
Exercise: Check the choice of phases in the propagator of the harmonic oscillator

$$K(q_\mathrm{f},q_\mathrm{i},t)=\sqrt{\frac{m\omega}{2\pi\hbar|\sin(\omega t)|}}\exp\left(\mathrm{i}\frac{m\omega}{2\hbar}\frac{(q_\mathrm{f}^2+q_\mathrm{i}^2)\cos(\omega t)-2q_\mathrm{i}q_\mathrm{f}}{\sin(\omega t)}-\mathrm{i}\left(\frac{\pi}{4}+n\frac{\pi}{2}\right)\right) \tag{1}$$

with $n=\lfloor\omega t/\pi\rfloor$ by combining two propagators over time intervals $3\pi/4\omega$ to a propagator over the time interval $3\pi/2\omega$ by means of the semigroup property.

Solution:

Inserting the explicit values for the time intervals, we have

$$K(q_\mathrm{f},q_\mathrm{i},\frac{3\pi}{4\omega})=\sqrt{\frac{m\omega}{\sqrt{2}\pi\hbar}}\exp\left[\mathrm{i}\frac{m\omega}{2\hbar}\left(-(q_\mathrm{f}^2+q_\mathrm{i}^2)-2\sqrt{2}q_\mathrm{i}q_\mathrm{f}\right)-\mathrm{i}\frac{\pi}{4}\right] \tag{2}$$

and

$$K(q_\mathrm{f},q_\mathrm{i},\frac{3\pi}{2\omega})=\sqrt{\frac{m\omega}{2\pi\hbar}}\exp\left[\mathrm{i}\frac{m\omega}{\hbar}q_\mathrm{i}q_\mathrm{f}-\mathrm{i}\frac{3\pi}{4}\right] \tag{3}$$

By means of the semigroup property, we get

$$\begin{aligned}K(q_\mathrm{f},q_\mathrm{i},\frac{3\pi}{2\omega})&=\int_{-\infty}^{+\infty}\mathrm{d}q'K(q_\mathrm{f},q',\frac{3\pi}{4\omega})K(q',q_\mathrm{i},\frac{3\pi}{4\omega})\\&=\frac{m\omega}{\sqrt{2}\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[\mathrm{i}\frac{m\omega}{2\hbar}\left(-q_\mathrm{f}^2-q_\mathrm{i}^2-2q'^2-2\sqrt{2}(q_\mathrm{f}+q_\mathrm{i})q'\right)-\mathrm{i}\frac{\pi}{2}\right]\\&=\frac{m\omega}{\sqrt{2}\pi\hbar}\exp\left[-\mathrm{i}\frac{m\omega}{2\hbar}(q_\mathrm{f}^2+q_\mathrm{i}^2)-\mathrm{i}\frac{\pi}{2}\right]\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[-\mathrm{i}\frac{m\omega}{\hbar}\left(q'+\frac{1}{\sqrt{2}}(q_\mathrm{f}+q_\mathrm{i})\right)^2\right]\\&\qquad\times\exp\left[\mathrm{i}\frac{m\omega}{2\hbar}(q_\mathrm{f}+q_\mathrm{i})^2\right]\,.\end{aligned} \tag{4}$$

The Fresnel integral becomes

$$\int_{-\infty}^{+\infty}\mathrm{d}q'\exp\left[-\mathrm{i}\frac{m\omega}{\hbar}\left(q'+\frac{1}{\sqrt{2}}(q_\mathrm{f}+q_\mathrm{i})\right)^2\right]=\sqrt{\frac{\pi\hbar}{m\omega}}\exp\left(-\mathrm{i}\frac{\pi}{4}\right)\,. \tag{5}$$

Thus, we finally arrive at the expected result

$$K(q_\mathrm{f},q_\mathrm{i},\frac{3\pi}{2\omega})=\sqrt{\frac{m\omega}{2\pi\hbar}}\exp\left[\mathrm{i}\frac{m\omega}{\hbar}q_\mathrm{i}q_\mathrm{f}-\mathrm{i}\frac{3\pi}{4}\right]\,. \tag{6}$$