Exercise: Show that the propagator for a particle on a ring

$$K(\phi_\mathrm{f},\phi_\mathrm{i},t)=R\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\sum_{n=-\infty}^{+\infty}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mR^2}{2}\frac{(\phi_\mathrm{f}-\phi_\mathrm{i}-2\pi n)^2}{t}\right) \tag{1}$$

obtained from the Feynman path integral is equivalent to the representation

$$K(\phi_\mathrm{f},\phi_\mathrm{i},t)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\exp\left(\mathrm{i}\ell(\phi_\mathrm{f}-\phi_\mathrm{i})-\mathrm{i}\frac{\hbar\ell^2}{2mR^2}t\right) \tag{2}$$

obtained by means of the eigenvalues and eigenfunctions of the Hamiltonian.

Solution:

The central step in going from (1) to (2) involves the Fourier representation of the δ-comb

$$\sum_{n=-\infty}^{+\infty}\delta(x-2\pi n)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\ell x}\,. \tag{3}$$

The propagator is a 2π-periodic function in $q_\mathrm{f}-q_\mathrm{i}$ so that we can express the sum in (1) in terms of an integral over a δ-comb:

$$K(\phi_\mathrm{f},\phi_\mathrm{i},t)=R\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\int_{-\infty}^{+\infty}\mathrm{d}\phi\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mR^2}{2t}\phi^2\right)\sum_{n=-\infty}^{+\infty}\delta\big(\phi-(\phi_\mathrm{f}-\phi_\mathrm{i}-2\pi n)\big)\,. \tag{4}$$

According to (3), we can write

$$\sum_{n=-\infty}^{+\infty}\delta\big(\phi-(\phi_\mathrm{f}-\phi_\mathrm{i}-2\pi n)\big)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\ell(\phi-\phi_\mathrm{f}+\phi_\mathrm{i})} \tag{5}$$

and thus obtain

$$K(\phi_\mathrm{f},\phi_\mathrm{i},t)=\frac{R}{2\pi}\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}(\phi_\mathrm{f}-\phi_\mathrm{i})}\int_{-\infty}^{+\infty}\mathrm{d}\phi\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mR^2}{2t}\phi^2+\mathrm{i}\ell\phi\right)\,. \tag{6}$$

Completing the square in the exponent of the integrand, we can evaluate the Fresnel integral.

$$\begin{aligned}K(\phi_\mathrm{f},\phi_\mathrm{i},t)&=\frac{R}{2\pi}\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}(\phi_\mathrm{f}-\phi_\mathrm{i})}\int_{-\infty}^{+\infty}\mathrm{d}\phi\exp\left[\mathrm{i}\frac{mR^2}{2\hbar t}\left(\phi+\frac{\hbar t}{mR^2}\ell\right)^2\right]\exp\left(-\mathrm{i}\frac{\hbar t}{2mR^2}\ell^2\right)\\&=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\exp\left(-\mathrm{i}(\phi_\mathrm{f}-\phi_\mathrm{i})\ell-\mathrm{i}\frac{\hbar t}{2mR^2}\ell^2\right)\end{aligned} \tag{7}$$

Replacing the summation index ℓ by $-\ell$, we arrive at the desired result (2).