Econometric Problem Set 1

Question 1

Prove the Chebyshev Inequality, and then the WLLN.

The Chebyshev Inequality is

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Proof: Let X be a random variable with mean μ and variance σ^2. Let $\epsilon > 0$ be any postive real number. I will show for the **discrete case**, but the continuous case is identical to this proof. Let $f(x)$ be the distribution function of X. Then the probability that X deviates from μ by at least ϵ is

$$P(|X - \mu| \geq \epsilon) = \sum_{|x-\mu| \leq \epsilon} f(x) \qquad (1)$$

Let us explicitly define the variance of X:

$$\begin{aligned}
\sigma^2 &= \sum_x (x - \mu)^2 f(x) && (2)\\
&\geq \sum_{|x-\mu|\leq\epsilon} (x - \mu)^2 f(x) && (3)\\
&\geq \sum_{|x-\mu|\geq\epsilon} \epsilon^2 f(x) = \epsilon^2 \sum_{|x-\mu|\geq\epsilon} f(x) && (4)\\
&= \epsilon^2 \, P(|X - \mu| \geq \epsilon). && (5)
\end{aligned}$$

In (4), since ϵ is a constant, we can factor it out of the expression. (1) is substituted in (4) to get (5).

From (2) and (5), we now have

$$\sigma^2 \geq \epsilon^2 \, P(|X - \mu \geq \epsilon),$$

which by simplifying is the Chebyshev Inequality

$$\frac{\sigma^2}{\epsilon^2} \geq P(|X - \mu \geq \epsilon).$$

□

An assumption to this inequality is that $\sigma^2 \neq \infty$.

Now, the **Weak Law of Large Numbers** says let $X_1, \ldots, X_n$ be a sequence of N independent and identically distributed (iid) random variables, with $E[X_i] = \mu < \infty$. Then,

$$\bar{X_N} \xrightarrow{\text{p}} \mu.$$

Which can also be said as for any $\epsilon > 0$ and n $\to \infty$,

$$P(|\frac{S_n}{n} - \mu) \geq \epsilon) \to 0.$$

Proof: We will use Chebyshev's Inequality (which was proved above) as this assumes mutual independence. Let $S_n = X_1, X_2, \ldots, X_n$ be a sequence of mutually independent random variables, with $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$.

Since $S_n = X_1, X_2, \ldots, X_n$ are independently distributed, we can start with the properties of $Var(S_n)$ and $E(S_n)$ from the sequence S_i, and find the properties of $Var(\frac{S_n}{n})$ and $E(\frac{S_n}{n})$ of the sequence $\frac{S_n}{n}$. Since $Var(S_i) = \sigma^2$, we have

$$\begin{aligned} Var(\frac{S_n}{n}) &= \frac{1}{n^2} Var(S_n) \\ &= \frac{1}{n^2} * n\sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned} \qquad (1)$$

And $E(S_i) = \mu$, so

$$\begin{aligned} E(\frac{S_n}{n}) &= \frac{1}{n} E(S_n) \\ &= \frac{1}{n} * n\mu \\ &= \mu \end{aligned} \qquad (2)$$

Now we substitute in (1) as $Var(\frac{S_n}{n})$ and (2) as $E(S_n)$ in the Chebyshev's Inequality, and we see for any $\epsilon > 0$,

$$P(|\frac{S_n}{n} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}.$$

So, for any constant ϵ and as $\mathbf{n} \rightarrow \infty$**,**

$$P(|\frac{S_n}{n} - \mu) \geq \epsilon) \rightarrow 0$$

□

Question 2

Show that if X and Y are continuous independent random variables, $\mathbf{E}[XY] = \mathbf{E}[X]E[Y]$.

Proof: If we were to prove with independent, **continuous random variables** X and Y, we know that $f_x y(x, y) = f_x(x) f_y(y)$, so following in a similar fashion as above:

$$\begin{aligned} \mathrm{E}(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_x(x) f_y(y) dx dy \\ &= (\int_{-\infty}^{\infty} x f_x(x) dx)(\int_{-\infty}^{\infty} x f_y(y) dy) \\ &= \mathrm{E}(X)\mathrm{E}(Y) \end{aligned}$$

If we were to prove with independent, **discrete random variables** X and Y, we know that $f_x y(x_i, y_j) = f_x(x_i) f_y(y_j)$, so following in a similar fashion as above:

$$\begin{aligned}
\mathrm{E}(XY) =& \sum_i \sum_j x_i y_j f_{xy}(x_i, y_j) \\
=& \sum_i \sum_j x_i y_j f_x(x_i) f_y(y_j) \\
=& (\sum_i x_i f_x(x_i))(\sum_j y_j f_y(y_j)) \\
=& \mathrm{E}(X)\mathrm{E}(Y)
\end{aligned}$$

□

Question 3

Linear regression model

Obtain the covariance matrix of $\mathbf{u} = (u_1, \ldots, u_T)'$ as a function of ρ and σ_u^2.

Work: Our linear regression model is $y_t = x_t\beta + u_t$ in the presence of residual autocorrelation of the form $u_t = \rho u_{t-1} + \epsilon_t$. $E(\epsilon_t) = 0$, $Var(\epsilon_t) = \sigma_\epsilon^2$, $Var(u_t) = \sigma_u^2$ for $t = 1, \ldots, T$, and $Cov(\epsilon_s, \epsilon_t) = 0$ for $s, t = 1, \ldots, T,$, where $s \neq t$.

$$E(\epsilon_t, \epsilon_{t+s}) = \begin{bmatrix} \sigma_\epsilon^2 & if & s = 0 \\ 0 & if & s \neq 0 \end{bmatrix}$$

Now, since we are given $u_t = \rho u_{t-1} + \epsilon_t$, we can see for $u_1, u_2, \ldots, u_T$:

$$\begin{aligned}
u_t =& \rho u_{t-1} + \epsilon_t \\
=& \rho(u_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
=& \rho^2(u_{t-3} + \epsilon_{t-2}) + \rho\epsilon_{t-1} + \epsilon_t \\
=& \vdots \\
=& \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \ldots \\
=& \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i}
\end{aligned}$$

Since $E(u_t) = 0$, taking the expected value of (3) $E(u_t)$, we get

$$\begin{aligned}
E(u_t^2) =& E(\epsilon_t^2) + \rho^2 E(\epsilon_{t-1}^2) + \rho^4 E(\epsilon_{t-2}^2) + \ldots \\
=& \sigma_{\epsilon^2}(1 + \rho^2 + \rho^4 + \ldots) \\
E(u_t^2) =& \sigma_u^2 = \frac{\sigma_\epsilon^2}{1 - \rho^2}
\end{aligned}$$

for all i.

$$\begin{aligned}
E(u_t u_{t-1}) =& E[(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \ldots) \times (\epsilon_{t-1} + \rho\epsilon_{t-2} + \rho^2\epsilon_{t-3} + \ldots)] \\
=& E[[\epsilon_t + \rho(\epsilon_{t-1} + \epsilon_{t-2} + \ldots)](\epsilon_{t-1} + \rho\epsilon_{t-2} + \rho^2\epsilon_{t-3} + \ldots)]] \\
=& \rho E[(\epsilon_{t-1} + \rho\epsilon_{t-2} + \ldots)^2] \\
=& \rho\sigma_u^2
\end{aligned}$$

as $E(\epsilon_t) = 0$. Also $E(u_t u_{t-2}) = \rho^2 \sigma_u^2$. In general, $E(u_t u_{t-s}) = \rho^s \sigma_u^2$, so the covariance-variance matrix can be obtained as:

$$E(uu') = \Omega = \sigma_u^2 \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$