Markov Decision Processes

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Markov Chain

Definition

A *Markov Chain* is a stochastic model that describes a sequence of possible events $(X_t)_{t\geq 0}$ that satisfy the *Markov Property*:

$$P(X_n|X_{n-1},\dots,X_0) = P(X_n|X_{n-1})$$

Markov Decision Process

Definition

A *Markov Decision Process* is a tuple (S, A, P_a, R_a, γ).

- S is the state space.
- A is the action space.
- $P_a(s, s') = Pr(s_{t+1} = s' | s_t = s, a_t = a)$ is the probability that action a in state s at time t will lead to state s' at time $t + 1$
- $R_a(s, s') = E[R_{t+1} | s_t = s, a_t = a]$ is the immediate reward received after transitioning from state s to state s', due to action a
- γ is the discount factor

Markov Decision Process

Definition

A policy π is a distribution over actions given states:

$$\pi(a|s) = P(A_t = a|S_t = s)$$

Definition

The return G_t is the total discounted reward from time-step t:

$$G_t = \sum_{t=0}^{\infty} \gamma R_{a_t}(s_t, s_{t+1})$$

Markov Decision Process

Definition

The *Value Function* is the expected return starting from state S and following policy π: $v_\pi(s) = E[G_t | s_t = s]$

Definition

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π:
$q_\pi(s, a) = E_\pi[G_t | S_t = s, A_t = a]$

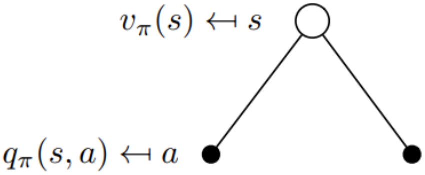
- Using the *Bellman equations*:

$$v_\pi(s) = E_\pi[R_{t+1} + \gamma v(S_{t+1}) | S_t = s]$$
$$q_\pi(s, a) = E_\pi[R_{t+1} + \gamma q(S_{t+1}, A_{t+1}) | S_t = s]$$

Intuition

- State s leads to two possible actions following probability π, hence:

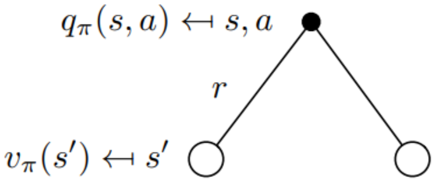
$$v_\pi(s) = \sum_{a \in A} \pi(a|s) q_\pi(s, a)$$



Intuition

- Action a, coming from state s, leads to two possible states following probability $P_a(s, s')$, hence:

$$q_\pi(s, a) = R_s^a + \gamma \sum_{s' \in A} P_a(s, s') v_\pi(s')$$



Optimization

- Combining the two equations:

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s' \in A} P_a(s,s') \sum_{a \in A} \pi(a|s) q_\pi(s,a)$$

- In order to know the quality of an action in *any* given state we are interested in finding the optimal q_*:

$$q_*(s,a) = \max_\pi q_\pi(s,a)$$

Examples

Partially Observable Markov Decision Process

- In a *Partially Observable Markov Decision Process* the agent can't directly observe the underlying state s. More formally:

Definition

A POMDP is a tuple $S, A, T, R, \Omega, O, \gamma$.

- $T(s'|s,a)$ is a set of conditional probabilities between states
- Ω is a set of observations
- $O(s'|s,a)$ is a set of conditional observation probabilities

Partially Observable Markov Decision Process

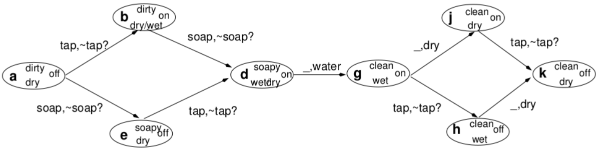
- The state isn't defined with certainty, hence data gathering is needed to update the *belief* $b(s)$ that the environment is in state s:

$$b(s) = \eta O(o|s,a) \sum_{s \in S} T(s'|s,a) b(s)$$

- The optimal policy is then chosen by maximizing the long-term reward:

$$\pi_* = \arg\max_{\pi} \sum_{t=0}^{\infty} \gamma^t E\left[R(s_t, a_t) | b_0, \pi\right]$$

Example



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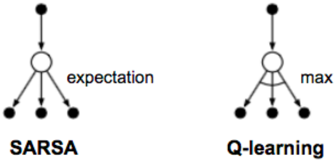
ABOUT THISSS

SARSA Vs Q-learning



SARSA Algorithm

Definition

SARSA is an ON-policy TD algorithm with parameters *step size* $\alpha \in (0, 1]$, *exploration rate* $\epsilon > 0$.

- Initialize $Q(s, a)$ for each state s and action a
- For each episode:
  - Initialize S
  - Choose A from S using policy derived from Q (ϵ-greedy)
  - For each state of the episode:
    - Take action A, observe (R, S')
    - Choose A' from S' using policy derived from Q (ϵ-greedy)
    - $Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma Q(S', A') - Q(S, A)]$
    - $S \leftarrow S'$, $A \leftarrow A'$
  - until S is terminal

SARSA Algorithm

Definition

SARSA is an OFF-policy TD algorithm with parameters *step size* $\alpha \in (0, 1]$, *exploration rate* $\epsilon > 0$.

- Initialize $Q(s, a)$ for each state s and action a
- For each episode:
    - Initialize S
    - For each state of the episode:
        - Choose A from S using policy derived from Q (ϵ-greedy)
        - Take action A, observe (R, S')
        - $Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma \max_a Q(S', a) - Q(S, A)]$
        - $S \leftarrow S'$
    - until S is terminal