Exercise: Derive the propagator of the harmonic oscillator from its stationary wave functions and eigenenergies by making use of the Mehler formula

$$\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{n!}\left(\frac{z}{2}\right)^n=\frac{1}{\sqrt{1-z^2}}\exp\left(\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right)\,. \tag{1}$$

Solution:

The stationary wave functions of a harmonic oscillator with mass m and frequency ω are given by

$$\psi_n(q)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right)\exp\left(-\frac{m\omega}{2\hbar}q^2\right)\qquad n=0,1,2,\ldots \tag{2}$$

with the eigenenergies

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)\,. \tag{3}$$

For convenience, we introduce the abbreviation

$$\xi_\mathrm{f}=\sqrt{\frac{m\omega}{\hbar}}q_\mathrm{f}\qquad \xi_\mathrm{i}=\sqrt{\frac{m\omega}{\hbar}}q_\mathrm{i}\,. \tag{4}$$

We start from the general expression of the propagator

$$K(q_\mathrm{f},q_\mathrm{i},t)=\sum_{n=0}^{\infty}\psi_n(q_\mathrm{f})\psi_n(q_\mathrm{i})\exp\left(-\frac{\mathrm{i}}{\hbar}E_nt\right) \tag{5}$$

and insert (2) and (3). Arranging the terms in way suitable for the application of the Mehler formula, we have

$$K(q_\mathrm{f},q_\mathrm{i},t)=\sqrt{\frac{m\omega}{\pi\hbar}}\sum_{n=0}^{\infty}\frac{H_n(\xi_\mathrm{f})H_n(\xi_\mathrm{i})}{n!}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{2}\right)^n\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t}\exp\left(-\frac{\xi_\mathrm{f}^2+\xi_\mathrm{i}^2}{2}\right)\,. \tag{6}$$

By means of the Mehler formula, setting $x=\xi_\mathrm{f}, y=\xi_\mathrm{i}$, and $z=\mathrm{e}^{-\mathrm{i}\omega t}$ we can express the propagator as

$$\begin{aligned}K(q_\mathrm{f},q_\mathrm{i},t)&=\sqrt{\frac{m\omega}{\pi\hbar(1-\mathrm{e}^{-2\mathrm{i}\omega t})}}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t}\exp\left(-\frac{\xi_\mathrm{f}^2+\xi_\mathrm{i}^2}{2}+\frac{2\xi_\mathrm{i}\xi_\mathrm{f}\mathrm{e}^{-\mathrm{i}\omega t}-(\xi_\mathrm{f}^2+\xi_\mathrm{i}^2)\mathrm{e}^{-2\mathrm{i}\omega t}}{1-\mathrm{e}^{-2\mathrm{i}\omega t}}\right)\\&=\sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}}\exp\left(\frac{\mathrm{i}}{2}(\xi_\mathrm{f}^2+\xi_\mathrm{i}^2)\frac{\cos(\omega t)}{\sin(\omega t)}-\mathrm{i}\frac{\xi_\mathrm{i}\xi_\mathrm{f}}{\sin(\omega t)}\right)\\&=\sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}}\exp\left(\mathrm{i}\frac{m\omega}{2\hbar}\frac{(q_\mathrm{f}^2+q_\mathrm{i}^2)\cos(\omega t)-2q_\mathrm{i}q_\mathrm{f}}{\sin(\omega t)}\right)\,.\end{aligned} \tag{7}$$

We thus arrive at the expression which we had obtained by evaluating the Feynman path integral for the harmonic oscillator.