# Microeconomics Problem Set 2

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**Question 1:**

Derivation to expressions needed to verify the Slutsky equation. To find the Marshallian demand function $x(p, w)$: we first set up the utility maximization problem

$$\max_{x_1, x_2} x_1 + 2\sqrt{x_2}$$
$$subject\,to\, p_1x_1 + p_2x_2 = w$$

Since there is no interior solution, the budget constraint is binding. Then we use the lagrangian:

$$Lagrangian\ function:$$
$$\mathcal{L} = x_1 + 2\sqrt{x_2} + \lambda[w - p_1x_1 - p_2x_1] \tag{1}$$
$$FOC\ (assuming\ no\ corner\ solutions):$$
$$\frac{\partial}{\partial x_1} = 1 - \lambda p_1 = 0 \tag{2}$$
$$\frac{\partial}{\partial x_2} = \frac{2}{2\sqrt{x_2}} - \lambda p_2 = 0 \tag{3}$$
$$\frac{\partial}{\partial \lambda} = w - p_1x_1 - p_2x_2 = 0 \tag{4}$$

From (2) and (3), we get $\sqrt{x_2} = \frac{p_1}{p_2}$ and $x_2 = \frac{p_1^2}{p_2^2}$. When we plug into our original budget constraint, we get $w = p_1x_1 + p_2(\frac{p_1^2}{p_2^2}) = p_1x_1 + \frac{p_1^2}{p_2}$.

If we solve for $x_1$ and $x_2$, we get the Marshallian Demands as

$$x_1(p, w) = \frac{w - \frac{p_1^2}{p_2}}{p_1} = \frac{p_2w - p_1^2}{p_1p_2}$$

$$x_2(p, w) = \frac{p_1^2}{p_2^2}$$

To find the indirect utility function, substitute $x_1$ and $x_2$ from (b) in the original $u(x_1, x_2)$: $V(x_1(p, w), x_2(p, w)) = \frac{p_2w-p_1^2}{p_1p_2} + \frac{2p_1}{p_2} = \frac{p_2w+p_1^2}{p_1p_2}$

From the indirect utility function, I can solve for $e(p, u)$ by fixing the value of utility as constant and solving for $w$. Since $V = \bar{u} = \frac{p_2w+p_1^2}{p_1p_2}$

$$\bar{u}p_1p_2 = p_2w + p_1^2$$
$$e(p, u) = w = \frac{\bar{u}p_1p_2 - p_1^2}{p_2}$$

Using the expenditure function, I can find the Hicksian demand function $x_1^h$ and $x_2^h$ by differentiating the expenditure function with respect to the associated prices:

$$x_1^h = \frac{\partial e}{\partial p_1} = \frac{1}{p_2} * (p_2 U - 2p_1) = \frac{p_2 U - 2p_1}{p_2}$$
$$x_2^h = \frac{\partial e}{\partial p_2} = -\frac{1}{p_2^2} * (p_1 p_2 U - p_1^2) + \frac{1}{p_2} * (p_1 U) = \frac{p_1^2}{p_2^2}$$

Moving forward, I will use the following expressions: $x_2 = \frac{p_1^2}{p_2^2}$, $x_2^h = \frac{p_1^2}{p_2^2}$, $x_1 = \frac{wp_2 - p_1^2}{p_1 p_2}$, $x_1^h = u - \frac{2p_1}{p_2}$

a) Verify Slutsky equation holds for cross-price changes:

**We want to verify a price change for $p_1$ on $x_2$ by:**

Total effect $= \frac{\partial x_2}{\partial p_1} = \frac{2}{p_2} * \frac{p_1}{p_2} = \frac{2p_1}{p_2^2}$

Substitution effect $= \frac{\partial x_2^h}{\partial p_1} = 2 * \frac{p_1}{p_2} \frac{1}{p_2} = \frac{2p_1}{p_2^2}$

Income effect $= -x_1 * \frac{\partial x_2}{\partial w} = -x_1 * 0 = 0$

We see that TE = SE - IE as $\frac{2p_1}{p_2^2} = \frac{2p_1}{p_2^2} + 0$

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**We want to verify a price change for $p_2$ on $x_1$ by:**

Total effect $= \frac{\partial x_1}{\partial p_2} = (-p_1)(-1)(p_2^{-2}) = \frac{p_1}{p_2^2}$

Income effect $= -x_2 * \frac{\partial x_1}{\partial p_2} = -x_2 * \frac{1}{p_1} = \frac{p_1^2}{p_2^2} * \frac{1}{p_1} = -\frac{p_1}{p_2^2}$ a Substitution effect $= \frac{\partial x_1^h}{\partial p_2} = \frac{2p_1}{p_2^2}$

We see that TE = SE - IE as $\frac{p_1}{p_2^2} = \frac{2p_1}{p_2^2} - \frac{p_1}{p_2^2}$

**The Slutsky equation holds for cross-price changes**

b) $p_1 \to p_1'$, with $p_1 < p_1'$ and $p_2$ no change.

$u_o$: initial utility

$u_1$: final utility

Hicksian demand function:

$$x^h = (u - \frac{2p_1}{p_2}, \frac{p_1^2}{p_2^2})$$

Expression for EV is:

$$EV(p, p', w) = \int_{p_1'}^{p_1} h_1(\rho, p_2, u') \partial\rho = \int_{p_1'}^{p_1} u - \frac{2\rho}{p_2} \partial\rho$$

Then $EV = u'(p_1 - p_1') - \frac{p_1^2}{p_2} + \frac{p_1'^2}{p_2}$

Expression for CV is:

$$CV(p,p',w) = \int_{p_1'}^{p_1} h_1(\rho, p_2, u^o)\partial\rho$$

Then $CV = u^o(p_1 - p_1') - \frac{p_1^2}{p_2} + \frac{p_1'^2}{p_2}$

They are almost the same, but knowing $p_1 < p_1'$ means $(p_1 - p_1') < 0$. This means the first term for both expressions $< 0$. Also, since wealth remained constant but the consumer faced a higher $p_1' > p_1$, this gives $u_o < u_1$. The first term of EV is a smaller negative number than the first term of CV as $u'(p_1 - p_1') > u_o(p_1 - p_1')$. Then, since the second term in each expression is the same, $EV(p,p',w) > CV(p,p',w)$. This is consistent with what we saw in class, as the EV $>$ CV is true for a normal good.

c) $p_2 \to p_2'$, with $p_2 < p_2'$ and $p_1$ no change.

Hicksian demand function:

$$x^h = (u - \frac{2p_1}{p_2}, (\frac{p_1}{p_2})^2)$$

Expression for EV is:

$$EV(p,p',w) = \int_{p_2'}^{p_2} h_2(p_1, \rho, u')\partial\rho = \int_{p_2'}^{p_2} \frac{p_1^2}{\rho^2}\partial\rho$$

Then

$$EV = \frac{p_1^2}{p_2'} - \frac{p_1^2}{p_2}$$

Expression for CV is:

$$CV(p,p',w) = \int_{p_2'}^{p_2} h_2(p_1, \rho, u^o)\partial\rho$$

Then $CV = \frac{p_1^2}{p_2'} - \frac{p_1^2}{p_2}$

The comparison differs from part (b) because now here, $CV = EV$. The income effect on $x_2$ is 0, as $x_2 = \frac{p_1^2}{p_2^2}$ is independent of income.

## Question 2

Look at changes not only in price, but also in wealth. $(p^o, w^o) \to (p^1, w^1)$

CV $= e(p^1, v(p^1, w^1)) - e(p^1, v(p^o, w^o))$

EV $= e(p^o, v(p^1, w^1)) - e(p^o, v(p^o, w^o))$

Expenditure function $= e(p, u) = \frac{u}{p_1^{-1} p_2^{-1}}$

$(p^o, w^o) = (1, 1)$ and $w^o = 1$

a) Indirect money-metric utility function: $e(\bar{p}, v(p, w))$. The indirect money-metric utility function uses $v(p, w)$ instead of $u(x_1, x_2)$.

I will use the identity $e(p, v(p, w)) = w$.

Replace $u = v(p, w)$ in $e(p, u) = \frac{u}{p_1^{-1} p_2^{-1}} = \frac{v(p,w)}{p_1^{-1} p_2^{-1}} = w$. Then solve for $v(p, w) = w(p_1^{-1} + p_2^{-1})$, to get $e(\bar{p}, v(p, w)) = \frac{w(p_1^{-1}+p_2^{-1})}{\bar{p}_1^{-1}+\bar{p}_2^{-1}}$.

b) Evaluate EV and CV if $p^1 = p^o$ and $w^o \to w^1 = 2$.

$CV = e(p^1, v(p^1, w^1)) - e(p^1, v(p^o, w^o)) = \frac{2(1^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} - \frac{1(1^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} = 1$

$EV = e(p^o, v(p^1, w^1)) - e(p^o, v(p^o, w^o))\frac{2(1^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} - \frac{1(1^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} = 1$

Prices did not change as $p^o = p^1$ so CV = EV.

c) Evaluate EV and CV if $p^o \to p^2 = (2, 1)$ and $w^o \to w^2 = 2.5$

$CV = e(p^1, v(p^1, w^1)) - e(p^1, v(p^o, w^o)) = \frac{2.5(2^{-1}+1^{-1})}{\bar{2}^{-1}+\bar{1}^{-1}} - \frac{1(1^{-1}+1^{-1})}{\bar{2}^{-1}+\bar{1}^{-1}} = \frac{7}{6}$

$EV = e(p^o, v(p^1, w^1)) - e(p^o, v(p^o, w^o)) = \frac{2.5(2^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} - \frac{1(1^{-1}+1^{-1})}{\bar{1}^{-1}+\bar{1}^{-1}} = \frac{7}{8}$

d) To determine which one the consumer prefers, we can look at the indirect utility function to see whether $v(p'w') > v(p^2w^2)$ or $v(p'w') < v(p^2w^2)$ is true.

$v(p, w) = w(p_1^{-1} + p_2^{-1})$

$v(p', w') = 2(1^{-1} + 1^{-1}) = 4$

$v(p^2, w^2) = 2.5(2^{-1} + 1^{-1}) = 3.75$

So $v(p'w') > v(p^2, w^2)$ which shows that the consumer received higher utility under $v(p', w')$, and this scheme is better than $v(p^2, w^2)$. The equivalent variation is the better measure of income change because it uses the same base prices, whereas compensating variation uses different base prices. In this case, the consumer has a better measure of the relativity in changes of wealth with her initial standing.

## Question 3

Agent chooses between two lotteries.

**Lottery 1**: $L_1$: fair coin is tossed once

- If it comes up heads, the agent wins $2
- If it comes up tails, the agent pays $2

**Lottery 2**: $L_2$: fair coin is tossed twice

- For each head that comes up, the agent wins $1
- For each tail that comes up, the agent pays $1

**Show that a risk averse agent will always prefer the second lottery**

*Lottery 1*: $L_1$: Pr(heads) = 0.50, Pr(tails) = 0.50

*Lottery 2*: $L_2$: Pr(heads, heads) = 0.25, Pr(heads, tails) = 0.25, Pr(tails, tails) = 0.25, Pr(tails, heads) = 0.25

I will use the von-Neumann Morgenstern utility functions. If the risk averse agent always prefers (i.e.: strictly prefers) $L_2$ to $L_1$, the following should be true:

$$\begin{aligned}
& 0.50 * u(2) + 0.50 * u(-2) < 0.25 * u(2) + 0.25 * u(0) + 0.25 * u(0) + 0.25 * (-2) \\
\Leftrightarrow\ & 0.25 * u(2) + 0.25 * u(-2) < 0.50 * u(0) \\
\Leftrightarrow\ & 0.50 * u(2) + 0.50 * u(-2) < u(0)
\end{aligned}$$

Since the RHS is the concave combination of the LHS (definition of concavity), we see the agent as being risk averse: will rather not engage in Lottery 1, and strictly prefer Lottery 2. She will require a certainty equivalent if she were to take Lottery 1, than the lottery itself.

## Question 4

$L_1$: \$0 with $Pr(\cdot) = \frac{1}{4}$, \$20 with $Pr(\cdot) = \frac{3}{4}$

$L_2$: \$12 with $Pr(\cdot) = 1$

Decision-maker is an expected utility maximizer with utility $u(w)$ and $w$: $\triangle wealth$.

$u(\cdot)$ **is continuous and strictly increasing.**

a) For $L_1$, $E(L_1) = 0 \times \frac{1}{4} + 20 \times \frac{3}{4} = 15$ dollars

For $L_2$, $E(L_2) = 12 \times 1 = 12$ dollars

The decision-maker is risk-averse: She will prefer $CE(L_1) < E_{L_2}(u)$ iff $u(CE(L_1)) < u(E_{L_2}(u))$. The certainty equivalent $\in (0, 15)$, and $> 12$, but we do not know the exact amount. So we cannot determine which lottery she will choose.

b) The decision-maker is risk-neutral:

She can look at expected values as $E_L[u] = u[E_L[x]]$. If she is willing to pay \$c for it, then she can still decide if she wants to take $L_1$ after gaining information. If not, she can choose to have $L_2$ (with certainty). But this outcome of \$12 replaces the \$0 she would have got under $L_1$. Since now, after paying for information, the investor's base wealth isn't \$0 anymore, it's \$12. Because by paying, she is able to increase his probability of getting a better outcome (if the info from the lottery outcome isn't good, then just go with $L_2$ with certainty, if not, then go with $L_1$).

The revised probability outcomes are: \$20 of $Pr(\cdot) = \frac{3}{4}$ and \$12 of $Pr(\cdot) = \frac{1}{4}$:

$$E(L_{info}) = \frac{3}{4} \times u(20) + \frac{1}{4} \times u(12) = \frac{3}{4} \times 20 + \frac{1}{4} \times 12 = 18$$

since risk-neutral investors do not derive additional utility from the expected outcomes so $u(w) = w$.

If no information, she has $E(L_{no\,info}) = 15$. So to be indifferent (since she is risk-neutral), she is willing to pay the difference in expected value of up to $c = 18 - 15 = 3$ dollars for the information.

c) The decision-maker is risk-loving:

If she does not buy information, she chooses $L_1$ over $L_2$ because the $u(\cdot)$ we know is strictly increasing and

$$E(L_1) = \frac{3}{4}u(20) + \frac{1}{4}u(0) > E(L_2) = u(12)$$

If she decides to buy information, she will get $E(L_{info}) = \frac{3}{4}u(20 - c) + \frac{1}{4}u(12 - c)$.

A risk-loving investor will be willing to pay for information if:

$$E(L_1) = \frac{3}{4}u(20) + \frac{1}{4}u(0) < E(L_{info}) = \frac{3}{4}u(20 - c) + \frac{1}{4}u(12 - c)$$

In this case, she will not choose $L_2$ as she derives utility from taking on the lottery ($L_1$) with uncertainty (so that is why it is $u(0)$ and not $u(12)$), rather than $L_2$ with certainty. If the price of information is small, then she is still willing to pay, since $u(20-c) \sim u(20)$ and $u(12-c) \geq u(0)$ and we know this to be true based on the shape of the $u(\cdot)$ function being continuous and increasing.

## Question 5

A weather forecaster paid based on performance. She forecasts the probabiltiy $q \in [0, 1]$ that it will rain the following day. The true probability is $p$ of whether it will rain, and when choosing her forecast $q$, cares only about maximizing her bonus for that day.

a) This is a risk neutral expected utility maximizer. She wants to maximize her expected value.

Her bonus is calculated as

$$= p(q \times 100) + (1-p) * ((1-q) \times 100) = 100pq + 100(1-p)(1-q) = 100(pq + (1-p)(1-q))$$

If we maximized this equation with respect to q, we get $\frac{\partial Bonus}{\partial q} = 100p + 100(1-p)(-1) = 100p - 100 + 100p = 0 \Leftrightarrow 200p = 100 \Leftrightarrow p = \frac{1}{2}$

Since she knows the true probability, depending on p, the forecaster should revise her prediction. Therefore, if $p < \frac{1}{2}$, she should pick $q = 0$. If it really does not rain, she will be paid \$100.

Similarly, for if $p > \frac{1}{2}$, she should pick $q = 1$, as it is more likely and if it rains, she will be paid \$100.

However, at $p = \frac{1}{2}$, it's a toss-up, so she would be indifferent on the value of q (1 or 0) she picks.

$$q = \begin{bmatrix} 0 & if & p < \frac{1}{2} \\ indifferent & if\, p & = \frac{1}{2} \\ 1 & if & p > \frac{1}{2} \end{bmatrix}$$

b) If it rains, her bonus would be a function of $= 100(1-(1-q)^2)$ and $= 100(1-q^2)$ if it does not rain. The risk neutral expected utility maximizer will want to maximize expected utility:

$$\begin{aligned} E(Bonus) =& p * 100(1-(1-q)^2) + (1-p) * 100(1-q^2) \\ \frac{\partial Bonus}{\partial q} =& (100p)(2(1-q)) + (100-100p)(-2q) \\ =& (100p)(2-2q) + (100-100p)(-2q) \\ =& 200p - 200pq - 200q + 200pq \\ =& 200p - 200q \Leftrightarrow q* = p \end{aligned}$$

c) If the forecaster was risk-averse, then we look again at the utility functions. She will maximize $E(Bonus) = p * u(100(1-(1-q)^2)) + (1-p) * u(100(1-q^2))$. In maximizing this we get:

$$\begin{aligned} E(Bonus) =& p * u(100(1-(1-q)^2)) + (1-p) * u((100(1-q^2)) \\ \frac{\partial Bonus}{\partial q} =& pu'[100(1-(1-q)^2)](2(1-q)) + (1-p)u'(100(1-q^2))(-2q) \\ \Leftrightarrow& 0 = 2p(1-q)u'(100(1-(1-q)^2)](2(1-q)) - 2q(1-p)u'(100(1-q^2)) \\ \Leftrightarrow& 2p(1-q)u'(100(1-(1-q)^2)](2(1-q)) = 2q(1-p)u'(100(1-q^2)) \end{aligned}$$

In our answer from b), if we look at for q = p, $\frac{2p(1-q)}{2q(1-p)} = 1$ and we have:

$$\begin{aligned}\frac{p(1-q)}{q(1-p)} =& \frac{u'(100(1-q^2))}{u'[100(1-(1-q)^2)]} \\ u'(100(1-q^2)) =& u'[100(1-(1-q)^2)](2(1-q))\end{aligned}$$

We know $u'(100(1-q^2)) = u'(100(1-(1-q)^2))(2(1-q))$ is equivalent to $(100(1-q^2)) = (100(1-(1-q)^2))(2(1-q))$ based on the $u'(\cdot)$ being strictly decreasing (shape of a utility function for a risk averse investor). Then $100(1-q^2) = 100(1-(1-q)^2) \Leftrightarrow 1-q^2 = 1-(1-2q+q^2)$ and we get $q* = \frac{1}{2}$. So we know in (c), the answer from b of $q = p$ would hold only at $q = \frac{1}{2}$. However, if $p \neq \frac{1}{2}$, the risk-averse forecaster will pick $q \in (p, \frac{1}{2})$ if $p < \frac{1}{2}$ and $q \in (\frac{1}{2}, p)$ if $p > \frac{1}{2}$, due to the shape of $u'(\dot{c})$, he prefers less risk.

d) If the scheme in part (b) is replaced with a randomized scheme that pays \$100 with Pr() = $(1-(1-q)^2)$ if it rains, \$100 with probability $(1-q)$ if it does not rain, and otherwise pay \$0.

She would maximize her expected bonus = $p(100(1-(1-q)^2)) + (1-p(100(1-q^2)))$. Similar to differentiating wrt q as in (b), she would choose $q = p$.

$$\begin{aligned}E(Bonus) =& 100p(1-(1-q)^2) + 100(1-p)(1-q^2) \\ \frac{\partial Bonus}{\partial q} =& (100p)(2(1-q)) + (100-100p)(-2q) \\ =& (100p)(2-2q) + (100-100p)(-2q) \\ =& 200p - 200pq - 200q + 200pq \\ =& 200p - 200q \Leftrightarrow q* = p\end{aligned}$$

e) If the forecaster was risk averse, he would now maximize the following:

$$\begin{aligned}E(Bonus) =& (1-(1-q)^2)u(100)p + (1-p)u(100)(1-q^2) \\ \frac{\partial Bonus}{\partial q} =& pu(100)(2(1-q)) - (1-p)u(100)(2q) = 0 \\ pu(100)(2(1-q)) =& 2(1-p)qu(100)200p \\ \Leftrightarrow \frac{q(1-p)}{(1-q)p} =& \frac{u(100)}{u(100)} = 1\end{aligned}$$

We see this holds for all values of $q = p$ as both sides will hold to evaluate to 1.