Exercises for Lecture Course on Optimal Control and Estimation (OCE)
Albert-Ludwigs-Universität Freiburg – Summer Term 2014

Exercise 3 - Gauss-Newton

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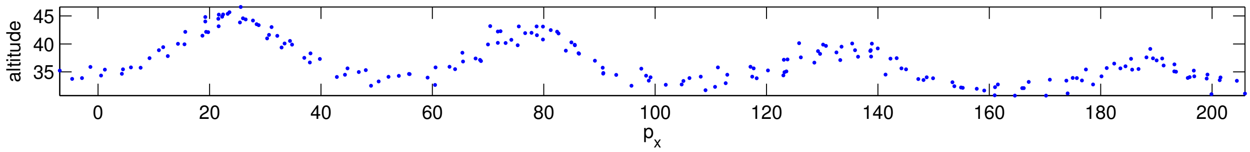
The aim of this exercise is to solve some parameter estimation problems with own solver instead of fmincon. You will warm up by fitting a model which can be solved analytically with least squares. You will then use Gauss-Newton to fit a simple model where derivative information is available. Last, you will solve the airplane parameter estimation problem from exercise 2 using your Gauss-Newton solver.

1 Paper Airplane Modeling

We will use the same data set as in exercise 2, which is available at
`https://github.com/ghorn/OCE-2014/blob/master/exercise2/airplane_data.m`



As before, this data set contains noisy position measurements $\hat{p}_{x,k}$ and $\hat{p}_{z,k}$ but not velocity.

Tasks

1. **Linear least squares**

   Assuming the solution trajectory has a polynomial form in time t:

$$\bar{p}_{x,k} = \sum_{j=0}^{10} a_j t_k^j \tag{1}$$

$$\bar{p}_{z,k} = \sum_{j=0}^{10} b_j t_k^j \tag{2}$$

   The optimization problem is:

$$\min_{a_0,\ldots,a_{10},b_0,\ldots,b_{10}} \sum_{k=0}^{N-1} (\bar{p}_{x,k}(a,b)) - \hat{p}_{x,k})^2 + (\bar{p}_{z,k}(a,b) - \hat{p}_{z,k})^2$$

   Formulate the problem as a linear least squares problem, that is:

$$\min_x \|Ax - b\|_2^2$$

   and solve using the well-known formula:

$$x_{\text{LS}} = (A^T A)^{-1} A^T b$$

   You should expect an imperfect fit, and possibly a badly conditioned matrix inverse. Numerically, it is certainly better to use the `MATLAB` `pinv` function to compute the Moore-Penrose pseudoinverse $A^+ = (A^\top A)^{-1} A^\top$.

   Plot p_x vs $-p_z$, $-p_z$ vs time, and p_x vs time.

2. **Gauss-Newton on simple model**

Assume the solution trajectory has the form:

$$\bar{p}_{x,k}(\theta_x) = \theta_{x,1} + t_k\theta_{x,2} + \theta_{x,3}\sin(\theta_{x,4} + t_k\theta_{x,5})\, e^{-\theta_{x,6}t_k} \tag{3}$$
$$\bar{p}_{z,k}(\theta_z) = \theta_{z,1} + t_k\theta_{z,2} + \theta_{z,3}\sin(\theta_{z,4} + t_k\theta_{z,5})\, e^{-\theta_{z,6}t_k} \tag{4}$$

The optimization problem is:

$$\min_{\theta_x,\theta_z} \sum_{k=0}^{N-1} (\bar{p}_{x,k}(\theta_x,\theta_z)) - \hat{p}_{x,k})^2 + (\bar{p}_{z,k}(\theta_x,\theta_z) - \hat{p}_{z,k})^2$$

Write down the objective function in the form of Gauss-Newton, that is:

$$\min_{\theta} \frac{1}{2} F(\theta)^T F(\theta) \tag{5}$$

Linearize $F(\theta)$ analytically to solve for F_0, J, where:

$$F(\theta) \approx F_0 + J\Delta\theta$$

Use Newton's method with the Gauss-Newton Hessian approximation to solve (5).

Plot p_x vs $-p_z$, $-p_z$ vs time, and p_x vs time. It will probably be very useful in debugging to plot each iteration of the algorithm.

3. **Gauss-Newton on simulation model**

For exercise 2, you used a RK4 integrator to minimize measurement errors, estimating α and initial state. Now put that probem in the Gauss-Newton form. Only write a `MATLAB` function for F, not J. A function for computing J from F is provided at:

`https://github.com/ghorn/OCE-2014/blob/master/exercise3/finite_difference_jacob.m`

You will call this function with a command something like:

```
[F0,J] = finite_difference_jacob(@(x)Fobj(x),x0);
```

Solve this problem using Newton's method with the Gauss-Newton Hessian approximation. For initial guess, use any initial state you want, and use $\alpha = 3°$.

Plot p_x vs $-p_z$, $-p_z$ vs time, and p_x vs time. It will probably be very useful in debugging to plot each iteration of the algorithm.