Exercise: Derive the propagator of the harmonic oscillator from its stationary wave functions and eigenenergies by making use of the Mehler formula

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \left(\frac{z}{2}\right)^n = \frac{1}{\sqrt{1-z^2}} \exp\left(\frac{2xyz - (x^2 + y^2)z^2}{1-z^2}\right) . \tag{1}$$

Solution:

The stationary wave functions of a harmonic oscillator with mass m and frequency ω are given by

$$\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right) \exp\left(-\frac{m\omega}{2\hbar}q^2\right) \qquad n = 0, 1, 2, \ldots \tag{2}$$

with the eigenenergies

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) . \tag{3}$$

For convenience, we introduce the abbreviation

$$\xi_{\mathrm{f}} = \sqrt{\frac{m\omega}{\hbar}}q_{\mathrm{f}} \qquad \xi_{\mathrm{i}} = \sqrt{\frac{m\omega}{\hbar}}q_{\mathrm{i}} . \tag{4}$$

We start from the general expression of the propagator

$$K(q_{\mathrm{f}}, q_{\mathrm{i}}, t) = \sum_{n=0}^{\infty} \psi_n(q_{\mathrm{f}})\psi_n(q_{\mathrm{i}}) \exp\left(-\frac{\mathrm{i}}{\hbar}E_n t\right) \tag{5}$$

and insert (2) and (3). Arranging the terms in way suitable for the application of the Mehler formula, we have

$$K(q_{\mathrm{f}}, q_{\mathrm{i}}, t) = \sqrt{\frac{m\omega}{\pi\hbar}} \sum_{n=0}^{\infty} \frac{H_n(\xi_{\mathrm{f}})H_n(\xi_{\mathrm{i}})}{n!} \left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{2}\right)^n \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t} \exp\left(-\frac{\xi_{\mathrm{f}}^2 + \xi_{\mathrm{i}}^2}{2}\right) . \tag{6}$$

By means of the Mehler formula, setting $x = \xi_{\mathrm{f}}, y = \xi_{\mathrm{i}}$, and $z = \mathrm{e}^{-\mathrm{i}\omega t}$ we can express the propagator as

$$\begin{aligned}
K(q_{\mathrm{f}}, q_{\mathrm{i}}, t) &= \sqrt{\frac{m\omega}{\pi\hbar(1 - \mathrm{e}^{-2\mathrm{i}\omega t})}} \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t} \exp\left(-\frac{\xi_{\mathrm{f}}^2 + \xi_{\mathrm{i}}^2}{2} + \frac{2\xi_{\mathrm{i}}\xi_{\mathrm{f}}\mathrm{e}^{-\mathrm{i}\omega t} - (\xi_{\mathrm{f}}^2 + \xi_{\mathrm{i}}^2)\mathrm{e}^{-2\mathrm{i}\omega t}}{1 - \mathrm{e}^{-2\mathrm{i}\omega t}}\right) \\
&= \sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}} \exp\left(\frac{\mathrm{i}}{2}(\xi_{\mathrm{f}}^2 + \xi_{\mathrm{i}}^2)\frac{\cos(\omega t)}{\sin(\omega t)} - \mathrm{i}\frac{\xi_{\mathrm{i}}\xi_{\mathrm{f}}}{\sin(\omega t)}\right) \\
&= \sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}} \exp\left(\mathrm{i}\frac{m\omega}{2\hbar}\frac{(q_{\mathrm{f}}^2 + q_{\mathrm{i}}^2)\cos(\omega t) - 2q_{\mathrm{i}}q_{\mathrm{f}}}{\sin(\omega t)}\right) .
\end{aligned} \tag{7}$$

We thus arrive at the expression which we had obtained by evaluating the Feynman path integral for the harmoni oscillator.