

## Exercise 1 - Discrete and Continuous Dynamic Systems (deadline: 12.5.2014)

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Aim of this exercise is to get used to dealing with dynamic systems. We will start with a discrete time population model. Then we will do a continuous-time nonlinear differential equation (a 2-dimensional aircraft model) and transform it into discrete time by numerical simulation.

### 1 Population Model

Consider a linear model of a population. State vector  $x \in \mathbb{R}^{100}$  represents the population of each age group. Let  $x_i(k)$  mean the number of people of age  $i$  during year  $k$ . For instance,  $x_6(2014)$  would be the number of people who are 6 years old in year 2014. Each year babies (0-year-olds) are formed depending on a linear birth rate:

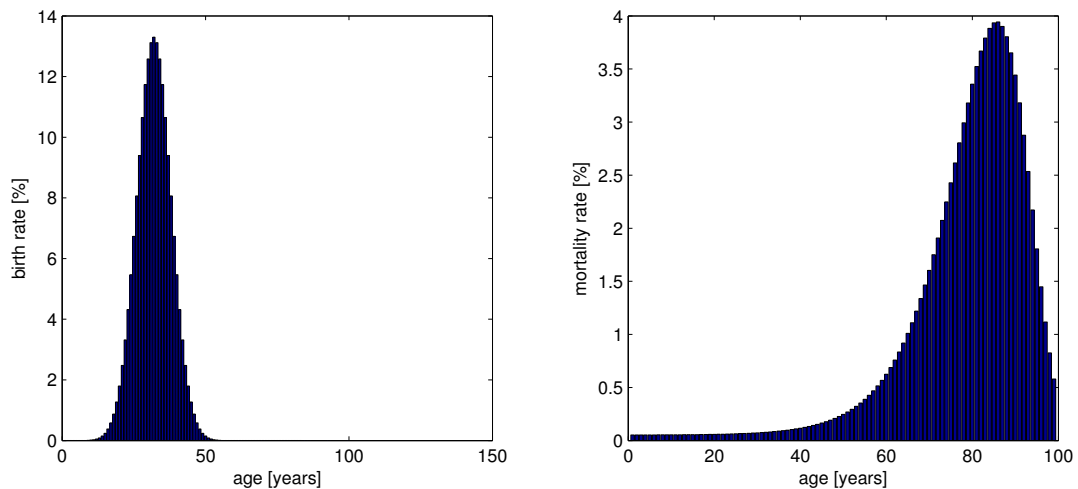
$$x_0(k+1) = \sum_{j=0}^{99} \beta_j x_j(k) \quad (1)$$

Each year most of the population ages by one year, except for a fraction who die according to mortality rate  $\mu$ :

$$x_{i+1}(k+1) = x_i(k) - \mu_i x_i(k) \quad i = 0, \dots, 98 \quad (2)$$

$\beta$  and  $\mu$  will be provided for you at

NUMOPCO/OCE-COURSE/exercises/exercisel/code/birth\_mortality\_rates.m  
and look like:



### Tasks

1. Write the discrete time model in the form of

$$x(k+1) = A x(k) \quad (3)$$

2. Lord of the Flies: Setting an initial population of 100 four-year-olds, and no other people, simulate the system for 150 years. Make a 3-d plot of the population, with axes {year, age, population}.
3. Eigen decomposition: Plot the eigenvalues of  $A$  in the complex plane. Plot the real part of the two eigenvectors of  $A$  which have largest eigenvalue magnitude

Is this system stable? What is the significance of these eigenvectors with large eigenvalues?

4. Run two simulations: in each simulation, use for  $x(0)$  the real part of an eigenvector from the previous question. What is the significance of this result?

## 2 Paper Airplane Modeling

Consider a two-dimensional model of an airplane with states  $x = [p_x, p_z, v_x, v_z]$  where position  $\vec{p} = [p_x, p_z]$  and velocity  $\vec{v} = [v_x, v_z]$  are vectors in the  $x - z$  directions. Since the TA for this class has an aerospace background, we will use the standard aerospace convention that  $\hat{x}$  is forward and  $\hat{z}$  is DOWN, so altitude is  $-p_z$ . The system has one control  $u = [\alpha]$ , where  $\alpha$  is the aerodynamic angle of attack in radians. The system dynamics are:

$$\frac{d}{dt} \begin{pmatrix} p_x \\ p_z \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_z \\ F_x/m \\ F_z/m \end{pmatrix} \quad (4)$$

where  $m = 2.0$  is the mass of the airplane. The forces  $\vec{F}$  on the airplane are

$$\vec{F} = \vec{F}_{\text{lift}} + \vec{F}_{\text{drag}} + \vec{F}_{\text{gravity}} \quad (5)$$

Lift force  $\vec{F}_{\text{lift}}$  is

$$\vec{F}_{\text{lift}} = \frac{1}{2} \rho \|\vec{v}\|^2 C_L(\alpha) S_{\text{ref}} \hat{e}_L \quad (6)$$

where lift direction  $\hat{e}_L = [v_z, -v_x]/\|\vec{v}\|$ , and lift coefficient  $C_L = 2\pi\alpha \frac{10}{12}$ .  $S_{\text{ref}}$  is the wing aerodynamic reference area. The drag force  $\vec{F}_{\text{drag}}$  is

$$\vec{F}_{\text{drag}} = \frac{1}{2} \rho \|\vec{v}\|^2 C_D(\alpha) S_{\text{ref}} \hat{e}_D \quad (7)$$

Drag direction  $\hat{e}_D = -\vec{v}/\|\vec{v}\|$ , and drag coefficient  $C_D = 0.01 + \frac{C_L^2}{AR\pi}$ . The gravitational force is

$$\vec{F}_{\text{gravity}} = [0, m g] \quad (8)$$

Use  $AR = 10$ ,  $\rho = 1.2$ ,  $g = 9.81$ ,  $S_{\text{ref}} = 0.5$ .

### Tasks

1. Write the continuous time model in the form of

$$\frac{d}{dt} x = f(x, u) \quad (9)$$

2. Simulate the system for 10 seconds using the `ode45` MATLAB function. Use  $\alpha = 3^\circ$ , and initial conditions  $p_x = p_z = v_z = 0$ ,  $v_x = 10$ . Plot  $p_x$ ,  $p_z$ ,  $v_x$ ,  $v_z$  vs. time, and  $p_x$  vs. altitude.
3. Convert the system to the discrete time form

$$x(k+1) = f_d(x(k), u(k)) \quad (10)$$

using a forward Euler integrator. Simulate this system and compare to `ode45`. Estimating the accuracy by eye, how small do you have to make the time step so that results are similar accuracy to `ode45`? Using the MATLAB functions `tic` and `toc`, how much time does `ode45` take compared to forward Euler for similar accuracy?

4. Re-do the previous item using 4th order Runge-Kutta (RK4) instead of forward Euler. Which is faster (for similar accuracy) among the three methods?
5. **Bonus question:** Linearize the discrete time RK4 system to make an approximate system

$$x(k+1) \approx A x(k) + B u(k) \quad (11)$$

using any linearization method you want.

Plot the Eigenvalues of  $A$  in the complex plane. Is the system stable? Is this a problem?