
Exercise: Check the semigroup property for the propagator of a free particle.

Solution:

The propagator of a free particle of mass m is given by

$$K(q_\mathrm{f}, q_\mathrm{i}, t) = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right)\,. \tag{1}$$

We want to show by an explicit calculation that this propagator fulfills the semigroup property

$$K(q_\mathrm{f}, q_\mathrm{i}, t_2+t_1) = \int_{-\infty}^{+\infty}\mathrm{d}q' K(q_\mathrm{f}, q', t_2)K(q', q_\mathrm{i}, t_1)\,. \tag{2}$$

We insert (1) into the right-hand side of (2), complete the square, and shift the integration variable q':

$$\begin{aligned}
\int_{-\infty}^{+\infty}\mathrm{d}q' K(q_\mathrm{f}, q', t_2)K(q', q_\mathrm{i}, t_1) &= \frac{m}{2\pi\mathrm{i}\hbar\sqrt{t_1t_2}}\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\frac{(q_\mathrm{f}-q')^2}{t_2}+\frac{(q'-q_\mathrm{i})^2}{t_1}\right)\right]\\
&= \frac{m}{2\pi\mathrm{i}\hbar\sqrt{t_1t_2}}\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\frac{t_1+t_2}{t_1t_2}q'^2-2\left(\frac{q_\mathrm{f}}{t_2}+\frac{q_\mathrm{f}}{t_2}\right)q'+\frac{q_\mathrm{f}^2}{t_2}+\frac{q_\mathrm{i}^2}{t_1}\right)\right]\\
&= \frac{m}{2\pi\mathrm{i}\hbar\sqrt{t_1t_2}}\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[\frac{\mathrm{i}m}{2\hbar}\frac{t_1+t_2}{t_1t_2}\left(q'-\frac{t_1t_2}{t_1+t_2}\left(\frac{q_\mathrm{f}}{t_2}+\frac{q_\mathrm{i}}{t_1}\right)\right)^2\right]\\
&\qquad\qquad\times\exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\frac{q_\mathrm{f}^2}{t_2}+\frac{q_\mathrm{i}^2}{t_1}-\frac{t_1t_2}{t_1+t_2}\left(\frac{q_\mathrm{f}}{t_2}+\frac{q_\mathrm{i}}{t_1}\right)\right)^2\right]\\
&= \frac{m}{2\pi\mathrm{i}\hbar\sqrt{t_1t_2}}\exp\left(\frac{\mathrm{i}m}{2\hbar}\frac{(q_\mathrm{f}-q_\mathrm{i})^2}{t_1+t_2}\right)\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left(\frac{\mathrm{i}m}{2\hbar}\frac{t_1+t_2}{t_1t_2}q'^2\right)\,.
\end{aligned} \tag{3}$$

With the Fresnel integral

$$\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(\mathrm{i}\alpha x^2\right) = \sqrt{\frac{\mathrm{i}\pi}{\alpha}} \tag{4}$$

we finally obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty}\mathrm{d}q' K(q_\mathrm{f}, q', t_2)K(q', q_\mathrm{i}, t_1) &= \sqrt{\frac{m}{2\pi\mathrm{i}\hbar(t_1+t_2)}}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(q_\mathrm{f}-q_\mathrm{i})^2}{2(t_1+t_2)}\right)\\
&= K(q_\mathrm{f}, q_\mathrm{i}, t_2+t_1)\,.
\end{aligned} \tag{5}$$

The propagator of the free particle satisfies the semigroup property as expected.