Exercise: Verify the relation

$$\sum_{n=-\infty}^{+\infty}\delta(x-2\pi n)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\ell x}\,. \tag{1}$$

Solution:

The left-hand side is a 2π-periodic function which can be expressed in terms of a Fourier series

$$f(x)=\sum_{\ell=-\infty}^{+\infty}c_\ell\mathrm{e}^{\mathrm{i}\ell x} \tag{2}$$

with coefficients

$$c_\ell=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x f(x)\mathrm{e}^{-\mathrm{i}\ell x}\,. \tag{3}$$

In our specific case, we have

$$f(x)=\sum_{n=-\infty}^{+\infty}\delta(x-2\pi n) \tag{4}$$

and obtain for the Fourier coefficients

$$\begin{aligned}c_\ell&=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\pi}^{\pi}\mathrm{d}x\delta(x-2\pi n)\mathrm{e}^{-\mathrm{i}\ell x}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x\delta(x)\mathrm{e}^{-\mathrm{i}\ell x}\\&=\frac{1}{2\pi}\,.\end{aligned} \tag{5}$$

The Fourier series representation of the δ-comb thus follows as

$$\sum_{n=-\infty}^{+\infty}\delta(x-2\pi n)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\ell x}\,. \tag{6}$$