girafe ai

Math Refresher for DS

Practical Session 4

Solving Systems of Linear Equations

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- A – $m{\times}n$ matrix (= m equations, n variables).

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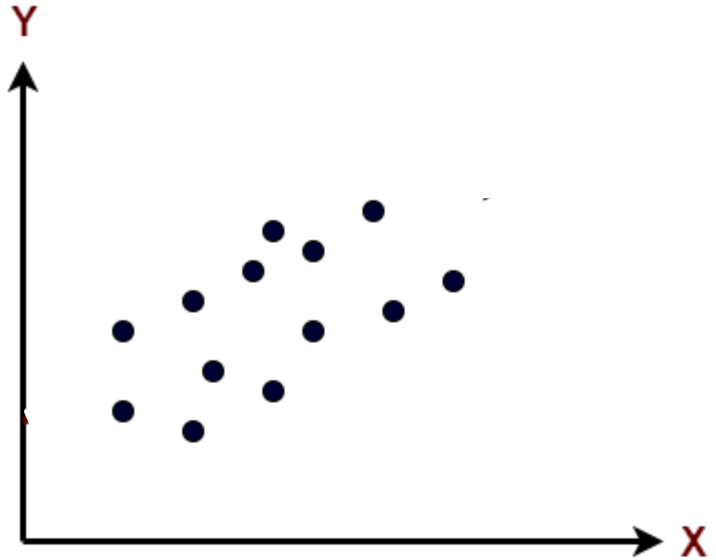
How to find a reasonable approximate solution?

Least Squares

Motivating Example

- Imagine that you have m observations:

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \dots , \qquad (x_m, y_m)$$



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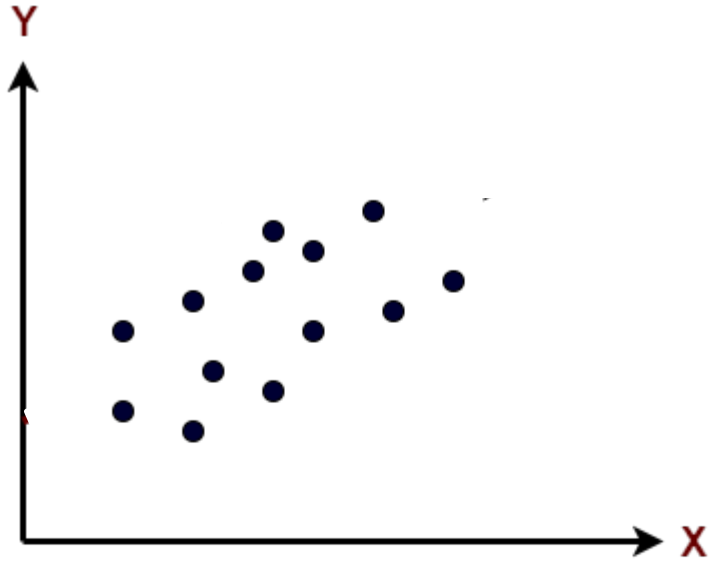

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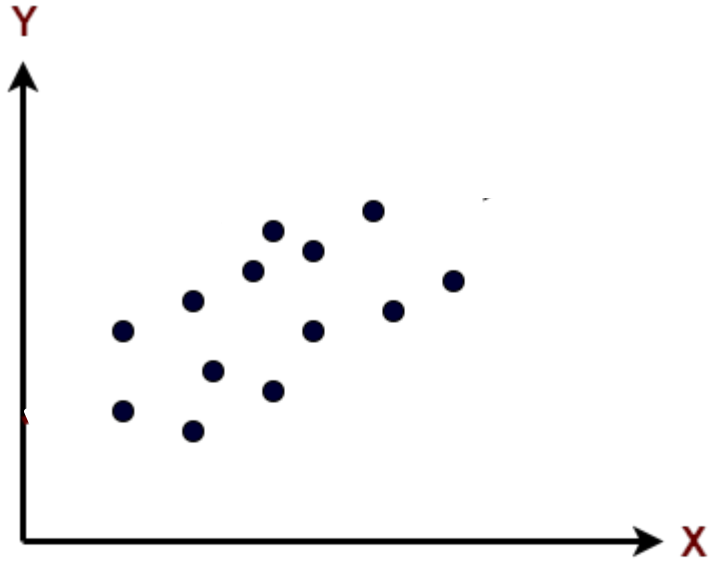

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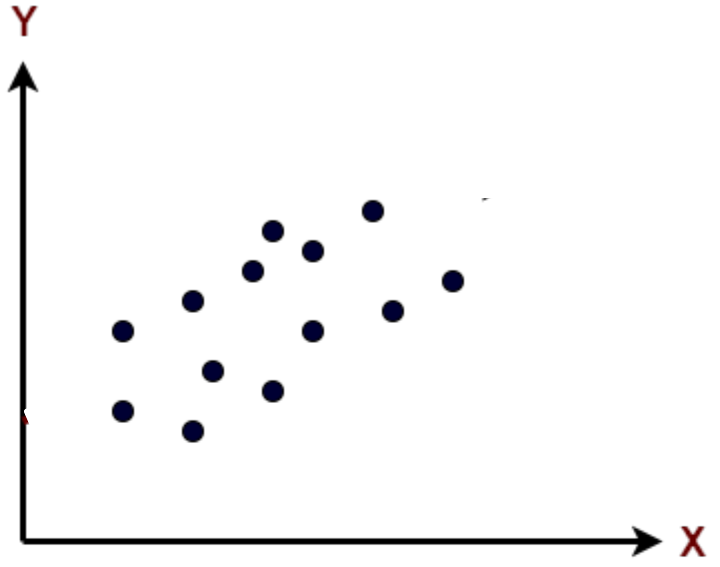

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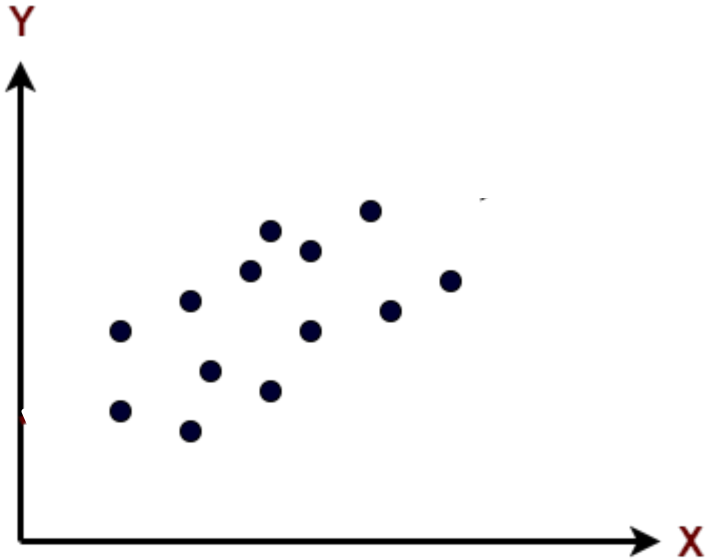

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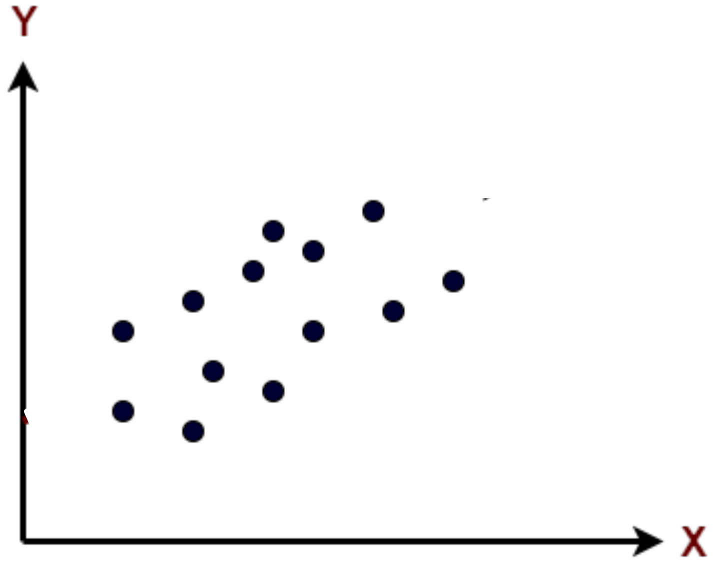

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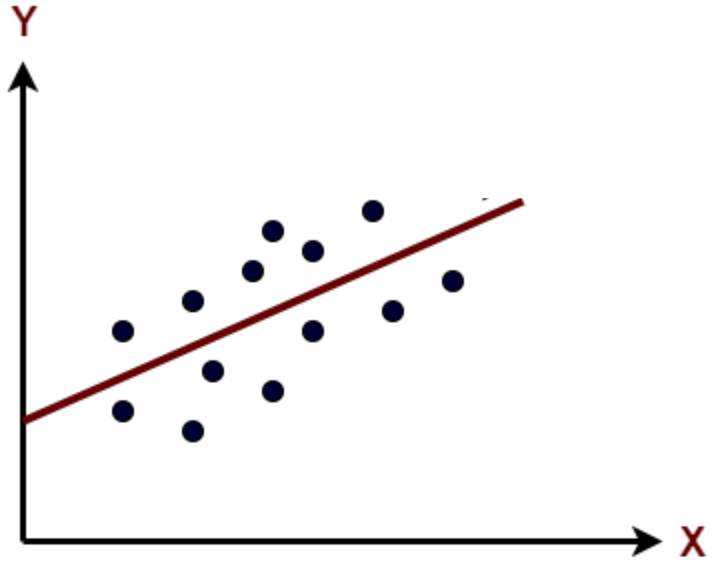

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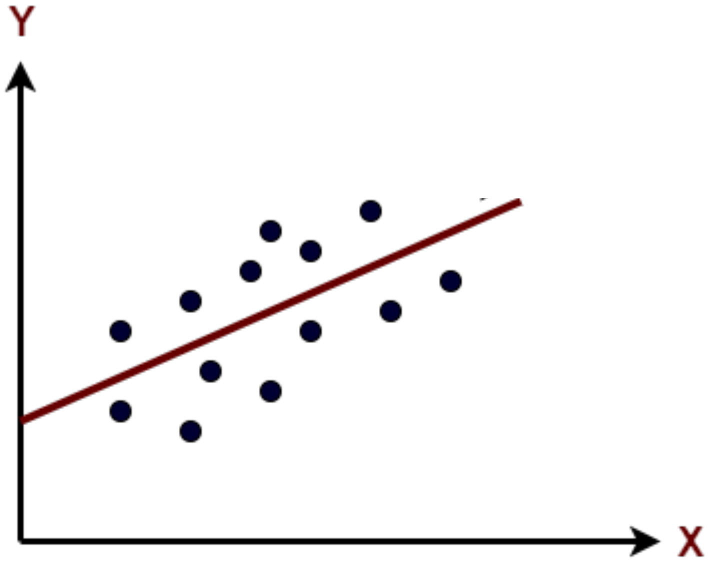

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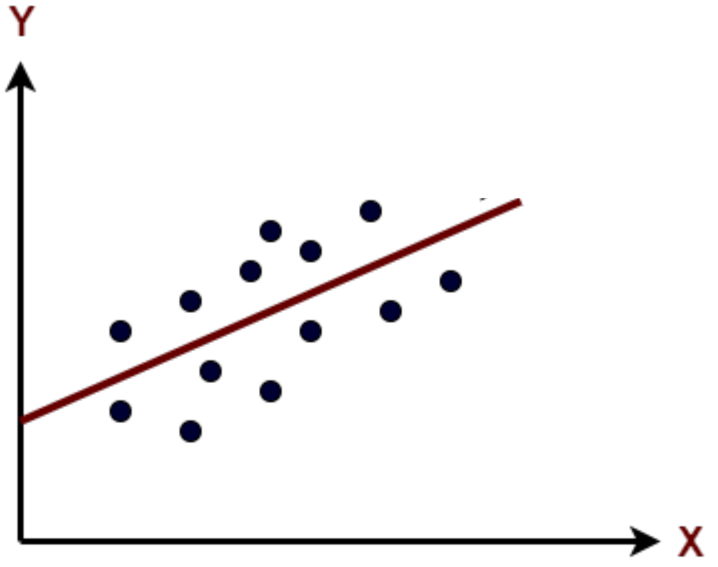

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- Let’s look at it from the Linear Algebra perspective.

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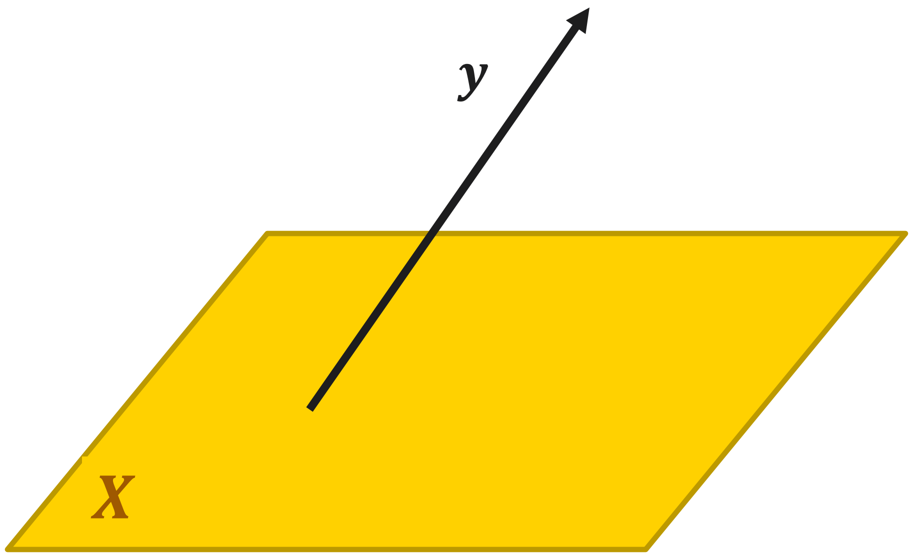

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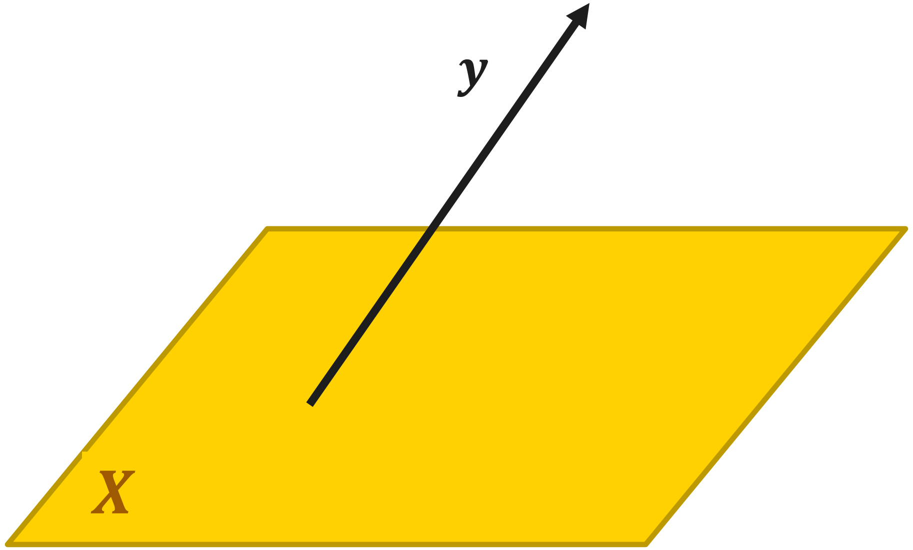

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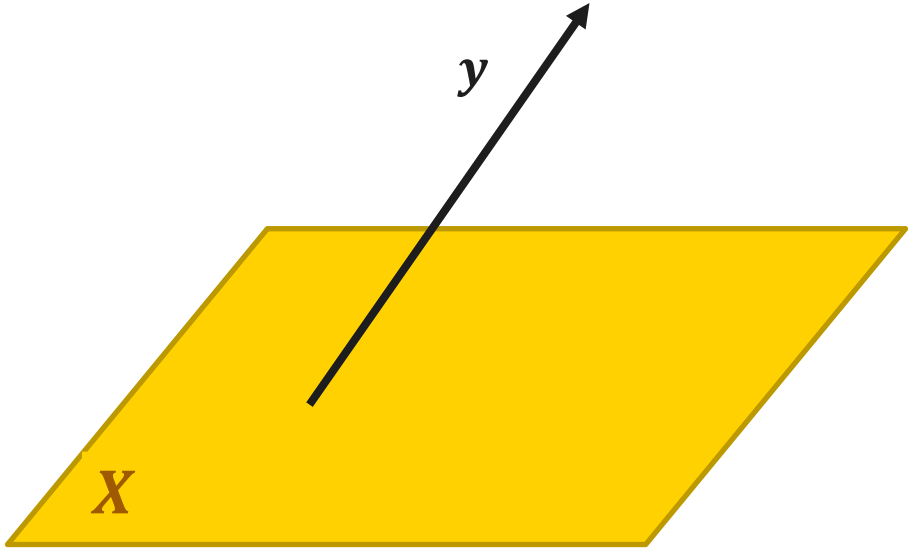

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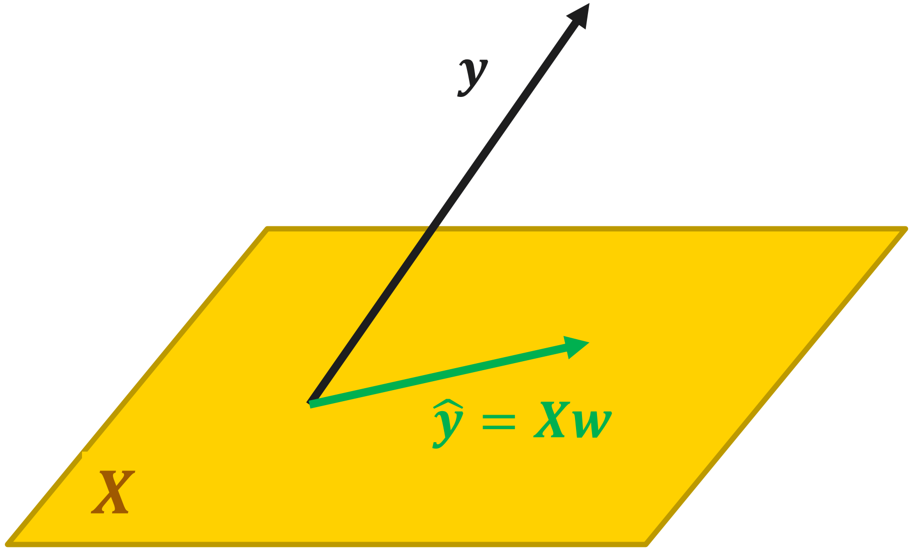

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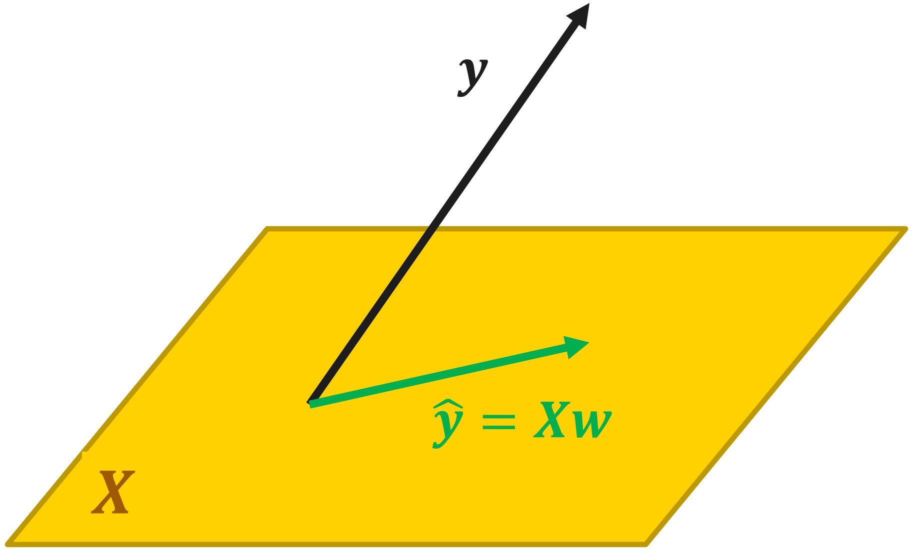

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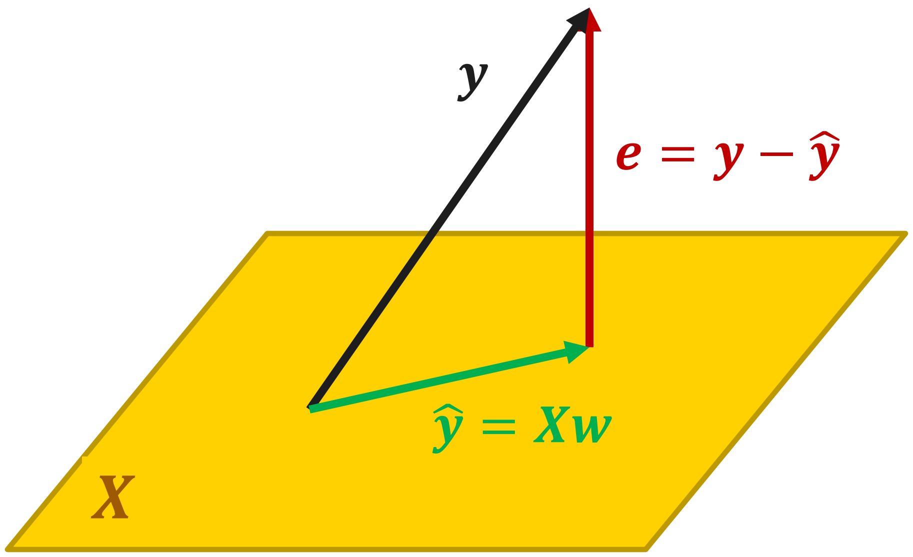

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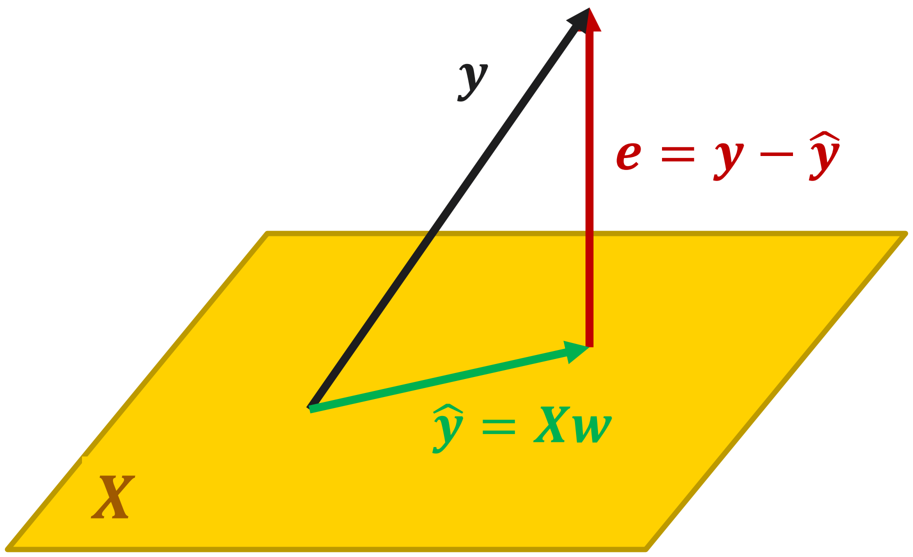

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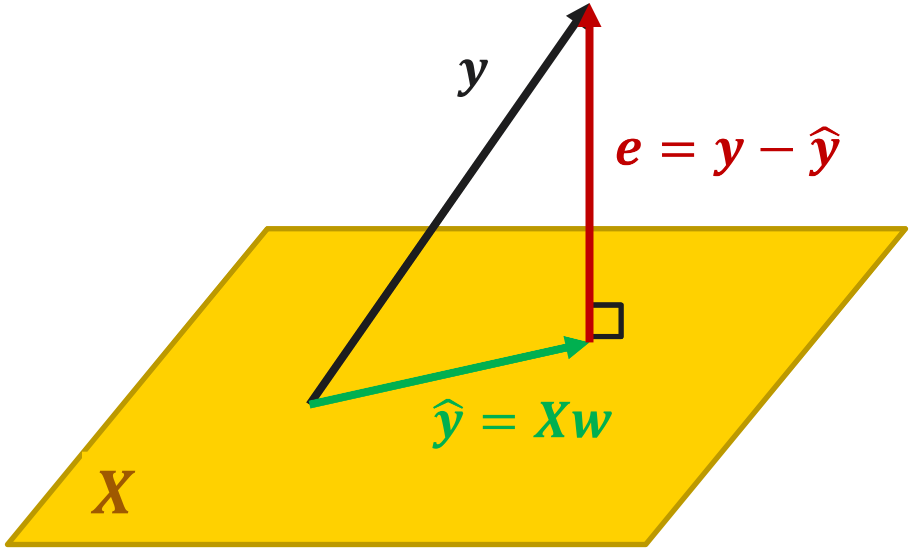

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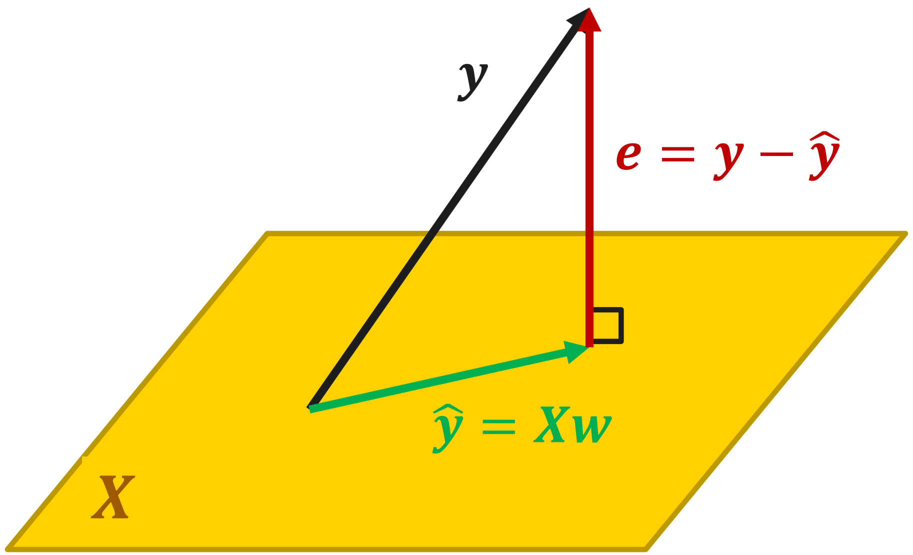

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Orthogonal Projections

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- Example:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \perp span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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$$\mathrm{V} = \mathbb{R}^3, \qquad \mathrm{W} = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \qquad \mathrm{W}_{\perp} = span\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

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- x_W is the closest vector to x in W.

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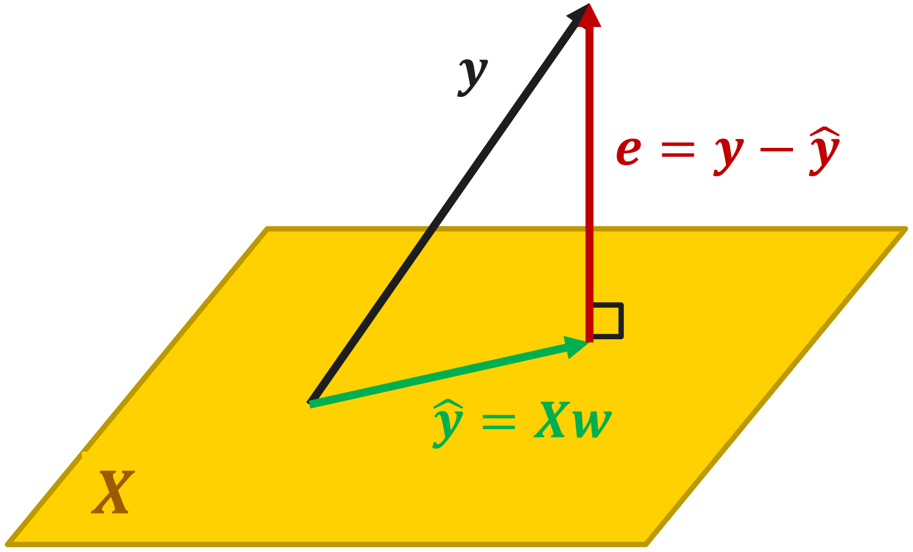

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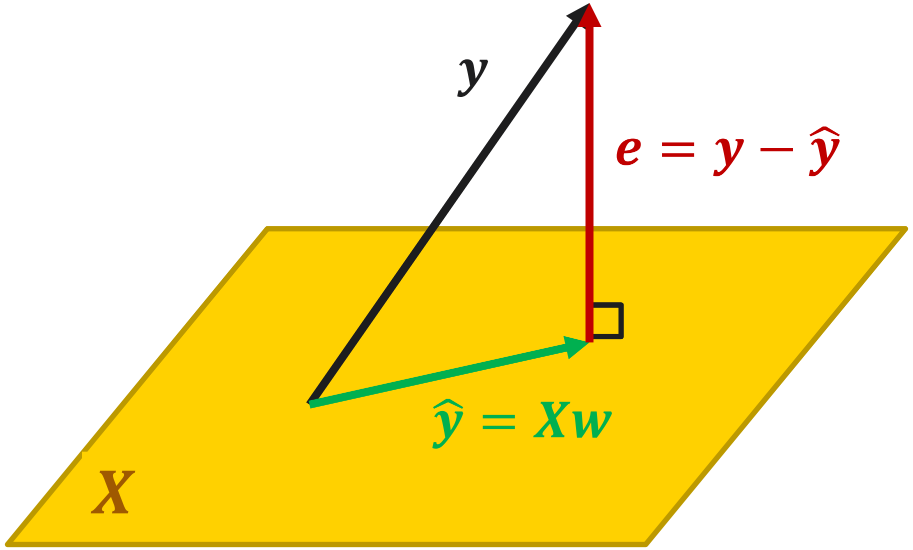

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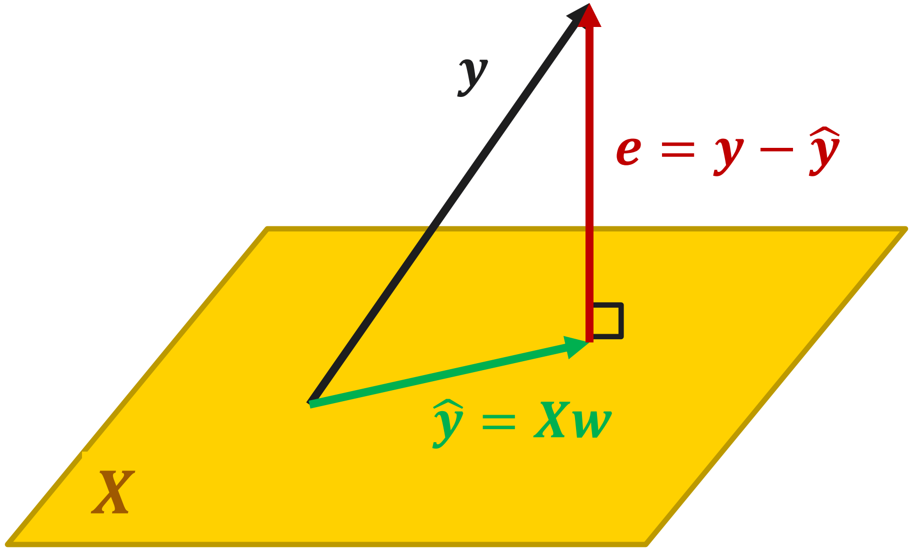

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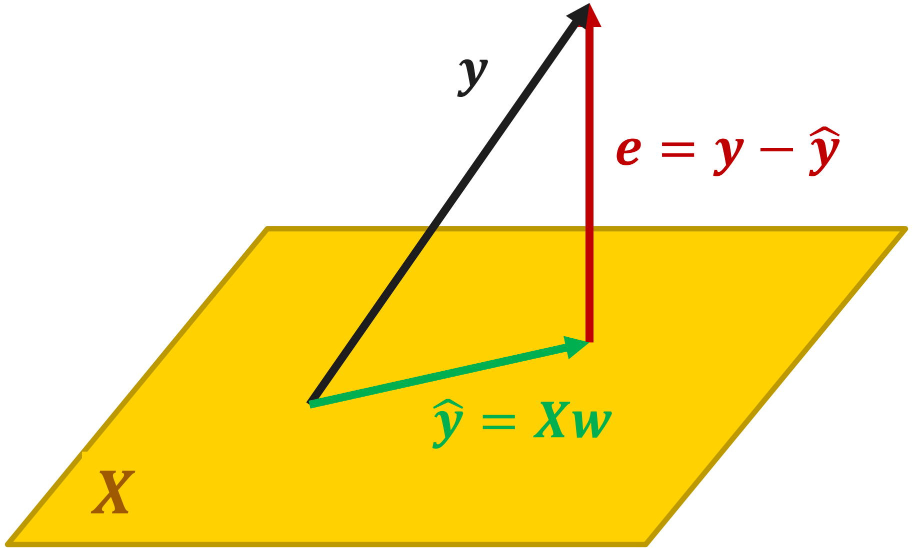
$$Xw^* = \hat{y} = y - e$$

$\hat{y}$ - orthogonal projection of y onto $col(X)$
$w^* = ?$ – optimal weights



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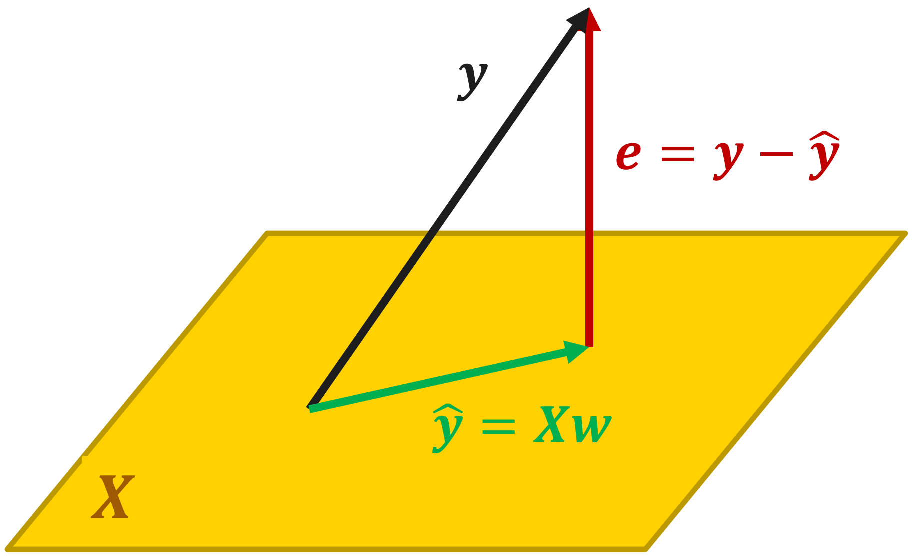

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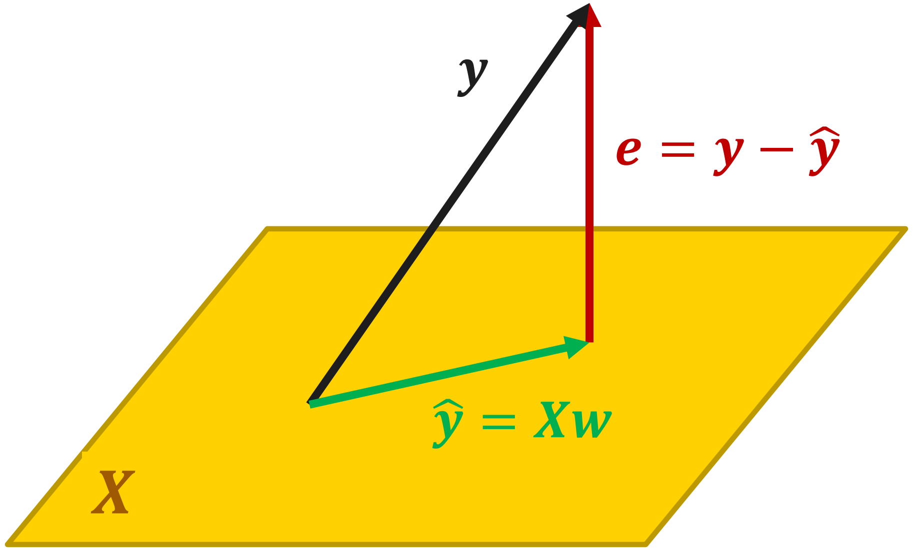

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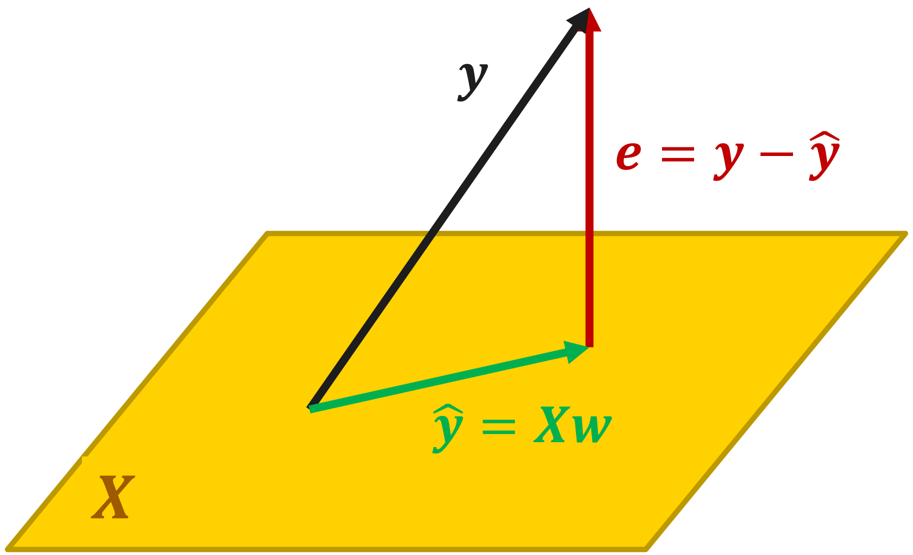

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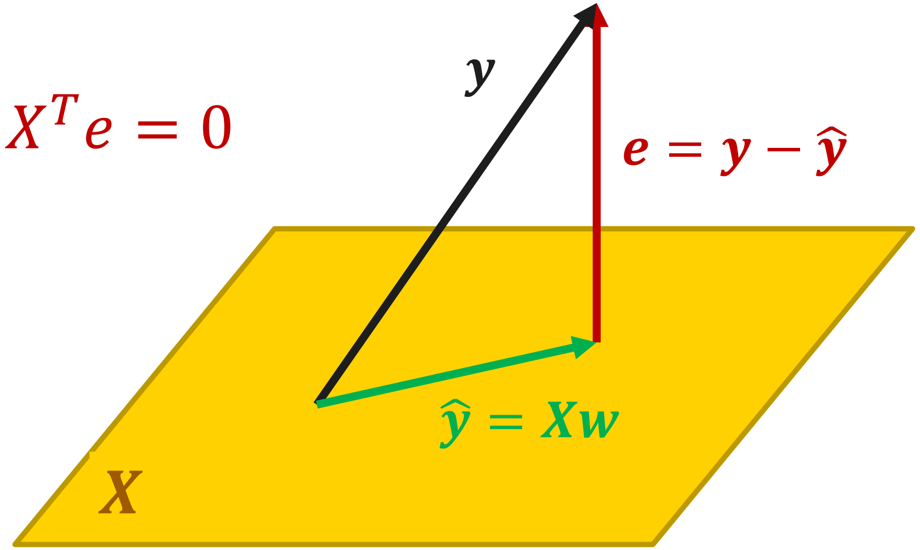

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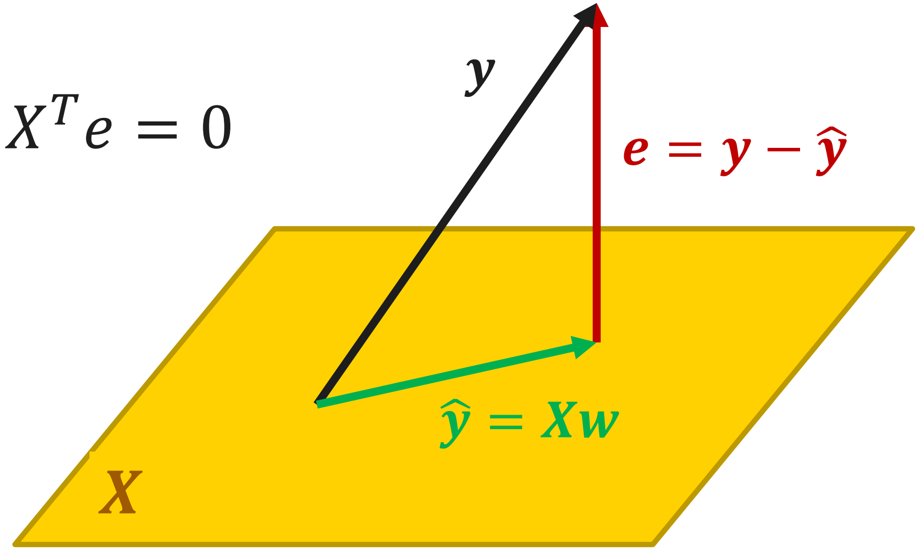
$$X^T X w^* = X^T y$$



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e is orthogonal to columns of $X \Rightarrow X^T e = 0$

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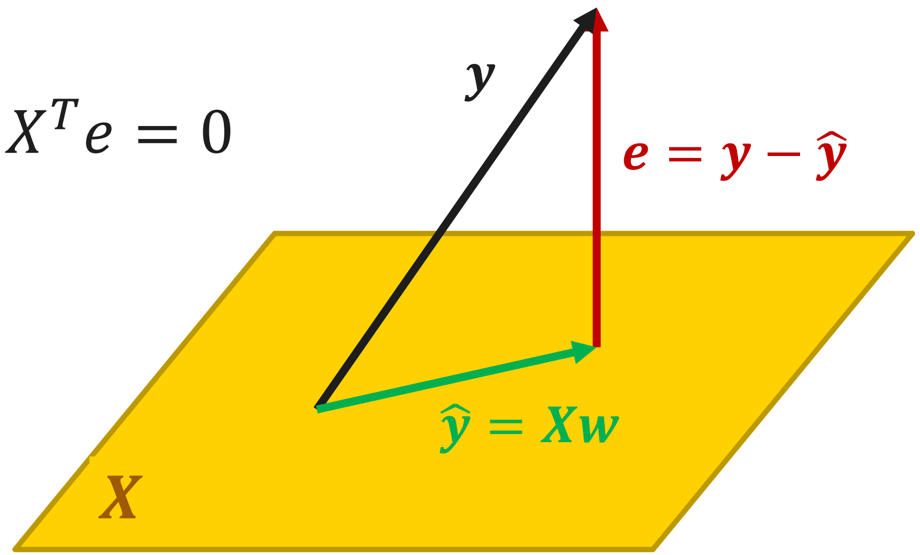

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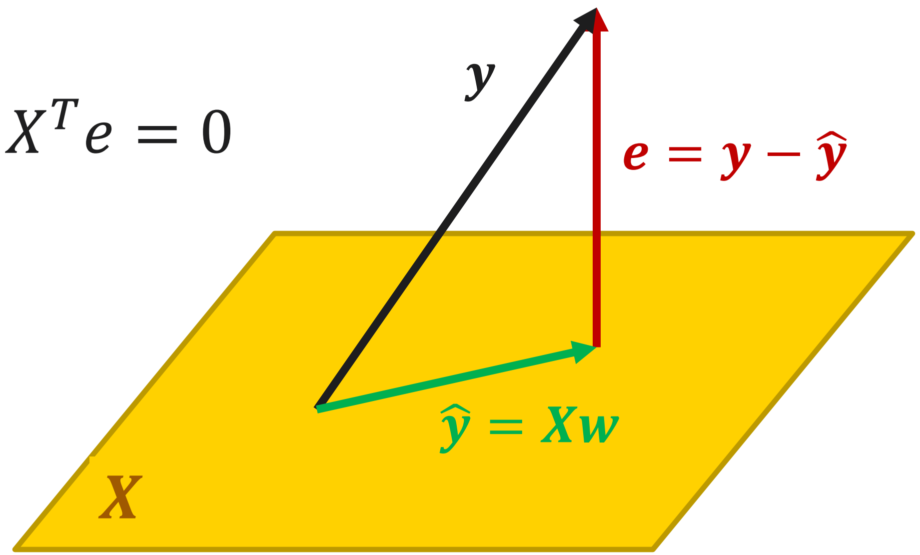

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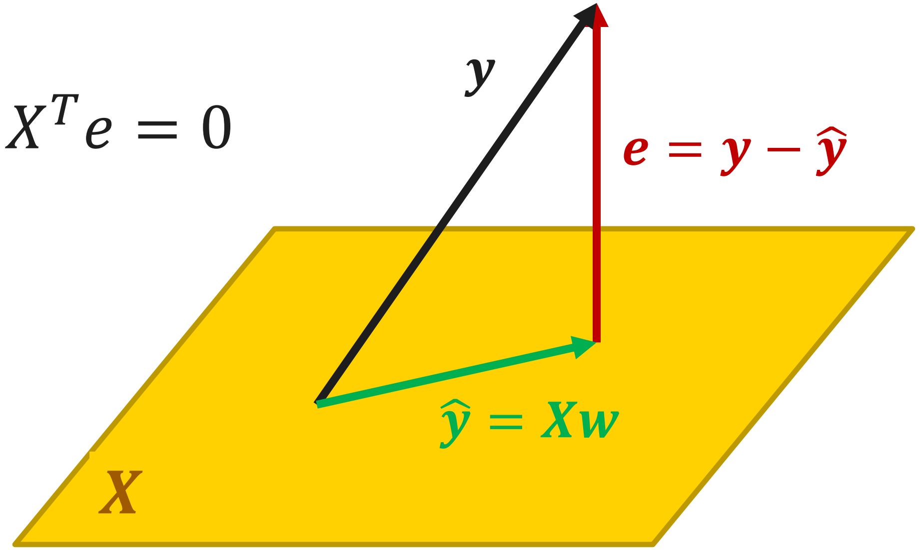

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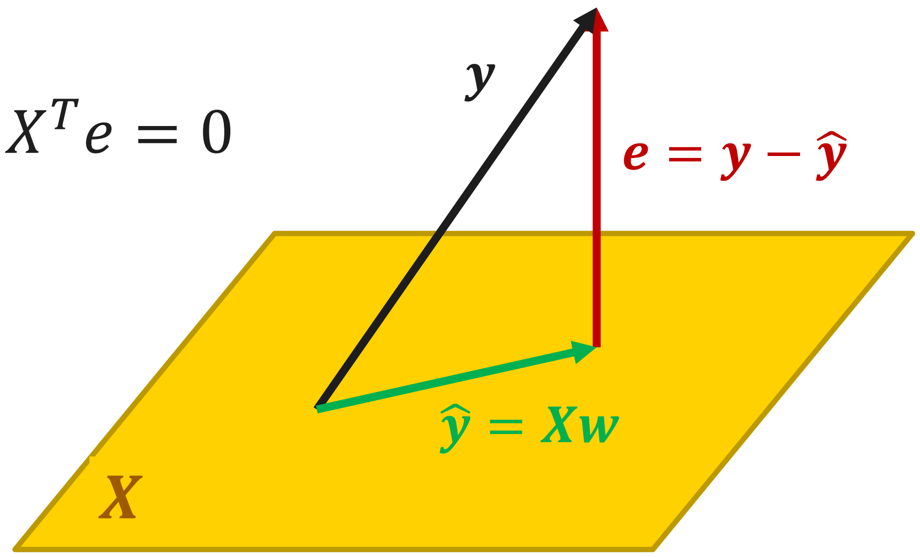

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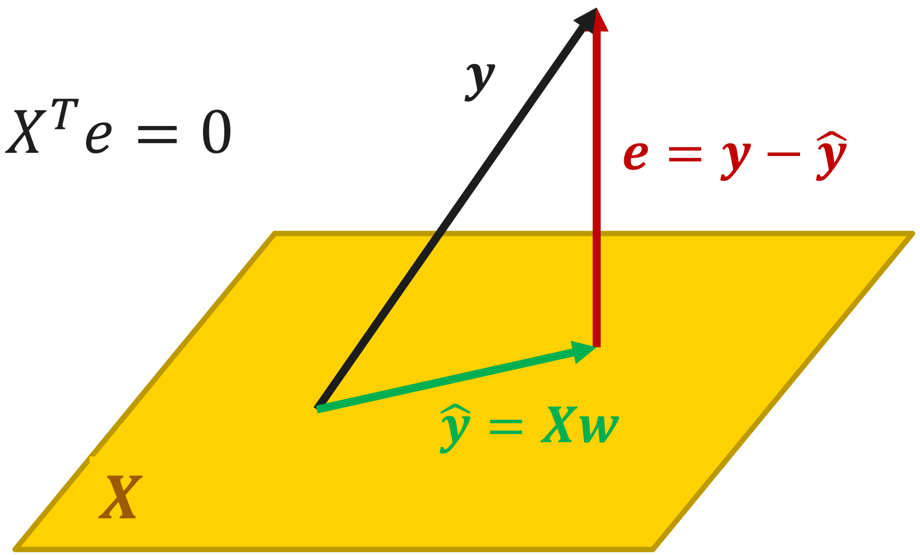

$$\hat{\boldsymbol{y}} = X w^* = \underbrace{\boldsymbol{X(X^T X)^{-1} X^T}}_{\text{projection matrix}} \boldsymbol{y}$$

Why Does $(X^T X)^{-1}$ Exist?

- We derived that $w = (X^T X)^{-1} X^T y$. But can we be sure $(X^T X)^{-1}$ exists?

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$\Longrightarrow$ contradiction.

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So, $(X^TX)^{-1}$ exists.

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Toy Example

- Observations (x_i, y_i):

$$(1,1), \qquad (2,3), \qquad (3,2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

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$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ → regression line $y = 1 + 0.5x$.

Let's practice more!

https://colab.research.google.com/drive/1v_dDH5aSx9pQG4SSCLzNjKuCk6rwNWzf?usp=sharing

Coordinates Change

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \ldots, b_n\}$ – current basis, $S = \{s_1, \ldots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

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$$\exists M^{-1} = M_{S \to B}, \qquad x_S = M^{-1} x_B$$

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- But vectors aren't the only things with coordinates...

Change of Basis for Linear Transforms

- Consider a linear transform A.
- It's defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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- $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – another basis.

How would A look like in this basis?

Change of Basis for Linear Transforms

- A – linear transform;
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Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM =$$

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$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x_S' = [A]_S \cdot x_S =$$

Change of Basis for Linear Transforms

$$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ [A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix},\ S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

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$$x_E =$$

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$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x'_E =$$

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Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} - \text{ new basis}$$

$$[A]_S = M^{-1}AM =$$

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$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

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$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

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$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

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- Another example:

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$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

We get a diagonal matrix, it’s easier to work with it!