Math Refresher for DS

Practical Session 2

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Change of Coordinates - Example

- $b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ are linearly independent.
- Therefore, $B = \{b_1, b_2, b_3\}$ - basis in $\mathbb{R}^3$.

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- How do we go from the standard basis $E = \left\{e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ to B?

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    - $x_E = [6, 9, 14\ ], \quad x_B = [x_1, x_2, x_3] =?$

Change of Coordinates

- There was a small typo in the lecture!

Coordinate Change: Matrix Notation

- Result obtained before:

  $e_1, \dots, e_n$ - old basis
  $e'_1, \dots, e'_n$ - new basis

  $x_{old} = [x_1, \dots, x_n], \qquad x_{new} = [x'_1, \dots, x'_n]$

$$e'_i$$

$$x_{old} \quad \begin{matrix} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{matrix}$$

$$x_{new}$$

- Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

$$x_{old} = A^T x_{new}$$

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Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A\cancel{^T} x_{new} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$

$$x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

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- $x_E = [6, 9, 14\ ]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

$$x_E = A_{E \to B} \cdot x_B$$

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$$A_{E \to B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

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$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \overset{(2)-(1)}{\Leftrightarrow} \begin{cases} \\ x_3 = 3 \\ \\ \end{cases} \Leftrightarrow$$

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Orthogonal Basis

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Orthogonal Basis

- There is more than one basis in a vector space.
- Some are more convenient than the other ones.
- Orthonormal basis = all vectors are pairwise orthogonal ($(e_i, e_j) = 0$) + of unit length ($\|e_i\| = 1$).
- Any basis can be transformed into orthonormal basis!

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

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2. Let's look for v_2 of the form $v_2 := b_2 + \alpha v_1,\ \ \alpha \in \mathbb{R}$.

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   How to chose α? v_1 and v_2 must be orthogonal!

$$0 = (v_1, v_2) = (v_1, b_2 + \alpha v_1) = (v_1, b_2) + \alpha (v_1, v_1) \Leftrightarrow$$

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$$0 = (v_1, v_2) = (v_1, b_2 + \alpha v_1) = (v_1, b_2) + \alpha(v_1, v_1) \Leftrightarrow \alpha = -\frac{(v_1, b_2)}{(v_1, v_1)}$$

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$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} v_1, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1+1+2}{1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

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Gram-Schmidt Process

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$$v_3 = b_3 + \alpha_1 v_1 + \alpha_2 v_2$$

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$$0 = (v_1, v_3) =$$

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$$v_3 = b_3 + \alpha_1 v_1 + \alpha_2 v_2 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2$$

$$0 = (v_1, v_3) = (v_1, b_3) + \alpha_1 (v_1, v_1) + \alpha_2 (v_1, v_2) = (v_1, b_3) + \alpha_1 (v_1, v_1) \iff \alpha_1 = -\frac{(v_1, b_3)}{(v_1, v_1)}$$

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$$v_3 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2 =$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

$$v_1 \coloneqq b_1, \qquad v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1+2+3}{1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-\frac{1}{3} - \frac{2}{3} + 2}{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} \cdot \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

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$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Orthogonal basis $V = \{v_1, v_2, v_3\}$ from B:

$$v_1 \coloneqq b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Gram-Schmidt Process: General Case

- Some basis $B = \{b_1, \dots, b_n\}$.
- Constructing orthogonal basis $V = \{v_1, \dots, v_n\}$, $\left(v_i, v_j\right) = 0$:

$$v_1 = b_1$$

$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} v_1$$

$$\vdots$$

$$v_k = b_k - \frac{(v_1, b_k)}{(v_1, v_1)} v_1 - \frac{(v_2, b_k)}{(v_2, v_2)} v_2 - \dots - \frac{(v_{k-1}, b_k)}{(v_{k-1}, v_{k-1})} v_{k-1}$$

- If we additionally normalize v_i , we get orthonormal basis.