{t_j}\right)^2 - (t_{j+1} - t_j)\right]^2 \\ &= \mathbb{E}\left[\left(W_{t_j+1} - W_{t_j}\right)^4 - 2\left(W_{t_j+1} - W_{t_j}\right)^2 (t_{j+1} - t_j) + (t_{j+1} - t_j)^2\right] \end{aligned}$$

Where $\mathbb{E}\left[\left(W_{t_j+1} - W_{t_j}\right)^4\right] = 3(t_{j+1} - t_j)^2$

$$= 2(t_{j+1} - t_j)^2$$

□

Bibliography

[1] Dave Applebaum. Lecture notes in measure and probability, 2016.

[2] Jaksa Cvitanic and Fernando Zapatero. *Introduction to the Economics and Mathematics of Financial Markets*. MIT Press, 2004. ISBN 0262033208.

[3] Larry Harris. *Trading and Exchanges: Market Microstructure for Practitioners*. Oxford University Press, 2002. ISBN 0195144708.

[4] Onésimo Hernández-Lerma. Probabilidad y procesos estocásticos, September 2009.

[5] John C. Hull. *Options Futures and Other Derivatives*. Prentice Hall/Pearson, 2014. ISBN 0133456315.

[6] Zbigniew Kozikowski. *Finanzas Internacionales*. McGraw Hill, 2013. ISBN 6071509777.

[7] Office of Investor Education and Advocacy. What are corporate bonds? URL `https://www.sec.gov/investor/alerts/ib_corporatebonds.pdf`. SEC Pub. No. 149.

[8] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag Berlin Heidelberg, 2003. ISBN 9783642143946.

[9] Luis Rincón. *Curso Intermedio de Probabilidad*. Facultad de Ciencias UNAM, 2007.

[10] Albert N. Shiryaev. *Probability-1*. Springer, 2016. ISBN 038772205X.

[11] Steven Shreve. *Stochastic Calculus for Finance II*. Springer-Verlag New York, 2004. ISBN 9780387401010.

[12] Comisión Nacional Bancaria y de Valores. *Glosario de Términos Portafolio de Información*. URL http://portafolioinfdoctos.cnbv.gob.mx/Documentacion/minfo/00_DOC_R1.pdf.