Theoretical Grounds and Market Adaptations of Financial Fx and Interest Rate Options

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Some Important Definitions

Definition (Security)

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A market is the place where buyers meet sellers to exchange securities.

Definition (Arbitrage)

We define an arbitrage as the probability of making a risk-free profit from a suitable market strategy.

$$\mathbb{P}(\text{Risk-free profit} > 0) = 1$$

Market Assumptions

- Many buyers and sellers (liquidity);
- possible to lend and borrow at the same rate interest rate r;
- Market participants take advantage of arbitrage opportunity, thereby correcting any deviation from the actual value of any asset.
- All participants have the same information.

The Instruments

- Bonds

A Bond is a debt obligation, its main function is to raise capital for the issuer of the bond. In turn, the buyer of the bond receives interest on the amount loaned.

The Instruments

- Bonds
- Stocks

A stock is a security that represents ownership on a fraction of a corporation. The return on the company for the owner of a stock is represented as a dividend.

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The Instruments

- Bonds
- Stocks
- Foreign Exchange Currencies

"One countrys currency freely convertible in the foreign exchange market." (Kozikowski, 2013)

The Instruments

- Bonds
- Stocks
- Foreign Exchange Currencies
- Derivatives

"[A derivative is] a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables." (Hull, 2014)

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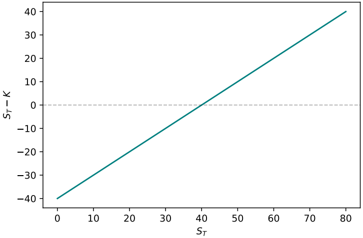
Derivatives: An Example

Definition (Forward)

A forward is a derivative contract that gives the buyer both the right, and the obligation to to purchase a specified amount of the stock at some future time T, at a price K. The value of the forward today is 0.

The payoff of the forward is $S_T - K$. What is the K such that the contract has zero value today and has no possibility of arbitrage?



Assume a continuously compounded interest rate r, denote S_t the value of the stock at time t. At $t = T$ the value of the forward is

$$S_T - K = 0$$

By no arbitrage, the present value of the strategy is

$$S_0 - Ke^{-rT} = 0 \implies K = S_0 e^{rT}$$

Therefore, $S_0 e^{rT}$ is the the value that guarantees no arbitrage.

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Derivatives: An Example (cont'd)

(Why?)
Assume $K' > K = S_0 e^{rT}$.

- At $t = 0$, borrow S_0; purchase the stock; and take a short position the forward contract.
- At $t = T$, receive K'; give the forward; and repay $S_0 e^{rT}$. Profit $K' - S_0 e^{rT}$.

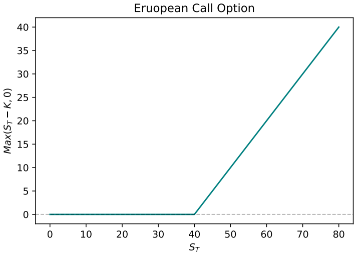
The same can be said for $K' < K$, by taking a long position.

Derivatives: Another Example

Definition (European Call Option)

A European call option is a derivative contract that gives the buyer the right, but not the obligation to purchase a specified amount of stock at some future time T, at a price K.

The payoff of the option is $\max\{S_T - K, 0\}$. What is the price of the option today that guarantees no arbitrage?



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We will work on the following probability space.

Definition (Probability Space: Kolmogorov's Axiom System)

An ordered tripe $(\Omega, \mathscr{F}, \mathbb{P})$ where

- Ω is a set of points ω;
- $\mathscr{F}$ is a σ-algebra of elements of Ω; and
- $\mathbb{P}$ is a probability on $\mathscr{F}$.

Definition (Random Variable)

$$X^{-1}(B) = \{\omega \in \Omega | X(\omega) \in B\} \ \forall \ B \subseteq \mathscr{B}(\mathbb{R}).$$

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Integrals

Definition (Integral v.1)

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, we define the Lebesgue integral (or integral)of the indicator function $\mathbb{1}_A$ w.r.t. μ as

$$\int_{\Omega} \mathbb{1}_A d\mu := \mu(A)$$

Integrals

Definition (Simple Function)

Let $(\Omega, \mathscr{F})$ a measurable space and X an $\mathscr{F}$-measurable function. It is said that X is a **simple function** if it takes only a unique finite number of values $\{x_i\}_{i=1}^n \in \mathbb{R}$ over measurable sets $\{A_i\}$. Then, X can be written as

$$X(\omega) = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$$

Definition (Integral v.2)

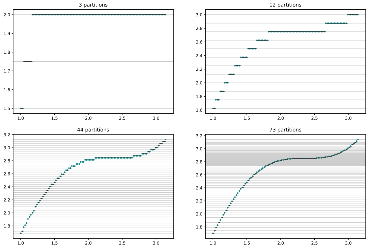
let X be a simple function, we define the integral of X w.r.t. μ as:

$$\int_{\Omega} X(\omega) d\mu(\omega) := \sum_{i=1}^{n} x_i \mu(A_i)$$

Definition (Integral v.3)

For any nonnegative function X, we define the integral of X w.r.t. a measure μ as

$$\int_\Omega X d\mu := \sup\left\{\int_\Omega h d\mu \mid 0 \leq h \leq X,\ h \text{ is simple}\right\}$$



If X is any measurable function, we define the positive and negative parts of X as:

$$X^+ := \max\{X(\omega), 0\} \tag{1}$$
$$X^- := \max\{-X(\omega), 0\} \tag{2}$$

Thus, $X = X^+ - X^-$, and $|X| = X^+ + X^-$. Since both parts of X are positive, their integrals are well defined.

Definition (L_1 spaces)

We denote by $L_1 \equiv L_1(\Omega, \mathscr{F}, \mu)$ the family of integrable functions w.r.t. μ. Note that $X \in L_1$ if and only if $|X| \in L_1$. i.e.,

$$\int_\Omega |X| d\mu = \int_\Omega X^+ d\mu + \int_\Omega X^- d\mu < \infty \tag{3}$$

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Proposition

For any X, Y $\mathscr{F}$-measurable functions in L_1; A, B members of $\mathscr{F}$:

$$X \geq 0 \implies \int_\Omega X d\mu \geq 0$$

If $X \leq Y$ (monotonicity),

$$\int_\Omega X d\mu \leq \int_\Omega Y d\mu$$

If $A \subseteq B$ and $X \geq 0$,

$$\int_A X d\mu \leq \int_B X d\mu$$

If α, $\beta \in \mathbb{R}$ (linearity),

$$\int_\Omega (\alpha X + \beta Y) d\mu = \alpha \int_\Omega X d\mu + \beta \int_\Omega Y d\mu$$

Theorem (Radon Nikodym Theorem)

Let μ and λ be σ-finite positive measures defined on $(\Omega, \mathscr{F})$ such that, for every $A \in \mathscr{F}$, $\lambda(A) = 0 \implies \mu(A) = 0$. Then, there exists a function $f : \Omega \to [0, \infty]$ such that

$$\mu(A) = \int_A f d\lambda.$$

Where the function f is defined up to sets with measure zero. f is sometimes called the Radon-Nikodym derivative and it can be written as $\frac{d\mu}{d\lambda}$

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Expectation

Definition

Let X be a r.v. on $(\Omega, \mathscr{F}, \mathbb{P})$, we define the expectation of X with respect to (w.r.t.) $\mathbb{P}$ as

$$\mathbb{E}[X] := \int_\Omega X(\omega) d\mathbb{P}(\omega)$$

Definition (Conditional Expectation w.r.t. a σ-algebra)

The conditional expectation of a nonnegative random variable X with respect to the σ-algebra $\mathscr{G}$ is a nonnegative random variable denoted $\mathbb{E}[X|\mathscr{G}]$ or $\mathbb{E}[X|\mathscr{G}](\omega)$ such that,

1. $\mathbb{E}[X|\mathscr{G}]$ is $\mathscr{G}$-measurable; and
2. For every $A \in \mathscr{G}$,

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}[X|\mathscr{G}] d\mathbb{P}$$

.

Proposition

1. *If* $a, b \in \mathbb{R}$, $\mathbb{E}[\alpha X + \beta Y|\mathscr{G}] = \alpha\mathbb{E}[X|\mathscr{G}] + \beta\mathbb{E}[Y|\mathscr{G}]$;
2. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}]] = \mathbb{E}[X]$;
3. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_1]$ *if* $\mathscr{F}_1 \subseteq \mathscr{F}_2$;
4. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_2]$ *if* $\mathscr{F}_2 \subseteq \mathscr{F}_1$; *and*
5. *Let* X *be a random variable such that* $X \perp \mathscr{G}$ *then,* $\mathbb{E}[X|\mathscr{G}] = \mathbb{E}[X]$. *Consequently, for any borel-measurable function* h, $\mathbb{E}[h(X)|\mathscr{G}] = \mathbb{E}[h(X)]$.

Stochastic Processes

Definition (**Stochastic Process**)

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A stochastic process is a set of random variables that takes values in some set S called the **state space**, and are indexed by some set T.

$$\{X_t \mid t \in T\}. \tag{4}$$

We can consider a stochastic process as a two variable function

$$X : T \times \Omega \to S$$

Where,

1. For for fixed $t \in T$, $\omega \to X_t(\omega)$ is random variable; and
2. for fixed $\omega \in \Omega$, $t \to X_t(\omega)$ is the path of the process.

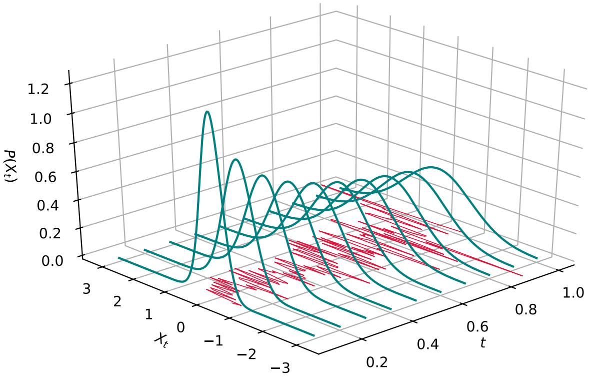

Figure: Distribution and Path of a Stochastic Process.

Definition (Martingale)

An stochastic process $\{X_t\}_{t\geq 0}$ defined over $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a martingale w.r.t. a filtration $\{\mathscr{F}_n\}_{n\geq 0}$ if, for all $t \geq 0$, and $t \leq s$,

1. X_t is $\mathscr{F}_t$-measurable;
2. X_t is in L_1 (i.e. is is measurable for all t); and
3. $\mathbb{E}[X_t|\mathscr{F}_s] = X_t$.

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We assume a simple market consisting of a stock and a cash bond.

- **The Stock**

$$\begin{tikzcd} & S_0 u \\ S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S_0 d \end{tikzcd}$$

Stock has value S_0 at outset. Between any two units of time, the stock can either go up by a factor u (with probability p), or down by a factor d (with probability $1 - p$).

One Step Binomial Model

We assume a simple market consisting of a stock and a cash bond.

- **The Stock**

$$\begin{tikzcd} & S_0 u \\ S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S_0 d \end{tikzcd}$$

  Stock has value S_0 at outset. Between any two units of time, the stock can either go up by a factor u (with probability p), or down by a factor d (with probability $1-p$).

- **The Cash Bond**

  Represents the time value of money. B_0 invested at $t=0$ grows to $B_0 e^{rT}$ one period thereafter.

This simple market yields the possibility for a derivatives' market, i.e., there could be a payoff with price f_0 today that derives its value from the possible directions the stock may take: either f_u if it goes up, or f_d otherwise.

$$
\begin{tikzcd}
 & f_u \\
f_0 \arrow[ur, "S_0 u"] \arrow[dr, "S_0 d"'] & \\
 & f_d
\end{tikzcd}
$$

(Why?) Consider the portfolio (ϕ, ψ) with ϕ number of stocks, and ψ amount in the bank. The value of the portfolio at $t = 0$ is

$$\phi S_0 + \psi B_0$$

We derive a payoff f contingent of the movement of the stock. At $t = \delta_t$ we have

$$\phi S_0 u + \psi B_0 e^{r\delta_t} = f_u$$
$$\phi S_0 d + \psi B_0 e^{r\delta_t} = f_d$$

Yields,

$$\phi = \frac{f_u - f_d}{S_0(u - d)} \tag{4}$$
$$\psi = e^{-r\delta_t} \frac{uf_d - df_u}{u - d} \tag{5}$$

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Therefore, buying the portfolio (ϕ, ψ) guarantees the payoff at $t = \delta_t$ (a risk-free strategy). Denote $\mathcal{V}$ the value of the portfolio at $t = 0$.

$$\mathcal{V} = e^{-r\delta_t}\left[f_u \frac{e^{r\delta_t} - d}{u - d} + f_d \frac{u - e^{r\delta_t}}{u - d}\right]. \tag{6}$$

$\mathcal{V}$ is the cost of a risk-free strategy that guarantees the payoff. Furthermore, showing that $d < \exp(r\delta_t) < u$ proves that there exists no possibility of arbitrage.

Denote

$$q := \frac{\exp r\delta_t - d}{u - d} \Longrightarrow 1 - q = \frac{u - \exp(r\delta_t)}{u - d}. \tag{7}$$

We rewrite (6) as

$$\mathcal{V} = \exp(-r\delta_t)[S_0 u \cdot q + S_0 d \cdot (1 - q)] = \mathbb{E}_{\mathbb{Q}}[\mathcal{V}_{\delta_t} | \mathscr{F}_0]. \tag{8}$$

$\mathcal{V}$ is the discounted expectation under a probability measure $\mathbb{Q}$, and a martingale.