## FULL CONDITIONAL DISTRIBUTION OF $z_{i,l}$

For simplicity we consider only 2 communities ($K = 2$) and suppose that our focus is on $l = l'$ and $s = s'$. It is also assumed that after removing the $i$-th element we have $[n_{l',*,1}, n_{l',*,2}]$, $[n_{l',1,1}, \dots, n_{l',S,1}]$ and $[n_{l',1,2}, \dots, n_{l',S,2}]$. We consider $\lambda_{l',k} = \exp\left(\mathbf{x}_{l'}^T \boldsymbol{\beta}_k\right)$ and $p_{l',k} = \frac{N}{N+\lambda_{l',k}}$. Then integrating out $\boldsymbol{\phi_k}$ we have that

$$p(z_{i,l'} = 1 \mid y_{i,l'} = s', \dots) \propto \prod_{k=1}^{K} \Bigg[ NB(n_{l',*,k} \mid \lambda_{l',k}, N) \times \int \left( \prod_{l=1}^{L} Multinomial([n_{l,1,k}, \dots, n_{l,S,k}] \mid n_{l,*,k}, \boldsymbol{\phi}_k) \right) \times \text{ Dirichlet } \left(\boldsymbol{\phi}_k \mid \gamma\right) d\boldsymbol{\phi}_k \Bigg]$$

The integral involving $\boldsymbol{\phi}_k$ is available in closed form. Furthermore, several elements in the equation above can be eliminated because they are constants. As a result, we obtain the following expression:

$$\begin{aligned}
&p(z_{i,l'} = 1 \mid y_{i,l'} = s', \dots) \\
&\propto \left[ \frac{\Gamma\left(n_{l',*,1} + 1 + N\right) p_{l',1}^{N} \left(1 - p_{l',1}\right)^{\left(n_{l',*,1}+1\right)}}{\Gamma(N) \left(n_{l',*,1} + 1\right)!} \right. \\
&\left. \times \frac{\Gamma\left(n_{l',*,2} + N\right) p_{l',2}^{N} \left(1 - p_{l',2}\right)^{\left(n_{l',*,2}\right)}}{\Gamma(N) n_{l^*,*,2}!} \right] \\
&\times \left( \prod_{l \neq l'} \frac{n_{l,*,1}!}{n_{l,1,1}! \dots n_{l,S,1}!} \right) \left( \frac{(n_{l',*,1} + 1)!}{n_{l',1,1}! \dots (n_{l',s,1} + 1)! \dots n_{l',S,1}!} \right) \\
&\times \frac{(n_{*,s',1} + 1 + \gamma_s) \prod_{s \neq s'} (n_{*,s,1} + \gamma_s)}{\Gamma(n_{*,*,1} + 1 + \sum_s \gamma_s)} \\
&\times \frac{n_{l,*,2}!}{n_{l,1,2}! \dots n_{l,S,2}!} \frac{\prod_{s=1} (n_{*,s,2} + \gamma_s)}{\Gamma(n_{*,*,2} + \sum_s \gamma_s)}.
\end{aligned}$$

We drop additional terms that are constants to obtain:

$$\begin{aligned}
&p(z_{i,l'} = 1 \mid y_{i,l'} = s', \dots) \\
&\propto \left[ \frac{\Gamma\left(n_{l',*,1} + 1 + N\right) \left(1 - p_{l',1}\right)^{\left(n_{l',*,1}+1\right)}}{\left(n_{l',*,1} + 1\right)!} \right. \\
&\left. \times \frac{\Gamma\left(n_{l',*,2} + N\right) \left(1 - p_{l',2}\right)^{\left(n_{l',*,2}\right)}}{n_{l^*,*,2}!} \right] \\
&\times \left( \frac{(n_{l',*,1} + 1)!}{n_{l',1,1}! \dots (n_{l',s,1} + 1)! \dots n_{l',S,1}!} \right) \frac{(n_{*,s',1} + 1 + \gamma_s)}{\Gamma(n_{*,*,1} + 1 + \sum_s \gamma_s)} \\
&\left( \frac{n_{l',*,2}!}{n_{l',1,2}! \dots n_{l',s',2}! \dots n_{l',S,2}!} \right) \frac{(n_{*,s',2} + \gamma_{s'})}{\Gamma\left(n_{*,*,2} + \sum_s \gamma_s\right)} \propto b_1 a_1
\end{aligned}$$

where:

$$a_1 = \left( \frac{(n_{l',*,1} + 1)!}{n_{l',1,1}! \dots (n_{l',s,1} + 1)! \dots n_{l',S,1}!} \right) \frac{(n_{*,s',1} + 1 + \gamma_s)}{\Gamma(n_{*,*,1} + 1 + \sum_s \gamma_s)} \times \left( \frac{n_{l',*,2}!}{n_{l',1,2}! \dots n_{l',s',2}! \dots n_{l',S,2}!} \right) \frac{(n_{*,s',2} + \gamma_{s'})}{\Gamma\left(n_{*,*,2} + \sum_s \gamma_s\right)} \text{ and}$$

$$b_1 = \left[ \frac{\Gamma\left(n_{l',*,1} + 1 + N\right) \left(1 - p_{l',1}\right)^{\left(n_{l',*,1}+1\right)}}{\left(n_{l',*,1} + 1\right)!} \times \frac{\Gamma\left(n_{l',*,2} + N\right) \left(1 - p_{l',2}\right)^{\left(n_{l',*,2}\right)}}{n_{l^*,*,2}!} \right].$$

Similarly, it can be shown that

$$p(z_{i,l'} = 2 \mid y_{i,l'} = s', \dots) \propto b_2 a_2$$

where

$$a_2 = \left( \frac{n_{l',*,1}!}{n_{l',1,1}! \dots n_{l',s',1}! \dots n_{l',S,1}!} \right) \frac{(n_{*,s',1} + \gamma_{s'})}{\Gamma\left(n_{*,*,1} + \sum_s \gamma_s\right)} \times \left( \frac{(n_{l',*,2} + 1)!}{n_{l',1,2}! \dots (n_{l',s,2} + 1)! \dots n_{l',S,2}!} \right) \frac{(n_{*,s',2} + 1 + \gamma_s)}{\Gamma(n_{*,*,2} + 1 + \sum_s \gamma_s)} \text{ and}$$

$$b_2 = \left[ \frac{\Gamma\left(n_{l',*,1} + N\right) \left(1 - p_{l',1}\right)^{\left(n_{l',*,1}\right)}}{n_{l^*,*,1}!} \times \frac{\Gamma\left(n_{l',*,2} + 1 + N\right) \left(1 - p_{l',2}\right)^{\left(n_{l'*,2}+1\right)}}{\left(n_{l',*,2} + 1\right)!} \right].$$

Because $z_{i,l'}$ is either equal to 1 or 2, we can divide both sizes by $b_1 a_1 + b_2 a_2$ and, using factorial and gamma function rules, we obtain:

$$\begin{aligned}
&p(z_{i,l'} = 1 \mid y_{i,l'} = s', \dots) \\
&\propto \left( \frac{\frac{\left(n_{l',*,1}+1\right)\left(n_{*,s',1}+\gamma_{s'}\right)}{\left(n_{l',s'}+1\right)\left(n_{*,*,1}+\sum_s \gamma_s\right)}}{\frac{\left(n_{l',*,1}+1\right)\left(n_{*,s',1}+\gamma_{s'}\right)}{\left(n_{l',s',1}+1\right)\left(n_{*,*,1}+\sum_s \gamma_s\right)} + \frac{\left(n_{l',*,2}+1\right)\left(n_{*,s',2}+\gamma_{s'}\right)}{\left(n_{l',s',2+1}\right)\left(n_{*,*,2}+\sum_s \gamma_s\right)}} \right) \\
&\times \left( \frac{\frac{\left(n_{l',*,1}+N\right)\left(1-p_{l',1}\right)}{\left(n_{l',*,1}+1\right)}}{\frac{\left(n_{l',*,1}+N\right)\left(1-p_{l',1}\right)}{\left(n_{l',*,1}+1\right)} + \frac{\left(n_{l',*,2}+N\right)\left(1-p_{l',2}\right)}{\left(n_{l',*,2}+1\right)}} \right) \\
&\propto \frac{\left(n_{l',*,1} + N\right)\left(n_{*s',1} + \gamma_{s'}\right)}{\left(n_{l's',1} + 1\right)\left(n_{*,*,1} + \sum_s \gamma_s\right)} \left(1 - p_{l',1}\right).
\end{aligned}$$

And finally we have that

$$z_{i,l} \mid y_{i,l} = s, \dots \sim \text{Categorical}\left( \left[ \frac{\frac{(n_{l,*,1}+N)(n_{*,s,1}+\gamma_s)}{(n_{l,s,1}+1)(n_{*,*,1}+\sum_s \gamma_s)}(1 - p_{l,1})}{\sum_{k=1}^{K} \frac{(n_{l,*,k}+N)(n_{*,s,k}+\gamma_s)}{(n_{l,s,k}+1)(n_{*,*,k}+\sum_s \gamma_s)}(1 - p_{l,k})}, \dots, \frac{\frac{(n_{l,*,K}+N)(n_{*,s,K}+\gamma_s)}{(n_{l,s,K}+1)(n_{*,*,K}+\sum_s \gamma_s)}(1 - p_{l,K})}{\sum_{k=1}^{K} \frac{(n_{l,*,k}+N)(n_{*,s,k}+\gamma_s)}{(n_{l,s,k}+1)(n_{*,*,k}+\sum_s \gamma_s)}(1 - p_{l,k})} \right] \right)$$

## MULTINOMIAL INTEGRATION IN $\phi_k$

To simplify the calculation of the conditional distribution of $z_{i,l}$ we can integrate out $\boldsymbol{\phi}_k$ as shown below.

$$
\begin{aligned}
&\prod_{k=1}^{K} \int \left[ \prod_{l=1}^{L} \text{Multinomial}\left( [n_{l,1,k}, \ldots, n_{l,S,k}] \mid n_{l,*,k}, \boldsymbol{\phi}_k \right) \right] \text{ Dirichlet } \left( \boldsymbol{\phi}_k \mid \gamma \right) d\boldsymbol{\phi}_k \\
&\propto \prod_{k=1}^{K} \int \left[ \prod_{l=1}^{L} \frac{n_{l,*,k}!}{n_{l,1,k}! \ldots n_{l,S,k}!} \phi_{k,1}^{n_{l,1,k}} \ldots \phi_{k,S}^{n_{l,S,k}} \right] \phi_{k,1}^{\gamma_1 - 1} \ldots \phi_{k,S}^{\gamma_S - 1} d\boldsymbol{\phi}_k \\
&\propto \prod_{k=1}^{K} \left( \prod_{l=1}^{L} \frac{n_{l,*,k}!}{n_{l,1,k}! \ldots n_{l,S,k}!} \right) \int \phi_{k,1}^{n_{*,1,k} + \gamma_1 - 1} \ldots \phi_{k,S}^{n_{*,S,k} + \gamma_S - 1} d\boldsymbol{\phi}_k \\
&\propto \prod_{k=1}^{K} \left( \prod_{l=1}^{L} \frac{n_{l,*,k}!}{n_{l,1,k}! \ldots n_{l,S,k}!} \right) \frac{\prod_{s=1}^{S} \left( n_{*,s,k} + \gamma_s \right)}{\Gamma\left( n_{*,*,k} + \sum_{s=1}^{S} \gamma_s \right)}
\end{aligned}
$$