Math Refresher for DS

Lecture 2

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Last Time

- Vector spaces
- Euclidian spaces (= vector spaces + dot product)
- Length of a vector
- Distances and angles between the vectors
- Orthogonality
- Orthogonal projections

Today

- Back to vector spaces
    - Linear independence
    - Basis

- Basic operations with matrices.

Back to Vector Spaces

(Reminder) Vector Space: Definition

- A real-valued **vector space** $(V, +, \cdot)$ is a set of vectors V with two operations

$$(1)\ +:\ \ V \times V \to V, \qquad (2)\ \cdot:\ \ \mathbb{R} \times V \to V$$

that satisfy the following properties (axioms):

	Property	Meaning
1.	**Associativity** of addition	$x + (y + z) = (x + y) + z$
2.	**Commutativity** of addition	$x + y = y + x$
3.	**Identity element** of addition	$\exists 0 \in V:\ \forall x \in V \quad 0 + x = x$
4.	**Identity element** of scalar multiplication	$\forall x \in V \quad 1 \cdot x = x$
5.	**Inverse element** of addition	$\forall x \in V\ \exists -x \in V:\ \ x + (-x) = 0$
6.	**Compatibility** of scalar multiplication	$\alpha(\beta x) = (\alpha\beta)x$
7.	**Distributivity**	$(\alpha + \beta)x = \alpha x + \beta x$
8.		$\alpha(x + y) = \alpha x + \alpha y$

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(Reminder) Examples of Vector Spaces

- $\mathbb{R}^n$ - a set of vectors with n real entries.
  $(\mathbb{R}^n, +, \cdot)$ is a vector space.

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- $\mathbb{R}^n$ - a set of vectors with n real entries.
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- $\mathbb{P}^n$ - a set of polynomials of degree $\leq n$ with real coefficients
  $(\mathbb{P}^n,\ +,\ \cdot)$ is also a vector space!
  "Vectors" here are polynomials.

Vector Subspaces

Vector Subspace

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Vector Subspace

- $V = (\mathbb{V}, +, \cdot)$ - a vector space.
- Consider $\mathbb{U} \neq \emptyset$ – a subset of $\mathbb{V}$ $(\mathbb{U} \subseteq \mathbb{V})$.
- $U = (\mathbb{U}, +, \cdot)$ - *a vector subspace* $(U \subseteq V)$ if U is a vector space with operations
    - $+ : \mathbb{U} \times \mathbb{U} \to \mathbb{U}$
    - $\cdot : \mathbb{R} \times \mathbb{U} \to \mathbb{U}$

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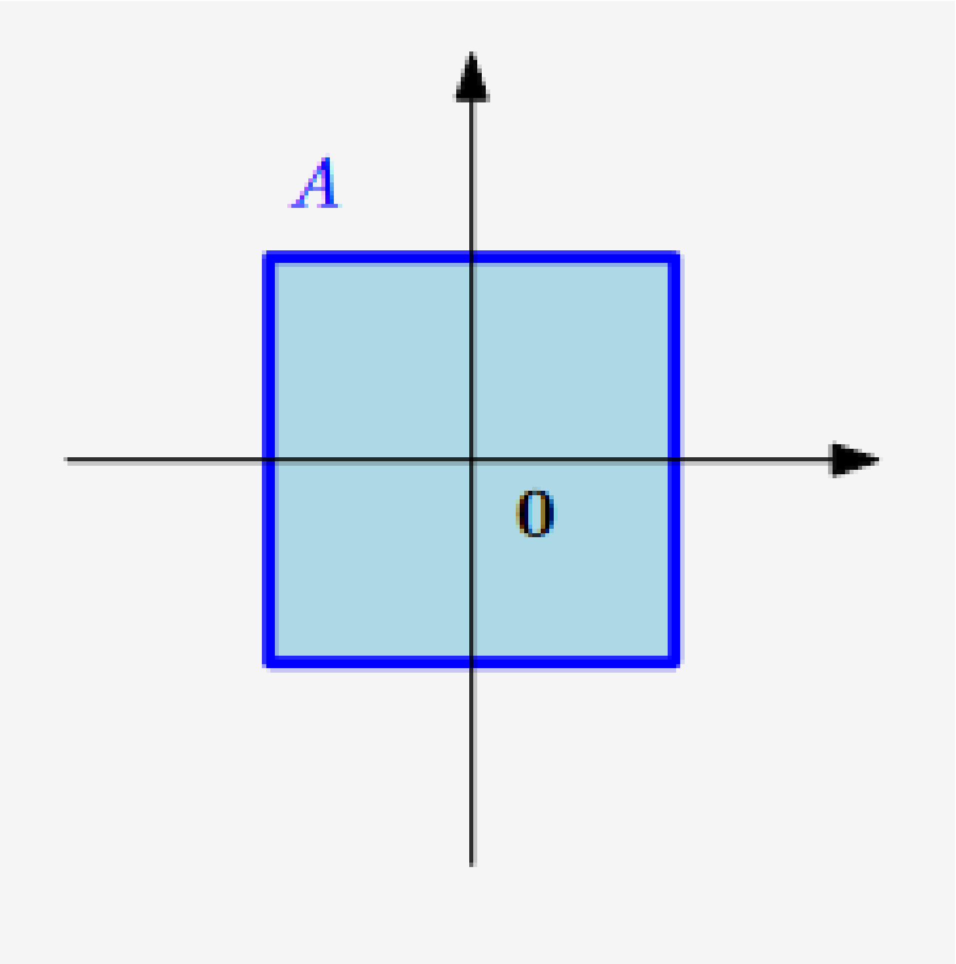
- In fact, we only need to check:

  1. that $\mathbf{0} \in \mathbb{U}$
  2. closure of $+$ and $\cdot$:
     - $\forall\, x, y \in \mathbb{U}\ \ x + y \in \mathbb{U}$
     - $\forall\, x \in \mathbb{U},\ \lambda \in \mathbb{R}\ \ \lambda x \in \mathbb{U}$

Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{A}$ a vector subspace?



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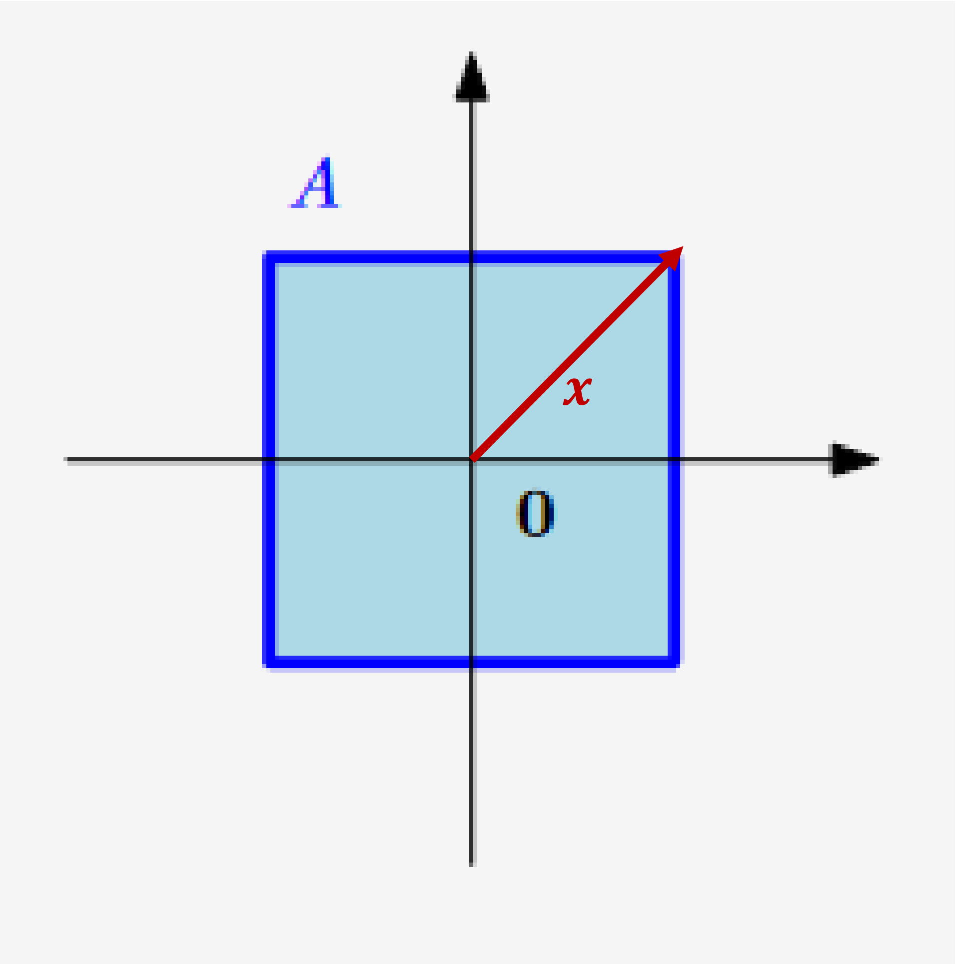

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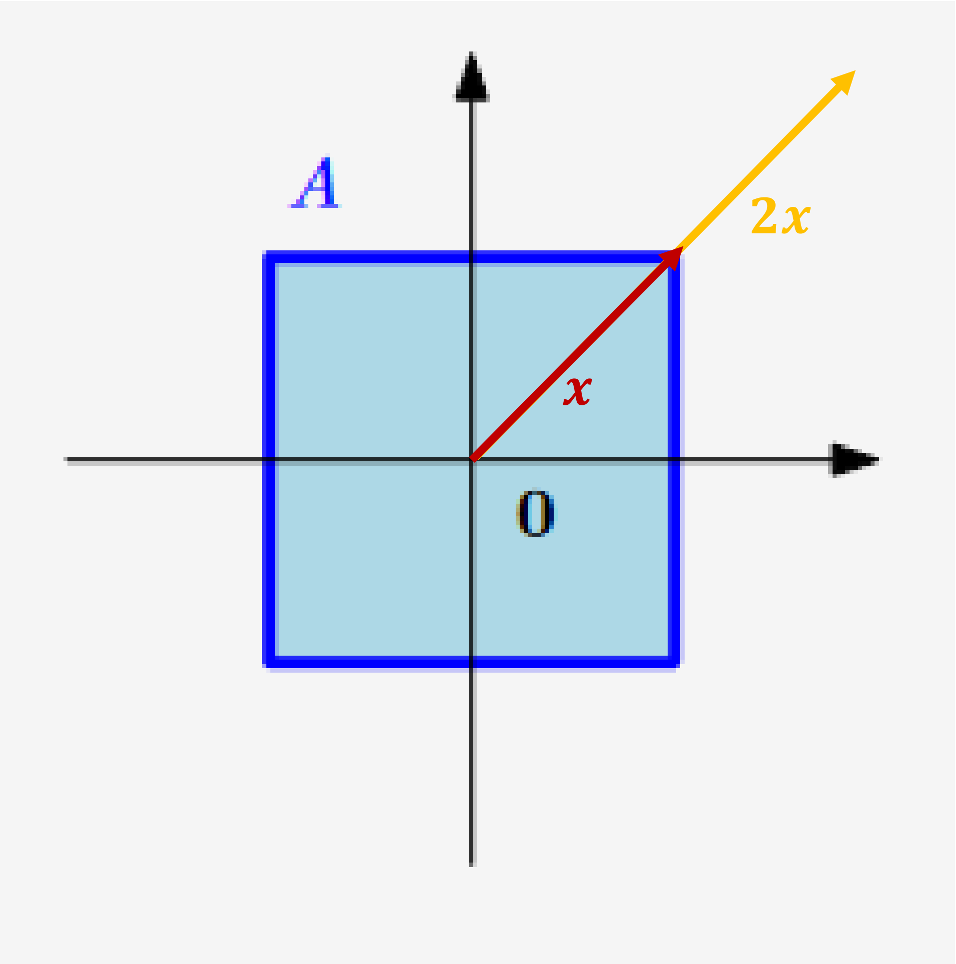
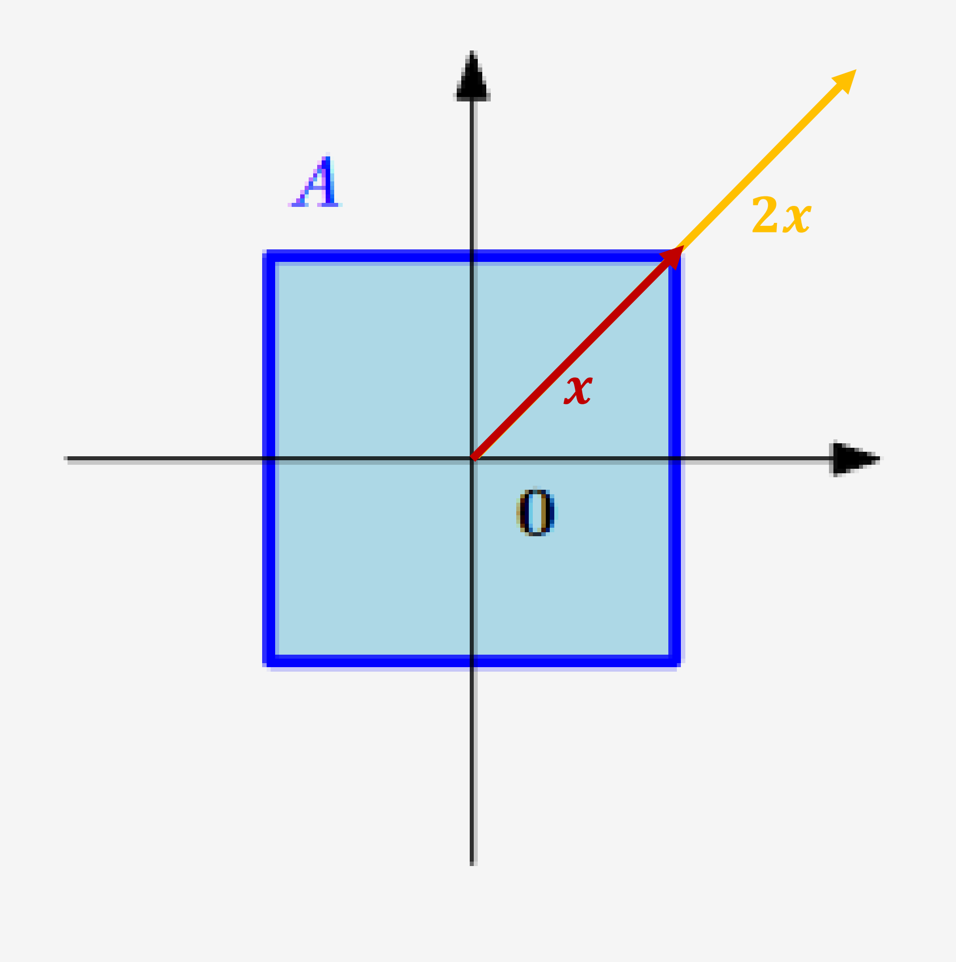


Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{A}$ a vector subspace?
- No!

$\boldsymbol{x} \in \boldsymbol{A}$ but $\mathbf{2}\boldsymbol{x} \notin \boldsymbol{A} \rightarrow$

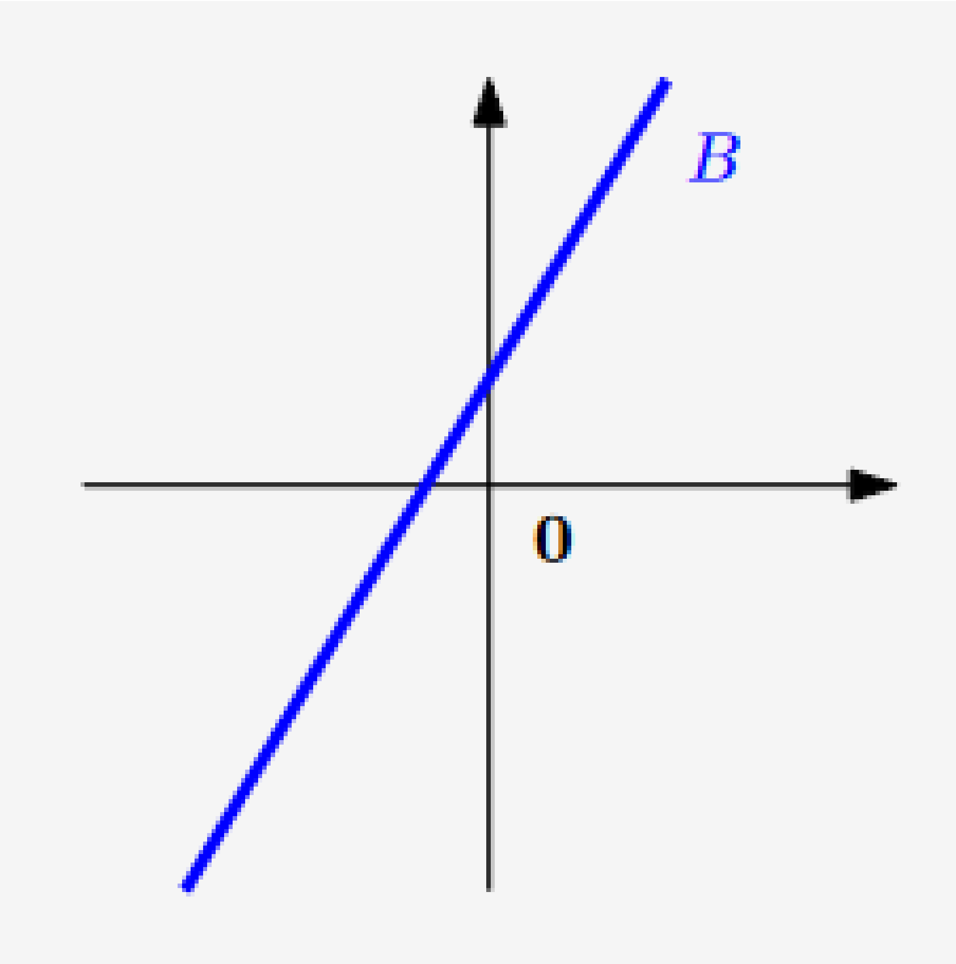
· operation isn't closed.



Vector Subspace: Examples

- Consider $\mathbb{R}^2$.

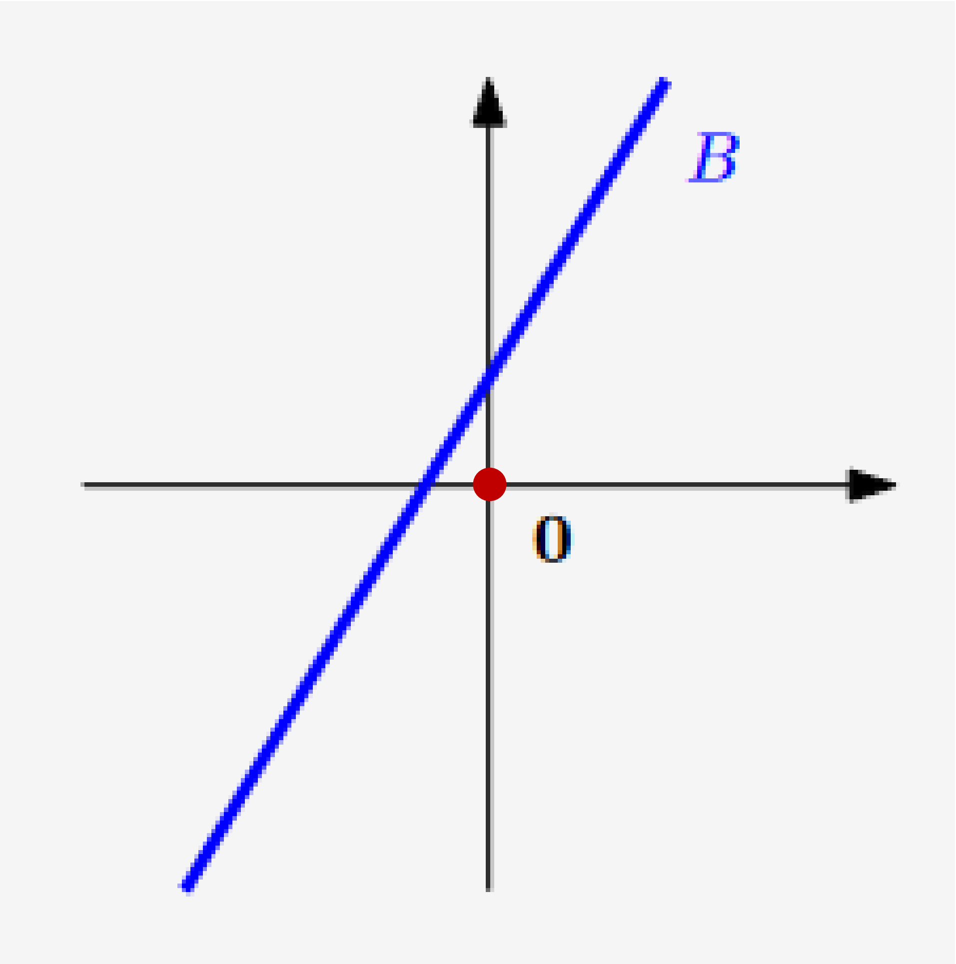
- Is $\boldsymbol{B}$ a vector subspace?



Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{B}$ a vector subspace?
- No!

$$0 \notin B$$



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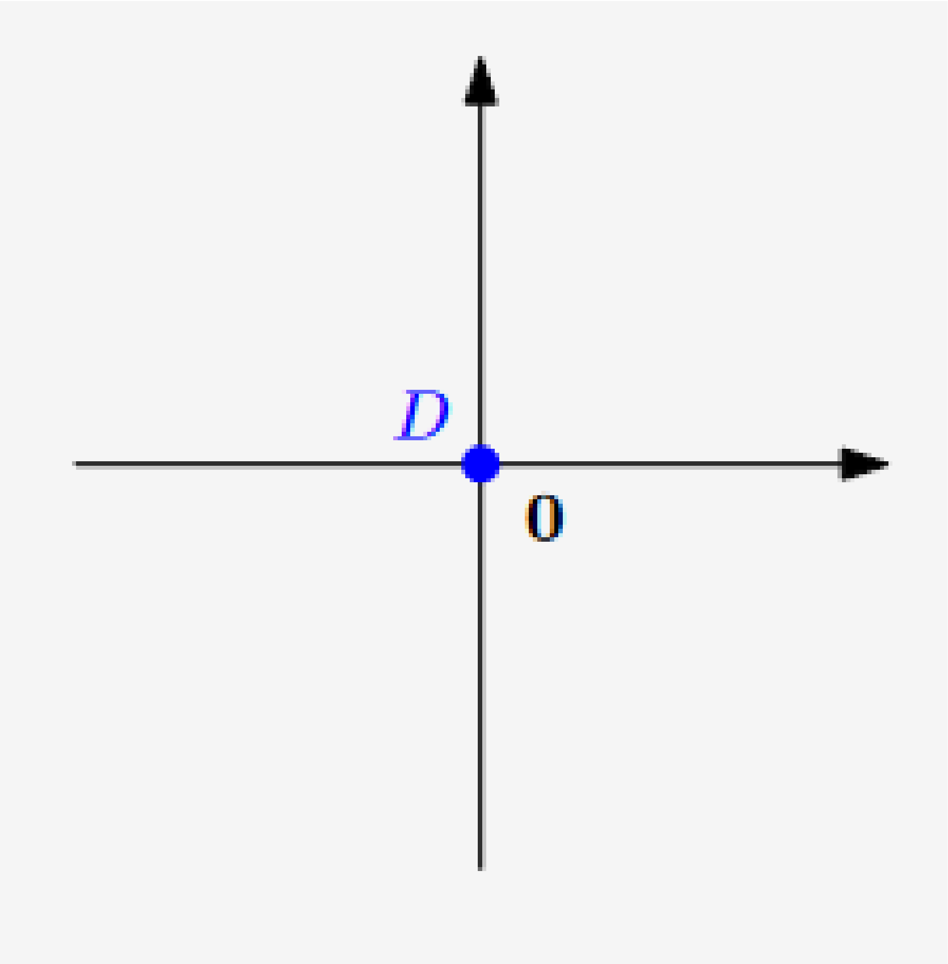

Vector Subspace: Examples

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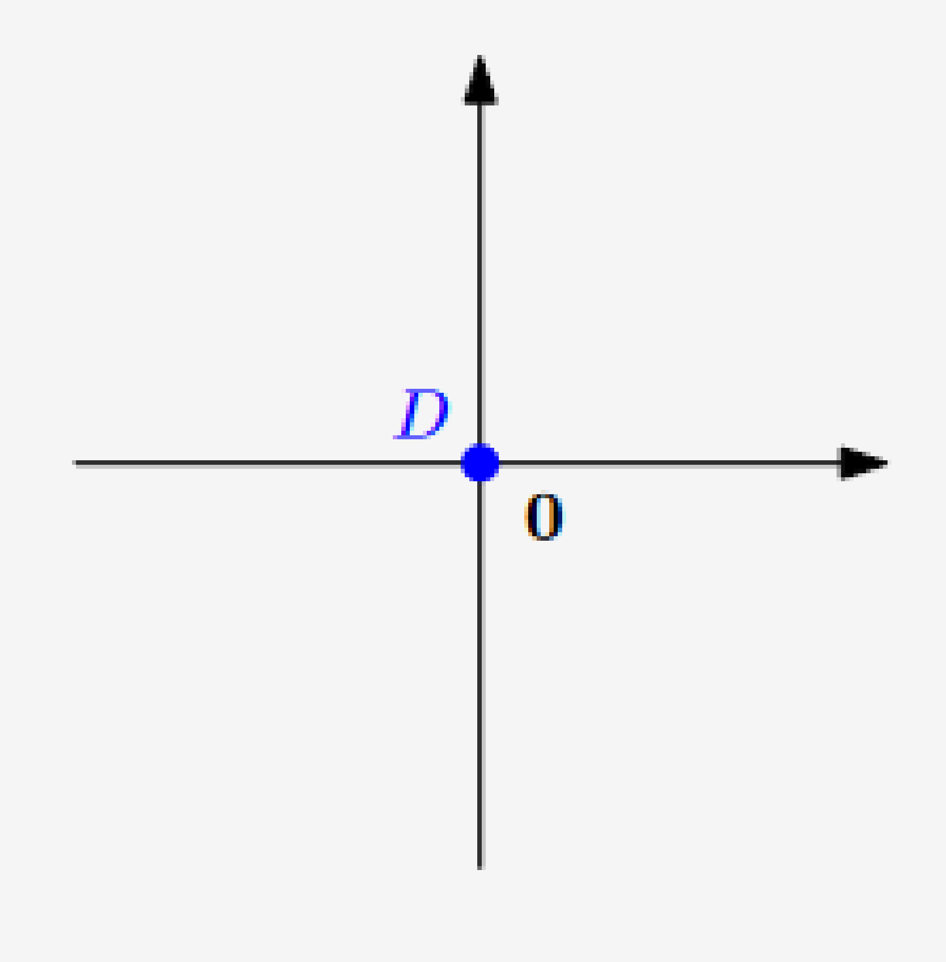
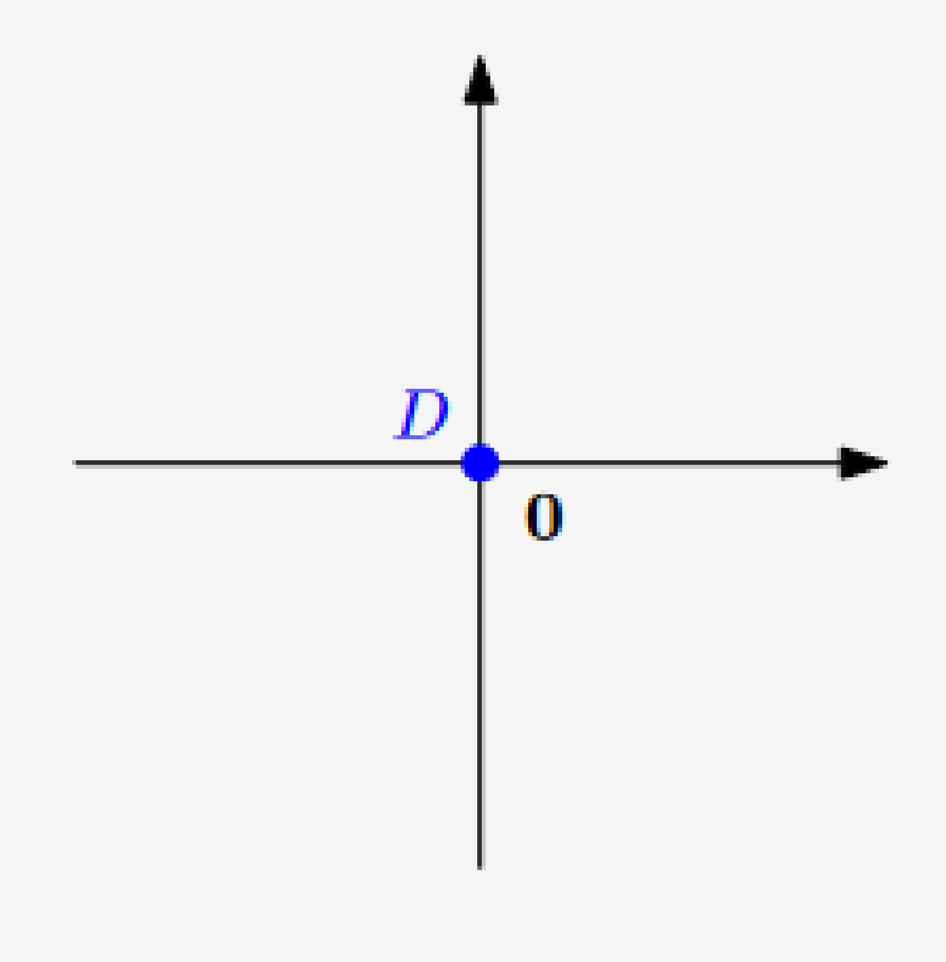


Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{D}$ a vector subspace?
- Yes!

$\{0, +, \cdot\}$ *is a trivial vector subspace of any vector space*



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Vector Subspace: Examples

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- $\forall\, m \leq n \;\; P^m = (\mathbb{P}^m, +, \ \cdot) \subseteq P^n$ - a vector subspace:
  - $0 \in \mathbb{P}^m$
  - Closure: when we add up polynomials of degree $m \leq n$ or multiply them by a scalar, we always get a polynomial of of degree $m \leq n$.

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$$v = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i \in V \text{ –}$$

a *linear combination* of $x_1, x_2, \dots, x_k$.

Linear Combinations: Examples

- In $(\mathbb{R}^2, +, \cdot)$, consider vectors $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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  $u = 2t^2 - t + 1 = 2e_2 - e_1 + e_0$ is a linear combination of e_0, e_1 and e_2.

  $v = 3t + 3 = 3e_1 + 3e_0$ is a linear combination of e_0 and e_1.

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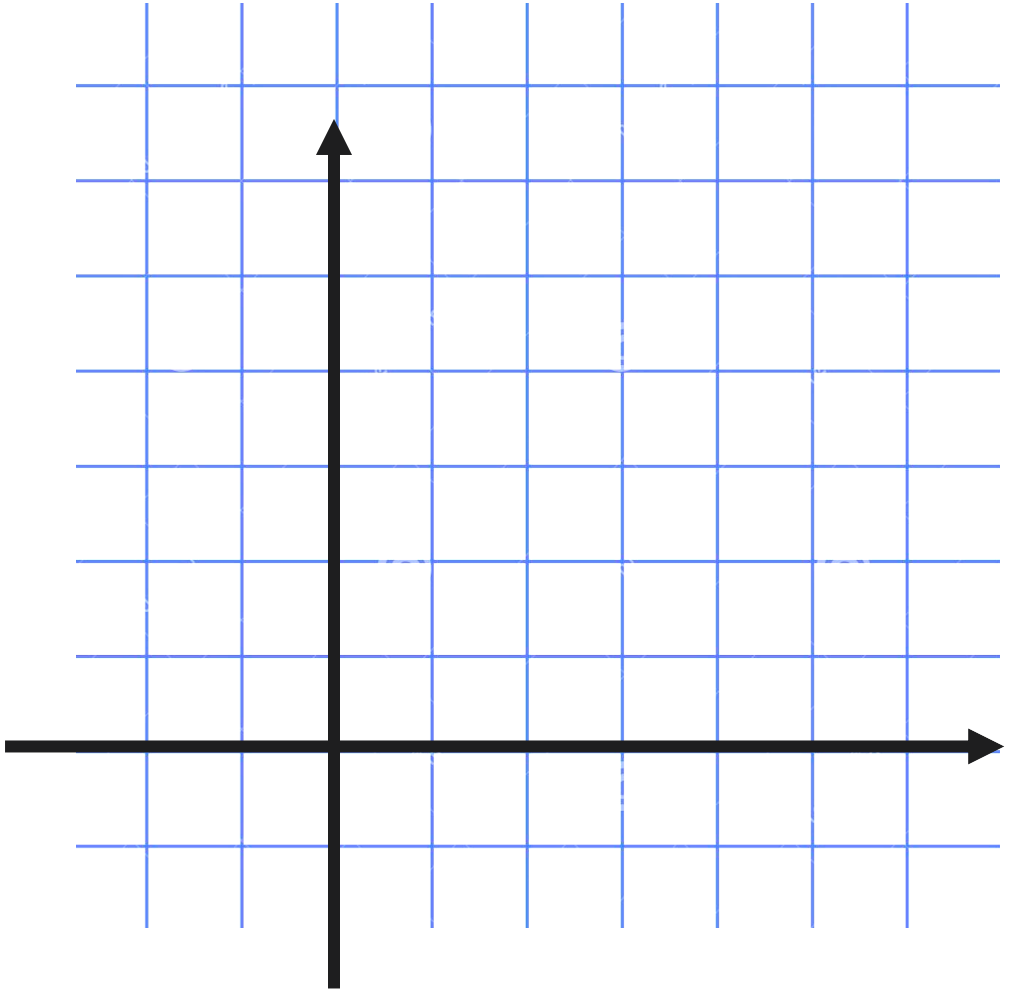
- If $\mathbb{A}$ spans vector space V, we write

$$V = span[\mathbb{A}] \text{ or } V = span[x_1, \dots, x_n].$$

Span: Example 1

- Consider $\mathbb{R}^2$.



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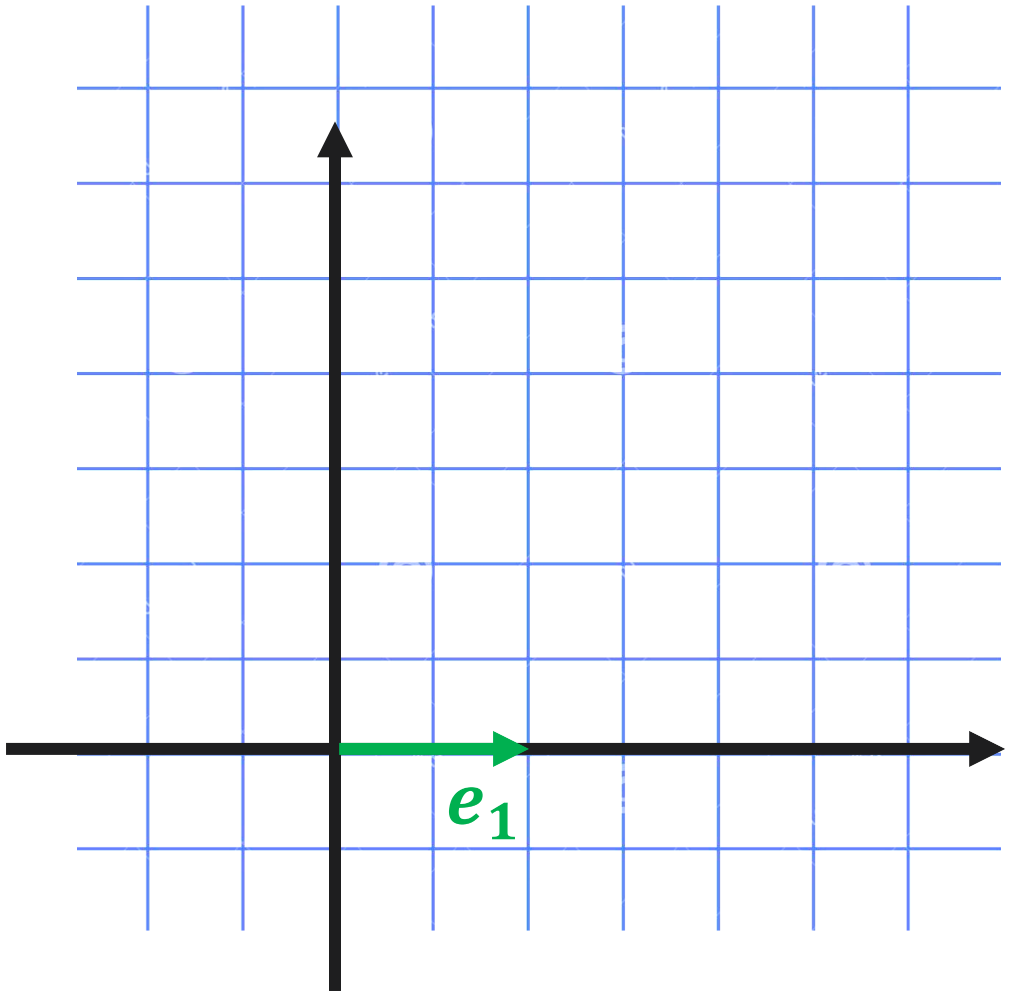

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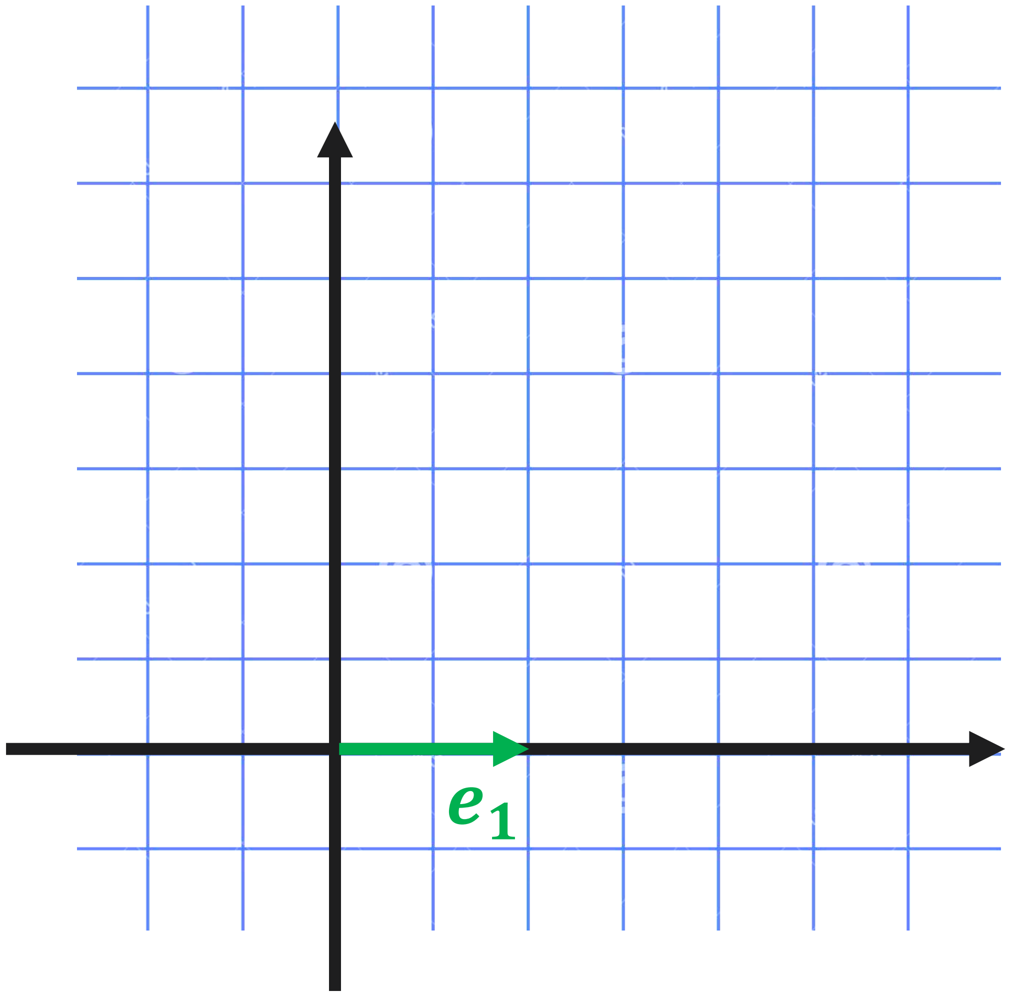
  $span[e_1] = \left\{ \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \mid \lambda \in \mathbb{R} \right\}$ - only vectors parallel to the first axis.



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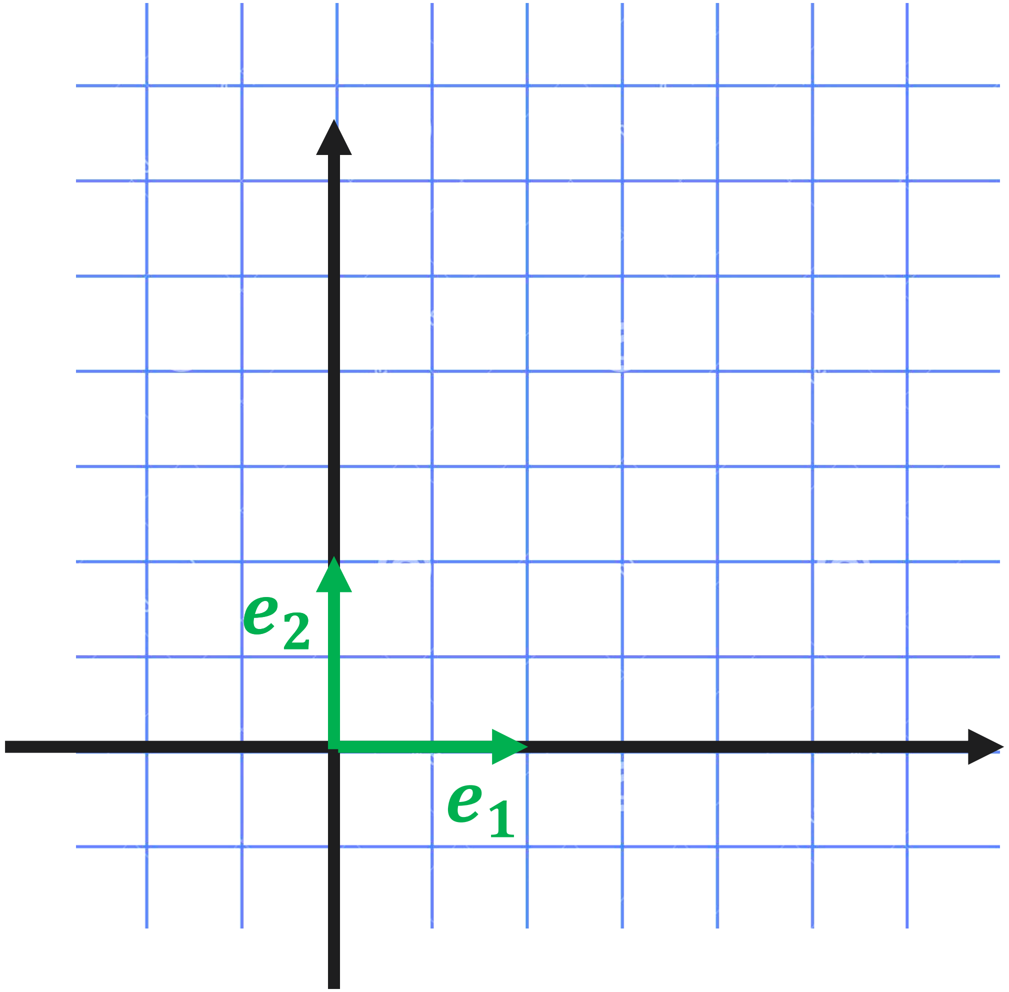
- $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



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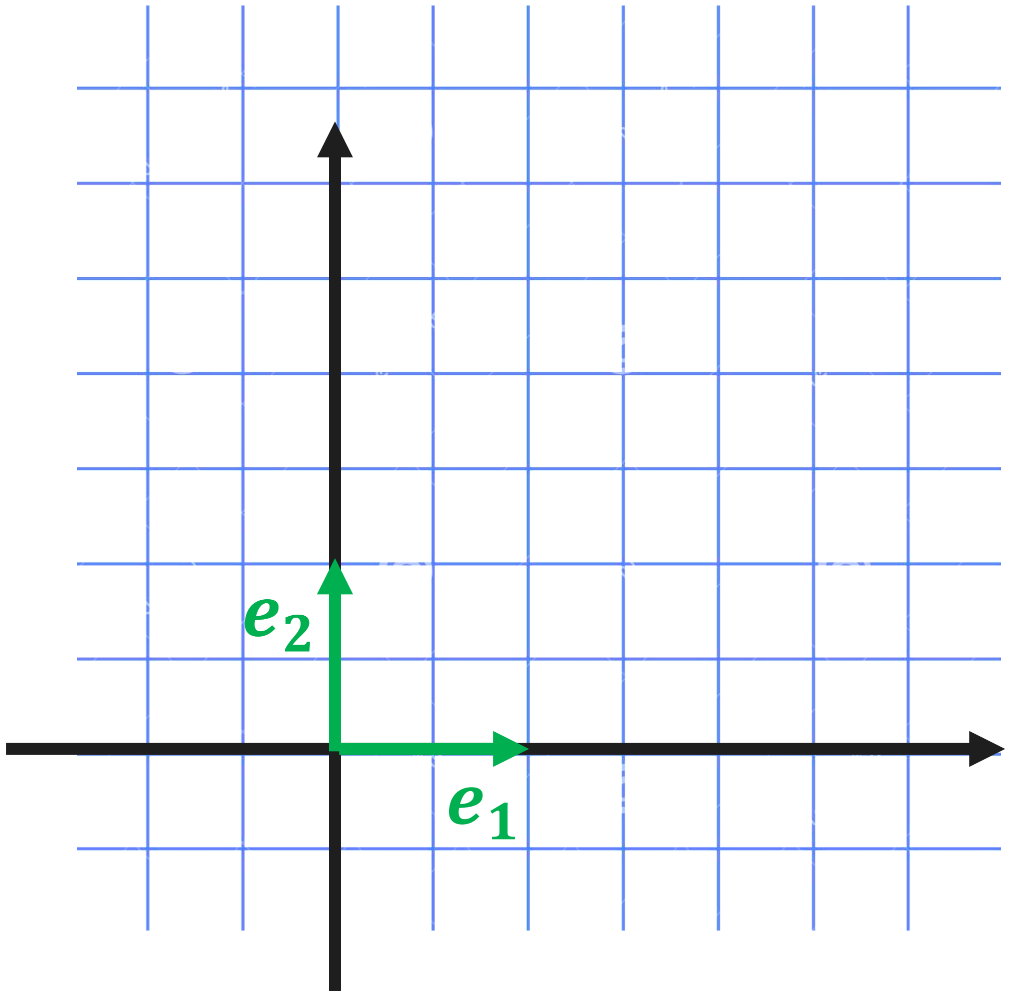
  $span[e_1] = \left\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \mid \lambda_{1,2} \in \mathbb{R} \right\} = \mathbb{R}^2.$



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Generating Set

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- If every vector $\mathbf{v} \in V$ can be expressed as a linear combination of $x_1, x_2, \dots, x_k$, $\mathbb{A}$ is called a *generating set* for V.

Linear independence

Linear Combinations

- A zero vector can always be represented as a trivial linear combination of $x_1, x_2, \dots, x_k$:

$$0 = \sum_{i=1}^{k} 0 \cdot x_i$$

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Linear Combinations

- A zero vector can always be represented as a trivial linear combination of $x_1, x_2, \dots, x_k$:

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- We are mostly interested in *non-trivial linear combinations* of $x_1, x_2, \dots, x_k$ where not all λ_i are $\mathbf{0}$.

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- If only trivial solution exists, vectors $x_1, x_2, \dots, x_k$ are *linearly independent.*

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$$\leftrightarrow$$

- A set of vectors $x_1, x_2, \dots, x_k$ is linearly dependent if and only if (at least) one of the vectors is a linear combination of the others

$$x_i = \alpha_1 x_1 + \cdots + \alpha_k x_k$$

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Why is this so? Try to prove this yourself.

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- Consider $\mathbb{R}^2$.
- Vectors $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ are not linearly independent: $u = -2v$.
- Vectors $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent:
  there are no $\lambda_1, \lambda_2 \in \mathbb{R}$ with at least one $\lambda_i \neq \mathbf{0}$ such that
  $\lambda_1 e_1 + \lambda_2 e_2 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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  $$\lambda_1 \boldsymbol{e_1} + \lambda_2 \boldsymbol{e_2} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  (Or: you cannot represent $\boldsymbol{e_1}$ as $\lambda \boldsymbol{e_2}$ or vice versa).

Linear (In)dependence: Example 2

- Consider $P = (\mathbb{P}^3, +, \ \cdot)$.
- $1, \ t, \ t^2 \in P$ – vectors. Are they linearly independent?

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- Consider $P = (\mathbb{P}^3, +, \ \cdot)$.
- $1, \ t, \ t^2 \in P$ – vectors. Are they linearly independent?

- Yes!

  There is no way we can represent one of those vectors as a linear combination of the others.

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Linear (In)dependence: Example 3

- Are the following vectors in $\mathbb{R}^4$ linearly independent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Linear (In)dependence: Example 3

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- Are there $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with at least one $\lambda_i \neq 0$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

Linear (In)dependence: Example 3

- Are the following vectors in $\mathbb{R}^4$ linearly independent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

- Are there $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with at least one $\lambda_i \neq 0$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \leftrightarrow$$

$$\lambda_1 \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 - \lambda_3 \\ 2\lambda_1 + \lambda_2 - 2\lambda_3 \\ -3\lambda_1 + \lambda_3 \\ 4\lambda_1 + 2\lambda_2 + \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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- Are there $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with at least one $\boldsymbol{\lambda_i \neq 0}$ such that

$$\begin{cases} \lambda_1 + \lambda_2 - \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 - 2\lambda_3 = 0 \\ -3\lambda_1 + \lambda_3 = 0 \\ 4\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{cases}$$

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Next lecture: a better way to solve such systems of equations

Dimension of a Linear Space

- The linear space V is called ***n**-dimensional* if the maximal number of linearly independent vectors in it is equal to n.

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  We denote this as $\mathbf{dim}(V) = n$.

Dimension: Example

- $\mathbb{R}^n$ is a n-dimensional vector space. Why?
- Consider n vectors $e_1, \dots, e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- $e_1, \dots, e_n$ are linearly independent → $\mathbf{dim}(\mathbb{R}^n) \geq n$.
- Can there be more than n linearly independent vectors in $\mathbb{R}^n$?

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  No!
  Explanation: next lecture.

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Dimension: Another Example

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  $\rightarrow \dim(P^3) = 4$.

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Basis

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- A set of n linearly independent vectors $e_1, e_2, \ldots, e_n$ in an n-dimensional space V is called a *basis* for V.

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- Basis is A set of vectors with which we can represent every vector in the vector space by adding them together and scaling them.

Basis: Example

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- $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^2$.

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- $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^2$.

- $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \dots, \ e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^n$.

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- $e_0 = 0, \ e_1 = t, \ \dots, e_n = t^n$ is a basis for P^n.

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Basis: Example

- Find the basis of a vector space spanned by vectors

$$x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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- Vector z is a linear combination of x and y: $z = x - y$.

- Therefore, $V = span[x, y, z] = span[x, y]$.
  $B = \{x, y\}$ - basis of V.

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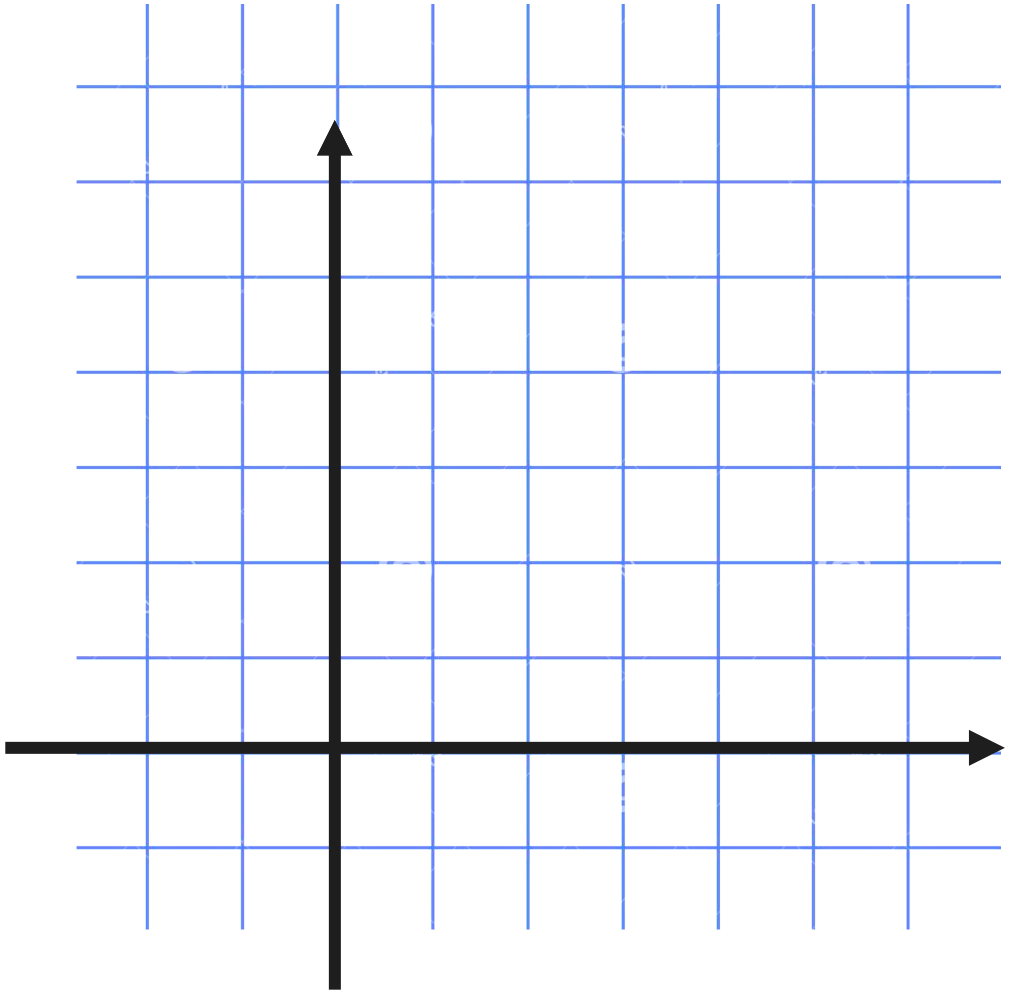
- $a_1, a_2, \dots, a_n$ - *coordinates* of the vector v in the basis $e_1, e_2, \dots, e_n$.

Coordinates: Example

- Consider $\mathbb{R}^2$.



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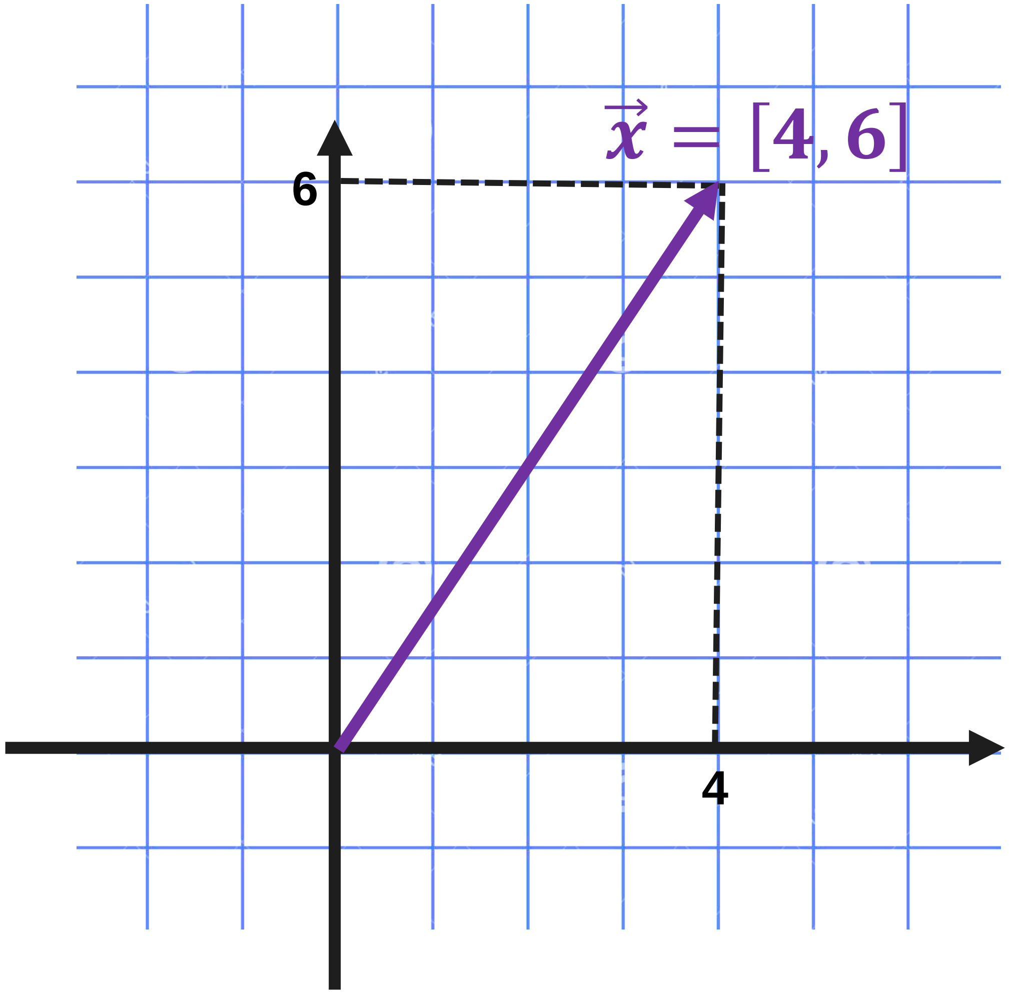
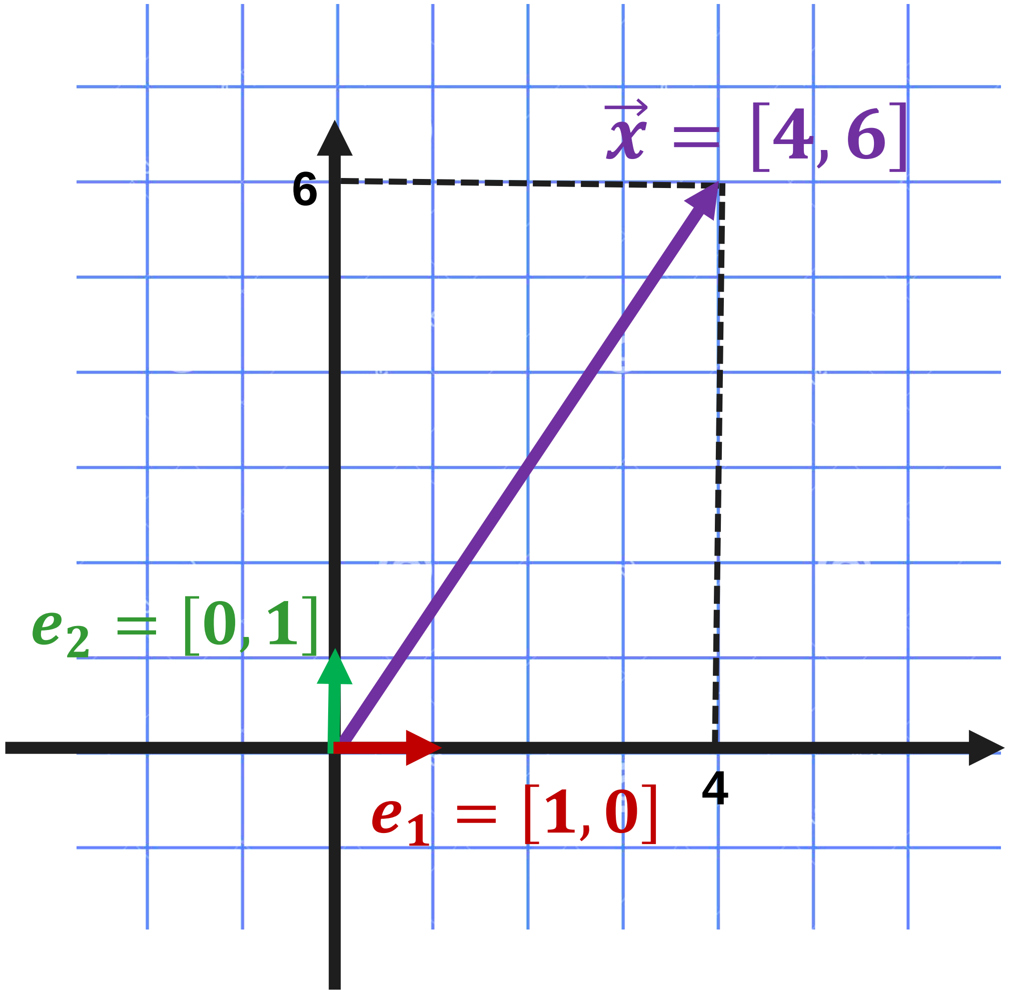
- $x = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$



Coordinates: Example

- Consider $\mathbb{R}^2$.
- $x = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$
- Canonical basis:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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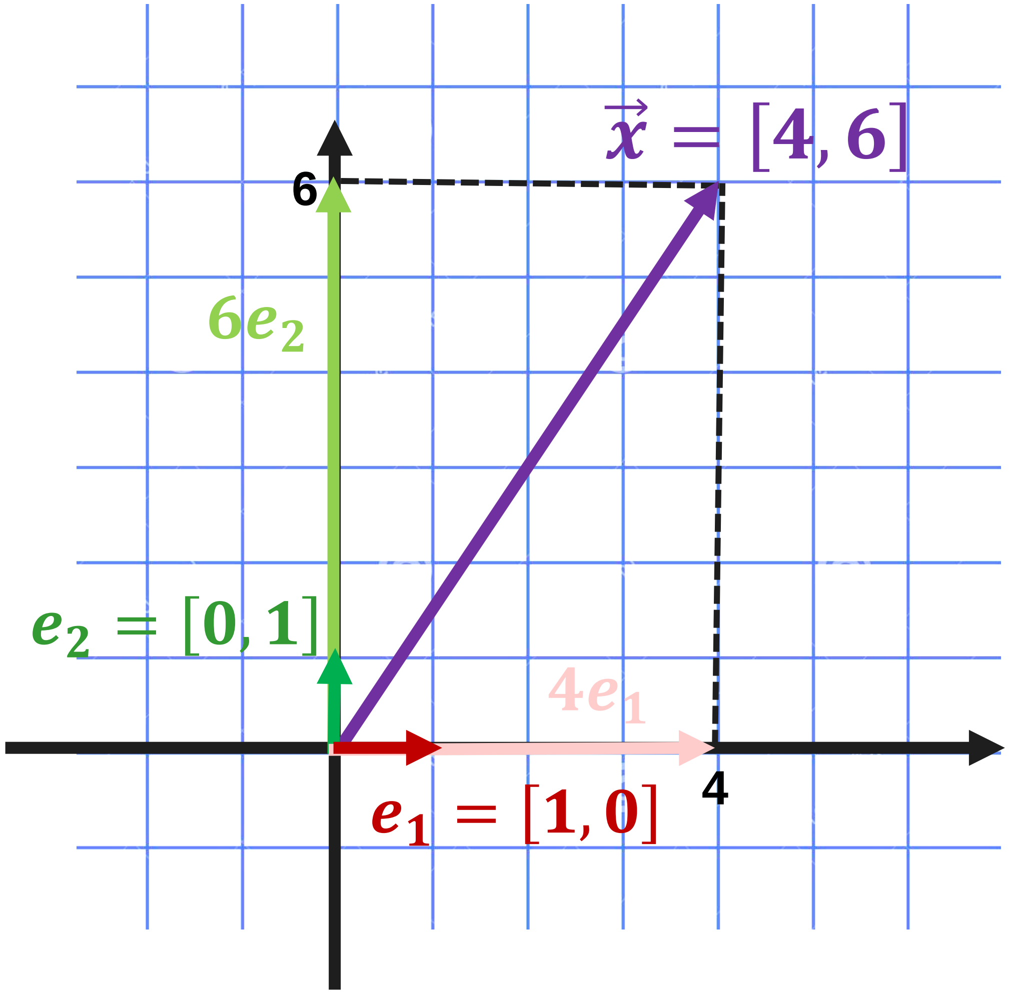
$$\boldsymbol{x} = 4\boldsymbol{e_1} + 6\boldsymbol{e_2}$$



Orthogonal Basis

- A basis $e_1, \ldots, e_n$ where $\forall i, j \;\; e_i \perp e_j$ is an *orthogonal basis*.

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  Example:

  $$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \dots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$ is an orthonormal basis of $\mathbb{R}^n$.

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- *Gram-Schmidt process*: a way to convert any basis to an orthogonal one. More details: practical session.

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Change of Basis

Basis

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  - $b = \left\{ b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ - yet another one.

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  - $b = \left\{ b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ - yet another one.
- Different basis = different coordinates.
  How exactly do they change?

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with canonical basis

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- What are the coordinates in the new basis?

$$x_{new} = ?$$

Coordinate Change: General Case

- Consider a vector space V with basis $e_1, e_2, \dots e_n$.

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- Imagine vector $x = [x_1, x_2, \dots, x_n] \in V$
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- New basis: $e'_1, e'_2, \dots e'_n$.

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  $x_1, x_2, \dots, x_n$ - coordinates in basis $e_1, e_2, \dots e_n$.
- New basis: $e'_1, e'_2, \dots e'_n$.
- What are the coordinates of x in this new basis?

$$x'_1, x'_2, \dots, x'_n = ?$$

Coordinate Change

- Old basis: $e_1, e_2, \dots e_n$
  New basis: $e'_1, e'_2, \dots e'_n$
- $x_{old} = [x_1, x_2, \dots, x_n]$, $x_{new} = [x'_1, x'_2, \dots, x'_n] = ?$
- Coordinates of the new basis in the old one:

Coordinate Change

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  New basis: $e'_1, e'_2, \dots e'_n$

- $x_{old} = [x_1, x_2, \dots, x_n]$, $\quad x_{new} = [x'_1, x'_2, \dots, x'_n\,] =?$

- Coordinates of the new basis in the old one:

$$
\begin{gathered}
e'_1 = \boldsymbol{\alpha_{11}} e_1 + \boldsymbol{\alpha_{21}} e_2 + \cdots + \boldsymbol{\alpha_{n1}} e_n \\
e'_2 = \boldsymbol{\alpha_{12}} e_1 + \boldsymbol{\alpha_{22}} e_2 + \cdots + \boldsymbol{\alpha_{n2}} e_n \\
\vdots \\
e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n \\
\vdots \\
e'_n = \boldsymbol{\alpha_{1n}} e_1 + \boldsymbol{\alpha_{2n}} e_2 + \cdots + \boldsymbol{\alpha_{nn}} e_n
\end{gathered}
$$

Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

Coordinate Change

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Remember: $e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

Remember: $e_i' = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

$$= x'_1 \cdot (\boldsymbol{\alpha_{11} e_1} + \boldsymbol{\alpha_{21} e_2} + \cdots + \boldsymbol{\alpha_{n1} e_n}) + \cdots + x_i' \cdot (\boldsymbol{\alpha_{1i} e_1} + \boldsymbol{\alpha_{2i} e_2} + \cdots + \boldsymbol{\alpha_{ni} e_n}) + \cdots$$
$$+ x_n' (\boldsymbol{\alpha_{1n} e_1} + \boldsymbol{\alpha_{2n} e_2} + \cdots + \boldsymbol{\alpha_{nn} e_n}) =$$

Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

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$e_1, \dots, e_n$ linearly independent -> coefficients in front of them should be the same on the both sides of the equality:

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Coordinate Change

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Coordinate Change

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$$\boldsymbol{x_{new}}$$

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$\boldsymbol{e'_i}$

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Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ \ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

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$$\begin{aligned} 2 &= 2x'_1 - 1x'_2 \\ -1 &= 1x'_1 - 1x'_2 \end{aligned}$$

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Coordinate Change

- Going from one basis to the other:

$\boldsymbol{e'_i}$

$$
\boldsymbol{x_{old}} \qquad
\begin{aligned}
x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\
x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\
&\ \ \vdots \\
x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn}
\end{aligned}
$$

$\boldsymbol{x_{new}}$

Coordinate Change

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$$\mathbf{x}_{new}$$

- There is a more compact way of writing this down using matrices.

Matrices

A Matrix

- $A \in \mathbb{R}^{m \times n}$ - a matrix with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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- *Examples:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j\right)$

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- Symmetric matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ij} = a_{ji}\right)$
- Triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $\left(a_{ij} = 0\ \forall i > j\ or\ \ \forall i < j\right)$

Vectors vs Matrices

- An n-dimensional vector can be considered a $n \times 1$ matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Operations with Matrices

Transpose of a Matrix

- Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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- Transpose = writing columns as rows:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, \dots, x_n]$$

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Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

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Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

- Transposing a symmetrical matrix = no changes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

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Multiplying by a Scalar

- We can multiply matrix by a scalar:

$$\lambda A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

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- Example:

$$5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Sum of Two Matrices

- We can sum up matrices of the same size:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

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- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

Matrices Also Form a Vector Space!

- $(\mathbb{R}^{m\times n}, +, \ \cdot)$ - a vector space.
  "Vectors" = matrices.

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- You can check yourself that the necessary axioms hold.

Matrix Multiplication

- Consider two matrices $A = \{a_{ij}\}_{m \times n}$ and $b = \{b_{ij}\}_{n \times p}$.
- $C = AB$ – product of two matrices.

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- Example $\mathbb{R}^{2\times 2}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Matrix Multiplication: Example

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 6 & 4 \\ 2 & 5 & 8 \\ 7 & 1 & 9 \end{bmatrix} =$$

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$$= \begin{bmatrix} 16 & 7 & 26 \\ 43 & 22 & 61 \end{bmatrix}$$

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Coordinate Change: Matrix Notation

- Result obtained before:

$e_1, \dots, e_n$ - old basis
$e'_1, \dots, e'_n$ - new basis

$$x_{old} = [x_1, \dots, x_n], \qquad x_{new} = [x'_1, \dots, x'_n]$$

$\boldsymbol{e'_i}$

$$\boldsymbol{x_{old}} \quad \begin{array}{l} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \qquad\qquad\qquad \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{array}$$

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  $x_{old} = [x_1, \ldots, x_n], \qquad x_{new} = [x'_1, \ldots, x'_n]$

$\boldsymbol{e'_i}$

$$\boldsymbol{x_{old}} \quad \begin{array}{l} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{array}$$

$\boldsymbol{x_{new}}$

- Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{bmatrix}$$

Coordinate Change: Matrix Notation

- Result obtained before:

$e_1, \dots, e_n$ - old basis
$e'_1, \dots, e'_n$ - new basis

$x_{old} = [x_1, \dots, x_n], \qquad x_{new} = [x'_1, \dots, x'_n]$

$\boldsymbol{e'_i}$

$$\boldsymbol{x_{old}} \quad \begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

$\boldsymbol{x_{new}}$

- Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

$$x_{old} = A^T x_{new}$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix},$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A^T x_{new} =$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A^T x_{new} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} =$$

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Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

- New basis: $e'_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $e'_2 = \begin{bmatrix}-1\\-1\end{bmatrix}$

- $x_{old} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$, $x_{new} = \begin{bmatrix}x'_1\\x'_2\end{bmatrix} = ?$

$$A = \begin{bmatrix}2 & -1\\1 & -1\end{bmatrix}, \qquad \begin{bmatrix}2\\-1\end{bmatrix} = x_{old} = A^T x_{new} = \begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}x'_1\\x'_2\end{bmatrix} = \begin{bmatrix}2x'_1 + x'_2\\-x'_1 - x'_2\end{bmatrix}$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A^T x_{new} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 2x'_1 + x'_2 \\ -x'_1 - x'_2 \end{bmatrix}$$

$$x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

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To Sum Up

- Vector spaces
  - Linear (in)dependence
  - Span
  - Basis
- Matrices
  - Matrix operations
  - Change of coordinates

Next Time

- More on matrices
- Systems of linear equations