


Introduction to Feynman path integrals

Gert-Ludwig Ingold
Universität Augsburg

 github.com/gertingold/feynman-intro



Introduction to
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integrals

Motivation

Propagator

Derivation of path
integral

Particle on a ring

Particle in a box

Harmonic
oscillator

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Classical mechanics: Hamilton vs. Lagrange

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Hamilton formalism

Hamiltonian $H(q, p)$

Poisson bracket $\{q, p\} = 1$
equations of motion
 $\dot{q} = \{q, H\}, \dot{p} = \{p, H\}$

||

canonical quantization $[\hat{q}, \hat{p}] = i\hbar$

⇓

Heisenberg, Schrödinger, ... (1925)

Schrödinger equation

$$\hat{H}|\Psi\rangle = i\hbar \frac{\partial}{\partial t} |\Psi\rangle$$

Lagrange formalism

Lagrangian $L(q, \dot{q})$

action $S = \int dt L(q, \dot{q})$
equation of motion from
Hamilton's principle $\delta S = 0$

||

?

⇓

Dirac (1933), Feynman (1948)

Path integral formulation
of quantum mechanics

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Why base quantum mechanics on Lagrange formalism?

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path integral formulation of quantum mechanics ...

- ...can provide an alternative view on physical problems

- ...does without operators
for fermionic fields, Grassmann variables are used

- ...is very well suited for relativistic field theories
action $S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$ is a Minkowski scalar

Hamiltonian density $\mathcal{H}(\phi, \pi)$ corresponds to the $0 - 0$
component of the energy-momentum tensor

- ...is very well suited to consider the semiclassical limit $S \gg \hbar$



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Schrödinger equation $i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle$

time evolution of a state

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle \quad \text{with} \quad U(t) = \exp\left(-\frac{i}{\hbar} H t\right)$$

in position representation

$$\langle q_f | \Psi(t) \rangle = \int dq_i \langle q_f | U(t) | q_i \rangle \langle q_i | \Psi(0) \rangle$$

$$\Psi(q_f, t) = \int dq_i K(q_f, q_i, t) \Psi(q_i, 0)$$

\Rightarrow propagator $K(q_f, q_i, t) = \langle q_f | U(t) | q_i \rangle$

Hamiltonian of the free particle

$$H = \frac{p^2}{2m}$$

propagator of the free particle

$$\begin{aligned} K(q_f, q_i, t) &= \langle q_f | e^{-\frac{i}{\hbar} H t} | q_i \rangle \\ &= \int_{-\infty}^{+\infty} dp \langle q_f | p \rangle \langle p | q_i \rangle e^{-\frac{i}{\hbar} \frac{p^2}{2m} t} \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp \exp \left[-\frac{i}{\hbar} \left(\frac{p^2}{2m} t - (q_f - q_i) p \right) \right] \\ &= \frac{1}{2\pi\hbar} \exp \left(\frac{i}{\hbar} \frac{m(q_f - q_i)^2}{2t} \right) \int_{-\infty}^{+\infty} dp \exp \left(-\frac{i}{\hbar} \frac{t}{2m} p^2 \right) \end{aligned}$$



Digression: Fresnel integral

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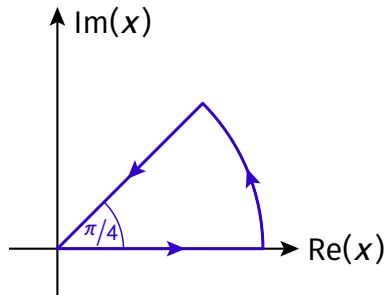
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$$\begin{aligned}\int_0^\infty dx e^{i\alpha x^2} &= e^{i\pi/4} \int_0^\infty du e^{-\alpha u^2} \\ &= \frac{1}{2} \sqrt{\frac{i\pi}{\alpha}}\end{aligned}$$

$$\Rightarrow \int_{-\infty}^{+\infty} dx e^{-i\alpha x^2} = \sqrt{\frac{\pi}{i\alpha}}$$



propagator

$$K(q_f, q_i, t) = \langle q_f | e^{-\frac{i}{\hbar} H t} | q_i \rangle = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{i}{\hbar} \frac{m(q_f - q_i)^2}{2t}\right)$$

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classical motion from position q_i to q_f in time t :

■ classical path $q(s) = q_i + (q_f - q_i) \frac{s}{t}$

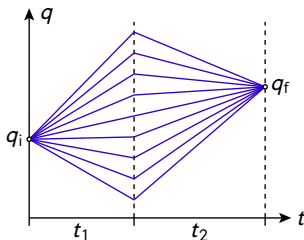
■ classical action $S_{cl} = \int_0^t ds \frac{m}{2} \dot{q}^2 = \frac{m(q_f - q_i)^2}{2t}$

propagator of free particle in terms of classical properties

$$K(q_f, q_i, t) = \sqrt{-\frac{1}{2\pi i \hbar} \frac{\partial^2 S_{cl}}{\partial q_i \partial q_f}} \exp\left(\frac{i}{\hbar} S_{cl}\right)$$

semigroup property

$$\begin{aligned}
 K(q_f, q_i, t_2 + t_1) &= \langle q_f | e^{-\frac{i}{\hbar} H t_2} e^{-\frac{i}{\hbar} H t_1} | q_i \rangle \\
 &= \int_{-\infty}^{+\infty} dq' \langle q_f | e^{-\frac{i}{\hbar} H t_2} | q' \rangle \langle q' | e^{-\frac{i}{\hbar} H t_1} | q_i \rangle \\
 &= \int_{-\infty}^{+\infty} dq' K(q_f, q', t_2) K(q', q_i, t_1)
 \end{aligned}$$



Exercise: Check the semigroup property for the propagator of a free particle.

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Strategy for the derivation of the Feynman path integral

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basic steps:

- use semigroup property to split the propagator into a large number of short-time propagators
- use the Baker-Campbell-Hausdorff formula to separate kinetic and potential contributions to the Hamiltonian in the time-evolution operator



Baker-Campbell-Hausdorff formula and Lie-Trotter formula

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application to a short-time propagator with operators T and V representing kinetic and potential energy, respectively

$$\exp\left(-\frac{i}{\hbar}(T + V)\Delta t\right) = \exp\left(-\frac{i}{\hbar}V\Delta t\right) \exp\left(-\frac{i}{\hbar}T\Delta t\right) \exp\left(-\frac{i}{\hbar^2}X\Delta t^2\right)$$

$$X = \frac{i}{2}[V, T] - \frac{1}{6\hbar}([V, [V, T]] - 2[T, [T, V]])\Delta t + \dots$$

for $\Delta t \rightarrow 0$, the last factor may be neglected

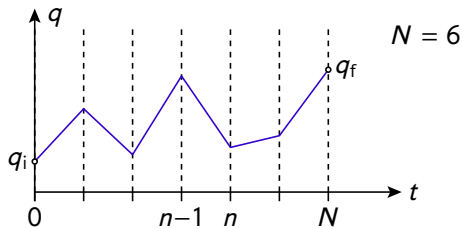
⇒ Lie-Trotter formula

$$\exp\left(-\frac{i}{\hbar}(T + V)t\right) = \lim_{N \rightarrow \infty} \left[\exp\left(-\frac{i}{\hbar}V\frac{t}{N}\right) \exp\left(-\frac{i}{\hbar}T\frac{t}{N}\right) \right]^N$$

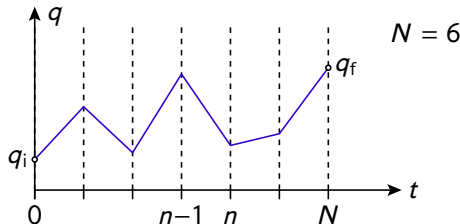
propagator $\Delta t = \frac{t}{N} \quad q_0 = q_i \quad q_N = q_f$

$$\langle q_f | U(t) | q_i \rangle =$$

$$\int_{-\infty}^{+\infty} \left(\prod_{n=1}^{N-1} dq_n \right) \langle q_f | \cdots | q_n \rangle \underbrace{\langle q_n | e^{-\frac{i}{\hbar} V \Delta t} e^{-\frac{i}{\hbar} T \Delta t} | q_{n-1} \rangle}_{\sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp \left[\frac{i}{\hbar} \left(\frac{m(q_n - q_{n-1})^2}{2\Delta t} - V(q_n) \Delta t \right) \right]} \langle q_{n-1} | \cdots | q_i \rangle$$



$$\langle q_f | U(t) | q_i \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-\infty}^{+\infty} \prod_{n=1}^{N-1} \left(\sqrt{\frac{m}{2\pi i \hbar \Delta t}} dq_n \right) \\ \times \exp \left[\frac{i}{\hbar} \sum_{n=1}^N \left(\frac{m}{2} \left(\frac{q_n - q_{n-1}}{\Delta t} \right)^2 - V(q_n) \right) \Delta t \right]$$



let us now take the continuum limit $N \rightarrow \infty$



$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \left(\frac{m}{2} \left(\frac{q_n - q_{n-1}}{\Delta t} \right)^2 - V(q_n) \right) \Delta t = \int_0^t ds \left(\frac{m}{2} \dot{q}(s) - V(q) \right)$$

in the exponent of the propagator, we find the action functional

$$S[q] = \int_0^t ds \left(\frac{m}{2} \dot{q}(s) - V(q(s)) \right)$$

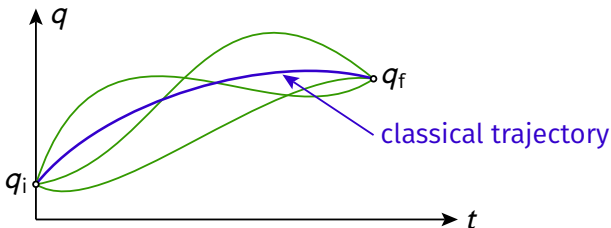
which takes a function $q(s)$ and returns a number S

⇒ we have obtained a connection between the
Lagrange formalism of classical mechanics and
the quantum mechanical propagator

propagator as functional integral (Feynman path integral)

$$K(q_f, q_i, t) = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q \exp\left(\frac{i}{\hbar} S[q]\right)$$

the integral runs over all trajectories $q(s)$ satisfying the boundary conditions $q(0) = q_i$ and $q(t) = q_f$

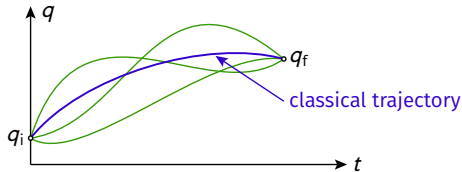


- evaluate discretized version as an N -dimensional integral
for linear systems, the following relation is useful:

$$\int_{-\infty}^{+\infty} d\mathbf{x}^N \exp(-\mathbf{x}^T \mathbf{A} \mathbf{x}) = \sqrt{\frac{\pi^N}{\det(\mathbf{A})}}$$

- expansion in a complete set of functions
→ see our later discussion of the harmonic oscillator

Exercise: Evaluate the Feynman path integral for the free particle by means of discretization. Why is the result independent of N ?



- All trajectories connecting initial to final point in the given time contribute to the propagator.
- Almost all of these trajectories are not solutions of the corresponding classical equation of motion.
- Each trajectory contributes with a phase factor depending on the action associated with the trajectory, cf. double slit.
- The nonclassical trajectories can be viewed as contributions of quantum fluctuations.

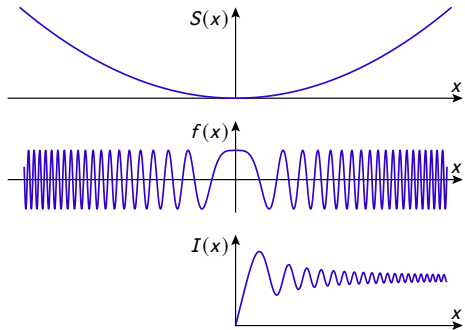
Conventional integral as an analogy:

$$S(x) = x^2$$

$x = 0$ corresponds to
classical trajectory

$$f(x) = \cos(S(x))$$

$$I(x) = \int_{-x}^x du \cos(S(u))$$



The dominant contributions come from stationary points of the action and fluctuations around it.



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particle of mass m on a ring of radius R

Hamiltonian
$$H = -\frac{\hbar^2}{2mR^2} \frac{d^2}{d\phi^2}$$

eigenfunctions with $\psi(0) = \psi(2\pi)$ and $\psi'(0) = \psi'(2\pi)$

$$\psi_\ell(\phi) = \frac{1}{\sqrt{2\pi}} e^{i\ell\phi} \quad \ell = 0, \pm 1, \pm 2, \dots$$

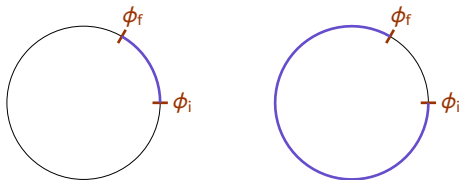
eigenenergies

$$E_\ell = \frac{\hbar^2 \ell^2}{2mR^2}$$

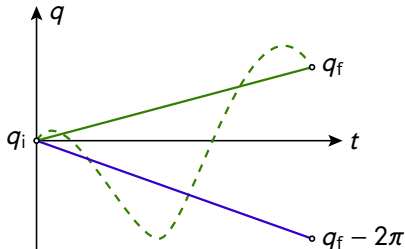
propagator

$$K(\phi_f, \phi_i, t) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} \exp \left(i\ell(\phi_f - \phi_i) - i \frac{\hbar \ell^2}{2mR^2} t \right)$$

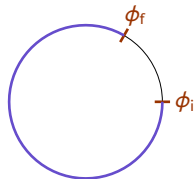
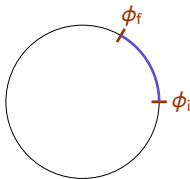
two ways to reach ϕ_f :



■ the two paths cannot be deformed into each other



two ways to reach ϕ_f :

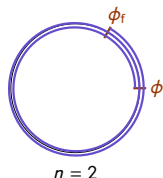
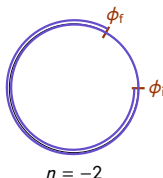
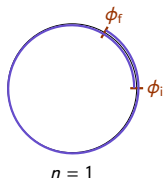


- the two paths cannot be deformed into each other
- two contributions:

$$R \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left(\frac{i}{\hbar} \frac{m R^2}{2} \frac{(\phi_f - \phi_i)^2}{t} \right) \quad \text{and}$$

$$R \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left(\frac{i}{\hbar} \frac{m R^2}{2} \frac{((\phi_f - 2\pi) - \phi_i)^2}{t} \right)$$

but there is more ...



- the angles ϕ_f and $\phi_f + 2\pi n$ have to be identified
- the propagator is a sum over all topologically distinct contributions

$$K(\phi_f, \phi_i, t) = R \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{n=-\infty}^{+\infty} \exp \left(\frac{i}{\hbar} \frac{m R^2}{2} \frac{(\phi_f - \phi_i - 2\pi n)^2}{t} \right)$$

two different expressions for the propagator

winding number

$$K(\phi_f, \phi_i, t) = R \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{n=-\infty}^{+\infty} \exp \left(\frac{i}{\hbar} \frac{m R^2}{2} \frac{(\phi_f - \phi_i - 2\pi n)^2}{t} \right)$$

angular momentum

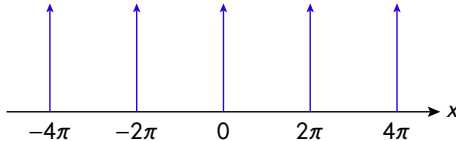
$$K(\phi_f, \phi_i, t) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} \exp \left(i \ell (\phi_f - \phi_i) - i \frac{\hbar \ell^2}{2m R^2} t \right)$$

compare with propagator of free particle

$$K(q_f, q_i, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dp}{\hbar} \exp \left(i \frac{p}{\hbar} (q_f - q_i) - i \frac{p^2}{2\hbar m} t \right)$$



Fourier representation of a periodic δ -comb



$$\sum_{n=-\infty}^{+\infty} \delta(x - 2\pi n) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} e^{i\ell x}$$

Exercise: Verify this relation.

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$$\begin{aligned} K(\phi_f, \phi_i, t) &= R \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{n=-\infty}^{+\infty} \exp\left(\frac{i}{\hbar} \frac{m R^2}{2} \frac{(\phi_f - \phi_i - 2\pi n)^2}{t}\right) \\ &= R \sqrt{\frac{m}{2\pi i \hbar t}} \int_{-\infty}^{+\infty} d\phi \exp\left(\frac{i}{\hbar} \frac{m R^2}{2t} \phi^2\right) \sum_{n=-\infty}^{+\infty} \delta(\phi - (\phi_f - \phi_i - 2\pi n)) \\ &= \frac{R}{2\pi} \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{\ell=-\infty}^{+\infty} \exp^{-i\ell(\phi_f - \phi_i)} \int_{-\infty}^{+\infty} d\phi \exp\left(\frac{i}{\hbar} \frac{m R^2}{2t} \phi^2 + i\ell\phi\right) \\ &= \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} \exp\left(i\ell(\phi_f - \phi_i) - i \frac{\hbar \ell^2}{2m R^2} t\right) \quad \checkmark \end{aligned}$$

Exercise: Fill in the missing details.



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particle of mass m in an infinitely deep potential well of width L

Hamiltonian $H = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2}$

eigenfunctions with $\psi(0) = \psi(L) = 0$

$$\psi_j(q) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi j}{L} q\right) \quad j = 1, 2, 3, \dots$$

eigenenergies

$$E_j = \frac{\hbar^2 \pi^2}{2mL^2} j^2$$

propagator

$$K(q_f, q_i, t) = \frac{2}{L} \sum_{j=1}^{\infty} \exp\left(-i \frac{\hbar \pi^2 j^2}{2mL^2} t\right) \sin\left(\frac{\pi j}{L} q_f\right) \sin\left(\frac{\pi j}{L} q_i\right)$$

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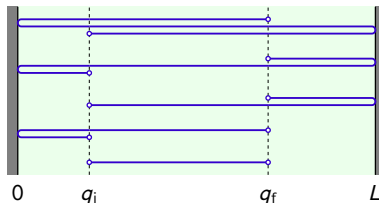
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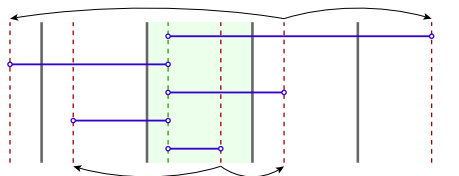
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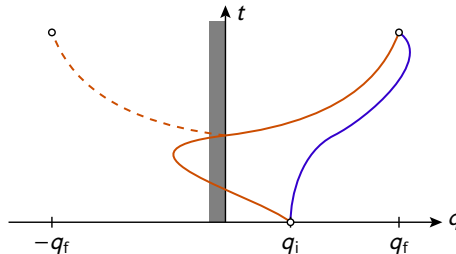
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unfolding of the trajectories



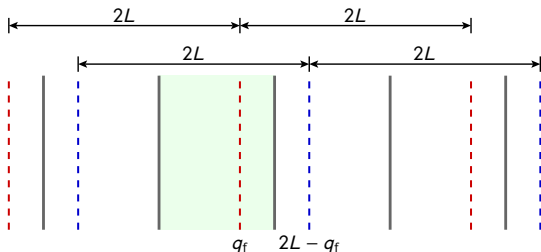
- sum over propagators of a free particle
- relative phase between the various contributions?



subtract the contribution of paths crossing the wall:

$$K_{\text{wall}}(q_f, q_i, t) = K_{\text{free}}(q_f, q_i, t) - K_{\text{free}}(-q_f, q_i, t)$$

- compare with method of images in electrostatics
- each reflection yields a factor -1



even number of reflections at $q = q_f + 2nL$

odd number of reflections at $q = 2nL - q_f$

propagator

$$K(q_f, q_i, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{n=-\infty}^{+\infty} \left[\exp\left(\frac{i}{\hbar} \frac{m(2nL + q_f - q_i)^2}{2t}\right) - \exp\left(\frac{i}{\hbar} \frac{m(2nL - q_f - q_i)^2}{2t}\right) \right]$$

again, we have obtained two dual versions of the propagator:

$$K(q_f, q_i, t) = \frac{2}{L} \sum_{j=1}^{\infty} \exp\left(-i \frac{\hbar \pi^2 j^2}{2mL^2} t\right) \sin\left(\frac{\pi j}{L} q_f\right) \sin\left(\frac{\pi j}{L} q_i\right)$$

and

$$K(q_f, q_i, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \sum_{n=-\infty}^{+\infty} \left[\exp\left(\frac{i}{\hbar} \frac{m(2nL + q_f - q_i)^2}{2t}\right) - \exp\left(\frac{i}{\hbar} \frac{m(2nL - q_f - q_i)^2}{2t}\right) \right]$$

Exercise: Show that the two representations of the propagator of a particle in a box agree. Follow the strategy which proved successful already for the particle on a ring.



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path integral representation of the propagator of the harmonic oscillator

$$K(q_f, q_i, t) = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q \exp \left[\frac{i}{\hbar} \int_0^t ds \frac{m}{2} \left(\left(\frac{dq}{ds} \right)^2 - \omega^2 q^2 \right) \right]$$

How do we integrate over all functions satisfying the boundary conditions $q(0) = q_i$ and $q(t) = q_f$?

- decompose path into two contributions:
 - a path satisfying the boundary conditions
 - fluctuations around this path described by a complete set of functions vanishing at the boundaries
- integrate over the expansion coefficients of the fluctuations

$$q(s) = \bar{q}(s) + \xi(s)$$

$$\text{with } q(0) = q_i, q(t) = q_f \quad \text{and} \quad \xi(0) = \xi(t) = 0$$

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action

$$\begin{aligned} S[q] &= \int_0^t ds \frac{m}{2} \left[\left(\frac{dq}{ds} \right)^2 - \omega^2 q^2 \right] \\ &= \int_0^t ds \frac{m}{2} \left[\left(\frac{d\bar{q}}{ds} \right)^2 - \omega^2 \bar{q}^2 + 2 \frac{d\bar{q}}{ds} \frac{d\xi}{ds} - 2\omega^2 \bar{q}\xi + \left(\frac{d\xi}{ds} \right)^2 - \omega^2 \xi^2 \right] \end{aligned}$$

integration by parts

$$= \int_0^t ds \frac{m}{2} \left[\left(\frac{d\bar{q}}{ds} \right)^2 - \omega^2 \bar{q}^2 - 2 \left(\frac{d^2 \bar{q}}{ds^2} + \omega^2 \bar{q} \right) \xi + \left(\frac{d\xi}{ds} \right)^2 - \omega^2 \xi^2 \right]$$

linear term in ξ vanishes if $\bar{q}(s)$ solves the equation of motion

$$q(s) = q_{\text{cl}}(s) + \xi(s)$$

action

$$\begin{aligned} S[q] &= \int_0^t ds \frac{m}{2} \left[\left(\frac{dq_{\text{cl}}}{ds} \right)^2 - \omega^2 q_{\text{cl}}^2 + \left(\frac{d\xi}{ds} \right)^2 - \omega^2 \xi^2 \right] \\ &= S_{\text{cl}} + \int_0^t ds \frac{m}{2} \left[\left(\frac{d\xi}{ds} \right)^2 - \omega^2 \xi^2 \right] \end{aligned}$$

- expansion around the classical path makes linear term in ξ disappear because $\delta S = 0$ for classical paths
- for the harmonic oscillator:
 - classical action S_{cl} contains the full dependence on q_i and q_f
 - fluctuation part is independent of q_i and q_f and only contributes to the prefactor

general solution of the equation of motion of the **harmonic oscillator**

$$q_{\text{cl}}(s) = A \cos(\omega s) + B \sin(\omega s)$$

boundary conditions

$$q_{\text{cl}}(0) = q_i = A$$

$$q_{\text{cl}}(t) = q_f = A \cos(\omega t) + B \sin(\omega t)$$

classical solution satisfying the boundary conditions

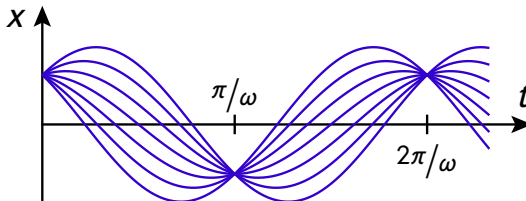
$$\begin{aligned} q_{\text{cl}}(s) &= q_i \cos(\omega s) + \frac{q_f - q_i \cos(\omega t)}{\sin(\omega t)} \sin(\omega s) \\ &= q_i \frac{\sin(\omega(t-s))}{\sin(\omega t)} + q_f \frac{\sin(\omega s)}{\sin(\omega t)} \end{aligned}$$

$$q_{\text{cl}}(s) = q_i \frac{\sin(\omega(t-s))}{\sin(\omega t)} + q_f \frac{\sin(\omega s)}{\sin(\omega t)}$$

What happens for $t = \frac{2\pi n}{\omega}$?

boundary value problems ...

- ...do not necessarily have a solution
- ...may have more than one solution



$$S_{\text{cl}} = \int_0^t ds \frac{m}{2} \left(\dot{q}_{\text{cl}}^2(s) - \omega^2 q_{\text{cl}}^2(s) \right)$$

integration by parts

$$\begin{aligned} &= \frac{m}{2} q_{\text{cl}}(s) \dot{q}_{\text{cl}}(s) \Big|_0^t - \int_0^t ds \frac{m}{2} q_{\text{cl}}(s) \left(\ddot{q}_{\text{cl}}(s) + \omega^2 q_{\text{cl}}(s) \right) \\ &= \frac{m}{2} (q_{\text{cl}}(t) \dot{q}_{\text{cl}}(t) - q_{\text{cl}}(0) \dot{q}_{\text{cl}}(0)) \end{aligned}$$

$$\dot{q}_{\text{cl}}(0) = \omega \frac{q_{\text{f}} - q_{\text{i}} \cos(\omega t)}{\sin(\omega t)} \quad \dot{q}_{\text{cl}}(t) = \omega \frac{q_{\text{f}} \cos(\omega t) - q_{\text{i}}}{\sin(\omega t)}$$

classical action of the harmonic oscillator

$$S_{\text{cl}} = \frac{m\omega}{2} \frac{(q_{\text{f}}^2 + q_{\text{i}}^2) \cos(\omega t) - 2q_{\text{f}}q_{\text{i}}}{\sin(\omega t)}$$

fluctuation part of the action

$$\begin{aligned}\int_0^t ds \frac{m}{2} (\dot{\xi}^2 - \omega^2 \xi^2) &= \underbrace{\frac{m}{2} \xi(s) \dot{\xi}(s)}_{=0} \Big|_0^t - \int_0^t ds \frac{m}{2} \xi (\ddot{\xi} + \omega^2 \xi) \\ &= - \int_0^t ds \frac{m}{2} \xi \left(\frac{d^2}{ds^2} + \omega^2 \right) \xi\end{aligned}$$

expand the fluctuations in the complete set of eigenfunctions of $\frac{d^2}{ds^2} + \omega^2$ vanishing at the boundaries:

$$\left(\frac{d^2}{ds^2} + \omega^2 \right) \xi_n(s) = \lambda_n \xi_n(s)$$

Motivation

Propagator

Derivation of path
integral

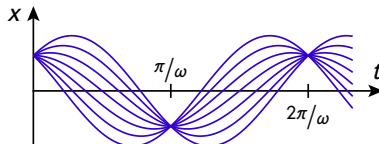
Particle on a ring

Particle in a box

Harmonic
oscillator

$$\xi_n(s) = \sqrt{\frac{2}{t}} \sin\left(\pi n \frac{s}{t}\right) \quad \lambda_n = -\frac{\pi^2 n^2}{t^2} + \omega^2 \quad n = 1, 2, \dots$$

- $0 \leq \omega t < \pi$: $\lambda_n > 0$ for all n
classical action is a minimum
- $\omega t = \pi$: $a_1 \xi_1(s)$ is a classical solution for all amplitudes a_1



- $\pi < \omega t < 2\pi$: $\lambda_1 < 0$, $\lambda_n > 0$ for $n = 2, 3, \dots$
classical action is a saddle point
- ...

action

$$S[q] = S_{\text{cl}} - \sum_{n=1}^{\infty} \frac{m}{2} \left(-\frac{\pi^2 n^2}{t^2} + \omega^2 \right) a_n^2$$

propagator

$$\begin{aligned} K(q_f, q_i, t) &\sim \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\prod_{n=1}^{\infty} da_n \right) \exp \left[\frac{i}{\hbar} \left(S_{\text{cl}} - \frac{m}{2} \sum_{n=1}^{\infty} \lambda_n a_n^2 \right) \right] \\ &\sim \left(\prod_{n=1}^{\infty} \lambda_n \right)^{-1/2} \exp \left(\frac{i}{\hbar} S_{\text{cl}} \right) \end{aligned}$$

fluctuation determinant

$$\det \left(\frac{d^2}{ds^2} + \omega^2 \right) = \prod_{n=1}^{\infty} \lambda_n$$

the prefactor unknown so far does not depend on ω
→ compare with propagator of the free particle

classical action

$$\lim_{\omega \rightarrow 0} \frac{m}{2} \underbrace{\frac{\omega}{\sin(\omega t)}}_{\rightarrow 1/t} \underbrace{((q_f^2 + q_i^2) \cos(\omega t) - 2q_f q_i)}_{\rightarrow (q_f - q_i)^2} = \frac{m}{2} \frac{(q_f - q_i)^2}{t} \quad \checkmark$$

prefactor

$$\sqrt{\prod_{n=1}^{\infty} \frac{\lambda_n^{(0)}}{\lambda_n}} \sqrt{\frac{m}{2\pi i \hbar t}}$$

with

$$\lambda_n = -\frac{\pi^2 n^2}{t^2} + \omega^2 \quad \text{and} \quad \lambda_n^{(0)} = -\frac{\pi^2 n^2}{t^2}$$

$$\prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^{(0)}} = \prod_{n=1}^{\infty} \left(1 - \left(\frac{\omega t}{\pi n} \right)^2 \right) = \frac{\sin(\omega t)}{\omega t}$$

propagator of the harmonic oscillator

$$\begin{aligned} K(q_f, q_i, t) &= \sqrt{\frac{m\omega}{2\pi i \hbar \sin(\omega t)}} \exp \left(i \frac{m\omega}{2\hbar} \frac{(q_f^2 + q_i^2) \cos(\omega t) - 2q_i q_f}{\sin(\omega t)} \right) \\ &= \sqrt{\frac{m\omega}{2\pi \hbar |\sin(\omega t)|}} \exp \left(i \frac{m\omega}{2\hbar} \frac{(q_f^2 + q_i^2) \cos(\omega t) - 2q_i q_f}{\sin(\omega t)} - i \left(\frac{\pi}{4} + n \frac{\pi}{2} \right) \right) \end{aligned}$$

n : number of zeros of the sine function crossed up to time t

This choice of the phase ensures the validity of the semigroup property, e.g.

$$K(q_f, q', \frac{3\pi}{2}) = \int_{-\infty}^{+\infty} dq' K(q_f, q', \frac{3\pi}{4}) K(q', q_i, \frac{3\pi}{4})$$

action

$$S[q] = \int_0^t ds \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$$

expansion around the classical solution: $q(s) = q_{\text{cl}}(s) + \xi(s)$

$$\begin{aligned} S &= \int_0^t ds \left(\frac{m}{2} (\dot{q}_{\text{cl}}^2 + 2\dot{q}_{\text{cl}}\dot{\xi} + \dot{\xi}^2) - V(q_{\text{cl}}) - V'(q_{\text{cl}})\xi - \frac{1}{2}V''(q_{\text{cl}})\xi^2 + \dots \right) \\ &= S_{\text{cl}} - \int_0^t ds \underbrace{(m\ddot{q}_{\text{cl}} + V'(q_{\text{cl}}))}_{=0} \xi - \frac{1}{2} \int_0^t ds \xi \left(m \frac{d^2}{ds^2} + V''(q_{\text{cl}}) \right) \xi + \dots \end{aligned}$$

- the quadratic contribution in the fluctuations generally depends on q_f and q_i
- it leads again to a determinant
- the expansion of ξ does not necessarily need to be done in terms of eigenfunctions

Motivation

Propagator

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oscillator

- the second-order term in the fluctuations determines their typical amplitude

$$\frac{\xi^2}{\hbar} \sim 1 \quad \rightarrow \quad \xi \sim \sqrt{\hbar}$$

- the third-order term is smaller by a factor $\sqrt{\hbar}$ and can be neglected in the semiclassical limit
- however, higher-order terms can become relevant when the eigenvalue of a fluctuation mode approaches zero
example: for a harmonic oscillator around $t = \pi/\omega$, weak anharmonicities of the potential may become relevant