Exercise: Show that the propagator for a particle in an infinitely deep potential well

$$K(q_\mathrm{f}, q_\mathrm{i}, t) = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}} \sum_{n=-\infty}^{+\infty} \left[\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(2nL+q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right) - \exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(2nL-q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right)\right] \tag{1}$$

obtained from the Feynman path integral agrees with the representation

$$K(q_\mathrm{f}, q_\mathrm{i}, t) = \frac{2}{L}\sum_{j=1}^{\infty} \sin\left(\frac{\pi j}{L}q_\mathrm{f}\right)\sin\left(\frac{\pi j}{L}q_\mathrm{i}\right)\exp\left(-\mathrm{i}\frac{\hbar\pi^2 j^2}{2mL^2}t\right) \tag{2}$$

based on the eigenfunctions and eigenvalues of the Hamiltonian.

Solution:

We follow the same strategy as for the propagator for a particle on a ring and consider the first part of the sum in (1). The second part then immediately follows by the replacement $q_\mathrm{f} \to -q_\mathrm{f}$.

While for the particle on a ring, we were dealing with a 2π-periodic function, we now consider a function of period $2L$. The Fourier representation of the corresponding δ-comb is given by

$$\sum_{n=-\infty}^{+\infty} \delta(x-2Ln) = \frac{1}{2L}\sum_{\ell=-\infty}^{+\infty} \exp\left(\mathrm{i}\frac{\pi}{L}\ell x\right) . \tag{3}$$

Going through the same steps as for a particle on a ring, we have

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(2nL+q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right) &= \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}q\,\delta(q-q_\mathrm{f}+q_\mathrm{i}-2nL)\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mq^2}{2t}\right)\\ &= \frac{1}{2L}\sum_{\ell=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}q\exp\left(\mathrm{i}\frac{\pi}{L}\ell(q-q_\mathrm{f}+q_\mathrm{i})\right)\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mq^2}{2t}\right)\\ &= \frac{1}{2L}\sum_{\ell=-\infty}^{+\infty}\exp\left(\mathrm{i}\frac{\pi}{L}\ell(q_\mathrm{f}-q_\mathrm{i})\right)\int_{-\infty}^{+\infty}\mathrm{d}q\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mq^2}{2t}-\mathrm{i}\frac{\pi}{L}\ell q\right) .\end{aligned} \tag{4}$$

In the last line, we have replaced the summation index ℓ by $-\ell$. The Fresnel integral can be evaluated to yield

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}q\exp\left(\frac{\mathrm{i}}{\hbar}\frac{mq^2}{2t}-\mathrm{i}\frac{\pi}{L}\ell q\right) &= \int_{-\infty}^{+\infty}\mathrm{d}q\exp\left[\mathrm{i}\frac{m}{2\hbar t}\left(q-\frac{\hbar\pi\ell}{mL}t\right)^2\right]\exp\left(-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right)\\ &= \sqrt{\frac{2\pi\mathrm{i}\hbar t}{m}}\exp\left(-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right) .\end{aligned} \tag{5}$$

Inserting this result into the previous expression, we obtain

$$\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\sum_{n=-\infty}^{+\infty}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(2nL+q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right) = \frac{1}{2L}\sum_{\ell=-\infty}^{+\infty}\exp\left(\mathrm{i}\frac{\pi}{L}\ell(q_\mathrm{f}-q_\mathrm{i})-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right) . \tag{6}$$

Replacing q_f by $-q_\mathrm{f}$, we find for the second contribution to (1)

$$\sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\sum_{n=-\infty}^{+\infty}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m(2nL-q_\mathrm{f}-q_\mathrm{i})^2}{2t}\right)=\frac{1}{2L}\sum_{\ell=-\infty}^{+\infty}\exp\left(\mathrm{i}\frac{\pi}{L}\ell(-q_\mathrm{f}-q_\mathrm{i})-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right)\,. \tag{7}$$

Subtracting the two contributions, we realize that the terms depending on q_f can be combined to a sine function

$$K(q_\mathrm{f},q_\mathrm{i},t)=\frac{\mathrm{i}}{L}\sum_{\ell=-\infty}^{+\infty}\sin\Big(\frac{\pi}{L}\ell q_\mathrm{f}\Big)\exp\left(-\mathrm{i}\frac{\pi}{L}\ell q_\mathrm{i}-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right)\,. \tag{8}$$

Since the term with $\ell=0$ vanishes, we can rewrite this expression as

$$\begin{aligned}K(q_\mathrm{f},q_\mathrm{i},t)&=\frac{\mathrm{i}}{L}\sum_{\ell=1}^{+\infty}\sin\Big(\frac{\pi}{L}\ell q_\mathrm{f}\Big)\Big[\exp\Big(-\mathrm{i}\frac{\pi}{L}\ell q_\mathrm{i}\Big)-\exp\Big(\mathrm{i}\frac{\pi}{L}\ell q_\mathrm{i}\Big)\Big]\exp\left(-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right)\\&=\frac{2}{L}\sum_{\ell=1}^{+\infty}\sin\Big(\frac{\pi}{L}\ell q_\mathrm{f}\Big)\sin\Big(\frac{\pi}{L}\ell q_\mathrm{i}\Big)\exp\left(-\mathrm{i}\frac{\hbar\pi^2\ell^2}{2mL^2}t\right)\end{aligned} \tag{9}$$

which precisely agrees with the representation (2) of the propagator.