Theoretical Grounds and Market Adaptations of Financial Fx and Interest Rate Options

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Some Important Definitions

Definition (Security)

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Definition (Arbitrage)

We define an arbitrage as the probability of making a risk-free profit from a suitable market strategy.

$$\mathbb{P}(\text{Risk-free profit} > 0) = 1$$

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Market Assumptions

- Many buyers and sellers (liquidity);
- possible to lend and borrow at the same rate interest rate r;
- Market participants take advantage of arbitrage opportunity, thereby correcting any deviation from the actual value of any asset.
- All participants have the same information.

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The Instruments

- Bonds

A Bond is a debt obligation, its main function is to raise capital for the issuer of the bond. In turn, the buyer of the bond receives interest on the amount loaned.

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The Instruments

- Bonds
- Stocks

A stock is a security that represents ownership on a fraction of a corporation. The return on the company for the owner of a stock is represented as a dividend.

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The Instruments

- Bonds
- Stocks
- Foreign Exchange Currencies

"One countrys currency freely convertible in the foreign exchange market." (Kozikowski, 2013)

The Instruments

- Bonds
- Stocks
- Foreign Exchange Currencies
- Derivatives

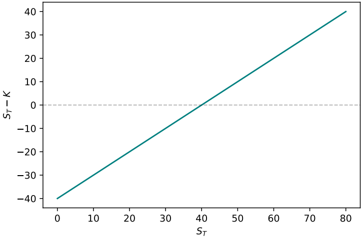
"[A derivative is] a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables." (Hull, 2014)

Derivatives: An Example

Definition (Forward)

A forward is a derivative contract that gives the buyer both the right, and the obligation to to purchase a specified amount of the stock at some future time T, at a price K. The value of the forward today is 0.

The payoff of the forward is $S_T - K$. What is the K such that the contract has zero value today and has no possibility of arbitrage?



Assume a continuously compounded interest rate r, denote S_t the value of the stock at time t. At $t = T$ the value of the forward is

$$S_T - K = 0$$

By no arbitrage, the present value of the strategy is

$$S_0 - Ke^{-rT} = 0 \implies K = S_0 e^{rT}$$

Therefore, $S_0 e^{rT}$ is the the value that guarantees no arbitrage.

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Derivatives: An Example (cont'd)

(Why?)
Assume $K' > K = S_0 e^{rT}$.

- At $t = 0$, borrow S_0; purchase the stock; and take a short position the forward contract.
- At $t = T$, receive K'; give the forward; and repay $S_0 e^{rT}$. Profit $K' - S_0 e^{rT}$.

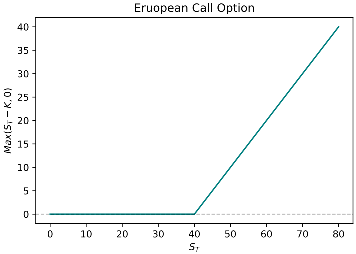
The same can be said for $K' < K$, by taking a long position.

Derivatives: Another Example

Definition (European Call Option)

A European call option is a derivative contract that gives the buyer the right, but not the obligation to purchase a specified amount of stock at some future time T, at a price K.

The payoff of the option is $\max\{S_T - K, 0\}$. What is the price of the option today that guarantees no arbitrage?



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Measures and σ-algebras

Definition (σ-algebra)

Let Ω a set of points ω. $\mathscr{F}$ is a σ-algebra if

1. $\Omega \in \mathscr{F}$;
2. if $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$; and
3. if $\{A_n\}_{n\geq 1} \in \mathscr{F}$, then $\bigcup_n A_n \in \mathscr{F}$.

Definition

Let $\emptyset \neq A \subseteq \Omega$, we define the σ-algebra generated by A as

$$\sigma(A) := \bigcap \{\mathscr{F} | \mathscr{F} \text{ is a } \sigma\text{-algebra and } A \subseteq \mathscr{F}\}$$

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Definition (Measurable Space)

An ordered tripe $(\Omega, \mathscr{F}, \mathbb{P})$ where

- Ω is a set of points ω;
- $\mathscr{F}$ is a σ-algebra of elements of Ω; and
- $\mathbb{P}$ is a probability on $\mathscr{F}$.

Definition (Random Variable)

$$X^{-1}(B) = \{\omega \in \Omega | X(\omega) \in B\} \; \forall \; B \subseteq \mathscr{B}(\mathbb{R}).$$

Integrals

Definition (Integral v.1)

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, we define the Lebesgue integral (or integral)of the indicator function $\mathbb{1}_A$ w.r.t. μ as

$$\int_\Omega \mathbb{1}_A d\mu := \mu(A)$$

Integrals

Definition (Simple Function)

Let $(\Omega, \mathscr{F})$ a measurable space and X an $\mathscr{F}$-measurable function. It is said that X is a **simple function** if it takes only a unique finite number of values $\{x_i\}_{i=1}^{n} \in \mathbb{R}$ over measurable sets $\{A_i\}$. Then, X can be written as

$$X(\omega) = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$$

Definition (Integral v.2)

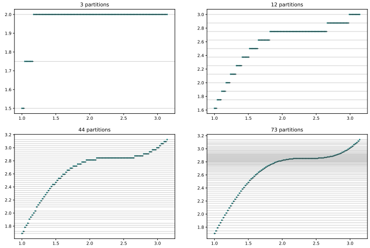
let X be a simple function, we define the integral of X w.r.t. μ as:

$$\int_{\Omega} X(\omega) d\mu(\omega) := \sum_{i=1}^{n} x_i \mu(A_i)$$

Definition (Integral v.3)

For any nonnegative function X, we define the integral of X w.r.t. a measure μ as

$$\int_\Omega X d\mu := \sup\left\{ \int_\Omega h d\mu \mid 0 \leq h \leq X,\ h \text{ is simple} \right\}$$



Proposition

For any X, Y $\mathscr{F}$-measurable functions in L_1; A, B members of $\mathscr{F}$:

$$X \geq 0 \implies \int_\Omega X d\mu \geq 0$$

If $X \leq Y$ (monotonicity),

$$\int_\Omega X d\mu \leq \int_\Omega Y d\mu$$

If $A \subseteq B$ and $X \geq 0$,

$$\int_A X d\mu \leq \int_B X d\mu$$

If α, $\beta \in \mathbb{R}$ (linearity),

$$\int_\Omega (\alpha X + \beta Y) d\mu = \alpha \int_\Omega X d\mu + \beta \int_\Omega Y d\mu$$

Theorem (Radon Nikodym Theorem)

Let μ and λ be σ-finite positive measures defined on $(\Omega, \mathscr{F})$ such that, for every $A \in \mathscr{F}$, $\lambda(A) = 0 \implies \mu(A) = 0$. Then, there exists a function $f : \Omega \to [0, \infty]$ such that

$$\mu(A) = \int_A f d\lambda.$$

Where the function f is defined up to sets with measure zero. f is sometimes called the Radon-Nikodym derivative and it can be written as $\frac{d\mu}{d\lambda}$

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Expectation

Definition

Let X be a r.v. on $(\Omega, \mathscr{F}, \mathbb{P})$, we define the expectation of X with respect to (w.r.t.) $\mathbb{P}$ as

$$\mathbb{E}[X] := \int_\Omega X(\omega) d\mathbb{P}(\omega)$$

Definition (Conditional Expectation w.r.t. a σ-algebra)

The conditional expectation of a nonnegative random variable X with respect to the σ-algebra $\mathscr{G}$ is a nonnegative random variable denoted $\mathbb{E}[X|\mathscr{G}]$ or $\mathbb{E}[X|\mathscr{G}](\omega)$ such that,

1. $\mathbb{E}[X|\mathscr{G}]$ is $\mathscr{G}$-measurable; and
2. For every $A \in \mathscr{G}$,

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}[X|\mathscr{G}] d\mathbb{P}$$

.

Proposition

1. *If* $a, b \in \mathbb{R}$, $\mathbb{E}[\alpha X + \beta Y|\mathscr{G}] = \alpha\mathbb{E}[X|\mathscr{G}] + \beta\mathbb{E}[Y|\mathscr{G}]$;
2. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}]] = \mathbb{E}[X]$;
3. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_1]$ *if* $\mathscr{F}_1 \subseteq \mathscr{F}_2$;
4. $\mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{F}_1] = [X|\mathscr{F}_2]$ *if* $\mathscr{F}_2 \subseteq \mathscr{F}_1$; *and*
5. *Let* X *be a random variable such that* $X \perp \mathscr{G}$ *then,* $\mathbb{E}[X|\mathscr{G}] = \mathbb{E}[X]$. *Consequently, for any borel-measurable function* h, $\mathbb{E}[h(X)|\mathscr{G}] = \mathbb{E}[h(X)]$.

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