Econometric Problem Set 1

Question 1

Prove the Chebyshev Inequality, and then the WLLN.

The Chebyshev Inequality is

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Proof: Let X be a random variable with mean μ and variance σ^2. Let $\epsilon > 0$ be any postive real number. I will show for the **discrete case**, but the continuous case is identical to this proof. Let $f(x)$ be the distribution function of X. Then the probability that X deviates from μ by at least ϵ is

$$P(|X-\mu| \geq \epsilon) = \sum_{|x-\mu| \leq \epsilon} f(x) \qquad (1)$$

Let us explicitly define the variance of X:

$$\begin{aligned}
\sigma^2 &= \sum_x (x-\mu)^2 f(x) && (2)\\
&\geq \sum_{|x-\mu| \leq \epsilon} (x-\mu)^2 f(x) && (3)\\
&\geq \sum_{|x-\mu| \geq \epsilon} \epsilon^2 f(x) = \epsilon^2 \sum_{|x-\mu| \geq \epsilon} f(x) && (4)\\
&= \epsilon^2 \, P(|X-\mu| \geq \epsilon). && (5)
\end{aligned}$$

In (4), since ϵ is a constant, we can factor it out of the expression. (1) is substituted in (4) to get (5).

From (2) and (5), we now have

$$\sigma^2 \geq \epsilon^2 \, P(|X-\mu \geq \epsilon),$$

which by simplifying is the Chebyshev Inequality

$$\frac{\sigma^2}{\epsilon^2} \geq P(|X-\mu \geq \epsilon).$$

$\square$

An assumption to this inequality is that $\sigma^2 \neq \infty$.

Now, the **Weak Law of Large Numbers** says let $X_1, \ldots, X_n$ be a sequence of N independent and identically distributed (iid) random variables, with $E[X_i] = \mu < \infty$. Then,

$$\bar{X_N} \xrightarrow{\text{p}} \mu.$$

Which can also be said as for any $\epsilon > 0$ and n $\rightarrow \infty$,

$$P(|\frac{S_n}{n} - \mu) \geq \epsilon) \rightarrow 0.$$

Proof: We will use Chebyshev's Inequality (which was proved above) as this assumes mutual independence. Let $S_n = X_1, X_2, \ldots, X_n$ be a sequence of mutually independent random variables, with $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$.

Since $S_n = X_1, X_2, \ldots, X_n$ are independently distributed, we can start with the properties of $Var(S_n)$ and $E(S_n)$ from the sequence S_i, and find the properties of $Var(\frac{S_n}{n})$ and $E(\frac{S_n}{n})$ of the sequence $\frac{S_n}{n}$. Since $Var(S_i) = \sigma^2$, we have

$$\begin{aligned} Var(\frac{S_n}{n}) =& \frac{1}{n^2} Var(S_n) \\ =& \frac{1}{n^2} * n\sigma^2 \\ =& \frac{\sigma^2}{n} \qquad (1) \end{aligned}$$

And $E(S_i) = \mu$, so

$$\begin{aligned} E(\frac{S_n}{n}) =& \frac{1}{n} E(S_n) \\ =& \frac{1}{n} * n\mu \\ =& \mu \qquad (2) \end{aligned}$$

Now we substitute in (1) as $Var(\frac{S_n}{n})$ and (2) as $E(S_n)$ in the Chebyshev's Inequality, and we see for any $\epsilon > 0$,

$$P(|\frac{S_n}{n} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}.$$

So, for any constant ϵ and as $\mathbf{n} \to \infty$**,**

$$P(|\frac{S_n}{n} - \mu) \geq \epsilon) \to 0$$

$\square$

Question 2

Show that if X and Y are continuous independent random variables, $\mathbf{E}[XY] = \mathbf{E}[X]E[Y]$.

Proof: Let X and Y be two independent continuous random variables. Independence implies the covariance between X and Y is 0: $\sigma_{X,Y} = 0$.

$$\begin{aligned} \sigma_{X,Y} &= E[(X - E(X))(Y - E(Y))] && (1) \\ &= E[XY - XE(Y) - YE(X) + E(X)E(Y)] && (2) \\ &= E(XY) - E(XE(Y)) - E(YE(X)) + E(E(X)E(Y)) && (3) \\ &= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) && (4) \\ &= E(XY) - E(X)E(Y) && (5) \end{aligned}$$

Since $\sigma_{X,Y} = 0$,

$$\begin{aligned} \sigma_{X,Y} &= E(XY) - E(X)E(Y) && (6) \\ 0 &= E(XY) - E(X)E(Y) && (7) \\ E(XY) &= E(X)E(Y), && (8) \end{aligned}$$

where (1) is the definition of covariance; (2) follows from distributing the terms inside the brackets; (3) is the communitivity property of expectation; (4) is the distributive law; (5) cancels out like terms; (7) is using the result of independence between 2 random variables, and (8) is our result.

If we were to prove with independent, **continuous random variables** X and Y, we know that $f_x y(x,y) = f_x(x)f_y(y)$, so following in a similar fashion as above:

$$\begin{aligned}\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{xy}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_x(x)f_y(y)dxdy \\ &=(\int_{-\infty}^{\infty} xf_x(x)dx)(\int_{-\infty}^{\infty} xf_y(y)dy) \\ &=\mathrm{E}(X)\mathrm{E}(Y)\end{aligned}$$

If we were to prove with independent, **discrete random variables** X and Y, we know that $f_x y(x_i,y_j) = f_x(x_i)f_y(y_j)$, so following in a similar fashion as above:

$$\begin{aligned}\mathrm{E}(XY) &= \sum_i\sum_j x_i y_j f_{xy}(x_i,y_j) \\ &= \sum_i\sum_j x_i y_j f_x(x_i)f_y(y_j) \\ &=(\sum_i x_i f_x(x_i))(\sum_j y_j f_y(y_j)) \\ &=\mathrm{E}(X)\mathrm{E}(Y)\end{aligned}$$

□

Question 3

Linear regression model

Obtain the covariance matrix of $\mathbf{u} = (u_1, \ldots, u_T)'$ as a function of ρ and σ_u^2.

Work: Our linear regression model is $y_t = x_t\beta + u_t$ in the presence of residual autocorrelation of the form $u_t = \rho u_{t-1} + \epsilon_t$. $E(\epsilon_t) = 0$, $Var(\epsilon_t) = \sigma_\epsilon^2$, $Var(u_t) = \sigma_u^2$ for $t = 1, \ldots, T$, and $Cov(\epsilon_s, \epsilon_t) = 0$ for $s, t = 1, \ldots, T,$, where $s \neq t$.

$$E(\epsilon_t, \epsilon_{t+s}) = \begin{bmatrix} \sigma_\epsilon^2 & if & s = 0 \\ 0 & if & s \neq 0 \end{bmatrix}$$

Now, since we are given $u_t = \rho u_{t-1} + \epsilon_t$, we can see for $u_1, u_2, \ldots, u_T$:

$$
\begin{aligned}
u_t =& \rho u_{t-1} + \epsilon_t \\
=& \rho(u_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
=& \rho^2(u_{t-3} + \epsilon_{t-2}) + \rho\epsilon_{t-1} + \epsilon_t \\
=& \vdots \\
=& \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \dots \\
=& \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i}
\end{aligned}
$$

Since $E(u_t) = 0$, taking the expected value of (3) $E(u_t)$, we get

$$
\begin{aligned}
E(u_t^2) =& E(\epsilon_t^2) + \rho^2 E(\epsilon_{t-1}^2) + \rho^4 E(\epsilon_{t-2}^2) + \dots \\
=& \sigma_{\epsilon^2}(1 + \rho^2 + \rho^4 + \dots) \\
E(u_t^2) =& \sigma_u^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}
\end{aligned}
$$

for all i.

$$
\begin{aligned}
E(u_t u_{t-1}) =& E[(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \dots) \times (\epsilon_{t-1} + \rho\epsilon_{t-2} + \rho^2\epsilon_{t-3} + \dots)] \\
=& E[[\epsilon_t + \rho(\epsilon_{t-1} + \epsilon_{t-2} + \dots)](\epsilon_{t-1} + \rho\epsilon_{t-2} + \rho^2\epsilon_{t-3} + \dots)]] \\
=& \rho E[(\epsilon_{t-1} + \rho\epsilon_{t-2} + \dots)^2] \\
=& \rho\sigma_u^2
\end{aligned}
$$

as $E(\epsilon_t) = 0$. Also $E(u_t u_{t-2}) = \rho^2\sigma_u^2$. In general, $E(u_t u_{t-s}) = \rho^s \sigma_u^2$, so the covariance-variance matrix can be obtained as:

$$
E(uu') = \Omega = \sigma_u^2 \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}
$$