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Math Refresher for DS

Practical Session 11

Today: Integrals

- Indefinite integrals

$$\int f(x)dx$$

- Definite integrals

$$\int_a^b f(x)dx$$

- Improper integrals

$$\int_{-\infty}^{+\infty} f(x)dx$$

Indefinite Integral

Antiderivatives

- Which function $F(x)$ should we differentiate to get

$$f(x) = x^4 + 3x - 9$$

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Antiderivatives

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$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + C, \qquad C \in \mathbb{R}$$

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Antiderivatives

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\, dx = F(x) + c, \qquad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

Indefinite Integral

$$\int f(x)\,dx$$

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Indefinite Integral: Example

$$\int x^n \, dx = \qquad\qquad\qquad\qquad , \qquad n \neq -1$$

$$\int \frac{1}{x} \, dx =$$

$$\int \sin x \, dx =$$

$$\int e^x dx =$$

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Indefinite Integral: Example

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{x} dx =$$

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Indefinite Integral: Example

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x + C$$

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$$\int e^x dx = e^x + C$$

Indefinite Integral: Example

- Determine the function $f(x)$ if

$$f'(x) = 4x^3 - 9 + 2\sin x + 7e^x, \qquad f(0) = 15$$

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Indefinite Integral: Example

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$$f'(x) = 4x^3 - 9 + 2\sin x + 7e^x, \qquad f(0) = 15$$

$$f(x) = \int 4x^3 - 9 + 2\sin x + 7e^x\,dx = x^4 - 9x \; - 2\cos x + 7e^x + C$$

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$$f(0) = 15 =$$

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$$f(0) = 15 = -2\cos 0 + 7e^0 = 5$$

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$$f'(x) = 4x^3 - 9 + 2\sin x + 7e^x, \qquad f(0) = 15$$

$$f(x) = \int 4x^3 - 9 + 2\sin x + 7e^x \, dx = x^4 - 9x \; - 2\cos x + 7e^x + C$$

$$f(0) = 15 = -2\cos 0 + 7e^0 = 5 \quad \rightarrow \quad C = 10$$

$$f(x) = x^4 - 9x \; - 2\cos x + 7e^x + 10$$

Integration techniques

Substitution Rule

- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3 + 5}\, dx =$$

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- Substitution rule:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \quad \text{where, } u = g(x)$$

Substitution Rule

- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx = \{u = 6x^3+5, \qquad du = \qquad\qquad \} =$$

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Substitution Rule

- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3+5}\,dx = \{u = 6x^3+5, \qquad du = 18x^2 dx\} =$$

$$= \int \sqrt[4]{u}\,du = \frac{4}{5}u^{5/4} + C$$

- Substitution rule:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du, \quad \text{where, } u = g(x)$$

Substitution Rule

- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx = \{u = 6x^3+5, \qquad du = 18x^2 dx\} =$$

$$= \int \sqrt[4]{u}\, du = \frac{4}{5} u^{5/4} + C = \frac{4}{5} \sqrt[4]{(6x^3+5)^5} + C.$$

- Substitution rule:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \quad \text{where, } u = g(x)$$

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Substitution Rule - Example

$$\int 3(8y-1)e^{4y^2-y}\,dy =$$

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Substitution Rule - Example

$$\int 3(8y-1)e^{4y^2-y}dy = \int 3e^{4y^2-y}d(4y^2-y) =$$

Substitution Rule - Example

$$\int 3(8y-1)e^{4y^2-y}dy = \int 3e^{4y^2-y}d(4y^2-y) =$$

$$= 3e^{4y^2-y} + C$$

Integration by Parts

- Consider the following integrals:

$$\int e^{x} dx =$$

$$\int x e^{x^2} dx =$$

$$\int x e^{6x} dx =$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx =$$

$$\int xe^{6x} dx =$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2} \int e^{x^2} dx^2 =$$

$$\int xe^{6x} dx =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int x e^{x^2} dx = \frac{1}{2} \int e^{x^2} dx^2 = \frac{1}{2} e^{x^2} + C$$

$$\int x e^{6x} dx =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx = \cdots ?$$

Integration by Parts

- Chain rule:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

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$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$u \cdot v = \int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u \cdot v' \, dx =$$

Integration by Parts

- Chain rule:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$u \cdot v = \int (u \cdot v)'\, dx = \int u' \cdot v\, dx + \int u \cdot v'\, dx = \int v\, du + \int u\, dv$$

Integration by Parts

- Chain rule:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$u \cdot v = \int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u \cdot v' \, dx = \int v \, du + \int u \, dv$$

$$\int u \, dv = u \cdot v - \int v \, du$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2} \int e^{x^2} dx^2 = \frac{1}{2} e^{x^2} + C$$

$$\int xe^{6x} dx =$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx = \frac{1}{6}\int x de^{6x} =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2} e^{x^2} + C$$

$$\int xe^{6x} dx = \frac{1}{6}\int x de^{6x} = \frac{1}{6} xe^{6x} - \frac{1}{6}\int e^{6x} dx =$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx =$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 =$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x \, dx =$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x \, dx = 0.1 \int x d \sin 10x = 0.1 \sin 10x - 0.1 \int \sin 10x \, dx$$

$$=$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x\, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x\, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x\, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x\, dx = 0.1 \int xd \sin 10x = 0.1 \sin 10x - 0.1 \int \sin 10x\, dx$$

$$= 0.1 \sin 10x + 0.01 \cos 10x + C.$$

Integration by Parts – Example 3

$$\int \ln x \, dx =$$

Integration by Parts – Example 3

$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x =$$

Integration by Parts – Example 3

$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x = x \ln x - \int x \cdot \frac{1}{x} dx =$$

Integration by Parts – Example 3

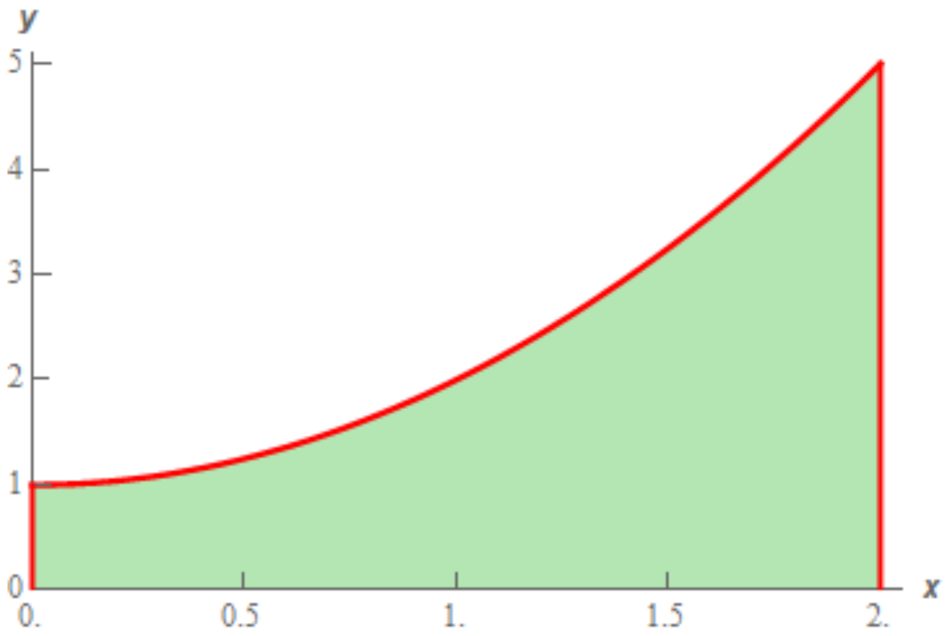
$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x = x \ln x - \int x \cdot \frac{1}{x} dx =$$

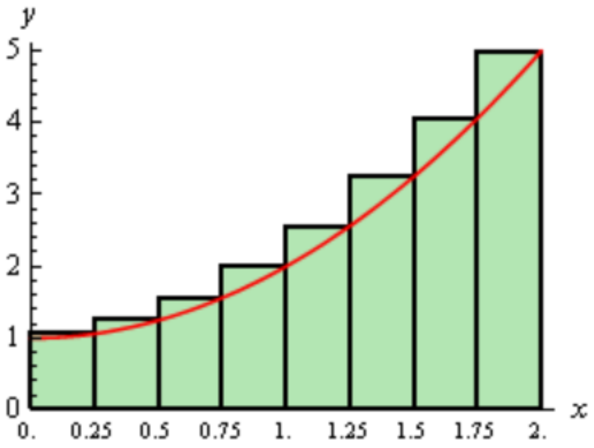
$$= x \ln x - x + C$$

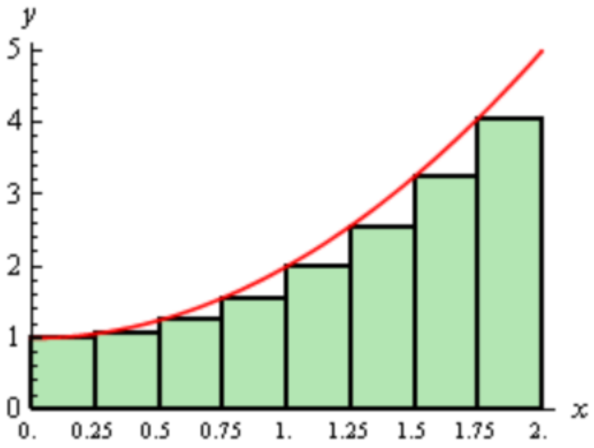
Definite Integral

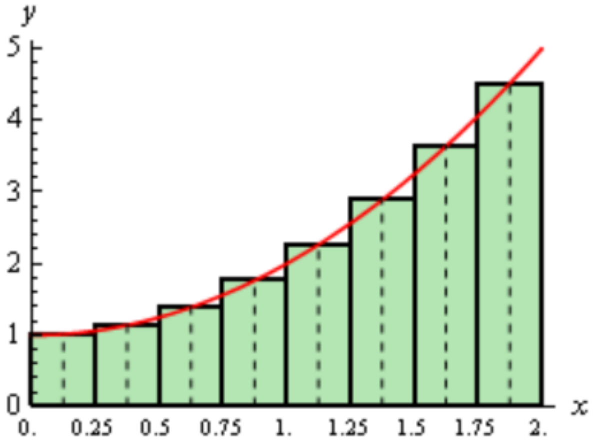
Definite Integral



Definite Integral

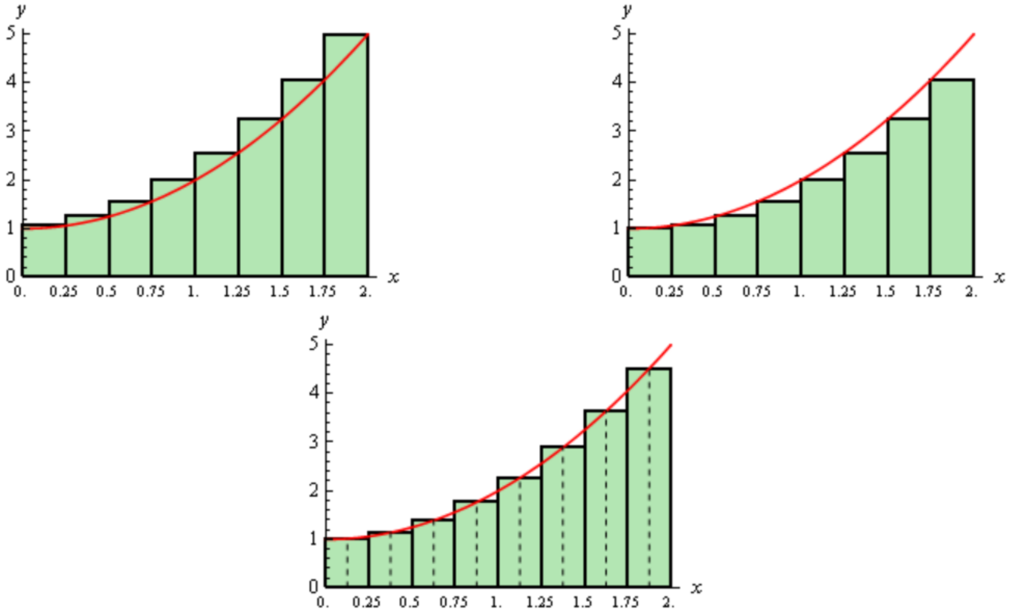






Definite Integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$



Definite Integral

- The fundamental theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

- Example:

$$\int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Definite Integral: Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a, c]$ and $[c, b]$. Note however that c doesn't need to be between a and b.

Definite Integral – Example 2

$$\int_0^1 2e^{-2x}dx =$$

Definite Integral – Example 2

$$\int_0^1 2e^{-2x}dx = -e^{-2x}\Big|_0^1 =$$

Definite Integral – Example 2

$$\int_0^1 2e^{-2x}dx = -e^{-2x}\Big|_0^1 = -e^{-2} + 1 = 1 - \frac{1}{e^2}$$

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Definite Integral – Example 3

$$\int_{-1}^{1} \frac{1}{x^2}\,dx =$$

Definite Integral – Example 3

$$\int_{-1}^{1} \frac{1}{x^2} dx =$$

$\frac{1}{x^2}$ isn't defined at 0

Not a definite integral!

Infinite interval

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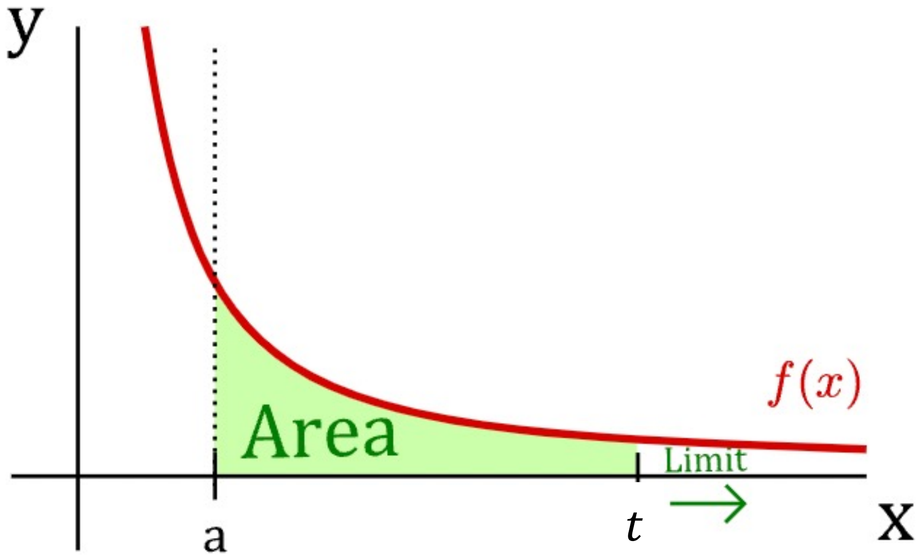
Definition

- If $f(x)$ is continuous on $[a;\ +\infty)$, then

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$



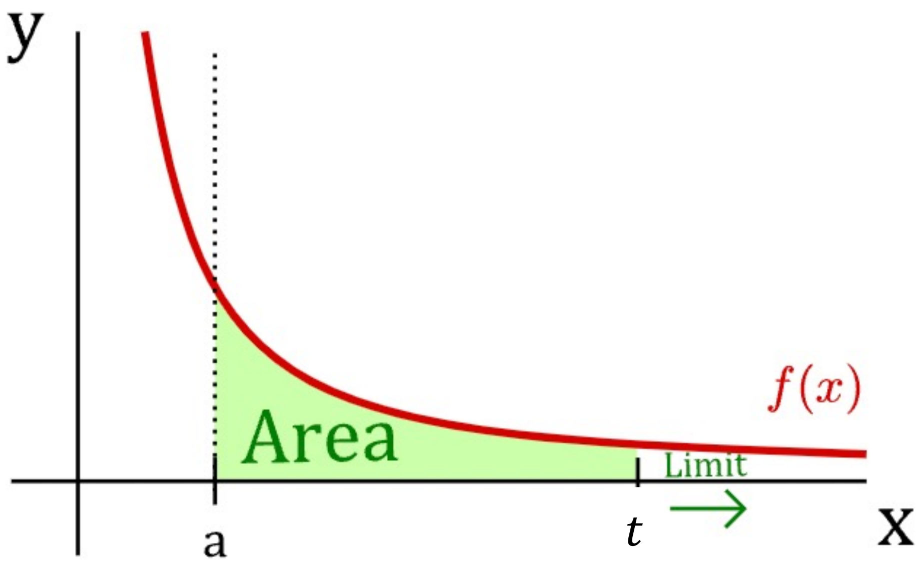
Definition

- If $f(x)$ is continuous on $[a;\ +\infty)$, then

$$\int_{a}^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_{a}^{t} f(x)dx$$

- If $f(x)$ is continuous on $(-\infty; b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$$



Example

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx =$$

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Example

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_1^{t} \frac{1}{x^2}\,dx =$$

Example

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to+\infty} \left(-\frac{1}{x}\Big|_{1}^{t} \right) =$$

Example

$$\int_1^{+\infty} \frac{1}{x^2} dx = \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx =$$

$$= \lim_{t \to +\infty} \left(-\frac{1}{x} \Big|_1^t \right) = \lim_{t \to +\infty} \left(-\frac{1}{t} \right) + 1 =$$

Example

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to+\infty} \left(-\frac{1}{x}\Big|_{1}^{t} \right) = \lim_{t\to+\infty} \left(-\frac{1}{t} \right) + 1 =$$

$$= 0 + 1 = 1$$

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Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.
- Example:

$$\int_{1}^{+\infty} \frac{1}{x} dx =$$

Divergent integrals

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- Example:

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{t \to +\infty} \int_1^{t} \frac{1}{x}\, dx =$$

Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.

- Example:

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\, dx =$$

$$= \lim_{t\to+\infty} \left(\log x \Big|_1^t \right) =$$

Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.
- Example:

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{t \to +\infty} \int_1^t \frac{1}{x} dx =$$

$$= \lim_{t \to +\infty} \left(\log x \Big|_1^t \right) =$$

$$= \lim_{t \to +\infty} \log t + 0$$

Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.
- Example: the following integral in divergent

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{t \to +\infty} \int_1^t \frac{1}{x} dx =$$

$$= \lim_{t \to +\infty} \left(\log x \Big|_1^t \right) =$$

$$= \lim_{t \to +\infty} \log t + 0 \to +\infty$$

One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_{a}^{+\infty} \frac{1}{x^p} dx =$$

One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p} dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \frac{1}{x^{p-1}}\bigg|_a^t =$$

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One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p} dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \frac{1}{x^{p-1}}\bigg|_a^t =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}} \right) =$$

One more example

- For which p is the following integral convergent $(a > 0)$?

$$\int_a^{+\infty} \frac{1}{x^p}\,dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t\to+\infty} \frac{1}{x^{p-1}}\Big|_a^t =$$

$$= -\frac{1}{p-1} \cdot \lim_{t\to+\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right) =$$

$$= \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}$$

when $p - 1 > 0 \Leftrightarrow p > 1.$

One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p}\,dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t\to+\infty} \frac{1}{x^{p-1}}\Big|_a^t =$$

$$= -\frac{1}{p-1} \cdot \lim_{t\to+\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right) =$$

$$= \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}$$

when $p - 1 > 0 \Leftrightarrow p > 1.$

If $p \leq 1$, the limit doesn't exist.

Two infinite limits

- If both $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{+\infty} f(x)dx$ are convergent, then the improper integral of f over $(-\infty;\ +\infty)$ is

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$$

Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}\,dx =$$

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Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx =$$

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Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx = \begin{Bmatrix} y = x^2 \\ dy = 2xdx \end{Bmatrix} =$$

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Comparison test

- There are many techniques to check if an integral is convergent or not.
- *Example:* comparison test

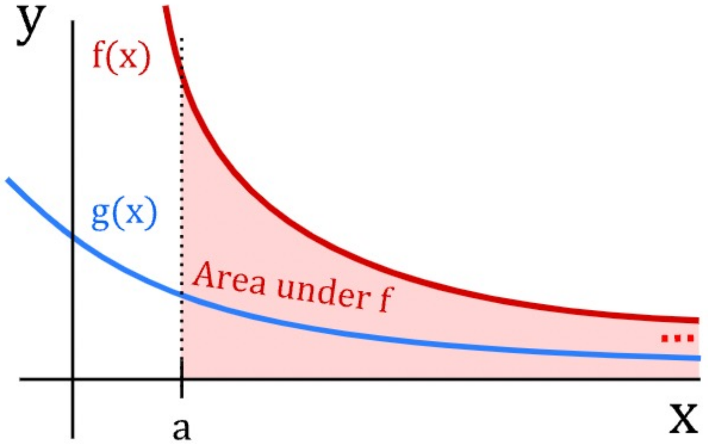
Suppose that $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

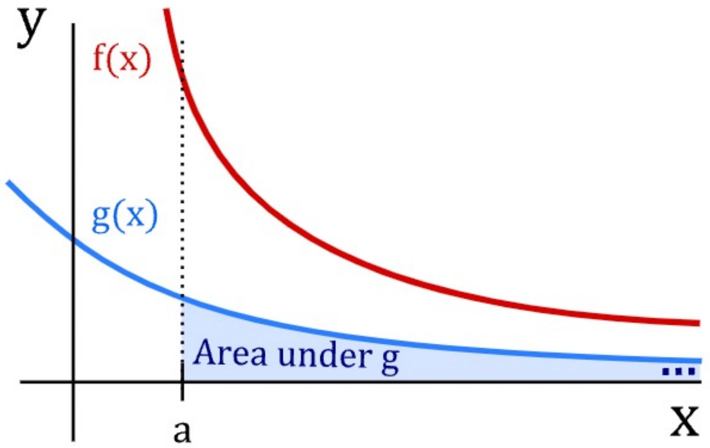
- if $\int_a^{+\infty} f(x)dx$ converges,

  $\int_a^{+\infty} g(x)dx$ also converges

- if $\int_a^{+\infty} f(x)dx$ diverges,

  $\int_a^{+\infty} g(x)dx$ also diverges





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Comparison test - Example

- Check if the following integral converges:

$$\int_{2}^{+\infty} \frac{\cos^2 x}{x^2} dx$$

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Discontinuous integrand

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Definition - 1

- If $f(x)$ is continuous on $(a;b]$, then the improper integral of f over $[a;b]$ is

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

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$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

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$$\int_0^1 \frac{1}{x^2}\,dx =$$

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$$= -1 + \lim_{t\to 0^+} \frac{1}{t} \to \infty$$

Definition - 2

- If $f(x)$ has a discontinuity at $x = c \in [a; b]$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example - 2

$$\int_0^{+\infty} \frac{1}{x^2}\,dx =$$

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Example - 2

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + \int_1^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + 1$$

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Example - 2

$$\int_{0}^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_{0}^{1} \frac{1}{x^2} dx + \int_{1}^{+\infty} \frac{1}{x^2} dx =$$

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