Exercise 1 - Discrete and Continuous Dynamic Systems

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Aim of this exercise is to get used to dealing with dynamic systems. We will start with a discrete time popoluation model. Then we will do a continuous-time nonlinear differential equation (a 2-dimensional aircraft model) and transform it into discrete time by numerical simulation.

1 Population Model

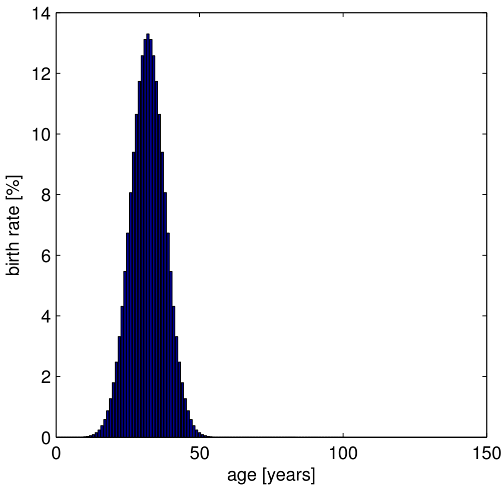
Consider a linear model of a population. State vector $x \in \mathbb{R}^{100}$ represents the population of each age group. Let $x_i(k)$ mean the number of people of age i during year k. For instance, $x_6(2014)$ would be the number of people who are 6 years old in year 2014. Each year babies (0-year-olds) are formed depending on a linear birth rate:

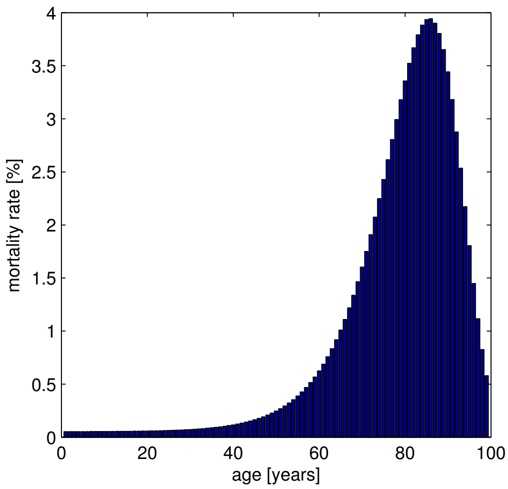
$$x_0(k+1) = \sum_{j=0}^{99} \beta_j x_j(k) \tag{1}$$

Each year most of the population ages by one year, except for a fraction who die according to mortality rate μ:

$$x_{i+1}(k+1) = x_i(k) - \mu_i x_i(k) \qquad i = 0, \dots, 98 \tag{2}$$

β and μ are provided for you at
`https://github.com/ghorn/OCE-2014/blob/master/exercise1/birth_mortality_rates.m`
and look like:





Tasks

1. Write the discrete time model in the form of

$$x(k+1) = A\,x(k) \tag{3}$$

2. Lord of the Flies: Setting an initial population of 100 four-year-olds, and no other people, simulate the system for 150 years. Make a 3-d plot of the population, with axes {year, age, population}.

3. Eigen decomposition: Plot the eigenvalues of A in the complex plane. Plot the real part of the two eigenvectors of A which have largest eigenvalue magnitude

   Is this system stable? What is the significance of these eigenvectors with large eigenvalues?

4. Run two simulations: in each simulation, use for $x(0)$ the real part of an eigenvector from the previous question. What is the significance of this result?

2 Paper Airplane Modeling

Consider a two-dimensional model of an airplane with states $x = [p_x, p_z, v_x, v_z]$ where position $\vec{p} = [p_x, p_z]$ and velocity $\vec{v} = [v_x, v_z]$ are vectors in the $x - z$ directions. Since the TA for this class has an aerospace background, we will use the standard aerospace convention that $\hat{x}$ is forward and $\hat{z}$ is DOWN, so altitude is $-p_z$. The system has one control $u = [\alpha]$, where α is the aerodynamic angle of attack in radians. The system dynamics are:

$$\frac{d}{dt}\begin{pmatrix} p_x \\ p_z \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_z \\ F_x/m \\ F_z/m \end{pmatrix} \tag{4}$$

where $m = 2.0$ is the mass of the airplane. The forces $\vec{F}$ on the airplane are

$$\vec{F} = \vec{F}_{\text{lift}} + \vec{F}_{\text{drag}} + \vec{F}_{\text{gravity}} \tag{5}$$

Lift force $\vec{F}_{\text{lift}}$ is

$$\vec{F}_{\text{lift}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_L(\alpha) S_{\text{ref}}\, \hat{e}_L \tag{6}$$

where lift direction $\hat{e}_L = [v_z, -v_x]/\|\vec{v}\|$, and lift coefficient $C_L = 2\pi\alpha\frac{10}{12}$. S_{ref} is the wing aerodynamic reference area. The drag force $\vec{F}_{\text{drag}}$ is

$$\vec{F}_{\text{drag}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_D(\alpha) S_{\text{ref}}\, \hat{e}_D \tag{7}$$

Drag direction $\hat{e}_D = -\vec{v}/\|\vec{v}\|$, and drag coefficient $C_D = 0.01 + \frac{C_L^2}{10\pi}$. The gravitational force is

$$\vec{F}_{\text{gravity}} = [0, m\, g] \tag{8}$$

Use $\rho = 1.2$, $g = 9.81$, $S_{\text{ref}} = 0.5$.

Tasks

1. Write the continuous time model in the form of

$$\frac{d}{dt}x = f(x, u) \tag{9}$$

2. Simulate the system for 10 seconds using the `ode45` MATLAB function. Use $\alpha = 3°$, and initial conditions $p_x = p_z = v_z = 0$, $v_x = 10$. Plot p_x, p_z, v_x, v_z vs. time, and p_x vs. altitude.

3. Convert the system to the discrete time form

$$x(k+1) = f_d(x(k), u(k)) \tag{10}$$

   using a forward Euler integrator. Simulate this system and compare to `ode45`. Estimating the accuracy by eye, how small do you have to make the time step so that results are similar accuracy to `ode45`? Using the MATLAB functions `tic` and `toc`, how much time does `ode45` take compared to forward Euler for similar accuracy?

4. Re-do the previous item using 4th order Runge-Kutta (RK4) instead of forward Euler. Which is faster (for similar accuracy) among the three methods?

5. **Bonus question:** Linearize the discrete time RK4 system to make an approximate system

$$x(k+1) \approx A\, x(k) + B\, u(k) \tag{11}$$

   using any linearization method you want.

   Plot the Eigenvalues of A in the complex plane. Is the system stable? Is this a problem?