Robust nonlinear control for the fully-actuated Hexa-rotor: theory and experiments

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Abstract

This auxiliary document presents a concise control algorithm to stabilize the error equilibrium point of the well-known rigid-body equations modeled in the SO(3) group. Furthermore, we assume that the system is affected by exogenous and unknown signals. The control algorithm is based on the geometric approach.

I. Problem setting

Let consider the following UAV 6-DOF mathematical model,

$$\Sigma : \begin{cases} \dot{x} = v \\ \dot{v} = ge_3 - \frac{f}{m} Re_3 + \Delta_v(t) \end{cases} \tag{1}$$

$$\Pi : \begin{cases} \dot{R} = R\hat{\Omega} \\ \dot{\Omega} = -J^{-1}\Omega \times J\Omega + J^{-1}\tau + \Delta_\Omega(t) \end{cases} \tag{2}$$

where $\hat{(\cdot)} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is $\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$ with $x = [x_1,\ x_2,\ x_3]^\intercal$ in which $\mathfrak{so}(3)$ is a Lie algebra in $SO(3)$. Such operation has an inverse given by $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ which basically transforms a skew-symetric matrix into a vector in $\mathbb{R}^3$. $(\Delta_v(t), \Delta_\Omega(t))$ are the unknown forces and moment due to unmodeled dynamics and exogenous disturbances. The rotation matrix R is given by [1]

$$R = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta. \end{pmatrix} \tag{3}$$

Assumption 1: The external disturbance $\Delta_\Omega(t)$ is bounded as,

$$\|\Delta_\Omega(t)\|_1 \leq c\|e_\Omega\|_1, \tag{4}$$

where $c \in \mathbb{R}_{>0}$.

II. Main result

A. Attitude control

The attitude control is based on the SO(3), and the aim is to stabilize both equations in subsystem Π to a desired attitude position and angular velocity given by

$$\dot{R}_d = R_d\hat{\Omega}_d, \text{ and } \hat{\Omega}_d = R_d^\intercal \dot{R}_d \tag{5}$$

respectively. Notice that we need the time-derivative of the given R_d to get $\hat{\Omega}_d$. In practice this is numerically computed in the autopilot.

Once we have defined the desired attitude states, to design the controller we define the following configuration error function, and tracking error functions, respectively [2]

$$\begin{aligned} \Psi_{\mathrm{SO(3)}}(R) &= \frac{1}{2}\operatorname{Tr}(I_3 - R_d^\intercal R) \in \mathbb{R}, \\ e_R &= \frac{1}{2}(R_d^\intercal R - R^\intercal R_d)^\vee = [e_R(1),\ e_R(2),\ e_R(3)]^\intercal \in \mathbb{R}^3, \\ e_\Omega &= \Omega - R^\intercal R_d \Omega_d = [e_\Omega(1),\ e_\Omega(2),\ e_\Omega(3)]^\intercal \in \mathbb{R}^3. \end{aligned} \tag{6}$$

It is clear that $\Psi_{\mathrm{SO(3)}}(R) = 0$, $e_R = e_\Omega = 0_{3\times3}$, where $0_{3\times3}$ is the zero vector in $\mathbb{R}^3$, when $R = R_d$, and $\Omega = \Omega_d$. Notice that we have used the right attitude error defined by $R_{e,r} = R_d^\intercal R$.

The attitude controller is presented in the following proposition.

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Proposition 1 (Main result): Consider the attitude aircraft dynamics given by Π in (2). The control algorithm

$$\tau = -J\left(K_R e_R + K_\Omega e_\Omega + K_3 \upsilon_R + K_4 \upsilon_\Omega\right) + \Omega \times J\Omega - J(\hat{\Omega} R^\intercal R_d \Omega_d - R^\intercal R_d \dot{\Omega}_d), \tag{7}$$

with vector signals and matrices $K_R = \operatorname{diag}[k_R,\ k_R,\ k_R]$, where $k_R \in \mathbb{R}_{>0}$,

$$\upsilon_R = \begin{pmatrix} |e_R(1)|^\alpha \operatorname{sgn} e_R(1) \\ |e_R(2)|^\alpha \operatorname{sgn} e_R(2) \\ |e_R(3)|^\alpha \operatorname{sgn} e_R(3) \end{pmatrix},\ \upsilon_\Omega = \begin{pmatrix} |e_\Omega(1)|^\alpha \operatorname{sgn} e_\Omega(1) \\ |e_\Omega(2)|^\alpha \operatorname{sgn} e_\Omega(2) \\ |e_\Omega(3)|^\alpha \operatorname{sgn} e_\Omega(3) \end{pmatrix} \tag{8}$$

with a positive real number $\alpha < 1$, exponentially stabilizes the zero equilibrium points (6).

Proof: Let us begin by computing the error dynamics of equations (6) as follows,

$$\begin{aligned} \dot{\Psi}_{\mathrm{SO(3)}} &= \frac{1}{2} e_\Omega^\intercal \left(R_d^\intercal R - R^\intercal R_d\right)^\vee = e_R^\intercal e_\Omega \\ \dot{e}_R &= \frac{1}{2}\left(R_d^\intercal R \hat{e}_\Omega + \hat{e}_\Omega R^\intercal R_d\right) = \underbrace{\frac{1}{2}\left(\operatorname{Tr}[R^\intercal R_d] I_3 - R^\intercal R_d\right)}_{C(R_d,R)} e_\Omega \\ \dot{e}_\Omega &= J^{-1}\tau - J^{-1}\Omega \times J\Omega + \left(\hat{\Omega} R^\intercal R_d \Omega_d - R^\intercal R_d \dot{\Omega}_d\right) + \Delta_\Omega(t). \end{aligned} \tag{9}$$

The last equation with the proposed control (7) results in:

$$\dot{e}_\Omega = -\left(K_R e_R + K_\Omega e_\Omega + K_3 \upsilon_R + K_4 \upsilon_\Omega\right) + \Delta_\Omega(t). \tag{10}$$

Let the candidate Lyapunov function

$$V(R, R_d, \Omega, \Omega_d) = \underbrace{e_R^\intercal e_\Omega}_{V_1} + \underbrace{\frac{1}{2} e_\Omega^\intercal J A e_\Omega}_{V_2} + \underbrace{\frac{d}{2}\operatorname{Tr}\left(I_3 - R_d^\intercal R\right)}_{V_3} + \underbrace{\operatorname{sgn} e_\Omega{}^\intercal B e_\Omega}_{V_4}, \tag{11}$$

where $A \in \mathbb{R}^{3\times 3}$ is a positive definite matrix, $B \in \mathbb{R}^{3\times 3}$ is a diagonal and positive definite matrix, and

$$\operatorname{sgn} e_R = \begin{pmatrix} \operatorname{sgn} e_R(1) \\ \operatorname{sgn} e_R(2) \\ \operatorname{sgn} e_R(3) \end{pmatrix},\ \operatorname{sgn} e_\Omega = \begin{pmatrix} \operatorname{sgn} e_\Omega(1) \\ \operatorname{sgn} e_\Omega(2) \\ \operatorname{sgn} e_\Omega(3) \end{pmatrix}. \tag{12}$$

Now, let us compute the time-derivative of V. For easy interpretation, let begin by computing $\dot{V}_1$ as follows

$$\dot{V}_1 = \dot{e}_R^\intercal e_\Omega + e_R^\intercal \dot{e}_\Omega = e_\Omega^\intercal C(R_d,R)^\intercal e_\Omega + e_R^\intercal\left(J^{-1}\tau - J^{-1}\Omega \times J\Omega + \left(\hat{\Omega} R^\intercal R_d \Omega_d - R^\intercal R_d \dot{\Omega}_d\right) + \Delta_\Omega(t)\right) \tag{13}$$

then, we substitute (7) in the previous equation,

$$\dot{V}_1 = e_\Omega^\intercal C(R_d,R)^\intercal e_\Omega + e_R^\intercal\Big(-K_R e_R - K_\Omega e_\Omega - K_3 \upsilon_R - K_4 \upsilon_\Omega + \Delta_\Omega(t)\Big) \tag{14}$$

and since $\|C(R_d,R)\|_2 \le 1$, it follows that

$$\dot{V}_1 \le \|e_\Omega\|_2^2 - e_R^\intercal K_R e_R - e_R^\intercal K_\Omega e_\Omega - |e_R|^\intercal K_3 |e_R|^\alpha + \|K_4\| |e_R|^\intercal |e_\Omega|^\alpha + e_R^\intercal \Delta_\Omega(t) \tag{15}$$

where

$$|e_R| = \begin{pmatrix} |e_R(1)| \\ |e_R(2)| \\ |e_R(3)| \end{pmatrix},\quad |e_R|^\alpha = \begin{pmatrix} |e_R(1)|^\alpha \\ |e_R(2)|^\alpha \\ |e_R(3)|^\alpha \end{pmatrix},\quad |e_\Omega| = \begin{pmatrix} |e_\Omega(1)| \\ |e_\Omega(2)| \\ |e_\Omega(3)| \end{pmatrix},\quad |e_\Omega|^\alpha = \begin{pmatrix} |e_\Omega(1)|^\alpha \\ |e_\Omega(2)|^\alpha \\ |e_\Omega(3)|^\alpha \end{pmatrix}. \tag{16}$$

Now, let us continue by computing the time-derivative of V_2 in (11) along the trajectories of (10) as follows,

$$\begin{aligned} \dot{V}_2 &= e_\Omega^\intercal J A \dot{e}_\Omega = e_\Omega^\intercal J A\left(-K_R e_R - K_\Omega e_\Omega - K_3 \upsilon_R - K_4 \upsilon_\Omega + \Delta_\Omega(t)\right) \\ &= -e_\Omega^\intercal (JAK_R) e_R - e_\Omega^\intercal (JAK_\Omega) e_\Omega - e_\Omega^\intercal (JAK_3) \upsilon_R - e_\Omega^\intercal (JAK_4) \upsilon_\Omega + e_\Omega^\intercal (JA) \Delta_\Omega(t) \\ &\le -e_R^\intercal (JAK_R) e_\Omega - e_\Omega^\intercal (JAK_\Omega) e_\Omega + |e_\Omega|^\intercal JAK_3 |e_R|^\alpha - |e_\Omega|^\intercal JAK_4 |e_\Omega|^\alpha + e_\Omega^\intercal (JA) \Delta_\Omega(t). \end{aligned} \tag{17}$$

Let us compute $\dot{V}_3$ as,

$$\dot{V}_3 = \frac{d}{2}\frac{d}{dt}\operatorname{Tr}\left(I_3 - R_d^\intercal R\right) = -\frac{d}{2}\operatorname{Tr}\left(\frac{d}{dt}\left(R_d^\intercal R\right)\right) \tag{18}$$

and since

$$\frac{d}{dt}\left(R_d^\intercal R\right) = \frac{d}{dt}(R_d^\intercal) R + R_d^\intercal \frac{d}{dt}(R) = \hat{\Omega}_d^\intercal R_d^\intercal R + R_d^\intercal R \hat{\Omega} = R_d^\intercal R\left(\underbrace{R^\intercal R_d \hat{\Omega}_d^\intercal R_d^\intercal R}_{-(R^\intercal R_d \Omega_d)^\wedge} + \hat{\Omega}\right), \tag{19}$$

where we have used (2) and (5) together with the property $R\hat{a}R^\intercal = (Ra)^\wedge$ where $a \in \mathbb{R}^3$ and $R \in SO(3)$, and the fact that $\hat{\Omega}_d^\intercal = -\hat{\Omega}_d$. Then, from the last equation of (6) it follows that,

$$\frac{d}{dt}\left(R_d^\intercal R\right) = R_d^\intercal R \hat{e}_\Omega. \tag{20}$$

Finally, from the last expression it is clear that,

$$\dot{V}_3 = -\frac{d}{2}\operatorname{Tr}\left(R_d^\intercal R \hat{e}_\Omega\right) = \frac{d}{2} e_\Omega^\intercal (R_d^\intercal R - R^\intercal R_d) = e_\Omega^\intercal D e_R = e_R^\intercal D e_\Omega, \tag{21}$$

where we have used the property $\operatorname{Tr}(A\hat{a}) = -a^\intercal (A - A^\intercal)^\vee$, where a is defined as above, and $A \in \mathbb{R}^{3\times 3}$, and the second equation of (6). The diagonal matrix D is chosen accordingly.

We finally compute the time derivative of V_4 as follows,

$$\begin{aligned} \dot{V}_4 &= \operatorname{sgn} e_\Omega{}^\intercal B\left(-K_R e_R - K_\Omega e_\Omega - K_3 \upsilon_R - K_4 \upsilon_\Omega + \Delta_\Omega(t)\right) \\ &\leq \operatorname{sgn} e_R{}^\intercal (BK_R) e_R - \operatorname{sgn} e_\Omega{}^\intercal (BK_\Omega) e_\Omega + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3) |e_R|^{\frac{\alpha}{2}} - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4)|e_\Omega|^{\frac{\alpha}{2}} + \|B\Delta_\Omega(t)\|. \end{aligned} \tag{22}$$

Then, we are ready to compute the following,

$$\begin{aligned} \dot{V} \leq{}& \|e_\Omega\|_2^2 - e_R^\intercal K_R e_R - e_R^\intercal K_\Omega e_\Omega - |e_R|^\intercal K_3 |e_R|^\alpha + \|K_4\| |e_R|^\intercal |e_\Omega|^\alpha + e_R^\intercal \Delta_\Omega(t) \\ & - e_R^\intercal (JAK_R) e_\Omega - e_\Omega^\intercal (JAK_\Omega) e_\Omega + |e_\Omega|^\intercal JAK_3 |e_R|^\alpha - |e_\Omega|^\intercal JAK_4 |e_\Omega|^\alpha + e_\Omega^\intercal (JA)\Delta_\Omega(t) \\ & + e_R^\intercal D e_\Omega + \operatorname{sgn} e_R{}^\intercal (BK_R) e_R - \operatorname{sgn} e_\Omega{}^\intercal (BK_\Omega) e_\Omega + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3)|e_R|^{\frac{\alpha}{2}} - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4)|e_\Omega|^{\frac{\alpha}{2}} + \|B\Delta_\Omega(t)\|. \end{aligned} \tag{23}$$

We group all the terms as follows,

$$\dot{V} \leq -e_R^\intercal K_R e_R - |e_R|^\intercal K_3 |e_R|^\alpha + \operatorname{sgn} e_R{}^\intercal (BK_R) e_R + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3)|e_R|^{\frac{\alpha}{2}} \tag{24}$$
$$- e_\Omega^\intercal (JAK_\Omega) e_\Omega - |e_\Omega|^\intercal JAK_4 |e_\Omega|^\alpha - \operatorname{sgn} e_\Omega{}^\intercal (BK_\Omega) e_\Omega - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4)|e_\Omega|^{\frac{\alpha}{2}} + \|e_\Omega\|_2^2 \tag{25}$$
$$- e_R^\intercal K_\Omega e_\Omega - e_R^\intercal (JAK_R) e_\Omega + e_R^\intercal D e_\Omega \tag{26}$$
$$+ e_R^\intercal \Delta_\Omega(t) + e_\Omega^\intercal (JA)\Delta_\Omega(t) + \|B\Delta_\Omega(t)\| \tag{27}$$
$$+ \|K_4\| |e_R|^\intercal |e_\Omega|^\alpha + |e_\Omega|^\intercal JAK_3 |e_R|^\alpha. \tag{28}$$

Let's begin with the positive terms of (24) by noting that

$$\begin{aligned} \operatorname{sgn} e_R{}^\intercal (BK_R) e_R + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3)|e_R|^{\frac{\alpha}{2}} &= (|e_R|^\alpha)^\intercal (BK_R)|e_R|^\beta + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3)|e_R|^{\frac{\alpha}{2}} \\ &\leq 2^\beta (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_R)|e_R|^{\frac{\alpha}{2}} + (|e_R|^{\frac{\alpha}{2}})^\intercal (BK_3)|e_R|^{\frac{\alpha}{2}} \\ &\leq (|e_R|^{\frac{\alpha}{2}})^\intercal (B[2^\beta K_R + K_3])|e_R|^{\frac{\alpha}{2}} \end{aligned} \tag{29}$$

where we assume that $\alpha + \beta = 1$, and the fact that $\||e_R|\| \leq 2$. And then, (24) is simplified to

$$-e_R^\intercal K_R e_R - |e_R|^\intercal K_3 |e_R|^\alpha + (|e_R|^{\frac{\alpha}{2}})^\intercal (B[2^\beta K_R + K_3])|e_R|^{\frac{\alpha}{2}} \tag{30}$$

We continue with (25):

$$-e_\Omega^\intercal (JAK_\Omega - I_{3\times 3}) e_\Omega - |e_\Omega|^\intercal JAK_4 |e_\Omega|^\alpha - \operatorname{sgn} e_\Omega{}^\intercal (BK_\Omega) e_\Omega - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4)|e_\Omega|^{\frac{\alpha}{2}} \tag{31}$$

Next, (26) can simplified as:

$$-e_R^\intercal K_\Omega e_\Omega - e_R^\intercal (JAK_R) e_\Omega + e_R^\intercal D e_\Omega = 0 \tag{32}$$

as long as we choose $D = K_\Omega + JAK_R$.

Then, (27) is simplified by taking into account assumption 1 as follows,

$$e_R^\intercal \Delta_\Omega(t) + e_\Omega^\intercal (JA)\Delta_\Omega(t) + \|B\Delta_\Omega(t)\| \leq c\|e_R^\intercal\| \|e_\Omega\| + \underbrace{e_\Omega^\intercal (cJA) e_\Omega}_{\text{1-norm}} + \underbrace{(\operatorname{sgn} e_\Omega)^\intercal cBe_\Omega}_{\text{1-norm}} \tag{33}$$

Finally, we simplify (28) by using the Young inequality, and considering that $\alpha = 1/2$:

$$\begin{aligned} |e_R|^\intercal K_4 |e_\Omega|^\alpha + |e_\Omega|^\intercal JAK_3 |e_R|^\alpha &\leq \|(|e_\Omega|^\alpha)^\intercal\|_1 \|K_4\|_1 \||e_R|\|_2 + \||e_\Omega|^\intercal\|_1 \|JAK_3\|_1 \||e_R|^\alpha\|_2 \\ &\leq \frac{\|K_4\|^2 |e_R|^\intercal |e_R|}{2} + \frac{(|e_\Omega|^{\frac{1}{2}})^\intercal |e_\Omega|^{\frac{1}{2}}}{2} + (|e_\Omega|^{\frac{1}{2}})^\intercal (2^\alpha \|JAK_3\|_1 I_{3\times 3})|e_\Omega|^{\frac{1}{2}}. \end{aligned} \tag{34}$$

We put together all the simplifications:

$$\begin{aligned}\dot{V} \leq & -e_R^\intercal K_R e_R - |e_R|^\intercal K_3 |e_R|^\alpha + (|e_R|^{\frac{\alpha}{2}})^\intercal (B[2^\beta K_R + K_3])|e_R|^{\frac{\alpha}{2}} \\ & - e_\Omega^\intercal (JAK_\Omega - I_{3\times 3})e_\Omega - |e_\Omega|^\intercal JAK_4 |e_\Omega|^\alpha - \operatorname{sgn} e_\Omega{}^\intercal (BK_\Omega)e_\Omega - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4)|e_\Omega|^{\frac{\alpha}{2}} \\ & + c\|e_R^\intercal\|\|e_\Omega\| + e_\Omega^\intercal (cJA)e_\Omega + (\operatorname{sgn} e_\Omega)^\intercal cBe_\Omega \\ & + \left(|e_\Omega|^{\frac{\alpha}{2}}\right)^\intercal (2\|K_4\|_1 I_{3\times 3})\left(|e_\Omega|^{\frac{\alpha}{2}}\right) + (\operatorname{sgn} e_\Omega)^\intercal (2^\alpha \|JAK_3\|_1 I_{3\times 3})e_\Omega,\end{aligned} \tag{35}$$

which can be rearranged as,

$$\begin{aligned}\dot{V} \leq & -e_R^\intercal K_R e_R - (|e_R|^{\frac{1}{2}(\alpha+1)})^\intercal K_3 |e_R|^{\frac{1}{2}(\alpha+1)} + (|e_R|^{\frac{\alpha}{2}})^\intercal (B[2^\beta K_R + K_3])|e_R|^{\frac{\alpha}{2}} \\ & - e_\Omega^\intercal (JAK_\Omega - I_{3\times 3} - cJA)e_\Omega - (|e_\Omega|^{\frac{1}{2}(\alpha+1)})^\intercal JAK_4 |e_\Omega|^{\frac{1}{2}(\alpha+1)} \\ & - (|e_\Omega|^{1/2})^\intercal (BK_\Omega - cB - 2^\alpha\|JAK_3\|_1 I_{3\times 3})|e_\Omega|^{1/2} - (|e_\Omega|^{\frac{\alpha}{2}})^\intercal (BK_4 - 2\|K_4\|_1 I_{3\times 3})|e_\Omega|^{\frac{\alpha}{2}} \\ & + c\|e_R^\intercal\|\|e_\Omega\|.\end{aligned} \tag{36}$$

The last two terms in the first row of the previous equation can be arranged as:

$$\begin{aligned}-(|e_R|^{\frac{1}{2}(\alpha+1)})^\intercal K_3 |e_R|^{\frac{1}{2}(\alpha+1)} + (|e_R|^{\frac{\alpha}{2}})^\intercal (B[2^\beta K_R + K_3])|e_R|^{\frac{\alpha}{2}} = -(|e_R|^{\frac{1}{2}(\alpha+1)})^\intercal \times \\ \left[K_3 - I_{3\times 3} \odot |e_R|^{-1} 1^\intercal B(2^\beta K_R + K_3)\right] |e_R|^{\frac{1}{2}(\alpha+1)}\end{aligned} \tag{37}$$

where

$$|e_R|^{-1} = \begin{pmatrix} |e_R(1)|^{-1} \\ |e_R(2)|^{-1} \\ |e_R(3)|^{-1} \end{pmatrix} \tag{38}$$

and $\odot$ is the Hadamard product, and $1^\intercal = [1,\ 1,\ 1]$. Notice that the RHS of (37), must be positive, and hence

$$\begin{aligned}K_3 &> I_{3\times 3} \odot |e_R|^{-1} 1^\intercal B(2^\beta K_R + K_3) \\ (K_3)(2^\beta K_R + K_3)^{-1}(B)^{-1} &> I_{3\times 3} \odot |e_R|^{-1} 1^\intercal \\ I_{3\times 3} \odot |e_R| 1^\intercal &> (B)(2^\beta K_R + K_3)(K_3)^{-1},\end{aligned} \tag{39}$$

which after simple computations it follows that,

$$|e_R(i)| > b_i \frac{2^\beta k_{R_i} + k_{3_i}}{k_{3_i}}, \tag{40}$$

for $i = \{1, 2, 3\}$. From (40) one can choose k_{3_i} arbitrarily large, and b_i arbitrarily small so that the right hand side of (40) is arbitrarily close to zero.

Thus,

$$\dot{V} \leq -\zeta^\intercal Z \zeta, \tag{41}$$

where

$$\zeta = \left(|e_R|, \quad |e_R|^{\frac{1}{2}(\alpha+1)}, \quad |e_\Omega|, \quad |e_\Omega|^{\frac{1}{2}(\alpha+1)}, \quad |e_\Omega|^{\frac{\alpha}{2}}, \quad |e_\Omega|^{\frac{1}{2}}\right)^\intercal, \tag{42}$$

and

$$Z = \begin{pmatrix} K_R - \frac{\|K_4\|^2}{2} & 0 & -\frac{c}{2} & 0 & 0 & 0 \\ 0 & K_3 - \epsilon I_{3\times 3} & 0 & 0 & 0 & 0 \\ -\frac{c}{2} & 0 & JAK_\Omega - I_{3\times 3} - cJA & 0 & 0 & 0 \\ 0 & 0 & 0 & JAK_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & BK_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & BK_\Omega - cB - (2^\alpha\|JAK_3\|_1 - \frac{1}{2})I_{3\times 3} \end{pmatrix}, \tag{43}$$

where ϵ is a small positive number comming from (40). Besides,

$$\begin{aligned}BK_\Omega - cB - \left(2^\alpha\|JAK_3\|_1 - \frac{1}{2}\right) I_{3\times 3} &> 0_{3\times 3} \\ B(K_\Omega - cI_{3\times 3}) &> \left(2^\alpha\|JAK_3\|_1 - \frac{1}{2}\right) I_{3\times 3} \\ K_\Omega &> B^{-1}\left(2^\alpha\|JAK_3\|_1 - \frac{1}{2} + cB\right) I_{3\times 3}\end{aligned} \tag{44}$$

and the following inequalities must be hold,

$$
\begin{aligned}
&K_R > \frac{\|K_4\|^2}{2} \\
&K_3 > \epsilon I_{3\times 3} \\
&K_\Omega > A^{-1}J^{-1}\left(cJA + I_{3\times 3}\right).
\end{aligned}
\tag{45}
$$

Achieving that all the above inequalities hold is possible, and thus it is clear that all the eigenvalues of the above matrix are positive. This implies that such a matrix is positive definite and the stability is achieved. ■

III. Conclusions

This note presented a geometric control algorithm capable of stabilizing the set of rigid body equations solving the trajectory tracking problem. The controller can be applied to a series of robotic systems that are modeled by the dynamic attitude equations, such as drones, marine vehicles, and terrestrial vehicles.

References

[1] B. Etkin and L. Reid, *Dynamics of Flight Stability and Control*. John Wiley and Sons, 1996.

[2] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*. New York, NY, USA: Springer-Verlag New York, 2005.